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Nonholonomic Variable Geometry Truss Assemblies I: Motion Control

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Abstract

We consider the nonholonomic motion planning problem for a novel class of snake-like modular mobile manipulators, where each module is implemented as a planar parallel manipulator with *idler* wheels. This assembly is actuated by shape changes of its modules, which, under the influence of the nonholonomic constraints on the wheels, induce a global motion of the assembly.

We formulate the kinematics for a generic assembly of this type and specialize to the 2–module case in order to study the motion planning problem in greater detail.

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1 Introduction

In this paper we consider a class of Variable Geometry Truss (VGT) assemblies (Miura, Furuya & Suzuki [1985]; Wada [1990]), which are structures consisting of longitudinal repetition of truss modules. In the present instance, each module is implemented as a planar parallel manipulator consisting of two platforms connected by legs whose lengths can vary under the control of linear actuators. Each platform is equipped with a pair of wheels, so that it can move on the plane that supports the structure (fig. 1.1). The wheels of each platform are free and not actuated and their motion is independent of each other, while we assume that the wheels roll without slipping on the plane. This imposes a nonholonomic constraint on the motion of each platform, namely the requirement that its velocity is perpendicular to the axis connecting the wheels. When the legs of the individual modules are expanded or contracted, the shape of the whole VGT assembly changes. As a consequence of the nonholonomic constraints imposed by the rolling–without–slipping assumption on the wheels, this shape change induces a global motion of the VGT assembly.

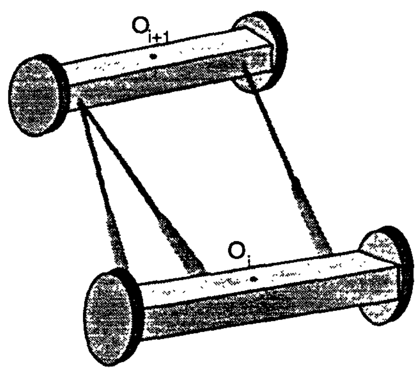


Fig. 1.1

The motion planning problem for such an assembly is of the nonholonomic variety. There is a significant body of research related to such problems (see e.g. (Latombe [1991]; Li & Canny [1993])), which in general assumes cart–type mobile robots moving under direct actuation of a set of wheels. The main difference in our case is the prominence of shape changes as the means which, together with the action of the nonholonomic constraints, induces global motion of the system. This is analogous to the idea of reorientation in free–floating multibody systems, induced by closed joint space trajectories under the nonholonomic constraint of conservation of angular momentum

(Krishnaprasad [1990]; Krishnaprasad & Yang [1991]; Marsden, Montgomery & Ratiu [1990]).

VGT assemblies of the type discussed here have been examined in the past (see (Chirikjian & Burdick [1991]; Wada [1990]) and references there), but the emphasis was on its capabilities as a redundant manipulator and on locomotion using snake–like motions, not on the special problems introduced by nonholonomic constraints. A system similar to the one described here was built by (Chirikjian & Burdick [1993]) *using castors instead of wheels* in the platforms of the modules and therefore the nonholonomic constraints that we consider here were not present. A visit to Burdick's lab in Caltech in 1992 was a source of inspiration for the present work.

In section 2, we examine the kinematics of a VGT assembly with ℓ modules. Consider the i–th module (fig. 1.1). Its *shape* can be described by the relative position and orientation of the coordinate frame centered at the point O_{i+1} with respect to the coordinate frame centered at the point O_i. Then, the shape of each module corresponds to an element of the Special Euclidean group $SE(2)$ that describes rigid motions on the plane and, as a result, the shape of the ℓ–module VGT can be described by ℓ elements of $SE(2)$. The *configuration* of the VGT assembly can be described by its shape and by the position and orientation of the assembly with respect to some fixed (world) coordinate system, thus by a total of $\ell+1$ elements of $SE(2)$. In (Brockett, Stokes & Park [1993]) a systematic way of deriving the kinematics of serial linkages is presented based on the "product of exponentials" formula, where the configuration of the system is described by an element of the appropriate $SE(n)$ group and is expressed as a product of its one–parameter subgroups, with one element of the product corresponding to each of the one–degree–of–freedom joints of the linkage. The VGT assembly that we consider here is a structure similar to the ones described there, but the joints are more complicated parallel manipulator modules with more than one degree–of–freedom each. Moreover, the whole assembly is not anchored to a base, but is free to move on a plane and, finally, nonholonomic constraints are present, in addition to the holonomic ones. However, an extension of the above method, allows us to systematically derive the kinematics of the VGT assembly as follows: Using the Wei–Norman representation of $SE(2)$, we express, in section 2.1, the shape of each module as a product of the one–parameter subgroups of $SE(2)$. Then, the configuration of the whole assembly can be expressed as a product of such one–parameter subgroups. Using the notion of the adjoint action of $SE(2)$ on its

Lie algebra, we determine, in section 2.2, how the motion of a module relates to the motion of the other modules of the assembly. We also express the nonholonomic constraints in a compact form that can be used to make explicit the dependence of the assembly configuration on the shape of its modules. This allows us to characterize the dependence of the global motion of the assembly on the *shape controls*, namely the changes in the shape of each module, which are expressed as elements of the Lie algebra of $SE(2)$. In section 2.3, we consider the implementation of each module as a planar parallel manipulator. The shape of each module is determined by the lengths of the legs of the parallel manipulator. From the velocity kinematics of the parallel manipulator we conclude that motion planning schemes for the VGT assembly can disregard the particular details of the implementation of the modules and only consider the shape of each module. Thus, instead of considering the changes in leg lengths as controls for the VGT assembly, we can use the corresponding shape controls of each module.

In section 3, we specialize the previous discussion to the 2–module VGT. Unlike the generic ℓ–module case, here we have exactly the number of nonholonomic constraints that we need in order to determine the position and orientation of the VGT assembly with respect to the world coordinate frame, based on a sequence of shape changes from a reference shape. As a result, we can demonstrate how shape changes of the VGT assembly induce a global snake–like motion due to the nonholonomic constraints. We consider the motion planning problem under a specific shape actuation scheme, where one of the two modules is responsible for the motion of the assembly through periodic changes of its shape and the other module is responsible for steering. We demonstrate how to generate primitive "straight line motion" and "turning" behaviors and we show by computer simulations how to synthesize these into more complex ones, like avoidance of obstacles.

In section 4, we discuss possible extensions of this work.

2 Kinematic Chains on Lie Groups

In section 2.1 we discuss the Wei–Norman representation of curves in $SE(2)$ and in section 2.2 we apply this representation to the derivation of the kinematics of the ℓ–module VGT. It is found convenient to employ the language of matrix Lie groups throughout. In section 2.3 we consider the implementation of a module of the VGT assembly as a planar parallel manipulator.

2.1 The Wei–Norman Representation of SE(2)

The instantaneous shape of a module of the VGT assembly or the position and orientation of the whole assembly with respect to the world coordinate system corresponds, as was discussed in section 1, to an element χ of the matrix Lie group $G = SE(2)$. Given a curve $\chi(.) \subset G = SE(2)$, there is a curve $V(.) \subset \mathcal{G} = se(2)$, the Lie algebra of $SE(2)$, such that:

$$\dot{\chi} = \chi V \ . \tag{1}$$

Let $\{\mathcal{A}_i,\ i = 1,2,3\}$ be the following basis of $\mathcal{G}$:

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2}$$

with $[\,,]$ being the usual Lie bracket on G. Then:

$$[\mathcal{A}_1 \ , \ \mathcal{A}_2] = \mathcal{A}_3 \ , \ [\mathcal{A}_1 \ , \ \mathcal{A}_3] = -\mathcal{A}_2 \ , \ [\mathcal{A}_2 \ , \ \mathcal{A}_3] = 0 \ . \tag{3}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}_i^\flat,\ i = 1,2,3\}$ be a basis of $\mathcal{G}^*$ such that $\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_i^j$, for $i,j = 1,2,3$, where δ_i^j is the Kronecker symbol. Then the curve $V(.) \subset \mathcal{G}$ can be represented as:

$$V = \sum_{i=1}^{3} v_i \mathcal{A}_i = \sum_{i=1}^{3} \mathcal{A}_i^\flat(V) \mathcal{A}_i \ . \tag{4}$$

Here v_i is a scalar function of t, the curve parametrization.

For $x, y \in \mathcal{G}$, define $(adx)y \stackrel{\text{def}}{=} [x,y]$, $(adx)^k y = [x, (adx)^{k-1}y]$. Then, from the Baker–Campbell–Hausdorff formula (Wei & Norman [1964]) we have:

$$e^x y e^{-x} = (e^{adx})y = y + [x,y] + \frac{1}{2!}[x,[x,y]] + \dots \ . \tag{5}$$

Proposition 2.1.1

Let $\chi(0) = I$, the identity of G. There exists a *global* representation of the curve $\chi(.) \subset G$ of the form:

$$\chi(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} e^{\gamma_3(t)\mathcal{A}_3} \ . \tag{6}$$

The coefficients $\gamma_i \in I\!R$ are related to the coefficients v_i in (4) by:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{7}$$

and, equivalently, by:

$$\begin{aligned} \gamma_1(t) &= \gamma_1(0) + \int_0^t v_1(\tau)d\tau \,, \\ \gamma_2(t) &= \gamma_2(0)C(t,0) + \gamma_3(0)S(t,0) + \int_0^t v_2(\tau)C(t,\tau)d\tau + \int_0^t v_3(\tau)S(t,\tau)d\tau \,, \\ \gamma_3(t) &= -\gamma_2(0)S(t,0) + \gamma_3(0)C(t,0) - \int_0^t v_2(\tau)S(t,\tau)d\tau + \int_0^t v_3(\tau)C(t,\tau)d\tau \,, \end{aligned} \tag{8}$$

where $C(t,\tau) \stackrel{\text{def}}{=} \cos\left(\int_\tau^t v_1(\sigma)d\sigma\right)$ and $S(t,\tau) \stackrel{\text{def}}{=} \sin\left(\int_\tau^t v_1(\sigma)d\sigma\right)$. For the initial condition $\chi(0) = I$, we have $\gamma_i(0) = 0, \; i = 1,2,3$.

