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Ph.D.

Noisy Precursors for Nonlinear System Instability with Application to Axial Flow Compressors

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ABSTRACT

Title of Dissertation: NOISY PRECURSORS FOR NONLINEAR SYSTEM INSTABILITY WITH APPLICATION TO AXIAL FLOW COMPRESSORS

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This dissertation addresses monitoring of nonlinear systems for detection and prediction of incipient instabilities. The analysis and design presented here rely on the influence of noise on system behavior near the onset of instability. The work is of relevance to high performance engineering systems, which are often operated with a low stability margin in order to maximize performance. In such a stressed operating mode, a small or moderate disturbance can result in loss of stability of the nominal operating condition. This can be followed by operation in a new lower performance mode, oscillatory behavior, or even system collapse. All of these conditions can be viewed as bifurcations in the underlying dynamical models. Prediction of the precise onset points of these instabilities is made difficult by the lack of accurate models for complex engineering systems. Thus,

in this thesis monitoring systems are proposed that can signal an approaching instability before it occurs, without requiring a precise system model.

The approach taken in this work is based on precursors to instability that are features of the power spectral density of a measured output signal. The noise in the system can be naturally occurring noise or can be intentionally injected noise. The output signal can be measured directly from the physical system or from the system with an augmented monitoring system. Design of appropriate augmented monitoring systems is a major topic of this work. These monitoring systems result in enhancing precursor signals and also allow control of the precursor by tuning external parameters. This tuning is important in that it adds confidence to the detection of an impending instability.

The methods developed on precursors for instablility are applied to models of axial flow compression systems. Existing results on bifurcations for such models and their relation to compressor stall provide a starting point for the analysis.

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by

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DEDICATION

To my parents

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Chapter 1

Introduction

In recent years, bifurcation and chaos have been a very active research area in which a good deal of theoretical and applied work has been done. Bifurcation theory has an important role in science, engineering and social science. The appearance of topologically nonequivalent phase portraits under variation of parameters is called a bifurcation [38]. Systems that upon analysis are found to be nonlinear, nonequilibrium, deterministic, dynamic, and that incorporate randomness so that they are sensitive to initial conditions, and have strange attractors are said to be *chaotic* [16].

Due to today's ever-increasing demands for performance, engineering systems are often required to operate very close to the limits of their operating envelopes. Exceeding these limits can result in degradation of performance, oscillatory behavior, a jump to an unacceptable new operating condition, or even system collapse.

Operation near the limits of a system envelope ("*stressed operation*") can still lead to these undesirable effects. This is because even a small or moderate external disturbance can push the system outside its operating envelope. Aircraft stall in supermaneuvers, blackout of a heavily loaded power system, and compressor stall are just a few examples of system failures that can occur in stressed operation.

Conventional linear models are incapable of explaining such catastrophic dynamical behavior. Nonlinear dynamic models can often be employed to explain transitions in system behavior. In particular, bifurcation theory has proved to be quite a useful tool for understanding qualitative changes in the behavior of nonlinear systems. This theory has become an indispensable tool in the analysis of many problems in many technological and scientific problems. In [7], [10], [41], [39], and [43], bifurcation theory has been successfully applied to axial flow compressor system models. It has also been employed to explain the dynamics of voltage collapse in electric power systems in [23], [4], [12], [18], [17], and [55].

The fact that bifurcations are usually considered as undesirable events in applications has motivated research on the control of bifurcations. There are three basic categories of control of bifurcations.

One is the use of a state feedback to relocate (delay) a bifurcation in parameter space. Linear state feedback control is often enough to achieve this. Application of linear state feedback to delay bifurcation in axial flow compressors is given in [50], [51].

Another category of bifurcation control involves modifying the stability of bifurcated solutions. In [5], Abed and Fu devised local feedback laws that ensure that the periodic orbit emerging at a Hopf bifurcation is stable. In [6], they extended this work to stationary bifurcation. These results were successfully applied to axial flow compressors [39], [40], to aircraft stall at high angle-of-attack [8] and to voltage collapse [54]. In [54] and [3], the issue of relocating a bifurcation while at the same time stabilizing the bifurcated solution has been addressed. In [9], the control design result of [1] was extended in a new dynamic feedback structure incorporating a washout filter. The advantage gained by employing washout filters in the feedback loop is the preservation of all system equilibria.

Mehra [45] and Mehra, Kessel, and Carroll [46] investigated the third category of bifurcation control, namely global removal of bifurcations by state feedback. Their results can apply only to stationary bifurcations, since it is obtained by appealing to a global implicit theorem. These publications were the first on any aspect of bifurcation control.

Even with significant control authority, it might not be possible to remove bifurcations altogether. In fact, much work in bifurcation control deals with simply delaying the bifurcation as much as possible and/or stabilizing the bifurcated solution instead of eliminating it. Applying control to delay or eliminate a bifurcation might be costly and unnecessary if the system normally operates far from conditions leading to bifurcation. For such cases, we may want to apply control only when the system approaches bifurcation. In some cases, an accurate model of the system is not available for control design, or physical means of actuation might not be sufficient for controlling a bifurcation. For these cases, it is imperative that the system (or system operator) recognizes during operation that the system is approaching a bifurcation. Once such a condition is detected, appropriate measures can be taken.

These considerations motivate us to develop monitoring techniques that can provide some type of warning signal that the system is close to a bifurcation. Wiesenfeld [57] has shown that the power spectrum of a measured output exhibits distinguishing features near a bifurcation for systems normally operating along a periodic solution. For many engineering systems, the normal operating condition is an an equilibrium point rather than a periodic solution. Thus, we extend this work to systems operating at an equilibrium point. This will form a basis for our design of monitoring systems that exhibit *precursors* of bifurcation for a nonlinear system operating at an equilibrium point.

It should be noted that the term "precursor" has also been used in the literature on stall of axial flow compressors [26], [21] and instabilities in combustion systems [52]. The precursors that have been suggested for these applications differ markedly in spirit to what is sought in this dissertation. In the cited references, the initiation of an instability is detected by examining the time signal as it begins to depart from the nominal operating state. In contrast, in this work we attempt to obtain warning signals for instabilities that provide an indication of nearness to instability as opposed to an indication of the onset of instability. The implications for system operability of having such precursors available are significant.

The remainder of the dissertation proceeds as follows. Some basic theoretical results of the bifurcations and related mathematical tools will be discussed in Chapter 2. The basic local codimension one bifurcations are presented.

In Chapter 3, the effects of noise on systems exhibiting bifurcation are investigated. First, we review the effect of noise on systems operating along a periodic solution and recall why the power spectrum can be used as a useful precursor for bifurcation of a periodic solution. Then, we extend this result to bifurcations from an equilibrium point.

The precursor of Chapter 3 are very useful for detecting Hopf bifurcation, but much less so for detecting stationary bifurcation. Chapter 4 addresses how to make stationary bifurcation more detectable by a power spectrum precursor. A new augmented state is presented as a solution. Theorem that states a new augmented state transformation renders stationary bifurcation into Hopf bifurcation without much knowledge on system dynamics is shown. The effects of a augmented states transformation on a nonlinear system are considered in this chapter. Stability of bifurcated periodic solution due to a transformation, effect of transformation on a system experiencing Hopf bifurcation, and reduced order transformation are also considered. Extension of an augmented state transformation to singularly perturbed systems is investigated. Finally, we suggest a another augmented state transformation to relax some of assumptions which we have made in a previous transformation.

In Chapter 5, we use the axial compression system as a example. Bifurcation analysis of an axial compression system model is performed. The model undergoes both stationary and Hopf bifurcation as a parameter varies. The compression model provides examination of a suggested precursor in a practical engineering system. We perform numerical simulation on axial compressor under mild assumption on the modeling part.

Conclusions and future research directions are given in Chapter 6.

Chapter 2

Preliminaries

In this chapter, we review some basic concepts and results from bifurcation theory, mulitilinear functions, and random process theory which will be used in subsequent chapters. The discussion follows [2], [53] and [28].

2.1 Local Bifurcations

In this section, we recall the statement of well known local bifurcation theorems. The term *bifurcation* refers to qualitative changes in the phase portraits of dynamical systems occurring with slight variation in the system parameters. There are many types of bifurcation. Here, we have particular interest in local bifurcations, i.e. bifurcations in the neighborhood of an equilibrium point. The nominal operating condition of an engineering system can often be taken to be an equilibrium point.

2.1.1 Low-dimensional example

Originally, Poincaré used the term bifurcation to describe the splitting of equilibrium solutions for a family of differential equations. Bifurcations involving only equilibrium points are known as stationary or static bifurcations. There are also bifurcations, such as Hopf bifurcation, which involve both an equilibrium point and a periodic solution.

Consider a system

$$\dot{x} = f(x, \mu) \tag{2.1}$$

where $x \in R^n$ is the system state and $\mu \in R^k$ denotes a k-dimensional parameter; k can be any positive integer. In this work, we limit k to be 1 so that μ is a scalar. The equilibrium solutions are given by the solutions of the equation $f(x, \mu) = 0$. By the implicit function theorem, as μ varies, these equilibria are smooth function of μ as long as $D_x f$, the Jacobian matrix of $f(x, \mu)$ with respect to x evaluated at the equilibrium, does not have a zero eigenvalue. An equilibrium point for a given parameter value is called a "stationary bifurcation point" if two or more equilibria join at that point. A necessary condition for (x_0, μ_0) to be a stationary bifurcation point is that the Jacobian $D_x f$ has a zero eigenvalue for $x = x_0$, $\mu = \mu_0$. Moreover, it is also true that a necessary condition for a local bifurcation any kind to occur is that $D_x f$ have at least one eigenvalue with a zero real part. For example, a pair of purely imaginary eigenvalues typically results in a Hopf bifurcation.

Bifurcations are often classified according to their codimension. The codimension number is the minimum number of parameters that are needed to make the description of the dynamics possible, in the vicinity of the singularity [47]. A rigorous definition of codimension can be found in [27] and [38]. After an appropriate linear coordinate transformation, $D_x f$ can be represented in blockdiagonal form

$$D_x f = \begin{pmatrix} A_c & 0 \\ 0 & A_s \end{pmatrix} \tag{2.2}$$

where A_c is the Jordan block corresponding to the critical eigenvalues (i.e. those with zero real part) and A_s involves the remaining eigenvalues. Bifurcations from an equilibrium of codimension one and two fall into one of the following situations.

Codimension 1 bifurcation:

1. $A_c = 0$, a scalar.

