ABSTRACT

Title of dissertation: KOTTWITZ'S NEARBY CYCLES CONJECTURE FOR A CLASS OF UNITARY SHIMURA VARIETIES

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This thesis proves that the nearby cycles complex on a certain family of PEL local models is *central* with respect to the convolution product of sheaves on the corresponding affine flag variety. As a corollary, the semisimple trace function defined using the action of Frobenius on that nearby cycles complex is, via the sheaf-function dictionary, in the center of the corresponding Iwahori-Hecke algebra. This is commonly referred to as *Kottwitz's conjecture*. The reductive groups associated to the PEL local models under consideration are unramified unitary similitude groups with even dimension. The proof follows the method of [15].

KOTTWITZ'S NEARBY CYCLES CONJECTURE FOR A CLASS OF UNITARY SHIMURA VARIETIES

by

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Chapter 0

Introduction

The object of study in this thesis is a certain projective $\mathbf{Z}_p$-scheme $\mathbf{M}^{\mathrm{loc}}$, called a *local model*, and the *nearby cycles complex* $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$ on $\mathbf{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$, a certain complex of étale ℓ-adic sheaves.

The purpose of local models is to give étale-local descriptions of various Shimura varieties in a way that uses only module-theoretic language and makes some questions and computations more tractable. A major step in computing the *Hasse-Weil zeta function* of a Shimura variety is the computation of the trace of the Frobenius element considered as a linear map on the stalks of the nearby cycles complex $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$ on $\mathbf{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$. In this thesis, I prove that the nearby cycles complexes on a certain class of local models coming from unitary-type division algebras are *central* with respect to a convolution product of étale ℓ-adic sheaf complexes. A corollary is that the trace function associated to the Frobenius element as above is a specific, effectively computable basis element in the center of the corresponding Iwahori-Hecke algebra. This is known as *Kottwitz's conjecture.* A description of the local models considered and a precise statement of the theorem appears below.

Let $\mathbb{A}_{\mathrm{f}}$ be the finite adeles over $\mathbf{Q}$. Let $\mathbf{G}$ be a linear algebraic group defined over $\mathbf{Q}$, let $h : \mathbf{C}^\times \to \mathbf{G}_{\mathbf{R}}$ be an algebraic cocharacter and let $\mathbf{K} \subset \mathbf{G}(\mathbb{A}_{\mathrm{f}})$ be a compact open subgroup. The triple $(\mathbf{G}, h, \mathbf{K})$ (under certain additional hypotheses) is called a *Shimura datum* and can be used to construct a Shimura variety Sh. This Shimura variety is defined over some number field $\mathbf{E}$, called the *reflex field*. Some Shimura varieties, for example those whose datum comes from a PEL datum (in particular, the case considered in this thesis), have an integral model, i.e. a scheme $\mathbf{Sh}$ over $\mathcal{O}_{\mathbf{E}}$ such that $\mathbf{Sh}_{\mathbf{E}}$ is the original Shimura variety Sh. The fibers over primes $\mathfrak{p} \subset \mathcal{O}_{\mathbf{E}}$ of such an integral model $\mathbf{Sh}$ are sometimes smooth (in which case $\mathbf{Sh}$ is said to have *good reduction* at $\mathfrak{p}$) and sometimes non-smooth (in which case $\mathbf{Sh}$ is said to have *bad reduction* at $\mathfrak{p}$). I now fix a prime $\mathfrak{p}$ and consider only PEL ("polarization, endomorphisms, level-structure") Shimura varieties with *parahoric* level-structure at $\mathfrak{p}$. Rapoport and Zink [30] constructed local models of many integral Shimura varieties $\mathbf{Sh}$ within an axiomatic framework. In some cases, the objects constructed by [30] were found to be unsatisfactory; some examples where $\mathbf{M}^{\mathrm{loc}}$ is not a flat scheme were provided by Pappas [23] in the ramified unitary case and Genestier in the even-dimensional orthogonal case. Modifications were made by Pappas and Rapoport in subsequent papers [24], [25], and [27], and evidence that these modifications produce flat models is supplied by Smithling's recent papers [35], [33] and [34], which specifically address both of the problematic examples previously mentioned. Nonetheless, it follows from [10] that the local models I consider (described below) are flat.

The Hasse-Weil zeta function $Z(s, Sh)$ is defined as a product over all primes $\mathfrak{p} \subset \mathbf{E}$ of local factors $Z_{\mathfrak{p}}(s, Sh)$, each of which is an alternating product of determinants involving a lift Frob to $\overline{\mathbf{Q}}_p$ of the Frobenius automorphism acting on $Sh_{\overline{\mathbf{Q}}_p}$. More precisely, one selects an arbitrary preimage Frob $\in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{E}_{\mathfrak{p}})$ of the Frobenius element in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_{\mathfrak{p}})$ and, via its action on $Sh_{\overline{\mathbf{Q}}_p}$, Frob induces an action on the inertia-fixed-points of the global étale ℓ-adic cohomology of $Sh_{\overline{\mathbf{Q}}_p}$.

I now briefly discuss the nearby cycles construction $\mathrm{R}\Psi$. Let X be a $\mathbf{Z}_p$-scheme and let $X_{\mathbf{Q}_p} \xrightarrow{p} X \xleftarrow{q} X_{\mathbf{F}_p}$ and $X_{\overline{\mathbf{Q}}_p} \xrightarrow{c} X_{\mathbf{Q}_p}$ be the canonical maps. Then the nearby cycles complex $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$ on $X_{\overline{\mathbf{F}}_p}$ is defined to be the derived complex $\overline{q}^*(R\overline{p}_*(c^*(\overline{\mathbf{Q}}_\ell)))$ (in the étale context), where $\overline{\mathbf{Q}}_\ell$ represents the constant étale ℓ-adic sheaf on $X_{\mathbf{Q}_p}$. The nearby cycles construction is a tool to transform the problem of calculating cohomology of $X_{\overline{\mathbf{Q}}_p}$ into the problem of calculating cohomology of $X_{\overline{\mathbf{F}}_p}$: if X is a *proper* $\mathbf{Z}_p$-scheme, then $\mathbb{H}^\bullet(X_{\overline{\mathbf{Q}}_p}; c^*(\overline{\mathbf{Q}}_\ell)) = \mathbb{H}^\bullet(X_{\overline{\mathbf{F}}_p}; \mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell))$, and the action by $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on the left is consistent with the action by $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on the right. This equality is a corollary of *base change for proper morphisms* (noting the trivial nature of the nearby cycles construction on the diagram $\mathrm{Spec}(\mathbf{Q}_p) \to \mathrm{Spec}(\mathbf{Z}_p) \leftarrow \mathrm{Spec}(\mathbf{F}_p)$).

If $\mathbf{Sh}$ is proper over $\mathcal{O}_{\mathbf{E}}$ then the previous paragraph and the Grothendieck-Lefschetz trace formula show that the local factor $Z_{\mathfrak{p}}(s, Sh)$ of the Hasse-Weil zeta function can be calculated by knowing the trace of all powers of Frob on the cohomology stalks over $Sh(\mathbf{F}_{\mathfrak{p}})$ of $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$. By the étale-local equivalence mentioned above, it is the same to calculate that trace on $\mathbf{M}^{\mathrm{loc}}(\mathbf{F}_{\mathfrak{p}})$ instead. The alternating sum of these traces defines a function $\tau : \mathbf{M}^{\mathrm{loc}}(\mathbf{F}_{\mathfrak{p}}) \to \overline{\mathbf{Q}}_\ell$. The inertia-fixed-points operation makes the trace function τ difficult to understand in general, and Rapoport suggests replacing τ by a similar function τ^{ss}, called the *semisimple trace function.* Assuming Deligne's weight-monodromy conjecture, τ can be reconstructed from τ^{ss}. See §2 of [29] for details of this reconstruction. It is important to note that the determination of τ^{ss} (but *not* the local factor $Z_{\mathfrak{p}}(s, Sh)$ as a whole) is *trivial* in the case of good reduction, since the nearby cycles complex is the constant sheaf. See Lemma 8.6 and the summary Theorem 10.1 of properties of the nearby cycles functor in [14] for more details.

Haines and Ngô [15] consider the split groups GL and GSp and the standard local models corresponding to these groups. They prove an instance of Kottwitz's conjecture, that the semisimple trace function τ^{ss} in this situation is essentially the Bernstein basis function z_μ in the center of the corresponding Iwahori-Hecke algebra (here μ is a certain cocharacter occurring in the precise definition of the local model, which is omitted here). In fact, [15] proves more–that every member of a family of functions, each of which is defined similarly to τ^{ss}, is a specific linear combination of Bernstein basis functions; see Theorem 11 in [15] for a precise statement. The strategy of the proof is:

1. Construct an ind-scheme $\mathbf{M}$ over $\mathbf{Z}_p$, which contains $\mathbf{M}^{\mathrm{loc}}$ as a closed subscheme and whose extension to $\mathbf{Q}_p$, resp. to $\mathbf{F}_p$, is the is the affine Grassmannian, resp. full affine flag variety. This requires finding alternate descriptions of $\mathbf{M}^{\mathrm{loc}}$ that are more compatible with the usual definitions of affine flag varieties as unions of sets of lattice-chains.

2. Via the embedding of $\mathbf{M}^{\mathrm{loc}}_{\mathbf{F}_p}$ into the full affine flag variety, prove that the semisimple trace function τ^{ss} is an element of the Iwahori-Hecke algebra $\mathcal{H}$ and construct products $*_{\overline{\mathbf{Q}}_p}$ and $*_{\overline{\mathbf{F}}_p}$ of complexes of étale ℓ-adic sheaves on $\mathbf{M}_{\overline{\mathbf{Q}}_p}$ and $\mathbf{M}_{\overline{\mathbf{F}}_p}$ such that (via the sheaf-function dictionary) $*_{\overline{\mathbf{F}}_p}$ categorifies

the convolution product in $\mathcal{H}$.

3. Show that the nearby-cycles functor $\mathrm{R}\Psi$ is a "homomorphism" with respect to these two products and that the product of the relevant complexes on $\mathbf{M}_{\overline{\mathbf{Q}}_p}$ is commutative. It follows that the product on $\mathbf{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$ of the relevant complexes is also commutative.

On the other hand, Gaitsgory [9] proves a similar result (albeit not in the context of Shimura varieties) for split connected reductive $\mathbf{F}_p((t))$-groups G. One of the objects occurring in [9] is an ind-scheme $\mathbf{Fl}_X$, reportedly due to Beilinson, defined over a smooth curve X such that one fiber is the full affine flag variety for G and every other fiber is essentially the affine Grassmannian for G; see Proposition 3 of [9] for a precise statement. The main result of [9] is that the nearby cycles functor on $\mathbf{Fl}_X$ induces the isomorphism (the composition of the Satake and Bernstein isomorphisms) from the special parahoric Hecke algebra $\mathcal{H}(G(\mathbf{F}_p((t))); G(\mathbf{F}_p[[t]]))$ to the center of the Iwahori-Hecke algebra; see Theorem 1 in [9] for a precise statement.

I now describe the specific contents of this thesis. The Shimura data that I consider are similar to those occuring in Kottwitz [20], except I consider *Iwahori* level structure rather than *hyperspecial maximal* level structure. Let $F \supset \mathbf{Q}$ be an imaginary quadratic extension with ring of integers $\mathcal{O}$. Let D be a central division F-algebra and suppose that D has a unitary (2nd kind) involution $*$. To this pair $(D, *)$ is attached a certain similitude group $\mathbf{G}$ defined over $\mathbf{Q}$. Let $2 \neq p \in \mathbf{Z}$ be a prime for which $G = \mathbf{G}_{\mathbf{Q}_p}$ is quasi-split and split over ${\mathbf{Q}_p}^{\mathrm{unr}}$. Since the case when p *splits* in $\mathcal{O}$ is known (see Haines [14]), I assume that p is *inert* in $\mathcal{O}$. After selecting a minuscule cocharacter μ, one can define the local model $\mathbf{M}^{\mathrm{loc}}$, an $\mathcal{O}_E$-scheme for a certain extension $E/\mathbf{Q}_p$ which is called the *reflex field* and depends on the $G(\overline{\mathbf{Q}}_p)$-conjugacy class of μ. By inertness, $F_p = F \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a *field*, the completion of F at p, and $D \otimes_{\mathbf{Q}} \mathbf{Q}_p = M_d(F_p)$. It follows that G is a unitary similitude group (possibly not quasi-split) and is quasi-split if and only if the involution $*_p$ is isomorphic to the standard one. See §1.2.2 for more details. Moreover, the reflex field E must be either F_p or $\mathbf{Q}_p$, and since F_p is also the splitting field of G, the case of $E = F_p$ reduces to the case of GL and I may assume without loss of generality that $E = \mathbf{Q}_p$. This also implies that the dimension d is *even* and that the *signature* of μ is $(d/2, d/2)$. See §1.2.3 for more details.

Görtz's idea that $\mathbf{M}^{\mathrm{loc}}$ can frequently be embedded into an appropriate affine flag variety holds in this case and the semisimple trace function τ^{ss} on $\mathbf{M}^{\mathrm{loc}}(\mathbf{F}_p)$ can therefore be interpreted as an element of the Iwahori-Hecke algebra $\mathcal{H}$ of GU_d. Kottwitz's conjecture is that this element of $\mathcal{H}$ is a certain scalar multiple of the Bernstein basis function z_μ associated to μ. By Haines's characterization (Theorem 5.8 in [13]) of minuscule Bernstein basis functions, Kottwitz's conjecture (in the case at hand) follows from the main theorem of this thesis:

Main Theorem. *Suppose $2 \neq p \in \mathbf{Z}$ is* inert *in $\mathcal{O}$ and let $\mathrm{M}^{\mathrm{loc}}$ be the local model over $\mathbf{Z}_p$ associated to the unitary-type division algebra datum $(D, *, \mu)$ as above, and suppose that the similitude group attached to $(D, *)$ is quasi-split.*

Then $\mathbf{M}^{\mathrm{loc}}$ is isomorphic to the standard local model corresponding to the (unramified) unitary similitude group GU_d associated to the extension $F_p/\mathbf{Q}_p$ and the *nearby cycles complex* $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$ *on* $\mathbf{M}^{\mathrm{loc}}_{\overline{\mathbf{F}}_p}$*, considered as a complex on the full affine flag variety* $\mathcal{F}\ell^{\mathrm{aff}}_{\overline{\mathbf{F}}_p}$ *of* GU_d*, is* central *with respect to the convolution product* $*$ *of sheaf complexes, i.e.* $\mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell) * \mathcal{C}^\bullet \cong \mathcal{C}^\bullet * \mathrm{R}\Psi(\overline{\mathbf{Q}}_\ell)$ *naturally for every perverse Iwahori-equivariant complex of étale* ℓ*-adic sheaves* $\mathcal{C}^\bullet$ *on* $\mathcal{F}\ell^{\mathrm{aff}}_{\overline{\mathbf{F}}_p}$.

Via the sheaf-function dictionary, the associated semisimple trace function τ^{ss} *is therefore a* central *element of the Iwahori-Hecke algebra of* GU_d*.*

Kottwitz's conjecture, whenever it holds, allows τ^{ss} to be computed explicitly: the Bernstein basis functions can be computed in a systematic way using only some well-known information about linear algebraic groups and Coxeter groups. This can be done on a computer or even by hand, in low rank cases.

Upon completion of this thesis, Pappas and Zhu released a preprint [28] which proves Kottwitz's nearby cycles conjecture in all cases where the group is unramified. This includes the unitary group cases considered in this thesis. This thesis and [28] constitute the first proofs of Kottwitz's conjecture in the *non-split* case.

I now give an outline of the thesis.

In Chapter 1, I choose the objects that will eventually be used to construct the local models, set some conventions, and recall various classical results about simple algebras, hermitian forms, involutions, etc. I recall and also prove some commutative algebra lemmas that are used throughout the thesis.

In Chapter 2, I recall the definition of the local model as it appears in [30] and rephrase parts of the definition in equivalent ways that are more obviously related to affine flag varieties. I analyze what happens to these conditions after applying

Morita equivalence to change the target categories of $\mathbf{M}^{\mathrm{loc}}$ from $M_d(\mathcal{O})$-Modules to $\mathcal{O}$-Modules. I define an ind-scheme $\mathbf{M}$, prove some basic properties about it, and prove that it is a degeneration from the affine Grassmannian over $\mathbf{Q}_p$ to the full affine flag variety over $\mathbf{F}_p$. I define an ind-group $\mathbf{J}$ which acts on $\mathbf{M}$ and which is similarly an interpolation between the special parahoric over $\mathbf{Q}_p$ to the Iwahori over $\mathbf{F}_p$. I prove that the subgroups comprising $\mathbf{J}$ are *smooth*, which is critical in order for the semisimple trace function to be an element of the Iwahori-Hecke algebra. I also give Schubert cell decompositions for the subschemes comprising $\mathbf{M}$.

In Chapter 3, I set notation and conventions for the sheaf theory I will use, I recall the precise definition for the semisimple trace function τ^{ss} that is the subject of this thesis, and I verify that τ^{ss} can be interpreted as an element of the Iwahori-Hecke algebra for GU_d. I then restate (without proof) the main theorem, prove that $\mathbf{M}^{\mathrm{loc}}$ is a flat scheme (using [10]), and show how Kottwitz's conjecture follows from the centrality of the nearby cycles complexes. Following a well-known general recipe from [22], I define several objects and morphisms, the totality of which is commonly referred to as a "convolution diagram", and prove various properties (representability, finite-type, etc.) about those objects and morphisms. I define an ind-group $\widetilde{\mathbf{J}}$, similar in spirit to $\mathbf{J}$, which acts on some objects in the convolution diagram, and I prove several critical and non-trivial facts about the action of $\widetilde{\mathbf{J}}$ on the objects in the convolution diagram. I also mention some important simplifications that occur over $\mathbf{Q}_p$ and $\mathbf{F}_p$. Finally, I use the convolution diagram to define a product operation between complexes of étale ℓ-adic sheaves, and I explain why this product induces the usual convolution product in the Hecke algebra.

In Chapter 4, I use the material from Chapter 3 to prove the main theorem.

The main difficulties in this proof occur in §2.5 and §3.3. A collection of properties (involving connectedness, smoothness, and transitivity) related to the groups $\mathbf{J}$ and $\widetilde{\mathbf{J}}$ are necessary in order to construct the convolution product and needed to be proved from scratch.

Chapter 1

Notations, conventions, and preliminary setup

1.1 Notation and conventions

Let F be a totally imaginary quadratic number field. Let $\mathcal{O}$ be the ring of integers in F. Let D be a central division F-algebra such that $\dim_F(D) = d^2$. Let $* : D \to D$ be a unitary (or "2nd kind") involution. Let $p \in \mathbf{Z}$ be a prime that is *inert* in $\mathcal{O}$. Denote by $\mathbf{F}_p$ the finite field with p elements. Fix field embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. I frequently use the following simple and related facts:

- $F_p \overset{\text{def}}{=} F \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is an unramified quadratic extension of $\mathbf{Q}_p$ and the automorphism of F_p induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$ is the same as that of the non-trivial element of $\mathrm{Gal}(F_p/\mathbf{Q}_p)$.

- $D_p \overset{\text{def}}{=} D \otimes_{\mathbf{Q}} \mathbf{Q}_p = D \otimes_F F_p$ is a central simple F_p-algebra, and $*$ induces a unitary involution $*_p$ on D_p.

- $\mathbf{F} \overset{\text{def}}{=} \mathbf{F}_p \otimes_{\mathbf{Z}} \mathcal{O}$ is the residue field of F_p, and the automorphism of $\mathbf{F}$ induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$ is the same as that of the non-trivial element of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_p)$.

All rings are assumed to have 1. I denote by $\mathbf{G}^{\mathrm{a}}$ and $\mathbf{G}^{\mathrm{m}}$ the additive and multiplicative algebraic groups $R \mapsto (R, +)$ and $R \mapsto (R^\times, \cdot)$. If R is any ring, the category of *commutative* R-algebras is denoted R-Algebras. I call any R-algebra that is also a field an "R-field". By a "geometric point" of an R-scheme X, I mean a K-point where K is a separably-closed R-field. I usually make no distinction between a morphism of schemes and its associated functor-of-points. In the later portion of this paper, I frequently use the trivial observation that, for a (commutative) ring R, any property that holds "Zariski-locally on $\mathrm{Spec}(R)$" holds, period, if R is a *local* ring.

Let K/k be a separable quadratic field extension with non-trivial Galois automorphism $x \mapsto \overline{x}$. The standard involution $X \mapsto \overline{X}^{\mathrm{tr}}$ on K-matrices is sometimes called $*_{\mathrm{std}}$. A function $\phi : K^d \times K^d \to K$ is a "K/k-hermitian form" if and only if it is k-bilinear and satisfies $\phi(xv, w) = x\phi(v, w) = \phi(v, \overline{x}w)$ for all $v, w \in K^d$, $x \in K$. I frequently denote by $\mathrm{id}^\vee$ the "anti-identity" matrix, the matrix with 1 in the anti-diagonal entries and 0 in all other entries, with dimension implied by context.

Assumption. $p \neq 2$.

Remark. *This assumption is used in the proof of Proposition 1.2.2.2 (page 15) and Proposition 2.5.2.2 (page 74)*

1.2 Description of the PEL datum

1.2.1 The global PEL datum

The starting point is to form a global PEL datum (B, ι, V, ψ) using the pair $(D, *)$. Consider the following data:

- finite-dimensional simple $\mathbf{Q}$-algebra $B \overset{\text{def}}{=} D^{\text{opp}}$

- *positive* involution $\iota : B \to B$ defined by

$$\iota(b) \overset{\text{def}}{=} \xi \cdot b^* \cdot \xi^{-1}$$

  for certain $\xi \in D$ satisfying $\xi^* = -\xi$ (see §5.2 of [14] for existence of such an element)

- finite-dimensional *left* B-module $V \overset{\text{def}}{=} D$ with B acting by *right*-multiplication:

$$b \star v \overset{\text{def}}{=} vb$$

- alternating $\mathbf{Q}$-bilinear form $\psi : V \times V \to \mathbf{Q}$ defined by

$$\psi(v, w) \overset{\text{def}}{=} \mathrm{Tr}^{\text{red}}_{D/\mathbf{Q}}(v \cdot \xi \cdot w^*)$$

  which automatically satisfies $\psi(b \star v, w) = \psi(v, \iota(b) \star w)$.

Remark. *Two* different *products $D \times D \to \mathbf{Q}$ as above can induce the* same *involution ι on B, due to the fact that ψ induces an involution on* $\mathrm{End}_{F\text{-lin}}(D)$*, of which B is only a proper subalgebra.*

From this data one can define an affine algebraic group G: for any (commutative) $\mathbf{Q}$-algebra R, set

$$G(R) \overset{\text{def}}{=} \{g \in D \otimes_{\mathbf{Q}} R \mid \exists c(g) \in R^\times \text{such that } \psi_R(g-, g-) = c(g)\psi_R(-, -)\}$$

(the "D" used to define G is really $\mathrm{End}_{B\text{-lin}}(V) = D$, i.e. D acting by *left*-multiplication on $V = D$; this action is $B = D^{\text{opp}}$-linear because of associativity of multiplication in D)

By definition of ψ (noting the previous parenthetical comment), another description of G is

$$G(R) = \{x \in D \otimes_{\mathbf{Q}} R \mid g^{*\otimes\mathrm{id}} \cdot g \in R^{\times}\}$$

I select also an $\mathbf{R}$-algebra homomorphism

$$h : \mathbf{C} \to \mathrm{End}_{B\text{-lin}}(V) \otimes_{\mathbf{Q}} \mathbf{R} = D \otimes_{\mathbf{Q}} \mathbf{R}$$

and require that h satisfies

- $h(\overline{z}) = h(z)^{*\otimes\mathrm{id}}$
- $B \otimes_{\mathbf{Q}} \mathbf{R} \to B \otimes_{\mathbf{Q}} \mathbf{R}$ defined by $b \mapsto h(i)^{-1} \cdot b^{*\otimes\mathrm{id}} \cdot h(i)$ is a *positive* involution

Any h satisfying the first property can be used to define a cocharacter

$$\mu = \mu_h : \mathbf{G}^{\mathrm{m}}_{\mathbf{C}} \to G_{\mathbf{C}}.$$

See §1.2.3 (page 21) for details. Set $B_p \stackrel{\mathrm{def}}{=} B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and similarly for $D, V, *, \iota, \psi$. Let $\mathcal{O}_B \subset B = D^{\mathrm{opp}}$ be a maximal order such that $\mathcal{O}_B \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is a maximal order $\mathcal{O}_{B_p}$ in B_p.

Assumption. $\iota_p(\mathcal{O}_{B_p}) = \mathcal{O}_{B_p}$.

This standard assumption guarantees that if Λ is an $\mathcal{O}_{B_p}$-submodule of V_p, then the dual module

$$\widehat{\Lambda} \stackrel{\mathrm{def}}{=} \{x \in V_p \mid \psi_p(\Lambda, x) \subset \mathbf{Z}_p\}$$

is again an $\mathcal{O}_{B_p}$-module. I also use this assumption in the proof of Proposition 1.2.2.2 (page 15).

I will also need to select a certain chain $(\cdots \subset \Lambda_0 \subset \Lambda_1 \subset \cdots)$ of $\mathcal{O}_{B_p}$-lattices in $V_p = D_p$. See §2.1.1 (page 28) for more details.

The datum used to define the local model is the tuple

$$(B, \iota, V, \psi, \mu, \mathcal{O}_B, F, \{\Lambda_i\}_{i \in \mathbf{Z}})$$

1.2.2 Theorems about unitary involutions, hermitian forms, etc.

Proposition 1.2.2.1. *The central simple F_p-algebra D_p is split, i.e. $D_p \cong M_d(F_p)$ as F_p-algebras.*

Proof. By Wedderburn's theorem, the central simple F_p-algebra D_p is a matrix algebra over some central division F_p-algebra. Since D_p has a unitary involution, Corollary 8.8.3 on page 306 of [31] states that this division algebra has a unitary involution also. But Albert's theorem (Theorem 10.2.2(ii) on page 353 of [31]) states that, over a local field, the only division F_p-algebra with a unitary involution is F_p itself. $\square$

Remark. *This result is explicitly part of Landherr's theorem (Theorem 10.2.4 on page 355 of [31]). In the above proof, the assumption that* p *is inert guarantees that $D \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a* simple *algebra.*

This means that $G_{\mathbf{Q}_p}$ is *always* a unitary similitude group, although depending on $*$, perhaps $G_{\mathbf{Q}_p}$ is not quasi-split.

Using Proposition 1.2.2.1, fix some isomorphism $D_p \cong M_d(F_p)$, and consider $*_p$ and ι_p as unitary involutions on $B_p = M_d(F_p)^{\mathrm{opp}}$ via this isomorphism.

In §2.2.2 (page 35), I will use the following proposition:

Proposition 1.2.2.2. *There exists an F_p-algebra automorphism (necessarily inner by Skolem-Noether) of $B_p = M_d(F_p)^{\mathrm{opp}}$ that identifies ι_p with $*_{\mathrm{std}}$ and $\mathcal{O}_{B_p}$ with $M_d(\mathcal{O}_{F_p})^{\mathrm{opp}}$.*

Proof. This is actually just Theorem 10.2.5 on page 355 of [31] but it is not clear just from the statement ("almost all primes"), so I make some additional comments.

Let K/k be a quadratic extension of global fields with non-trivial Galois automorphism $x \mapsto \overline{x}$. Let A be a central simple K-algebra ($\dim_K(A) = n^2$), and I a unitary involution on A. The assertion of Theorem 10.2.5 is that for all but finitely many primes $\mathfrak{p}$ of k, there is a K-algebra isomorphism $A \otimes_K K_{\mathfrak{p}} \cong M_n(K_{\mathfrak{p}})$ such that I becomes identified to $*_{\mathrm{std}}$.

A careful reading of the proof shows that, for a particular $\mathfrak{p}$, such an isomorphism exists provided that:

1. $\mathfrak{p}$ is non-archimedean
2. $\mathrm{char}(\mathcal{O}_k/\mathfrak{p}) \neq 2$
3. $\mathfrak{p}$ is not ramified in K
4. $A \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathfrak{p}}$ is split, i.e. a matrix algebra
5. I stabilizes a maximal order in $A \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathfrak{p}}$

The key point is that the initial setup of this proof is unnecessarily restrictive: [31] chooses a single *global* order $\Lambda \subset A$, creates the I-stable order $\Lambda \cap I(\Lambda)$, considers only those $\mathfrak{p}$ for which the completion at $\mathfrak{p}$ of this new I-stable order is *maximal*, and notes that there are only finitely many exceptions. Really, all that is needed is that for each $\mathfrak{p}$ (with only finitely many exceptions), there is some $I_{\mathfrak{p}}$-stable maximal order (possibly depending on $\mathfrak{p}$).

My situation assumes (1) and (3), I have explicitly assumed (2) and (5), and Proposition 1.2.2.1 provides (4), so the first part of the proposition is proven.

It is obvious from the proof that the isomorphism sends the I-stable maximal order to $M_d(\mathcal{O}_{F_p})$. □

Fix an isomorphism

$$(B_p, \iota_p, \mathcal{O}_{B_p}) \cong (M_d(F_p)^{\mathrm{opp}}, *_{\mathrm{std}}, M_d(\mathcal{O}_{F_p})^{\mathrm{opp}})$$

as in Proposition 1.2.2.2.

Recall the following classification theorem for unitary involutions:

Classification of Unitary Involutions (Theorem 8.7.4 on pages 301-302 of [31]). *Let K/k be a quadratic extension with non-trivial Galois automorphism $x \mapsto \overline{x}$. Let A be a central simple K-algebra and fix a K/k-unitary involution I.*

1. *if $b \in A^{\times}$ satisfies $b = I(b)$, then the function $\mathrm{Inn}(b) \circ I$ is a K/k-unitary involution*

2. *if $J : A \to A$ is a K/k-unitary involution, then there is an $b \in A^{\times}$ satisfying $b = I(b)$ such that $J = \mathrm{Inn}(b) \circ I$, and this b is unique up to scalar in $k^{\times}$*

3. *there exists an isomorphism $(A, \mathrm{Inn}(a) \circ I) \xrightarrow{\sim} (A, \mathrm{Inn}(b) \circ I)$ if and only if there exists $c \in A$ and $\alpha \in K$ such that $b = \alpha(c \cdot a \cdot I(c))$*

Recall also the correspondence between unitary involutions and hermitian forms in the split-algebra case (by Proposition 1.2.2.1 (page 14), this is the case of interest):

Correspondence between Unitary Involutions and Hermitian Forms. *Let K/k be a quadratic extension with non-trivial Galois automorphism $x \mapsto \overline{x}$ and let $\phi_H : K^d \times K^d \to K$ be the (necessarily K/k-hermitian) form defined by*

$$\phi_H(v, w) \overset{\text{def}}{=} v^{\text{tr}} \cdot H \cdot \overline{w}$$

and let $*_H : M_d(K) \to M_d(K)$ be the (necessarily K/k-unitary) involution defined by

$$X^{*_H} \overset{\text{def}}{=} H \cdot \overline{X}^{\text{tr}} \cdot H^{-1}$$

***Assertion**:*

1. *any K/k-hermitian form $K^d \times K^d \to K$ is can be written as ϕ_H for some H satisfying $\overline{H}^{\text{tr}} = H$*

2. *any K/k-unitary involution $M_d(K) \to M_d(K)$ can be written as $*_H$ for some H satisfying $\overline{H}^{\text{tr}} = H$*

3. *the involution induced by ϕ_H on $M_d(K)$ is exactly $*_H$*

4. *the function $\phi_H \mapsto *_H$ descends to a bijection between isometry classes of K/k-hermitian forms*

$$K^d \times K^d \longrightarrow K$$

and K/k-unitary involutions

$$M_d(K) \longrightarrow M_d(K)$$

Assertions (1) and (3) are trivial. Assertion (2) is a special case of the "Classification of Unitary Involutions". To verify (4), first note that if the vector space K^d is transformed by $A \in \mathrm{GL}_d(K)$ then the hermitian form ϕ_H is transformed into

$$(v,w) \longmapsto (A(v))^{\mathrm{tr}} \cdot H \cdot \overline{A(w)} = v^{\mathrm{tr}} \cdot (A^{\mathrm{tr}} \cdot H \cdot \overline{A}) \cdot \overline{w}$$

In other words, ϕ_H is transformed into $\phi_{A^{\mathrm{tr}} \cdot H \cdot \overline{A}}$. Second, note that if $M_d(K)$ is transformed by $\mathrm{Inn}(A)$, then the involution $*_H$ is transformed into

$$X \longmapsto A(H(\overline{A^{-1}XA})^{\mathrm{tr}}H^{-1})A^{-1} = (AH\overline{A}^{\mathrm{tr}})\overline{X}^{\mathrm{tr}}(AH\overline{A}^{\mathrm{tr}})^{-1}$$

In other words, $*_H$ is transformed to into $*_{A \cdot H \cdot \overline{A}^{\mathrm{tr}}}$. This proves that the function $\phi_H \mapsto *_H$ descends to a function from isometry classes of hermitian forms to isomorphism classes of unitary involutions. By parts (2) and (3) of this Correspondence, the function is *surjective*. By part (3) of the "Classification of Unitary Involutions", it is *injective*: if two hermitian forms induce two involutions that are isomorphic, then part (3) of the Classification guarantees a certain element "c", and this element can be applied to K^d in order to transform one hermitian form into the other.

When $k = \mathbf{Q}_{p'}$ (including $p' = 2$), these classifications can be made much more specific and it is well-known (see §10.6.5 of [31]) that for each dimension there are exactly 2 isometry classes of hermitian forms, and if that dimension is *even* (as in my situation), then these hermitian forms yield 2 *non-isomorphic* unitary similitude groups, only one of which is quasi-split (if the dimension is *odd*, then the unitary groups associated to the two hermitian forms are isomorphic and quasi-split). It is clear that whether or not $G_{\mathbf{Q}_p}$ is quasi-split depends on $*_p$. I address this question at the end of this subsection.

I recall a correspondence between certain bilinear forms occurring frequently in the theory of PEL local models and certain hermitian forms. Fix an element $\zeta \in K$ such that $\overline{\zeta} = -\zeta$. Let ${}^0\psi : K^d \times K^d \to k$ be an alternating k-bilinear form. I call such a ${}^0\psi$ *internally hermitian* if and only if for all $A \in M_d(K)$ and $v, w \in K^d$ it is true that

$$ {}^0\psi(A(v), w) = {}^0\psi(v, \overline{A}^{\mathrm{tr}}(w)) $$

I use the following correspondence in §2.3 (page 45):

Extraction of Hermitian Forms. *The function*

$$ {}^0\phi \longmapsto \{(v, w) \mapsto \mathrm{Tr}_{K/k}(\zeta \cdot {}^0\phi(v, w))\} $$

is a bijection between K/k-hermitian forms $K^d \times K^d \to K$ and internally-hermitian alternating k-bilinear forms $K^d \times K^d \to k$. The inverse function is

$$ {}^0\psi \longmapsto \left\{(v, w) \mapsto \frac{\zeta^{-1} \cdot {}^0\psi(v, w) + {}^0\psi(\zeta^{-1} \cdot v, w)}{2}\right\} $$

Fix such a hermitian form ${}^0\phi$. Note that the involution ^{0*} induced by ${}^0\phi$ on $M_d(K)$ is the same as the one induced by ${}^0\psi$.

Let ${}^0G$ be the usual similitude group associated to ^{0*}, the functor assigning to each (commutative) k-algebra R the group

$$ {}^0G(R) \overset{\mathrm{def}}{=} \{g \in M_d(K \otimes_k R) \mid g^{{}^0* \otimes \mathrm{id}} \cdot g \in R^\times\} $$

The following lemma will be used in §2.3 (page 45) to simplify and concretize the description of certain lattice-chains:

Lemma 1.2.2.3. *Consider the following 4 statements:*

1. *${}^0G$ is quasi-split*

2. *there is a K-algebra automorphism of $M_d(K)$ transforming ${}^0{*}$ into the standard unitary involution $X \mapsto \overline{X}^{\mathrm{tr}}$*

3. *there is a K-linear automorphism of K^d transforming ${}^0\phi$ into the standard K/k-hermitian form $(v, w) \mapsto v^{\mathrm{tr}} \cdot \overline{w}$*

4. *the involution ${}^0{*}$ on $M_d(K)$ stabilizes a maximal order*

When $k = \mathbf{Q}_{p'}$ and $p' \neq 2$, these are equivalent.

