TECHNICAL RESEARCH REPORT

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Bilinear Systems

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T.R. 96-63

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January 11, 1994

1   Introduction

The study of robust nonlinear control has attracted increasing interest over the last few years. Progress has been aided by the recent entension [FM91, Jam92] of the linear quadratic results [Jac73, Whi81] linking the theories of L_2 gain control (nonlinear H_∞ control), differential games, and the stochastic risk sensitive control. Most of the previous research conducted in the area of robust nonlinear control has focused on the case where full state information is available. Thus, previously little attention has been given to the problem of robust nonlinear control via output feedback. In this paper we address the problem of robust H_∞ output feedback control for the special case of bilinear systems.

As a generalization of the results from linear theory, the solution to the output feedback problem has been postulated to involve a nonlinear observer combined with a controlled dissipation inequality for an augmented system. By postulating such a structure and solving an augmented game problem, several researchers [BHW91, IA92, vdS93] have established results yielding sufficient conditions for the existence of a solution to the output feedback robust control problem. In [JBE93c] a theoretical solution is obtained for the finite-horizon partially observed dynamic game problem for discrete-time nonlinear systems. The approach taken there is motivated by ideas from stochastic control; in fact, the controller is obtained as an asymptotic limit of the controller for the risk sensitive stochastic control problem. In [JB93] this work is extended to the infinite horizon case and stability issues are addressed. In the latter paper a purely deterministic viewpoint is maintained. Both papers present results which are necessary and sufficient. The continuous time problem can be solved, in principle, using a similar approach [JBE93a]; however, these results have not yet been proved rigorously for general systems.

The novelty of the approach of [JBE93a, JBE93c] is the reformulation of the partially observed dynamic game problem as a fully observed problem in terms of an *information state*. The information state is derived

using only past outputs and is determined as the solution of a dynamic programming equation. Thus using this approach, the original problem is restated in terms of an equivalent problem in which there is full knowledge of a new state. This new state is called the *information state*. Although common in stochastic control, this type of separation approach had not been previously applied to deterministic game problems. For linear systems, the information state can be related to the "past stress" used by Whittle [Whi81] in solving the risk sensitive problem for linear systems (see also [Whi91]). Başar and Bernhard [BB91] have also employed the "past stress" in their solution to the game problem.

In this paper we address the problem of robust control of bilinear systems using output feedback. This paper is organized as follows. In Section 2, a statement of the robust output feedback control problem for bilinear systems is given. In Sections 3 and 4 respectively descriptions are given for the optimal solution of the problem due to James, Baras, and Elliott, and the Certainty Equivalence solution of the problem due to Başar and Bernhard. The implementations of these two solutions are compared in Section 5. The goal of the comparison is to determine under what conditions the Certainty Equivalence Control (CEC) is optimal. In addition, examples are given which demonstrate that the optimal controller is stabilizing and robust to input noise as well as modeling noise.

2 Problem Statement

Consider the class of bilinear systems described by the following set of state space equations

$$
\Sigma \quad \begin{cases} \dot{x}(t) &=& [A + \sum_{i=1}^{m} B_i u_i(t)]x(t) + w(t), \qquad x(t_0) = x_0, \\[2mm] y(t) &=& Cx(t) + v(t), \\[2mm] z(t) &=& \begin{pmatrix} \sqrt{Q}x(t) \\ \sqrt{R}u(t) \end{pmatrix}. \end{cases} \tag{2.1}
$$