Proof

Since $\mathcal{G} = se(2)$ is solvable, the existence of a global representation of the form (6) is immediate by (Wei & Norman [1964]). Differentiating (6) we have:

$$\begin{aligned} \frac{d\chi}{dt} &= \dot{\gamma}_1 e^{\gamma_1 \mathcal{A}_1} \mathcal{A}_1 e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} + \dot{\gamma}_2 e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} \mathcal{A}_2 e^{\gamma_3 \mathcal{A}_3} + \dot{\gamma}_3 e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} \mathcal{A}_3 \\ &= e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} [\dot{\gamma}_1 e^{-\gamma_3 \mathcal{A}_3} e^{-\gamma_2 \mathcal{A}_2} \mathcal{A}_1 e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} + \dot{\gamma}_2 e^{-\gamma_3 \mathcal{A}_3} \mathcal{A}_2 e^{\gamma_3 \mathcal{A}_3} + \dot{\gamma}_3 \mathcal{A}_3] \\ &= \chi \, [\dot{\gamma}_1 e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 + \dot{\gamma}_2 e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 + \dot{\gamma}_3 \mathcal{A}_3] \end{aligned} \tag{9}$$

We now compute the RHS of (9). From (3) and (5) :

$$\begin{aligned} & e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 = \mathcal{A}_2 \,, \\ & e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 = e^{ad(-\gamma_3 \mathcal{A}_3)} (\mathcal{A}_1 + \gamma_2 \mathcal{A}_3) = \mathcal{A}_1 - \gamma_3 \mathcal{A}_2 + \gamma_2 \mathcal{A}_3 \,. \end{aligned} \tag{10}$$

From (1), (4), (9) and (10) we have:

$$\begin{aligned} V &= v_1 \mathcal{A}_1 + v_2 \mathcal{A}_2 + v_3 \mathcal{A}_3 \\ &= \dot{\gamma}_1 (\mathcal{A}_1 - \gamma_3 \mathcal{A}_2 + \gamma_2 \mathcal{A}_3) + \dot{\gamma}_2 \mathcal{A}_2 + \dot{\gamma}_3 \mathcal{A}_3 = \dot{\gamma}_1 \mathcal{A}_1 + (\dot{\gamma}_2 - \gamma_3 \dot{\gamma}_1) \mathcal{A}_2 + (\dot{\gamma}_3 + \gamma_2 \dot{\gamma}_1) \mathcal{A}_3 \,. \end{aligned}$$

Since $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a basis, we have:

$$v_1 = \dot{\gamma}_1 \ , \ \ v_2 = \dot{\gamma}_2 - \gamma_3\dot{\gamma}_1 \ , \ \ v_3 = \dot{\gamma}_3 + \gamma_2\dot{\gamma}_1 \ .$$

Solving for the $\dot{\gamma}_i$'s we get (7), which can be rewritten as:

$$\dot{\gamma}_1 = v_1 \ ,$$

$$\begin{pmatrix} \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 0 & v_1 \\ -v_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix} + \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \ .$$

This system can be solved by quadratures, giving (8).

∎

For $\chi \in G$, $V \in \mathcal{G}$, define the *adjoint action* of G on $\mathcal{G}$ denoted $Ad_\chi : \mathcal{G} \to \mathcal{G}$ by:

$$Ad_\chi V \stackrel{\text{def}}{=} \chi V \chi^{-1} \ . \tag{11}$$

From (4) we have:

$$Ad_\chi V = \sum_{i=1}^{3} v_i Ad_\chi \mathcal{A}_i = \sum_{i=1}^{3} \mathcal{A}_i^\flat(V) Ad_\chi \mathcal{A}_i \ . \tag{12}$$

Proposition 2.1.2

Consider the Wei–Norman representation (6) of χ determined by (7) and (8). Then:

$$\begin{aligned} Ad_{\chi^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ , \\ Ad_{\chi^{-1}}\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3 \ , \\ Ad_{\chi^{-1}}\mathcal{A}_3 &= \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ . \end{aligned} \tag{13}$$

Proof

From (11) we have for $i = 1, 2, 3$:

$$Ad_{\chi^{-1}}\mathcal{A}_i = e^{ad(-\gamma_3\mathcal{A}_3)} e^{ad(-\gamma_2\mathcal{A}_2)} e^{ad(-\gamma_1\mathcal{A}_1)} \mathcal{A}_i \ .$$

From this and (5) we have for $\mathcal{A}_1$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1$$

$$e^{ad(-\gamma_2\mathcal{A}_2)} e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = e^{ad(-\gamma_2\mathcal{A}_2)}\mathcal{A}_1 = \mathcal{A}_1 + \gamma_2\mathcal{A}_3$$

$$e^{ad(-\gamma_3\mathcal{A}_3)} e^{ad(-\gamma_2\mathcal{A}_2)} e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = e^{ad(-\gamma_3\mathcal{A}_3)}(\mathcal{A}_1 + \gamma_2\mathcal{A}_3) = \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ .$$

For $\mathcal{A}_2$:

$$
\begin{aligned}
&e^{ad(-\gamma_1 \mathcal{A}_1)}\mathcal{A}_2 = \cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3 \\
&e^{ad(-\gamma_2 \mathcal{A}_2)}e^{ad(-\gamma_1 \mathcal{A}_2)}\mathcal{A}_2 = e^{ad(-\gamma_2 \mathcal{A}_2)}(\cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3) = \cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3 \\
&e^{ad(-\gamma_3 \mathcal{A}_3)}e^{ad(-\gamma_2 \mathcal{A}_2)}e^{ad(-\gamma_1 \mathcal{A}_1)}\mathcal{A}_2 = e^{ad(-\gamma_3 \mathcal{A}_3)}(\cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3 \ .
\end{aligned}
$$

For $\mathcal{A}_3$:

$$
\begin{aligned}
&e^{ad(-\gamma_1 \mathcal{A}_1)}\mathcal{A}_3 = \sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3 \\
&e^{ad(-\gamma_2 \mathcal{A}_2)}e^{ad(-\gamma_1 \mathcal{A}_3)}\mathcal{A}_3 = e^{ad(-\gamma_2 \mathcal{A}_2)}(\sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3) = \sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3 \\
&e^{ad(-\gamma_3 \mathcal{A}_3)}e^{ad(-\gamma_2 \mathcal{A}_2)}e^{ad(-\gamma_1 \mathcal{A}_1)}\mathcal{A}_3 = e^{ad(-\gamma_3 \mathcal{A}_3)}(\sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3 \ .
\end{aligned}
$$

$\blacksquare$

Using the definition of the basis $\{\mathcal{A}_i\}$, we have from (6) :

$$
\chi(t) = e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\ \sin\gamma_1 & \cos\gamma_1 & \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\ 0 & 0 & 1 \end{pmatrix} . \tag{14}
$$

Define:

$$
\begin{aligned}
x &\stackrel{\text{def}}{=} \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \ , \\
y &\stackrel{\text{def}}{=} \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \ , \\
\phi &\stackrel{\text{def}}{=} \gamma_1 \ .
\end{aligned} \tag{15}
$$

Differentiating and using (7) we have:

$$
\begin{aligned}
\dot{x} &= v_2\cos\gamma_1 - v_3\sin\gamma_1 \ , \\
\dot{y} &= v_2\sin\gamma_1 + v_3\cos\gamma_1 \ , \\
\dot{\phi} &= v_1 \ .
\end{aligned} \tag{16}
$$

2.2 The ℓ-module Variable Geometry Truss (VGT)

We consider a chain of ℓ modules of the type shown in fig. 1.1 and 2.1. Each module consists of a planar parallel manipulator with one pair of wheels per platform and with each wheel rotating independently from the other around its axis, both forward and backwards. Neither wheel pair is actuated and we assume that the wheels roll without slipping. This system has $n = 3(\ell + 1)$ degrees–of–freedom, its configuration space is $Q = \underbrace{SE(2) \times \cdots \times SE(2)}_{\ell+1 \text{ times}}$ and it is subject to 3ℓ holonomic constraints from the parallel manipulator legs and to $p = \ell + 1$ nonholonomic constraints from the rolling–without–slipping wheel motion. The configuration of the assembly can be determined by its shape (which is an element of the shape space $S = \underbrace{SE(2) \times \cdots \times SE(2)}_{\ell \text{ times}}$) *and* by the position and orientation of the assembly with respect to the world coordinate system (which is an element of $G = SE(2)$) . Then $Q = G \times S$.

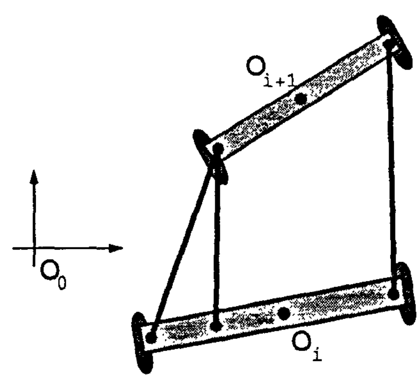


Fig. 2.1

Consider a world coordinate system centered at O_0 and platform coordinate systems centered at O_i, $i = 1, \ldots, \ell + 1$. Let $\chi_i \in G = SE(2)$ be the configuration matrix of the i–th platform with respect to the world coordinate system. Define $V_i \in \mathcal{G} = se(2)$ by:

$$\dot{\chi}_i = \chi_i V_i \ , \ i = 1, \ldots, \ell + 1 \ . \tag{17}$$

From (4) we have:

$$V_i = \sum_{j=1}^{3} v^i_j \mathcal{A}_j = \sum_{j=1}^{3} \mathcal{A}^\flat_j(V_i) \mathcal{A}_j \ . \tag{18}$$

Also define $v^i = (\, v^i_1 \, v^i_2 \, v^i_3 \,)^\top = (\, \mathcal{A}^\flat_1(V_i) \; \mathcal{A}^\flat_2(V_i) \; \mathcal{A}^\flat_3(V_i) \,)^\top$.