Proof. $\boxed{(1) \Leftarrow (2)}$ This is obvious from the definition of ${}^0G$. $\boxed{(2) \Leftrightarrow (3)}$ This is immediate from the above "Correspondence between Unitary Involutions and Hermitian Forms". $\boxed{(2) \Rightarrow (4)}$ This is trivial: by the Skolem-Noether theorem, ${}^0{*}$ stabilizes the maximal order $H \cdot M_d(\mathcal{O}) \cdot H^{-1}$ for some H. $\boxed{(1) \Rightarrow (3)}$ This follows from the discussion following the above "Classification of Unitary Involutions": in the case of $k = \mathbf{Q}_{p'}$, the isomorphism class of the quasi-split unitary similitude group corresponds to the isometry class of the standard hermitian form. $\boxed{(2) \Leftarrow (4)}$ This is just Proposition 1.2.2.2 (page 15). $\square$

Remark. *Of course, several of the implications in the lemma are true without one or both assumptions.*

Because of this lemma, the following assumption will simplify the description of the local model considerably:

Assumption. $G_{\mathbf{Q}_p}$ is quasi-split.

Remark. *This is a contextual assumption; see the introduction. See §2.3 (page 45) for the application of this assumption and the lemma.*

1.2.3 Determining the reflex field

Let $D_{\mathbf{C}} \stackrel{\text{def}}{=} D \otimes_{\mathbf{Q}} \mathbf{C}$ and let D_+ and D_- be the $+i$ and $-i$ eigenspaces in $D_{\mathbf{C}}$ of the linear operator $h(i)$. Since h is $\mathbf{R}$-linear, the minimal polynomial of $h(i)$ divides $T^2 + 1$ and so $D_{\mathbf{C}} = D_+ \oplus D_-$. Note that acting by $h(z)$ on $D_{\mathbf{C}}$ is the same as acting by $(z, \overline{z})$ on $D_+ \oplus D_-$.

Define $h_{\mathbf{C}} : \mathbf{C} \times \mathbf{C} \to D_{\mathbf{C}}$ to be the composite

$$\mathbf{C} \times \mathbf{C} \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{h \otimes \mathrm{id}} (D \otimes_{\mathbf{Q}} \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = D_{\mathbf{C}}$$

Note that $h(z) \in G(\mathbf{R})$ and $h_{\mathbf{C}}(z, 1) \in G(\mathbf{C})$ for $z \in \mathbf{C}^\times$.

Define $\mu(z) \stackrel{\text{def}}{=} h_{\mathbf{C}}(z, 1)$. As an operator on $D_{\mathbf{C}}$, this $\mu(z)$ acts by z on D_+ and by 1 on D_-. In particular, μ is *minuscule*. The set of all μ coming from all the h in a $G(\mathbf{R})$-conjugacy class is a $G(\mathbf{C})$-conjugacy class of cocharacters $\mathbf{G}^{\mathrm{m}}_{\mathbf{C}} \to G_{\mathbf{C}}$, and this conjugacy class is defined over some finite extension $\mathbf{E} \supset \mathbf{Q}$, called the *reflex field*.

On the other hand, if $\epsilon, \overline{\epsilon}$ are the two embeddings $F \hookrightarrow \mathbf{C}$

$$D_{\mathbf{C}} \xrightarrow{\sim} (D \otimes_{\epsilon} \mathbf{C}) \times (D \otimes_{\overline{\epsilon}} \mathbf{C})^{\mathrm{opp}}$$

$$d \otimes z \longmapsto (d \otimes z, d^* \otimes z)$$

(as $\mathbf{C}$-algebras) and of course each of these $D \otimes_F \mathbf{C}$ is isomorphic as an $\mathbf{R}$-algebra to $M_d(\mathbf{C})$, with the $\mathbf{C}$-action depending on which embedding is used.

Recalling that $G(\mathbf{C}) \cong \mathrm{GL}_d(\mathbf{C}) \times \mathbf{C}^\times$, I can choose within the conjugacy class of cocharacters one whose image is in the diagonal torus. Using this cocharacter, the eigenspace D_+ is, with respect to the decomposition

$$D_{\mathbf{C}} = M_d(\mathbf{C}) \times M_d(\mathbf{C})^{\mathrm{opp}},$$

the subspace consisting of the entries making up the top r rows (this is the *definition* of r) of the $M_d(\mathbf{C})$ factor together with the entries making up the bottom $s := d - r$ rows of the $M_d(\mathbf{C})^{\mathrm{opp}}$ factor. Then

$$rd = \dim_{\mathbf{C}}(D_+ \cap (D \otimes_\epsilon \mathbf{C})) = \dim_{\mathbf{C}}(D_- \cap (D \otimes_{\overline{\epsilon}} \mathbf{C})^{\mathrm{opp}})$$

$$sd = \dim_{\mathbf{C}}(D_- \cap (D \otimes_\epsilon \mathbf{C})) = \dim_{\mathbf{C}}(D_+ \cap (D \otimes_{\overline{\epsilon}} \mathbf{C})^{\mathrm{opp}})$$

By page 274 in [30], one way to construct $\mathbf{E}$ is to adjoin to $\mathbf{Q}$ the traces $\mathrm{Tr}_{\mathbf{C}}(x; D_-)$ for all $x \in D$. Pick $x \in D$ and, recalling the classification of involutions over an algebraically-closed field, let $(X, \overline{X}^{\mathrm{tr}})$ be the image of x under

$$D \longrightarrow D_{\mathbf{C}} = M_d(\mathbf{C}) \times M_d(\mathbf{C})^{\mathrm{opp}}$$

With an appropriate choice of basis, the $\mathbf{C}$-linear operation of x on D is given by the matrix

$$X \oplus \cdots \oplus X \ \ (d \text{ times})$$

on $M_d(\mathbf{C}) \hookrightarrow D_{\mathbf{C}}$ and by the matrix

$$\overline{X}^{\mathrm{tr}} \oplus \cdots \oplus \overline{X}^{\mathrm{tr}} \ \ (d \text{ times})$$

on $M_d(\mathbf{C})^{\mathrm{opp}} \hookrightarrow D_{\mathbf{C}}$. By the choice of μ and the corresponding representation of D_+ and D_-,

$$\mathrm{Tr}_{\mathbf{C}}(x; D_-) = s\,\mathrm{Tr}_{\mathbf{C}}(X) + r\,\mathrm{Tr}_{\mathbf{C}}(\overline{X}^{\mathrm{tr}}) = s\,\mathrm{Tr}_{\mathbf{C}}(X) + r\overline{\mathrm{Tr}_{\mathbf{C}}(X)}$$

Since $x \in D$, $X \in M_d(F) \subset M_d(\mathbf{C})$ and so $\mathbf{E} \subset F$. It follows that if $r = s$ then $\mathrm{Tr}_{\mathbf{C}}(x; D_-) \in \mathbf{Q}$ and $\mathbf{E} = \mathbf{Q}$. Conversely, if $r \neq s$ then there are certainly $x \in D$ for which $\mathrm{Tr}_{\mathbf{C}}(x; D_-) \notin \mathbf{Q}$, and $\mathbf{E} = F$. Note that the assumption $\mathbf{E} = \mathbf{Q}$ forces d even.

Assumption. $\mathbf{E} = \mathbf{Q}$.

Remark. *This assumption is justified because the case of* $\mathbf{E} = F$ *can be reduced to the case of* GL*, which is known by [15]; see the proof of Lemma 3.1.4.1 (page 89).*

By using the embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, the cocharacter μ defines a cocharacter $\mathbf{G}^{\mathrm{m}}_{\overline{\mathbf{Q}}_p} \to G_{\overline{\mathbf{Q}}_p}$. The $G(\overline{\mathbf{Q}}_p)$-conjugacy class of this μ is (at least) defined over the completion E (still called the reflex field) of $\mathbf{E}$ at the prime corresponding to $\mathbf{E} \hookrightarrow \overline{\mathbf{Q}}_p$. The cocharacter μ itself is split by some (possibly non-trivial) extension $E' \supset E$ and defines a similar weight decomposition

$$M_d(F_p) \otimes_{\mathbf{Q}_p} E' = M_d(F_p)_+ \oplus M_d(F_p)_-$$

as above.

1.3 Some commutative algebra lemmas

I will use the following several times:

Local Criteria for Projectivity (Theorem 1 on page 109 of [4])**.** *Let $\mathcal{R}$ be a commutative ring and let M be an $\mathcal{R}$-module. The following are equivalent:*

1. *M is finitely-generated and projective.*

2. *M is finitely-generated and for every prime $\mathfrak{p} \subset \mathcal{R}$, the module of fractions $M_{\mathfrak{p}}$ is a free finite-rank $\mathcal{R}_{\mathfrak{p}}$-module and the function*

$$\begin{aligned} \operatorname{Spec}(\mathcal{R}) &\longrightarrow \mathbf{N} \\ \mathfrak{p} &\longmapsto \operatorname{rank}_{\mathcal{R}_{\mathfrak{p}}}(M_{\mathfrak{p}}) \end{aligned}$$

is locally constant with respect to the Zariski topology.

3. *There are $s_1, \ldots, s_n \in \mathcal{R}$ generating the trivial ideal $\mathcal{R}$ such that each principal module of fractions M_{s_i} is a free finite-rank $\mathcal{R}_{s_i}$-module.*

I frequently refer to the function in (2) as the "projective rank function" of M. I frequently express the property in characterization (3) by saying that M is "Zariski-locally on $\operatorname{Spec}(\mathcal{R})$" a free and finite-rank $\mathcal{R}$-module or something similar.

Permanence of Finite-Presentedness (Lemma 9 on page 21 of [4])**.** *Let A be a (commutative) ring and*

$$0 \to N \to M \to Q \to 0$$

an exact sequence of A-modules. If M is finitely-generated *and Q is finitely*-presented, *then N is finitely*-generated.

In other words, any *finite set of generators of a finitely-presented module is automatically a finite-presentation.*

The following handles a slight complication special to the case of an *unramified* unitary group (see the proof of Proposition 2.4.6.1 (page 63)):

Lemma 1.3.0.1. *Let A and B be commutative rings, $f : A \to B$ a ring homomorphism, and M a B-module. Regard all B-modules as A-modules via f.*

If *(1) B is a finitely-generated A-module, (2) M is a finitely-generated A-module, (3) M is a projective A-module, and (4) f is faithfully flat,* ***then*** *M is a projective B-module.*

Proof. It is equivalent (see Corollary 2 on page 111 of [4]) to prove that M is a flat and finitely-presented B-module. Projective modules are flat so by (3) and (4), M is then B-flat. By (2) and the definition of the A-action, M is finitely-generated over B. Let $B^k \twoheadrightarrow M$ be a B-module presentation and let N be the kernel (a B-module). By (1), B^k is finitely-generated over A. Finitely-generated projective modules are finitely-presented, so by "Permanence of Finite-Presentedness" N is finitely-generated over A. As before, this means that N is finitely-generated over B. This means that M is a finitely-presented B-module. $\square$

Localization of Hom-Sets (Proposition 2.10 on page 68 of [8])**.** *Let A be a (commutative) ring, $\mathfrak{p} \subset A$ a prime, and M and N two A-modules.* ***If*** *M is finitely-presented* ***then*** *the natural $A_{\mathfrak{p}}$-linear map*

$$\begin{aligned} \mathrm{Hom}_{A\text{-lin}}(M,N)_{\mathfrak{p}} &\longrightarrow \mathrm{Hom}_{A_{\mathfrak{p}}\text{-lin}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) \\ \frac{f}{s} &\longmapsto \left\{\frac{m}{t} \mapsto \frac{f(m)}{st}\right\} \end{aligned}$$

is an isomorphism.

More generally, if A' is an A-algebra which is flat as an A-module and M is a finitely-presented A-module, then

$$\mathrm{Hom}_{A\text{-lin}}(M,N) \otimes_A A' \longrightarrow \mathrm{Hom}_{A'\text{-lin}}(M \otimes_A A', N \otimes_A A')$$

is an A'-linear isomorphism. If M is free, then the flatness hypothesis is (obviously) *unnecessary.*

In many papers local models, a submodule is sometimes assumed to be "Zariski-locally a direct summand" (or something similar). Actually, this assumption is usually equivalent to the assumption that the submodule be a direct summand, period:

Lemma 1.3.0.2. *Let A be a (commutative) ring and let*

$$0 \to N \to M \to Q \to 0$$

be a short-exact-sequence of R-modules. Assume that Q is finitely-presented. ***Assertion****: the sequence splits if and only if for every prime $\mathfrak{p} \subset A$ the localized sequence*

$$0 \to N_{\mathfrak{p}} \to M_{\mathfrak{p}} \to Q_{\mathfrak{p}} \to 0$$

also splits.

In particular, if M is free and finite-rank, and if N is finitely-generated, then $N \subset M$ is a direct summand if and only if it is a direct summand Zariski-locally on $\operatorname{Spec}(A)$ *(any Zariski-local property implies the corresponding local property).*

Proof. $\boxed{\Rightarrow}$ This is trivial. $\boxed{\Leftarrow}$ The short-exact-sequence is split if and only if the induced homomorphism $\operatorname{Hom}_{A\text{-lin}}(Q, M) \to \operatorname{Hom}_{A\text{-lin}}(Q, Q)$ is surjective. Since surjectivity is a local property, this homomorphism is surjective if and only if all the localized homomorphisms $\operatorname{Hom}_{A\text{-lin}}(Q, M)_{\mathfrak{p}} \to \operatorname{Hom}_{A\text{-lin}}(Q, Q)_{\mathfrak{p}}$ are surjective. By Localization of Hom-Sets, the localized homomorphisms are really the same as the induced homomorphisms $\operatorname{Hom}_{A_{\mathfrak{p}}\text{-lin}}(Q_{\mathfrak{p}}, M_{\mathfrak{p}}) \to \operatorname{Hom}_{A_{\mathfrak{p}}\text{-lin}}(Q_{\mathfrak{p}}, Q_{\mathfrak{p}})$. The hypothesis is exactly that these localized homomorphisms are surjective. $\square$

Remark. *This lemma can also be proved using the "Local Criteria for Projectivity". I do this implicitly in §2.1.3 (page 31).*

I frequently use the following extremely useful consequence of Nakayama's lemma:

Linear Independence of Minimal Generating Sets (Exercise 15 in Chapter 3 of [1])**.** *Let A be a (commutative) ring and let M be a rank n free A-module. If $x_1, \ldots, x_n$ generates M then it is automatically a basis.*

Chapter 2

Local models and affine flag varieties

2.1 Definition of the local model

Recall that I have identified

$$B \otimes_{\mathbf{Q}} \mathbf{Q}_p = M_d(F_p)^{\mathrm{opp}}$$

$$\mathcal{O}_B \otimes_{\mathbf{Z}} \mathbf{Z}_p = M_d(\mathcal{O}_{F_p})^{\mathrm{opp}}$$

$$\iota_p = *_{\mathrm{std}}$$

Now that the relationship between $(D, *)$ and G is clear and I have restricted attention to $E = \mathbf{Q}_p$, *I will no longer refer to the global objects.* For simplicity of notation, refer to $F_p, \mathcal{O}_{F_p}, *_p, \psi_p$ etc. simply as $F, \mathcal{O}, *, \psi$ etc. I will now use $r = s = d/2$ without warning.

2.1.1 The base lattice chain

I need to fix a chain $(\cdots \subset \Lambda_0 \subset \Lambda_1 \subset \cdots)$ of $\mathcal{O}_B$-lattices in V, i.e. a chain of *left* $M_d(\mathcal{O})^{\mathrm{opp}}$-submodules of $M_d(F)$, each of which spans $M_d(F)$ as an F-vector space. I further require that:

- The chain is "periodic", in the sense that for any Λ in the chain, $p\Lambda$ is also in the chain.

- The chain is "ψ-self-dual", in the sense that for every Λ in the chain, $\widehat{\Lambda}$ (see §1.2, page 11) is also in the chain.

See Definition 3.1 on page 69-70 and Definition 3.18 on page 77-78 of [30]. I will define a particularly convenient such lattice chain in §2.3 (page 45). For now, assume that such a lattice chain exists.

2.1.2 Definition of the local model

Here I recall the definition of the local model $\mathbf{M}^{\text{loc}} : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$. The fact that the domain is $\mathbf{Z}_p$-Algebras is due to the fact that $E = \mathbf{Q}_p$, which implicitly depends on the cocharacter μ.

The following comment is needed to make sense of the definition of a PEL local model.

Let R be a (commutative) ring, let M be a finitely-generated *projective* R-module, and let $T : M \to M$ be an R-linear endomorphism. Recall from the "Local Projectivity Criteria" (page 23) that M is Zariski-locally free. If $S \subset R$ is a multiplicative subset such that $S^{-1}M$ is a free $S^{-1}R$-module, then the endomorphism $S^{-1}T : S^{-1}M \to S^{-1}M$ induced by T has a determinant,

$$\det_{S^{-1}R}(S^{-1}T; S^{-1}M) \in S^{-1}R.$$

Covering $\operatorname{Spec}(R)$ by multiplicative subsets S as above gives a collection of determinants which patch together into a global determinant

$$\det_R(T; M) \in R.$$

Definition: The Original Local Model (Definition 3.27 on page 89 of [30]).

Define the functor

$$\mathbf{M}^{\mathrm{loc}} : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set of commutative diagrams

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_0 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \cdots & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_d \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \cdots \\
\cdots & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \cdots \\
\cdots & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_d & \longrightarrow & \cdots
\end{array}$$

of $M_d(\mathcal{O} \otimes_{\mathbf{Z}_p} R)^{\mathrm{opp}}$-modules satisfying:

1. ***OLM1***

   For each i, if $p\Lambda_i = \Lambda_j$, then the isomorphism $-\cdot p : \Lambda_i \xrightarrow{\sim} \Lambda_j$ descends to an isomorphism $T_i \xrightarrow{\sim} T_j$

2. ***OLM2***

   each $\Lambda_i \otimes_{\mathbf{Z}_p} R \to T_i$ is surjective

3. ***OLM3***

   each T_i is Zariski-locally on $\mathrm{Spec}(R)$ a free finite rank R-module

4. ***OLM4***

   for each i, $\det_R(b; T_i) = \det_{E'}(b; M_d(F)_-)$ for all $b \in M_d(\mathcal{O})$.

   (Recall that E' acts on $M_d(F)_-$ since it is a submodule of $M_d(F) \otimes_{\mathbf{Q}_p} E'$.)

5. ***OLM5***

   For each $\Lambda = \Lambda_i$, the composite

$$T_\Lambda^\vee \longrightarrow (\Lambda \otimes_{\mathbf{Z}_p} R)^\vee \cong \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \longrightarrow T_{\widehat{\Lambda}}$$

is the 0 *map. (here* $\vee = \mathrm{Hom}_{R\text{-lin}}(-, R)$*)*

2.1.3 Changing from quotients to kernels

Notice that in the definition of $\mathbf{M}^{\mathrm{loc}}$, specifying the object T_i is the same as specifying the kernel K_i of $\Lambda_i \otimes_{\mathbf{Z}_p} R \twoheadrightarrow T_i$. To more closely match the description of affine flag varieties, I need to express the points of $\mathbf{M}^{\mathrm{loc}}$ using the K_i instead of the T_i. I need to determine what conditions equivalent to **OLM** must be imposed on the K_i.

$\boxed{\text{Changing } \mathbf{OLM3}}$ By the “Local Projectivity Criteria”, **OLM3** is equivalent to finitely-generated projectivity, so each inclusion $K_i \subset \Lambda_i \otimes_{\mathbf{Z}_p} R$ has an R-linear splitting. Conversely, if $K_i \subset \Lambda_i \otimes_{\mathbf{Z}_p} R$ splits R-linearly, then $\Lambda_i \otimes_{\mathbf{Z}_p} R \twoheadrightarrow T_i$ does also, T_i is projective, and by the “Local Projectivity Criteria”, **OLM3** holds.

$\boxed{\text{Changing } \mathbf{OLM4}}$ It is clear from the definition that

$$\det_{E'}(b; M_d(F)) = \det_{E'}(b; M_d(F)_-) \cdot \det_{E'}(b; M_d(F)_+)$$
$$\det_R(b; \Lambda_i \otimes_{\mathbf{Z}_p} R) = \det_R(b; T_i) \cdot \det_R(b; K_i)$$

so **OLM4** is equivalent to

$$\det_R(b; K_i) = \det_{E'}(b; M_d(F)_+)$$

$\boxed{\text{Changing } \mathbf{OLM5}}$ An element of $T_\Lambda^\vee$ is the same as an R-linear map $f : \Lambda \to R$ whose kernel contains K_Λ. Since Λ is a *lattice*, any such functional f is of the form $\psi_R(-, \lambda)$ for some $\lambda \in \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R$. Define

$$K_\Lambda^\perp \stackrel{\text{def}}{=} \{\lambda \in \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \mid \psi_R(K_\Lambda, \lambda) = 0\}.$$

Then the requirement that $K_\Lambda \subset \ker(f)$ is equivalent to the requirement that $\lambda \in K_\Lambda^\perp$. The last condition in $\mathbf{M}^{\mathrm{loc}}$ is therefore equivalent to the statement that $K_{\widehat{\Lambda}} \supset K_\Lambda^\perp$. The reverse inclusion $K_{\widehat{\Lambda}} \subset K_\Lambda^\perp$ will follow from:

Lemma 2.1.3.1. *There exists a short-exact-sequence*

$$0 \longrightarrow \mathrm{Hom}_{R\text{-lin}}(K_\Lambda, R) \longrightarrow \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \xrightarrow{s} K_\Lambda^\perp \longrightarrow 0$$

of R-modules such that s is a splitting of $K_\Lambda^\perp \subset \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R$.

In particular, $K_\Lambda^\perp$ is R-projective and has the same projective rank function as $K_{\widehat{\Lambda}}$.

Note that the short-exact-sequence only proves that the projective ranks of K_Λ and $K_\Lambda^\perp$ are complementary, but the projective rank of K_Λ is always $\frac{1}{2}\,\mathrm{rank}_{\mathbf{Z}_p}(\Lambda)$.

Proof. Let p_T be the composition

$$\Lambda \otimes_{\mathbf{Z}_p} R \twoheadrightarrow T_\Lambda \hookrightarrow \Lambda \otimes_{\mathbf{Z}_p} R,$$

using R-projectivity of T_Λ. Interpreting $f \in \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R$ as an R-linear functional $\Lambda \otimes_{\mathbf{Z}_p} R \to R$, the composition $f \circ p_T$ is 0 on K_Λ and so the function

$$f \mapsto f \circ p_T$$

defines an R-linear map

$$\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \longrightarrow K_\Lambda^\perp \tag{2.1}$$

It is immediate from the definition that (2.1) is an R-linear splitting of $K_\Lambda^\perp \subset \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R$ (in particular, (2.1) is surjective).

Let p_K be the splitting

$$\Lambda \otimes_{\mathbf{Z}_p} R \twoheadrightarrow K_\Lambda$$

and define an R-linear map

$$\mathrm{Hom}_{R\text{-lin}}(K_\Lambda, R) \longrightarrow \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \tag{2.2}$$

by the rule $f \mapsto f \circ p_K$. Since p_K is a splitting of the inclusion, (2.2) is injective. It remains to show middle-exactness. Obviously, $p_K \circ p_T = 0$. On the other hand, if $f \in \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R$ and $f \circ p_T = 0$, then precomposing with the splitting $T_\Lambda \hookrightarrow \Lambda \otimes_{\mathbf{Z}_p} R$ gives that $f|_{T_\Lambda} = 0$ also, and so f is the inflation along p_K of a functional on K_Λ. $\square$

By Lemma 2.1.3.1, $(\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R)/K_\Lambda^\perp$ is R-projective and has the same projective rank function as $(\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R)/K_{\widehat{\Lambda}}$. Localizing and using "Linear Independence of Minimal Generating Sets", the canonical surjection

$$(\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R)/K_\Lambda^\perp \twoheadrightarrow (\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R)/K_{\widehat{\Lambda}}$$

is an *isomorphism*. A typical "five-lemma" argument then shows that $K_\Lambda^\perp = K_{\widehat{\Lambda}}$.

Therefore, $\mathbf{M}^{\mathrm{loc}}(R)$ can be expressed as all those commutative diagrams

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_0 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \cdots & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \Lambda_d \otimes_{\mathbf{Z}_p} R & \xrightarrow{\mathrm{inc}\otimes\mathrm{id}} & \cdots \\
\cdots & & \uparrow & & \uparrow & & \cdots & & \uparrow & & \cdots \\
\cdots & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & \cdots & \longrightarrow & K_d & \longrightarrow & \cdots
\end{array}$$

of $M_d(\mathcal{O} \otimes_{\mathbf{Z}_p} R)$-modules satisfying:

- For each i, if $p\Lambda_i = \Lambda_j$, then the isomorphism $- \cdot p : \Lambda_i \xrightarrow{\sim} \Lambda_j$ restricts to an isomorphism $K_i \xrightarrow{\sim} K_j$

- Each $K_i \to \Lambda_i \otimes_{\mathbf{Z}_p} R$ is *injective*

- Each $K_i \hookrightarrow \Lambda_i \otimes_{\mathbf{Z}_p} R$ splits R-linearly

- For each i, $\det_R(b; K_i) = \det_{E'}(b; M_d(F)_+)$ for all $b \in M_d(\mathcal{O})$

- $K_{\widehat{\Lambda}} = K_{\Lambda}^{\perp}$ for each $\Lambda = \Lambda_i$.

2.2 Compressing the local model with Morita equivalence

The definition of the local model is a somewhat bloated. My goal in this section is to use Morita equivalence to work within $\mathcal{O}$-Modules instead of $M_d(\mathcal{O})$-Modules. The conditions **OLM4** and **OLM5** are not directly digestible by Morita equivalence, so it is not completely trivial to determine what conditions should be imposed in their place after the equivalence has been applied.

2.2.1 Recalling the Morita equivalence

Let $\mathcal{R}$ be a (commutative) ring. I only need the simplest case of Morita equivalence, where the "progenerator" is $\mathcal{R}^d$. Morita equivalence is the statement that the functor

$$\mathrm{Mrta}_{\mathcal{R}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{R}\text{-lin}}(\mathcal{R}^d, -) : \mathcal{R}\text{-Modules} \to (\text{right})M_d(\mathcal{R})\text{-Modules}$$

is an equivalence of categories (the action of $M_d(\mathcal{R})$ is by precomposition). An explicit quasi-inverse to Morita equivalence is the functor

$$\mathrm{Mrta}_{\mathcal{R}}^{-1} \stackrel{\mathrm{def}}{=} - \otimes_{M_d(\mathcal{R})} \mathcal{R}^d$$

I want to replace the local model by the functor

$$\mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

$$R \longmapsto \{\mathrm{Mrta}^{-1}_{\mathcal{O}\otimes_{\mathbf{Z}_p}R}(\Delta) \mid \text{all } \Delta \in \mathbf{M}^{\mathrm{loc}}(R)\}$$

Obviously, it will be necessary to know the following:

Lemma 2.2.1.1. *Let $\Lambda \subset F^d$ be an $\mathcal{O}$-lattice. Then*

$$\mathrm{Mrta}_{\mathcal{O}}(\Lambda) \otimes_{\mathbf{Z}_p} R = \mathrm{Mrta}_{\mathcal{O}\otimes_{\mathbf{Z}_p}R}(\Lambda \otimes_{\mathbf{Z}_p} R)$$

Proof. This is the trivial case of "Localization of Hom-Sets". □

2.2.2 Morita equivalence and bilinear products

Let R be a commutative $\mathbf{Z}_p$-algebra and S a (commutative) R-algebra. Let $s \mapsto \overline{s}$ be an involution on S such that $\overline{r} = r$ for all $r \in R$. Let $\iota : M_d(S) \to M_d(S)$ be an involution (assumed to be multiplication-reversing) such that $\iota(s) = \overline{s}$ for all $s \in S$.

Let M and M' be *right* $M_d(S)$-modules (or equivalently, *left* $M_d(S)^{\mathrm{opp}}$-modules). Let $\psi : M \times M' \to R$ be an R-bilinear form such that $\psi(x \cdot b, y) = \psi(x, y \cdot \iota(b))$ for all $b \in M_d(S)$ and $x, y \in M$.

Suppose $M = \mathrm{Mrta}_S(N)$ and $M' = \mathrm{Mrta}_S(N')$, for S-modules N, N'. I want to construct an R-bilinear product ${}^0\psi : N \times N' \to R$ naturally corresponding to ψ. It is equivalent to construct the corresponding R-linear map $N \to \mathrm{Hom}_{R\text{-lin}}(N', R)$.

Give $\mathrm{Hom}_{R\text{-lin}}(M', R)$ a *right*-$M_d(S)$-module structure by the rule

$$(F \cdot b)(m') \overset{\mathrm{def}}{=} F(m' \cdot \iota(b))$$

Then the R-linear adjoint map

$$\psi^{\mathrm{ad}} : M \longrightarrow \mathrm{Hom}_{R\text{-lin}}(M', R) \tag{2.3}$$

induced by ψ is automatically *right*-$M_d(S)$-linear:

$$F_{m\cdot b}(m') \stackrel{\text{def}}{=} \psi(m \cdot b, m') = \psi(m, m' \cdot \iota(b)) \stackrel{\text{def}}{=} F_m(m' \cdot \iota(b)) = (F_m \cdot b)(m')$$

Give $\mathrm{Hom}_{R\text{-lin}}(N', R)$ an S-module structure by the rule

$$(s \cdot g)(n') \stackrel{\text{def}}{=} g(\overline{s}n')$$

Give $\mathrm{Hom}_{R\text{-lin}}(\mathrm{Hom}_{S\text{-lin}}(S^d, N'), R)$ a *right* $M_d(S)$-module structure by the rule

$$(F \cdot b)(\varphi) \stackrel{\text{def}}{=} F(\varphi \circ \overline{b}^{\mathrm{tr}})$$

The following lemma says roughly that $\mathrm{Mrta}_S(N^\vee) = \mathrm{Mrta}_S(N)^\vee$, where in both cases $\vee = \mathrm{Hom}_{R\text{-lin}}(-, R)$:

Lemma 2.2.2.1. *Using the above actions, the function*

$$\begin{aligned} \mathrm{Hom}_{S\text{-lin}}(S^d, \mathrm{Hom}_{R\text{-lin}}(N', R)) &\longrightarrow \mathrm{Hom}_{R\text{-lin}}(\mathrm{Hom}_{S\text{-lin}}(S^d, N'), R) \\ f &\longmapsto F_f \stackrel{\text{def}}{=} \left\{ \varphi \mapsto \sum_{i=1}^{d} f(e_i)[\varphi(e_i)] \right\} \end{aligned} \tag{2.4}$$

is a right-$M_d(S)$*-linear isomorphism.*

Proof. It is easy to verify that the function is a group isomorphism. The linearity is proved by letting $b_{i,j} \in S$ be the entries of b and checking directly:

$$\begin{aligned}
F_{f \cdot b}(\varphi) &\overset{\text{def}}{=} \sum_{i=1}^{d} (f \cdot b)(e_i)[\varphi(e_i)] \\
&= \sum_{i=1}^{d} f\left(\sum_{j=1}^{d} b_{j,i} e_j\right)[\varphi(e_i)] \\
\text{(because } f \text{ is } S\text{-linear)} &= \sum_{i=1}^{d}\left(\sum_{j=1}^{d} (b_{j,i} \cdot f(e_j))[\varphi(e_i)]\right) \\
(S \text{ acting on } \mathrm{Hom}_{R\text{-lin}}(N', R)) &= \sum_{i=1}^{d}\left(\sum_{j=1}^{d} f(e_j)[\overline{b}_{j,i} \cdot \varphi(e_i)]\right) \\
\text{(because } \varphi \text{ is } S\text{-linear)} &= \sum_{j=1}^{d} f(e_j)\left[\varphi\left(\sum_{i=1}^{d} \overline{b}_{j,i} e_i\right)\right] \\
&= \sum_{j=1}^{d} f(e_j)[\varphi(\overline{b}^{\mathrm{tr}}(e_j))] \\
&= F_f(\varphi \circ \overline{b}^{\mathrm{tr}}) \\
&\overset{\text{def}}{=} (F_f \cdot b)(\varphi)
\end{aligned}$$

□

Because of Proposition 1.2.2.2 (page 15), I may assume for my applications that $\iota(b) = \overline{b}^{\mathrm{tr}}$. In that case, the codomains of (2.3) and (2.4) are identical as *right*-$M_d(S)$-modules, and composing (2.3) with the inverse of (2.4) produces the right-$M_d(S)$-linear map

$$\mathrm{Hom}_{S\text{-lin}}(S^d, N) \longrightarrow \mathrm{Hom}_{S\text{-lin}}(S^d, \mathrm{Hom}_{R\text{-lin}}(N', R)) \tag{2.5}$$

Because $\mathrm{Mrta}_S = \mathrm{Hom}_{S\text{-lin}}(S^d, -)$ is fully-faithful, (2.5) is the image of a unique S-linear map

$$N \to \mathrm{Hom}_{R\text{-lin}}(N', R)$$

By definition of the action of S on $\mathrm{Hom}_{R\text{-lin}}(N', R)$, the product

$$^0\psi : N \times N' \to R$$

induced by $N \to \mathrm{Hom}_{R\text{-lin}}(N', R)$ is R-bilinear and satisfies

$$^0\psi(sn, n') = {}^0\psi(n, \bar{s}n')$$

for all $s \in S$ and $n \in N$, $n' \in N'$.

Note that the only non-canonical input was the original product ψ. Because Mrta_S is exact, the adjoint map $m \mapsto \psi(m, -)$ is injective or surjective if and only if $n \mapsto {}^0\psi(n, -)$ is, so ψ is non-degenerate or perfect if and only if $^0\psi$ is.

2.2.3 Morita equivalence and lattices

Let

$$^0\psi : F^d \times F^d \longrightarrow \mathbf{Q}_p$$

be the $\mathbf{Q}_p$-bilinear form guaranteed by the previous subsection for the choices

$$\begin{aligned} R &= \mathbf{Q}_p \\ S &= F \\ (R \to S) &= (\mathbf{Q}_p \subset F) \\ M, M' &= M_d(F) \\ \iota &= \text{the usual } \iota : M_d(F) \to M_d(F) \\ \psi &= \text{the usual } \psi : M_d(F) \times M_d(F) \to \mathbf{Q}_p \end{aligned}$$

Let Λ be an $M_d(\mathcal{O})^{\mathrm{opp}}$-lattice in $M_d(F)$. Set

$$^0\Lambda \stackrel{\mathrm{def}}{=} \mathrm{Mrta}_{\mathcal{O}}^{-1}(\Lambda)$$

Consider F^d also as an $\mathcal{O}$-module, consider $M_d(F)$ as a left-$M_d(\mathcal{O})^{\mathrm{opp}}$-module, fix permanently an identification $\mathrm{Mrta}_{\mathcal{O}}(F^d) = M_d(F)$, and use the embedding $^0\Lambda \hookrightarrow F^d$ that is the image under $\mathrm{Mrta}_{\mathcal{O}}^{-1}$ of the inclusion $\Lambda \subset M_d(F)$. Define

$$\widehat{^0\Lambda} \stackrel{\mathrm{def}}{=} \{x \in F^d \mid {}^0\psi({}^0\Lambda, x) \subset \mathbf{Z}_p\}$$

(note that I need an inclusion $^0\Lambda \hookrightarrow F^d$ in order to make the preceding definition)

Proposition 2.2.3.1. $\mathrm{Mrta}_{\mathcal{O}}(\widehat{^0\Lambda}) = \widehat{\Lambda}$.

Proof. This is true simply because $\mathrm{Mrta}_{\mathcal{O}}$ is an equivalence of categories and the dual lattices can be expressed categorically. In more detail, by definition $\widehat{\Lambda}$ is the image of the $M_d(\mathcal{O})^{\mathrm{opp}}$-linear map

$$\mathrm{Hom}_{\mathbf{Z}_p\text{-lin}}(\Lambda, \mathbf{Z}_p) \stackrel{\text{extend}}{\longrightarrow} \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(M_d(F), \mathbf{Q}_p) \cong M_d(F) \tag{2.6}$$

(the 'extend' map uses the fact that Λ spans $M_d(F)$, and the isomorphism is that induced by the *perfect* form ψ).