We assume that the state $x(t) \in \mathbb{R}^n$ is not directly measurable and the initial condition $x_0 \in \mathbb{R}^n$ is unknown. All knowledge of the state is restricted to be obtained through the observations $y(t) \in \mathbb{R}^p$. The additional output $z(t) \in \mathbb{R}^{n+m}$ is a performance measure. The parameters $Q = Q' > 0$ and $R = R' > 0$ are design parameters which depend on the specific control system design objectives. The admissible controls $u = [u_1, ...u_m]'$ are restricted to take values in $U = \mathbb{R}^m$ and to be non-anticipating functions of the observation path y. Let $\mathcal{O}_{t_0, t_f}$ denote the class of all such controls on the interval $[t_0, t_f]$. For notational convenience, we define $Bu \stackrel{\triangle}{=} \sum_{i=1}^{m} B_i u_i(t)$. The disturbances w and v are assumed to be finite energy signals on the interval $[t_0, t_f]$, i.e., $w \in L^2([t_0, t_f], \mathbb{R}^n)$ and $v \in L^2([t_0, t_f], \mathbb{R}^p)$.

The Robust H_∞ Output Feedback Control Problem is to determine an admissible control u such that the resulting closed loop system Σ^u has finite L_2 gain from the disturbance inputs w, v to the performance output z. This is stated precisely in Problem 2.1.

Problem 2.1 Robust H_∞ Output Feedback Control
Given $\mu > 0$, find a control $u \in \mathcal{O}_{t_0, t_f}$ such that for all initial conditions $x_0 \in \mathbb{R}^n$, there exists $\beta_{t_f}^u$ finite such that

$$
||z||^2 \leq \frac{1}{\mu}(||w||^2 + ||v||^2) + \beta_{t_f}^u
$$

for all $w, v \in L_2([t_o, t_f])$.

Here we chose to use the gain parameter μ in order to remain consistent with the risk sensitive stochastic

control problem from which our solution was derived. The more common gain parameter γ, consistent with the linear H_∞ problem, is related to μ by $\gamma^2 = \frac{1}{\mu}$.

The solution to the robust H_∞ output feedback control problem is equivalent to the solution to a related zero sum game problem [FM91, Jam92]. Consider the cost functional

$$J(u) \triangleq J_{t_f}(u) = \sup_{w,v \in L^2} \sup_{x_0 \in \mathbb{R}^n} \{ \quad \bar{p}(x_0) + \frac{1}{2} \int_{t_0}^{t_f} (<x(s), Qx(s)> + <u(s), Ru(s)>$$
$$-\frac{1}{\mu}(||w(s)||^2 + ||v(s)||^2))ds + \Phi(x(t_f)) \}. \tag{2.2}$$

It can be shown that Σ^u is finite gain on $[t_0, t_f]$ if and only if there exists a finite quantity $\beta_{t_f}^u$ such that

$$J_t(u) \leq \beta_{t_f}^u \qquad \forall t \in [t_0, t_f].$$

Problem 2.2 Bilinear Partially Observed Game
Given $\mu > 0$, under the assumptions

(A1) there exists $\mu > 0$ and $u \in \mathcal{O}_{t_0, t_f}$ such that $J(u)$ is finite,

(A2) the initial cost satisfies

$$\bar{p}(x) = \bar{\phi} - \frac{1}{2\mu} <(x - \bar{x}), \bar{P}^{-1}(x - \bar{x})>$$

where $\bar{x} \in \mathbb{R}^n$, $\bar{P} \in \mathbb{R}^{n \times n} > 0$ and $\bar{\phi} \in \mathbb{R}$, and

(A3) the terminal cost $\Phi(x) \geq 0$ is locally Lipschitz continuous with at most quadratic growth,

find $u \in \mathcal{O}_{t_0, t_f}$ to minimize the cost functional for the bilinear systems described by equation (2.1).

By making use of the equivalence of Problems 2.1 and 2.2 we are able to use all the established tools of optimal control theory.