Let $\chi_{j,i+1} \in G$ be the configuration matrix of the $(i+1)$–th platform with respect to the coordinate system of the j–th platform. Define $V_{j,i+1} \in \mathcal{G}$ by:

$$\dot{\chi}_{j,i+1} = \chi_{j,i+1} V_{j,i+1} \ , \ \text{ for } i = 1, \ldots, \ell \text{ and } 1 \le j < i+1 \ . \tag{19}$$

From (4) we have:

$$V_{j,i+1} = \sum_{k=1}^{3} v_k^{j,i+1} \mathcal{A}_k = \sum_{k=1}^{3} \mathcal{A}_k^\flat(V_{j,i+1}) \mathcal{A}_k \ . \tag{20}$$

Also define $v^{j,i+1} = (\, v_1^{j,i+1} \, v_2^{j,i+1} \, v_3^{j,i+1} \,)^\top = (\, \mathcal{A}_1^\flat(V_{j,i+1}) \ \mathcal{A}_2^\flat(V_{j,i+1}) \ \mathcal{A}_3^\flat(V_{j,i+1}) \,)^\top$. Observe that the $\chi_{j,i+1}$'s and the $V_{j,i+1}$'s are the *shape variables* of the chain.

The *shape* of the VGT assembly is determined by $\{\chi_{i,i+1},\ i = 1, \ldots, \ell\}$. The velocities $\{V_{i,i+1},\ i = 1, \ldots, \ell\}$ are called *shape controls* for reasons to become obvious at the end of this section.

By $(6), (15)$ and (16) we have:

$$\chi_i(t) = e^{\gamma_1^i(t)\mathcal{A}_1} e^{\gamma_2^i(t)\mathcal{A}_2} e^{\gamma_3^i(t)\mathcal{A}_3} = \begin{pmatrix} R_i & T_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi_i & -\sin\phi_i & x_{O_i} \\ \sin\phi_i & \cos\phi_i & y_{O_i} \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

and

$$\begin{aligned} \chi_{j,i+1}(t) &= e^{\gamma_1^{j,i+1}(t)\mathcal{A}_1} e^{\gamma_2^{j,i+1}(t)\mathcal{A}_2} e^{\gamma_3^{j,i+1}(t)\mathcal{A}_3} \\ &= \begin{pmatrix} R_{j,i+1} & T_{j,i+1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_{j,i+1} & -\sin\theta_{j,i+1} & x_{j,i+1} \\ \sin\theta_{j,i+1} & \cos\theta_{j,i+1} & y_{j,i+1} \\ 0 & 0 & 1 \end{pmatrix} \ , \end{aligned} \tag{22}$$

where the γ's and the corresponding v's are related by (7) and (8). By the system kinematics we have:

$$\chi_{i+1} = \chi_i \chi_{i,i+1} = \chi_1 \chi_{1,2} \cdots \chi_{i,i+1} \ , \ i = 1, \ldots, \ell \tag{23}$$

and

$$\chi_{j,i+1} = \chi_{j,i} \cdots \chi_{i,i+1} = \chi_{j,j+1} \cdots \chi_{i,i+1} \ , \ \text{for } i = 1, \ldots, \ell \text{ and } 1 \le j < i+1 \ . \tag{24}$$

Define $\chi_{j,j} = I$, where I is the identity in G.

Equations (23) with (21), (22) can be seen as a generalization of the "product–of–exponentials" formula of (Brockett, Stokes & Park [1993]) for kinematics chains with more than one degree–of–freedom per joint.

Proposition 2.2.1

The velocities of the $(i+1)$–th module depend on the velocities of the previous modules as follows:

$$\begin{aligned}
V_{i+1} &= Ad_{\chi^{-1}_{i,i+1}} V_i + V_{i,i+1} \\
&= Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{1,2}} V_1 + Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{2,3}} V_{1,2} \\
&\qquad + \cdots + Ad_{\chi^{-1}_{i,i+1}} V_{i-1,i} + V_{i,i+1} \ , \\
V_{j,i+1} &= Ad_{\chi^{-1}_{i,i+1}} V_{j,i} + V_{i,i+1} \\
&= Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{j+1,j+2}} V_{j,j+1} + Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{j+2,j+3}} V_{j+1,j+2} \\
&\qquad + \cdots + Ad_{\chi^{-1}_{i,i+1}} V_{i-1,i} + V_{i,i+1} \ .
\end{aligned} \tag{25}$$

Proof

From (17) and (23):

$$\begin{aligned}
\dot{\chi}_{i+1} &= \dot{\chi}_i \chi_{i,i+1} + \chi_i \dot{\chi}_{i,i+1} = \chi_i V_i \chi_{i,i+1} + \chi_i \chi_{i,i+1} V_{i,i+1} \\
&= \chi_i \chi_{i,i+1} [\chi^{-1}_{i,i+1} V_i \chi_{i,i+1} + V_{i,i+1}] \\
&= \chi_{i+1} [Ad_{\chi^{-1}_{i,i+1}} V_i + V_{i,i+1}] \ ,
\end{aligned} \tag{26}$$

$$\begin{aligned}
\dot{\chi}_{i+1} &= \dot{\chi}_1 \chi_{1,2} \cdots \chi_{i,i+1} + \chi_1 \dot{\chi}_{1,2} \cdots \chi_{i,i+1} + \cdots + \chi_1 \chi_{1,2} \cdots \dot{\chi}_{i,i+1} \\
&= \chi_{i+1} [Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{1,2}} V_1 + Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{2,3}} V_{1,2} \\
&\qquad + \cdots + Ad_{\chi^{-1}_{i,i+1}} V_{i-1,i} + V_{i,i+1}] \ .
\end{aligned} \tag{27}$$

Then (25) follows from (17), (26) and (27). The equation for $V_{j,i+1}$ is derived similarly.

∎

Corollary 2.2.2

Equations (25) induce the following relationships between the positions and velocities of the $(i+1)$–th module and those of the previous modules:

$$\begin{aligned} v_1^{i+1} &= v_1^i + v_1^{i,i+1} , \\ v_2^{i+1} &= -v_1^i \gamma_3^{i,i+1} + v_2^i \cos\gamma_1^{i,i+1} + v_3^i \sin\gamma_1^{i,i+1} + v_2^{i,i+1} , \\ v_3^{i+1} &= v_1^i \gamma_2^{i,i+1} - v_2^i \sin\gamma_1^{i,i+1} + v_3^i \cos\gamma_1^{i,i+1} + v_3^{i,i+1} . \end{aligned} \tag{28}$$

$$\begin{aligned} v_1^{j,i+1} &= v_1^{j,i} + v_1^{i,i+1} , \\ v_2^{j,i+1} &= -v_1^{j,i} \gamma_3^{i,i+1} + v_2^{j,i} \cos\gamma_1^{i,i+1} + v_3^{j,i} \sin\gamma_1^{i,i+1} + v_2^{i,i+1} , \\ v_3^{j,i+1} &= v_1^{j,i} \gamma_2^{i,i+1} - v_2^{j,i} \sin\gamma_1^{i,i+1} + v_3^{j,i} \cos\gamma_1^{i,i+1} + v_3^{i,i+1} . \end{aligned} \tag{29}$$

Moreover:

$$\begin{aligned} \gamma_1^{i+1} &= \gamma_1^i + \gamma_1^{i,i+1} , \\ \gamma_2^{i+1} &= \gamma_2^i \cos\gamma_1^{i,i+1} + \gamma_3^i \sin\gamma_1^{i,i+1} + \gamma_2^{i,i+1} , \\ \gamma_3^{i+1} &= -\gamma_2^i \sin\gamma_1^{i,i+1} + \gamma_3^i \cos\gamma_1^{i,i+1} + \gamma_3^{i,i+1} , \end{aligned} \tag{30}$$

$$\begin{aligned} \gamma_1^{j,i+1} &= \gamma_1^{j,i} + \gamma_1^{i,i+1} , \\ \gamma_2^{j,i+1} &= \gamma_2^{j,i} \cos\gamma_1^{i,i+1} + \gamma_3^{j,i} \sin\gamma_1^{i,i+1} + \gamma_2^{i,i+1} , \\ \gamma_3^{j,i+1} &= -\gamma_2^{j,i} \sin\gamma_1^{i,i+1} + \gamma_3^{j,i} \cos\gamma_1^{i,i+1} + \gamma_3^{i,i+1} . \end{aligned} \tag{31}$$

Proof

From $(4),(25),(12)$ and (13), observing that $\{\mathcal{A}_i\}$ form a basis, we get equations (28) and (29). From (14) and (23) we get equations (30) and (31).