2. A_c is 2×2 and has a pair of pure imaginary eigenvalues.

Codimension 2 bifurcation:

1. A_c is 2×2 and is nondiagonalizable with a double zero eigenvalue, that is

$$A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.3}$$

2. A_c is 2×2 and is diagonalizable with a double zero eigenvalue, that is

$$A_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4}$$

3. A_c is 3×3 and has one zero eigenvalue and one pair of pure imaginary eigenvalue, that is

$$A_c = \begin{pmatrix} 0 & -\omega_c & 0 \\ \omega_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

4. A_c is 4×4 and has two pairs of pure imaginary eigenvalues, that is

$$A_c = \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix} \tag{2.6}$$

By employing what is known as center manifold reduction [38] and a normal form transformation, system (2.1) can be reduced to a lower order simplified system called the normal forms. The normal form preserves the qualitative properties of the solution near bifurcation. Analyzing the dynamics of normal forms yields a qualitative picture of the solution for each type of bifurcation. The normal forms of codimension one bifurcations are summarized as follows:

i) Saddle-node bifurcation: The normal form is given by

$$\dot{x} = \mu - x^2 \tag{2.7}$$

where x is a scalar variable. Equilibrium solutions exist only for $\mu > 0$ and are given by $x = \pm\sqrt{\mu}$. The branch, $x = \sqrt{\mu}$, is stable while the other branch, $x = -\sqrt{\mu}$, is unstable.

ii) Transcritical bifurcation: The normal form is given by

$$\dot{x} = \mu x - x^2 \tag{2.8}$$

where x is a scalar variable. The nominal equilibrium point is the origin for all values of μ. The bifurcated equilibrium solution is $x = \mu$, and exists for both $\mu > 0$ and $\mu < 0$. For $\mu > 0$ (resp. $\mu < 0$), the bifurcated branch is stable (resp. unstable).

iii) Pitchfork bifurcation: The normal form (for the *supercritical* case) is given by

$$\dot{x} = \mu x \pm x^3 \tag{2.9}$$

where x is a scalar variable. Again, the origin exists as the nominal equilibrium point for all value of μ. There are two bifurcated equilibrium branches. For "+" case in (2.9), these branches are $x = \pm\sqrt{\mu}$ for $\mu > 0$, and they are both stable. The opposite is true in the "-" case.

iv) Hopf bifurcation: The normal form is two dimensional and given by

$$\begin{aligned} \dot{x} &= -y + x(\mu \pm (x^2 + y^2)) \\ \dot{y} &= -x + y(\mu \pm (x^2 + y^2)) \end{aligned} \tag{2.10}$$

where x and y are scalar variables. The associated bifurcated solution is a non-constant periodic trajectory. In the "-"case in (2.10), the periodic solution occurs for $\mu > 0$ and is stable, while in the "+" case it exists for $\mu < 0$ and is unstable.

2.1.2 Bifurcation theorems

We have the following codimension one bifurcation theorems for the system (2.1).

Theorem 2.1 *(Stationary Bifurcation Theorem) Suppose $f(x, \mu)$ of system (2.1) is sufficiently smooth with respect to both x and μ, and $f(0, \mu) = 0$ for all μ, and the Jacobian of $f(x, \mu)$, $A_\mu := D_x f(0, \mu)$, possesses a simple eigenvalue $\lambda(\mu)$ such that at the critical parameter value $\mu_c = 0$,*

$$\lambda'(0) := \frac{d\lambda}{d\mu}|_{\mu=0} \neq 0 \tag{2.11}$$

and all the remaining eigenvalues of A_0 have strictly negative real part. Then:

i) there is an $\epsilon_0 > 0$ *and a function*

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + O(\epsilon^3) \tag{2.12}$$

such that if $\mu_1 \neq 0$*, there is a nontrivial equilibrium* $x(\mu)$ *near* $x = 0$ *for each* $\epsilon \in \{[-\epsilon_0, 0) \bigcup (0, \epsilon_0)\}$*; if* $\mu_1 = 0$ *and* $\mu_2 > 0$ *(resp.* $\mu < 0$*), there are two equilibrium points* $x_\pm(\mu)$ *near* $x = 0$ *for each* $\mu \in (0, \epsilon_0]$ *(resp.* $\mu \in [-\epsilon_0, 0)$*)*

ii) Exactly one eigenvalue $\beta(\epsilon)$ *of the Jacobian evaluated with respect to each of the nontrivial equilibrium points in (i) approaches 0 as* $\epsilon \downarrow 0$ *and it is given by a real function*

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + O(\epsilon^3) \tag{2.13}$$

The coefficient β_1 *of this function satisfies* $\beta_1 = -\lambda'(0)\mu_1$*. The nontrivial equilibrium* x_- *(resp.* x_+*) is stable (resp. unstable) if* $\beta_1\epsilon < 0$ *and is unstable (resp. stable) if* $\beta_1\epsilon > 0$*. Nevertheless, the bifurcation point itself is unstable. If* $\beta_1 = 0$*, then* $\beta_2 = -2\lambda'(0)\mu_2$*, and the nontrivial equilibria are asymptotically stable if* $\beta_2 < 0$ *but are unstable if* $\beta_2 > 0$*.*

The assumptions of the theorem above are not generic for the case of a single parameter μ. A result that is generic Theorem 2.2 below, which gives conditions for saddle node bifurcation. In this bifurcation, nominal, stable equilibrium merges with another, unstable equilibrium.

To state the theorem on saddle node bifurcation, we consider system (2.1) where f is sufficiently smooth and $f(0, 0) = 0$. Express the expansion of $f(x, \mu)$ in a Taylor series about $x = 0, \mu = 0$ in the form

$$f(x, \mu) = Ax + b\mu + Q(x, x) + \cdots \tag{2.14}$$

Note that $A = D_x f(0,0)$ is simply the Jacobian matrix of f at the origin for $\mu = 0$.

(SN1) The Jacobian A possesses a simple zero eigenvalue.

(SN2) $lb \neq 0$ and $lQ(r,r) \neq 0$ where r (resp. l) is the right column (resp. row) eigenvector of the Jacobian A associated with the zero eigenvalue, with r and l normalized by setting the first component of r to 1 and then choosing l such that $lr = 1$.

Theorem 2.2 *(Saddle Node Bifurcation Theorem) If (SN1) and (SN2) hold, then there is an* $\epsilon_0 > 0$ *and a function*

$$\mu(\epsilon) = \mu_2 \epsilon^2 + O(\epsilon^3) \tag{2.15}$$

such that $\mu_2 \neq 0$ *and for each* $\epsilon \in (0, \epsilon_0]$*, (2.1) has a nontrivial equilibrium* $x(\epsilon)$ *near* 0 *for* $\mu = \mu(\epsilon)$*. The equilibrium point* $x = 0$ *is unstable at bifurcation, i.e., for* $\mu = 0$*.*

Suppose that instability of the nominal equilibrium $x(\mu_0)$ is the result of a pair of eigenvalues of the system linearization crossing the imaginary axis in the complex plane. Then, as is well known [34], [42], [47], generically it will be the case that a small amplitude periodic orbit of (2.1) emerges from the equilibrium $x_0(\mu)$ at the critical parameter value. We have the following theorem for the case.

Theorem 2.3 *(C^r-Hopf Bifurcation Theorem) [34] Suppose the system (2.1) satisfies the following conditions:*

i) $f_\mu(0) = 0$ *for* μ *in an open interval containing* 0*, and* $0 \in R^n$ *is an isolated equilibrium point of* f*.*

ii) All partial derivatives of the components f_μ^l of the vector f of orders $l \leq r$ $r \geq 4$ (including the partial derivative with respect to μ) exist and are continuous in x and μ in a neighborhood of $(0,0)$ in $R^n \times R^1$ space.

iii) $A_\mu := D_x f(0,\mu)$ has a complex conjugate pair of eigenvalues λ and $\bar{\lambda}$ such that $\lambda(\mu) = \alpha(\mu) + j\omega(\mu)$, where $w_0 := w(0) > 0$, $\alpha(0) = 0$, and

$$\alpha'(0) := \frac{d\alpha}{d\mu}|_{\mu=0} \neq 0 \tag{2.16}$$

iv) The remaining eigenvalues of A_0 have strictly negative real parts.

Then:

i) There exist an $\epsilon_p > 0$ and C^{r-1} function

$$\mu(\epsilon) = \sum_{i=1}^{[\frac{r-2}{2}]} \mu_{2i}\epsilon^{2i} + O(\epsilon^{r-1}) \qquad (0 < \epsilon < \epsilon_p) \tag{2.17}$$

such that for each $\epsilon \in (0, \epsilon_p)$ there exists a nonconstant periodic solution $p_\epsilon(t)$ with period

$$T(\epsilon) = \frac{2\pi}{\omega_0}[1 + \sum_{i=1}^{[\frac{r-2}{2}]} \tau_{2i}\epsilon^{2i}] + O(\epsilon^{r-1}) \qquad (0 < \epsilon < \epsilon_p) \tag{2.18}$$

occurring for $\mu = \mu(\epsilon)$.

ii) There exists a neighborhood η of $x = 0$ and an open interval ϑ containing 0 such that for any $\mu \in \vartheta$, the only nonconstant periodic solutions that lie in η are members of the family $p_\epsilon(t)$.

iii) Exactly two of the Floquet exponents of $p_\epsilon(t)$ approach 0 as $\epsilon \downarrow 0$. One is 0 identically, and the other is a C^{r-1} function

$$\beta(\epsilon) = \sum_{i=1}^{[\frac{r-2}{2}]} \beta_{2i}\epsilon^{2i} + O(\epsilon^{r-1}) \qquad (0 < \epsilon < \epsilon_p) \tag{2.19}$$

The periodic solution $p_\epsilon(t)$ is orbitally asymptotically stable if $\beta(\epsilon) < 0$, and is unstable if $\beta(\epsilon) > 0$. If there is a first nonvanishing coefficient μ_{2k}^p, then the first nonvanishing coefficient in (2.19) is given by

$$\beta_{2k} = -2\lambda'(0)\mu_{2k} \tag{2.20}$$

Moreover, there is then an $\epsilon_1 \in (0, \epsilon_p)$ such that

$$sgn[\beta(\mu)] = sgn[\beta_2] \tag{2.21}$$

for $\mu \in \{\mu | 0 < \mu/\mu_{2k} < \mu(\epsilon_1)/\mu_{2k}\}$. Here, sgn denotes the sign of a real number.