3   Optimal Controller

The information state can be thought of as a deterministic *sufficient statistic* in that it contains all the information needed to control the system with respect to the given performance measure. The information state is *not* a state estimator. For Problem 2.2 the information state p is given by

$$p_t(x) = \sup_{w \in L^2} \sup_{x_0 \in \mathbb{R}^n} \{ \quad \bar{p}(x_0) + \frac{1}{2} \int_{t_0}^{t} (<x(s), Qx(s)> + <u(s), Ru(s)>$$
$$-\frac{1}{\mu}(||w(s)||^2 + ||Cx(s) - y(s)||^2))ds : x(t) = x \}$$

where past observations and controls $\{u(s), y(s) : s \in [t_0, t]\}$ are known. From the definition it is clear that the information state is the cost accumulated up to the time t, consistent with the available information at time t, and assuming the state at time t is x. Thus the information state evaluates the accrued cost for

each possible state x at time t. Using dynamic programming it can be shown that the information state is the solution to the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial p_t}{\partial t} &= F(p_t, u(t), y(t)) \\ p_0 &= \bar{p}, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F(p, u, y) &\triangleq& -\sup_{w \in \mathbf{R}^n} \{ <\nabla_x p, (A + Bu)x + w> +\tfrac{1}{2\mu}(||w||^2 + ||Cx - y||^2) \\ && -\tfrac{1}{2}(<x, Qx> + <u, Ru>) \} \\ &=& -<\nabla_x p, (A + Bu)x> +\tfrac{\mu}{2}||\nabla_x p||^2 + \tfrac{1}{2}(<x, Qx> + <u, Ru>) - \tfrac{1}{2\mu}||Cx - y||^2. \end{aligned}$$

The maximizing disturbance in this case is $\hat{w} = -\mu \nabla_x p$.

The cost (equation (2.2)) can be rewritten in terms of the information state

$$J(u) = \sup_{y \in L^2} (p_{t_f}, \Phi)$$

where the "sup pairing" $(\cdot, \cdot)$ is defined by

$$(p, q) \triangleq \sup_{x \in \mathbf{R}^n} \{ p(x) + q(x) \}.$$

Thus the value of the game is given by

$$W(p, t) = \inf_{u \in \mathcal{O}_{t, t_f}} \sup_{y \in L^2} \{ (p_{t_f}, \Phi) : p_t = p \}.$$

Applying dynamic programming, we can determine an optimal control $u(t)$ which is a function of the past observations $\{ y(s) : s \in [t_0, t] \}$ through the information state p. The dynamic programming equation is (c.f., [JBE93b])

$$\begin{cases} \frac{\partial W}{\partial t} + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \{ <\nabla_p W, F(p, u, y)> \} = 0 \\ W(p, t_f) = (p, \Phi) \end{cases}$$

The optimal control is given by the minimizing value of u in this equation.

In general the information state is infinite dimensional, i.e., $p_t = p_t(x)$ evolves in a general class of functions which cannot be parameterized by finite dimensional quantities. In [Jam93] James describes a class of nonlinear systems for which the special form of the dynamics and cost function allow the information state to be described in terms of finite dimensional quantities. Not only do bilinear systems with quadratic cost fall into this class, in this case the information state is also a quadratic. The exact form of the quadratic is explored in Theorem 3.6.

The fact that the quadratic information state satisfies equation (3.3) can be shown by direct differentiation and substitution. Before we show this, however, we must first define some pertinent operators and present a lemma. Let A, $B \in \mathbb{R}^{n \times m}$ where $A = [a_{ij}]$ and $B = [b_{ij}]$.

Definition 3.3 *The matrix dot product* $[\cdot, \cdot] : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \longrightarrow \mathbb{R}$ *is defined by*

$$[A, B] \triangleq \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}.$$

4

Definition 3.4 *The derivative of a function* $W(\cdot) : \mathbb{R}^{n \times m} \longrightarrow \mathbb{R}$ *with respect to its matrix argument is defined by*

$$\nabla_A W \triangleq \left[\frac{\partial W}{\partial a_{ij}}\right].$$

Thus $\nabla_A W$ is the matrix of first partial derivatives with respect to the elements of A.