∎

The nonholonomic constraint of rolling–without–slipping on the wheels of each platform can be expressed using (16) as:

$$v_2^i = \mathcal{A}_2^\flat(V_i) = \dot{x}_{O_i} \cos\phi_i + \dot{y}_{O_i} \sin\phi_i = 0 , \quad i = 1, \ldots, \ell+1 . \tag{32}$$

Define the composite *velocity vector* of the VGT assembly:

$$\begin{aligned} v &\stackrel{\text{def}}{=} (\, v^1 \,|\, v^{1,2} \,|\, \cdots \,|\, v^{\ell,\ell+1} \,)^\top \\ &= (\, v_1^1 v_2^1 v_3^1 \,|\, v_1^{1,2} \cdots \,|\, v_1^{\ell,\ell+1} v_2^{\ell,\ell+1} v_3^{\ell,\ell+1} \,)^\top \\ &= \left(\, \mathcal{A}_1^\flat(V_1)\, \mathcal{A}_2^\flat(V_1)\, \mathcal{A}_3^\flat(V_1) \,|\, \mathcal{A}_1^\flat(V_{1,2}) \cdots \,|\, \mathcal{A}_1^\flat(V_{\ell,\ell+1})\, \mathcal{A}_2^\flat(V_{\ell,\ell+1})\, \mathcal{A}_3^\flat(V_{\ell,\ell+1}) \,\right)^\top . \end{aligned}$$

Proposition 2.2.3

The $p = \ell + 1$ nonholonomic constraints (32) can be written in matrix form as:

$$A(\chi_{1,2}, \cdots, \chi_{\ell,\ell+1})\, v = 0 \,, \tag{33}$$

where A is a function of only the shape of the VGT assembly $\{\chi_{i,i+1}, i = 1, \cdots, \ell\}$ and is a block lower triangular matrix of maximal rank of the form:

$$A(\chi_{1,2}, \cdots, \chi_{\ell,\ell+1}) = \begin{pmatrix} *_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \cdots & 0 & 0 \\ *_{1,i+1} & *_{2,i+1} & \cdots & *_{i,i+1} & *_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \ddots & 0 \\ *_{1,\ell+1} & *_{2,\ell+1} & \cdots & *_{i,\ell+1} & *_{i+1,\ell+1} & \cdots & *_{\ell,\ell+1} & *_{\ell+1,\ell+1} \end{pmatrix} \tag{34}$$

with the k–th block of the $(i+1)$–th line defined as:

$$\begin{aligned} *_{k,i+1} &= \Big(\mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,i+1}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,i+1}} \mathcal{A}_2) \;\; \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,i+1}} \mathcal{A}_3) \Big) \\ &= \Big(\mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{k,k+1}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{k,k+1}} \mathcal{A}_2) \\ &\qquad \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{k,k+1}} \mathcal{A}_3) \Big) \\ &= \big(-\gamma_3^{k,i+1} \;\; \cos\gamma_1^{k,i+1} \;\; \sin\gamma_1^{k,i+1} \big) \,, \qquad \text{for } k < i+1 \,, \\ *_{k,k} &= \Big(\mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,k}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,k}} \mathcal{A}_2) \;\; \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{k,k}} \mathcal{A}_3) \Big) \\ &= (0\;1\;0) \,, \qquad \text{for } k = i+1 \,, \\ *_{k,i+1} &= (0\;0\;0) \,, \qquad \text{for } k > i+1 \,. \end{aligned}$$

Proof

From (25), (18) and (20), we get for $i = 1, \ldots, \ell$:

$$\begin{aligned} 0 = \mathcal{A}_2^\flat(V_{i+1}) &= \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{1,2}} V_1) + \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} \cdots Ad_{\chi^{-1}_{2,3}} V_{1,2}) \\ &\quad + \cdots + \mathcal{A}_2^\flat(Ad_{\chi^{-1}_{i,i+1}} V_{i-1,i}) + \mathcal{A}_2^\flat(V_{i,i+1}) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{A}_2^\flat\big(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{1,2}^{-1}}(\sum_{j=1}^{3} \mathcal{A}_j^\flat(V_1)\mathcal{A}_j)\big) \\
&\qquad + \mathcal{A}_2^\flat\big(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{2,3}^{-1}}(\sum_{j=1}^{3} \mathcal{A}_j^\flat(V_{1,2})\mathcal{A}_j)\big) \\
&\qquad\qquad + \cdots + \mathcal{A}_2^\flat\big(Ad_{X_{i,i+1}^{-1}}(\sum_{j=1}^{3} \mathcal{A}_j^\flat(V_{i-1,i})\mathcal{A}_j)\big) + \mathcal{A}_2^\flat(V_{i,i+1}) \\
&= \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{1,2}^{-1}}\mathcal{A}_j)\mathcal{A}_j^\flat(V_1) \\
&\qquad + \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{2,3}^{-1}}\mathcal{A}_j)\mathcal{A}_j^\flat(V_{1,2}) \\
&\qquad\qquad + \cdots + \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}}\mathcal{A}_j)\mathcal{A}_j^\flat(V_{i-1,i}) + \mathcal{A}_2^\flat(V_{i,i+1}) \\
&= \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{1,2}^{-1}}\mathcal{A}_j)v_j^1 \\
&\qquad + \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}} \cdots Ad_{X_{2,3}^{-1}}\mathcal{A}_j)v_j^{1,2} \\
&\qquad\qquad + \cdots + \sum_{j=1}^{3} \mathcal{A}_2^\flat(Ad_{X_{i,i+1}^{-1}}\mathcal{A}_j)v_j^{i-1,i} + v_2^{i,i+1}\,.
\end{aligned}
\tag{35}
$$

Thus, the $(i+1)$–th nonholonomic constraint can be expressed as a linear combination of the v^1's and the shape velocities $v^{1,2}, \ldots, v^{i,i+1}$, for $i = 1, \ldots, \ell$. Then (34) follows. From its form, and especially from the expressions for the diagonal blocks $*_{k,k}$, we can easily see that A has always maximal rank $p = \ell + 1$.

∎

Since A has always maximal rank p, its null space $\mathcal{N}(A)$ has dimension $m \stackrel{\text{def}}{=} n - p = 2p$.

Proposition 2.2.4

Assume that the velocities v can be reordered, so that the matrix A is partitioned as $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$, with A_1 a $p \times m$ matrix and A_2 a locally invertible $p \times p$ matrix.

Let the corresponding partition of the velocity vector be $v = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$, with $\nu_1 \in \mathbb{R}^m$ and $\nu_2 \in \mathbb{R}^p$. Then, there exists an $n \times m$ matrix B such that:

$$v = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = B\,\nu_1 \,. \tag{36}$$

Proof

The matrix $B = \begin{pmatrix} I_{m\times m} \\ -A_2^{-1}A_1 \end{pmatrix}$ works.

∎

Notice that, since A depends only on the shape, so does A_2. In addition, the choice of the locally invertible matrix A_2 is dictated by the choice of the splitting of v into ν_1 and ν_2. For a particular choice, as the shape is altered, A_2 will become singular at certain shapes. The corresponding configurations of the VGT assembly shall be referred to as *nonholonomic singularities.* It will be apparent that these singularities are not removable by merely choosing alternative splittings of v or alternative parametrizations of the configuration space (e.g. how we assign orderings to the kinematic chain at hand). In these configurations, the system kinematics together with the shape control may not provide sufficient information to determine its motion and the dynamics of the system may have to be considered.

Remarks

1) Unlike previous work on nonholonomic motion planning, in our case the ν_1's of equation (36) do not correspond directly to the controls of the system and, thus, are not at our disposal to alter at will. Our real controls are the leg velocities $\dot{\sigma}$ of the parallel manipulator modules (see section 2.3). However, as we will see in section 2.3, *off* the *kinematic singularities* of the parallel manipulators, the shape controls can easily determine the corresponding leg velocities. Therefore, in order to simplify the discussion of motion planning, once the partitioning of the velocity vector v as $\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ is done in such a way that all the ν_1's are controllable from the $\dot{\sigma}$'s, we will disregard the particulars of the implementation of the modules and only consider the shape controls. Note that when $\ell = 1$, such a partitioning of the velocity vector cannot be done.

2) The 3ℓ holonomic constraints imposed by the legs of the modules of the ℓ–module VGT assembly determine the shape $\chi_{1,2}, \chi_{2,3}, \ldots, \chi_{\ell,\ell+1}$ of the assembly. In order to determine completely its configuration, we also need to specify the position and orientation of the assembly with respect to the world coordinate frame, given by an element (χ_1 in this case) of $G = SE(2)$. Thus, we need 3 more constraints. These come from the $\ell + 1$ nonholonomic constraints provided by the platform wheels. Consider now some special cases: *i*) If $\ell = 1$, we have 3 holonomic and 2 nonholonomic constraints, but we need to determine 6 degrees–of–freedom, thus we do not have enough kinematic constraints to determine the motion of the assembly. We either have to consider its dynamics or we need to impose additional constraints (e.g. unidirectional wheel motion). *ii*) If $\ell = 2$, we have 6 holonomic and 3 nonholonomic constraints and we need to determine 9 degrees–of–freedom, thus we have exactly the required number of constraints. *iii*) If $\ell > 2$, we have 3ℓ holonomic and $\ell + 1$ nonholonomic constraints and we need to determine $3(\ell + 1)$ degrees–of–freedom, thus we have $3\ell + (\ell + 1) - 3(\ell + 1) = \ell - 2$ extra constraints that have to be satisfied. Therefore, from the 3ℓ shape velocities, *only* $3\ell - (\ell - 2) = 2(\ell + 1)$ can be determined independently and will be elements of the vector ν_1 in equation (36). The remaining $\ell - 2$ shape velocities will be determined by the $\ell - 2$ extra constraints, i.e. they will be elements of the vector ν_2, together with the velocities that characterize the global motion of the assembly (here they are the v^1's). Observe that if there exists k such that $\ell - 2 = 3k$, we can choose $\ell - k$ of the ℓ modules and alter their shape at will, while the shape of the remaining k modules will be determined by the extra constraints. In brief, for $\ell > 2$, the problem is over–constrained.