Similarly, by definition $\widehat{^0\Lambda}$ is the image of the $\mathcal{O}$-linear map

$$\mathrm{Hom}_{\mathbf{Z}_p\text{-lin}}({}^0\Lambda, \mathbf{Z}_p) \stackrel{\text{extend}}{\longrightarrow} \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(F^d, \mathbf{Q}_p) \cong F^d \tag{2.7}$$

(the 'extend' map uses the fact that Λ spans $M_d(F)$, and the isomorphism is that induced by the *necessarily perfect* form $^0\psi$).

Equation (2.4) in the previous subsection applied to the case of

$$R = \mathbf{Z}_p$$

$$S = \mathcal{O}$$

$$(R \to S) = (\mathbf{Z}_p \subset \mathcal{O})$$

and a slightly different version

$$\mathrm{Hom}_{\mathcal{O}\text{-lin}}(\mathcal{O}^d, \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(F^d, \mathbf{Q}_p)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(\mathrm{Hom}_{\mathcal{O}\text{-lin}}(\mathcal{O}^d, F^d), \mathbf{Q}_p)$$

of that same isomorphism together with the fact that $\mathrm{Mrta}_{\mathcal{O}}({}^0\psi) = \psi$ (in the sense of the previous subsection) imply that if you apply $\mathrm{Mrta}_{\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}\text{-lin}}(\mathcal{O}^d, -)$ to (2.7) then you get (2.6). The fact that $\mathrm{Mrta}_{\mathcal{O}}$ is an equivalence of categories means that the images, $\mathrm{Mrta}_{\mathcal{O}}({}^0\widehat{\Lambda})$ on the one hand and $\widehat{\Lambda}$ on the other, correspond. $\square$

2.2.4 Translating the PEL duality condition

Let Λ be an $M_d(\mathcal{O})^{\mathrm{opp}}$-lattice in $M_d(F)$. In this subsection, redefine

$${}^0\psi : {}^0\Lambda \times {}^0\widehat{\Lambda} \longrightarrow \mathbf{Z}_p$$

to be the $\mathbf{Z}_p$-bilinear form guaranteed by §2.2.2 for the choices

$$R = \mathbf{Z}_p$$

$$S = \mathcal{O}$$

$$(R \to S) = (\mathbf{Z}_p \subset \mathcal{O})$$

$$\iota = \text{the restriction to } M_d(\mathcal{O}) \text{ of the usual } \iota$$

$$\psi = \text{the restriction to } M_d(\mathcal{O}) \times M_d(\mathcal{O}) \text{ of the usual } \psi$$

(here is another appearance of the assumption on ι). Note that by Proposition 2.2.3.1 ${}^0\psi$ is the same as the restriction to ${}^0\Lambda \times {}^0\widehat{\Lambda}$ of the ${}^0\psi$ defined in §2.2.3.

Let R be a commutative $\mathbf{Z}_p$-algebra and set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$. Let $K_\Lambda \subset \Lambda \otimes_{\mathbf{Z}_p} R$ be an $M_d(\mathcal{R})^{\mathrm{opp}}$-submodule and set

$$ {}^0K_\Lambda \stackrel{\mathrm{def}}{=} \mathrm{Mrta}_{\mathcal{R}}^{-1}(K_\Lambda) $$

Define

$$ {}^0K_\Lambda^\perp \stackrel{\mathrm{def}}{=} \{\lambda \in {}^0\widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \mid {}^0\psi_R({}^0K_\Lambda, \lambda) = 0\}. $$

Proposition 2.2.4.1. $\mathrm{Mrta}_{\mathcal{R}}({}^0K_\Lambda^\perp) = K_\Lambda^\perp$.

Proof. Taking into account Proposition 2.2.3.1 (page 39), this is true simply because $\mathrm{Mrta}_{\mathcal{R}}$ is an equivalence of categories (and therefore commutes with the kernel operator) and the operation $K_\Lambda \mapsto K_\Lambda^\perp$ can be expressed as

$$ K_\Lambda^\perp = \ker(\widehat{\Lambda} \longrightarrow \mathrm{Hom}_{\mathbf{Z}_p\text{-lin}}(\Lambda, \mathbf{Z}_p)) $$

and similarly for ${}^0K_\Lambda \mapsto {}^0K_\Lambda^\perp$. □

2.2.5 Translating the determinant condition

Let R be a (commutative) $\mathbf{Z}_p$-algebra and for simplicity of notation, set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$. Choose a point $\{K_i\}_{i\in\mathbf{Z}} \in \mathbf{M}^{\mathrm{loc}}(R)$. Condition **OLM4** certainly forces the projective rank function associated to each K_i to be the constant function

$$ \begin{aligned} \mathrm{Spec}(R) &\longrightarrow \mathbf{N} \\ \mathfrak{p} &\longmapsto d(2s) = d^2 \end{aligned} $$

I claim that this rank requirement is sufficient, i.e. that *any* $M_d(\mathcal{R})^{\mathrm{opp}}$-submodule K_i of $\Lambda_i \otimes_{\mathbf{Z}_p} R$ with constant projective rank function $\mathfrak{p} \mapsto d^2$ automatically has the required generic determinant. This answers the question in §2.2.1 since a rank requirement is easily translated by Morita equivalence.

In fact, a general statement including the above claim can be proven:

Proposition 2.2.5.1. *Let $\mathcal{R}$ be a (commutative) $\mathcal{O}$-algebra. Let M and N be left-$M_d(\mathcal{R})$-modules that are projective as $\mathcal{R}$-modules.* ***If*** *M and N have the same projective rank functions* $\mathrm{Spec}(\mathcal{R}) \to \mathbf{N}$ ***then*** *for any $b \in M_d(\mathcal{R})$,*

$$\det_{\mathcal{R}}(b; M) = \det_{\mathcal{R}}(b; N)$$

To prove the above claim using this proposition, let i be arbitrary, set $M = K_i$ and $N = \mathcal{R}^d \oplus \cdots \oplus \mathcal{R}^d$ ($d/2$ times) and use Lemma 1.3.0.1 (page 24) to supply the projectivity hypothesis on M. This proposition implies that for $b \in M_d(\mathcal{O})$, $\det_{\mathcal{R}}(b; K_i)$ is just the product of $d/2$ copies of the "ordinary" $\mathcal{O}$-valued determinant of b. One can easily compute $\det_{E'}(b; M_d(F)_+)$ and see that the two are equal.

Proof. Let M' and N' be $\mathcal{R}$-modules such that $\mathrm{Mrta}_{\mathcal{R}}(M') = M$ and $\mathrm{Mrta}_{\mathcal{R}}(N') = N$. Since $M = M' \oplus \cdots \oplus M'$ and $N = N' \oplus \cdots \oplus N'$ as $\mathcal{R}$-modules, it is true that M' and N' are also projective as $\mathcal{R}$-modules. Let $S \subset \mathcal{R}$ be a multiplicative subset such that $S^{-1}M'$ and $S^{-1}N'$ are both free. The hypothesis implies that

$$\mathrm{rank}_{S^{-1}\mathcal{R}}(S^{-1}M') = \mathrm{rank}_{S^{-1}\mathcal{R}}(S^{-1}N')$$

(since Mrta multiplies ranks by d), so $S^{-1}M'$ and $S^{-1}N'$ are isomorphic as $S^{-1}\mathcal{R}$-modules. This means that $S^{-1}M$ and $S^{-1}N$ are isomorphic as right-$M_d(S^{-1}\mathcal{R})$-

modules, hence also as right-$M_d(\mathcal{R})$-modules (via restriction-of-scalars). Therefore, for any $b \in M_d(\mathcal{R})$,

$$\det_{S^{-1}\mathcal{R}}(b; S^{-1}M) = \det_{S^{-1}\mathcal{R}}(b; S^{-1}N)$$

Varying S to get an open cover of $\mathrm{Spec}(\mathcal{R})$, and patching the generic determinants together finishes the proof. $\square$

Remark. *The claim can also be proven by a short matrix computation.*

2.2.6 Morita equivalence and similitude groups

Consider the vector space F^d and the $\mathbf{Q}_p$-bilinear form ${}^0\psi : F^d \times F^d \to \mathbf{Q}_p$ as in §2.2.3 (page 38). Since ψ is non-degenerate and perfect, ${}^0\psi$ is also and therefore induces an involution

$${}^0* : \mathrm{End}_{F\text{-lin}}(F^d) \longrightarrow \mathrm{End}_{F\text{-lin}}(F^d)$$

In this subsection, I prove that the similitude group $G_{\mathbf{Q}_p}$ defined in §1.2 (page 11) can be expressed using only the data F^d, ${}^0\psi$, ^{0*}, etc. rather than $M_d(F)$, ψ, $*$, etc.

Recall from §1.2 (page 11) that $M_d(F)$ acts by *left*-multiplication on $V = M_d(F)$ and the involution induced by ψ on this left-acting $M_d(F)$ is none other than $*$, i.e. $\psi(bx, y) = \psi(x, b^* y)$ for all $b, x, y \in M_d(F)$. If $b \in M_d(F)$ is considered as an F-linear map $F^d \to F^d$, then $\mathrm{Mrta}_F(b)$ is a right-$M_d(F)$-linear map $M_d(F) \to M_d(F)$, and this map is none other than multiplication by b on the left of $M_d(F)$ (linearity follows from associativity of multiplication in $M_d(F)$).

A careful inspection of the construction of ${}^0\psi$ shows that ψ can be expressed using ${}^0\psi$ in a very simple way: if $x, y \in M_d(F)$, then

$$\psi(x, y) = \sum_{i=1}^{d} {}^0\psi(x_i, y_i)$$

where x_i and y_i are the ith columns of the matrices x, y. This is verified as follows. Writing a matrix $x \in M_d(F) = \mathrm{Mrta}_F(F^d)$ as a d-tuple $(x_1, \ldots, x_d)$ of column vectors corresponds to writing $\mathrm{Mrta}_F(F^d) = \mathrm{Hom}_{F\text{-lin}}(F^d, F^d)$ as the d-fold product of $\mathrm{Hom}_{F\text{-lin}}(F, F^d) \cong F^d$. Denote by

$${}^0\psi^{\mathrm{ad}} : F^d \longrightarrow \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(F^d, \mathbf{Q}_p)$$

the morphism induced by ${}^0\psi$. Writing all matrices as d-tuples of column vectors as above, and considering the isomorphism from Lemma 2.2.2.1 (page 36), the map

$$\psi^{\mathrm{ad}} : M_d(F) \longrightarrow \mathrm{Hom}_{\mathbf{Q}_p\text{-lin}}(M_d(F), \mathbf{Q}_p)$$

induced by ψ is written as

$$\psi^{\mathrm{ad}}(x_1, \ldots, x_d) = \left((y_1, \ldots, y_d) \longmapsto \sum_{i=1}^{d} {}^0\psi^{\mathrm{ad}}(x_i)[y_i] \right)$$

Since ${}^0\psi^{\mathrm{ad}}(x_i)[y_i]$ is just another way to write ${}^0\psi(x_i, y_i)$, the claim is proved.

The preceding relationship trivially implies the following: if $x, y \in F^d$, and if $X, Y \in M_d(F)$ are the matrices whose ith columns are x, y (respectively) and 0 in all other entries, then $\psi(X, Y) = {}^0\psi(x, y)$. A consequence of this is that the involution ^{0*} induced by ${}^0\psi$ on $\mathrm{End}_{F\text{-lin}}(F^d) = M_d(F)$ is *identical to* the involution $*$ induced by ψ on $M_d(F)$. A further consequence of this is that the group $G_{\mathbf{Q}_p}$ introduced in §1.2 (page 11) can also benefit from the simplification afforded by

Morita equivalence: $G_{\mathbf{Q}_p}$ can also be described as the the functor assigning to any (commutative) $\mathbf{Q}_p$-algebra R the group

$$G_{\mathbf{Q}_p}(R) = \{g \in \mathrm{End}_{F\text{-lin}}(F^d) \otimes_{\mathbf{Q}_p} R \mid g^{{}^0\!*\otimes\mathrm{id}} \cdot g \in R^\times\}$$

Remark. *One of the reasons for expressing $G_{\mathbf{Q}_p}$ using ^{0*} rather than $*$ is that it is not obvious that the assumption of quasi-splitness implies anything convenient about ψ (this is related to the remark made in §1.2 (page 11) that ι does not determine ψ). However, once the above simplification is made, Lemma 1.2.2.3 (page 19) can be used.*

Now that I no longer need to refer to the objects ψ, $*$, ϕ, etc. I abuse notation by dropping the superscript "0" from ${}^0\psi$, ^{0*}, ${}^0\phi$, etc.

2.3 Most simplified description of the local model

I make three final simplifications before stating the most condensed definition of the local model.

Because of Lemma 1.2.2.3 (page 19), the assumption made on page 20, and the description of $G_{\mathbf{Q}_p}$ in the previous subsection, I know that $G_{\mathbf{Q}_p}$ is the similitude group of the hermitian form defined by the identity matrix. I claim that I can replace this hermitian form by the the hermitian form defined by the *anti*-identity matrix. This is possible because, by the classification of hermitian forms over local fields, an isomorphism class of hermitian forms is determined (see 1.6(ii) on page 351 in [31]) by its dimension (which is the same for both), and the image of its determinant (meaning the determinant of the Gram matrix defining the form) in the norm class group $\mathbf{Q}_p^\times / N_{F/\mathbf{Q}_p}(F^\times)$. The determinant of the anti-identity matrix is ± 1 and for unramified extensions the norm map $N_{F/\mathbf{Q}_p} : \mathcal{O}^\times \to \mathbf{Z}_p^\times$ is *surjective*, so both these hermitian forms are in the same class. *I now use ϕ to refer to the hermitian form defined by the anti-identity matrix.*

The second simplification is to notice that all concepts of duality and orthogonality can be rewritten to use ϕ instead of ψ. The explicit correspondence between alternating forms like ψ and hermitian forms ϕ on page 19 implies that

$$\widehat{\Lambda} = \{w \in F^d \mid \phi(\Lambda, w) \subset \mathcal{O}\}$$

$$K_\Lambda^\perp = \{\lambda \in \widehat{\Lambda} \otimes_{\mathbf{Z}_p} R \mid \phi_R(K_\Lambda, \lambda) = 0\}$$

where Λ is an $\mathcal{O}$-lattice in F^d, R is a (commutative) $\mathbf{Z}_p$-algebra, and K_Λ is an $(\mathcal{O} \otimes_{\mathbf{Z}_p} R)$-submodule of $\Lambda \otimes_{\mathbf{Z}_p} R$.

Because of the previous two simplifications, I can fulfill my promise from §2.1.1 (page 28) to construct a self-dual chain of left-$M_d(\mathcal{O})^{\mathrm{opp}}$-lattices in $M_d(F)$. Because ϕ is defined by the *anti*-identity matrix, extending

$$\Lambda_i = p^{-1}\mathcal{O}^i \oplus \mathcal{O}^{d-i} \ (0 \leq i \leq d)$$

periodically produces periodic ϕ-self-dual lattice-chain in F^d (this is the usual lattice-chain used for local models associated to unitary groups). By Proposition 2.2.3.1, applying $\mathrm{Mrta}_\mathcal{O}$ to $\{\Lambda_i\}_{i \in \mathbf{Z}}$ gives the lattice-chain I promised in §2.1.1 (although I no longer care about that lattice chain, provided that it exists).

Remark. *One reason to change to the hermitian form defined by the anti-identity matrix is that the construction of a periodic self-dual lattice chain is obvious.*

The third simplification is to replace each point

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\text{inc}\otimes\text{id}} & \Lambda_0 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\text{inc}\otimes\text{id}} & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\text{inc}\otimes\text{id}} & \cdots & \xrightarrow{\text{inc}\otimes\text{id}} & \Lambda_d \otimes_{\mathbf{Z}_p} R & \xrightarrow{\text{inc}\otimes\text{id}} & \cdots \\
\cdots & & \uparrow & & \uparrow & & \cdots & & \uparrow & & \cdots \\
\cdots & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & \cdots & \longrightarrow & K_d & \longrightarrow & \cdots
\end{array}$$

of $\mathbf{M}^{\text{loc}}(R)$ with the finite diagram

$$\begin{array}{ccccccc}
\Lambda_0 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\text{inc}\otimes\text{id}} & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \xrightarrow{\text{inc}\otimes\text{id}} & \cdots & \xrightarrow{\text{inc}\otimes\text{id}} & \Lambda_{d/2} \otimes_{\mathbf{Z}_p} R \\
\uparrow & & \uparrow & & \cdots & & \uparrow \\
K_0 & \longrightarrow & K_1 & \longrightarrow & \cdots & \longrightarrow & K_{d/2}
\end{array}$$

The original diagram can be reconstructed from this by the properties

$$\Lambda_{-i} = \widehat{\Lambda}_i$$

$$\Lambda_i = p\Lambda_{i+d}$$

$$K_{-i} = K_i^{\perp}$$

$$K_i = pK_{i+d}$$

The only requirements that survive on the finite diagram are

$$K_0 = K_0^{\perp}$$

$$K_{d/2}^{\perp} = K_{-d/2} \ (= pK_{d/2})$$

So, the most condensed description of my local model is:

Definition: The Simplified Description of the Local Model. *The functor* $\mathbf{M}^{\text{loc}}$ *can also be described as assigning to each (commutative)* $\mathbf{Z}_p$*-algebra* R *the set* *of all commutative diagrams*

$$\begin{array}{ccccccccc}
\Lambda_0 \otimes_{\mathbf{Z}_p} R & \longrightarrow & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \longrightarrow & \cdots & \longrightarrow & \Lambda_{d/2-1} \otimes_{\mathbf{Z}_p} R & \longrightarrow & \Lambda_{d/2} \otimes_{\mathbf{Z}_p} R \\
\uparrow & & \uparrow & & \cdots & & \uparrow & & \uparrow \\
K_0 & \longrightarrow & K_1 & \longrightarrow & \cdots & \longrightarrow & K_{d/2-1} & \longrightarrow & K_{d/2}
\end{array}$$

of $(\mathcal{O} \otimes_{\mathbf{Z}_p} R)$*-modules such that*

1. ***SLM1***

   each $K_i \to \Lambda_i \otimes_{\mathbf{Z}_p} R$ *is injective*

2. ***SLM2***

   each inclusion $K_i \hookrightarrow \Lambda_i \otimes_{\mathbf{Z}_p} R$ *splits* R*-linearly*

3. ***SLM3***

   the projective rank function $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each* K_i *is the constant function* $\mathfrak{p} \mapsto d$

   (note that projectivity follows from ***SLM2*** *and that* $\mathrm{rank}_R(\Lambda_i) = 2d$*)*

4. ***SLM4***

   $K_0^\perp = K_0$ *and* $K_{d/2}^\perp = pK_{d/2}$ *with respect to the restrictions* $\Lambda_0 \times \Lambda_0 \to \mathcal{O}$ *and* $\Lambda_{d/2} \times p\Lambda_{d/2} \to \mathcal{O}$ *of* ϕ.

2.4 Definition of the enlarged models

2.4.1 Alternate description of the local model using a single base lattice

This subsection rephrases the definition of $\mathbf{M}^{\mathrm{loc}}$ so that its points consist of submodules of the single lattice Λ_0 and it will then be easier to see how to enlarge the models satisfactorily to a degeneration of the affine Grassmannian to the full affine flag variety.

For $i = 1, \ldots, d$, let $\alpha_i : F^d \to F^d$ be the F-linear map

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{i-1}, px_i, x_{i+1}, \ldots, x_d)$$

Then α_i induces an $\mathcal{O}$-module isomorphism $\Lambda_i \xrightarrow{\sim} \Lambda_{i-1}$. Define

$$\alpha_{[i]} \stackrel{\mathrm{def}}{=} \alpha_i \circ \cdots \circ \alpha_1$$

Then $\alpha_{[i]}$ induces an $\mathcal{O}$-module isomorphism $\Lambda_i \xrightarrow{\sim} \Lambda_0$. Define

$$\alpha^{[i]} \stackrel{\mathrm{def}}{=} \alpha_d \circ \cdots \circ \alpha_{d-i+1}$$

Pick a (commutative) $\mathbf{Z}_p$-algebra R, and consider a point

$$\begin{array}{ccccccc} \Lambda_0 \otimes_{\mathbf{Z}_p} R & \longrightarrow & \Lambda_1 \otimes_{\mathbf{Z}_p} R & \longrightarrow & \cdots & \longrightarrow & \Lambda_{d/2} \otimes_{\mathbf{Z}_p} R \\ \cup & & \cup & & \cdots & & \cup \\ K_0 & \longrightarrow & K_1 & \longrightarrow & \cdots & \longrightarrow & K_{d/2} \end{array}$$

of $\mathbf{M}^{\mathrm{loc}}(R)$.

Specifying $K_i \subset \Lambda_i \otimes_{\mathbf{Z}_p} R$ is the same as specifying $\alpha_{[i]}(K_i) \subset \Lambda_0 \otimes_{\mathbf{Z}_p} R$, so define

$$L_i \stackrel{\text{def}}{=} \alpha_{[i]}(K_i)$$

(note that $L_0 = K_0$). The condition that $\Lambda_i \otimes_{\mathbf{Z}_p} R \to \Lambda_{i+1} \otimes_{\mathbf{Z}_p} R$ restrict to $K_i \to K_{i+1}$ is equivalent to the condition that $\alpha_{i+1}(L_i) \subset L_{i+1}$ (note that $\alpha_{i+1} \circ \alpha_{[i]} = \alpha_{[i+1]}$). So, a point of $\mathbf{M}^{\text{loc}}(R)$ is equivalent to a tuple $(L_0, \ldots, L_{d/2})$ satisfying $\alpha_{i+1}(L_i) \subset L_{i+1}$ for each $0 \leq i < d/2$ and the implicit equivalents of **SLM**.

For **SLM2**, note that if s is an R-linear splitting of $K_i \hookrightarrow \Lambda_i \otimes_{\mathbf{Z}_p} R$ then $\alpha_{[i]}^{-1} \circ s \circ \alpha_{[i]}$ is an R-linear splitting of $L_i \subset \Lambda_0 \otimes_{\mathbf{Z}_p} R$, so that condition is identical: require that each inclusion $L_i \subset \Lambda_0 \otimes_{\mathbf{Z}_p} R$ split R-linearly. It is obvious that the projective rank condition **SLM3** is the same for the K_i as for the L_i.

For **SLM4**, note that the condition $K_0^\perp = K_0$ is equivalent to the condition $L_0^\perp = L_0$ tautologically. The fact that

$$\phi(\alpha_{[i]}(x), y) = \phi(x, \alpha^{[i]}(y))$$

and

$$\alpha_{[d/2]} \circ \alpha^{[d/2]} = p$$

suggests the following:

Lemma 2.4.1.1. *$K_{d/2}^\perp = pK_{d/2}$ $(\subset p\Lambda_{d/2})$ with respect to $\phi : \Lambda_{d/2} \times p\Lambda_{d/2} \to \mathcal{O}$* ***if and only if*** *$L_{d/2}^\perp = L_{d/2}$ with respect to $\phi : \Lambda_0 \times \Lambda_0 \to \mathcal{O}$.*

Proof. $\boxed{\Rightarrow}$ $\boxed{\subset}$ If $y \in \Lambda_0$ then $\alpha^{[d/2]}(y) \in p\Lambda_{d/2}$, and since

$$\phi_R(L_{d/2}, y) = \phi_R(\alpha_{[d/2]}(K_{d/2}), y) = \phi_R(K_{d/2}, \alpha^{[d/2]}(y)),$$

the fact that $\phi(L_{d/2}, y) = 0$ implies that $\alpha^{[d/2]}(y) \in K_{d/2}^{\perp} = pK_{d/2}$. This means that

$$py = \alpha_{[d/2]}(\alpha^{[d/2]}(y)) \in \alpha_{[d/2]}(pK_{d/2}) = pL_{d/2}$$

and so $y \in L_{d/2}$. $\boxed{\supset}$ This is easy to verify: if $x \in L_{d/2}$ then $x = \alpha_{[d/2]}(x')$ for some $x' \in K_{d/2}$ so

$$\phi_R(L_{d/2}, x) = \phi_R(\alpha_{[d/2]}(K_{d/2}), \alpha_{[d/2]}(x')) = p\phi_R(K_{d/2}, x') = 0$$

and so $x \in L_{d/2}^{\perp}$. $\boxed{\Leftarrow}$ $\boxed{\subset}$ If $y \in p\Lambda_{d/2}$ is such that $\phi_R(K_{d/2}, y) = 0$ then write $y = py'$ for some $y' \in \Lambda_{d/2}$ so that

$$\phi_R(L_{d/2}, \alpha_{[d/2]}(y')) = \phi_R(K_{d/2}, \alpha^{[d/2]}(\alpha_{[d/2]}(y'))) = \phi_R(K_{d/2}, y) = 0$$

using the same ideas as before. Since $\alpha_{[d/2]}(y') \in \Lambda_0$, the hypothesis gives $\alpha_{[d/2]}(y') \in L_{d/2}$ and so $y' \in K_{d/2}$. $\boxed{\supset}$ This is similar. $\square$

Definition: The Alternate Description of the Local Model. *According to the previous subsection, the functor* $\mathbf{M}^{\mathrm{loc}}$ *can also be described as assigning to each (commutative)* $\mathbf{Z}_p$ *algebra* R*, the set of all tuples* $(L_0, L_1, \ldots, L_{d/2})$ *of* $(\mathcal{O} \otimes_{\mathbf{Z}_p} R)$*-submodules of* $\Lambda_0 \otimes_{\mathbf{Z}_p} R$ *satisfying:*

1. ***ALM1***

   $\alpha_{i+1}(L_i) \subset L_{i+1}$ *for all* i

2. ***ALM2***

   each inclusion $L_i \subset \Lambda_0 \otimes_{\mathbf{Z}_p} R$ *splits* R*-linearly*

3. ***ALM3***

   the projective rank function $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each* L_i *is the constant function* $\mathfrak{p} \mapsto d$

4. ***ALM4***

$L_0^\perp = L_0$ and $L_{d/2}^\perp = L_{d/2}$ with respect to the restriction $\phi : \Lambda_0 \times \Lambda_0 \to \mathcal{O}$.

This description is the key to constructing larger schemes analogous to $\mathbf{M}^{\mathrm{loc}}$.

2.4.2 Definition of the larger models

In this subsection, I will define a family of functors

$$\mathbf{M}^{(m,n)} : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$$

over all $m, n \in \mathbf{N}$ such that $\mathbf{M}^{(0,1)} = \mathbf{M}^{\mathrm{loc}}$, and such that the generic (resp. special) fibers form an increasing and exhaustive filtration of the affine Grassmannian (resp. full affine flag variety). Fix $m, n \in \mathbf{N}$.

The anti-identity matrix $\mathrm{id}^\vee$ induces a non-degenerate hermitian $\mathbf{Z}_p[t]$-bilinear form

$$\phi : \frac{t^{-m}\mathcal{O}[t]^d}{t^n\mathcal{O}[t]^d} \times \frac{t^{-m}\mathcal{O}[t]^d}{t^n\mathcal{O}[t]^d} \longrightarrow \frac{t^{-2m}\mathcal{O}[t]}{t^{n-m}\mathcal{O}[t]}$$

by the rule $(v, w) \mapsto v^{\mathrm{tr}} \cdot \mathrm{id}^\vee \cdot \overline{w}$, where $w \mapsto \overline{w}$ is induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q}_p)$. For an $\mathcal{R}[t]$-submodule $L \subset t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ define $L^\perp$ in the usual way:

$$L^\perp \overset{\mathrm{def}}{=} \{v \in t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d \mid \phi_R(L, v) = 0\}$$

(note that "0" here refers to the zero-element of the codomain $t^{-2m}\mathcal{O}[t]/t^{n-m}\mathcal{O}[t]$)

Note that when $(m, n) = (0, 1)$ this recovers "$\phi : \Lambda_0 \times \Lambda_0 \to \mathcal{O}$" and the concept of "$\perp$" from the previous subsection via the identification $\mathcal{O}[t]^d/t\mathcal{O}[t]^d = \Lambda_0$. Therefore, the abuse of notation is acceptable.

The rule

$$(x_1, \dots, x_d) \mapsto (x_1, \dots, x_{i-1}, (t+p)x_i, x_{i+1}, \dots, x_d)$$

induces an $\mathcal{O}[t]$-linear map

$$\alpha_i : \frac{t^{-m}\mathcal{O}[t]^d}{t^n\mathcal{O}[t]^d} \longrightarrow \frac{t^{-m}\mathcal{O}[t]^d}{t^n\mathcal{O}[t]^d}$$

When $(m, n) = (0, 1)$, this map is the restriction to Λ_0 of the "α_i" from the previous subsection via the identification $\mathcal{O}[t]^d/t\mathcal{O}[t]^d = \Lambda_0$, so the abuse of notation is acceptable. As before, define

$$\alpha_{[i]} \stackrel{\text{def}}{=} \alpha_i \circ \cdots \circ \alpha_1$$

The following definition is based on **ALM** and is strongly analogous to the symplectic case in [15]:

Definition: The Enlarged Model (preliminary). *Define the functor*

$$\mathbf{M}^{(m,n)} : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set (for simplicity of notation, set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$) of tuples $(L_0, L_1, \dots, L_{d/2})$ of $\mathcal{R}[t]$-submodules of $t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ satisfying:

- $\alpha_{i+1}(L_i) \subset L_{i+1}$ *for all* $0 \leq i < d/2$
- *each inclusion* $L_i \subset t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ *splits R-linearly*
- *the projective rank function* $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each L_i is the constant function* $\mathfrak{p} \mapsto d(m+n)$

  (note that $\mathrm{rank}_R(t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d) = 2d(m+n)$*)*

- $L_0^\perp = L_0$ *and* $L_{d/2}^\perp = L_{d/2}$ *with respect to* ϕ_R.

It is clear that $\mathbf{M}^{(0,1)} = \mathbf{M}^{\mathrm{loc}}$.

Remark. *I will later embed* $\mathbf{M}_{\mathbf{F}_p}^{(m,n)}$ *into a full affine flag variety* $\mathcal{F}\ell_{\mathbf{F}_p}^{\mathrm{aff}}$ *and the freedom of two parameters* m, n *is necessary in order to exhaust* $\mathcal{F}\ell_{\mathbf{F}_p}^{\mathrm{aff}}$.

I now reformulate the definition of $\mathbf{M}^{(m,n)}$ so that it is more obviously a degeneration from the affine Grassmannian over $\mathbf{Q}_p$ to the full affine flag variety over $\mathbf{F}_p$. Now that the variable t and the parameters m, n have been introduced, I am in a sense returning to the point of view used for **SLM**.

Define $\mathcal{V} = \mathcal{O}[t, t^{-1}, (t+p)^{-1}]^d$ and submodules

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{O}[t]^d \\
\mathcal{V}_1 &= (t+p)^{-1}\mathcal{O}[t] \oplus \mathcal{O}[t]^{d-1} \\
&\vdots \\
\mathcal{V}_{d-1} &= (t+p)^{-1}\mathcal{O}[t]^{d-1} \oplus \mathcal{O}[t] \\
\mathcal{V}_d &= (t+p)^{-1}\mathcal{O}[t]^d
\end{aligned}$$

Note that the $\alpha_{[i]}$, as $\mathcal{O}[t]$-linear maps of $\mathcal{V}$, induce isomorphisms

$$\alpha_{[i]} : \frac{t^{-m}\mathcal{V}_i}{t^n\mathcal{V}_i} \xrightarrow{\sim} \frac{t^{-m}\mathcal{O}[t]^d}{t^n\mathcal{O}[t]^d} \tag{2.8}$$

(these are the generalizations of the isomorphisms "$\alpha_{[i]} : \Lambda_i \xrightarrow{\sim} \Lambda_0$" from §2.4.1)

Let R be a (commutative) $\mathbf{Z}_p$-algebra and consider $(L_0, L_1, \ldots, L_{d/2}) \in \mathbf{M}^{(m,n)}(R)$. Let $\overline{\mathcal{L}}_i$ be the submodule of $t^{-m}\mathcal{V}_i/t^n\mathcal{V}_i$ such that $\alpha_{[i]}(\overline{\mathcal{L}}_i) = L_i$. Let $\mathcal{L}_i$ be the submodule of $\mathcal{V}(R)$ satisfying

$$t^n\mathcal{V}_i(R) \subset \mathcal{L}_i \subset t^{-m}\mathcal{V}_i(R) \tag{2.9}$$

which corresponds to $\overline{\mathcal{L}}_i$. The requirement that $\alpha_{i+1}(L_i) \subset L_{i+1}$ for all $0 \leq i < d/2$ is equivalent to the requirement that $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{d/2}$.

The requirement that each inclusion $L_i \subset t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ split R-linearly is equivalent to the requirement that each inclusion $\overline{\mathcal{L}}_i \subset t^{-m}\mathcal{V}_i(R)/t^n\mathcal{V}_i(R)$ split R-linearly.

The anti-identity matrix $\mathrm{id}^\vee$ defines a non-degenerate hermitian $\mathbf{Z}_p[t, t^{-1}, (t+p)^{-1}]$-bilinear product

$$\phi : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{O}[t, t^{-1}, (t+p)^{-1}]$$

by the rule $(v, w) \mapsto v^{\mathrm{tr}} \cdot \mathrm{id}^\vee \cdot \overline{w}$, where $w \mapsto \overline{w}$ is induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q}_p)$. Since this ϕ induces (restrict and descend) the previously defined ϕ, this abuse of notation is acceptable.

For $0 \leq i \leq d/2$, define

$$\phi^{[i]} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{O}[t, t^{-1}, (t+p)^{-1}]$$

by $\phi^{[i]}(x, y) \stackrel{\mathrm{def}}{=} \phi(\alpha_{[i]}(x), \alpha_{[i]}(y))$. Note that $\phi^{[d/2]} = (t+p)\phi$.

For $0 \leq i \leq d/2$, define

$$\widehat{\mathcal{L}}_i \stackrel{\mathrm{def}}{=} \{x \in \mathcal{V}(R) \mid \phi_R^{[i]}(\mathcal{L}_i, x) \in t^{n-m}\mathcal{R}[t]\}$$

(I am abusing notation: the concept of duality here depends on i). This concept of duality is specifically designed to match up with the concept of "$\perp$" above:

Lemma 2.4.2.1. *1. $L_0^\perp = L_0$ if and only if $\widehat{\mathcal{L}}_0 = \mathcal{L}_0$, and*

2. $L_{d/2}^\perp = L_{d/2}$ if and only if $\widehat{\mathcal{L}}_{d/2} = \mathcal{L}_{d/2}$.