2.1.3 Singularly perturbed Hopf bifurcation

Since we are also going to consider the case of Hopf bifurcation under singular perturbation in this thesis, discussion of a basic theorem on this topic is in order. The next theorem gives conditions that ensure persistence of Hopf bifurcation under singular perturbation of a dynamical system. Let the full system be

$$\begin{aligned} \dot{x} &= f(x, y, \mu, \epsilon) \\ \epsilon\dot{y} &= g(x, y, \mu, \epsilon) \end{aligned} \tag{2.22}$$

where $x \in R^n, y \in R^m, \mu, \epsilon \in R$ and ϵ is small but nonzero. Here, μ is the bifurcation parameter and a ϵ is the singular perturbation parameter. The associated reduced system , obtained by formally setting $\epsilon = 0$ in (2.22), is

$$\begin{aligned} \dot{x} &= f(x, y, \mu, 0) \\ 0 &= g(x, y, \mu, 0) \end{aligned} \tag{2.23}$$

Let $J(\mu, \epsilon)$ denote the Jacobian matrix

$$J(\mu, \epsilon) := \begin{pmatrix} D_x f & D_y f \\ \epsilon^{-1} D_x g & \epsilon^{-1} D_y g \end{pmatrix} \tag{2.24}$$

of the full system evaluated at the equilibrium point $x_0(\mu, \epsilon), y_0(\mu, \epsilon)$ of interest. Introduce the following assumptions

(S1) f, g are $C^r \quad (r \geq 5)$ in x, y, μ, ϵ in a neighborhood of $(x_0, y_0, 0, 0)$.

(S2) det $D_y g(x_0, y_0, 0, 0) \neq 0$.

(S3) The reduced system (2.23) undergoes a Hopf bifurcation from the equilibrium $m_0 = (x_0, y_0)$ for the critical parameter value $\mu = 0$.

(S4) No eigenvalue of $D_2 g(x_0, y_0, 0, 0)$ has zero real part.

Under conditions **(S1)-(S4)**, Abed [1] has proved the following theorem.

Theorem 2.4 *(Persistence Under Singular Perturbation) Let (S1)-(S4) above hold. Then, there is an $\epsilon_0 > 0$ and for each $\epsilon \in (0, \epsilon_0]$ the full system (2.22) undergoes a Hopf bifurcation at an equilibrium $m_0^{\mu_c^\epsilon, \epsilon}$ near m_0 for a critical parameter value μ_c^ϵ near 0. We also have*

$$\lim_{\epsilon \to 0} \frac{d}{d\mu} Re(\lambda_1(\mu_c^\epsilon, \epsilon)) = \alpha'(0) \tag{2.25}$$

where $\lambda_1(\mu, \epsilon), \bar{\lambda}_1(\mu, \epsilon)$ are the complex-conjugate eigenvalues of $J(\mu, \epsilon)$ which cross the imaginary axis for $\mu = \mu_c^\epsilon$, and where $\alpha(\mu)$ is the real part of λ_1.

2.2 Multilinear Functions and the Fredholm Alternative

2.2.1 Multilinear functions

Multivariable Taylor series expansions are used in the derivation of local stability conditions for bifurcations. The notation of multilinear functions is convenient in dealing with multivariable Taylor series. The following are some basic definitions and properties of mulitilinear functions.

Definition 2.1 *Let $V_1, V_2, \ldots, V_k$ and W be vector spaces over the same field. A map $\psi : V_1 \times V_2 \times \cdots \times V_k \mapsto W$ is said to be multilinear (or k-linear) if it is linear in each of its variables. That is ([15], p. 76), for arbitrary $v^i, \tilde{v}^i \in V_i, i = 1, \ldots, k$, and for arbitrary scalars $a, \tilde{a}$, we have*

$$\begin{aligned} \psi(v^1, \cdots, av^i + \tilde{a}\tilde{v}^i, \cdots, v^k) &= a\psi(v^1, \cdots, v^i, \cdots, v^k) \\ &+ \tilde{a}\psi(v^1, \cdots, \tilde{v}^i, \cdots, v^k) \end{aligned} \tag{2.26}$$

We refer to k as the degree of the multilinear function ψ. In particular, multilinear functions of degree two, three, and four are referred as *bilinear, trilinear,* and *tetralinear functions*, respectively.

In the sequel, we shall deal exclusively with mulitilinear functions whose domain is the product space of k identical vector spaces $V_1 = V_1 = \cdots = V_k = V$. For such multilinear functions, we have the following notion of symmetry.

Definition 2.2 *A k-linear function $\psi : V \times V \times \cdots \times V \to W$ is symmetric if, for any $v^i \in V, i = 1, \ldots, k$ the vector*

$$\psi(v^1, v^2, \cdots, v^k) \tag{2.27}$$

is invariant under arbitrary permutations of the argument vectors v^i.

With an arbitrary multilinear function ψ, we associate a symmetric multilinear function ψ_s by the *symmetrization operation* ([15], pp. 88-89). Given a multilinear function $\psi(x^1, x^2, \cdots, x^k)$, define a new (symmetric) multilinear function ψ_s as follows:

$$\psi_s(x^1, x^2, \cdots, x^k) := \frac{1}{k!} \sum_{i_1, i_2, \cdots, i_k} \psi(x^{i_1}, x^{i_2}, \cdots, x^{i_k}) \tag{2.28}$$

where the sum is taken over the $k!$ permutation of the integer $1, 2, \ldots, k$.

An important property of homogeneous functions represented in terms of multilinear fictions is given next.

Proposition 2.1 *Let $\psi : (R^m)^k \to R^m$ be a symmetric k-linear function. For any vector $v \in R^n$,*

$$D\psi(\eta, \eta, \cdots, \eta) \cdot v = k\psi(\eta, \eta, \cdots, \eta, v) \tag{2.29}$$

2.2.2 The Fredholm Alternative

Next, we consider the Fredholm Alternative. Consider the system of linear equations

$$Ax = b \tag{2.30}$$

where A is a real $n \times n$ matrix and $b \in R^n$. If the coefficient matrix A is nonsingular, then (2.30) has a unique solution, given by $A^{-1}b$. Existence of solutions for cases in which A is singular is the subject of the Fredholm Alternative. Below, we first present this result for the general case of a singular coefficient matrix A, and then employ it for the particular case in which A has a simple zero eigenvalue.

Theorem 2.5 *Let $N(A)$ be k-dimensional, with basis $r^1, \cdots, r^k$, and dual basis $l^1, \cdots, l^k$. Then (2.30) has at least one solution in R^n if and only if $l^i b = 0$ for $i = 1, \ldots, k$. Moreover, in such a case, the general solution of (2.30) has the representation*

$$x = x^0 + \sum_{i=1}^{k} \alpha_i r^i \tag{2.31}$$

where x^0 is any particular solution of (2.30) and the α_i are arbitrary real scalar.

Suppose now that A has a simple zero eigenvalue. Let r and l denote right and left eigenvectors of A, respectively, corresponding to the zero eigenvalue, and require that these be chosen to satisfy $lr = 1$. Under these conditions, the Fredholm Alternative asserts that (2.30) has a solution if and only if $lb = 0$. Moreover, the Fredholm Alternative also implies that, if (2.30) has a solution x^0, then the totality of solutions is given by the one-parameter family $x = x^0 + \alpha r$ where $\alpha \in R$ is arbitrary. The solution is rendered unique upon imposing a normalization condition which specifies the value of lx.

Introduce subspaces $E^c, E^s \in R^n$ as follows: E^c is the one-dimensional subspace

$$E^c := span\{r\} \tag{2.32}$$

and E^s is the $(n-1)$-dimensional subspace

$$E^s := \{x \in R^n | lx = 0\} \tag{2.33}$$

From the foregoing, we have in particular that if $lb = 0$ then system $Ax = b,\ lx = 0$ has a unique solution. Equivalently, (2.30) has a unique solution in E^s for any vector $b \in E^s$. This proves that the restriction $A|_{E^s}$ of the linear map defines an invertible (one-to-one and onto) map. In the next result, we exhibit the unique solution which lies in E^s of the system $Ax = 0, lx = 0$.

Proposition 2.2 *Suppose A has a simple zero eigenvalue. Then the unique solution of $Ax = b, lx = 0$ given that $lb = 0$ is*

$$x = (A^T A + l^T l)^{-1} A^T b \tag{2.34}$$

Therefore, the inverse of the restricted map $A|_{E^s}$ exists and is given by

$$(A|_{E^s})^{-1} = (A^T A + l^T l)^{-1} A^T \tag{2.35}$$

2.3 Stability of Bifurcated Solutions

Abed and Fu [5],[6] derived formulas for determining the local stability of bifurcated solutions. We review their method here since we will employ it later to study the stability of the bifurcated solution of a system system monitored to detect possible nearness to bifurcation.

Consider a one-parameter family of nonlinear autonomous systems

$$\dot{x} = f(x, \mu) \tag{2.36}$$

where $x \in R^n$ is the vector state and μ is real-valued parameter. Let $f(x, \mu)$ be sufficiently smooth in x and μ and let x_μ^0 be the nominal equilibrium point of the system as a function of the parameter μ.

2.3.1 Stability calculation for stationary bifurcation

First, we consider the case of stationary bifurcation. The next hypothesis is in force throughout this section.

(S) The Jacobian matrix of system (2.36) at the equilibrium x_μ^0 has a simple zero eigenvalue $\lambda_1(\mu)$ with $\lambda_1'(0) \neq 0$, and the remaining eigenvalues lie in the open left half of the complex plane for $\mu_c = 0$.

Theorem 2.1 asserts that hypothesis (S) implies a stationary bifurcation from x_0^0 at $\mu = 0$ for (2.36). That is, a new equilibrium point bifurcates from x_0^0 at $\mu = 0$. Recall that near the point $(x_0^0, 0)$ of the $(n+1)$-dimensional (x, μ)-space, there exists a parameter ϵ and a locally unique curve of critical points $(x(\epsilon), \mu(\epsilon))$, distinct from x_μ^0 and passing through $(x_0^0, 0)$, such that for all sufficient small $|\epsilon|$, $x(\epsilon)$ is an equilibrium point of (2.36) when $\mu = \mu(\epsilon)$.