3) The 2–module VGT is "canonical" in the following sense: Suppose that we change the ℓ–module VGT architecture (for $\ell > 2$) so that we have wheels on only 3 platforms and castors on the remaining $\ell - 2$ platforms. Then we have only 3 nonholonomic constraints (instead of $\ell + 1$). Such a system has exactly the number of constraints needed to determine its motion, which, in our original VGT assembly is the case only for $\ell = 2$. The motion planning strategies that will be determined for the 2–module VGT will work also for this ℓ–module assembly. Moreover, this ℓ–module assembly possesses $3\ell - 6$ degrees–of–freedom more than the 2–module VGT and those can be exploited for other purposes (e.g. for motion in a constrained environment, or for generating a richer *repertoire* of motions of the type described in (Chirikjian & Burdick [1991])).

2.3 Implementation of a VGT module as a Parallel Manipulator

Consider the i–th module of the VGT assembly, implemented as a planar parallel manipulator (Fichter [1986]). This consists of the i–th and $(i+1)$–th platforms, which are connected by three legs of variable length. One possible architecture is shown in fig. 2.1.

In previous sections we saw that the shape of the i–th module is determined by $\chi_{i,i+1} \in SE(2)$. Let ${}^{p}P_k^i$, for $k = i, i+1$, $j = 1,2,3$ be the position of the j–th joint of the k–th platform with respect to the coordinate system of this platform. Consider the following quantities, defined with respect to the coordinate system of the i–th platform: the position P_j^{i+1} of the $(i+1)$–th platform, the vector S_j^i and the length σ_j^i of the j–th leg of the module. Let σ^i be the vector of the leg lengths of the i–th module, i.e. $\sigma^i \stackrel{\text{def}}{=} (\sigma_1^i\ \sigma_2^i\ \sigma_3^i)^\top$.

The *inverse kinematic map* $\mathcal{F}^{-1} : SE(2) \rightarrow \mathbb{R}_+^3$ specifies the leg lengths of the module as functions of its shape $\chi_{i,i+1}$. We can easily see that for the j–th leg:

$$\begin{pmatrix} S_j^i \\ 1 \end{pmatrix} = \chi_{i,i+1} \begin{pmatrix} {}^{p}P_j^{i+1} \\ 1 \end{pmatrix} - \begin{pmatrix} {}^{p}P_j^i \\ 1 \end{pmatrix} . \tag{37}$$

Then the leg lengths of the module are given by

$$\sigma_j^i = ||S_j^i|| = \sqrt{< S_j^i, S_j^i >}\ , \quad \text{for } j = 1,2,3\ , \tag{38}$$

where $< .\,, . >$ is the inner product in $\mathbb{R}^n$ and $||\,.\,||$ is the corresponding norm. Then the inverse kinematic map is:

$$\mathcal{F}^{-1}(\chi_{i,i+1}) = \sigma^i\ . \tag{39}$$

Proposition 2.3.1 (Velocity Kinematics)

The body velocities of the i–th module $v^{i,i+1}$ and the changes of length of its legs $\dot{\sigma}^i \stackrel{\text{def}}{=} (\dot{\sigma}_1^i\ \dot{\sigma}_2^i\ \dot{\sigma}_3^i)^\top$ are related by:

$$\Sigma(\sigma^i)\dot{\sigma}^i = J(\chi_{i,i+1})v^{i,i+1}\ , \tag{40}$$

where $\Sigma(\sigma^i) = diag\{\sigma_1^i\,, \sigma_2^i\,, \sigma_3^i\}$ and $J(\chi_{i,i+1}) = \left[{S_j^i}^\top R_{i,i+1}\, \widehat{{}^{p}P_j^{i+1}}^\top \mid {S_j^i}^\top R_{i,i+1} \right]_j$.

Proof

From (38) : $(\sigma_j^i)^2 =< S_j^i, S_j^i >$. Differentiating both sides and defining $\Omega \stackrel{\text{def}}{=} v_1^{i,i+1}$ and $\Xi \stackrel{\text{def}}{=} \begin{pmatrix} v_2^{i,i+1} \\ v_3^{i,i+1} \end{pmatrix}$, we get:

$$\sigma_j^i \dot{\sigma}_j^i =< S_j^i, \dot{S}_j^i >=< S_j^i, R_{i,i+1}(\Xi + \hat{\Omega}\, {}^pP_j^{i+1}) >$$

$$=< R_{i,i+1}{}^{\mathsf{T}} S_j^i, \Xi + \widehat{{}^pP_j^{i+1}}^{\mathsf{T}} \Omega >=< {}^pP_j^{i+1} R_{i,i+1}{}^{\mathsf{T}} S_j^i, \Omega > + < R_{i,i+1}{}^{\mathsf{T}} S_j^i, \Xi > ,$$

where $R_{i,i+1}$ was defined in (22). Then (40) follows.

∎

Configurations where $J(\chi)$ is singular are called *kinematic singularities.* Those do not have anything to do with nonholonomic singularities.

Corollary 2.3.2

Suppose the configuration $\chi_{i,i+1} \in SE(2)$ of the i–th module is not a kinematic singularity and that the corresponding leg lengths are σ^i. Then:

$$v^{i,i+1} = J^{-1}(\chi_{i,i+1})\Sigma(\sigma^i)\dot{\sigma}^i \tag{41}$$

and

$$\dot{\sigma}^i = \Sigma^{-1}(\sigma^i)J(\chi_{i,i+1})v^{i,i+1} \,. \tag{42}$$

Observe that after we specify the shape controls $v^{i,i+1}$, the corresponding leg length changes can be easily determined from (42). Therefore, our discussion of the motion planning problem can disregard, without loss of generality, the particular implementation details of the modules and consider only the shape controls.

3 The 2–module VGT

In section 3.1 we consider the kinematics of the 2-module VGT (fig. 3.1) as a special case of the kinematics of the ℓ-module VGT. We show that the velocity vector can be partitioned in two parts, one of which constitutes the independent shape controls and the other being velocities dependent on the shape variables, which characterize the motion of the assembly with respect to the world coordinate system. In section 3.2 we examine the motion planning problem for the 2-module VGT assembly. We show that shape actuation strategies, where one module keeps its shape fixed and the other varies it periodically, induce a rotation of the VGT assembly around the instantaneous center of rotation of the first module. If the platforms of the *fixed shape* module are parallel, the induced motion is a translation along a direction perpendicular to the platforms. We allow the shape of the second module to describe a closed path in shape space and show that a net displacement of the VGT assembly with respect to the world coordinate system is induced after each traversal of the shape space path.

3.1 Kinematics of the 2–module VGT

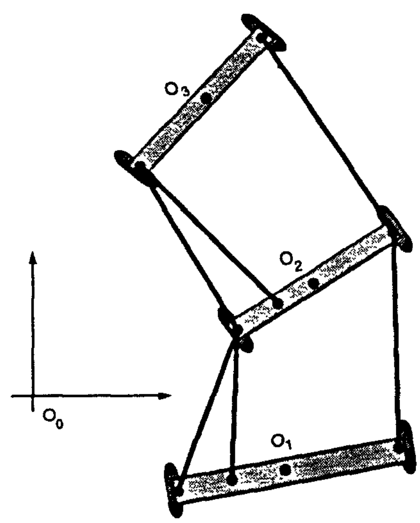


Fig. 3.1

In the assembly of fig. 3.1, we consider a chain of $\ell = 2$ VGT modules. This system has $n = 9$ degrees–of–freedom, its configuration space is $Q = G \times S$, where $G = SE(2)$ and the shape space is $S = SE(2) \times SE(2)$ and it is subject to 6 holonomic constraints from the parallel manipulator legs and to $p = 3$ nonholonomic constraints from the rolling–without–slipping assumption on the wheels. From the system kinematics we have (specializing the results of section 2):

$$\begin{aligned} \chi_2 &= \chi_1 \chi_{1,2} \,, \\ \chi_3 &= \chi_2 \chi_{2,3} = \chi_1 \chi_{1,2} \chi_{2,3} \,, \\ \chi_{13} &= \chi_{1,2} \chi_{2,3} \,. \end{aligned} \tag{1}$$

From (2.25) we get for the corresponding velocities:

$$\begin{aligned} V_2 &= Ad_{\chi_{1,2}^{-1}} V_1 + V_{1,2} \\ V_3 &= Ad_{\chi_{2,3}^{-1}} V_2 + V_{2,3} = Ad_{\chi_{2,3}^{-1}} Ad_{\chi_{1,2}^{-1}} V_1 + Ad_{\chi_{2,3}^{-1}} V_{1,2} + V_{2,3} \\ V_{1,3} &= Ad_{\chi_{2,3}^{-1}} V_{1,2} + V_{2,3} \end{aligned} \tag{2}$$

The nonholonomic constraint of rolling–without–slipping on the wheels of each platform can be expressed using (2.32) as:

$$v_2^i = \mathcal{A}_2^\flat(V_i) = \dot{x}_{O_i} \cos\phi_i + \dot{y}_{O_i} \sin\phi_i = 0\ , \quad i = 1,2,3\ . \tag{3}$$

The 3 nonholonomic constraints can be put in the matrix form of (2.33) :

$$A(\chi_{1,2}, \chi_{2,3})\, v = 0\ , \tag{4}$$

where $v = \begin{pmatrix} v^1 \\ v^{1,2} \\ v^{2,3} \end{pmatrix}$. The matrix A is a function of only the shape variables $\chi_{1,2}$, $\chi_{2,3}$ of the chain and is a block lower triangular matrix of the form:

$$\begin{aligned} A(\chi_{1,2}, \chi_{2,3}) &= \begin{pmatrix} *_{1,1} & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 \\ *_{1,3} & *_{2,3} & *_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_3) & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_3) & \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_3) & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} & 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} & -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{5}$$