Lemma 3.5 *Suppose* $A, B \in \mathbb{R}^{n \times n}$ *and A is symmetric, then*

(i) $\nabla_A < x, A^{-1}x > = -A^{-1}xx'A^{-1}$, *and*

(ii) $[\nabla_A < x, A^{-1}x >, B] = - < A^{-1}x, BA^{-1}x >$.

Theorem 3.6 *For Problem 2.2, the information state is given by*

$$p_t(x) = \phi(t) - \frac{1}{2\mu} < x(t) - \widehat{x}(t), P^{-1}(t)(x(t) - \widehat{x}(t)) >$$

where $P = P' > 0$ *and* $\widehat{x}(t)$, $P(t)$ *and* $\phi(t)$ *satisfy the ODE's*

$$
\begin{aligned}
\dot{\widehat{x}}(t) &= (A + Bu(t) + \mu P(t)Q)\widehat{x}(t) + P(t)C'(y(t) - C\widehat{x}(t)) \\
\widehat{x}(0) &= \bar{x}
\end{aligned}
$$

$$
\begin{aligned}
\dot{P}(t) &= P(t)(A + Bu(t))' + (A + Bu(t))P(t) - P(t)(C'C - \mu Q)P(t) + I \\
P(0) &= \bar{P}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\phi}(t) &= \tfrac{1}{2}(< \widehat{x}(t), Q\widehat{x}(t) > + < u(t), Ru(t) > -\tfrac{1}{\mu}\|y(t) - C\widehat{x}(t)\|^2) \\
\phi(0) &= \bar{\phi}
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\text{LHS} \;\triangleq\; & \frac{\partial p}{\partial t} \\
= \; & < \nabla_{\widehat{x}}p, \dot{\widehat{x}} > + [\nabla_P p, \dot{P}] + < \nabla_\phi p, \dot{\phi} > \\
= \; & \tfrac{1}{\mu} < P^{-1}(x - \widehat{x}), (A + Bu + \mu PQ)\widehat{x} + PC'(y - C\widehat{x}) > \\
& + \tfrac{1}{2\mu} < (x - \widehat{x})'P^{-1}, (P(A + Bu)' + (A + Bu)P - P(C'C - \mu Q)P + I)P^{-1}(x - \widehat{x}) > \\
& + \tfrac{1}{2}(< \widehat{x}, Q\widehat{x} > + < u, Ru > -\tfrac{1}{\mu}\|y - C\widehat{x}\|^2) \\
= \; & \tfrac{1}{\mu}(x - \widehat{x})'P^{-1}(A + Bu)x + \tfrac{1}{2\mu}(x - \widehat{x})'P^{-2}(x - \widehat{x}) \\
& + \tfrac{1}{2}(< x, Qx > + < u, Ru > -\tfrac{1}{\mu}\|y - Cx\|^2)
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} \;\triangleq\; & - < \nabla_x p, (A + Bu)x > + \tfrac{\mu}{2}\|\nabla_x p\|^2 + \tfrac{1}{2}(< x, Qx > + < u, Ru > -\tfrac{1}{\mu}\|y - Cx\|^2) \\
= \; & \tfrac{1}{\mu}(x - \widehat{x})'P^{-1}(A + Bu)x + \tfrac{1}{2\mu}(x - \widehat{x})'P^{-2}(x - \widehat{x}) \\
& + \tfrac{1}{2}(< x, Qx > + < u, Ru > -\tfrac{1}{\mu}\|y - Cx\|^2) \\
= \; & LHS
\end{aligned}
$$