The parameter ϵ may be chosen such that $x(\epsilon), \mu(\epsilon)$ are smooth. The series expansions of $x(\epsilon), \mu(\epsilon)$ can be written as

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + \cdots \tag{2.37}$$

$$x(\epsilon) = x_\mu^0 + x_1\epsilon + x_2\epsilon^2 + \cdots \tag{2.38}$$

If $\mu_1 \neq 0$, the system undergoes a transcritical bifurcation from x_μ^0 at $\mu = 0$. That is, there is a second equilibrium point besides x_μ^0 for both positive and negative values of μ with $|\mu|$ small. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the system undergoes a pitchfork bifurcation for $|\mu|$ sufficiently small. That is, there are two new equilibrium points existing simultaneously, either for positive or for negative values of μ with $|\mu|$ small. The new equilibrium points have an eigenvalue $\beta(\epsilon)$ which vanishes at $\mu = 0$. The series expansion $\beta(\epsilon)$ is given by

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \cdots \tag{2.39}$$

with

$$\beta_1 = -\mu_1\lambda'(0) \tag{2.40}$$

and, in case $\beta_1 = 0$, β_2 is given by

$$\beta_2 = -2\mu_2\lambda'(0) \tag{2.41}$$

The stability coefficients β_1 and β_2 can be determined solely by eigenvector computations and the coefficients of the series expansion of the vector field. System (2.36) can be written in the series form

$$\begin{aligned}\dot{\tilde{x}} &= L_\mu \tilde{x} + Q_\mu(\tilde{x}, \tilde{x}) + C_\mu(\tilde{x}, \tilde{x}, \tilde{x}) + \cdots \\ &= L_0 \tilde{x} + \mu L_1 \tilde{x} + \mu^2 L_2 \tilde{x} + \cdots \\ &\quad + Q_0(\tilde{x}, \tilde{x}) + \mu Q_1(\tilde{x}, \tilde{x}) + \cdots \\ &\quad + C_0(\tilde{x}, \tilde{x}, \tilde{x}) + \cdots \end{aligned} \tag{2.42}$$

where $\tilde{x} = x - x_0^0, L_\mu, L_1, L_2$ are $n \times n$ matrices, $Q_\mu(x,x), Q_0(x,x), Q_1(x,x)$ are vector-valued quadratic forms generated by symmetric bilinear forms, and $C_\mu(x,x,x), C_0(x,x,x)$ are vector-valued cubic forms generated by symmetric trilinear forms.

By assumption, the Jacobian matrix L_0 has only one simple zero eigenvalue with the remaining eigenvalues stable. Denote by l and r the left (row) and right (column) eigenvectors of the matrix L_0 associated with the simple zero eigenvalue, respectively, where first component of r is set to be 1 and the left eigenvector l is chosen such that $lr = 1$. It is well known that

$$\lambda'(0) = l L_1 r \tag{2.43}$$

A stability criterion for the bifurcated equilibria of system (2.42) is given in the following two lemmas.

Lemma 2.1 *The two bifurcated equilibrium points of (2.42) near x_μ^0 for μ near 0, which appear only for $\mu > 0$ (resp. $\mu < 0$) when $lL_1r > 0$ (resp. $lL_1r < 0$), are asymptotically stable if $\beta_1 = 0$ and $\beta < 0$, and unstable if $\beta_1 = 0$ and $\beta > 0$.*

Here,

$$\beta_1 = lQ_0(r,r) \tag{2.44}$$

$$\beta_2 = 2l\{2Q_0(r,x_2) + C_0(r,r,r)\} \tag{2.45}$$

with x_2 satisfies the following equation:

$$L_0 x_2 = -Q_0(r,r) \tag{2.46}$$

Lemma 2.2 *Suppose the value of β_1 given in (2.44) is negative. Then the bifurcated solution occurring for for $\mu > 0$ (resp. $\mu < 0$) is asymptotically stable when $lL_1 r > 0$ (resp. $lL_1 r < 0$).*

The criterion given in Lemma 2.1 corresponds to pitchfork bifurcation, with the one in Lemma 2.2 is for transcritical bifurcation.

2.3.2 Stability calculation for Hopf bifurcation

Now consider system (2.36) under for the following hypothesis, which implies occurrence of Hopf bifurcation

(H) The Jacobian matrix of system (2.36) at the equilibrium x^0_μ has a pair of pure imaginary eigenvalues $\lambda_1(\mu_c) = i\omega_c$ and $\bar{\lambda_1}(\mu_c) = -i\omega_c$ with $\omega_c \neq 0$, the transversality condition $\frac{\partial Re[\lambda(\mu_c)]}{\partial \mu} \neq 0$ is satisfied, and all the remaining eigenvalues lie in the open left half complex plane.

Theorem 2.3, the Hopf bifurcation theorem asserts the existence of a one-parameter family $p_\epsilon, 0 < \epsilon \leq \epsilon_0$ of nonconstant periodic solutions of system

(2.36) emerging from $x = x^0_{\mu_c}$ at the parameter value μ_c for sufficiently small $|\mu - \mu_c|$. Exactly one of the characteristic exponents of p_ϵ governs the asymptotic stability and is given by a real, smooth and even function

$$\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots \tag{2.47}$$

Specifically, p_ϵ is orbitally stable if $\beta(\epsilon) < 0$ but is unstable if $\beta(\epsilon) > 0$. Generically the local stability of the bifurcated periodic solution p_ϵ is decided by the sign of the coefficient β_2. Note the sign of β_2 also determines the stability of the critical equilibrium point $x^0_{\mu_c}$. An algorithm for computing the stability coefficient β_2 is given as follows:

Step1 Express (2.36) in the Taylor series form (2.42). Let r be the right eigenvector of L_0 corresponding to eigenvalue $i\omega_c$ with the first component of r equal to 1. Let l be the left eigenvector of L_0 corresponding to the eigenvalue $i\omega_c$, normalized such that $lr = 1$.

Step2 Solve the equations

$$L_0 a = -\frac{1}{2}Q_0(r, \bar{r}) \tag{2.48}$$

$$(2i\omega_c I - L_0)b = \frac{1}{2}Q_0(r, r) \tag{2.49}$$

for a and b.

Step 3 The stability coefficient β_2 is given by

$$\beta_2 = 2Re[2lQ_0(r, a) + lQ_0(\bar{r}, b) + \frac{3}{4}lC(r, r, \bar{r})] \tag{2.50}$$

2.4 Washout Filters in Nonlinear Control

Washout filters are used commonly in control systems for power systems and aircraft. The main advantage of using these filters is the resulting robustness of the system operating point to model uncertainty and to other control actions which may be used. In this section, we give a brief discussion of these filters, their use in control of parameterized systems and especially system with bifurcation.

A washout filter (or washout circuit) is a stable high pass filter with transfer function [25]

$$\begin{aligned} G(s) &= \frac{y(s)}{x(s)} = \frac{s}{s+a} \\ &= 1 - \frac{a}{s+a} \end{aligned} \tag{2.51}$$

Here, $a > 0$ is the inverse of the filter time constant. Letting

$$z(s) := \frac{1}{s+a} x(s) \tag{2.52}$$

the dynamics of the filter can be written as

$$\dot{z} = x - az \tag{2.53}$$

and

$$y = x - az \tag{2.54}$$

Feedback through washout filters results in equilibrium preservation in the presence of system uncertainties and other control mechanisms that they inherently offer. Indeed, the most striking property of a system controlled through a washout filter is that all the original equilibria are preserved. Equilibrium points represent, in some sense, a system's capability to perform in a certain manner at steady state. There are situations in which such a capability should not be altered by the introduced control strategy, such as in the lateral control design for an aircraft.

Washout filters reject steady-state inputs, while passing transient inputs. At steady-state,

$$z = \frac{x}{a} \tag{2.55}$$

the output $y = 0$, and the steady-state input signal has been washed out.

Consider a system

$$\dot{x} = f(x, u) \tag{2.56}$$

with

$$f(x_0, 0) = 0 \tag{2.57}$$

where u is the control input and x_0 is an equilibrium point for the system with zero input. Let the control input u be a function of y (denote it $u = h(y)$), and let h satisfy

$$h(0) = 0 \tag{2.58}$$

From (2.53)-(2.54), it is clear that y vanishes in steady state. Hence

$$f(x_0, h(y_0)) = f(x_0, 0) = 0 \tag{2.59}$$

and x_0 remains an equilibrium point of the closed-loop system. This shows that by incorporating a washout filter in the feedback, the equilibrium points of the original system are preserved.

2.5 Background on the Power Spectral Density

In this section, we review the definition of the power spectral density of a random process and discuss related computational aspects. We begin by recalling basic definitions from the theory of random processes. The definitions given next follow Gray and Davisson [28], and the discussion of computation of the power spectral density follows Jenkins and Watts [37].

A random process or stochastic process is an indexed family of random variables $\{X_t : t \in I\}$ defined on a common probability space (Ω, F, P) where Ω is sample space, F is a sigma field of a sample space Ω, and P is a probability measure on a measurable space (Ω, F). The process said to be discrete-time if I is discrete and continuous-time if the index set I is continuous.

Given a random process $\{X_t : t \in I\}$, the autocorrelation function $R_X(t,s) : t, s \in I$ is defined by

$$R_X(t,s) = < X_t X_s > \tag{2.60}$$

The autocovariance function $K_X(t,s) : t, s \in I$ is defined by

$$\begin{aligned} K_X(t,s) &= cov(X_t, X_s) \\ &= R_X(t,s) - (< X_t >< X_s >) \end{aligned} \tag{2.61}$$

where *cov* denotes covariance. Note that the autocorrelation and autocovariance functions are equal if the process has zero mean, that is, if $< X_t >= 0$ for all t.

The random process is said to be *weakly stationary* (or *wide sense stationary*) if

$$< X_t >=< X_s > \tag{2.62}$$

for all $t, s \in I$ and

$$R_X(t, t+\tau) = R_x(\tau) \tag{2.63}$$

for all t, τ such that $t, t+\tau \in I$.

For a given weakly stationary random process $\{X_t\}$ with autocovariance function $K_X(\tau)$, the *power spectral density* $S_X(\eta)$ is defined as the Fourier transform of the autocovariance function, that is,

$$S_X(\eta) = \begin{cases} \sum_k K_X(k)e^{-j2\pi\eta}, & \text{in discrete time} \\ \int K_X(\tau)e^{-j2\pi\eta\tau}d\tau, & \text{in continuous time} \end{cases} \tag{2.64}$$

A weakly stationary random process $\{X_t\}$ is said to be *white* if its power spectral density is a constant for all f, that is

$$S_X(\eta) = \sigma_X^2 \tag{2.65}$$

for all f.

Let $\{X_t\}$ be a weakly stationary random process, and suppose that X_t is the input signal to a linear time-invariant system with transfer function $H(\eta)$ which represents a frequency response of a linear time-invariant system to sinusoidal signal input. If $\{X_t\}$ has spectral density $S_X(\eta)$, then the output random process $\{Y_t\}$ has spectral density

$$S_Y(\eta) = |H(\eta)|^2 S_X(\eta) \tag{2.66}$$

To estimate the power spectrum in an experimental or computer simulation setting, there are two important factors to consider [37]. First, the sampling rate should be chosen such that the estimated power spectrum is valid within the frequency range of interest $0 \leq \eta \leq \eta_0$. Second, the length of the record that is needed to detect a peak in the power spectrum of width w is inversely proportional to w. This means that a longer time history is needed for detecting a peak of smaller width. For a fixed length of the time record, there are techniques for improving the resolution [37, pp. 239-248]. The power spectral densities given in the examples later in the dissertation are computed by simulation using the Simulink package. In each case, the results were verified by checking that variation of the record length and sampling rate did not influence the calculated spectrum.