Equation (4) can be put in the form (2.36) by partitioning v as $v = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ with

$$\nu_1 = \begin{pmatrix} v^{1,2} \\ v^{2,3} \end{pmatrix} \quad \text{and} \quad \nu_2 = v^1 \tag{6}$$

and by partitioning A as $(A_1\ A_2)$ with

$$
\begin{aligned}
A_1(\chi_{2,3}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{2,3}^{-1}}\mathcal{A}_3) & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
A_2(\chi_{1,2},\chi_{2,3}) &= \begin{pmatrix} 0 & 1 & 0 \\ \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_3) \\ \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_3) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} \end{pmatrix} .
\end{aligned}
\tag{8}
$$

Then the velocity of the 2–module VGT assembly with respect to the world coordinate system, as it is characterized by v^1, can be expressed as a function of only the shape variables of the assembly:

$$
v^1 = -A_2^{-1}(\chi_{1,2},\chi_{2,3})A_1(\chi_{2,3})\begin{pmatrix} v^{1,2} \\ v^{2,3} \end{pmatrix} . \tag{9}
$$

The nonholonomic singularities of the system can be specified by considering the determinant of the matrix A_2 :

$$
\begin{aligned}
det(A_2) &= \mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_1)\mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_3) - \mathcal{A}_2^\flat(Ad_{\chi_{1,2}^{-1}}\mathcal{A}_1)\mathcal{A}_2^\flat(Ad_{\chi_{1,3}^{-1}}\mathcal{A}_3) \\
&= -\gamma_3^{1,3}\sin\gamma_1^{1,2} + \gamma_3^{1,2}\sin\gamma_1^{1,3} .
\end{aligned}
\tag{10}
$$

Observe that when $\sin\gamma_1^{1,2} \neq 0$ and $\sin\gamma_1^{1,3} \neq 0$:

$$
\begin{aligned}
det(A_2) &= \sin\gamma_1^{1,2}\sin\gamma_1^{1,3}\left[\left(-\frac{\gamma_3^{1,3}}{\sin\gamma_1^{1,3}}\right) - \left(-\frac{\gamma_3^{1,2}}{\sin\gamma_1^{1,2}}\right)\right] \\
&= \sin\gamma_1^{1,2}\sin\gamma_1^{1,3}\left[\Delta x_{O_{1,3}} - \Delta x_{O_{1,2}}\right] ,
\end{aligned}
\tag{11}
$$

where $\Delta x_{O_{1,j}} \stackrel{\text{def}}{=} -\frac{\gamma_3^{1,j}}{\sin\gamma_1^{1,j}}$, $j = 2,3$, is the distance of the intersection $O_{1,j}$ of the axis of platform 1 with the axis of platform j from the point O_1, as shown in fig. 3.2. It will

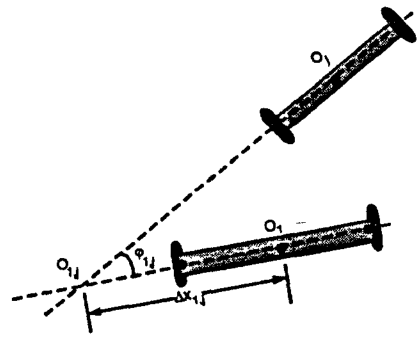
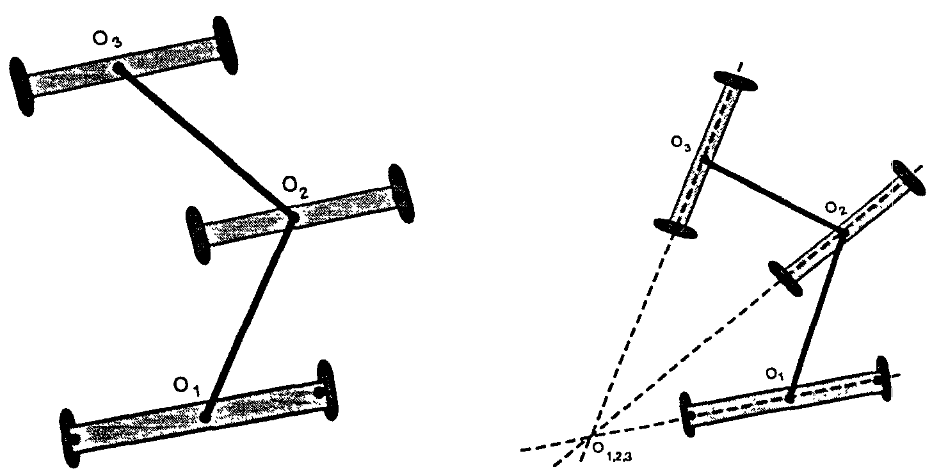


Fig. 3.2

be shown in the next section that the point $O_{1,j}$ coincides with the instantaneous axis of rotation of the module composed by the platforms 1 and j.

Then, the matrix A_2 is singular whenever the points $O_{1,2}$ and $O_{1,3}$ coincide (the point $O_{1,2,3}$ may be at infinity as in fig. 3.3.a). Even in this case the 3 nonholonomic constraints *remain independent* (c.f. equation (5), where $rank(A) = 3$), but, since the platforms have a common instantaneous center of rotation, equation (4) cannot be recast in the form of (9). Therefore, motion with respect to point $O_{1,2,3}$ cannot be controlled by the system shape variables alone and the dynamics of the system ought to be considered. This is analogous to what practising engineers refer to as loss of control authority.



(a) (b)

Fig. 3.3

From (8) :

$$A_2^{-1} = \frac{1}{det(A_2)} \begin{pmatrix} \sin\gamma_1^{2,3} & -\sin\gamma_1^{1,3} & \sin\gamma_1^{1,2} \\ \gamma_3^{1,2}\sin\gamma_1^{1,3} - \gamma_3^{1,3}\sin\gamma_1^{1,2} & 0 & 0 \\ -\gamma_3^{1,2}\cos\gamma_1^{1,3} + \gamma_3^{1,3}\cos\gamma_1^{1,2} & -\gamma_3^{1,3} & \gamma_3^{1,2} \end{pmatrix} \qquad (12)$$

Then, from (9) we get:

$$v^1 = \frac{-1}{det(A_2)} \begin{pmatrix} -\sin\gamma_1^{1,2}\gamma_3^{2,3} & -\sin\gamma_1^{1,3} + \sin\gamma_1^{1,2}\cos\gamma_1^{2,3} & \sin\gamma_1^{1,2}\sin\gamma_1^{2,3} \\ 0 & 0 & 0 \\ -\gamma_3^{1,2}\gamma_3^{2,3} & -\gamma_3^{1,3} + \gamma_3^{1,2}\cos\gamma_1^{2,3} & \gamma_3^{1,2}\sin\gamma_1^{2,3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sin\gamma_1^{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & \gamma_3^{1,2} & 0 \end{pmatrix} \begin{pmatrix} v^{1,2} \\ v^{2,3} \end{pmatrix} \tag{13}$$

Observe that the partitioning of v in equation (6) is the only one that assigns to ν_1 shape controls which can be affected by leg length changes in the parallel manipulator modules. Moreover, it assigns to ν_2 the velocities that characterize the global motion of the VGT assembly with respect to the world coordinate system.

From $\dot{\chi}_1 = \chi_1 V_1$ and the definition of the basis $\{\mathcal{A}_i\}$ of $\mathcal{G}$, it is easy to see that, away from the nonholonomic singularities, *controllability* is guaranteed for a generic set of shape controls $\begin{pmatrix} v^{1,2} \\ v^{2,3} \end{pmatrix}$ whenever the 1st and 3rd rows of the matrix $A_2^{-1}A_1$ are linearly independent. However, the latter always holds away from the nonholonomic singularities.

3.2 Motion Planning for the 2–module VGT

There are several possible *actuation strategies* for the 2–module VGT. We will consider a simple one where the first module "steers" the system, while the second provides the translation mechanism through periodic variations of its shape parameters.

From (13) we observe that the motion of the system is determined completely, at least away from the nonholonomic singularities, by the *shape controls* $v^{1,2}$ and $v^{2,3}$. Here we will consider the special case of motions that are generated by keeping the shape of the first module fixed, i.e. $v^{1,2} = 0$, and vary the shape controls $v^{2,3}$ of the second module periodically. Then from (2.7) :

$$\dot{\gamma}^{1,2} = \begin{pmatrix} \dot{\gamma}_1^{1,2} \\ \dot{\gamma}_2^{1,2} \\ \dot{\gamma}_3^{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^{1,2} & 1 & 0 \\ -\gamma_2^{1,2} & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{1,2} \\ v_2^{1,2} \\ v_3^{1,2} \end{pmatrix} = 0 \quad \Rightarrow \quad \gamma^{1,2} = \gamma^{1,2}(0) \,. \tag{14}$$

Also :

$$\dot{\gamma}^{2,3} = \begin{pmatrix} \dot{\gamma}_1^{2,3} \\ \dot{\gamma}_2^{2,3} \\ \dot{\gamma}_3^{2,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^{2,3} & 1 & 0 \\ -\gamma_2^{2,3} & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{2,3} \\ v_2^{2,3} \\ v_3^{2,3} \,. \end{pmatrix} \tag{15}$$

Then from (2.8) :

$$\gamma_1^{2,3}(t) = \gamma_1^{2,3}(0) + \int_0^t v_1^{2,3}(\tau)d\tau \ ,$$

$$\gamma_2^{2,3}(t) = \gamma_2^{2,3}(0)\cos\left(\int_0^t v_1^{2,3}(\sigma)d\sigma\right) + \gamma_3^{2,3}(0)\sin\left(\int_0^t v_1^{2,3}(\sigma)d\sigma\right)$$

$$+ \int_0^t v_2^{2,3}(\tau)\cos\left(\int_\tau^t v_1^{2,3}(\sigma)d\sigma\right)d\tau + \int_0^t v_3^{2,3}(\tau)\sin\left(\int_\tau^t v_1^{2,3}(\sigma)d\sigma\right)d\tau \ ,$$

$$\gamma_3^{2,3}(t) = -\gamma_2^{2,3}(0)\sin\left(\int_0^t v_1^{2,3}(\sigma)d\sigma\right) + \gamma_3^{2,3}(0)\cos\left(\int_0^t v_1^{2,3}(\sigma)d\sigma\right)$$

$$- \int_0^t v_2^{2,3}(\tau)\sin\left(\int_\tau^t v_1^{2,3}(\sigma)d\sigma\right)d\tau + \int_0^t v_3^{2,3}(\tau)\cos\left(\int_\tau^t v_1^{2,3}(\sigma)d\sigma\right)d\tau \ . \tag{16}$$