∎

Theorem 3.6 implies that for bilinear systems with quadratic cost the information state can be identified with the finite dimensional quantity $\rho = (\widehat{x}, P, \phi)$. Thus the output feedback robust control problem is equivalent to a new finite dimensional state feedback game with state $\rho = (\widehat{x}, P, \phi)$. We now regard the value function W as a function of ρ:

$$W(\rho, t) = W(\widehat{x}, P, \phi, t) = W(p_\rho, t),$$

where $p_\rho = \phi - \frac{1}{2\mu} < x - \widehat{x}, P^{-1}(x - \widehat{x}) >$. The dynamic programming equation is now

$$
\left\{
\begin{array}{ll}
\frac{\partial W}{\partial t} & + \ \sup_{y \in \mathbf{R}^p} \inf_{u \in U} \{ < \nabla_{\widehat{x}} W, (A + Bu(t) + \mu P(t)Q)\widehat{x}(t) + P(t)C'(y(t) - C\widehat{x}(t)) > \\[2mm]
& + \ [\nabla_P W, P(t)(A + Bu(t))' + (A + Bu(t))P(t) - P(t)(C'C - \mu Q)P(t) + I] \\[2mm]
& + \ \frac{1}{2}(< \widehat{x}(t), Q\widehat{x}(t) > + < u(t), Ru(t) >) - \frac{1}{2\mu}\|y(t) - C\widehat{x}(t)\|^2 \} = 0 \\[2mm]
W(\rho, t_f) & = \ (p_{\rho_{t_f}}, \Phi)
\end{array}
\right.
\tag{3.4}
$$

where y plays the role of a competing disturbance. Note that since Isaacs condition is satisfied the order in which the inf and sup are applied is inconsequential.

An interesting and novel feature of this problem is that the value function need not be finite for all values of ρ, t. In the linear case, this is closely related to the coupling condition [YJ93]. Let us write

$$\mathcal{D} = \{ (\widehat{x}, P, \phi, t) \in \mathbb{R}^n \times \mathbf{S}^n \times \mathbb{R} \times [0, T] \ : \ W(\widehat{x}, P, \phi, t) \text{ is finite } \}.$$

In general, $\mathcal{D}$ is a nontrivial subset of $R^n \times \mathbf{S}^n \times \mathcal{R} \times [0, T]$; see the numerical examples in Section 5.

The partially observed game problem can now be solved using the dynamic programming equation (3.4) [Fri71, BB91, FH93], as stated in the following theorem.

Theorem 3.7 (Verification) *Assume there exists a smooth solution $W \in C^1(\mathcal{D})$ of the Hamilton-Jacobi equation (3.4). Then the control $u^*(\rho, t)$ which attains the infimum in (3.4) defines an optimal controller $u^* \in \mathcal{O}_{t_0, t_f}$ which minimizes the cost functional (2.2) for the partially observed game. In this case u^* is given by*

$$u^*(\rho, t) = -R^{-1}k(\rho, t)$$

where k is defined by

$$k_i(\rho, t) \overset{\triangle}{=} \ < \nabla_{\widehat{x}} W(\rho, t), B_i \widehat{x} > + [\nabla_P W(\rho, t), PB_i'] + [\nabla_P W(\rho, t), B_i P], \qquad i = 1, \cdots m$$

and the optimal control at time t is $u_t^* = u^*(\rho_t, t)$.

In general, the value function need not be C^1, and equation (3.4) must be interpreted in a generalized sense. This is typically the case in optimal control and game theory. However, it can happen that W is smooth in certain regions, as in the next theorem.

[This theorem may have to be restated. However, it is not as important as the other theorems.]

Theorem 3.8 *There exists $t' \in (t_0, t_f]$ (possibly small) such that the solution W to the Hamilton-Jacobi equation (3.4) is smooth for $t \in [t_0, t']$. Thus, by the Verification Theorem, u^* is optimal on $[t_0, t']$.*

Proof: ...using characteristic equations ■

6

4   Certainty Equivalence

Systems with finite dimensional information state are important because they allow direct implementation of the optimal controller. For problems which have an information state which is not finite dimensional direct implementation is impossible. However, theoretical knowledge of the optimal solution can be used for guidance in the design of a suboptimal controller. In this paper we examine a particular suboptimal controller, the Certainty Equivalence Controller (CEC) of Başar and Bernhard. We compare the optimal controller obtained in Section 3 with the CEC in an effort to investigate the optimality of the CEC.