Proof. $\boxed{\text{case (1)}}$ $\boxed{\Rightarrow}$ For $\lambda \in t^{-m}\mathcal{V}_0(R)$, denote by $\overline{\lambda}$ the image in $t^{-m}\mathcal{V}_0(R)/t^n\mathcal{V}_0(R) = t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$. $\boxed{\subset}$ Suppose $\lambda \in \widehat{\mathcal{L}}_0$, i.e. suppose that $\lambda \in \mathcal{V}(R)$ satisfies $\phi_R^{[0]}(\mathcal{L}_0, \lambda) \in t^{n-m}\mathcal{R}[t]$. Since $\mathcal{L}_0$ satisfies containments (2.9), so does $\widehat{\mathcal{L}}_0$ and so $\lambda \in t^{-m}\mathcal{R}[t]^d$. Altogether, $\overline{\lambda} \in L_0^{\perp}$ (the previous containment shows that $\overline{\lambda}$ is in the domain of the hermitian form defining "$L_0^{\perp}$"). By hypothesis, $\overline{\lambda} \in \mathcal{L}_0$ and so $\lambda \in \mathcal{L}_0$. $\boxed{\supset}$ This is obvious: if $\lambda \in \mathcal{L}_0$ then $\overline{\lambda} \in L_0$ and by hypothesis $\phi_R(L_0, \overline{\lambda}) = 0$ so $\phi_R^{[0]}(\mathcal{L}_0, \lambda) \in t^{n-m}\mathcal{R}[t]$ since both hermitian forms use the same Gram matrix. $\boxed{\Leftarrow}$ $\boxed{\subset}$ Suppose $\overline{\lambda} \in L_0^{\perp}$, i.e. suppose that $\overline{\lambda} \in t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ satisfies $\phi_R(L_0, \overline{\lambda}) = 0$. Let $\lambda \in \mathcal{L}_0$ be any representative of $\overline{\lambda}$. Then $\phi_R^{[0]}(\mathcal{L}_0, \lambda) \in t^{n-m}\mathcal{R}[t]$ and by hypothesis, $\lambda \in \mathcal{L}_0$ so $\overline{\lambda} \in L_0$. $\boxed{\supset}$ This is obvious: if $\overline{\lambda} \in L_0$ then $\lambda \in \mathcal{L}_0$ and by hypothesis $\phi_R^{[0]}(\mathcal{L}_0, \lambda) \in t^{n-m}\mathcal{R}[t]$ so $\phi_R(L_0, \overline{\lambda}) = 0$ since both hermitian forms use the same Gram matrix. $\boxed{\text{case (2)}}$ This proof is nearly identical. □

The previous discussion proves the following:

Definition: The Enlarged Model (final). *The functor $\mathbf{M}^{(m,n)}$ can also be described as assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set of tuples*

$$(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{d/2})$$

of $(\mathcal{O} \otimes_{\mathbf{Z}_p} R)[t]$-submodules of $\mathcal{V}(R)$ satisfying

1. ***ELM1***

   $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{d/2}$

2. ***ELM2***

   $t^n\mathcal{V}_i(R) \subset \mathcal{L}_i \subset t^{-m}\mathcal{V}_i(R)$ *for all i*

3. ***ELM3***

   each inclusion $\mathcal{L}_i/t^n\mathcal{V}_i(R) \hookrightarrow t^{-m}\mathcal{V}_i(R)/t^n\mathcal{V}_i(R)$ *splits* R*-linearly.*

4. ***ELM4***

   the projective rank function $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each*

$$\mathcal{L}_i/t^n\mathcal{V}_i(R)$$

   is the constant function $\mathfrak{p} \mapsto d(m+n)$

   (note that $\mathrm{rank}_R(t^{-m}\mathcal{V}_i(R)/t^n\mathcal{V}_i(R)) = 2d(m+n)$*)*

5. ***ELM5***

   $\widehat{\mathcal{L}}_0 = \mathcal{L}_0$ *and* $\widehat{\mathcal{L}}_{d/2} = \mathcal{L}_{d/2}$

   (these concepts of duality were defined on page 55)

For future use, define

$$\mathcal{V}_{\mathrm{inf}} = t^n\mathcal{O}[t]^d$$

$$\mathcal{V}_{\mathrm{sup}} = t^{-m}(t+p)^{-1}\mathcal{O}[t]^d$$

$$\overline{\mathcal{V}}_{\mathrm{sup}} = \mathcal{V}_{\mathrm{sup}}/\mathcal{V}_{\mathrm{inf}}$$

$$\overline{\mathcal{V}}_i = \mathcal{V}_i/\mathcal{V}_{\mathrm{inf}}$$

The first two are the largest (resp. smallest) modules contained in (resp. containing) all the modules used in **ELM2**.

2.4.3 The enlarged models are projective schemes

For each $0 \leq i \leq d/2$, let

$$\mathrm{Gr}_i : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$$

denote the (ordinary) Grassmannian of direct summands of

$$t^{-m}\mathcal{V}_i/t^n\mathcal{V}_i \cong \mathbf{Z}_p^{2d(m+n)}$$

with constant projective rank function $d(m+n)$. Then **ELM3** and **ELM4** yield a closed embedding

$$\mathbf{M}^{(m,n)} \hookrightarrow \mathrm{Gr}_0 \times \cdots \times \mathrm{Gr}_{d/2}$$

Take $(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$. Consider another pair $m', n' \in \mathbf{N}$. If $(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m',n')}(R)$, then necessarily $m - n = m' - n'$ since $\mathcal{L}_0$ can only be self-dual with respect to $t^N\phi$ for one N. On the other hand, if $m' \geq m$ and $n' \geq n$, then requirement **ELM2** for $\mathbf{M}^{(m',n')}(R)$ trivially follows from requirement **ELM2** for $\mathbf{M}^{(m,n)}(R)$.

These 2 requirements on m, n, m', n' already imply that $\mathbf{M}^{(m,n)} \subset \mathbf{M}^{(m',n')}$: the R-linear splitting of the short-exact-sequence

$$0 \to t^n\mathcal{V}_i(R)/t^{n'}\mathcal{V}_i(R) \to \mathcal{L}_i/t^{n'}\mathcal{V}_i(R) \to \mathcal{L}_i/t^n\mathcal{V}_i(R) \to 0$$

shows that **ELM3** is satisfied, and this sequence also shows that

$$\begin{aligned}
\mathrm{rank}_R(\mathcal{L}_i/t^{n'}\mathcal{V}_i(R)) &= \mathrm{rank}_R(t^n\mathcal{V}_i(R)/t^{n'}\mathcal{V}_i(R)) + \mathrm{rank}_R(\mathcal{L}_i/t^n\mathcal{V}_i(R)) \\
&= 2(n'-n)d + (m+n)d \\
&= (m'-m)d + (n'-n)d + (m+n)d \\
&= (m'+n')d
\end{aligned}$$

(the "2" here comes from the fact that the coefficients of the polynomials are in $\mathcal{R}$ and $\mathrm{rank}_R(\mathcal{R}) = 2$)

In summary, for each $\Delta \in \mathbf{Z}$, the set of $(m,n) \in \mathbf{N} \times \mathbf{N}$ such that $n - m = \Delta$ is totally-ordered and

$$\mathbf{M}^{(0,\Delta)} \subset \mathbf{M}^{(1,1+\Delta)} \subset \mathbf{M}^{(2,2+\Delta)} \subset \cdots$$

Remark. *In §2.4.6 (page 63), I will embed $\mathbf{M}^{(m,n)}_{\mathbf{F}_p}$ into an affine flag variety, and from that perspective, the chain associated to a particular $\Delta \in \mathbf{Z}$ exhausts the corresponding connected component of the affine flag variety. See Theorem 5.1 in [26] for a way to calculate the component group of an affine flag variety in the non-split case.*

Notice that if $m = n$ then the trivial tuple $(\mathcal{V}_0(R), \ldots, \mathcal{V}_{d/2}(R))$ satisfies all the conditions necessary for membership in $\mathbf{M}^{(m,m)}(R)$, the assumption being required for the rank. From the point of view of the affine flag variety, this is because the identity component of the affine flag variety is indexed by $\Delta = 0$.

2.4.4 Equivalent characterizations of Zariski-lattices

The description of the functor-of-points of an affine flag variety uses a certain definition of lattice, but other characterizations are needed to embed local models into affine flag varieties. The following list of characterizations is summarized as:

Equivalent Characterizations of Lattices (Lemma 2.11 in [11]). *Let $\mathcal{R}$ be a commutative ring and let $M \subset \mathcal{R}((t))^d$ be an $\mathcal{R}[[t]]$-submodule. The following 4 sets of conditions are equivalent:*

1. *(a) there is some N such that $t^N\mathcal{R}[[t]]^d \subset M \subset t^{-N}\mathcal{R}[[t]]^d$*

   (b) as an $\mathcal{R}$-module, the quotient $M/t^N\mathcal{R}[[t]]^d$ is projective

2. *(a) the product $M \otimes_{\mathcal{R}[[t]]} \mathcal{R}((t)) \to \mathcal{R}((t))^d$ is an isomorphism*

   (b) as an $\mathcal{R}[[t]]$-module, M is finitely-generated and projective

   (c) the projective rank function $\mathrm{Spec}(\mathcal{R}[[t]]) \to \mathbf{N}$ associated to M is the constant function $\mathfrak{p} \mapsto d$

3. *(a) the product $M \otimes_{\mathcal{R}[[t]]} \mathcal{R}((t)) \to \mathcal{R}((t))^d$ is an isomorphism*

   (b) Zariski-locally on $\mathrm{Spec}(\mathcal{R})$, M is a free $\mathcal{R}[[t]]$-module

4. *(a) the product $M \otimes_{\mathcal{R}[[t]]} \mathcal{R}((t)) \to \mathcal{R}((t))^d$ is an isomorphism*

   (b) fpqc-locally on $\mathrm{Spec}(\mathcal{R})$, M is a free $\mathcal{R}[[t]]$-module

I call such an M a "Zariski-lattice".

Conditions (3b) and (4b) require some clarification. Condition (3b) means that there are elements $r_1, \ldots, r_n$ generating the trivial ideal R (i.e. a system of principal open sets covering $\mathrm{Spec}(R)$) such that each fraction module $M[r_i^{-1}]$ is a free $R_{r_i}[[t]]$-module. A similar comment holds for (4b).

2.4.5 The full affine flag variety over $\mathbf{F}_p$

The setup here is: I use the unramified quadratic extension $\mathbf{F}_p((t)) \subset \mathbf{F}((t))$, the vector space $\mathbf{F}((t))^d$, and the standard hermitian form $\Phi : \mathbf{F}((t))^d \times \mathbf{F}((t))^d \to \mathbf{F}((t))$ defined by the anti-identity matrix. Note that Φ induces ϕ from previous sections.

Definition: The Affine Flag Variety. *The full affine flag variety over $\mathbf{F}_p$ is the functor $\mathcal{F}\ell^{\mathrm{aff}}$ that assigns to any (commutative) $\mathbf{F}_p$-algebra R (for simplicity of notation, set $\mathcal{R} := R \otimes_{\mathbf{F}_p} \mathbf{F}$) the set of all tuples $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ of $\mathcal{R}[[t]]$-submodules of $\mathcal{R}((t))^d$ satisfying:*

1. ***AFV1***

   it is periodic, in the sense that $\mathcal{F}_i = t \cdot \mathcal{F}_{i+d}$ for all i

2. ***AFV2***

   it forms a chain, $\cdots \subset \mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \cdots$

3. ***AFV3***

   each $\mathcal{F}_i$ is a Zariski-lattice

4. ***AFV4***

   Zariski-locally on $\mathrm{Spec}(R)$, each quotient $\mathcal{F}_{i+1}/\mathcal{F}_i \cong \mathcal{R}$ as $\mathcal{R}$-modules

5. ***AFV5***

Zariski-locally on $\operatorname{Spec}(R)$*, there exists* $u(t) \in R((t))^\times$ *such that*

$$\mathcal{F}_{-i} = u(t) \cdot \widehat{\mathcal{F}}_i$$

for all $i \in \mathbf{N}$*. Here*

$$\widehat{\mathcal{F}} \stackrel{\text{def}}{=} \{w \in \mathcal{R}((t))^d : \Phi_R(\mathcal{F}, w) \subset \mathcal{R}[[t]]^d\}$$

For a commutative ring homomorphism $R \to S$, the function

$$\mathcal{F}\ell^{\text{aff}}(R) \to \mathcal{F}\ell^{\text{aff}}(S)$$

is the application of the completed tensor product $-\widehat{\otimes}_R S$, which is designed so that $R[[t]]\widehat{\otimes}_R S = S[[t]]$ etc.

Remark. *Note that in* ***AFV5****, the Zariski-local cover of* $\operatorname{Spec}(R)$ *and the corresponding similitudes are independent of* i*.*

As in §2.3, a point $(\mathcal{F}_i)_{i \in \mathbf{Z}} \in \mathcal{F}\ell^{\text{aff}}(R)$ is completely determined by the finite chain $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{d/2}$ as follows: recover the Zariski-local cover of $\operatorname{Spec}(R)$ and the common degree k of the similitudes in **AFV5** by comparing $\mathcal{F}_0$ to $\widehat{\mathcal{F}}_0$, define $\mathcal{F}_{-i}$ for $0 < i < d/2$ Zariski-locally by $\mathcal{F}_{-i} \stackrel{\text{def}}{=} t^k \widehat{\mathcal{F}}_i$, and extend periodically. It is automatic from the definition that $\cdots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0$. Using the finite chain, **AFV1** disappears and the only part of **AFV5** that survives is: there exists Zariski-locally on $\operatorname{Spec}(R)$ a $u(t) \in R((t))^\times$ such that

$$\mathcal{F}_0 = u(t) \cdot \widehat{\mathcal{F}}_0$$

$$\mathcal{F}_{d/2} = t^{-1} u(t) \cdot \widehat{\mathcal{F}}_{d/2}$$

2.4.6 The special fibers are subschemes of the full affine flag variety

Let R be an $\mathbf{F}_p$-algebra. Consider $(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$. First, note that

$$\begin{aligned}
\mathcal{V}(\mathbf{F}_p) &= \mathbf{F}[t, t^{-1}]^d \subset \mathbf{F}((t))^d \\
\mathcal{V}_0(\mathbf{F}_p) &= \mathbf{F}[t]^d \\
\mathcal{V}_1(\mathbf{F}_p) &= t^{-1}\mathbf{F}[t] \oplus \mathbf{F}[t]^{d-1} \\
&\vdots \\
\mathcal{V}_{d-1}(\mathbf{F}_p) &= t^{-1}\mathbf{F}[t]^{d-1} \oplus \mathbf{F}[t] \\
\mathcal{V}_d(\mathbf{F}_p) &= t^{-1}\mathbf{F}[t]^d
\end{aligned}$$

So each $\mathcal{L}_i$ is an $\mathcal{R}[t]$-submodule of $\mathcal{V}(R) \subset \mathcal{R}((t))^d$ satisfying

$$t^n\mathcal{R}[t]^d = \mathcal{V}_{\inf}(R) \subset \mathcal{L}_i \subset \mathcal{V}_{\sup}(R) = t^{-(m+1)}\mathcal{R}[t]^d \tag{2.10}$$

Such modules are in canonical bijection with $\mathcal{R}[[t]]$-submodules $\mathcal{F}_i$ of $\mathcal{R}((t))^d$ satisfying

$$t^n\mathcal{R}[[t]]^d \subset \mathcal{F}_i \subset t^{-(m+1)}\mathcal{R}[[t]]^d \tag{2.11}$$

Let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{d/2}$ be the modules satisfying equation (2.11) corresponding to the modules $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}$ in equation (2.10).

Proposition 2.4.6.1. $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{d/2}) \in \mathcal{F}\ell^{\mathrm{aff}}(R)$.

Once this is proven, it is obvious that $\mathbf{M}^{(m,n)}(R) \to \mathcal{F}\ell^{\mathrm{aff}}(R)$ is *injective*. By §2.4.3 (page 58), $\mathbf{M}^{(m,n)}$ is a proper $\mathbf{Z}_p$-scheme, so $\mathbf{M}^{(m,n)} \hookrightarrow \mathcal{F}\ell^{\mathrm{aff}}$ is a proper morphism. By Corollary 12.92 of [12], $\mathbf{M}^{(m,n)} \hookrightarrow \mathcal{F}\ell^{\mathrm{aff}}$ is a *closed embedding*.

Proof. $\boxed{\textbf{AFV2}}$ This is automatic. $\boxed{\textbf{AFV3}}$ For this, it is obviously most convenient to use characterization (1) of Zariski-lattices (page 60). Equation (2.11) gives part (a) of the characterization. Now notice that the coefficient ring of the power series here is $\mathcal{R} = R \otimes_{\mathbf{F}_p} \mathbf{F}$, not R. This means that in this situation, condition (b) actually requires $\overline{\mathcal{L}_i}$ to be a projective as an $\mathcal{R}$-module, but **ELM3** and **ELM4** together imply only projectivity over R, but since $R \to \mathcal{R}$ makes $\mathcal{R}$ a finitely-generated R-module and a faithfully-flat R-algebra, Lemma 1.3.0.1 (page 24) says that $\overline{\mathcal{L}_i}$ is in fact a projective $\mathcal{R}$-module. $\boxed{\textbf{AFV4}}$ By construction, it suffices to prove the same fact for the quotient $\mathcal{L}_{i+1}/\mathcal{L}_i$. It is easy to prove but the notation becomes truly oppressive. For clarity, I prove this as a lemma following the end of the current proof. $\boxed{\textbf{AFV5}}$ I claim that $\widehat{\mathcal{F}}_0 = t^{m-n}\mathcal{F}_0$ and $\widehat{\mathcal{F}}_{d/2} = t^{m-n+1}\mathcal{F}_{d/2}$. This is clear because the products used in **ELM5** for $\mathcal{L}_0$ and $\mathcal{L}_{d/2}$ are just the standard ones multiplied by t^{m-n} and $t^{m-n}(t+p)$, and the concept of duality is the same. $\square$

Remark. *Note that the need for Lemma 1.3.0.1 (page 24) does not occur in the case of a* ramified *unitary group, since in that case the coefficient ring does not change (the* uniformizer *changes).*

Lemma 2.4.6.2. *Consider an $\mathcal{R}$-module diagram*

$$\begin{array}{ccc} \mathcal{L}_1 & \subset & V_1 \\ \cap & & \cap \\ \mathcal{L}_2 & \subset & V_2 \end{array}$$

If

1. *both quotients $V_1/\mathcal{L}_1$ and $V_2/\mathcal{L}_2$ are finitely-generated projective R-modules*

2. *the projective rank functions of* $V_1/\mathcal{L}_1$ *and* $V_2/\mathcal{L}_2$ *are constant on* $\mathrm{Spec}(R)$ *and the two constants are equal*

3. *the quotient* V_2/V_1 *is a free* $\mathcal{R}$*-module of rank* 1.

then $\mathcal{L}_2/\mathcal{L}_1$ *is a projective* $\mathcal{R}$*-module with constant projective rank function* 1.

To use this in the proof of Proposition 2.4.6.1, use the diagram

$$\begin{array}{ccc} \mathcal{L}_i & \subset & t^{-m}\mathcal{V}_i(R) \\ \cap & & \cap \\ \mathcal{L}_{i+1} & \subset & t^{-m}\mathcal{V}_{i+1}(R) \end{array}$$

from **ELM2**. The vertical inclusion on the right is valid (i.e. there are no problems due to non-flatness) because $\mathcal{V}_i \subset \mathcal{V}_{i+1}$ can be written as the obvious inclusion $\mathcal{O}^{\mathbf{N}} \subset \mathcal{O}^{\mathbf{N}} \oplus \mathcal{O}$.

Hypothesis (1) in this lemma is implied by **ELM3** and hypothesis (2) follows from **ELM4**. Hypothesis (3) is obvious from the definition of the $\mathcal{V}_i$.

Proof. By hypothesis (1), the short-exact-sequence

$$0 \to (\mathcal{L}_2/\mathcal{L}_1) \to (V_2/\mathcal{L}_1) \to (V_2/\mathcal{L}_2) \to 0$$

of $\mathcal{R}$-modules splits R-linearly. By hypothesis (3), the short-exact-sequence

$$0 \to (V_1/\mathcal{L}_1) \to (V_2/\mathcal{L}_1) \to (V_2/V_1) \to 0$$

of $\mathcal{R}$-modules splits $\mathcal{R}$-linearly. These splittings produce an R-module isomorphism

$$(\mathcal{L}_2/\mathcal{L}_1) \oplus (V_2/\mathcal{L}_2) \cong (V_1/\mathcal{L}_1) \oplus (V_2/V_1) \tag{2.12}$$

By hypotheses (1) and (3), the right-hand-side of equation (2.12) is a projective R-module, which shows that $\mathcal{L}_2/\mathcal{L}_1$ is a projective R-module. By Lemma 1.3.0.1, $\mathcal{L}_2/\mathcal{L}_1$ is a projective $\mathcal{R}$-module. Counting projective ranks over R on both sides of equation (2.12) and using hypothesis (2) shows that the projective rank function $\mathrm{Spec}(R) \to \mathbf{N}$ of $\mathcal{L}_2/\mathcal{L}_1$ is the constant function $\mathfrak{p} \mapsto \mathrm{rank}_R(V_2/V_1)$. Hypothesis (3) finishes the proof (use $\mathrm{rank}_R(\mathcal{R}) = 2$). $\square$

2.4.7 The special fibers exhausts the full affine flag variety

Let R be a (commutative) $\mathbf{F}_p$-algebra and set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$. Choose $(\mathcal{F}_i)_{i \in \mathbf{Z}} \in \mathcal{F}\ell^{\mathrm{aff}}(R)$. Let $u(t) \in \mathbf{F}_p((t))$ be the similitude for $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ occurring in **AFV5**. Denote by Δ the degree of $u(t)$. From the verification of **AFV5** in the previous subsection I know that Δ will be the future value of $m - n$. So, let $m, n \in \mathbf{N}$ be such that $m - n = \Delta$ and such that

$$t^n \lambda_i(R) \subset \mathcal{F}_i \subset t^{-m} \lambda_i(R) \tag{2.13}$$

for all $i = 0, \ldots, d/2$. From the discussion in §2.4.3 (page 58), I know that there is no danger in choosing m, n too large.

By passing through the quotient, define $\mathcal{L}_i$ to be the $\mathcal{R}[t]$-module satisfying

$$t^n \mathcal{V}_i(R) \subset \mathcal{L}_i \subset t^{-m} \mathcal{V}_i(R)$$

corresponding to $\mathcal{F}_i$ for each $i = 0, \ldots, d/2$. It is obvious that $\mathcal{L}_0 \subset \cdots \subset \mathcal{L}_{d/2}$. For each i, I have the short-exact-sequence of $\mathbf{F}[[t]]$-modules

$$0 \to \mathcal{F}_{i+1}/\mathcal{F}_i \to t^{-m}\mathcal{R}[[t]]^d/\mathcal{F}_i \to t^{-m}\mathcal{R}[[t]]^d/\mathcal{F}_{i+1} \to 0$$

By **AFV3** and **AFV4**,

$$\operatorname{rank}_{\mathcal{R}}(t^{-m}\mathcal{R}[[t]]^d/\mathcal{F}_i) = \operatorname{rank}_{\mathcal{R}}(t^{-m}\mathcal{R}[[t]]^d/\mathcal{F}_{i+1}) + 1$$

which implies that

$$\operatorname{rank}_{\mathcal{R}}(\mathcal{L}_i/t^n\mathcal{V}_i(R)) = \operatorname{rank}_{\mathcal{R}}(\mathcal{L}_{i+1}/t^n\mathcal{V}_{i+1}(R))$$

(note that the quotient on the right is by a slightly larger module than on the left)

To verify **ELM4**, it now suffices to show that the projective rank function $\operatorname{Spec}(R) \to \mathbf{N}$ of $\mathcal{F}_0/t^n\lambda_0(R)$ is the constant function $d(m+n)$. Dualizing equation (2.13) yields

$$t^m\lambda_0(R) \subset \widehat{\mathcal{F}}_0 \subset t^{-n}\lambda_0(R)$$

The quotient $t^{-n}\lambda_0(R)/\widehat{\mathcal{F}}_0$ is a projective $\mathcal{R}$-module and has the same (constant) projective rank function as $\mathcal{F}_0/t^n\lambda_0(R)$. This means that

$$\begin{aligned}\operatorname{rank}_{\mathcal{R}}(\mathcal{F}_0/t^n\lambda_0(R)) &= \operatorname{rank}_{\mathcal{R}}(t^{-n}\lambda_0(R)/\widehat{\mathcal{F}}_0)\\ &= \operatorname{rank}_{\mathcal{R}}(t^{-m}\lambda_0(R)/t^{n-m}\widehat{\mathcal{F}}_0) = \operatorname{rank}_{\mathcal{R}}(t^{-m}\lambda_0(R)/\mathcal{F}_0)\end{aligned}$$

The first and last ranks are equal and must sum to $d(m+n)$, so the claim is proven (recall that $\operatorname{rank}_R(\mathcal{R}) = 2$).

It is automatic that **ELM5** is satisfied.

2.4.8 The affine Grassmannian over $\mathbf{Q}_p$

For the affine Grassmannian, the setup is: I use the unramified quadratic extension $\mathbf{Q}_p((t)) \subset F((t))$, the vector space $F((t))^d$, and the standard hermitian form $\Phi : F((t))^d \times F((t))^d \to F((t))$ defined by the anti-identity matrix.

Definition: The Affine Grassmannian. *The affine Grassmannian* $\mathcal{G}r^{\mathrm{aff}}$ *is the functor that assigns to any (commutative)* $\mathbf{Q}_p$*-algebra* R *(for simplicity of notation, set* $\mathcal{R} := R \otimes_{\mathbf{Q}_p} F$*) the set of all* $\mathcal{R}[[t]]$*-submodules* $\mathcal{F}$ *of* $\mathcal{R}((t))^d$ *satisfying:*

1. ***AG1***

   $\mathcal{F}$ *is a Zariski-lattice*

2. ***AG2***

   Zariski-locally on $\mathrm{Spec}(R)$*, there exists* $u(t) \in R((t))^\times$ *such that*

$$\mathcal{F} = u(t) \cdot \widehat{\mathcal{F}}.$$

2.4.9 The generic fibers are subschemes of the affine Grassmannian

Let R be a (commutative) $\mathbf{Q}_p$-algebra. Consider a point $(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$. As before, note that

$$\begin{aligned}
\mathcal{V}(\mathbf{Q}_p) &= F[t, t^{-1}, (t+p)^{-1}]^d \subset F((t))^d \\
\mathcal{V}_0(\mathbf{Q}_p) &= F[t]^d \\
\mathcal{V}_1(\mathbf{Q}_p) &= (t+p)^{-1}F[t] \oplus F[t]^{d-1} \\
&\vdots \\
\mathcal{V}_{d-1}(\mathbf{Q}_p) &= (t+p)^{-1}F[t]^{d-1} \oplus F[t] \\
\mathcal{V}_d(\mathbf{Q}_p) &= (t+p)^{-1}F[t]^d
\end{aligned}$$

Since $t+p$ is a unit in $\mathbf{Q}_p[[t]]$, the discussion in §2.4.2 (page 52) implies that over $\mathbf{Q}_p$ the "chain" $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{d/2}$ collapses and I may ignore all of them except one, say $\mathcal{L} := \mathcal{L}_0$.

Using a similar argument as in §2.4.6, notice that passing through the isomorphism of quotients

$$\frac{t^{-m}(t+p)^{-1}\mathcal{R}[t]^d}{t^n\mathcal{R}[t]^d} \xrightarrow{\sim} \frac{t^{-m}\mathcal{R}[[t]]^d}{t^n\mathcal{R}[[t]]^d}$$

(this uses the fact that $t+p$ is a *unit* in $\mathbf{Q}_p[[t]]$) associates to $\mathcal{L}$ an $\mathcal{R}[[t]]$-submodule $\mathcal{F}$ of $\mathcal{R}((t))^d$ satisfying

$$t^n\mathcal{R}[[t]]^d \subset \mathcal{F} \subset t^{-m}\mathcal{R}[[t]]^d$$

I claim that $\mathcal{L} \mapsto \mathcal{F}$ defines an injective function $\mathbf{M}^{(m,n)}(R) \hookrightarrow \mathcal{G}r^{\text{aff}}_{\mathbf{Q}_p}$ and that the collection over all $\mathbf{Q}_p$-algebras R of these functions defines a natural transformation $\mathbf{M}^{(m,n)}_{\mathbf{Q}_p} \to \mathcal{G}r^{\text{aff}}_{\mathbf{Q}_p}$.

$\boxed{\text{Proof of } \mathbf{AG1}}$ The proof here is the same as the proof of **AFV3** (page 61): apply Lemma 1.3.0.1 (page 24) to $R \to R \otimes_{\mathbf{Q}_p} F$.

$\boxed{\text{Proof of } \mathbf{AG2}}$ The proof here is exactly the same as the proof of **AFV5** (page 61), noticing that $t+p$ is a unit in $\mathbf{Q}_p[[t]]$ etc.

2.4.10 The generic fibers exhaust the affine Grassmannian

The proof of the fact that the schemes $\mathbf{M}^{(m,n)}_{\mathbf{Q}_p}$ exhaust $\mathcal{G}r^{\text{aff}}_{\mathbf{Q}_p}$ is nearly identical to the case of the special fibers exhausting the full affine flag variety, noticing as usual that $t+p$ is a unit in $\mathbf{Q}_p[[t]]$ etc.

2.5 An automorphism group for the enlarged models

2.5.1 Definition

Define the functor

$$\mathbf{J}^{(m,n)} : \mathbf{Z}_p\text{-Algebras} \to \text{Groups}$$

by assigning to any (commutative) $\mathbf{Z}_p$-algebra R the group of all $\mathcal{R}[t]$-linear automorphisms g of the quotient $\overline{\mathcal{V}}_{\sup}(R)$ that stabilize each of the subspaces $\overline{\mathcal{V}}_i(R)$ and are similitudes with respect to the product

$$\overline{\phi}_R : \overline{\mathcal{V}}_{\sup}(R) \times \overline{\mathcal{V}}_{\sup}(R) \longrightarrow \frac{t^{-2m}(t+p)^{-2}\mathcal{R}[t]}{t^{n-m}(t+p)^{-1}\mathcal{R}[t]}$$

with multiplier $c(g) \in R[t]$ representing an element of $(R[t]/t^{m+n}(t+p)R[t])^{\times}$. In particular, $c(g) \in R[[t]]^{\times}$ and $c(g)(0) \in R^{\times}$.

Remark. *The "precision" $t^{m+n}(t+p)$ for $c(g)$ is sufficient because the codomain of $\overline{\phi}$ is isomorphic as an $\mathcal{R}[t]$-module to $\mathcal{R}[t]/t^{m+n}(t+p)\mathcal{R}[t]$.*

Lemma 2.5.1.1. *This functor $\mathbf{J}^{(m,n)}$ is a finite-type affine group $\mathbf{Z}_p$-scheme and the rule*

$$(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{d/2}) \longmapsto (g(\mathcal{L}_0), g(\mathcal{L}_1), \ldots, g(\mathcal{L}_{d/2}))$$

defines an action of $\mathbf{J}^{(m,n)}$ on $\mathbf{M}^{(m,n)}$.

Proof. $\boxed{\text{affine}}$ The condition that any $g \in \mathbf{J}^{(m,n)}$ must stabilize the filtration $\overline{\mathcal{V}}_i$ together with the condition that it be a similitude present $\mathbf{J}^{(m,n)}$ as a closed subscheme of $\mathrm{Aut}_{\mathcal{O}[t]\text{-lin}}(\overline{\mathcal{V}}_{\sup})$. The condition that $g \in \mathrm{Aut}_{\mathcal{O}[t]\text{-lin}}(\overline{\mathcal{V}}_{\sup})$ be $\mathcal{O}[t]$-linear rather than simply $\mathcal{O}$-linear presents $\mathrm{Aut}_{\mathcal{O}[t]\text{-lin}}(\overline{\mathcal{V}}_{\mathrm{sup}})$ as a closed subscheme of $\mathrm{GL}_{2d(m+n)}$ (this condition is the same as requiring that g commute with the operator t). $\boxed{\text{finite-type}}$ This is obvious from the proof of affine-ness. $\boxed{\text{action}}$ Let R be a (commutative) $\mathbf{Z}_p$-algebra and set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$. Take $(\mathcal{L}_i)_{i=0}^{d/2} \in \mathbf{M}^{(m,n)}(R)$ and take $g \in \mathbf{J}^{(m,n)}(R)$. It is obvious from the definition that $(g(\mathcal{L}_i))_{i=0}^{d/2}$ satisfies **ELM1**, **ELM2**, **ELM3** and **ELM4**. To prove **ELM5**, note that restricting the above $\overline{\phi}$ to $t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ agrees with the ϕ used in §2.4.2 (page 52). Also note that the image of $\mathcal{L}_0$ in $t^{-m}\mathcal{R}[t]^d/t^n\mathcal{R}[t]^d$ is identical to what was called L_0 in §2.4.2 (page 52). Therefore, by Lemma 2.4.2.1 (page 55), I need to show that $\phi_R(g(L_0), x) = 0$ if and only if $x \in g(L_0)$. $\boxed{\Rightarrow}$ Since $0 = \phi_R(g(L_0), x) = c(g)\phi_R(L_0, g^{-1}(x))$, the definition of $c(g)$ implies that $\phi_R(L_0, g^{-1}(x)) = 0$ also. So $g^{-1}(x) \in L_0$ and $x \in g(L_0)$. $\boxed{\Leftarrow}$ This is trivial since $c(g) \in R[t]$. The case of $\mathcal{L}_{d/2}$ is similar.

It is trivial that this is a group action and that these actions form a natural transformation $\mathbf{J}^{(m,n)} \times \mathbf{M}^{(m,n)} \to \mathbf{M}^{(m,n)}$. $\square$

This $\mathbf{J}^{(m,n)}$ is a degeneration of the special parahoric $\mathcal{K}$ over $\mathbf{Q}_p[[t]]$ to the Iwahori subgroup $\mathcal{I}$ over $\mathbf{F}_p[[t]]$, much in the same way that $\mathbf{M}^{(m,n)}$ degenerates the affine Grassmannian $\mathcal{G}r^{\mathrm{aff}}$ over $\mathbf{Q}_p$ to the full affine flag variety $\mathcal{F}\ell^{\mathrm{aff}}$ over $\mathbf{F}_p$: as verified above, the generic and special fibers of $\mathbf{M}^{(m,n)}$ are subschemes of $\mathcal{G}r^{\mathrm{aff}}$ and $\mathcal{F}\ell^{\mathrm{aff}}$ respectively, and the generic and special fibers of $\mathbf{J}^{(m,n)}$ are essentially the quotients one gets by restricting $\mathcal{K}$ and $\mathcal{I}$ to these subschemes.

2.5.2 The automorphism group is smooth

In this subsection, I prove that $\mathbf{J}^{(m,n)}$ is a *smooth* $\mathbf{Z}_p$-scheme. This is necessary to connect the equivariant sheaf theory of $\mathbf{M}_{\mathbf{Q}_p}^{(m,n)}$ to the equivariant sheaf theory on $\mathbf{M}_{\mathbf{F}_p}^{(m,n)}$. More precisely, it is needed in order to apply base-change for pullbacks.

By Lemma 2.5.1.1 (page 70), $\mathbf{J}^{(m,n)}$ is finite-type, so to show that

$$\mathbf{J}^{(m,n)} \longrightarrow \mathrm{Spec}(\mathbf{Z}_p)$$

is smooth, it suffices to verify the "infinitesimal lifting property" (formal smoothness):

$\mathbf{J}^{(m,n)} \to \mathrm{Spec}(\mathbf{Z}_p)$ *is formally smooth if and only if for each (commutative)* $\mathbf{Z}_p$*-algebra* R *and each nilpotent ideal* $I \subset R$ *(equivalently, one satisfying* $I^2 = 0$*), the group homomorphism* $\mathbf{J}^{(m,n)}(R) \to \mathbf{J}^{(m,n)}(R/I)$ *is surjective.*

Set

$$S \stackrel{\text{def}}{=} R[t]/t^{m+n}(t+p)R[t]$$

Set

$$\overline{S} \stackrel{\text{def}}{=} (R/I)[t]/t^{m+n}(t+p)(R/I)[t]$$

For simplicity of notation, set $\mathcal{R} := \mathcal{O} \otimes_{\mathbf{Z}_p} R$. Let $\mathcal{I}$ be the extension of the ideal I in $\mathcal{R}$. Set

$$\mathcal{S} \stackrel{\text{def}}{=} \mathcal{R}[t]/t^{m+n}(t+p)\mathcal{R}[t]$$

and

$$\overline{\mathcal{S}} \stackrel{\text{def}}{=} (\mathcal{R}/\mathcal{I})[t]/t^{m+n}(t+p)(\mathcal{R}/\mathcal{I})[t]$$

Let I_t be the extension of I in S and let $\mathcal{I}_t$ be the extension of $\mathcal{I}$ in $\mathcal{S}$. I use without warning the equalities

$$\overline{S} = S/I_t$$

$$\mathcal{S} = \mathcal{O} \otimes_{\mathbf{Z}_p} S$$

$$\overline{\mathcal{S}} = \mathcal{O} \otimes_{\mathbf{Z}_p} \overline{S} = \mathcal{S}/\mathcal{I}_t$$

Let $\overline{g} \in \mathbf{J}^{(m,n)}(R/I)$ be arbitrary. By definition, $\overline{g}$ is an $\overline{\mathcal{S}}[t]$-linear automorphism of $\overline{\mathcal{V}}_{\text{sup}}(R/I)$. I make the obvious $\mathcal{R}[t]$-linear identification

$$\overline{\mathcal{V}}_{\text{sup}}(R/I) = \overline{\mathcal{S}}^d \tag{2.14}$$

so that $\overline{g}$ is identified with an automorphism of $\overline{\mathcal{S}}^d$ and the hermitian form $\overline{\phi}_{R/I}$ used in the definition (page 70) of $\mathbf{J}^{(m,n)}(R/I)$ is identified with the standard hermitian form $\overline{\mathcal{S}}^d \times \overline{\mathcal{S}}^d \to \overline{\mathcal{S}}$ defined by the anti-identity matrix.