From (2.31), (10) and (14) :

$$\begin{aligned} det(A_2) &= -\gamma_3^{1,3}\sin\gamma_1^{1,2}(0) + \gamma_3^{1,2}(0)\sin\gamma_1^{1,3} \\ &= -\left[\gamma_3^{2,3} - \gamma_2^{1,2}(0)\sin\gamma_1^{2,3} + \gamma_3^{1,2}(0)\cos\gamma_1^{2,3}\right]\sin\gamma_1^{1,2}(0) \\ &\qquad + \gamma_3^{1,2}(0)\sin(\gamma_1^{1,2}(0) + \gamma_1^{2,3}) \end{aligned} \tag{17}$$

From (13) :

$$v^1 = \frac{-1}{det(A_2)}\begin{pmatrix} 0 & \sin\gamma_1^{1,2}(0) & 0 \\ 0 & 0 & 0 \\ 0 & \gamma_3^{1,2}(0) & 0 \end{pmatrix} v^{2,3} = \frac{-v_2^{2,3}}{det(A_2)}\begin{pmatrix} \sin\gamma_1^{1,2}(0) \\ 0 \\ \gamma_3^{1,2}(0) \end{pmatrix} \tag{18}$$

and from (2.7) :

$$\dot\gamma^1 = \begin{pmatrix} \dot\gamma_1^1 \\ \dot\gamma_2^1 \\ \dot\gamma_3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^1 & 1 & 0 \\ -\gamma_2^1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1^1 \\ 0 \\ v_3^1 \end{pmatrix} \tag{19}$$

Then from (2.8) :

$$\gamma_1^1(t) = \gamma_1^1(0) + \int_0^t v_1^1(\tau)d\tau \ ,$$

$$\gamma_2^1(t) = \gamma_2^1(0)\cos\left(\int_0^t v_1^1(\sigma)d\sigma\right) + \gamma_3^1(0)\sin\left(\int_0^t v_1^1(\sigma)d\sigma\right)$$
$$+ \int_0^t v_3^1(\tau)\sin\left(\int_\tau^t v_1^1(\sigma)d\sigma\right)d\tau \ , \quad (20)$$

$$\gamma_3^1(t) = -\gamma_2^1(0)\sin\left(\int_0^t v_1^1(\sigma)d\sigma\right) + \gamma_3^1(0)\cos\left(\int_0^t v_1^1(\sigma)d\sigma\right)$$
$$+ \int_0^t v_3^1(\tau)\cos\left(\int_\tau^t v_1^1(\sigma)d\sigma\right)d\tau \ .$$

Proposition 3.2.1

i) If $v^{1,2} = 0$ and $\gamma_1^{1,2} = 0$, the 2–module VGT *translates* along an axis perpendicular to platforms 1 and 2.

ii) If $v^{1,2} = 0$ and $\gamma_1^{1,2} \neq 0$, the 2–module VGT *rotates* around the intersection of the axes of platforms 1 and 2.

Proof

i) Translation : Let $\gamma_1^{1,2} = 0$. From (17) and (18):

$$v_1^1 = 0 \ ,$$
$$v_2^1 = 0 \ , \quad (21)$$
$$v_3^1 = -\frac{v_2^{2,3}}{\sin\gamma_1^{2,3}} \ .$$

From (15):

$$\left.\begin{array}{l} \dot\gamma_1^1 = 0 \\ \dot\gamma_2^1 = 0 \\ \dot\gamma_3^1 = v_3^1 \end{array}\right\} \Rightarrow \begin{array}{l} \gamma_1^1(t) = \gamma_1^1(0) \ , \\ \gamma_2^1(t) = \gamma_2^1(0) \ , \\ \gamma_3^1(t) = \gamma_3^1(0) + \int_0^t v_3^1(\tau)d\tau \ . \end{array} \quad (22)$$

From (2.16):

$$\begin{pmatrix} \dot{x}_{O_1} \\ \dot{y}_{O_1} \\ \dot{\phi}_1 \end{pmatrix} = v_3^1 \begin{pmatrix} -\sin\gamma_1^1(0) \\ \cos\gamma_1^1(0) \\ 0 \end{pmatrix} . \tag{23}$$

Thus, platform 1 translates along the axis that passes through the point (x_{O_1}, y_{O_1}) and is parallel to the vector $(-\sin\gamma_1^1(0), \cos\gamma_1^1(0))$, i.e. perpendicular to the platform. This is a constant vector, thus the point O_1 traces a straight line. Moreover, since ϕ_1 is constant, the whole platform translates along this line.

ii) Rotation : Let $\gamma_1^{1,2}(t) = \gamma_1^{1,2}(0) \neq 0$. The *instantaneous center of rotation (ICR)* of the velocity distribution of equation (18) (Bottema & Roth [1979]) can be proven to be the point $O_{1,2}$, where the axes of platform 1 and 2 intersect. To see this, assume that the ICR has coordinates (x_P^1, y_P^1) with respect to the fixed world coordinate system and $(x_P^{1,2}, y_P^{1,2})$ with respect to the moving coordinate system of platform 1. Define $x_P = (x_P^1\ y_P^1\ 1)^\top$ and $X_P = (x_P^{1,2}\ y_P^{1,2}\ 1)^\top$. By its definition, $\dot{X}_P = 0$. Then:

$$x_P = \chi_1 X_P \;\Rightarrow\; \dot{x}_P = \dot{\chi}_1 X_P + \chi_1 \dot{X}_P = \chi_1 V_1 X_P . \tag{24}$$

The ICR is defined as the point where $\dot{x}_P = 0$. From this and (24) we get:

$$V_1 X_P = 0 \;\Rightarrow\; x_P^{1,2} = -\frac{\gamma_3^{1,2}}{\sin\gamma_1^{1,2}} \;,\; y_P^{1,2} = 0 . \tag{25}$$

From fig. 3.2, it is easy to see that this is point $O_{1,2}$, the intersection of the axes of platform 1 and 2.

Furthermore, it is possible to show that the ICR is constant, not only with respect to the coordinate system of platform 1, as is immediately evident from the expressions for $(x_P^{1,2}, y_P^{1,2})$ in (25), but also with respect to the world coordinate system. Therefore, the motion of the system, is indeed a rotation around this point. To see this, consider the vector x_P as a function of time t and expand in Taylor series around a fixed time instant t_0. Then, defining $\Delta t = t - t_0$, we get:

$$x_P(t) = x_P(t_0) + \frac{dx_P(t_0)}{dt}\Delta t + \frac{1}{2!}\frac{d^2x_P(t_0)}{dt^2}\Delta t^2 + \dots .$$

By definition: $\frac{dx_P}{dt} = \dot{\chi}_1 X_P + \chi_1 \dot{X}_P = \chi_1 V_1 X_P = 0$. Moreover, from (18) and (25) :

$$\frac{d^2x_P}{dt^2} = \ddot{\chi}_1 X_P + \dot{\chi}_1 \dot{X}_P + \dot{\chi}_1 \dot{X}_P + \chi_1 \ddot{X}_P = \ddot{\chi}_1 X_P = (\dot{\chi}_1 V_1 + \chi_1 \dot{V}_1) X_P = 0 .$$

Similarly, all the higher derivatives of x_P are zero under the given velocity distribution. Thus the ICR is a constant point.

∎

In the case where $v^{1,2} = 0$, consider the following periodic shape controls for the second module:

$$\begin{aligned} v_1^{2,3} &= \alpha_1 \omega \cos \omega t \;, \\ v_2^{2,3} &= \alpha_2 \omega \sin \omega t \cos \gamma_1^{2,3} \;, \\ v_3^{2,3} &= -\alpha_2 \omega \sin \omega t \sin \gamma_1^{2,3} \;. \end{aligned} \tag{26}$$

From (16) :

$$\begin{aligned} \gamma_1^{2,3}(t) &= \gamma_1^{2,3}(0) + \alpha_1 \sin \omega t \;, \\ \gamma_2^{2,3}(t) &= \gamma_2^{2,3}(0) \cos(\alpha_1 \sin \omega t) + \gamma_3^{2,3}(0) \sin(\alpha_1 \sin \omega t) + \alpha_2 (1 - \cos \omega t) \cos \gamma_1^{2,3} \;, \\ \gamma_3^{2,3}(t) &= -\gamma_2^{2,3}(0) \sin(\alpha_1 \sin \omega t) + \gamma_3^{2,3}(0) \cos(\alpha_1 \sin \omega t) - \alpha_2 (1 - \cos \omega t) \sin \gamma_1^{2,3} \;. \end{aligned} \tag{27}$$

From (2.15) and (27) we see that those shape controls correspond to a closed elliptical path in $(x_{2,3}, \phi_{2,3})$–space.

We attempt to specify the global motion of the VGT assembly induced by the shape controls, as characterized by the position and orientation γ^1 of platform 1. We saw that instantaneously this motion is a translation whenever $\gamma_1^{1,2} = 0$ or a rotation whenever $\gamma_1^{1,2} \neq 0$. We want to find out if, after a period $T = \frac{2\pi}{\omega}$ of the shape controls, there is a net motion $\Delta\gamma^1 \stackrel{\text{def}}{=} \gamma^1(T) - \gamma^1(0)$ of the VGT assembly. This is equivalent to the geometric phase idea of (Krishnaprasad [1990]).