For the case of linear systems it has been shown that the Certainty Equivalence Controller (CEC) is optimal [BB91]. For nonlinear systems, some general conditions have been found under which a Certainty Equivalence Principle (CEP) [Whi81, BB91, JBE93a] holds, i.e., the CEC is optimal. The CEC is defined in terms of a minimum stress estimate $\bar{x}_t$ and the optimal state feedback controller $\tilde{u}(x,t)$:

$$u_{CE}(t) = \tilde{u}(\bar{x}_t, t). \tag{4.5}$$

The value function of the full state feedback game is the solution to the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial V}{\partial t} &= -\inf_u \sup_w \{< \nabla_x V, (A+Bu)x + w > + \frac{1}{2}(< x, Qx > + < u, Ru > -\frac{1}{\mu}||w||^2)\} \\ V_{t_f} &= \Phi, \end{cases} \tag{4.6}$$

and $\tilde{u}(x,t)$ is the value of u achieving the minimum in (4.6), assuming V is sufficiently smooth. Indeed, $\tilde{u}(x,t) = -R^{-1}\tilde{k}(x,t)$, where $\tilde{k}_i(x,t) \overset{\triangle}{=} < \nabla_x V(x,t), B_i x >$, $i = 1, \cdots m$, and the optimal disturbance is $\tilde{w}(x,t) = \mu \nabla_x V(x,t)$. The *minimum stress estimate* [Whi81] of the state is given by

$$\bar{x}(\rho, t) = \arg \max_x \{p_\rho(x) + V_t(x)\}. \tag{4.7}$$

Thus $\bar{x}_t = \bar{x}(\rho_t, t)$. Note that in general, $\bar{x}$ is set valued.

When the minimum stress estimate is unique and the information state and value function are sufficiently smooth, it can be shown by applying the verification principle that the CEC is optimal. This gives an alternative proof to the ones given in [BB91, DBB93].

Theorem 4.9 *If*

(i)   the minimum stress estimate $\bar{x}(\rho, t)$ is unique for all $(\rho, t) \in \mathcal{D}$, and

(ii)   the full state information value function V satisfying (4.6) is continuously differentiable,

then the function

$$W_{CE}(\rho, t) = (p_\rho, V_t) \tag{4.8}$$

is a solution to the dynamic programming equation (3.4) and the Certainty Equivalence Controller (4.5) is optimal.

Proof:

Consider $(\rho, t) \in \mathcal{D}$, then by assumption (i) the minimum stress estimate $\bar{x}$ is unique. This together with assumption (ii), that the full state information value function V is continuously differentiable, allow us to equate $W_{CE} = p_\rho(\bar{x}) + V_t(\bar{x})$ and to differentiate naturally with $\bar{x}$ as a parameter.

By first order optimality condition $\nabla_x V(\bar{x}) = -\nabla_x p(\bar{x}) = \frac{1}{\mu} P^{-1}(\bar{x} - \hat{x})$. First calculate y^* and u^* when $W = W_{CE}$.

$$y^* = C\hat{x} + \mu C P \nabla_{\hat{x}} W_{CE} = C\bar{x}$$

$$u^*(\rho, t) = -R^{-1}k(\rho, t)$$

where k is defined by

$$
\begin{aligned}
k_i(\rho, t) &= <\nabla_{\hat{x}} W_{CE}(\rho,t), B_i \hat{x}> +[\nabla_P W_{CE}(\rho,t), P B_i'] + [\nabla_P W_{CE}(\rho,t), B_i P], \qquad i = 1, \cdots m \\
&= \frac{1}{\mu} < P^{-1}(\bar{x} - \hat{x}), B_i \hat{x}> + \frac{1}{2\mu} < P^{-1}(\bar{x} - \hat{x}), (B_i P)P^{-1}(\bar{x} - \hat{x})> \\
&\quad + \frac{1}{2\mu} < P^{-1}(\bar{x} - \hat{x}), (PB_i')P^{-1}(\bar{x} - \hat{x})> \\
&= \frac{1}{\mu} < P^{-1}(\bar{x} - \hat{x}), B_i \bar{x}> \\
&= k_i(\bar{x}, t)
\end{aligned}
$$