Chapter 3

Noisy Precursors for Nonlinear System Instabilities

In this chapter, we first summarize the work of Wiesenfeld [57],[58],[60] on properties of nonlinear systems with noise inputs in the vicinity of bifurcation, and then extend these results both in scope and in concept.

The original motivation for Wiesenfeld's work was the determination of the precise value of bias current in a nonlinear circuit for which the onset of a period doubling instability occurs. The power spectrum of a system variable was measured. Near a period doubling bifurcation, the power spectrum would show a new peak at half the fundamental frequency of the nominal periodic solution when period doubling bifurcation occurred. However, substantial broad-band noise also centered at half the fundamental frequency made precise determination of the onset of instability impossible. This observation motivated the development of theory which could be applied to all codimension one bifurcations of a periodic solution.

Wiesenfeld showed that the power spectrum of a measured output for a system operating at a periodic steady state and forced by small white Gaussian noise perturbations exhibits a sharply growing peak near the fundamental frequency as the system nears a bifurcation. Then the suggested analysis was successfully applied in signal amplification area such as a resonantly driven silicon $p-n$ junction [36] and a modulated semiconductor laser [59].

Based on this observation, we will introduce the measured output power spectrum as a precursor that could predict proximity to bifurcation.

3.1 Noisy Precursor for System Operating at Limit Cycle

In this section,we will briefly review work of Wiesenfeld on precursors for bifurcation for system operating at a limit cycle. We do not give the detail derivation, since a similar derivation will be used in next section. Consider the system

$$\dot{x} = f(x, \mu) \tag{3.1}$$

where $x \in R^n$ and μ is a scalar bifurcation parameter. Suppose the system has an asymptotically stable $T-$periodic solution x_p for that changes with μ (T can depend on μ). Thus,

$$x_p(t+T) = x_p(t) \tag{3.2}$$

For small perturbations, the dynamical system behavior can be described by the linearized equation of (3.1) about x_p. Suppose that the noise enters as an additive forcing term. The linearized system with noise forcing term becomes

$$\dot{\tilde{x}} = Df(x_p)\dot{\tilde{x}} + n(t) \tag{3.3}$$

where $\tilde{x} := x - x_p$ and $n(t) \in R^n$ is a vector white Gaussian noise with mean zero and correlation

$$\langle n_i(t) n_j(t+\tau) \rangle = \psi_{ij} \delta(\tau) \tag{3.4}$$

where the ψ_{ij} are finite constants for all i, j. Then, we can solve equation (3.3) by using the fundamental matrix and Floquet theory. A detailed derivation is not presented here since it is analogous to the derivation of the measured output power spectrum for nonlinear systems operating on equilibrium point which is presented in next section.

Using this solution and assuming that the real parts of the non critical Floquet exponents of the fundamental matrix are significantly larger (in magnitude) than real parts of the critical Floquet exponents for a system near bifurcation, Wiesenfeld [57] approximated the solution of equation (3.3) in terms of only the imaginary axis crossing Floquet exponents.

Finally, Wiesenfeld [57] studied the power spectrum of the approximate solution, focusing on changes that might occur near a bifurcation. Moreover, he showed that different types of bifurcation correspond to different power spectrum change.

Figure 3.1 from [57] shows the expected effects in the power spectrum for each type of codimension one bifurcation. This figure concerns only bifurcation from a limit cycle, not from an equilibrium point. Note that the power spectrum for saddle-node or transcritical bifurcation does not exhibit new peaks, but rather has an increasing bump around the fundamental frequency (defined $\frac{2\pi}{T}$ where T is period). Here, fundamental frequency is assumed to be 1 except pitchfork bifurcation that has 0.5 fundamental frequency. The other bifurcations such as pitchfork, Hopf and period doubling bifurcation show the increasing power spectrum peaks at new locations.

3.2 Noisy Precursor for System Operating at an Equilibrium Point

In this section, we consider the effect of noise on systems operating at an equilibrium point under condition that could give rise to a bifurcation. In the foregoing section, we discussed the effect of noise in the analogous situation of a system operating at a periodic solution. It was suggested that the power spectrum could be used to indicate the closeness to bifurcation. Moreover, we saw that the power spectrum could also be used to discern the bifurcation type such as saddle-node, pitchfork, and Hopf bifurcation. The conclusions motivate our work this section, where we study be the effect of noise on the power spectrum of system outputs when an operating condition is near bifurcation. We follow closely the method used in [57].

Consider the autonomous dynamic system

$$\dot{\tilde{x}} = f(\tilde{x}, \mu) \tag{3.5}$$

where $\tilde{x} \in R^n$ and μ is a bifurcation parameter. For small perturbations or noise, the dynamical behavior of the system can be described by the linearized equation near the equilibrium point x_0. The linearized system corresponding to (3.5) with a small noise forcing $N(t)$ is given by

$$\dot{x} = Df(x_0, \mu)x + N(t) \tag{3.6}$$

where $x = \tilde{x} - x_0$ and $N(t) \in R^n$ is a vector white Gaussian noise having zero mean. For the results of the linearized analysis to have any bearing on the

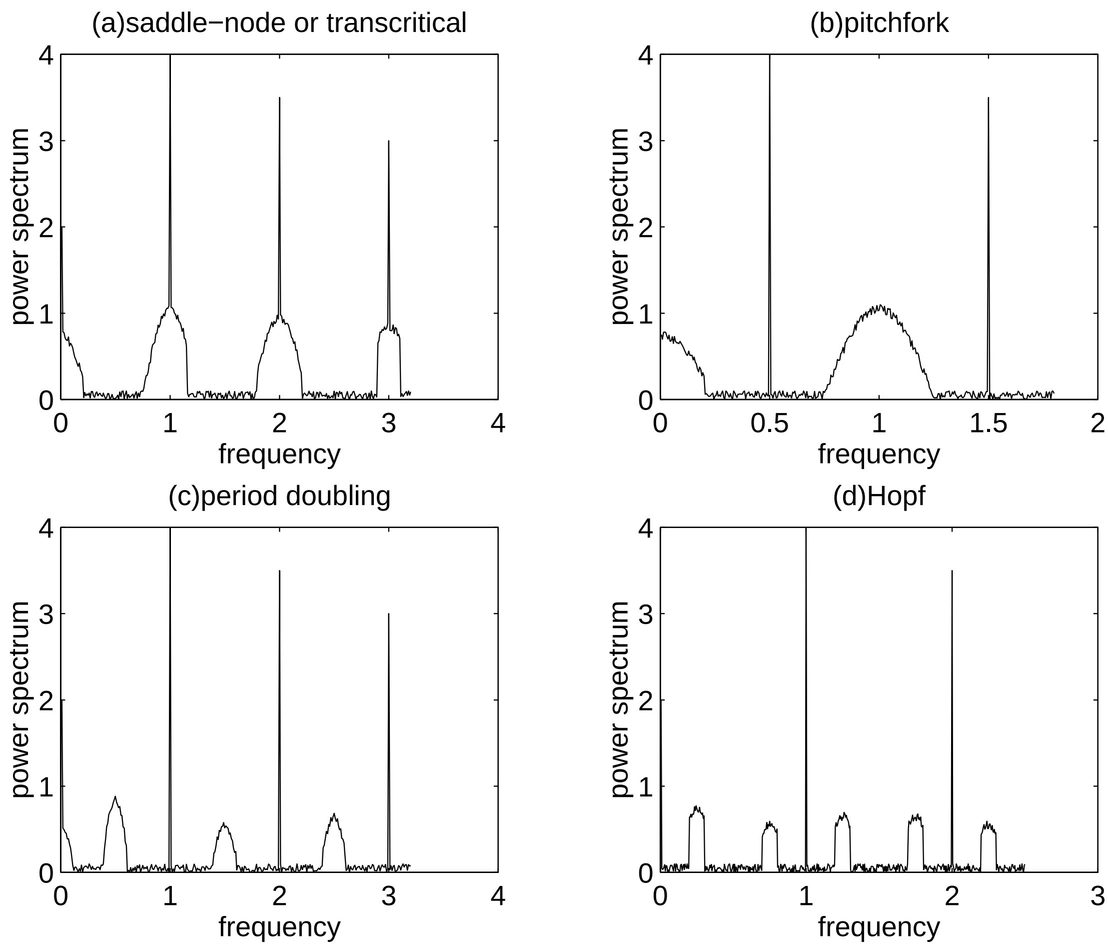


Figure 3.1: Power spectrum for system operating at a limit cycle near bifurcation

original nonlinear model, we must assume that the noise is of small amplitude. This assumption of small noise will be made clear below, in terms of smallness of correlation and cross-correlation coefficients.

The noise $N(t)$ can occur naturally or can be injected using available controls. To facilitate consideration of cases in which the noise is intentionally injected, we write $N(t)$ in the general form

$$N(t) = Bn(t) \tag{3.7}$$

where $B \in R^{n \times m}$ and $n(t) \in R^m$ is a vector white Gaussian noise. This notation allows us to easily consider cases in which noise is injected in different equations through available actuation means. The noise vector $N(t)$ has zero mean as long as $n(t)$ has zero mean.

We view the system (3.6) as being in steady state and driven only by the noise process. Thus, we solve for the evolution of the state assuming a zero initial condition. The solution of equation (3.6) with a zero initial condition is

$$x(t) = e^{At} \int_0^t e^{-As} N(s) ds \tag{3.8}$$

where $A := DF(x_0)$. For our analysis, we assume that x_0 is an asymptotically stable equilibrium point, i.e., all the eigenvalues of A have negative real part. We can express (3.8) in terms of the eigenvectors and eigenvalues of A:

$$\begin{aligned} x(t) &= \sum_{j=1}^{n} l^j \int_0^t \sum_{k=1}^{n} l^k N(s) e^{\lambda_k s} r^k ds e^{\lambda_j t} r^j \\ l^i r^j &= \delta_{ij} \end{aligned}$$

where r^i and l^i are right and left eigenvectors of A corresponding to eigenvalue

λ_i, respectively and where δ_{ij} is the Kronecker delta,

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{3.9}$$

Thus, the i-th component of $x(t)$ is given by

$$x_i(t) = \sum_{j=1}^{n} e^{\lambda_j t} r_i^j \int_0^t \sum_{k=1}^{n} l^k N(s) e^{-\lambda_k s} r^k ds \tag{3.10}$$

Since the power spectrum is the Fourier transformation of the auto-covariance function, we calculate auto-covariance for $x_i(t)$.

$$\begin{aligned} \langle x_i(t) x_i(t+\tau) \rangle &= \sum_{j=1}^{n} \sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^{t+\tau} \int_0^t e^{-\lambda_j s_1} e^{-\lambda_k s_2} \\ &\cdot \sum_{o=1}^{n} \sum_{p=1}^{n} l_o^j l_p^k \langle N_o(s_1) N_p(s_2) \rangle ds_1 ds_2 \end{aligned}$$