Translation: Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) = 0$.
From (21) and (22) :

$$\begin{aligned} \Delta\gamma_1^1(t) &= \gamma_1^1(t) - \gamma_1^1(0) = 0 \;, \\ \Delta\gamma_2^1(t) &= \gamma_2^1(t) - \gamma_2^1(0) = 0 \;, \\ \Delta\gamma_3^1(t) &= \gamma_3^1(t) - \gamma_3^1(0) = \int_0^t v_3^1(\tau) d\tau = -\alpha_2 \int_0^t \frac{\omega \sin \omega\tau}{\tan(\alpha_3 + \alpha_1 \sin \omega\tau)} d\tau \;, \end{aligned} \tag{28}$$

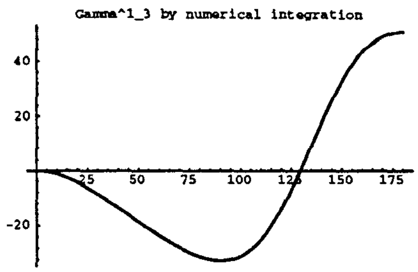


Fig. 3.4

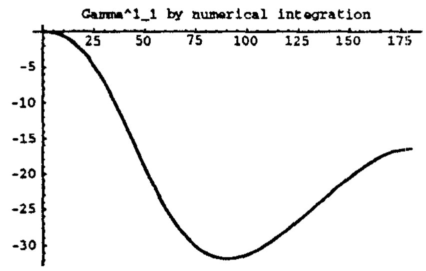


Fig. 3.5

where $\alpha_3 \overset{\text{def}}{=} \gamma_1^{2,3}(0)$.

From (2.15) and (28) we have:

$$
\begin{aligned}
\phi_1 &= \gamma_1^1(0)\,, \\
x_{O_1} &= \gamma_2^1(0)\cos\gamma_1^1(0) - \gamma_3^1 \sin\gamma_1^1(0)\,, \\
y_{O_1} &= \gamma_2^1(0)\sin\gamma_1^1(0) + \gamma_3^1 \cos\gamma_1^1(0)\,.
\end{aligned}
\tag{29}
$$

From this we get:

$$
y_{O_1} = \frac{\gamma_2^1(0)}{\sin\gamma_1^1(0)} - \frac{1}{\tan\gamma_1^1(0)} x_{O_1}\,. \tag{30}
$$

Thus the locus of the point O_1 is the straight line given by equation (30), which is perpendicular to the axis of platform 1. Using Mathematica, we can integrate (28) numerically and verify that after a period of the shape controls, the 2–module VGT assembly has moved forward by a distance specified by $\Delta\gamma_3^1(\frac{2\pi}{\omega}) = \gamma_3^1(\frac{2\pi}{\omega}) - \gamma_3^1(0)$ (fig. 3.4). If we trace the closed shape–space path of (26) in the reverse direction, the assembly will move backwards by the same distance.

Rotation: Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) \neq 0$.

In Prop. 3.2.1 we saw that the instantaneous motion of the 2–module VGT in this case is a rotation around the point $O_{1,2}$. The position of the assembly with respect to this point can be characterized by the angle γ_1^1.

From (2.31) and (27) :

$$\gamma_1^{1,3}(t) = \gamma_1^{1,2} + \gamma_1^{2,3} = \gamma_1^{1,2}(0) + \gamma_1^{2,3}(0) + \alpha_1 \sin \omega t \ ,$$

$$\begin{aligned} \gamma_3^{1,3}(t) &= \gamma_3^{2,3} - \gamma_2^{1,2} \sin \gamma_1^{2,3} + \gamma_3^{1,2} \cos \gamma_1^{2,3} \\ &= -\gamma_2^{2,3}(0) \sin(\alpha_1 \sin \omega t) + \gamma_3^{2,3}(0) \cos(\alpha_1 \sin \omega t) \\ &\qquad - \left[\gamma_2^{1,2}(0) + \alpha_2(1 - \cos \omega t)\right] \sin\left(\gamma_1^{2,3}(0) + \alpha_1 \sin \omega t\right) \\ &\qquad + \gamma_3^{1,2}(0) \cos\left(\gamma_1^{2,3}(0) + \alpha_1 \sin \omega t\right) \ . \end{aligned} \tag{31}$$

Then, from (17) :

$$det(A_2(t)) = -\gamma_3^{1,3}(t) \sin \gamma_1^{1,2}(0) + \gamma_3^{1,2}(0) \sin \gamma_1^{1,3}(t) \ . \tag{32}$$

From (18) :

$$v_1^1(t) = -\sin \gamma_1^{1,2}(0) \frac{v_2^{2,3}(t)}{det(A_2(t))} \tag{33}$$

and from (20) :

$$\Delta\gamma_1^1(t) = \gamma_1^1(t) - \gamma_1^1(0) = -\sin \gamma_1^{1,2}(0) \int_0^t \frac{v_2^{2,3}(\tau)}{det(A_2(\tau))} d\tau \ . \tag{34}$$

Using Mathematica, we can integrate (34) numerically and verify that for e.g. $\gamma_1^{1,2} = -\frac{\pi}{4}$, after a period of the shape controls, the 2–module VGT assembly rotates clockwise around the point $O_{1,2}$ by an angle specified by $\Delta\gamma_1^1(\frac{2\pi}{\omega}) = \gamma_1^1(\frac{2\pi}{\omega}) - \gamma_1^1(0)$ (fig. 3.5). If we trace the closed shape–space path of (26) in the reverse direction, the assembly will rotate counter–clockwise by the same angle.

The 2–module VGT nonholonomic kinematics were simulated on a Silicon Graphics IRIS 4D/120 graphics workstation. The primitive straight line and rotational motions described above are shown in fig. 3.6 and 3.7. Those primitive motions can be synthesized to display more complex behaviors of the system, like obstacle avoidance, as shown in fig. 3.8.

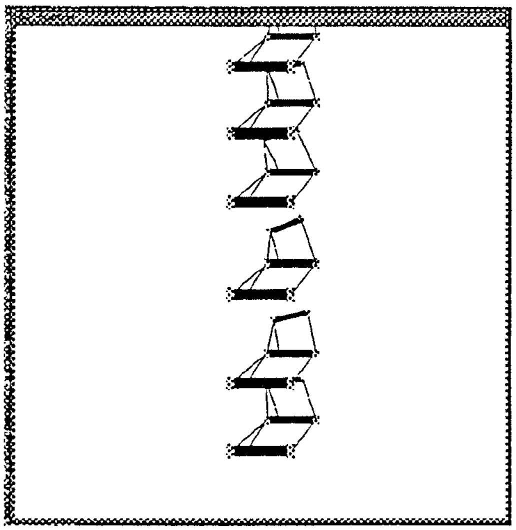


Fig. 3.6

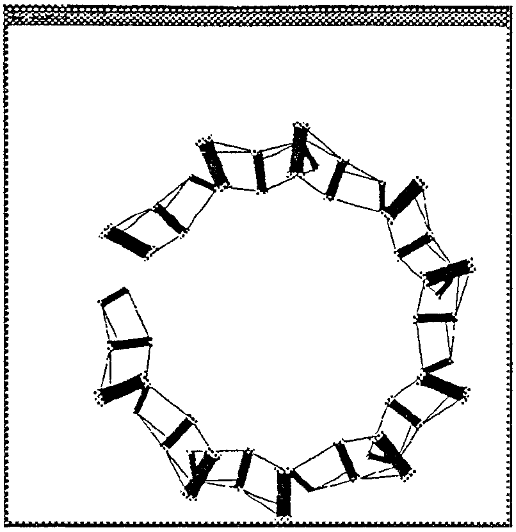


Fig. 3.7

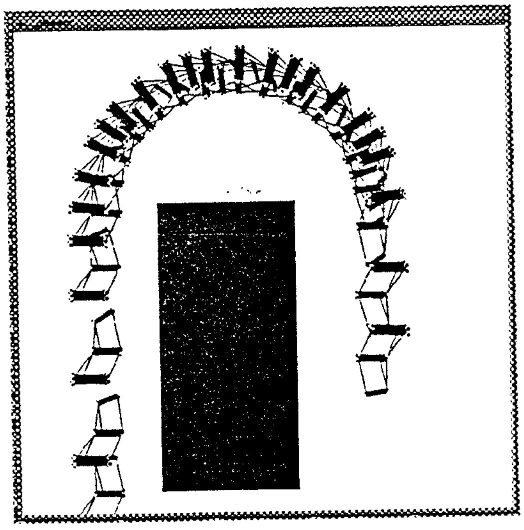


Fig. 3.8

4 Conclusions

In this paper we introduced Variable Geometry Truss assemblies *with nonholonomic constraints*. We derived their kinematics and examined motion planning by showing how periodic shape changes induce global translation or rotation of the assembly under the influence of the nonholonomic constraints. The framework discussed here is an instance of a class of nonholonomic systems that we refer to as *G–snakes*. An outline of this was discussed by one of us at a recent workshop (Krishnaprasad [1993]). Details appear in a forthcoming paper.

Further exploration of motion planning through alternative shape change strategies can be based on the above framework. In (Tsakiris & Krishnaprasad [1993]), configuration space trajectories for parallel manipulators that minimize curvature–squared cost criteria were shown to generalize the usual straight line and circular arc configuration space paths. These can be used to create a richer family of closed shape–space paths than the ones presented in Section 3.2. Moreover, shape changes like those presented in (Chirikjian & Burdick [1991]) can be explored for the ℓ–module VGT architecture (c.f. Remark 3, Section 2.2).

Finally, we can consider integration of this mechanical system with various obstacle avoidance (Latombe [1991]) and sensory information mediation schemes, in order to explore motion in a constrained or dynamic environment.

5 Acknowledgements

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