To construct a lift $g \in \mathbf{J}^{(m,n)}(R)$, I use the following point of view: having g is equivalent to having $v_1, \dots, v_d \in \mathcal{S}^d$ such that

- $v_1, \dots, v_d$ is an $\mathcal{S}$-module basis of $\mathcal{S}^d$
- $v_i \in \mathcal{S}^i \oplus (t+p)\mathcal{S}^{d-i}$ for all i.
- there is $c \in S^\times$ such that

$$\overline{\phi}_R(v_i, v_j) = \begin{cases} c & i+j = d+1 \\ 0 & i+j \neq d+1 \end{cases}$$

The link between the two points of view is

$$v_i = g(e_i)$$

$$c = c(g)$$

Note that it is automatic from $R[t]$-linearity that g stabilizes each $\mathcal{S}^i \oplus (t+p)\mathcal{S}^{d-i}$.

To simplify notation further, set

$$M{:=}\mathcal{S}^d$$

$$\sigma{:=}(t+p)$$

$$N_i{:=}\mathcal{S}^i \oplus \sigma\mathcal{S}^{d-i} \ \ (i = 0, \dots, d)$$

Let $\overline{M}$ denote $M \otimes_{\mathcal{S}} (\mathcal{S}/\mathcal{I}_t) = M/\mathcal{I}_t M$. Let $\overline{N}_i$ be the image of N_i in $\overline{M}$.

First, note that lifting is easy if the similitude condition is not involved:

Lemma 2.5.2.1. *If $\overline{v}_1, \dots, \overline{v}_d$ is an $(\mathcal{S}/\mathcal{I}_t)$-module basis for $\overline{M}$ such that $\overline{v}_i \in \overline{N}_i$ for all i, then there exists an $\mathcal{S}$-module basis $v_1, \dots, v_d$ for M such that such that $v_i \in N_i$ and*

$$\overline{v}_1 \equiv v_1 \bmod \mathcal{I}_t M.$$

Proof. Let $v_i \in N_i$ be *arbitrary* lifts of $\overline{v}_i$. Nakayama's lemma implies that since $\overline{v}_1, \dots, \overline{v}_d$ generates $\overline{M}$, the set $v_1, \dots, v_d$ generates M (to apply Nakayama's lemma, note that $\mathcal{I}_t^2 = 0$). By "Linear Independence of Minimal Generating Sets", $v_1, \dots, v_d$ must be a basis. □

Now I extend this lemma to handle the similitude condition:

Proposition 2.5.2.2. *Let $e_1, \ldots, e_d$ be the standard basis for $\mathcal{S}^d$ and set $\overline{v}_i := \overline{g}(\overline{e}_i)$.*

***Assertion**: There is $c \in S^\times$ and $w_i \in N_i$ ($i = 1, \ldots, d$) that form an $\mathcal{S}$-module basis for M and that satisfy*

- $w_1 \equiv \overline{v}_1 \mod \mathcal{I}_t M$, *and*
- $\overline{\phi}_R(w_i, w_j) = \begin{cases} c & i+j = d+1 \\ 0 & i+j \neq d+1 \end{cases}$

Proof. Let $\jmath$ denote the involution of $\mathcal{S}$ induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q}_p)$. Note that the ideal $\mathcal{I}_t$ is $\jmath$-stable because it was extended from $I \subset R$. Let $v_1, \ldots, v_d$ be the basis guaranteed by Lemma 2.5.2.1 and choose a representative $c \in S^\times$ of $c(\overline{g})$ such that $\jmath(c) = c$ (this is possible because $c(\overline{g}) \in \overline{S}^\times$ by definition).

By assumption, the similitude condition holds modulo $\mathcal{I}$, i.e.

$$\overline{\phi}_{R/I}(\overline{v}_i, \overline{v}_j) = \begin{cases} c(\overline{g}) & i+j = d+1 \\ 0 & i+j \neq d+1 \end{cases}$$

More succinctly,

$$\overline{\phi}_{R/I}(\overline{v}_i, \overline{v}_j) = c(\overline{g})\delta_{i,d+1-j}$$

where δ is the Kronecker delta. This means that there are $x_{i,j} \in \mathcal{I}_t$ such that

$$\overline{\phi}_R(v_i, v_j) = c\delta_{i,d+1-j} + x_{i,j} \tag{2.15}$$

Since c is independent of i, j and since $\jmath(c) = c$, it is true that $x_{j,i} = \jmath(x_{i,j})$:

$$\begin{aligned} x_{j,i} &= \overline{\phi}_R(v_j, v_i) - c\delta_{j,d+1-i} \\ (\overline{\phi} \text{ is hermitian}) &= \jmath(\overline{\phi}_R(v_i, v_j)) - c\delta_{j,d+1-i} \\ (\text{because } \delta_{i,d+1-j} = \delta_{j,d+1-i}) &= \jmath(\overline{\phi}_R(v_i, v_j)) - c\delta_{i,d+1-j} \\ (\text{because } \jmath(c) = c) &= \jmath(x_{i,j}) \end{aligned}$$

By bi-additivity,

$$\overline{\phi}_R(v_i + m_i, v_j + m_j) = \overline{\phi}_R(v_i, v_j) + \overline{\phi}_R(m_i, v_j) + \overline{\phi}_R(v_i, m_j) + \overline{\phi}_R(m_i, m_j)$$

Because of this and the equality $x_{j,i} = \jmath(x_{i,j})$, it suffices to find $m_1, \ldots, m_d \in \mathcal{I}_t M$ such that $m_i \in N_i$ and $\overline{\phi}_R(m_i, v_j) = -\frac{1}{2}x_{i,j}$ and then to take $w_i \stackrel{\text{def}}{=} v_i + m_i$, since then

$$\overline{\phi}_R(w_i, w_j) = c\delta_{i,d+1-j} + x_{i,j} - \frac{1}{2}x_{i,j} - -\frac{1}{2}\jmath(x_{j,i}) = c\delta_{i,d+1-j}$$

(note that $\mathcal{I}_t^2 = 0$ implies $\overline{\phi}_R(m_i, m_j) = 0$).

Remark. *Here I have used the assumption that $p \neq 2$.*

Note that

$$N_{d-i} = \{m \in M \mid \overline{\phi}(m, N_i) \subset \sigma\mathcal{S}\} \tag{2.16}$$

Fix i. Consider the $\mathcal{S}$-linear functional $M \to \mathcal{S}$ defined by $v_j \mapsto -\frac{1}{2}x_{i,j}$. Since $\overline{\phi}_R$ is perfect, this functional is $\overline{\phi}_R(m_i, -)$ for some $m_i \in M$. In fact, $m_i \in \mathcal{I}_t M$ since $x_{i,j} \in \mathcal{I}_t$. I claim that this functional automatically sends N_{d-i} into $\sigma\mathcal{S}$. It then follows from inclusion "$\supset$" of duality (2.16) that $m_i \in N_i$, and the proof will be finished.

Since

$$v_1, \ldots, v_{d-i}, \sigma v_{d-i+1}, \ldots, \sigma v_d$$

is an $\mathcal{S}$-linear basis for N_{d-i}, it suffices to show that

$$\overline{\phi}_R(m_i, v_j) = -\frac{1}{2} x_{i,j} \in \sigma \mathcal{S}$$

for the subset $1 \leq j \leq d-i$ of indices. This inequality implies that $i \neq d+1-j$ and the defining relation (2.15) gives

$$\overline{\phi}_R(m_i, v_j) = -\frac{1}{2} \overline{\phi}_R(v_i, v_j)$$

So it suffices to show that $\overline{\phi}_R(v_i, v_j) \in \sigma \mathcal{S}$. But this is just inclusion "$\subset$" of duality (2.16). $\square$

Remark. *This proof is a variant of Proposition A.13 from [30] extended to handle flags.*

2.5.3 Conventions for Weyl groups, cocharacters, etc.

In this subsection, I set some conventions for Weyl groups and related objects that will be used in §2.5.5 and §2.5.7.

Consider the unitary similitude group GU_d associated to the quadratic extension $\mathbf{F}((t))/\mathbf{F}_p((t))$, the vector space $\mathbf{F}((t))^d$, and the standard hermitian form defined by the anti-identity matrix $\mathrm{id}^\vee$. Let A be the usual maximally $\mathbf{F}_p((t))$-split diagonal torus of GU_d. Since GU_d is quasi-split, the centralizer $C_{\mathrm{GU}_d}(A)$ is a maximal torus T (also consisting of diagonal elements) and is defined over $\mathbf{F}_p((t))$. The *relative extended affine Weyl group* of GU_d with respect to A is the quotient

$$\widetilde{W} \stackrel{\text{def}}{=} N(\mathbf{F}_p((t)))/T(\mathbf{F}_p((t)))_0$$

where N is the normalizer $N_{\mathrm{GU}_d}(A)$ and $T(\mathbf{F}_p((t)))_0$ is the unique maximal compact open subgroup of $T(\mathbf{F}_p((t)))$.

Let $W = N/T$ be the *relative finite Weyl group* of G with respect to A and $X_*(A)$ the abelian group of algebraic group homomorphisms $A \to \mathbf{G}^{\mathrm{m}}$. The extended affine Weyl group $\widetilde{W}$ has a semidirect product decomposition

$$\widetilde{W} = X_*(A) \rtimes W$$

and parametrizes the double cosets of $\mathrm{GU}_d(\mathbf{F}_p((t)))$ modulo an Iwahori subgroup (this parametrization is called the *Bruhat-Tits decomposition*). Implicit in this parametrization is the fact that elements of $\widetilde{W}$ can be represented by $\mathbf{F}_p((t))$-points, and therefore $\widetilde{W}$ can be considered as a subset (usually not a subgroup) in many different ways of $\mathrm{GU}_d(\mathbf{F}_p((t)))$. In more detail, elements of W can be represented by elements of $\mathrm{GU}_d(\mathbf{F}_p((t)))$, and I consider $X_*(A)$ also as a subset of $A(\mathbf{F}_p((t))) \subset \mathrm{GU}_d(\mathbf{F}_p((t)))$ via the map $\lambda \mapsto \lambda(t)$. I fix such an inclusion

$$\widetilde{W} \hookrightarrow \mathrm{GU}_d(\mathbf{F}_p((t)))$$

and use it without warning from now on.

Remark. *In general, the group which parametrizes the Bruhat-Tits decomposition relative to an Iwahori subgroup is the* Iwahori-Weyl group*, defined completely generally in [16] using the Kottwitz homomorphism from [21]; see Definition 7, Proposition 8, and Remark 9 of [16]. But since* GU_d *here is* unramified*, the Iwahori-Weyl* *group coincides with what is usually called the extended affine Weyl group and the Kottwitz homomorphism also has a more direct definition; see Remark 10 of [16], and Lemma 3.0.1(III), Corollary 11.1.2(c), and Proposition 11.1.4 of [17].*

Finally, I denote by Φ_{aff} the affine root system for GU_d as described in §1.6 of [36]. Let $W_{\text{aff}} \subset \widetilde{W}$ be the subgroup generated by reflections across the kernels of elements of Φ_{aff}. Fix a Chevalley-Bruhat partial order $\leq$ and length function ℓ on W_{aff}, which I require to be consistent with the Iwahori subgroup defined in the next subsection. I consider ℓ on $\widetilde{W}$ by extending trivially.

Similar conventions are in place for the unitary similitude group associated to the (unramified) quadratic extension $F((t))/\mathbf{Q}_p((t))$.

2.5.4 Description of the Iwahori subgroup

Return to the setup of §2.4.5 (page 61), i.e. denote by GU_d the unitary similitude group associated to the (unramified) quadratic extension $\mathbf{F}((t))/\mathbf{F}_p((t))$, the vector space $\mathbf{F}((t))^d$, and the standard hermitian form Φ defined by the anti-identity matrix $\mathrm{id}^\vee$. Define the "standard" periodic Φ-self-dual chain $\{\lambda_i\}_{i\in\mathbf{Z}}$ of $\mathbf{F}[[t]]$-lattices in $\mathbf{F}((t))^d$ by extending

$$\lambda_i \overset{\text{def}}{=} t^{-1}\mathbf{F}[[t]]^i \oplus \mathbf{F}[[t]]^{d-i} \ (0 \leq i \leq d)$$

periodically. Notice that

$$\lambda_i \otimes_{\mathbf{F}_p[[t]]} \mathbf{F}_p = \Lambda_i \otimes_{\mathbf{Z}_p} \mathbf{F}_p$$

Let

$$\mathcal{I} : \mathbf{F}_p[[t]]\text{-Algebras} \to \text{Groups}$$

be the Iwahori subgroup scheme of GU_d associated to the lattice chain $(\lambda_i)_{i\in\mathbf{Z}}$, in the sense of [16] or [5], [6]. Then by Equality 8.0.1 and Remark 8.0.2 in [17]),

$$\mathcal{I}(\mathbf{F}_p[[t]]) = \{g \in \mathrm{GU}_d(\mathbf{F}_p((t))) \mid g(\lambda_i) = \lambda_i \text{ and } \kappa(g) = 1\}$$

(here κ is the "Kottwitz homomorphism" associated to GU_d). Note that if $g \in \mathcal{I}(\mathbf{F}_p[[t]])$ then $c(g) \in \mathbf{F}_p[[t]]^\times$ since g and g^{-1} both stabilize $\lambda_0 = \mathbf{F}[[t]]^d$.

2.5.5 A "Bruhat-Tits decomposition" of the enlarged models

I claim that there is a group homomorphism

$$\mathcal{I}(\mathbf{F}_p[[t]]) \longrightarrow \mathbf{J}^{(m,n)}(\mathbf{F}_p) \tag{2.17}$$

such that acting by $\mathcal{I}(\mathbf{F}_p[[t]])$ on the image of the embedding

$$\mathbf{M}^{(m,n)}(\mathbf{F}_p) \hookrightarrow \mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p)$$

is the same as acting directly on $\mathbf{M}^{(m,n)}(\mathbf{F}_p)$ via (2.17).

If $g \in \mathcal{I}(\mathbf{F}_p[[t]])$ then by definition g descends to an $\mathbf{F}[t]$-linear automorphism $\overline{g}$ of the quotient

$$t^{-(m+1)}\mathbf{F}[[t]]^d/t^n\mathbf{F}[[t]]^d = t^{-(m+1)}\mathbf{F}[t]^d/t^n\mathbf{F}[t]^d = \overline{\mathcal{V}}_{\mathrm{sup}}(\mathbf{F}_p)$$

and stabilizes the subquotients

$$\lambda_i/\mathcal{V}_{\mathrm{inf}}(\mathbf{F}_p) = \overline{\mathcal{V}}_i(\mathbf{F}_p)$$

Modulo t^{m+n+1}, the element $c(g)$ becomes a multiplier $c(\overline{g})$ as in the definition of $\mathbf{J}^{(m,n)}(\mathbf{F}_p)$. This shows that $g \mapsto \overline{g}$ is a function $\mathcal{I}(\mathbf{F}_p[[t]]) \to \mathbf{J}^{(m,n)}(\mathbf{F}_p)$. It is clear that this is a group homomorphism and that the actions of $\mathbf{J}^{(m,n)}(\mathbf{F}_p)$ on $\mathbf{M}^{(m,n)}(\mathbf{F}_p)$ and $\mathcal{I}(\mathbf{F}_p[[t]])$ on $\mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p)$ are compatible.

Let $\widetilde{W}$ be the Iwahori-Weyl group from §2.5.3 (page 77) and recall the notation and conventions there. For each $w \in \widetilde{W}$, there is the affine Schubert cell

$$C_w \stackrel{\mathrm{def}}{=} \mathcal{I}(\mathbf{F}_p[[t]]) \cdot w \cdot \mathcal{I}(\mathbf{F}_p[[t]])$$

These Schubert cells can be enriched to $\mathbf{F}_p$-schemes: define the functor

$$C_w : \mathbf{F}_p\text{-Algebras} \to \text{Sets}$$

to be the fpqc-sheafification of the functor that assigns to any (commutative) $\mathbf{F}_p$-algebra R the $\mathcal{I}(R[[t]])$-orbit of w_R in $\mathrm{GU}_d(R((t)))/\mathcal{I}(R[[t]])$ (here w_R denotes the image of w under $\mathrm{GU}_d(\mathbf{F}_p((t))) \to \mathrm{GU}_d(R((t)))$).

Because of (2.17), the subset $\mathbf{M}^{(m,n)}(\mathbf{F}_p) \subset \mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p)$ is $\mathcal{I}(\mathbf{F}_p[[t]])$-stable and there is a Bruhat-Tits decomposition of $\mathbf{M}^{(m,n)}(\mathbf{F}_p)$:

$$\mathbf{M}^{(m,n)}(\mathbf{F}_p) = \coprod_{w \in \widetilde{W}^{(m,n)}} C_w(\mathbf{F}_p)$$

for a certain finite subset $\widetilde{W}^{(m,n)} \subset \widetilde{W}$.

Remark. *In the trivial case $\mathbf{M}^{(0,1)}_{\mathbf{F}_p} = \mathbf{M}^{\mathrm{loc}}_{\mathbf{F}_p}$, the decomposition consists of the cells C_w for w in the "admissible set" $\mathrm{Adm}(\mu)$, which is the closure (in the sense of the combinatorially-defined Bruhat-Chevalley partial order) of the orbit of μ under the finite Weyl group. See [14] for some pictures of admissible sets.*

2.5.6 Description of the special parahoric subgroup

Return to the setup of §2.4.8 (page 67), i.e. denote by GU_d the unitary similitude group associated to the (unramified) quadratic extension $F((t))/\mathbf{Q}_p((t))$, the vector space $F((t))^d$, and the standard hermitian form Φ defined by the anti-identity matrix $\mathrm{id}^\vee$. Define the "standard" periodic Φ-self-dual chain $\{\lambda_i\}_{i\in\mathbf{Z}}$ of $F[[t]]$-lattices in $F((t))^d$ by extending

$$\lambda_i \stackrel{\text{def}}{=} t^{-1}F[[t]]^i \oplus F[[t]]^{d-i} \ (0 \leq i \leq d)$$

periodically and let

$$\mathcal{K} : \mathbf{Q}_p[[t]]\text{-Algebras} \longrightarrow \text{Groups}$$

be the special parahoric subgroup scheme of GU_d associated to the lattice λ_0. Then

$$\mathcal{K}(\mathbf{Q}_p[[t]]) = \{g \in \mathrm{GU}_d(\mathbf{Q}_p((t))) \mid g(\lambda_0) = \lambda_0 \text{ and } \kappa(g) = 1\}$$

(here κ is the "Kottwitz homomorphism"). As before, if $g \in \mathcal{K}(\mathbf{Q}_p[[t]])$ then $c(g) \in \mathbf{Q}_p[[t]]^\times$.

2.5.7 A "Cartan decomposition" of the enlarged models

As in §2.5.5 (page 80), I claim that there is a group homomorphism

$$\mathcal{K}(\mathbf{Q}_p[[t]]) \longrightarrow \mathbf{J}^{(m,n)}(\mathbf{Q}_p) \tag{2.18}$$

such that acting by $\mathcal{K}(\mathbf{Q}_p[[t]])$ on the image of the embedding

$$\mathbf{J}^{(m,n)}(\mathbf{Q}_p) \hookrightarrow \mathcal{G}r^{\mathrm{aff}}(\mathbf{Q}_p)$$

is the same as acting directly on $\mathbf{M}^{(m,n)}(\mathbf{Q}_p)$ via (2.18). The group homomorphism is defined in the same way: any $g \in \mathcal{K}(\mathbf{Q}_p[[t]])$ restricts to an automorphism $\overline{g}$ of $\overline{\mathcal{V}}_{\sup}(\mathbf{Q}_p)$ which stabilizes the subquotient

$$\lambda_0/\mathcal{V}_{\inf}(\mathbf{Q}_p) = \overline{\mathcal{V}}_0(\mathbf{Q}_p)$$

and, because $t+p$ is a unit in $\mathbf{Q}_p[[t]]$, also stabilizes the other $\overline{\mathcal{V}}_i(\mathbf{Q}_p)$. The automorphism $\overline{g}$ automatically satisfies the similitude condition necessary for membership in $\mathbf{J}^{(m,n)}(\mathbf{Q}_p)$.

Recall the notation and conventions in §2.5.3 (page 77). Let O_λ denote the $\mathbf{Q}_p$-subschemes of $\mathcal{G}r^{\text{aff}}$ forming the Cartan decomposition. Similar to §2.5.5, (2.18) implies that that $\mathbf{J}^{(m,n)}(\mathbf{Q}_p)$ is a $\mathcal{K}(\mathbf{Q}_p[[t]])$-stable subset of $\mathcal{G}r^{\text{aff}}(\mathbf{Q}_p)$, so there is a Cartan decomposition:

$$\mathbf{M}^{(m,n)}(\mathbf{Q}_p) = \coprod_{\lambda \in X_*^{(m,n)}} O_\lambda(\mathbf{Q}_p)$$

for a certain finite set $X_*^{(m,n)}$ of (dominant) cocharacters $\lambda \in X_*(A)$.

Remark. *In the case of* $(m,n) = (0,1)$*, this decomposition is a singleton:* $\mathbf{M}^{(0,1)}_{\mathbf{Q}_p} = \mathbf{M}^{\text{loc}}_{\mathbf{Q}_p}$ *consists of the single cell* O_μ*.*

Chapter 3

The trace function and the convolution product

3.1 The trace function

3.1.1 Generalities on nearby cycles and trace functions

Let X be a separated finite-type $\mathbf{Z}_p$-scheme. Let $\mathcal{C}^\bullet$ be a complex of étale ℓ-adic sheaves on $X_{\mathbf{Q}_p}$. The pullback $\overline{\mathcal{C}}^\bullet$ of $\mathcal{C}^\bullet$ along

$$X_{\overline{\mathbf{Q}}_p} \longrightarrow X_{\mathbf{Q}_p}$$

has a natural continuous action by

$$\Gamma \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$$

i.e. a collection for all $\gamma \in \Gamma$ of functor isomorphisms

$$\gamma_*(\overline{\mathcal{C}}^\bullet) \xrightarrow{\sim} \overline{\mathcal{C}}^\bullet$$

consistent with composition (by abuse of notation, γ also denotes the induced $\mathbf{Q}_p$-scheme automorphism of $X_{\overline{\mathbf{Q}}_p}$). Define

$$\mathrm{R}\Psi(\mathcal{C}^\bullet) \stackrel{\text{def}}{=} \imath^*(\mathrm{R}\jmath_*(\overline{\mathcal{C}}^\bullet)),$$

the *nearby cycles* complex of étale ℓ-adic sheaves on $X_{\overline{\mathbf{F}}_p}$, where

$$X_{\overline{\mathbf{F}}_p} \xrightarrow{\imath} X_{\overline{\mathbf{Z}}_p} \xleftarrow{\jmath} X_{\overline{\mathbf{Q}}_p}$$

are the structure morphisms. The complex $\mathrm{R}\Psi(\mathcal{C}^\bullet)$ on $X_{\overline{\mathbf{F}}_p}$ inherits the action by Γ. In particular, each $\gamma \in \Gamma$ induces an endomorphism of the cohomology stalk

$$\mathcal{H}^i(\mathrm{R}\Psi(\mathcal{C}^\bullet))_x$$

for every $x \in X(\mathbf{F}_p)$ and $i \in \mathbf{Z}$.

Grothendieck's quasi-unipotent inertia theorem (see the Proposition in the Appendix of [32]) applies to the continuous representation of Γ on the finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space $\mathcal{H}^i(\mathrm{R}\Psi(\mathcal{C}^\bullet))_x$ to yield "semisimplifications"

$$\mathrm{ss}(\mathcal{H}^i(\mathrm{R}\Psi(\mathcal{C}^\bullet))_x)$$

on which the inertia subgroup Γ_0 via a *finite* quotient (Γ acts on the semisimplification by acting individually on each summand). See §3 of [15] for a detailed discussion of these semisimplifications.

The action of Γ on $\mathrm{ss}(\mathcal{H}^i(\mathrm{R}\Psi(\mathcal{C}^\bullet))_x)^{\Gamma_0}$ factors through $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, and one defines

$$\tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathcal{C}^\bullet)}(x) \overset{\mathrm{def}}{=} \sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob}; \mathrm{ss}(\mathcal{H}^i(\mathrm{R}\Psi(\mathcal{C}^\bullet))_x)^{\Gamma_0})$$

for all $x \in X(\mathbf{F}_p)$. The exactness of the fixed-points functor $V \mapsto V^G$ for a *finite* group G makes this function more well-behaved; see for example the proof of Lemma 10 in [15].

More generally, define

$$\tau^{\mathrm{ss}}_{\mathcal{C}^\bullet} : X(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

for any complex of étale ℓ-adic sheaves $\mathcal{C}^\bullet$ on $X_{\overline{\mathbf{F}}_p}$ which has an action by $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ consistent with the action of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on $X_{\overline{\mathbf{F}}_p}$. For example, $\mathcal{C}^\bullet$ could be the pullback to $X_{\overline{\mathbf{F}}_p}$ of a complex on $X_{\mathbf{F}_p}$ with $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acting via the composition

$$\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{unr}}/\mathbf{Q}_p) \longrightarrow \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$$

(in which case the semisimplification is vacuous).

3.1.2 Definition of the main trace functions

Fix $m, n \in \mathbf{N}$. Let O_λ be a cell from the Cartan decomposition of $\mathbf{M}_{\mathbf{Q}_p}^{(m,n)}$. Let IC_λ be the (perverse) étale ℓ-adic intersection complex on $\overline{O}_\lambda$ (the reduced closure). Then applying the construction from the previous subsection to this special case yields

$$\tau_\lambda^{\mathrm{ss}} \stackrel{\mathrm{def}}{=} \tau_{\mathrm{R}\Psi(\mathrm{IC}_\lambda)}^{\mathrm{ss}} : \mathbf{M}^{(m,n)}(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

By the embedding in §2.4.6 (page 63), I can extend by 0 and consider $\tau_\lambda^{\mathrm{ss}}$ to be a function on $\mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p)$.

3.1.3 The trace function is Iwahori-invariant

In order to show that $\tau_\lambda^{\mathrm{ss}}$ is in the Iwahori-Hecke algebra $\mathcal{H}$ of $\mathrm{GU}(\mathbf{F}((t))^d, \Phi)(\mathbf{F}_p((t)))$ with respect to $\mathcal{I}(\mathbf{F}_p[[t]])$, I must show that it is invariant under left-translations by $\mathcal{I}(\mathbf{F}_p[[t]])$ (invariance under right-translations is automatic from the domain of $\mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p)$). Because of the group homomorphism $\mathcal{I}(\mathbf{F}_p[[t]]) \to \mathbf{J}^{(m,n)}(\mathbf{F}_p)$ (see §2.5.5 (page 80)) and the definition of $\tau_\lambda^{\mathrm{ss}}$, it suffices to show that $\mathrm{R}\Psi(\mathrm{IC}_\lambda)$ is $\mathbf{J}_{\overline{\mathbf{F}}_p}^{(m,n)}$-equivariant, in the sense that there is a isomorphism

$$\mathrm{ac}_{\overline{\mathbf{F}}_p}^*(\mathrm{R}\Psi(\mathrm{IC}_\lambda)) \xrightarrow{\sim} \mathrm{pr}_{\overline{\mathbf{F}}_p}^*(\mathrm{R}\Psi(\mathrm{IC}_\lambda)) \tag{3.1}$$

of étale sheaf complexes subject to a "cocycle" (group action axiom) condition. Here

$$\mathrm{ac}, \mathrm{pr} : \mathbf{J}^{(m,n)} \times_{\mathrm{Spec}(\mathbf{Z}_p)} \mathbf{M}^{(m,n)} \to \mathbf{M}^{(m,n)}$$

are the left-action (see §2.5.1.1) and projection morphisms.

By Proposition 2.5.2.2 (page 74), the morphism $\mathbf{J}^{(m,n)} \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth, so the projection, which is the morphism $\mathbf{J}^{(m,n)} \times_{\mathrm{Spec}(\mathbf{Z}_p)} \mathbf{M}^{(m,n)} \to \mathbf{M}^{(m,n)}$ supplied by the fiber product, is also smooth (since smoothness is preserved under base-change). It follows from "smooth base change" (the fact that pullback by a smooth morphism commutes with (derived) pushforward in a base-change diagram), that

$$\mathrm{pr}^*_{\overline{\mathbf{F}}_p}(\mathrm{R}\Psi(\mathrm{IC}_\lambda)) \cong \mathrm{R}\Psi(\mathrm{pr}^*_{\mathbf{Q}_p}(\mathrm{IC}_\lambda)) \tag{3.2}$$

On the other hand, the functor endomorphism of $\mathbf{J}^{(m,n)} \times \mathbf{M}^{(m,n)}$ defined by $(g, x) \mapsto (g, g(x))$ is an *automorphism* (over $\mathbf{Z}_p$). Since the action morphism ac is the composition of this automorphism with the (smooth) projection pr, this shows that the action morphism ac is smooth, and so by the same reasoning as for pr,

$$\mathrm{ac}^*_{\overline{\mathbf{F}}_p}(\mathrm{R}\Psi(\mathrm{IC}_\lambda)) \cong \mathrm{R}\Psi(\mathrm{ac}^*_{\mathbf{Q}_p}(\mathrm{IC}_\lambda)) \tag{3.3}$$

The intersection complex IC_λ is $\mathbf{J}^{(m,n)}_{\mathbf{Q}_p}$-equivariant by definition so combining (3.2) and (3.3) yields (3.1).

3.1.4 Statement of theorem

By the previous subsection, τ_μ^{ss} is identified with an element of the Iwahori-Hecke algebra $\mathcal{H}$.

Main Theorem. $\tau_\mu^{\mathrm{ss}} \in Z(\mathcal{H})$.

The remainder of the paper develops the tools needed to prove this theorem, and the proof of theorem itself occurs in Chapter 4 (page 138).

Remark. *Although the theorem only directly concerns μ minuscule, in which case* IC_μ *is, up to a shift, the constant sheaf (since $\overline{O}_\mu = O_\mu$), the commutativity property involves other $\tau_\lambda^{\mathrm{ss}}$ so those parts of §3.1 concerning non-minuscule λ are not useless.*

Assumption. The *complement* in $\mathbf{M}^{\mathrm{loc}}$ of the Schubert cells $C_w \subset \mathbf{M}^{\mathrm{loc}}_{\mathbf{F}_p}$ such that

$$\mathrm{codim}_{\mathbf{M}^{\mathrm{loc}}}(C_w) > 0$$

is a smooth $\mathbf{Z}_p$-scheme.

This assumption is surely already satisfied, but I have not yet verified it rigorously.

Corollary. τ_μ^{ss} *is the scaled Bernstein basis function* $(-1)^{\ell(\mu)} q(\mu)^{\frac{1}{2}} z_\mu$.

The value $q(\mu)$ is defined as the index $[\mathcal{I}\mu\mathcal{I} : \mathcal{I}]$. The sign $(-1)^{\ell(\mu)}$ is due to the shift by $-\dim(O_\mu) = -\ell(\mu)$ imposed on the intersection complexes to make them perverse.

Proof. By definition, $\tau_\mu^{\mathrm{ss}} : \mathcal{F}\ell^{\mathrm{aff}}(\mathbf{F}_p) \to \overline{\mathbf{Q}}_\ell$ is supported on $\mathbf{M}^{\mathrm{loc}}(\mathbf{F}_p)$. By the lemma following this proof, $\mathbf{M}^{\mathrm{loc}}$ is flat, so the cells occurring in the Bruhat-Tits decomposition for $\mathbf{M}^{\mathrm{loc}}(\mathbf{F}_p)$ are the admissible set $\mathrm{Adm}(\mu)$. By the assumption immediately before this corollary and Lemma 8.6 in [14], $\tau_\mu^{\mathrm{ss}}(C_\mu) = (-1)^{\ell(\mu)}$. It is clear from the definition of the Bernstein basis functions that the value of z_μ on C_μ is $q(\mu)^{-\frac{1}{2}}$, and Theorem 5.8 in [13] characterizes the (normalized) Bernstein basis functions as those which are central, supported on $\mathrm{Adm}(\mu)$ and have value 1 on the dominant cell C_μ. $\square$

Remark. *At first glance, Theorem 5.8 in [13] appears to work only in the split case, since the Hecke algebras considered use constant parameter systems. However, the theorem holds generally.*

Lemma 3.1.4.1. $\mathbf{M}^{\mathrm{loc}}$ *is a flat* $\mathbf{Z}_p$*-scheme.*

Proof. If $R \to S$ is a *faithfully-flat* homomorphism of (commutative) rings and M is an R-module such that $M \otimes_R S$ is a flat S-module, then M is a flat R-module. Since flatness of a morphism of schemes is local with respect to the domain, this means that it suffices to verify that $\mathbf{M}^{\mathrm{loc}}_{\mathcal{O}}$ is a flat $\mathcal{O}$-scheme. To see this, first note that since $- \otimes_{\mathbf{Z}_p} R = (- \otimes_{\mathbf{Z}_p} \mathcal{O}) \otimes_{\mathcal{O}} R$ for any (commutative) $\mathcal{O}$-algebra R, the set $\mathbf{M}^{\mathrm{loc}}(R)$ is the same as the R-points of the local model associated to the datum consisting of the base field F (instead of $\mathbf{Q}_p$), the vector space $F^d \otimes_{\mathbf{Q}_p} F$, the lattices $\Lambda_i \otimes_{\mathbf{Z}_p} \mathcal{O}$, etc. but which also satisfy the PEL condition **SLM4**. The function $X \otimes \alpha \mapsto (\alpha X, \overline{\alpha} X)$ defines a $\mathbf{Q}_p$-algebra isomorphism $M_d(F) \otimes_{\mathbf{Q}_p} F \xrightarrow{\sim} M_d(F) \times M_d(F)$ which transforms the involution $*_{\mathrm{std}} \otimes \mathrm{id}$ into the involution $(X, Y) \mapsto (\overline{Y}^{\mathrm{tr}}, \overline{X}^{\mathrm{tr}})$, and similar decompositions hold for $F^d \otimes_{\mathbf{Q}_p} F$, $\Lambda_i \otimes_{\mathbf{Z}_p} \mathcal{O}$, etc. These decompositions allow each module K_i occurring in a point of $\mathbf{M}^{\mathrm{loc}}(R)$ to be identified with a sum $K_i^{(1)} \oplus K_i^{(2)}$ inside $(\Lambda_i \otimes_{\mathcal{O}} R) \oplus (\Lambda_i \otimes_{\mathcal{O}} R)$. Because of the definition of the hermitian form ϕ and the reversal that occurs in the involution on $M_d(F) \times M_d(F)$, the PEL condition **SLM4** is equivalent to the condition $K_i^{(2)} = (K_{-i}^{(1)})^\perp$ (in particular, the modules $K_i^{(2)}$ are completely determined by the modules $K_i^{(1)}$). This means that

$\mathbf{M}^{\mathrm{loc}}_{\mathcal{O}}$ is isomorphic to a standard GL local model, which by [10] is flat. $\square$

Remark. *A similar collapsing of a PEL local model to an EL local model occurs in §6.3.3 of [14].*

3.2 The convolution diagram

3.2.1 The full affine flag variety over $\mathbf{Z}_p$

Here I recall the definition of the affine flag variety and Iwahori subgroup over $\mathbf{Z}_p$ as limits of projective $\mathbf{Z}_p$-schemes in a way that is compatible with the definition of $\mathbf{M}^{(m,n)}$ and $\mathbf{J}^{(m,n)}$. By extending scalars, this integral affine flag variety gives the usual affine flag varieties over $\mathbf{Q}_p$ and $\mathbf{F}_p$. The construction is just a slight variation on the previous theme.

Definition: The Integral Affine Flag Variety. *Fix $\mu, \nu \in \mathbf{N}$. Define the functor*

$$\mathbf{Fl}^{(\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set of all tuples $(\mathcal{F}_0, \ldots, \mathcal{F}_{d/2})$ of $\mathcal{R}[t]$-submodules of $\mathcal{V}(R)$ such that

- $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{d/2}$
- $(t+p)^{\nu}\mathcal{V}_i(R) \subset \mathcal{F}_i \subset (t+p)^{-\mu}\mathcal{V}_i(R)$ *for each* $0 \le i \le d/2$
- *each inclusion* $\mathcal{F}_i/(t+p)^{\nu}\mathcal{V}_i(R) \hookrightarrow (t+p)^{-\mu}\mathcal{V}_i(R)/(t+p)^{\nu}\mathcal{V}_i(R)$ *splits R-linearly*
- *the projective rank function* $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each*

$$\mathcal{F}_i/(t+p)^{\nu}\mathcal{V}_i(R)$$

is the constant function $\mathfrak{p} \mapsto d(\mu+\nu)$

- *Zariski-locally on* $\mathrm{Spec}(R)$, $\widehat{\mathcal{F}}_0 = \mathcal{F}_0$ *with respect to* $(t+p)^{\mu-\nu}\phi_R$ *and* $\widehat{\mathcal{F}}_{d/2} = \mathcal{F}_{d/2}$ *with respect to* $(t+p)^{\mu-\nu+1}\phi_R$.