Thus when $W = W_{CE}$,

$$u^*(\rho, t) = \tilde{u}(\bar{x}) = u_{CE}.$$

Now by direct differentiation and substitution it can be shown that $W_{CE} = V(\bar{x}) + p(\bar{x})$ satisfies equation (3.4). Now

$$
\begin{aligned}
\text{LHS} &\overset{\triangle}{=} \frac{\partial W_{CE}}{\partial t} \\
\\
&= \frac{\partial V}{\partial t}(\bar{x}) \\
\\
&= -\frac{1}{\mu} < P^{-1}(\bar{x} - \hat{x}), (A + B\tilde{u}(\bar{x}))\bar{x} + P^{-1}(\bar{x} - \hat{x})> -\frac{1}{2}(<\bar{x}, Q\bar{x}> + <\tilde{u}(\bar{x}), R\tilde{u}(\bar{x})>) \\
&\quad + \frac{1}{2\mu} < (\bar{x} - \hat{x}), P^{-2}(\bar{x} - \hat{x})>
\end{aligned}
$$

and also

$$
\begin{aligned}
\text{RHS} &\overset{\triangle}{=} - <\nabla_{\hat{x}} W_{CE}, (A + Bu^* + \mu PQ)\hat{x} + P(t)C'(y^*(t) - C\hat{x}(t))> \\
&\quad -[\nabla_P W_{CE}, P(A + Bu^*)' + (A + Bu^*)P - P(C'C - \mu Q)P + I] \\
&\quad -\frac{1}{2}(<\hat{x}, Q\hat{x}> + <u^*, Ru^*>) + \frac{1}{2\mu}\|y^* - C\hat{x}\|^2 \\
\\
&= -\frac{1}{\mu} < P^{-1}(\bar{x} - \hat{x}), (A + Bu^* + \mu PQ)\hat{x} + P(t)C'(y^*(t) - C\hat{x}(t))> \\
&\quad -\frac{1}{2\mu} < P^{-1}(\bar{x} - \hat{x}), (P(A + Bu^*)' + (A + Bu^*)P - P(C'C - \mu Q)P + I)P^{-1}(\bar{x} - \hat{x})> \\
&\quad -\frac{1}{2}(<\hat{x}, Q\hat{x}> + <u^*, Ru^*>) + \frac{1}{2\mu}\|y^* - C\hat{x}\|^2 \\
\\
&= -\frac{1}{\mu} < P^{-1}(\bar{x} - \hat{x}), (A + Bu^*)\bar{x} + P^{-1}(\bar{x} - \hat{x})> -\frac{1}{2}(<\bar{x}, Q\bar{x}> + <u^*, Ru^*>) \\
&\quad + \frac{1}{2\mu} < (\bar{x} - \hat{x}), P^{-2}(\bar{x} - \hat{x})>
\end{aligned}
$$

Thus since $u^* = \tilde{u}(\bar{x})$ when $W = W_{CE}$ we are done. ∎

The conditions given in Theorem 4.9 are essentially those of [BB91], and are difficult to verify in general. The key difficulty is the uniqueness of the minimum stress estimate.

[Is it possible to prove uniqueness of $\bar{x}$ in some cases? Presumably V will be smooth for small times.???]
....

....

5 Examples

In this section we present several examples of systems which are designed to demonstrate two different properties of the solution. First, we consider a bilinear system which is unstable without control in order to demonstrate that the optimal controller is stablizing and robust to input noise as well as modeling noise. Second, we consider a bilinear system which is stable without control in order to clarify the issue of the domain $\mathcal{D}$ of the optimal value function W. In addition we address the issue of determining when the CEC is optimal.

6 Conclusions

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