Let the noise have correlation characteristic

$$\langle N_i(t) N_j(t+\tau) \rangle = \nu_{ij} \delta(\tau) \tag{3.11}$$

where $\delta(\cdot)$ is the Dirac delta function and the ν_{ij} are constants for all i, j. Moreover, ν_{ij} should be small enough such that linearized analysis is valid. Again, (3.11) is satisfied for the linear affine input system as long as $n(t)$ satisfies

$$\langle n_i(t) n_j(t+\tau) \rangle = \psi_{ij} \delta(\tau) \tag{3.12}$$

where the ψ_{ij} are constants for all i, j. Then

$$\begin{aligned} \langle x_i(t) x_i(t+\tau) \rangle &= \sum_{j=1}^{n} \sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^{t+\tau} \int_0^t e^{-\lambda_j s_1} e^{-\lambda_k s_2} \\ &\cdot \sum_{o=1}^{n} \sum_{p=1}^{n} l_o^j l_p^k \nu_{op} \delta(s_1 - s_2) ds_1 ds_2 \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} e^{\lambda_j t} e^{\lambda_k (t+\tau)} r_i^j r_i^k \int_0^t e^{-\lambda_j s} e^{-\lambda_k s} \\ &\cdot \sum_{o=1}^{n} \sum_{p=1}^{n} l_o^j l_p^k \nu_{op} ds \end{aligned} \tag{3.13}$$

For a dynamic system which depends on a single parameter, there are two types of typical bifurcation from a nominal equilibrium point. One is stationary bifurcation in which case a new equilibrium emerges or the original equilibrium point suddenly disappears at the bifurcation. The other is Hopf bifurcation, where a periodic orbit emerges from the equilibrium point at bifurcation. For stationary bifurcation, a real eigenvalue of the linearized system becomes zero as the parameter varies. For Hopf bifurcation, a complex conjugate pair of eigenvalues crosses the imaginary axis.

Consider the Hopf bifurcation case first. Assume that a complex conjugate pair of eigenvalues (denote them as $\lambda \equiv \lambda_1, \bar{\lambda} \equiv \lambda_2$) close to the imaginary axis has relatively smaller negative real part in absolute value compared to other system eigenvalues:

$$|Re(\lambda_1)|, |Re(\lambda_2)| \ll |Re(\lambda_i)| \tag{3.14}$$

for $i = 3, \ldots, n$. Since the integrand in (3.13) is the product of decaying exponentials (due to the asymptotic stability assumption) and a bounded value, terms involving λ_1 and λ_2 dominate (3.13) for large t:

$$\begin{aligned}
\langle x_i(t)x_i(t+\tau)\rangle \;\approx\; & e^{\lambda_1(2t+\tau)}(r_i^1)^2 \int_0^t e^{-2\lambda_1 s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{ij}\, ds \\
+\; & e^{\lambda_2(2t+\tau)}(r_i^2)^2 \int_0^t e^{-2\lambda_2 s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^2 l_k^2 \nu_{ij}\, ds \\
+\; & e^{\lambda_1(t+\tau)+\lambda_2 t} r_i^1 r_i^2 \int_0^t e^{-(\lambda_1+\lambda_2)s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{ij}\, ds \\
+\; & e^{\lambda_2(t+\tau)+\lambda_1 t} r_i^1 r_i^2 \int_0^t e^{-(\lambda_1+\lambda_2)s} \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{ij}\, ds
\end{aligned}$$

The power spectrum is measured with the use of a spectrum analyzer and most practical spectrum analyzers perform both an ensemble average and a time average. Thus, the final auto-covariance function is

$$
\begin{aligned}
C_{ii}(\tau) &\equiv Re[\overline{\langle x_i(t)x_i(t+\tau)\rangle}^t] && (3.15)\\
&= \Xi \cdot \frac{2e^{-\epsilon\tau}}{\epsilon}\cos(\omega\tau) + \Upsilon \cdot [\frac{e^{-\epsilon\tau}(\epsilon\cos(\omega\tau) - \omega\sin(\omega\tau))}{2(\epsilon^2+\omega^2)}] && (3.16)
\end{aligned}
$$

where $\overline{<\cdot>}^t$ denotes the mean over time t, and we replaced $\lambda_1 = -\epsilon + i\omega$ and $\lambda_2 = -\epsilon - i\omega$ with $\epsilon, \omega > 0$. Also, Ξ and Υ are

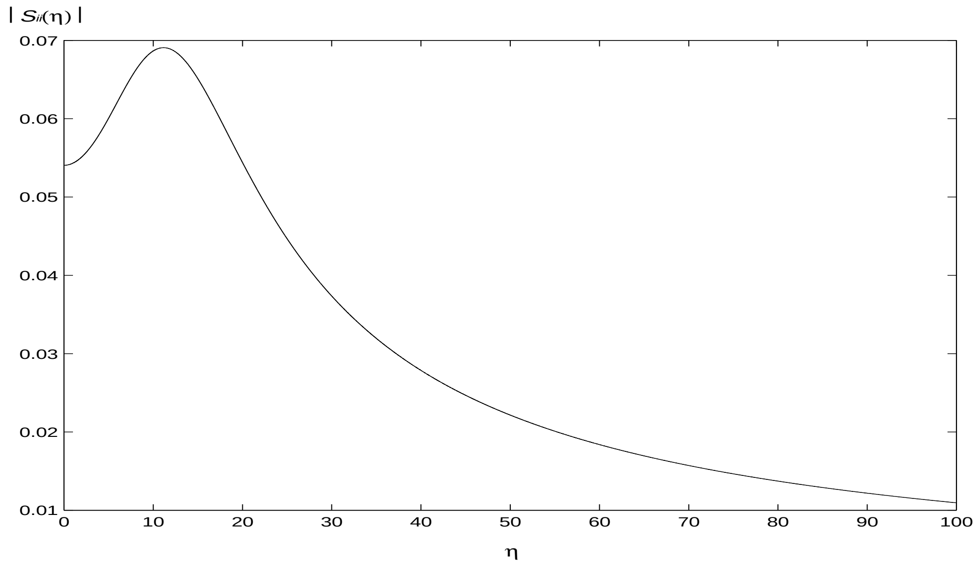
$$
\begin{aligned}
\Xi &:= \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^2 \nu_{jk} r_i^1 r_i^2 \\
\Upsilon &:= \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^1 l_k^1 \nu_{jk} (r_i^1)^2 + \sum_{j=1}^{n}\sum_{k=1}^{n} l_j^2 l_k^2 \nu_{jk} (r_i^2)^2
\end{aligned}
$$

Finally, taking the Fourier transform of equation (3.16) yields the desired power spectrum:

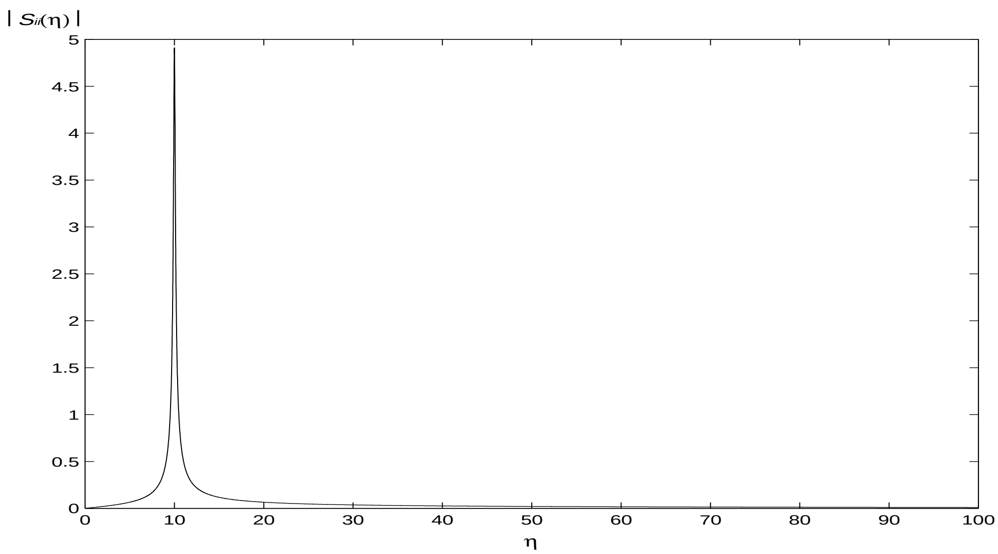
$$
\begin{aligned}
S_{ii}(\eta) &= \Xi\frac{(j\eta+\epsilon)}{(j\eta+\epsilon)^2+\omega^2} \\
&+ \Upsilon[\frac{\epsilon(j\eta+\epsilon)}{(\epsilon^2+\omega^2)((j\eta+\epsilon)^2+\omega^2)} - \frac{1}{(\epsilon^2+\omega^2)((j\eta+\epsilon)^2+\omega^2)}] && (3.17)
\end{aligned}
$$

The magnitude of $S_{ii}(\eta)$ is maximum at $\eta = \omega$ and the maximum grows without bound as $\epsilon \to 0$. Moreover, as noise power (as measured by the ν_{ij}) increases, the magnitude of $S_{ii}(\eta)$ also increases. However, since Ξ and Υ affect $S_{ii}(\eta)$ linearly and uniformly over frequency η, the shape of the magnitude $S_{ii}(\eta)$ doesn't change with increasing different noise power. Of course, we have assumed that the noise is of small amplitude, so we cannot actually allow the ν_{ij} to increase without bound.