(The duality occurring here is similar to the one occurring in **ELM5**: it is required that $\mathcal{F}_0$ be *exactly* the elements $x \in \mathcal{V}(R)$ such that

$$\phi_R(\mathcal{F}_0, x) \subset (t+p)^{\nu-\mu}\mathcal{R}[t]$$

and similarly for $\mathcal{F}_{d/2}$.)

Define

$$\mathcal{U}_{\inf} \stackrel{\text{def}}{=} (t+p)^{\nu}\mathcal{O}[t]^d$$

$$\mathcal{U}_{\sup} \stackrel{\text{def}}{=} (t+p)^{-\mu-1}\mathcal{O}[t]^d$$

$$\overline{\mathcal{U}}_{\sup} \stackrel{\text{def}}{=} \mathcal{U}_{\sup}/\mathcal{U}_{\inf}$$

For the purpose of this subsection, redefine $\overline{\phi}$ to be the hermitian $\mathbf{Z}_p[t]$-bilinear form

$$\overline{\phi} : \overline{\mathcal{U}}_{\sup} \times \overline{\mathcal{U}}_{\sup} \longrightarrow \frac{(t+p)^{-2(\mu+1)}\mathcal{O}[t]}{(t+p)^{\nu-\mu-1}\mathcal{O}[t]}$$

defined by the anti-identity matrix, and redefine $\overline{\mathcal{V}}_i$ to be the image of $\mathcal{V}_i$ in $\overline{\mathcal{U}}_{\sup}$.

Definition: The Integral Iwahori Subgroup. *Fix* $\mu, \nu \in \mathbf{N}$. *Define the functor*

$$\mathbf{Iw}^{(\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Groups}$$

by assigning to each (commutative) $\mathbf{Z}_p$*-algebra* R *the group of all* $\mathcal{R}[t]$*-linear automorphisms* g *of* $\overline{\mathcal{U}}_{\sup}$ *that stabilize each* $\overline{\mathcal{V}}_i$ *and are similitudes with respect to the product* $\overline{\phi}_R$ *with multiplier* $c(g) \in R[t]$ *representing a* unit *in* $R[t]/(t+p)^{\mu+\nu+1}R[t]$.

Like $\mathbf{J}^{(m,n)} \to \operatorname{Spec}(\mathbf{Z}_p)$, each of these $\mathbf{Iw}^{(\mu,\nu)} \to \operatorname{Spec}(\mathbf{Z}_p)$ is a smooth affine algebraic group scheme.

These schemes have a few purposes. First, the full affine flag varieties over $\mathbf{Q}_p$ and $\mathbf{F}_p$ are just the fibers of

$$\mathbf{Fl}^{\text{aff}} \stackrel{\text{def}}{=} \bigcup_{(\mu,\nu)} \mathbf{Fl}^{(\mu,\nu)},$$

hence the name.

Second, $\mathbf{Fl}^{\text{aff}}$ has a Bruhat-Tits decomposition $\mathbf{Fl}^{\text{aff}} = \coprod C_w$ over $\mathbf{Z}_p$, i.e. the Schubert cells C_w are $\mathbf{Z}_p$-schemes (The abuse of notation is acceptable because the extension to $\mathbf{F}_p$ of this C_w is the "C_w" from §2.5.5 (page 80)).

Third, one can define the étale ℓ-adic intersection complex IC_w on $\overline{C}_w$ and the restrictions $\mathrm{IC}_w|_{\mathbf{Fl}^{\text{aff}}_{\overline{\mathbf{F}}_p}}$ and $\mathrm{IC}_w|_{\mathbf{Fl}^{\text{aff}}_{\overline{\mathbf{Q}}_p}}$ are just the corresponding étale intersection complexes on the affine flag varieties over $\mathbf{F}_p$ and $\mathbf{Q}_p$. Because of all this, §5.2 of [15] shows that

$$\overline{\mathrm{IC}}_w|_{\mathbf{Fl}^{\text{aff}}_{\overline{\mathbf{F}}_p}} \xrightarrow{\sim} \mathrm{R}\Psi(\mathrm{IC}_w|_{\mathbf{Fl}^{\text{aff}}_{\overline{\mathbf{Q}}_p}})$$

Remark. *§5.2 of [15] applies because the fields involved here are algebraically-closed: by an argument similar to that given in the proof of Lemma 3.1.4.1 (page 89), the schemes used here simplify to the* GL *case after passing to the algebraic closure.*

3.2.2 Group-like schemes to act on $\mathbf{M}^{(m,n)}$ and $\mathbf{Fl}^{(\mu,\nu)}$

Fix $m, n, \mu, \nu \in \mathbf{N}$. I define two $\mathbf{Z}_p$-schemes, $\widetilde{\mathbf{M}}^{(m,n)}$ and $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}$, and morphisms $\widetilde{\mathbf{M}}^{(m,n)} \to \mathbf{M}^{(m,n)}$ and $\widetilde{\mathbf{Fl}}^{(\mu,\nu)} \to \mathbf{Fl}^{(\mu,\nu)}$. These schemes and morphisms play the role, in the truncated case, of the algebraic group acting on its affine flag variety by left-multiplication.

Fix $m, n, \mu, \nu \in \mathbf{N}$. Define

$$\overline{\mathcal{W}}_{\text{sup}} \overset{\text{def}}{=} \frac{t^{-m}(t+p)^{-\mu-1}\mathcal{O}[t]^d}{t^n(t+p)^{\nu}\mathcal{O}[t]^d}$$

For this section, redefine

$$\overline{\phi} : \overline{\mathcal{W}}_{\text{sup}} \times \overline{\mathcal{W}}_{\text{sup}} \longrightarrow \frac{t^{-2m}(t+p)^{-2(\mu+1)}\mathcal{O}[t]}{t^{n-m}(t+p)^{\nu-\mu-1}\mathcal{O}[t]}$$

be the hermitian $\mathbf{Z}_p[t]$-bilinear form defined by the anti-identity matrix, and redefine $\overline{\mathcal{V}}_i$ to be the image of $\mathcal{V}_i$ in $\overline{\mathcal{W}}_{\text{sup}}$.

Remark. *This* $\overline{\mathcal{W}}_{\text{sup}}$ *is designed to be a universal container for all modules occurring in the definitions of both* $\mathbf{M}^{(m,n)}$ *and* $\mathbf{Fl}^{(\mu,\nu)}$.

Define the functor

$$\widetilde{\mathbf{M}}^{(m,n)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R, the set of all $\mathcal{R}[t]$-linear maps $g : \overline{\mathcal{W}}_{\text{sup}}(R) \to \overline{\mathcal{W}}_{\text{sup}}(R)$ such that

1. each $\overline{\mathcal{L}}_i \overset{\text{def}}{=} g(t^{-m}\overline{\mathcal{V}}_i(R))$ satisfies

$$t^n\overline{\mathcal{V}}_i(R) \subset \overline{\mathcal{L}}_i \subset t^{-m}\overline{\mathcal{V}}_i(R)$$

2. each inclusion $\overline{\mathcal{L}}_i/t^n\overline{\mathcal{V}}_i(R) \hookrightarrow t^{-m}\overline{\mathcal{V}}_i(R)/t^n\overline{\mathcal{V}}_i(R)$ splits R-linearly

3. the projective rank function $\text{Spec}(R) \to \mathbf{N}$ associated to each

$$\overline{\mathcal{L}}_i/t^n\overline{\mathcal{V}}_i(R)$$

is the constant function $\mathfrak{p} \mapsto d(m+n)$

4. there exists a $c(g) \in R[t]$ representing a unit in $R[t]/t^{m+n}(t+p)^{\mu+\nu+1}R[t]$ such that

$$\overline{\phi}_R(g(x), g(y)) = c(g)t^{m+n}\overline{\phi}_R(x, y)$$

for all $x, y \in \overline{\mathcal{W}}_{\text{sup}}(R)$

5. Setting

$$t^{-m}\widetilde{\mathcal{V}}_i \stackrel{\text{def}}{=} \frac{t^{-m}(t+p)^{-1}\mathcal{O}[t]^i \oplus t^{-m}\mathcal{O}[t]^{d-i}}{t^n(t+p)^{\mu+\nu+1}\mathcal{O}[t]^d}$$

$$\widetilde{\mathcal{L}}_i \stackrel{\text{def}}{=} \frac{\mathcal{L}_i}{t^{m+n}(t+p)^{\mu+\nu+1}\mathcal{L}_0}$$

(note that $\widetilde{\mathcal{L}}_0 \subset \widetilde{\mathcal{L}}_1 \subset \cdots$), there exists Zariski-locally on $\operatorname{Spec}(R)$ an $\mathcal{R}[t]$-linear isomorphism

$$\widetilde{g} : t^{-m}\widetilde{\mathcal{V}}_0(R) \xrightarrow{\sim} \widetilde{\mathcal{L}}_0$$

inducing the restriction $g : t^{-m}\overline{\mathcal{V}}_0(R) \to \overline{\mathcal{L}}_0$ such that

$$\widetilde{g}((t+p)t^{-m}\widetilde{\mathcal{V}}_i(R)) = (t+p)\widetilde{\mathcal{L}}_i$$

and such that

$$\overline{\phi}_R(\widetilde{g}(x), \widetilde{g}(y)) = c(g)t^{m+n}\overline{\phi}_R(x, y)$$

for all $x, y \in t^{-m}\widetilde{\mathcal{V}}_0(R)$ (by the duality condition **ELM5**, the ordinary product $\overline{\phi}_R$ is well-defined on $\widetilde{\mathcal{L}}_0$)

(the meaning of "Zariski-locally" in the last condition is that there are multiplicative subsets $S_1, \ldots, S_n$ covering $\operatorname{Spec}(R)$ and $S_i^{-1}\mathcal{R}[t]$-linear isomorphisms

$$g_i : S_i^{-1}t^{-m}\widetilde{\mathcal{V}}_0(R) = t^{-m}\widetilde{\mathcal{V}}_0(S_i^{-1}R) \xrightarrow{\sim} S_i^{-1}\widetilde{\mathcal{L}}_0$$

inducing $S_i^{-1}g$)

Remark. *I am abusing notation by not including μ and ν in the symbol "$\widetilde{\mathbf{M}}^{(m,n)}$".*

It is clear that the tuple $(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_{d/2})$ has all the properties necessary to be the image in $t^{-m}\overline{\mathcal{V}}_i(R)/t^n\overline{\mathcal{V}}_i(R) = t^{-m}\mathcal{V}_i(R)/t^n\mathcal{V}_i(R)$ of a point $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$ except possibly the duality condition **ELM5**. That condition follows from the similitude condition (notice that $\overline{\phi}$ here restricts, descends and retracts to the $\overline{\phi}$ from §2.4.1 (page 49)), taking into account that the factor of t^{m+n} comes from a normalization: $t^{m-n}\overline{\phi}_R$ and $t^{2m}\overline{\phi}_R$ send $\overline{\mathcal{L}}_0 \times \overline{\mathcal{L}}_0$ and $t^{-m}\overline{\mathcal{V}}_0(R) \times t^{-m}\overline{\mathcal{V}}_0(R)$ (respectively) into $\mathcal{R}[t]$, and $\widetilde{g}$ should identify $t^{2m}\overline{\phi}_R$ to $t^{m-n}\overline{\phi}_R$, hence the above requirement.

Therefore, I may define

$$\widetilde{\mathbf{M}}^{(m,n)}(R) \longrightarrow \mathbf{M}^{(m,n)}(R)$$

$$g \longmapsto (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2})$$

Remark. *Recall that if $m = n$ then the trivial tuple $(\mathcal{V}_0(R), \ldots, \mathcal{V}_{d/2}(R))$ is an element of $\mathbf{M}^{(m,m)}(R)$, so the map $w \mapsto t^m w$ is a sort of "identity element" of $\widetilde{\mathbf{M}}^{(m,m)}$. As before, this is related to the identity component of the affine flag varieties.*

Proposition 3.2.2.1. *$\widetilde{\mathbf{M}}^{(m,n)}$ is a finite-type $\mathbf{Z}_p$-scheme.*

Proof. Let $\widetilde{\mathbf{M}}^{(m,n)}_{\text{weak}} : \mathbf{Z}_p\text{-Algebras} \to \text{Sets}$ be the functor defined only by conditions (1), (2), (3) and (4). Conditions (1) and (4) obviously define a finite-type scheme, and Lemma 18 from [15] handles conditions (2) and (3), so $\widetilde{\mathbf{M}}^{(m,n)}_{\text{weak}}$ is a finite-type scheme. Now consider $\widetilde{\mathbf{M}}^{(m,n)} \subset \widetilde{\mathbf{M}}^{(m,n)}_{\text{weak}}$. It is clear that the similitude part of condition (5) is no problem, so I now check the Zariski-local existence statement.

Let R be a (commutative) $\mathbf{Z}_p$-algebra and temporarily fix $g \in \widetilde{\mathbf{M}}^{(m,n)}_{\text{weak}}(R)$. The set of *possibly-non-invertible* $\widetilde{g}$ inducing g *globally* on $\mathrm{Spec}(R)$ is clearly the R-points of a finite-dimensional affine space (choose additional matrix entries from $\widetilde{\mathcal{L}}_0/\overline{\mathcal{L}}_0$). Let k be the dimension of this affine space. Then for fixed g, the condition of invertibility can be expressed (by Cramer's rule) as a polynomial equation by using the associated k-variable determinant $\det_g$. Now I extend this to the Zariski-local case.

Define for each $n \in \mathbf{N}$ and $\ell_1, \ldots, \ell_n, m_1, \ldots, m_n \in \mathbf{N}$ the ideal $I(n; \{\ell_i\}; \{m_j\})$ in

$$\mathbf{Z}_p[W_1, \ldots, W_n; X_1, \ldots, X_n; Y_1, \ldots, Y_n; Z_1^{(1)}, \ldots, Z_k^{(1)}, \ldots, Z_1^{(n)}, \ldots, Z_k^{(n)}]$$

by the equations

$$\begin{aligned} W_1Y_1 + \cdots + W_nY_n &= 1 \\ Y_1^{\ell_1} \cdot (\det\nolimits_g(Z_1^{(1)}, \ldots, Z_k^{(1)}) \cdot X_1 - Y_1^{m_1} \cdot 1) &= 0 \\ &\vdots \\ Y_n^{\ell_n} \cdot (\det\nolimits_g(Z_1^{(n)}, \ldots, Z_k^{(n)}) \cdot X_n - Y_n^{m_n} \cdot 1) &= 0 \end{aligned}$$

In any R-valued solution to this system,

- the values $Y_1, \ldots, Y_n$ will, because of the first equation, be generators of the trivial ideal R, i.e. a principal open cover of $\mathrm{Spec}(R)$ (the values W_i are auxiliary),

- the values $Z_1^{(i)}, \dots, Z_k^{(i)}$ define the (at the moment possibly-non-invertible) Zariski-local lift of g over the principal open subset $\mathrm{Spec}(R_{Y_i})$, and

- the last n equations exactly express (by definition of the fraction ring R_{Y_i}) that the determinant of the Zariski-local lift is a unit, i.e. that each Zariski-local lift is invertible.

Since $\widetilde{\mathbf{M}}^{(m,n)}_{\mathrm{weak}}$ is already known to be a scheme, it is clear that the above system of equations can be extended (simply add more variables and the ideal defining $\widetilde{\mathbf{M}}^{(m,n)}_{\mathrm{weak}}$) to eliminate the assumption that g is fixed. Taking I to be the sum of all the above ideals in the obvious countably-generated polynomial ring, it is then clear that the subfunctor $\widetilde{\mathbf{M}}^{(m,n)} \subset \widetilde{\mathbf{M}}^{(m,n)}_{\mathrm{weak}}$ representable (it is the image under the forgetful morphism $(\widetilde{g}, g) \mapsto g$ of scheme defined by the ideal I). This proves representability, and finite-type is then obvious. $\square$

Similarly, define the functor

$$\widetilde{\mathbf{Fl}}^{(\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set of all $\mathcal{R}[t]$-linear maps $g : \overline{\mathcal{W}}_{\mathrm{sup}}(R) \to \overline{\mathcal{W}}_{\mathrm{sup}}(R)$ such that

- each $\overline{\mathcal{F}}_i \stackrel{\mathrm{def}}{=} g((t+p)^{-\mu}\overline{\mathcal{V}}_i(R))$ satisfies

$$(t+p)^{\nu}\overline{\mathcal{V}}_i(R) \subset \overline{\mathcal{F}}_i \subset (t+p)^{-\mu}\overline{\mathcal{V}}_i(R)$$

- each inclusion $\overline{\mathcal{F}}_i/(t+p)^{\nu}\overline{\mathcal{V}}_i(R) \hookrightarrow (t+p)^{-\mu}\overline{\mathcal{V}}_i(R)/(t+p)^{\nu}\overline{\mathcal{V}}_i(R)$ splits R-linearly

- the projective rank function $\mathrm{Spec}(R) \to \mathbf{N}$ associated to each $\overline{\mathcal{F}}_i/(t+p)^{\nu}\overline{\mathcal{V}}_i(R)$ is the constant function $\mathfrak{p} \mapsto d(\mu+\nu)$

- there exists a $c(g) \in R[t]$ representing a unit in $R[t]/t^{m+n}(t+p)^{\mu+\nu+1}R[t]$ such that

$$\overline{\phi}_R(g(x), g(y)) = c(g)(t+p)^{\mu+\nu}\overline{\phi}_R(x,y)$$

for all $x, y \in \overline{\mathcal{W}}_{\sup}(R)$

- Setting

$$(t+p)^{-\mu}\widetilde{\mathcal{V}}_i \stackrel{\mathrm{def}}{=} \frac{(t+p)^{-\mu-1}\mathcal{O}[t]^i \oplus (t+p)^{-\mu}\mathcal{O}[t]^{d-i}}{t^{m+n+1}(t+p)^{\nu}\mathcal{O}[t]^d}$$
$$\widetilde{\mathcal{F}}_i \stackrel{\mathrm{def}}{=} \frac{\mathcal{F}_i}{t^{m+n}(t+p)^{\mu+\nu+1}\mathcal{F}_0}$$

(note that $\widetilde{\mathcal{F}}_0 \subset \widetilde{\mathcal{F}}_1 \subset \cdots$), there exists Zariski-locally on $\mathrm{Spec}(R)$ an $\mathcal{R}[t]$-linear isomorphism

$$\widetilde{g} : (t+p)^{-\mu}\widetilde{\mathcal{V}}_0(R) \xrightarrow{\sim} \widetilde{\mathcal{F}}_0$$

inducing the restriction $g : (t+p)^{-\mu}\overline{\mathcal{V}}_0(R) \to \overline{\mathcal{F}}_0$ such that

$$\widetilde{g}((t+p)^{-\mu+1}\widetilde{\mathcal{V}}_i(R)) = (t+p)\widetilde{\mathcal{F}}_i$$

and such that

$$\overline{\phi}_R(\widetilde{g}(x), \widetilde{g}(y)) = c(g)(t+p)^{\mu+\nu}\overline{\phi}_R(x,y)$$

for all $x, y \in (t+p)^{-\mu}\widetilde{\mathcal{V}}_0(R)$ (by the duality condition **ELM5**, the ordinary product $\overline{\phi}_R$ is well-defined on $\widetilde{\mathcal{F}}_0$)

(the meaning of the last condition is as in the case of $\widetilde{\mathbf{M}}^{(m,n)}$)

Remark. *I am abusing notation by not including m and n in the symbol "$\widetilde{\mathbf{Fl}}^{(\mu,\nu)}$".*

As before,

Proposition 3.2.2.2. *$\widetilde{\mathbf{Fl}}^{(\mu,\nu)}$ is a finite-type $\mathbf{Z}_p$-scheme.*

Proof. This is nearly identical to the proof for $\widetilde{\mathbf{M}}^{(m,n)}$ (page 95). □

As before, I may define

$$\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(R) \longrightarrow \mathbf{Fl}^{(\mu,\nu)}(R)$$

$$g \longmapsto (g((t+p)^{-\mu}\overline{\mathcal{V}}_0(R)), \ldots, g((t+p)^{-\mu}\overline{\mathcal{V}}_{d/2}(R)))$$

Remark. *As in the case of $\widetilde{\mathbf{M}}^{(m,n)}$, notice that if $\mu = \nu$ then $\mathbf{Fl}^{(\mu,\mu)}$ and $\widetilde{\mathbf{Fl}}^{(\mu,\mu)}$ have an "identity element".*

Define

$$p_1 : \widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \longrightarrow \mathbf{M}^{(m,n)} \times \mathbf{Fl}^{(\mu,\nu)}$$

to be the product of the above morphisms.

3.2.3 The convolution scheme

I now define a $\mathbf{Z}_p$-scheme $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$ and a "twisted action" morphism $p_2 : \widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \longrightarrow \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$. The purpose of this scheme $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$ is to perform the summation operation that occurs in the ordinary convolution of functions in the Iwahori-Hecke algebra. See §3.4.2 (page 135).

Definition: The Convolution Scheme. *Define the functor*

$$\mathbf{Conv}^{(m,n\,;\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$*-algebra* R *the set of all tuples*

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$$

of $\mathcal{R}[t]$*-submodules of* $\mathcal{V}(R)$ *satisfying*

- $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$

- *each* $\mathcal{K}_i$ *satisfies*

  $$(t+p)^{\nu}\mathcal{L}_i \subset \mathcal{K}_i \subset (t+p)^{-\mu}\mathcal{L}_i$$

- *each inclusion* $\mathcal{K}_i/(t+p)^{\nu}\mathcal{L}_i \hookrightarrow (t+p)^{-\mu}\mathcal{L}_i/(t+p)^{\nu}\mathcal{L}_i$ *splits R-linearly*

- *the projective rank function* $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each*

  $$\mathcal{K}_i/(t+p)^{\nu}\mathcal{L}_i$$

  is the constant function $\mathfrak{p} \mapsto d(\mu + \nu)$

  (note that the previous property implies R-projectivity)

- $\mathcal{K}_0$ *is self-dual with respect to* $t^{m-n}(t+p)^{\mu-\nu}\phi_R$ *and* $\mathcal{K}_{d/2}$ *is self-dual with respect to* $t^{m-n}(t+p)^{\mu-\nu+1}\phi_R$

  (the ϕ *used here has domain* $\mathcal{V} \times \mathcal{V}$*)*

(The duality occurring here is similar to the one occurring in **ELM5**: it is required that $\mathcal{K}_0$ be *exactly* the elements $x \in \mathcal{V}(R)$ such that

$$\phi_R(\mathcal{K}_0, x) \subset t^{n-m}(t+p)^{\nu-\mu}\mathcal{R}[t]$$

and similarly for $\mathcal{K}_{d/2}$.)

Similar to $\mathbf{M}^{(m,n)}$ (see §2.4.3 (page 58)), this $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$ is a closed subscheme of a product of (ordinary) Grassmannians. In particular,

$$\mathbf{Conv}^{(m,n\,;\,\mu,\nu)} \to \mathrm{Spec}(\mathbf{Z}_p)$$

is *proper.*

Define

$$p_2 : \widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \longrightarrow \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$$

by

$$(g,h) \longmapsto (g(t^{-m}\overline{\mathcal{V}}_i(R))_{i=0}^{d/2}\ ;\ g(h(t^{-m}(t+p)^{-\mu}\overline{\mathcal{V}}_i(R)))_{i=0}^{d/2})$$

(these images of g and h are technically submodules of $\overline{\mathcal{W}}_{\mathrm{sup}}(R)$, but as usual I replace them by their corresponding submodules of $\mathcal{V}(R)$).

To verify that the codomain of this morphism really is correct, note that the 1st coordinate is simply the previously verified action morphism $\widetilde{\mathbf{M}}^{(m,n)} \to \mathbf{M}^{(m,n)}$ and that by definition of h,

$$(t+p)^{\nu}t^{-m}\overline{\mathcal{V}}_i(R) \subset h(t^{-m}(t+p)^{-\mu}\overline{\mathcal{V}}_i(R)) \subset (t+p)^{-\mu}t^{-m}\overline{\mathcal{V}}_i(R) \tag{3.4}$$

and this chain is transformed by g to the chain

$$(t+p)^{\nu}\overline{\mathcal{L}}_i \subset g(h(t^{-m}(t+p)^{-\mu}\overline{\mathcal{V}}_i(R))) \subset (t+p)^{-\mu}\overline{\mathcal{L}}_i \tag{3.5}$$

Finally, I define a $\mathbf{Z}_p$-scheme $\mathbf{P}^{(m,n\,;\,\mu,\nu)}$ essentially as a target for the 2nd projection from $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$:

Definition: The Convolution Base. *Define the functor*

$$\mathbf{P}^{(m,n\,;\,\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$*-algebra* R *the set of all tuples* $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$ *of* $\mathcal{R}[t]$*-submodules of* $\mathcal{V}(R)$ *satisfying*

- *each* $\mathcal{K}_i$ *satisfies*

$$t^n(t+p)^\nu \mathcal{V}_i(R) \subset \mathcal{K}_i \subset t^{-m}(t+p)^{-\mu}\mathcal{V}_i(R)$$

- *each inclusion* $\mathcal{K}_i/t^n(t+p)^\nu\mathcal{V}_i(R) \hookrightarrow t^{-m}(t+p)^{-\mu}\mathcal{V}_i(R)/t^n(t+p)^\nu\mathcal{V}_i(R)$ *splits* R*-linearly*

- *the projective rank function* $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each*

$$\mathcal{K}_i/t^n(t+p)^\nu\mathcal{V}_i(R)$$

  is the constant function $\mathfrak{p} \mapsto (m+n+\mu+\nu)d$

  (the rank here is bigger than the rank in the definition of the convolution scheme because the quotient here is also bigger)

- $\mathcal{K}_0$ *is self-dual with respect to* $t^{m-n}(t+p)^{\mu-\nu}\phi_R$ *and* $\mathcal{K}_{d/2}$ *is self-dual with respect to* $t^{m-n}(t+p)^{\mu-\nu+1}\phi_R$

(The duality occurring here is similar to the one occurring in **ELM5**: it is required that $\mathcal{K}_0$ be *exactly* the elements $x \in \mathcal{V}(R)$ such that

$$\phi_R(\mathcal{K}_0, x) \subset t^{n-m}(t+p)^{\nu-\mu}\mathcal{R}[t]$$

and similarly for $\mathcal{K}_{d/2}$.)

Remark. *Notice that if* $\mu = \nu$ *then* $\mathbf{M}^{(m,n)} \subset \mathbf{P}^{(m,n;\mu,\mu)}$*: all the conditions are obviously satisfied except possibly the rank condition, which is true because, Zariski-*

locally on $\mathrm{Spec}(R)$,

$$\mathrm{rank}_R(\mathcal{L}_i/t^n(t+p)^\nu\mathcal{V}_i(R)) = (m+n)d + 2\nu d = (m+n)d + (\mu+\nu)d.$$

This is related to the "identity element" $1 \in \widetilde{\mathbf{Fl}}^{(\mu,\mu)}$*: if* $g \in \widetilde{\mathbf{M}}^{(m,n)}$ *maps to* $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}$*, then considered as an element of* $\mathbf{P}^{(m,n;\mu,\mu)}$*,*

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) = m(p_2(g,1)).$$

Similar to $\mathbf{M}^{(m,n)}$ (see §2.4.3 (page 58)), this $\mathbf{P}^{(m,n;\mu,\nu)}$ is a closed subscheme of a product of (ordinary) Grassmannians. In particular, $\mathbf{P}^{(m,n;\mu,\nu)} \to \mathrm{Spec}(\mathbf{Z}_p)$ is *proper.*

Define

$$m : \mathbf{Conv}^{(m,n;\mu,\nu)} \longrightarrow \mathbf{P}^{(m,n;\mu,\nu)}$$

by

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \longmapsto (\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$$

The verification that this function has codomain $\mathbf{P}^{(m,n;\mu,\nu)}$ is very easy: the duality condition is identical for both schemes, the containment relations are verified by concatenating the containment relations satisfied by the $\mathcal{L}_i$ (i.e. **ELM2**) onto those satisfied by the $\mathcal{K}_i$, and it is easy to see that the rank condition is then satisfied.

The morphism m is automatically proper due to the fact that the domain and codomain are proper schemes.

3.2.4 Convolution diagram construction

Combining all the above objects and morphisms gives, at long last, the convolution diagram:

$$\mathbf{M}^{(m,n)} \times \mathbf{Fl}^{(\mu,\nu)} \xleftarrow{p_1} \widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \xrightarrow{p_2} \mathbf{Conv}^{(m,n\,;\,\mu,\nu)} \xrightarrow{m} \mathbf{P}^{(m,n\,;\,\mu,\nu)}$$

3.2.5 The "reversed" convolution diagram

I now construct a "reversed" convolution diagram, which is used to construct a "reversed" convolution product product (see the end of §3.4.1 (page 130). The statement that the convolution product of two particular functions is commutative is equivalent to the statement that the convolution product and reversed convolution product of the corresponding sheaf complexes are equal.

Definition: The Reversed Convolution Scheme. *Fix* $m, n, \mu, \nu \in \mathbf{N}$. *Define the functor*

$${}^{\mathrm{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$*-algebra* R *the set of all tuples*

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$$

of $\mathcal{R}[t]$*-submodules of* $\mathcal{V}(R)$ *satisfying*

- $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{Fl}^{(\mu,\nu)}(R)$
- *each* $\mathcal{L}_i$ *satisfies*

$$t^n \mathcal{K}_i \subset \mathcal{L}_i \subset t^{-m} \mathcal{K}_i$$

- *each inclusion $\mathcal{L}_i/t^n\mathcal{K}_i \hookrightarrow t^{-m}\mathcal{K}_i/t^n\mathcal{K}_i$ splits R-linearly*

- *the projective rank function* $\mathrm{Spec}(R) \to \mathbf{N}$ *associated to each $\mathcal{L}_i/t^n\mathcal{K}_i$ is the constant function $\mathfrak{p} \mapsto d(m+n)$*

- *$\mathcal{L}_0$ is self-dual with respect to $t^{m-n}(t+p)^{\mu-\nu}\phi_R$ and $\mathcal{L}_{d/2}$ is self-dual with respect to $t^{m-n}(t+p)^{\mu-\nu+1}\phi_R$*

  (the ϕ used here has domain $\mathcal{V} \times \mathcal{V}$)

I define a morphism

$$^{\mathrm{rev}}m : {}^{\mathrm{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)} \longrightarrow \mathbf{P}^{(m,n\,;\,\mu,\nu)}$$

as in the non-reversed case by

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \longmapsto (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2})$$

(just as for m, it is easy to verify that this function has codomain $\mathbf{P}^{(m,n\,;\,\mu,\nu)}$). Note that the codomain of m and $^{\mathrm{rev}}m$ are *identical*.

I define a morphism

$$^{\mathrm{rev}}p_2 : \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \times \widetilde{\mathbf{M}}^{(m,n)} \longrightarrow {}^{\mathrm{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)}$$

as in the non-reversed case by

$$(g, h) \longmapsto (g((t+p)^{-\mu}\overline{\mathcal{V}}_i(R))_{i=0}^{d/2}\ ;\ g(h(t^{-m}(t+p)^{-\mu}\overline{\mathcal{V}}_i(R)))_{i=0}^{d/2})$$

(notice that the 1st coordinate of $^{\mathrm{rev}}p_2$ is just $\widetilde{\mathbf{Fl}}^{(\mu,\nu)} \to \mathbf{Fl}^{(\mu,\nu)}$ from before)

Substituting the new convolution object and morphisms yields a "reversed" convolution diagram:

$$\mathbf{Fl}^{(\mu,\nu)} \times \mathbf{M}^{(m,n)} \xleftarrow{p_1} \widetilde{\mathbf{Fl}}^{(\mu,\nu)} \times \widetilde{\mathbf{M}}^{(m,n)} \xrightarrow{^{\mathrm{rev}}p_2} {}^{\mathrm{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)} \xrightarrow{^{\mathrm{rev}}m} \mathbf{P}^{(m,n\,;\,\mu,\nu)}$$

3.3 Properties of the convolution diagram

Recall that for a point $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R)$ it is not the case that the tuple $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$ is a point of $\mathbf{Fl}^{(\mu,\nu)}(R)$ (indeed, that is the whole point). However,

Lemma 3.3.0.1. *Let R be a* local *(commutative) $\mathbf{Z}_p$-algebra. If*

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R)$$

and

$$g \in \widetilde{\mathbf{M}}^{(m,n)}(R)$$

is such that

$$\begin{aligned} \widetilde{\mathbf{M}}^{(m,n)}(R) &\longrightarrow \mathbf{M}^{(m,n)}(R) \\ g &\longmapsto (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \end{aligned}$$

then *there exists a*

$$(\mathcal{F}_0, \ldots \mathcal{F}_{d/2}) \in \mathbf{Fl}^{(\mu,\nu)}(R)$$

such that $g(\overline{\mathcal{F}}_i) = \overline{\mathcal{K}}_i$ for all i.

Remark. *The point of this lemma is roughly that given $g \in \widetilde{\mathbf{M}}^{(m,n)}$, one can revert (3.5) back to (3.4) (page 101), even though an "h" may not exist. This is useful for deriving statements about p_2 from similar statements about p_1.*

Proof. By definition,

$$(t + p)^{\nu}\mathcal{L}_i \subset \mathcal{K}_i \subset (t + p)^{-\mu}\mathcal{L}_i$$

Quotient by $t^{m+n}(t+p)^{\mu+\nu+1}\mathcal{L}_0$ (so that the leftmost and rightmost modules involve $\widetilde{\mathcal{L}}_i$, in the sense of $\widetilde{\mathbf{M}}^{(m,n)}$), apply $\widetilde{g}^{-1}$ (since R is assumed local, $\widetilde{g}$ exists globally on $\mathrm{Spec}(R)$), scale by t^m, and quotient by $t^n(t+p)^\nu\mathcal{R}[t]^d$ to get a module $\overline{\mathcal{F}}_i$ satisfying

$$(t+p)^\nu\overline{\mathcal{V}}_i(R) \subset \overline{\mathcal{F}}_i \subset (t+p)^{-\mu}\overline{\mathcal{V}}_i(R) \tag{3.6}$$

Because $\widetilde{g}$ induces g, it is true that $g(\overline{\mathcal{F}}_i) = \overline{\mathcal{K}}_i$.

Recall the conditions for membership in $\mathbf{Fl}^{(\mu,\nu)}(R)$. The containments (3.6) are one of those conditions. Because $\mathcal{K}_0 \subset \cdots \subset \mathcal{K}_{d/2}$, it is also true that $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{d/2}$. The projectivity condition and rank condition are satisfied because $\widetilde{g}$ is an isomorphism and because of the similar properties of $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$. The duality condition is not totally obvious, but follows from the similitude property of $\widetilde{g}$ and the similar duality property (page 90) of $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$:

$$\begin{aligned}
\phi_R(\mathcal{F}_0, \mathcal{F}_0) &= \phi_R(t^m\widetilde{g}^{-1}(\mathcal{K}_0), t^m\widetilde{g}^{-1}(\mathcal{K}_0)) \\
&= t^{2m}t^{-(m+n)}c(g)^{-1}\phi_R(\mathcal{K}_0) \\
&= t^{m-n}c(g)^{-1}\phi_R(\mathcal{K}_0, \mathcal{K}_0)
\end{aligned}$$

so

$$\phi_R(\mathcal{K}_0, \mathcal{K}_0) \subset t^{n-m}(t+p)^{\nu-\mu}\mathcal{R}[t] \iff \phi_R(\mathcal{F}_0, \mathcal{F}_0) \subset (t+p)^{\nu-\mu}\mathcal{R}[t]$$

etc. □

The following is the analogue of Lemma 19 from [15]:

Proposition 3.3.0.2. *The morphisms p_1 and p_2 are smooth and for any $\mathbf{Z}_p$-field K, the functions $p_1(K)$ and $p_2(K)$ are surjective.*

Proof. $\boxed{\text{Smoothness}}$ To prove that $\widetilde{\mathbf{M}}^{(m,n)} \to \mathbf{M}^{(m,n)}$ is smooth, I must show that for

- a (commutative) $\mathbf{Z}_p$-algebra R
- a nilpotent ideal $I \subset R$
- $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(R)$
- $g_{R/I} \in \widetilde{\mathbf{M}}^{(m,n)}(R/I)$ such that

  $$g_{R/I}(t^{-m}\overline{\mathcal{V}}_i(R/I)) = \overline{\mathcal{L}}_i \otimes_R (R/I)$$

  for each $0 \leq i \leq d/2$

there exists a $g_R \in \widetilde{\mathbf{M}}^{(m,n)}(R)$ such that $g_R(t^{-m}\overline{\mathcal{V}}_i(R)) = \overline{\mathcal{L}}_i$ for each $0 \leq i \leq d/2$ and $g_R \mapsto g_{R/I}$ under $\widetilde{\mathbf{M}}^{(m,n)}(R) \to \widetilde{\mathbf{M}}^{(m,n)}(R/I)$. By the proof of Corollary 4.5 in Chapter 1 §4 of [7], I may assume that R is *local*, in which case $\widetilde{g}_{R/I}$ exists globally on $\mathrm{Spec}(R)$.

Some notation. Set

$$\mathcal{S} \stackrel{\text{def}}{=} \mathcal{R}[t]/t^{m+n}(t+p)^{\mu+\nu+1}\mathcal{R}[t]$$

and let $\mathcal{I}_t$ be the extension in $\mathcal{S}$ of the ideal I. I use $\mathcal{L}_{i,R}$ and $\mathcal{L}_{i,R/I}$ refer to $\mathcal{L}_i$ and $\mathcal{L}_i \otimes_R (R/I)$, and so on. Let $\jmath : \mathcal{S} \to \mathcal{S}$ be the involution induced by the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q}_p)$.

First, a partial result:

Lemma 3.3.0.3. *With $R, I, (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}), g_{R/I}$ and the notation as above, $\widetilde{\mathcal{L}}_0$ is free over $\mathcal{S}$ and the hermitian form $\widetilde{\mathcal{L}}_0 \times \widetilde{\mathcal{L}}_0 \to \mathcal{S}$ induced by $t^{m-n}\phi_R$ is perfect.*

Note that these assertions are not automatic because it is not known *a priori* that $(\mathcal{L}_0, \dots, \mathcal{L}_{d/2})$ is the image of some g_R.

Proof. After normalization, I can assume that the domain of $\widetilde{g}$ is $(\mathcal{S}/\mathcal{I}_t)^d$. Let $\widetilde{v}_1, \dots, \widetilde{v}_d$ be arbitrary lifts to $\widetilde{\mathcal{L}}_0$ of the basis $\widetilde{g}(e_1), \dots, \widetilde{g}(e_d)$. By Nakayama's lemma, $\widetilde{v}_1, \dots, \widetilde{v}_d$ generates $\widetilde{\mathcal{L}}_0$. Let

$$0 \longrightarrow K \longrightarrow \mathcal{S}^d \longrightarrow \widetilde{\mathcal{L}}_{0,R} \longrightarrow 0 \tag{3.7}$$

be the presentation so defined. Since $\overline{\mathcal{L}}_{0,R}$ is R-projective by **ELM5** (page 57), and since the kernel of $\widetilde{\mathcal{L}}_{0,R} \twoheadrightarrow \overline{\mathcal{L}}_{0,R}$ is identified R-linearly with $t^{-m}(t+p)^{-\mu}\mathcal{V}_0(R)/\mathcal{L}_{0,R}$, which is also R-projective by **ELM5**, it follows that $\widetilde{\mathcal{L}}_{0,R}$ is R-projective. So (3.7) splits and

$$0 \longrightarrow K \otimes_R R/I \longrightarrow \mathcal{S}^d \otimes_R R/I \longrightarrow \widetilde{\mathcal{L}}_{0,R} \otimes_R R/I \longrightarrow 0 \tag{3.8}$$

is still exact. The middle module is just $(\mathcal{S}/\mathcal{I}_t)^d$ and, again by R-projectivity of $\widetilde{\mathcal{L}}_{0,R}$, the rightmost module is just $\widetilde{\mathcal{L}}_{0,R/I}$. This means that the presentation (3.8) is just the one given by the *isomorphism* $\widetilde{g}_{R/I}$, which means $K \otimes_R R/I = K/IK = 0$. By Nakayama's lemma (note that K is finitely-generated by the splitting of (3.7)), $K = 0$ and so the lift $\mathcal{S}^d \longrightarrow \widetilde{\mathcal{L}}_{0,R}$ of $\widetilde{g}_{R/I}$ is an isomorphism.

For perfection, recall that the form is perfect if and only if the associated adjoint map

$$\widetilde{\mathcal{L}}_{0,R} \longrightarrow \mathrm{Hom}_{\mathcal{S}\text{-lin}}(\widetilde{\mathcal{L}}_{0,R}, \mathcal{S}) \tag{3.9}$$

is *surjective*. By definition of $\widetilde{g}_{R/I}$, the corresponding form modulo I *is* perfect, i.e.

the adjoint map

$$\widetilde{\mathcal{L}}_{0,R/I} \longrightarrow \mathrm{Hom}_{(\mathcal{S}/\mathcal{I}_t)\text{-lin}}(\widetilde{\mathcal{L}}_{0,R/I}, \mathcal{S}/\mathcal{I}_t)$$

is surjective. Since $\widetilde{\mathcal{L}}_{0,R}$ is a free $\mathcal{S}$-module and $\widetilde{\mathcal{L}}_{0,R/I} = \widetilde{\mathcal{L}}_{0,R} \otimes_R R/I$,

$$\mathrm{Hom}_{(\mathcal{S}/\mathcal{I}_t)\text{-lin}}(\widetilde{\mathcal{L}}_{0,R/I}, \mathcal{S}/\mathcal{I}_t) = \mathrm{Hom}_{\mathcal{S}\text{-lin}}(\widetilde{\mathcal{L}}_{0,R}, \mathcal{S}) \otimes_R R/I$$

(this is the trivial case of "Localization of hom-sets") so Nakayama's lemma implies that (3.9) must also be surjective. $\square$

I return to the proof of Proposition 3.3.0.2. Let

$$\widetilde{w}_1, \ldots, \widetilde{w}_d \in \widetilde{\mathcal{L}}_{0,R/I}$$

be the images (necessarily a basis) under $\widetilde{g}_{R/I}$ of the standard basis. Let

$$\widetilde{v}_1, \ldots, \widetilde{v}_d \in \widetilde{\mathcal{L}}_{0,R}$$

be lifts of $\widetilde{w}_1, \ldots, \widetilde{w}_d$ such that $(t+p)\widetilde{v}_i \in (t+p)\widetilde{\mathcal{L}}_{i,R}$ (this is possible because of the hypotheses on $\widetilde{g}$). By Lemma 3.3.0.3, $\widetilde{v}_1, \ldots, \widetilde{v}_d$ is a *basis* of $\widetilde{\mathcal{L}}_{0,R}$. The normalized hermitian form $t^{m-n}\phi_R : \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{R}[t]$ descends to $\widetilde{\mathcal{L}}_{0,R} \times \widetilde{\mathcal{L}}_{0,R}$ and takes values in $\mathcal{S}$. By Lemma 3.3.0.3 again, it is *perfect*.

Now that I have the basic ingredients of freeness and perfection, I can use the same method used to prove that $\mathbf{J}^{(m,n)} \to \mathrm{Spec}(\mathbf{Z}_p)$ was smooth (page 74).

Let $c \in \mathcal{S}$ be any representative of $c(g_{R/I})$. Set

$$C_{R/I} \stackrel{\text{def}}{=} t^{m+n}(t+p)^{2\mu+2}c(g_{R/I})$$

and

$$C_R \stackrel{\text{def}}{=} t^{m+n}(t+p)^{2\mu+2}c$$

Since

$$\phi_{R/I}(\widetilde{w}_i, \widetilde{w}_j) = C_{R/I}\delta_{i,d+1-j} = \phi_{R/I}(w_i, w_j)$$

there are $x_{i,j} \in \mathcal{I}_t$ such that

$$\phi_R(\widetilde{v}_i, \widetilde{v}_j) = C_R\delta_{i,d+1-j} + x_{i,j} = \phi_R(v_i, v_j) \tag{3.10}$$

For each i, use freeness to define an $\mathcal{S}$-linear functional

$$f_i : \widetilde{\mathcal{L}}_{0,R} \longrightarrow \mathcal{S}$$

by $f_i(\widetilde{v}_j) = -\frac{1}{2}x_{i,j}$. Using perfection, there is an $\widetilde{m}_i \in \widetilde{\mathcal{L}}_{0,R}$ such that

$$f_i = t^{m-n}\phi_R(-, \widetilde{m}_i).$$

Let v_i and m_i be the images of $\widetilde{v}_i$ and $\widetilde{m}_i$ in $\overline{\mathcal{L}}_0$. Automatically, $m_i \in \mathcal{I}_t\overline{\mathcal{W}}_{\sup}(R)$ (since $\mathrm{im}(f_i) \subset \mathcal{I}_t\mathcal{S}$). It is automatic from definition that $x_{j,i} = \jmath(x_{i,j})$ so

$$\phi_R(\widetilde{v}_i + \widetilde{m}_i, \widetilde{v}_j + \widetilde{m}_j) = C_R\delta_{i,d+1-j} = \phi_R(v_i + m_i, v_j + m_j)$$

I must verify that $(t+p)m_i \in (t+p)\overline{\mathcal{L}}_i$ for each i. The proof will then be finished by defining $\widetilde{g}_R$ and g_R to be the maps sending the respective standard bases to $\{\widetilde{v}_i + \widetilde{m}_i\}$ and $\{t^{-(\mu+1)}(v_i + m_i)\}$.

Since $v_1, \ldots, v_i, (t+p)v_{i+1}, \ldots, (t+p)v_d$ generates $(t+p)\overline{\mathcal{L}}_i$ for each i, and since **ELM5** implies that

$$\overline{\mathcal{L}}_i = \{x \in \overline{\mathcal{W}}_{\sup}(R) \mid \phi_R(x, \overline{\mathcal{L}}_{d-i}) \subset t^{n-m}(t+p)\mathcal{S}\},$$

it suffices to show that

$$\phi_R(v_1, (t+p)m_i), \ldots, \phi_R(v_{d-i}, (t+p)m_i) \in t^{n-m}(t+p)^2\mathcal{S}$$

Note that the containments for $\phi_R((t+p)v_j,(t+p)m_i)$ are automatic since the defining relation (3.10) implies that $x_{i,j}$, and therefore $\phi_R(v_j,m_i)$, belongs to $t^{n-m}\mathcal{S}$.

Since $1 \le j \le d-i$ implies that $i+j \neq d+1$, the defining equality (3.10) implies that

$$\phi_R(v_j,m_i) = -\frac{1}{2}\phi_R(v_j,v_i) \;\; j=1,\dots,d-i$$

It is now automatic from the above duality that $\phi_R(v_j,m_i)$ satisfies the necessary condition.

The proof that $\widetilde{\mathbf{Fl}}^{(\mu,\nu)} \to \mathbf{Fl}^{(\mu,\nu)}$ is smooth is nearly identical.

Using Lemma 3.3.0.1 (recall that R is assumed local), smoothness of p_2 is essentially a *formal consequence* of smoothness of the individual factors of p_1. Let R be a (commutative) $\mathbf{Z}_p$-algebra and $I \subset R$ a nilpotent ideal. I must show that for all

- $(\mathcal{L}_0,\dots\mathcal{L}_{d/2};\mathcal{K}_0,\dots\mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n;\mu,\nu)}(R)$
- $(g_{R/I},h_{R/I}) \in \widetilde{\mathbf{M}}^{(m,n)}(R/I) \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(R/I)$

satisfying

$$g_{R/I}(-) = \overline{\mathcal{L}}_i \otimes_R R/I$$

$$g_{R/I}(h_{R/I}(-)) = \overline{\mathcal{K}}_i \otimes_R R/I$$

there exists $(g_R,h_R) \in \widetilde{\mathbf{M}}^{(m,n)}(R) \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(R)$ such that

$$g_R(-) = \overline{\mathcal{L}}_i$$

$$g_R(h_R(-)) = \overline{\mathcal{K}}_i$$

and $(g_R, h_R) \mapsto (g_{R/I}, h_{R/I})$.

Invoke smoothness of $\widetilde{\mathbf{M}}^{(m,n)} \to \mathbf{M}^{(m,n)}$ with the data $\{R, I, g_{R/I}, (\mathcal{L}_i)\}$ to get g_R. Let $(\mathcal{F}_0, \ldots, \mathcal{F}_{d/2}) \in \mathbf{Fl}^{(\mu,\nu)}(R)$ be the point guaranteed by Lemma 3.3.0.1 (page 106). Invoke smoothness of $\widetilde{\mathbf{Fl}}^{(\mu,\nu)} \to \mathbf{Fl}^{(\mu,\nu)}$ with the data $\{R, I, h_{R/I}, (\mathcal{F}_i)\}$ to get h_R. By Lemma 3.3.0.1,

$$g_R(h_R(t^{-m}(t+p)^{-\mu}\overline{\mathcal{V}}_i(R))) = \overline{\mathcal{K}}_i$$

so this pair (g_R, h_R) satisfies the requirements.

$\boxed{\text{Surjectivity}}$ I now prove that $\widetilde{\mathbf{M}}^{(m,n)}(K) \to \mathbf{M}^{(m,n)}(K)$ is surjective for any $\mathbf{Z}_p$-field K. Suppose first that $\mathrm{char}(K) = p$. Choose $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}) \in \mathbf{M}^{(m,n)}(K)$ and let $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ be the corresponding point of $\mathcal{F}\ell^{\mathrm{aff}}(K)$. By **AFV3** (page 61), each $\mathcal{F}_i$ is a rank d free $K[[t]]$-submodule of $K((t))^d$. On the other hand, each scaled hermitian form

$$t^{m-n+1}\Phi_K : \mathcal{F}_i \times \mathcal{F}_{d-i} \to K[[t]]$$

is *perfect*: Since $K[[t]]$ is a local principal-ideal-domain, there is a basis $f_1, \ldots, f_d$ of $K((t))^d$ and $n_1, \ldots, n_d \in \mathbf{N}$ such that $t^{n_1} f_1, \ldots, t^{n_d} f_d$ is a basis for $\mathcal{F}_i$. This implies that any functional $\mathcal{F}_i \to K[[t]]$ extends to a functional $K((t))^d \to K((t))$. This latter functional must be evaluation at some $x \in K((t))^d$ (the form is obviously perfect on the vector space $K((t))^d$), and the self-duality **AFV5** (page 61) forces $x \in \mathcal{F}_{d-i}$.

By Proposition A.43 ("every polarized chain is Zariski-locally isomorphic to the trivial polarized chain") on page 166 of [30] (recall that $K[[t]]$ is already a local ring), there are mutually compatible $K[[t]]$-linear isomorphisms

$$g_i : t^{-(m+1)}K[[t]]^i \oplus t^{-m}K[[t]]^{d-i} \xrightarrow{\sim} \mathcal{F}_i$$

such that

$$t^{m-n+1}\Phi_K(g_i(x), g_i(y)) = t^{2m+1}\Phi_K(x, y)$$

for all x, y in the domain of g_i. Each g_i sends $K[[t]]^d$ into $t^m\mathcal{F}_i$, so I have

$$t^{-(m+1)}K[[t]]^i \oplus t^{-m}K[[t]]^{d-i}/t^{n+\mu+\nu+1}K[[t]]^d \longrightarrow \mathcal{F}_i/t^{m+n+\mu+\nu+1}\mathcal{F}_i \tag{3.11}$$

Counting K-dimensions shows that (3.11) is a $K[[t]]$-linear *isomorphism*. As usual, notice that the domain is canonically identified with $t^{-m}\mathcal{V}_i(K)/t^{n+\mu+\nu+1}K[t]^d$ and the codomain with $\widetilde{\mathcal{L}}_i$ (recall the notation for $\widetilde{\mathbf{M}}^{(m,n)}$). Isomorphism (3.11) induces

$$t^{-(m+1)}K[[t]]^i \oplus t^{-m}K[[t]]^{d-i}/t^{n+\nu}K[[t]]^d \longrightarrow \mathcal{F}_i/t^{n+\nu}K[[t]]^d$$

The compatible family of these maps $t^{-m}\overline{\mathcal{V}}_i(K) \to \overline{\mathcal{L}}_i$ can obviously be lifted to a single $K[t]$-linear map $g : \overline{\mathcal{W}}_{\text{sup}}(K) \to \overline{\mathcal{W}}_{\text{sup}}(K)$. The isomorphisms (3.11) supply the necessary lift $\widetilde{g}$ so $g \in \widetilde{\mathbf{M}}^{(m,n)}(K)$ and $g \mapsto (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2})$.

The proof that $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K) \to \mathbf{Fl}^{(\mu,\nu)}(K)$ is surjective is nearly identical. This proves surjectivity for $p_1(K)$.

Using Lemma 3.3.0.1 (note that K is local), surjectivity of $p_2(K)$ is essentially a *formal consequence* of the surjectivity of the individual factors of $p_1(K)$. For

$$(\mathcal{L}_0, \ldots \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots \mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n;\mu,\nu)}(K),$$

invoke surjectivity of $\widetilde{\mathbf{M}}^{(m,n)}(K) \to \mathbf{M}^{(m,n)}(K)$ to get g, use Lemma 3.3.0.1 (page 106) to get $(\mathcal{F}_0, \ldots \mathcal{F}_{d/2}) \in \mathbf{Fl}^{(\mu,\nu)}(K)$, and use surjectivity of $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K) \to \mathbf{Fl}^{(\mu,\nu)}(K)$

to get h. It follows that that

$$p_2(g,h) = (\mathcal{L}_0, \dots \mathcal{L}_{d/2}; \mathcal{K}_0, \dots \mathcal{K}_{d/2}).$$

If $\mathrm{char}(K) = 0$ then use the Chinese remainder theorem as in the proof of Lemma 3.3.2.1 (page 116) to decompose all relevant rings and modules so that in each factor/summand, the relevant quotient involves either a power of (t) or a power of $(t+p)$, but not both. In each situation, the above proof applies (the importance of characteristic p in the above proof is only that the quotients are modulo a power of a single prime ideal). $\square$

Remark. *Applications:*

- *Smoothness of p_1 is used in §3.4.1 (page 130) and in the proof of Lemma 23 in [15] (which I invoke).*

- *Smoothness of p_2 is used in the proof of Lemma 21 in [15] (page 131 here) and in the proof of Lemma 23 in [15] (which I invoke).*

- *The surjectivity statement for p_1 is not used.*

- *The surjectivity statement for p_2 implies that the underlying map of topological spaces is surjective, which is used to satisfy the hypotheses of Lemma 21 in [15] (page 131 here).*

3.3.1 The special fiber of the convolution diagram

Two obvious but important simplifications occur over $\mathbf{F}_p$ in the convolution diagram.

First, if $(m,n) = (\mu,\nu)$ then the definitions of $\mathbf{M}^{(m,n)}$ and $\mathbf{Fl}^{(m,n)}$ are equal modulo p, i.e.

$$\mathbf{Fl}_{\mathbf{F}_p}^{(m,n)} = \mathbf{M}_{\mathbf{F}_p}^{(m,n)}$$

Similarly,

$$\widetilde{\mathbf{M}}_{\mathbf{F}_p}^{(m,n)} = \widetilde{\mathbf{Fl}}_{\mathbf{F}_p}^{(m,n)}$$

$$\mathbf{J}_{\mathbf{F}_p}^{(m,n)} = \mathbf{Iw}_{\mathbf{F}_p}^{(m,n)}$$

Second, for any $m, n, \mu, \nu \in \mathbf{N}$ it is immediate by looking at the definition that

$$\mathbf{P}_{\mathbf{F}_p}^{(m,n\,;\,\mu,\nu)} = \widetilde{\mathbf{Fl}}_{\mathbf{F}_p}^{(m+\mu,n+\nu)}$$

These observations are important for understanding why the convolution product of sheaf complexes (not yet defined) induces the convolution product of functions.

3.3.2 The generic fiber of the convolution diagram

The following result (an almost identical copy of Lemma 24 from [15]) is unique to the generic fiber, because of the fact that (t) and $(t+p)$ are *comaximal* in $\mathbf{Q}_p[t]$ and therefore the Chinese remainder theorem can be applied.

Lemma 3.3.2.1. *The extended morphisms $m_{\mathbf{Q}_p}$ and ${}^{\mathrm{rev}}m_{\mathbf{Q}_p}$ are isomorphisms, and there are isomorphisms $i, {}^{\mathrm{rev}}i$ such that the square formed by these 4 morphisms is commutative:*

$$\begin{array}{ccc}
\mathbf{M}_{\mathbf{Q}_p}^{(m,n)} \times \mathbf{Fl}_{\mathbf{Q}_p}^{(\mu,\nu)} & \xleftarrow{i} & \mathbf{Conv}_{\mathbf{Q}_p}^{(m,n\,;\,\mu,\nu)} \\
{}^{\mathrm{rev}}i \uparrow & \circ & \downarrow m \\
{}^{\mathrm{rev}}\mathbf{Conv}_{\mathbf{Q}_p}^{(\mu,\nu\,;\,m,n)} & \xrightarrow{{}^{\mathrm{rev}}m} & \mathbf{P}_{\mathbf{Q}_p}^{(m,n\,;\,\mu,\nu)}
\end{array}$$

(I mean the non-trivial commutativity you get by allowing these morphisms to be inverted)

Proof. Because the ideals (t) and $(t+p)$ are comaximal in $F[t]$, the Chinese remainder theorem implies that the $F[t]$-module $\overline{\mathcal{W}}_{\mathrm{sup}}(\mathbf{Q}_p)$ can be written as the direct sum

$$\overline{\mathcal{W}}_{\mathrm{sup}}(\mathbf{Q}_p) \cong \frac{t^{-m}F[t]^d}{t^nF[t]^d} \oplus \frac{(t+p)^{-\mu-1}F[t]^d}{(t+p)^{\nu}F[t]^d} \tag{3.12}$$

Similarly decompose each $\overline{\mathcal{V}}_i(\mathbf{Q}_p)$ into

$$\overline{\mathcal{V}}_i(\mathbf{Q}_p) \cong \overline{\mathcal{V}}_i^{(t)}(\mathbf{Q}_p) \oplus \overline{\mathcal{V}}_i^{(t+p)}(\mathbf{Q}_p)$$

Let R be a (commutative) $\mathbf{Q}_p$-algebra. Take

$$(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R).$$

Denote by $\overline{\mathcal{L}}_i$ and $\overline{\mathcal{K}}_i$ the images in $\overline{\mathcal{W}}_{\mathrm{sup}}(R)$. In particular,

$$t^n\overline{\mathcal{V}}_i(R) \subset \overline{\mathcal{L}}_i \subset t^{-m}\overline{\mathcal{V}}_i(R) \tag{3.13}$$

$$(t+p)^{\nu}\overline{\mathcal{L}}_i \subset \overline{\mathcal{K}}_i \subset (t+p)^{-\mu}\overline{\mathcal{L}}_i \tag{3.14}$$

Decompose each

$$\overline{\mathcal{L}}_i \cong \overline{\mathcal{L}}_i^{(t)} \oplus \overline{\mathcal{L}}_i^{(t+p)}$$

$$\overline{\mathcal{K}}_i \cong \overline{\mathcal{K}}_i^{(t)} \oplus \overline{\mathcal{K}}_i^{(t+p)}$$

Since the images under the 2nd projection from (3.12) of $t^k\overline{\mathcal{V}}_i(R)$ is the same, always equal to $((t+p)^{-1}\mathcal{R}[t]/\mathcal{R}[t])^i$, regardless of $k \in \mathbf{Z}$, the inclusions in (3.13) force

$$\overline{\mathcal{L}}_i^{(t+p)} = \overline{\mathcal{V}}_i^{(t+p)}(R) \tag{3.15}$$

Similarly, applying the 1st projection of from (3.12) to the inclusions in (3.14) shows that

$$\overline{\mathcal{K}}_i^{(t)} = \overline{\mathcal{L}}_i^{(t)} \tag{3.16}$$

In other words, the function

$$m_R(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) = (\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$$

is injective. Conversely, for any $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{P}^{(m,n\,;\,\mu,\nu)}(R)$, *defining* $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2})$ by (3.15) and (3.16) yields a point of $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R)$. So m_R is an *isomorphism* for any (commutative) $\mathbf{Q}_p$-algebra R.

Take $(\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) \in \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R)$. Using the preceding argument, write $(\mathcal{K}_0, \ldots, \mathcal{K}_{d/2})$ as

$$(\mathcal{L}_0^{(t)} \oplus \mathcal{K}_0^{(t+p)}, \ldots, \mathcal{L}_{d/2}^{(t)} \oplus \mathcal{K}_{d/2}^{(t+p)})$$

I claim that the chain $\overline{\mathcal{K}}_i^{(t+p)}$ (i.e. discarding the 1st summand from each $\mathcal{K}_i$) is an element of $\mathbf{Fl}^{(\mu,\nu)}(R)$. This is clear by applying the 2nd projection from (3.12) to the inclusions in (3.14) and then using the equality in (3.15) (this shows that $\overline{\mathcal{K}}_i^{(t+p)}$ has the necessary bounds, and the other properties are automatic from the definitions).

I claim that the function

$$\begin{aligned} i_R : \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(R) &\longrightarrow \mathbf{M}^{(m,n)}(R) \times \mathbf{Fl}^{(\mu,\nu)}(R) \\ (\mathcal{L}_0, \ldots, \mathcal{L}_{d/2}; \mathcal{K}_0, \ldots, \mathcal{K}_{d/2}) &\longmapsto \Big((\overline{\mathcal{L}}_0^{(t)}, \ldots, \overline{\mathcal{L}}_{d/2}^{(t)}), (\overline{\mathcal{K}}_0^{(t+p)}, \ldots, \overline{\mathcal{K}}_{d/2}^{(t+p)}) \Big) \end{aligned}$$

is a bijection. Injectivity is obvious because the discarded summand $\mathcal{L}_i^{(t)}$ in $\mathcal{K}_i$ is not truly discarded by i: it is retained by the 1st coordinate. Surjectivity is also obvious for the same reason: for any $((\overline{\mathcal{L}}_i), (\overline{\mathcal{F}}_i)) \in \mathbf{M}^{(m,n)}(R) \times \mathbf{Fl}^{(\mu,\nu)}(R)$, simply supply the respective missing summands $\overline{\mathcal{V}}_i^{(t+p)}(R)$ and $\overline{\mathcal{L}}_i^{(t)}$.

The proof for ${}^{\mathrm{rev}}m_{\mathbf{Q}_p}$ and ${}^{\mathrm{rev}}i$ is nearly identical, and the fact that the square commutes is then obvious. □

3.3.3 An automorphism group for the convolution diagram

I also need a $\mathbf{Z}_p$-group $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$ that acts on both $\widetilde{\mathbf{M}}^{(m,n)}$ and $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}$ and factors through both $\mathbf{J}^{(m,n)}$ and $\mathbf{Iw}^{(\mu,\nu)}$. The definition is a straightforward enlargement of the definition of $\mathbf{J}^{(m,n)}$ plus a lifting condition: define the functor

$$\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} : \mathbf{Z}_p\text{-Algebras} \longrightarrow \text{Sets}$$

by assigning to each (commutative) $\mathbf{Z}_p$-algebra R the set of all $\mathcal{R}[t]$-linear automorphisms γ of $\overline{\mathcal{W}}_{\mathrm{sup}}(R)$ satisfying:

- $\gamma(\overline{\mathcal{V}}_i(R)) = \overline{\mathcal{V}}_i(R)$ for all i
- there exists a *unit* $c(g) \in R[t]/t^{m+n}(t+p)^{\mu+\nu+1}R[t]$ such that

  $$\overline{\phi}_R(\gamma(x), \gamma(y)) = c(g)\overline{\phi}_R(x, y)$$

  for all $x, y \in \overline{\mathcal{W}}_{\mathrm{sup}}(R)$.
- γ is induced Zariski-locally on $\mathrm{Spec}(R)$ by some $\mathcal{R}[t]$-linear automorphism $\widetilde{\gamma}$ of

  $$t^{-m}(t+p)^{-\mu-1}\mathcal{R}[t]^d/t^{m+n}(t+p)^{\mu+\nu}\mathcal{R}[t]^d \tag{3.17}$$

  (of which $\overline{\mathcal{W}}_{\mathrm{sup}}(R)$ is a quotient) (the meaning of "Zariski-locally" in the last condition is as in $\widetilde{\mathbf{M}}^{(m,n)}$)

Notice that any $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R)$ restricts and descends from $\overline{\mathcal{W}}_{\sup}(R)$ to $\mathcal{R}[t]$-linear automorphisms $\gamma_{\mathcal{V}}$ of $\overline{\mathcal{V}}_{\sup}(R)$ and $\gamma_{\mathcal{U}}$ of $\overline{\mathcal{U}}_{\sup}(R)$. It is clear that these induced automorphisms $\gamma_{\mathcal{V}}, \gamma_{\mathcal{U}}$ are similitudes for the appropriate forms $\overline{\phi}_R$, and the multiplier $c(\gamma)$ specified from $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R)$ is also appropriate, so $\gamma \mapsto \gamma_{\mathcal{V}}$ and $\gamma \mapsto \gamma_{\mathcal{U}}$ define morphisms

$$\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} \longrightarrow \mathbf{J}^{(m,n)}$$

$$\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} \longrightarrow \mathbf{Iw}^{(\mu,\nu)}$$

of $\mathbf{Z}_p$-group schemes.

Remark. *The purpose of this group is to express equivariance properties of sheaves uniformly regardless of which object in the convolution diagram supports the sheaves.*

By the exact same process used in §2.5.5 (page 80) to define $\mathcal{I}(\mathbf{F}_p[[t]]) \to \mathbf{J}^{(m,n)}(\mathbf{F}_p)$, one has a group homomorphism

$$\mathcal{I}(\mathbf{F}_p[[t]]) \longrightarrow \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(\mathbf{F}_p)$$

(note that the existence of "$\widetilde{\gamma}$" is trivial) such that the composition

$$\mathcal{I}(\mathbf{F}_p[[t]]) \to \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(\mathbf{F}_p) \to \mathbf{J}^{(m,n)}(\mathbf{F}_p)$$

is exactly the group homomorphism from §2.5.5.

To define the convolution product, I will need two group actions of

$$\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} \times \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$$

on

$$\widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}$$

one tailored to p_1 and one tailored to p_2.

Take $g \in \widetilde{\mathbf{M}}^{(m,n)}(R)$. Since any $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}(R)$ is a similitude of $\overline{\mathcal{W}}_{\sup}(R)$ and stabilizes all $\overline{\mathcal{V}}_i$, it is clear that $g \circ \gamma$ satisfies all the conditions of $\widetilde{\mathbf{M}}^{(m,n)}(R)$ except possibly the lifting property (the existence of a certain "$\widetilde{g \circ \gamma}$"). For this, note that

$$t^{-m}(t+p)^{-\mu-1}\mathcal{R}[t]^d / t^{m+n}(t+p)^{\mu+\nu}\mathcal{R}[t]^d$$

is a subquotient of the domain of $\widetilde{\gamma}$, and define the desired lift of $g \circ \gamma$ by composing $\widetilde{g}$ with the automorphism induced by $\widetilde{\gamma}$ on that subquotient. By a nearly identical argument, $h \circ \gamma \in \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(R)$ for any $h \in \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(R)$ and $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}(R)$.

I define the 1st action α_1 by the rule

$$\alpha_1(\gamma, \eta; g, h) \stackrel{\text{def}}{=} (g \circ \gamma^{-1}, h \circ \eta^{-1})$$

Because elements $\widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}$ stabilize all $\overline{\mathcal{V}}_i$, this action stabilizes p_1-fibers.

I define the 2nd action α_2 by the rule

$$\alpha_2(\gamma, \eta; g, h) \stackrel{\text{def}}{=} (g \circ \gamma^{-1}, \gamma \circ h \circ \eta^{-1})$$

This action stabilizes p_2-fibers for the same reason, since the 2nd coordinate of p_2 uses $(g \circ \gamma^{-1}) \circ (\gamma \circ h \circ \eta^{-1}) = g \circ h \circ \eta^{-1}$.

Proposition 3.3.3.1. $\widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}$ *is a finite-type* $\mathbf{Z}_p$*-scheme.*

Proof. The proof is nearly identical (in fact easier since the codomain of the Zariski-local lifts is not varying) to that given for $\widetilde{\mathbf{M}}^{(m,n)}$ (page 95). □

3.3.4 The automorphism group is smooth

Proposition 3.3.4.1. $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} \to \mathrm{Spec}(\mathbf{Z}_p)$ *is smooth.*

Proof. Let $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}}$ be the $\mathbf{Z}_p$-group scheme defined using only the 1st and 2nd conditions (i.e. *excluding* the Zariski-local lifting condition). It is obvious that $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}}$ is finite-type, so to show that $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}}$ is smooth, it suffices to verify the infinitesimal lifting property (formal smoothness), and for this, the proof that $\mathbf{J}^{(m,n)} \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth, Proposition 2.5.2.2 (page 74), works almost verbatim: for a (commutative) $\mathbf{Z}_p$-algebra R and an ideal $I \subset R$ satisfying $I^2 = 0$, simply use

- $M := \overline{\mathcal{W}}_{\text{sup}}(R)$
- $S := R[t]/t^{m+n}(t+p)^{\mu+\nu+1}R[t]$
- $\overline{\phi}$ to be the product on $\overline{\mathcal{W}}_{\text{sup}}$
- continue to use $\sigma := (t+p)$

and define all ideals and submodules as before using the new objects just listed. This proves that $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}} \to \mathrm{Spec}(\mathbf{Z}_p)$ is smooth. By Proposition 3.3.3.1, $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$ is finite-type, so it again suffices to verify the infinitesimal lifting property. Suppose

$$\gamma_{R/I} \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R/I)$$

Since $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R/I) \subset \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}}(R/I)$, smoothness implies that there is

$$\gamma_R \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\text{weak}}(R)$$

such that $\gamma_R \mapsto \gamma_{R/I}$. By the proof of Corollary 4.5 in Chapter 1 §4 of [7], I may assume that R is *local*, in which case, to show that in fact $\gamma_R \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R)$, I must show the existence of $\widetilde{\gamma}_R$ globally on $\mathrm{Spec}(R)$. Let $\widetilde{\gamma}_R$ be an arbitrary lift of γ_R to an $\mathcal{R}[t]$-linear endomorphism of $t^{-m}(t+p)^{-\mu-1}\mathcal{R}[t]^d/t^{m+n}(t+p)^{\mu+\nu}\mathcal{R}[t]^d$. Since $\gamma_{R/I} \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R/I)$ (note that $\widetilde{\gamma}_{R/I}$ exists globally on $\mathrm{Spec}(R)$), Nakayama's lemma implies that $\gamma_{R/I}$ is *surjective*, and since the domain and codomain are the same rank d free module, "Linear independence of minimal generating sets" then implies that $\gamma_{R/I}$ is an *automorphism*. $\square$

3.3.5 Properties of the actions on the convolution diagram

The following is the analogue of Lemma 20 from [15]:

Proposition 3.3.5.1. *The action α_1 (resp. α_2) is transitive on the fibers of p_1 (resp. p_2) over K-points for any $\mathbf{Z}_p$-field K. The stabilizer subscheme under the action α_1 (resp. α_2) of any K-point is smooth for any $\mathbf{Z}_p$-field K. If K is separably-closed, then the stabilizer subscheme under the action α_1 (resp. α_2) of a K-point of $\widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}$ is also connected.*

For the 1st coordinate of the action α_1, the stabilizers in question are defined as follows: for any $\mathbf{Z}_p$-algebra A, choose $g \in \widetilde{\mathbf{M}}^{(m,n)}(A)$ and define

$$\mathrm{Stab}_g : A\text{-Algebras} \longrightarrow \mathrm{Groups}$$

to be the functor assigning to any A-algebra R the subgroup of all $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(R)$ such that $g_R \circ \gamma^{-1} = g_R$, where g_R is the image of g under $\widetilde{\mathbf{M}}^{(m,n)}(A) \to \widetilde{\mathbf{M}}^{(m,n)}(R)$. The stabilizers used for the 2nd coordinate of α_1 and the action α_2 have similar definitions.