Fig. 3.2 shows the magnitude of $S_{ii}(\eta)$ for $\omega = 10$. Note the sharp peak around $\omega = 10$ that appears as $\epsilon \to 0$. From this observation, we can conclude that the power spectrum peak near the bifurcation located at ω, and the magnitude of this peak grows as ϵ approaches to zero. This property will be used as a precursor signaling the closeness to Hopf bifurcation.



a. $\epsilon = 10$



b. ϵ= 0.1

Figure 3.2: Power spectrum magnitude for Hopf bifurcation when $\omega = 10$ for two values of ϵ

To study the impact of noise near a stationary bifurcation, assume that a real eigenvalue occurs close to zero (denote it as $\lambda \equiv \lambda_1$) and that it has relatively smaller negative real part in absolute value compared to the other system eigenvalues:

$$|\lambda_1| \ll |Re(\lambda_i)| \tag{3.18}$$

for $i = 2, \dots, n$. Due to (3.18), terms with $j = 1$ and $k = 1$ dominate the expression (3.13) for large t, so that

$$\langle x_i(t)x_i(t+\tau)\rangle \approx e^{\lambda_1(2t+\tau)}(r_i^1)^2 \int_0^t e^{-2\lambda_1 s} \sum_{j=1}^n \sum_{k=1}^n l_j^1 l_k^1 \nu_{ij} ds$$

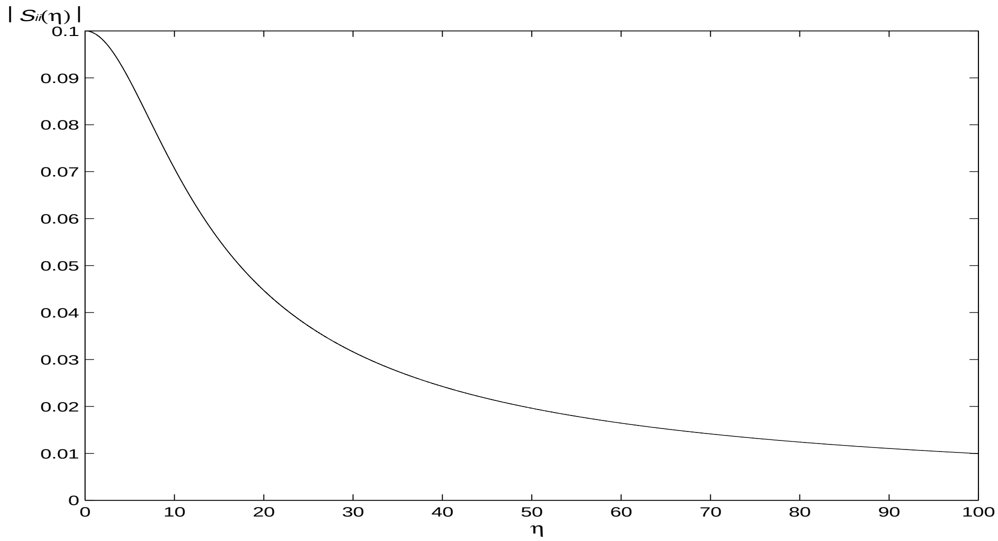
Taking the time average, we get auto-covariance function

$$\begin{aligned} C_{ii}(\tau) &:= \overline{\langle x_i(t)x_i(t+\tau)\rangle}^t & (3.19) \\ &= [\sum_{j=1}^n \sum_{k=1}^n l_j^1 l_k^1 \nu_{jk}](r_i^1)^2 \frac{e^{-\epsilon\tau}}{2\epsilon} & (3.20) \end{aligned}$$

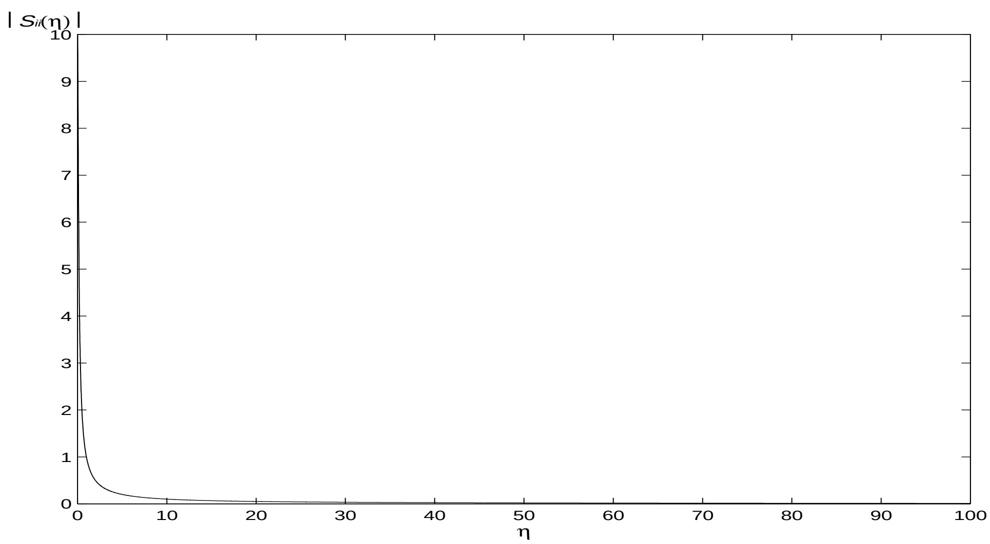
Fourier transformation of (3.20) gives the desired power spectrum:

$$S_{ii}(\eta) = [\sum_{j=1}^n \sum_{k=1}^n l_j^1 l_k^1 \nu_{jk}](r_i^1)^2 \frac{1}{2\epsilon(\epsilon + j\eta)} \tag{3.21}$$

This equation shows that the magnitude of the power spectrum peak grows as ϵ approaches to zero and the location of this peak is $\eta = 0$. Fig. 3.3 shows the magnitude of $S_{ii}(\eta)$ (3.21). Note the sharp growing peak around $\omega = 0$ as $\epsilon \to 0$.



a. $\epsilon = 10$



b. ϵ= 0.1

Figure 3.3: Power spectrum magnitude for stationary bifurcation for two values of ϵ

Chapter 4

Monitoring System for Detection of Proximity to Stationary and Hopf Bifurcation

As seen in the proceeding chapter, we can expect to observe a growing peak in the power spectrum of a measured output of a nonlinear system with white Gaussian noise input as the system approaches a bifurcation. In the case of Hopf bifurcation, the location of the power spectrum peak coincides with the imaginary axis crossing frequency of the critical eigenvalues. In the case of stationary bifurcation the power spectrum peak occurs at zero frequency. In this chapter, we start with these observations and develop monitoring system that yields useful precursors if proximity to bifurcation.

4.1 Transforming Stationary Bifurcation to Hopf Bifurcation

In this section, introduce a the monitoring system that , when augmented to a system undergoing stationary bifurcation, results in a new system that undergoes Hopf bifurcation. Hopf bifurcation is easier to detect than stationary bifurcation with the and of noisy precursors. Hopf bifurcation shows a power spectrum peak at a non zero frequency.

Consider the system

$$\dot{x} = f(x, \mu) \tag{4.1}$$

and suppose following assumptions hold.

(A1) The origin is an equilibrium point of system (4.1) for all values of μ.

(A2) System (4.1) undergoes stationary bifurcation at $\mu = \mu_c$. (i.e. there is a simple eigenvalue $\lambda(\mu)$ of $Df(x^0(\mu), \mu)$ such that for some value $\mu = \mu_c$, $\lambda(\mu_c) = 0$ and $\frac{d\lambda(\mu_c)}{d\mu} \neq 0$)

(A3) All other eigenvalues of $Df(0, \mu_c)$ are in the open left half complex plane.

Let the augmented system corresponding to (4.1) be

$$\begin{aligned} \dot{x}_i &= f(x, \mu) - cy_i \\ \dot{y}_i &= cx_i \end{aligned} \tag{4.2}$$

Here, $y \in R^n$, $c \in R$ and $i = 1, 2, \ldots, n$.

Proposition 4.1 *Under assumptions (A1)-(A3), the augmented system (4.2) experiences Hopf bifurcation at $\mu = \mu_c$. Moreover, if for any value of μ the origin of the original system (4.1) is asymptotically stable (resp. unstable), then the origin is asymptotically stable (resp. unstable) for the augmented system (4.2).*

Proof: Denote by A the Jacobian matrix for system (4.1) at the origin. Clearly, the origin $(0, 0)$ in R^{2n} is an equilibrium point of the augmented system (4.2).

The Jacobian matrix of the augmented system (4.2) at the origin is

$$D = \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{4.3}$$

Let α be any eigenvalue of A and r the corresponding right eigenvector. Also, denote by λ any eigenvalue of D and the associated eigenvector by $v = [v_1 \quad v_2]^T$. Then,

$$\lambda v_1 = A v_1 - c v_2 \tag{4.4}$$

$$\lambda v_2 = c v_1 \tag{4.5}$$

We seek a solution for which $v_1 = r$. From (4.5), we have

$$v_2 = \frac{c}{\lambda} r \tag{4.6}$$

Substituting (4.6) into (4.4) and using $r \neq 0$, we get

$$\lambda^2 - \alpha\lambda + c^2 = 0 \tag{4.7}$$

Thus, any eigenvalue α of A has corresponding to it two eigenvalues of D, which are the solutions of quadratic equation, above:

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 4c^2}}{2} \tag{4.8}$$

Thus, the eigenvalues of the that Jacobian matrix of the augmented system (4.2) are

$$\lambda_{2i-1,2i} = \frac{\alpha_i \pm \sqrt{\alpha_i^2 - 4c^2}}{2} \qquad i = 1, 2, \ldots, n \tag{4.9}$$

where α_i, $i = 1, \ldots, n$ are the eigenvalues of A. Let the eigenvalue of A that becomes 0 be α_1. At $\mu = \mu_c$, the eigenvalues of the augmented system associated with α_1 are (using (4.7)) a pair of pure imaginary eigenvalues at μ_c

$$\lambda_1, \lambda_2 = \pm cj \tag{4.10}$$

Note that the pair of pure imaginary eigenvalues (4.10) depends on c.

For a Hopf bifurcation to occur, the crossing complex conjugate pair of eigenvalues should satisfy the transversality condition. From the quadratic equation (4.7) and using the fact that $\alpha_1 = 0$ at $\mu = \mu_c$, we have

$$\frac{dRe(\lambda_1)}{d\mu} = \frac{1}{2}\frac{d\alpha_1}{d\mu} \tag{4.11}$$

At $\mu = \mu_c$. Since $\alpha_1 = 0$ and $\frac{d\alpha_1}{d\mu} \neq 0$ at $\mu = \mu_c$ from assumption **(A2)**, (4.11) implies $\frac{dRe(\lambda_1)}{d\mu} = \frac{1}{2}\frac{d\alpha_1}{d\mu} \neq 0$ (i.e., the transversality condition hold for system (4.2)). Therefore, the augmented system (4.2) undergoes Hopf bifurcation from the origin at $\mu = \mu_c$.