Proof. $\boxed{\text{Transitivity}}$ I now show that for any $\mathbf{Z}_p$-field K, the action by $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ is transitive on fibers of $\widetilde{\mathbf{M}}^{(m,n)}(K) \to \mathbf{M}^{(m,n)}(K)$. Suppose $g, h \in \widetilde{\mathbf{M}}^{(m,n)}(K)$ are in the same fiber. Since K is local, the isomorphisms $\widetilde{g}$ and $\widetilde{h}$ exist globally on $\mathrm{Spec}(K)$, and both have the same codomain so $\widetilde{h}^{-1} \circ \widetilde{g}$ is a $K[t]$-linear automorphism of

$$t^{-m}K[t]^d/t^n(t+p)^{\mu+\nu}K[t]^d$$

Let $\widetilde{\gamma}$ be an arbitrary $K[t]$-linear endomorphism of

$$t^{-m}(t+p)^{-\mu-1}K[t]^d/t^{m+n}(t+p)^{\mu+\nu}K[t]^d$$

inducing $\widetilde{h}^{-1} \circ \widetilde{g}$. Since $\widetilde{h}^{-1} \circ \widetilde{g}$ is injective, $\widetilde{\gamma}$ is also injective, and since $\widetilde{\gamma}$ is in particular a K-linear operator on a finite-dimensional K-vector space, it is a $K[t]$-linear *auto*morphism. This $\widetilde{\gamma}$ induces a $K[t]$-linear automorphism of the quotient $\overline{\mathcal{W}}_{\mathrm{sup}}(K)$ and by construction, $g \circ \gamma^{-1} = h$. It is obvious that γ stabilizes the chain $\overline{\mathcal{V}}_i(K)$ from the fact that g, h are in the same fiber, so to show $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$, I must only verify the similitude condition:

$$\overline{\phi}_K(\gamma(v), \gamma(w)) = c(h)^{-1}t^{-(m+n)}c(g)t^{m+n}\overline{\phi}_K(v,w) = c(h)^{-1}c(g)\overline{\phi}_K(v,w)$$

i.e. $c(\gamma) = c(h)^{-1}c(g)$.

A nearly identical proof shows that $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ is transitive on fibers of $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K) \to \mathbf{Fl}^{(\mu,\nu)}(K)$. This proves the statement for α_1 and p_1.

The proof for α_2 and p_2 is a formal consequence. If

$$(g_1, g_2), (h_1, h_2) \in \widetilde{\mathbf{M}}^{(m,n)}(K) \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K)$$

are such that

$$p_2(g_1, g_2) = p_2(h_1, h_2)$$

then I need

$$(\gamma, \eta) \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K) \times \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$$

such that

$$g_1 \circ \gamma^{-1} = h_1$$

$$\gamma \circ g_2 \circ \eta^{-1} = h_2$$

Use the previously-proved transitivity of $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ on fibers of

$$\widetilde{\mathbf{M}}^{(m,n)}(K) \to \mathbf{M}^{(m,n)}(K)$$

to get $\gamma \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ such that

$$g_1 \circ \gamma^{-1} = h_1.$$

Then form $\gamma^{-1} \circ h_2 \in \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K)$ and use the previously-proved transitivity of $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ on fibers of

$$\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K) \to \mathbf{Fl}^{(\mu,\nu)}(K)$$

to get $\eta \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ such that

$$g_2 \circ \eta^{-1} = \gamma^{-1} \circ h_2.$$

$\boxed{\text{Connectedness}}$ I now show that for any separably-closed $\mathbf{Z}_p$-field K, the stabilizer subscheme (a K-scheme) in $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$ of an element of $\widetilde{\mathbf{M}}^{(m,n)}(K)$ is connected. First assume that $\mathrm{char}(K) = p$ and set $N := m+n+\mu+\nu+1$. Fix $g \in \widetilde{\mathbf{M}}^{(m,n)}(K)$ and define

$$\mathcal{T}_g : K\text{-Algebras} \longrightarrow \text{Sets}$$

to be the functor assigning to any (commutative) K-algebra R the set of all $R[[t]]$-linear maps $R[[t]]^d \to R[[t]]^d$ that induce g_R after identifying $\overline{\mathcal{W}}_{\sup}(R) = R[t]^d/t^N R[t]^d$. Any $h \in \mathcal{T}_g(R)$ can be written as a $d \times d$ matrix with entries in $R[[t]]$. The requirement that h induce g_R simply specifies the first N terms of each entry, and there are no requirements whatsoever on the remaining terms. This means that $\mathcal{T}_g$ is affine space over K with countable dimension.

Let R be a *local* (commutative) K-algebra with maximal ideal $\mathfrak{m}$. Let $h \in \mathcal{T}_g(R)$ be arbitrary and consider it as an $R((t))$-linear map $h_{((t))} : R((t))^d \to R((t))^d$. I claim that $h_{((t))}$ is *invertible*. By **ELM2** (page 56), there is some $k < N$ such that $\mathrm{im}(g_R)$ contains the submodule $t^k R[t]^d/t^N R[t]^d$. Since $R[[t]]$ is local with maximal ideal (= Jacobson radical) $\mathfrak{m} + (t)$, and since h induces g_R, Nakayama's lemma implies that $\mathrm{im}(h)$ contains the submodule $t^k R[[t]]^d$. Since t is a unit in $R((t))$, $h_{((t))}$ must be surjective. By "Linear Independence of Minimal Generating Sets", it is *invertible.*

For any $h, k \in \mathcal{T}_g(R)$, consider the $R((t))$-linear map $\gamma \stackrel{\text{def}}{=} h^{-1}_{((t))} \circ k_{((t))}$. The fact that both h and k induce g_R implies that γ stabilizes $R[[t]]^d$. The same is true for the reversed composition $k^{-1}_{((t))} \circ h_{((t))}$ so γ restricts to a $R[[t]]$-linear *automorphism* of $R[[t]]^d$. It therefore induces an $R[t]$-linear automorphism $\overline{\gamma}$ of $\overline{\mathcal{W}}_{\sup}(R)$ and by construction $\overline{\gamma} \in \mathrm{Stab}_g(R)$ (in particular, $c(\overline{\gamma}) = 1$).

Altogether, for any *local* (commutative) K-algebra R, I have the function

$$\mathcal{T}_g(R) \times \mathcal{T}_g(R) \longrightarrow \mathrm{Stab}_g(R)$$

$$(h, k) \longmapsto \overline{\gamma}$$

which is obviously functorial whenever it can be. It is easy to see that each of these functions is surjective: for any $\gamma \in \mathrm{Stab}_g(R)$, let $h, k : R[[t]]^d \to R[[t]]^d$ be arbitrary lifts of $g_R, g_R \circ \gamma \in \widetilde{\mathbf{M}}^{(m,n)}(R)$ (recall from the beginning of this section that $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$ acts on the right of $\widetilde{\mathbf{M}}^{(m,n)}$) so by definition $h, k \in \mathcal{T}_g(R)$ and $(h,k) \mapsto \gamma$.

Now, assume that K is separably-closed. It then suffices to show that $\mathrm{Stab}_g(K)$ is a connected algebraic variety (recall that such a stabilizer is a closed subscheme of $\widetilde{\mathbf{J}}_K^{(m,n\,;\,\mu,\nu)}$, which itself is a locally-closed subscheme of finite-dimensional affine space by the same argument used in Lemma 2.5.1.1 (page 70)). But by the surjection from the previous paragraph, this is true.

This proves connectivity for the 1st coordinate of the action α_1. The proof for the 2nd coordinate is nearly identical.

The proof for the α_2-action is essentially a formal consequence. By definition, $(\gamma, \eta) \in \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K) \times \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}(K)$ fixes $(g,h) \in \widetilde{\mathbf{M}}^{(m,n)}(K) \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(K)$ under the α_2 action if and only if

$$g \circ \gamma^{-1} = g$$
$$\eta \circ h \circ \gamma^{-1} = h$$

so define $\mathcal{T}_g$ and $\mathcal{T}_h$ as before and define

$$\begin{aligned} \mathcal{T}_g(K) \times \mathcal{T}_g(K) \times \mathcal{T}_h(K) \times \mathcal{T}_h(K) &\longrightarrow \alpha_2\text{-}\mathrm{Stab}_{(g,h)}(K) \\ (g_1, g_2; h_1, h_2) &\longmapsto (g_2^{-1} \circ g_1 \;\; ; \;\; h_2 \circ (g_2^{-1} \circ g_1) \circ h_1^{-1}) \end{aligned}$$

(some notation, similar to the proof for α_1, is suppressed here: on the right, "$^{-1}$" means as $K((t))$-linear automorphisms and the resulting maps are really those induced on $\overline{\mathcal{W}}_{\sup}(K)$).

As before, the domain is the set of K-points of affine space of countable dimension and the function is a surjection: for $\gamma, \eta \in \alpha_2\text{-Stab}_{(g,h)}(K)$ take

$$g_1,\ g_2,\ h_1,\ h_2 : K[[t]]^d \longrightarrow K[[t]]^d$$

to be arbitrary lifts of

$$g \circ \gamma,\ g,\ \eta^{-1} \circ h \circ \gamma,\ h$$

If $\mathrm{char}(K) = 0$ then use the Chinese remainder theorem as in the proof of Lemma 3.3.2.1 (page 116) to decompose all relevant rings and modules so that in each factor/summand, the relevant quotient involves either a power of (t) or a power of $(t+p)$, but not both. In each situation, the above connectivity proof applies (the importance of characteristic p in the above proof is only that the quotients are modulo a power of a single prime ideal).

$\boxed{\text{Smoothness}}$ I now show that for any $\mathbf{Z}_p$-field K, the stabilizer subscheme in $\widetilde{\mathbf{J}}^{(m,n;\mu,\nu)}$ of an element of $\widetilde{\mathbf{M}}^{(m,n)}(K)$ is a smooth K-scheme. Fix $g \in \widetilde{\mathbf{M}}^{(m,n)}(K)$. By Proposition 3.2.2.1 (page 95), $\widetilde{\mathbf{M}}^{(m,n)}$ is finite-type, so Stab_g is also finite-type and it suffices to verify the infinitesimal lifting property (formal smoothness) for $\mathrm{Stab}_g \to \mathrm{Spec}(K)$. Let R be a K-algebra and $I \subset R$ a nilpotent ideal. By the proof of Corollary 4.5 in Chapter 1 §4 of [7], I may assume that R is *local*. In that case, there is the commutative square

$$\begin{array}{ccc} \mathcal{T}_g(R) \times \mathcal{T}_g(R) & \to & \mathrm{Stab}_g(R) \\ \downarrow & \circ & \downarrow \\ \mathcal{T}_g(R/I) \times \mathcal{T}_g(R/I) & \to & \mathrm{Stab}_g(R/I) \end{array}$$

From the connectedness proof, both horizontal arrows are surjections. Since $\mathcal{T}_g$

is affine space over K with countable dimension, it obviously has the infinitesimal lifting property, so the left vertical arrow is a surjection. This means that $\mathrm{Stab}_g(R) \to \mathrm{Stab}_g(R/I)$ must be surjective.

This proves smoothness for the 1st coordinate of the α_1 action. The proof for the 2nd coordinate is nearly identical. The proof for the α_2 action is essentially a formal consequence. □

Remark. *Applications:*

- *The transitivity statement for α_1 is used in §3.4.1 (page 130).*
- *The transitivity statement for α_2 is used in the proof of Lemma 21 of [15] (page 131 here).*
- *The connectivity statement for α_1 is not used.*
- *The connectivity statement for α_2 is used in the proof of Lemma 21 of [15] (page 131 here).*
- *The smoothness statement for α_1 is not used.*
- *The smoothness statement for α_2 is used in the proof of Lemma 21 of [15] (page 131 here).*

All of this section's results involving p_2 and m are also true of ${}^{\mathrm{rev}}p_2$ and ${}^{\mathrm{rev}}m$:

- ${}^{\mathrm{rev}}p_2$ is smooth
- ${}^{\mathrm{rev}}p_2(K)$ is surjective for any $\mathbf{Z}_p$-field K

- the action via α_2 stabilizes fibers of ${}^{\mathrm{rev}}p_2$, and is transitive on fibers over K-points for any $\mathbf{Z}_p$-field K

- ${}^{\mathrm{rev}}m$ is proper

3.4 The convolution product of sheaf complexes

3.4.1 Construction of the convolution product

I use the following general Proposition 4.2.5 on page 109 of [2]

Beilinson-Bernstein-Deligne Proposition 4.2.5. *If $f : X \to Y$ is a smooth morphism of schemes with relative dimension n and the fibers of f over geometric points are connected, then the shifted pullback $f^*[n]$ is a fully-faithful functor from perverse sheaves on Y to perverse sheaves on X.*

Fix a $\mathbf{Z}_p$-field K and pairs $m, n \in \mathbf{N}$ and $\mu, \nu \in \mathbf{N}$.

Let $\mathcal{A}$ be a perverse $\mathbf{J}_K^{(m,n)}$-equivariant ℓ-adic sheaf on $\mathbf{M}_K^{(m,n)}$ and $\mathcal{B}$ a perverse $\mathbf{Iw}_K^{(\mu,\nu)}$-equivariant ℓ-adic sheaf on $\mathbf{Fl}_K^{(\mu,\nu)}$. Because of the morphisms from §3.3.3 (page 119), I can unify these equivariance properties by saying that both are $\widetilde{\mathbf{J}}_K^{(m,n\,;\,\mu,\nu)}$-equivariant.

The external tensor product $\mathcal{A} \boxtimes_K \mathcal{B}$ (ordinary derived tensor product of the pullbacks along both projections) is another perverse $\widetilde{\mathbf{J}}_K^{(m,n\,;\,\mu,\nu)}$-equivariant ℓ-adic sheaf on $\mathbf{M}_K^{(m,n)} \times \mathbf{Fl}_K^{(\mu,\nu)}$. By Proposition 3.3.0.2 (page 107), p_1 is smooth.

The action by $\widetilde{\mathbf{J}}_K^{(m,n\,;\,\mu,\nu)}$ comes from the loop group $R \mapsto \mathcal{P}(R[[t]])$ of a parahoric $\mathcal{P}$ (depending on $\mathrm{char}(K)$), which is *connected*, so by the transitivity statement of Proposition 3.3.5.1 (page 123) the geometric fibers of p_1 are connected. By Proposition 4.2.5 of [2], the pullback $p_1^*(\mathcal{A} \boxtimes_K \mathcal{B})$ is a perverse ℓ-adic sheaf.

Since the action of $\widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)} \times \widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}$ by α_1 stabilizes p_1-fibers, $p_1^*(\mathcal{A} \boxtimes_K \mathcal{B})$ is *trivially* α_1-equivariant: $p_1 \circ \alpha_1 = p_1 \circ \mathrm{pr}$ already. Since the difference between α_1 and α_2 is the action

$$\widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)} \times (\widetilde{\widetilde{\mathbf{M}}}^{(m,n)} \times \widetilde{\widetilde{\mathbf{Fl}}}^{(\mu,\nu)}) \longrightarrow (\widetilde{\widetilde{\mathbf{M}}}^{(m,n)} \times \widetilde{\widetilde{\mathbf{Fl}}}^{(\mu,\nu)})$$

$$(\gamma, (g, h)) \longmapsto (g, \gamma \circ h)$$

the initial assumption that $\mathcal{A}$ and $\mathcal{B}$ were $\widetilde{\mathbf{J}}^{(m,n\,;\mu,\nu)}$-equivariant implies that $p_1^*(\mathcal{A} \boxtimes_K \mathcal{B})$ is α_2-equivariant.

I use the following general Lemma 21 from [15]:

Haines-Ngô Lemma 21. *Let $\pi : X \to Y$ be a morphism of finite-type $\mathbf{Z}_p$-schemes. Let G be a $\mathbf{Z}_p$-group scheme. Let $a_X : G \times X \to X$ a group action over* $\mathrm{Spec}(\mathbf{Z}_p)$. *Let G act trivially on Y. Let $\mathcal{F}$ be a perverse a_X-equivariant étale ℓ-adic sheaf on X.*

Assume *that π is smooth and surjective on the level of topological spaces, that G is smooth, that for any $\mathbf{Z}_p$-field K the action of G on X is transitive on fibers of π over K-points, G_K is connected, the stabilizer subscheme of a K-point of X is a smooth subgroup of G_K, and that if K is separably-closed, then those stabilizer subschemes are also connected.* ***Conclusion****: There is a unique perverse ℓ-adic sheaf $\mathcal{G}$ on Y such that $\mathcal{F} \cong \pi^*(\mathcal{G})$.*

A supplement to the original proof. The morphism "$a : G_Y \to X$" occurring in [15] is the composition

$$G \times Y \xrightarrow{\mathrm{id} \times s} G \times X \xrightarrow{a_X} X$$

where "s" in [15] is assumed temporarily to be a section of π.

Let

$$\mathrm{pr}_X : G \times X \to X$$

be the projection. To get the isomorphism

$$\text{“}a^*\pi^*s^*(\mathcal{F}) \cong a^*(\mathcal{F})\text{”}$$

in [15], take the assumed equivariance isomorphism $a_X^*(\mathcal{F}) \cong \mathrm{pr}_X^*(\mathcal{F})$, apply the functor $(\mathrm{id} \times s)^*$, and note that

$$s \circ \pi \circ a_X \circ (\mathrm{id} \times s) = \mathrm{pr}_X \circ (\mathrm{id} \times s)$$

since G stabilizes π-fibers:

$$\begin{aligned} a^*\mathcal{F} &= (a_X \circ (\mathrm{id} \times s))^*\mathcal{F} \\ &\cong (\mathrm{pr}_X \circ (\mathrm{id} \times s))^*\mathcal{F} \\ &= (s \circ \pi \circ a_X \circ (\mathrm{id} \times s))^*\mathcal{F} \\ &= a^*\pi^*s^*\mathcal{F} \end{aligned}$$

The proof relies on the fact that both a and $\pi \circ a$ satisfy the 2 requirements of Proposition 4.2.5 in [2]. It is easy to see that the fiber of $\pi \circ a$ over any R-point $y \in Y$ is simply $G_R \times \{y\}$, which is connected by hypothesis when R is a separably-closed $\mathbf{Z}_p$-field. It is similarly easy to see that the fiber of a over any R-point $x \in X$ is the coset $\mathrm{Stab}_G(x)g$ for any $g \in G$ satisfying $g \cdot s(\pi(x)) = x$. This is again connected by hypothesis when R is a separably-closed $\mathbf{Z}_p$-field. Since π is assumed smooth, it only remains to verify that a is smooth. By Proposition 8 in §2.4 of [3], it suffices to check that a is a flat morphism and that its fibers over K-points are smooth for all $\mathbf{Z}_p$-fields K. As before, the fiber of a over a point x is the coset $\mathrm{Stab}_G(x)g$ for any g satisfying $g \cdot s(\pi(x)) = x$, and by assumption this is smooth whenever R is a $\mathbf{Z}_p$-field.

Now I verify that a is flat. Since $\pi \circ a$ is flat (it is just the projection $\mathrm{pr}_2 : G \times Y \to Y$), Corollary 14.25 of [12] implies that it is sufficient to verify that all the morphisms a_y induced by a between fibers over $y \in Y$ (via $\pi \circ a = \mathrm{pr}_2$ and π, respectively) are flat. In more detail, if K is an arbitrary $\mathbf{Z}_p$-field and $y : \mathrm{Spec}(K) \to Y$ is an arbitrary K-point, I must show that the morphism

$$a_y : G_y \overset{\text{def}}{=} (G \times Y) \times_Y \mathrm{Spec}(K) \longrightarrow X \times_Y \mathrm{Spec}(K) \overset{\text{def}}{=} X_y$$

over $\mathrm{Spec}(K)$ induced by the fiber product is flat. Since G_y is a group and flatness of a morphism is local with respect to the domain, Corollary 10.85 of [12] ("sufficiently finite morphisms to integral schemes are flat almost everywhere") implies that I must only show that X_y is an *integral* K-scheme, i.e. that it is reduced and irreducible. That the π-fiber X_y is reduced is trivial since π is smooth by assumption. By the transitivity hypothesis, a_y is surjective on the level of topological spaces, so it suffices to show that G_y is irreducible. But this is obvious since G_y is simply $G_K \times \{y\}$, which is smooth and connected by hypothesis.

The sheaf $\mathcal{G}$ is, in the *global* section case, $s^*(\mathcal{F})$. The general proof is accomplished by using the topological surjectivity hypothesis to cover Y by étale local sections (see Proposition 14 on page 43 of [3] for existence of these), using the previous global result, and patching the complexes together. $\square$

To apply this to my case, use

$$X = \widetilde{\mathbf{M}}^{(m,n)} \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}$$

$$Y = \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$$

$$G = \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)} \times \widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}$$

$$\pi = p_2$$

$$a_X = \alpha_2$$

$$\mathcal{F} = p_1^*(\mathcal{A} \boxtimes_K \mathcal{B})$$

The assumption on π is provided by Proposition 3.3.0.2. The assumptions on G are provided by Proposition 3.3.4.1 (page 122) and the discussion in §3.4.1. The two assumptions on a_X are provided by Proposition 3.3.5.1 (page 123). The assumption on $\mathcal{F}$ was verified in the discussion at the beginning of this subsection.

In my case, denote the object $\mathcal{G}$ from the lemma by

$$\mathcal{A} \odot_K \mathcal{B}$$

Note that Proposition 3.3.0.2 (page 107) and Lemma 3.3.2.1 (page 116) together imply that p_1 and p_2 have the same relative dimension over each component of $\mathbf{Conv}^{(m,n\,;\,\mu,\nu)}$ and $\mathbf{M}^{(m,n)} \times \mathbf{Fl}^{(\mu,\nu)}$: smoothness of p_1 and p_2 imply *constant* relative dimension, but at the same time $\mathbf{Conv}_{\mathbf{Q}_p}^{(m,n\,;\,\mu,\nu)} \cong \mathbf{M}_{\mathbf{Q}_p}^{(m,n)} \times \mathbf{Fl}_{\mathbf{Q}_p}^{(\mu,\nu)}$. This means that $\mathcal{A} \odot_K \mathcal{B}$ is already perverse (i.e. no shift is needed).

At long last, define the convolution product $*_K$ by

$$\mathcal{A} *_K \mathcal{B} \stackrel{\text{def}}{=} Rm_!(\mathcal{A} \odot_K \mathcal{B})$$

Note that $\mathcal{A} *_K \mathcal{B}$ is a complex of ℓ-adic sheaves on $\mathbf{P}_K^{(m,n\,;\,\mu,\nu)}$. Note also that $m_* = m_!$ since m is a proper morphism.

Repeating the above discussion using the *reversed* convolution diagram from §3.2.5 (page 104) produces the product $\mathcal{B} \odot_K \mathcal{A}$ on ${}^{\text{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)}$, and the "reversed" convolution product $\mathcal{B} *_K \mathcal{A}$ on $\mathbf{P}_K^{(m,n\,;\,\mu,\nu)}$ is defined by

$$\mathcal{B} *_K \mathcal{A} \stackrel{\text{def}}{=} R({}^{\text{rev}}m_!)(\mathcal{B} \odot_K \mathcal{A})$$

There is no ambiguity between the original and reversed convolution products because the complexes $\mathcal{A}$ and $\mathcal{B}$ have different bases.

3.4.2 The convolution product of sheaf complexes categorifies the convolution product of functions

It is natural to ask exactly how the convolution product of sheaf complexes is related to the convolution product of functions in the Hecke algebra. This is apparently well-known, but since I have not seen it in print, I explain it.

Let $m, n, \mu, \nu \in \mathbf{N}$ be arbitrary. Let $\mathcal{A}$ and $\mathcal{B}$ be (bounded, constructible) complexes of ℓ-adic sheaves on $\mathbf{M}_{\overline{\mathbf{F}}_p}^{(m,n)}$ and $\mathbf{Fl}_{\overline{\mathbf{F}}_p}^{(\mu,\nu)}$ equipped with actions of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ that are consistent with the action of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on $\mathbf{M}_{\overline{\mathbf{F}}_p}^{(m,n)}$ and $\mathbf{Fl}_{\overline{\mathbf{F}}_p}^{(\mu,\nu)}$. Then $\mathcal{A} *_{\overline{\mathbf{F}}_p} \mathcal{B}$ is a (bounded, constructible) complex of ℓ-adic sheaves on $\mathbf{P}_{\overline{\mathbf{F}}_p}^{(m,n\,;\,\mu,\nu)}$ with all the same properties.

By §3.3.1 (page 115),

$$\mathbf{Fl}^{(\mu,\nu)}_{\mathbf{F}_p} = \mathbf{M}^{(\mu,\nu)}_{\mathbf{F}_p}$$

$$\mathbf{P}^{(m,n\,;\,\mu,\nu)}_{\mathbf{F}_p} = \mathbf{M}^{(m+\mu,n+\nu)}_{\mathbf{F}_p}$$

so the associated trace functions under consideration are:

$$\tau^{\mathrm{ss}}_{\mathcal{A}} : \mathbf{Fl}^{(m,n)}(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

$$\tau^{\mathrm{ss}}_{\mathcal{B}} : \mathbf{Fl}^{(\mu,\nu)}(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

$$\tau^{\mathrm{ss}}_{\mathcal{A}*\mathcal{B}} : \mathbf{Fl}^{(m+\mu,n+\nu)}(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

Let $x \in \mathbf{Fl}^{(m+\mu,n+\nu)}(\mathbf{F}_p)$ be arbitrary and set

$$\mathfrak{f} \stackrel{\text{def}}{=} \{y \in \mathbf{Fl}^{(m,n)}(\mathbf{F}_p) \mid (y,x) \in \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(\mathbf{F}_p)\}$$

($\mathfrak{f}$ is essentially the fiber $m(\mathbf{F}_p)^{-1}(x) \subset \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}(\mathbf{F}_p)$)

Because of the semisimplification done in §3.1.2 (page 86), the Γ_0-invariants operation is exact, and the following general Proposition 10 from [15] results:

Haines-Ngô Proposition 10. *Let $f : X \to Y$ be a morphism of $\mathbf{F}_p$-schemes and let $\mathcal{C}$ be a complex of ℓ-adic sheaves on X with an action $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ compatible with that on $X(\mathbf{F}_p)$. Then for any $y \in Y(\mathbf{F}_p)$,*

$$\tau^{\mathrm{ss}}_{f_!(\mathcal{C})}(y) = \sum_{\substack{x \in X(\mathbf{F}_p) \\ f(x)=y}} \tau^{\mathrm{ss}}_{\mathcal{C}}(x)$$

This implies that

$$\tau^{\mathrm{ss}}_{\mathcal{A}*\mathcal{B}}(x) = \sum_{y \in \mathfrak{f}} \tau^{\mathrm{ss}}_{\mathcal{A}\odot\mathcal{B}}(y,x)$$

For any $y \in \mathfrak{f}$, if $(h,k) \in \widetilde{\mathbf{Fl}}^{(m,n)}(\mathbf{F}_p) \times \widetilde{\mathbf{Fl}}^{(\mu,\nu)}(\mathbf{F}_p)$ is such that $p_2(h,k) = (y,x)$ (such elements exist by Proposition 3.3.0.2 (page 107))) then setting $(y,z) := p_1(h,k)$ (recall that the first coordinates p_1 and p_2 are the same), it is true that

$$\tau^{\mathrm{ss}}_{\mathcal{A}\odot\mathcal{B}}(y,x) = \tau^{\mathrm{ss}}_{\mathcal{A}}(y) \cdot \tau^{\mathrm{ss}}_{\mathcal{B}}(z)$$

(this follows from general sheaf theory: the way the operations $\boxtimes$ (external tensor product), p_1^* and p_2^* interact with stalks)

Let $\mathfrak{e} \subset \mathbf{Fl}^{(\mu,\nu)}(\mathbf{F}_p)$ be the set of all z occurring in this way. Then the above sum can be rewritten

$$\tau^{\mathrm{ss}}_{\mathcal{A}*\mathcal{B}}(x) = \sum_{\substack{y\in\mathfrak{f}\\ z\in\mathfrak{e}}} \tau^{\mathrm{ss}}_{\mathcal{A}}(y) \cdot \tau^{\mathrm{ss}}_{\mathcal{B}}(z)$$

To see this as a convolution, inflate $\tau^{\mathrm{ss}}_{\mathcal{A}}$ and $\tau^{\mathrm{ss}}_{\mathcal{B}}$ to $\widetilde{\mathbf{Fl}}^{(m,n)}(\mathbf{F}_p)$ and $\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(\mathbf{F}_p)$ and recall the "twisting" that occurs in the 2nd coordinate of p_2. Then for any $x \in \mathbf{Fl}^{(m+\mu,n+\nu)}(\mathbf{F}_p)$,

$$\tau^{\mathrm{ss}}_{\mathcal{A}*\mathcal{B}}(x) = \sum_{\substack{h\in\widetilde{\mathbf{Fl}}^{(m,n)}(\mathbf{F}_p)\\ k\in\widetilde{\mathbf{Fl}}^{(\mu,\nu)}(\mathbf{F}_p)\\ h(k(-))=x}} \tau^{\mathrm{ss}}_{\mathcal{A}}(h) \cdot \tau^{\mathrm{ss}}_{\mathcal{B}}(k)$$

(so h plays the role of "y" and k plays the role of "$y^{-1}x$" in the expression "$(f * g)(x) = \sum_y f(y)g(y^{-1}x)$")

Chapter 4

Proof of the main theorem

Let $w \in \widetilde{W}$ be arbitrary. There exists $\mu, \nu \in \mathbf{N}$ (infinitely many, all with the same difference $\Delta = \mu - \nu$) such that the Schubert variety $\overline{C}_w$ is contained in $\mathbf{Fl}^{(\mu,\nu)}_{\overline{\mathbf{F}}_p}$. Let $\overline{\mathrm{IC}}_w$ be the (perverse) étale ℓ-adic intersection complex associated to the cell C_w in the Bruhat-Tits decomposition of $\mathbf{Fl}^{(\mu,\nu)}_{\overline{\mathbf{F}}_p}$. The function

$$\tau^{\mathrm{ss}}_{\overline{\mathrm{IC}}_w} : \mathbf{Fl}^{(m,n)}(\mathbf{F}_p) \longrightarrow \overline{\mathbf{Q}}_\ell$$

is also an element of the Iwahori-Hecke algebra $\mathcal{H}$. By the main theorems of [18] and [19], the set of these functions $\tau^{\mathrm{ss}}_{\overline{\mathrm{IC}}_w}$ for all $w \in \widetilde{W}$ forms a vector-space basis for $\mathcal{H}$.

Therefore, to show that $\tau^{\mathrm{ss}}_\mu \in Z(\mathcal{H})$, it suffices to show that

$$\tau^{\mathrm{ss}}_\mu * \tau^{\mathrm{ss}}_{\overline{\mathrm{IC}}_w} \stackrel{?}{=} \tau^{\mathrm{ss}}_{\overline{\mathrm{IC}}_w} * \tau^{\mathrm{ss}}_\mu$$

(ordinary convolution of functions) for every $w \in \widetilde{W}$.

Remark. *Notice that none of the $\mathbf{Z}_p$-schemes $\mathbf{M}^{(m,n)}$ are genuinely needed in the proof except the support $\mathbf{M}^{(0,1)} = \mathbf{M}^{\mathrm{loc}}$ of τ^{ss}_μ (all $\mathbf{Fl}^{(\mu,\nu)}$ are needed, however). But it is not much harder to prove things for a general $\mathbf{M}^{(m,n)}$ than for $\mathbf{M}^{(0,1)}$, and it is interesting in any case to have such a degeneration in the unramified unitary case. For similar applications to trace functions $\tau^{\mathrm{ss}}_\lambda$ for* non-minuscule *λ, as is the case in [15], the larger schemes $\mathbf{M}^{(m,n)}$ really are necessary.*

Recall from §3.2.1 (page 90) that if IC_w is the (perverse) étale ℓ-adic intersection complex associated to the cell C_w in the Bruhat-Tits decomposition of $\mathcal{F}\ell^{\mathrm{aff}}_{\mathbf{Q}_p}$ (note the base field here), then $\overline{\mathrm{IC}}_w \xrightarrow{\sim} \mathrm{R}\Psi(\mathrm{IC}_w)$. Recall from §3.1.2 (page 86) that *by definition* if IC_μ is the (perverse) étale ℓ-adic intersection complex associated to the cell O_μ in the *Cartan* decomposition of $\mathcal{G}r^{\mathrm{aff}}_{\mathbf{Q}_p}$ then $\tau^{\mathrm{ss}}_\mu = \tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathrm{IC}_\mu)}$. Using these two identities, it suffices to prove that

$$\tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathrm{IC}_\mu)} * \tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathrm{IC}_w)} \stackrel{?}{=} \tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathrm{IC}_w)} * \tau^{\mathrm{ss}}_{\mathrm{R}\Psi(\mathrm{IC}_\mu)}$$

By §3.4.2 (page 135), the convolution product of sheaves induces the convolution product of functions, so it suffices to show that

$$\mathrm{R}\Psi(\mathrm{IC}_\mu) *_{\overline{\mathbf{F}}_p} \mathrm{R}\Psi(\mathrm{IC}_w) \stackrel{?}{=} \mathrm{R}\Psi(\mathrm{IC}_w) *_{\overline{\mathbf{F}}_p} \mathrm{R}\Psi(\mathrm{IC}_\mu)$$

Remark. *Note that the reversed convolution product occurs on the right-hand-side here.*

By the general Lemma 23 of [15] ("nearby cycles commutes with convolution product"), this equality is equivalent to the equality

$$\mathrm{R}\Psi(\mathrm{IC}_\mu *_{\mathbf{Q}_p} \mathrm{IC}_w) \stackrel{?}{=} \mathrm{R}\Psi(\mathrm{IC}_w *_{\mathbf{Q}_p} \mathrm{IC}_\mu) \tag{4.1}$$

Remark. *Lemma 23 of [15] applies because the fields involved here are algebraically-closed: by an argument similar to that given in the proof of Lemma 3.1.4.1 (page 89), the schemes used here simplify to the* GL *case after passing to the algebraic closure.*

Remark. *Lemma 23 in [15] uses "smooth base-change" for p_1 and p_2, and therefore requires the appropriate analogue of Proposition 3.3.0.2 (page 107).*

The following lemma implies that this last isomorphism (4.1) is true.

Lemma 4.0.2.1. *Recall the isomorphisms from Lemma 3.3.2.1 (page 116):*

$$\begin{array}{ccc} \mathbf{M}^{(m,n)}_{\mathbf{Q}_p} \times \mathbf{Fl}^{(\mu,\nu)}_{\mathbf{Q}_p} & \xleftarrow{i} & \mathbf{Conv}^{(m,n\,;\,\mu,\nu)}_{\mathbf{Q}_p} \\ {}^{\mathrm{rev}}i \uparrow & \circ & \downarrow m \\ {}^{\mathrm{rev}}\mathbf{Conv}^{(\mu,\nu\,;\,m,n)}_{\mathbf{Q}_p} & \xrightarrow{{}^{\mathrm{rev}}m} & \mathbf{P}^{(m,n\,;\,\mu,\nu)}_{\mathbf{Q}_p} \end{array}$$

***Assertion**: if $\mathcal{A}$ and $\mathcal{B}$ are complexes of $\widetilde{\mathbf{J}}^{(m,n\,;\,\mu,\nu)}_{\mathbf{Q}_p}$-equivariant ℓ-adic sheaves on $\mathbf{M}^{(m,n)}_{\mathbf{Q}_p}$ and $\mathbf{Fl}^{(\mu,\nu)}_{\mathbf{Q}_p}$ respectively, then*

$$i^*(\mathcal{A} \boxtimes_{\mathbf{Q}_p} \mathcal{B}) \xrightarrow{\sim} \mathcal{A} \odot_{\mathbf{Q}_p} \mathcal{B}$$

$${}^{\mathrm{rev}}i^*(\mathcal{A} \boxtimes_{\mathbf{Q}_p} \mathcal{B}) \xrightarrow{\sim} \mathcal{B} \odot_{\mathbf{Q}_p} \mathcal{A}$$

Applying $Rm_!$ and $R({}^{\mathrm{rev}}m_!)$ to these isomorphisms and using commutativity of the above square implies that

$$\mathcal{A} *_{\mathbf{Q}_p} \mathcal{B} \cong \mathcal{B} *_{\mathbf{Q}_p} \mathcal{A}.$$

Proof. The proof is nearly identical to the one occurring for the 2nd part of Lemma 24 in [15], replacing the objects and morphisms used there by the slightly modified objects and morphisms used in this paper for Lemma 3.3.2.1 (page 116). □

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