The last step in the proof consists in showing that all other eigenvalues of the matrix D are in the open left half complex plane. Any pair of eigenvalues of D can be obtained from (4.9). For a real eigenvalue of A, it is clear from (4.9) that the corresponding pair of eigenvalues of D have negative real part if $\alpha_i < 0$ since $\alpha_i < Re[\sqrt{\alpha_i^2 - 4c^2}]$. For a complex conjugate pair of eigenvalues of A (denote them as $\gamma, \bar{\gamma}$), we have following the two equations:

$$\lambda^2 - \gamma\lambda + c^2 = 0 \tag{4.12}$$

$$\lambda^2 - \bar{\gamma}\lambda + c^2 = 0 \tag{4.13}$$

Multiply (4.12) and (4.13) to get the following fourth order equation:

$$\lambda^4 - (\gamma + \bar{\gamma})\lambda^3 + (2c^2 + \gamma\bar{\gamma})\lambda^2 - c^2(\gamma + \bar{\gamma})\lambda + c^4 = 0 \tag{4.14}$$

Denoting $\gamma = a + bj$ and $\bar{\gamma} = a - bj$, equation (4.14) simplifies to

$$\lambda^4 - 2a\lambda^3 + (2c^2 + a^2 + b^2)\lambda^2 - 2ac^2\lambda + c^4 = 0 \tag{4.15}$$

Applying the Routh-Hurwitz criterion [25] to (4.15), we obtain the Routh array

$$\begin{array}{cccc}
s^4 & 1 & 2c^2+a^2+b^2 & c^4 \\
s^3 & -2a & -2ac^2 & 0 \\
s^2 & 3c^2+a^2+b^2 & c^4 & 0 \\
s^1 & \frac{-2ac^2(2c^2+a62+b^2)}{3c^2+a^2+b^2} & 0 & 0 \\
s^0 & c^4 & 0 & 0
\end{array}$$

From assumption (A3), $a < 0$. This guarantees that all the entries in the first column of Routh array are positive. Therefore, all eigenvalues of the Jacobian matrix of the augmented system have negative real part.

From forging discussion, it is also clear that any eigenvalue of A has positive real part, then the corresponding eigenvalues of D also have positive real part. This proves that if the original system is unstable, then the augmented system is also unstable. $\square$

Note that by changing the value c in equation (4.2), we can control the crossing frequency of the complex conjugate pair of eigenvalues of the augmented system. Thus, for detecting stationary bifurcation, we only need to monitor a frequency band around the value c , which we control.

There are some other advantages of our monitoring system. The augmented system (4.2) has the same critical parameter value (μ_c) as the original system. This is actually not a luxury but a necessity for the system to be practically useful. In addition, the final part of the proof shows that augmenting the state y_i and applying the feedback cy_i to the original system does not change stability of the system. Moreover, use of design does not require knowledge of the original system. However, there are some restriction to apply to suggested system to general nonlinear system. We will discuss this in the some of the following sections.

We assumed that in the original system the stationary bifurcation, the transversality condition is satisfied. It is not the case for saddle node bifurcation. For a saddle-node bifurcation, the augmented system of this section results in a degenerate Hopf bifurcation. The possible bifurcation diagrams for degenerate Hopf bifurcation are more complex than for Hopf bifurcation [47], [32]. However, for the purposes of the power spectrum precursor it is enough that a new peak in the power spectrum appears near bifurcation.

4.2 Detection Hopf Bifurcation using Monitoring System

In this section, we consider the effect of the monitoring system of the proceeding section on a system which undergoes Hopf bifurcation instead of stationary bifurcation. Consider again the system (4.1), repeated here for convenience:

$$\dot{x} = f(x, \mu) \tag{4.16}$$

(A1)$'$ The origin is an equilibrium point of (4.16) for all values of μ.

(A2)$'$ System (4.16) undergoes a Hopf bifurcation from the origin for $\mu = \mu_c$.

(A3)$'$ All other eigenvalues of $Df(0, \mu_c)$ are in the open left half complex plane.

Let the augmented system be

$$\begin{aligned} \dot{x}_i &= f_i(x, \mu) - cy_i \\ \dot{y}_i &= cx_i \end{aligned} \tag{4.17}$$

where $x \in R^n$, $y \in R^n$, $c \in R$, and $i = 1, 2, \ldots, n$.

Proposition 4.2 *Under the assumptions (A1)$'$-(A3)$'$, the augmented system (4.17) undergoes a codimension two bifurcation at $\mu = \mu_c$, in which two complex conjugate pair of eigenvalues cross the imaginary axis. Moreover, for any value of μ if the origin of the original system is asymptotically stable (resp. unstable), then the origin is asymptotically stable (resp. unstable) for the augmented system.*

Proof: First, we show that the augmented system has two pair of pure imaginary eigenvalues at the origin for $\mu = \mu_c$, and that these eigenvalues satisfy the transversality condition.

From assumption $(\mathbf{A2})'$, the Jacobian matrix of the original system at the origin has a pair of pure imaginary conjugate eigenvalues (denote them by $j\omega, -j\omega$) for $\mu = \mu_c$. From the proof of Proposition 4.1, it is clear that each of these eigenvalues results in a pair of eigenvalues for the augmented system which are the solutions of the following equations

$$\lambda^2 - j\omega\lambda + c^2 = 0 \tag{4.18}$$

$$\lambda^2 + j\omega\lambda + c^2 = 0 \tag{4.19}$$

By multiplying the equations above, we get a fourth order equation the solutions of which are eigenvalues of augmented system:

$$\lambda^4 + (2c^2 + \omega^2)\lambda^2 + c^4 = 0 \tag{4.20}$$

The four solutions of the equation above are given by

$$\lambda = \pm\frac{\sqrt{-2c^2 - \omega^2 \pm \sqrt{4c^2\omega^2 + \omega^4}}}{\sqrt{2}} \tag{4.21}$$

Note that $2c^2 + \omega^2 > \sqrt{4c^2\omega^2 + \omega^4}$ for all $c, \omega \in R$. Therefore, the Jacobian matrix of the augmented system has two pairs of pure imaginary conjugate eigenvalues at the critical parameter value.

To check the transversality condition, consider the eigenvalues for μ near μ_c. Near $\mu = \mu_c$, we have the following fourth order equation the solutions of which result from the pair of complex conjugate eigenvalues $(\alpha \pm \omega j)$ of the original system (see (4.14)):

$$\lambda^4 - 2\alpha\lambda^3 + (2c^2 + \alpha^2 + \omega^2)\lambda^2 - 2c^2\alpha + c^4 = 0 \tag{4.22}$$

Since μ is close to μ_c, by continuity it follows from above that this equation has two pairs of complex conjugate eigenvalues as its solutions for μ near μ_c. Denote these as $e \pm fj, g \pm hj$. Then, we have (4.24) from relationship between solutions of equations and its coefficients.

$$\begin{aligned} -2\alpha &= \sum_{i=1}^{4} \lambda_i = e + g && (4.23)\\ -2c^2\alpha &= \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4} \lambda_i\lambda_j\lambda_k = 2g(e^2 + f^2) + 2e(g^2 + h^2) && (4.24) \end{aligned}$$

where $\lambda_i, \lambda_j, \lambda_k$ are roots of (4.22). Take the derivative of both sides of the equations above with respect to μ and evaluate at $\mu = \mu_c$ ($e = g = 0$, $\beta = 0$,$\gamma^2 = c - a$), giving

$$\begin{aligned} \frac{de}{d\mu} + \frac{dg}{d\mu} &= -2\frac{d\alpha}{d\mu} \\ h^2\frac{de}{d\mu} + f^2\frac{dg}{d\mu} &= -c^2\frac{d\alpha}{d\mu} \end{aligned} \tag{4.25}$$

Solve equations (4.25) for $\frac{dg}{d\mu}, \frac{de}{d\mu}$. Also, h and f are not 0 at the critical point from (4.21) and $f \neq h$ at the critical point if $c \neq 0$. These conditions guarantee that if $\frac{d\alpha}{d\mu} \neq 0$, then $\frac{dg}{d\mu} \neq 0, \frac{de}{d\mu} \neq 0$.

The last step in the proof consists in showing that all other eigenvalues of Jacobian matrix D of (4.17) in the open left half complex plane. Note that we have same form of matrix D as in proof of Proposition 4.1.

$$D = \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{4.26}$$

where all noncritical eigenvalues of A have negative real part. We can use the same procedure as in Proposition 4.1 to prove that if all noncritical eigenvalues of A have negative real part, then all corresponding eigenvalues of D have negative real part.

It is also clear that if any eigenvalue of A has positive real part, then the corresponding eigenvalues of D also have positive real part. This proves that if the original system is unstable, then the augmented system is also unstable. □

Since two pairs of eigenvalues of the augmented system cross the imaginary axis at the critical parameter value, we can expect to see two peaks in the power spectrum as the system nears the bifurcation point. From (4.21), we see that values of the pairs of imaginary eigenvalues at criticality depend on c. Hence, we can change the location of the power spectrum peaks by changing c. Moreover, we can predict the exact locations of the peaks if the imaginary axis crossing pair of eigenvalues of the original system are known.

However, the augmented system undergoes a codimension two bifurcation. Thus, the post-bifurcation behavior is complicated. The nature of the bifurcation depends strongly on $f(x,\mu)$. The detailed possible bifurcation diagrams for the corresponding normal form can be found in [13], [27], [38] and [31].

4.3 Reduced Order Monitoring System

There are some drawbacks of the monitoring system approach presented in the preceding two sections. Most crucial of these is that it requires that all of the original system states be measured and fed back through the augmented system. This is not be practical for many systems. In this section, we show that for some classes of nonlinear systems we can apply a similar monitoring system without needing feedback of all the original system states. This is achieved by supposing the original system has an inherent time-scale structure, i.e., that it is in singularly perturbed form or the system with affine input has linearly decoupled Jacobian matrix.

4.3.1 Monitoring system for singularly perturbed system

Consider a general singular perturbation problem first. Let the full system be of the form

$$\begin{aligned} \dot{x} &= f(x,z,\mu,\epsilon) \\ \epsilon\dot{z} &= g(x,z,\mu,\epsilon) \end{aligned} \tag{4.27}$$

where $x \in R^n$, $z \in R^m$, $\mu, \epsilon \in R$ and ϵ is small but positive. The reduced system is obtained by formally setting $\epsilon = 0$ in (4.27), giving

$$\begin{aligned} \dot{x} &= f(x,z,\mu,0) \\ 0 &= g(x,z,\mu,0) \end{aligned} \tag{4.28}$$

Let $m_0 = (0, z_0)$ be an equilibrium point of the reduced system. Also, assume

(H1) $m_0 = (0, z_0)$ is an equilibrium point of (4.28) for all values of μ.

(H2) f, g, are $C^r (r \geq 5)$ in x, z, μ, ϵ in a neighborhood of $(m_0, 0, 0)$.

(H3) $\det(D_2 g(0, z_0, 0, 0)) \neq 0$.

(H4) The reduced system undergoes a stationary bifurcation at m_0 for the critical parameter value $\mu = \mu_c$.

(H5) No eigenvalue of $D_z g(0, z_0, 0, 0)$ has zero real part.

Let the augmented system corresponding to (4.27) be

$$
\begin{aligned}
\dot{x} &= f(x, z, \mu, \epsilon) - cy \\
\dot{y} &= cx \\
\epsilon \dot{z} &= g(x, z, \mu, \epsilon)
\end{aligned}
\tag{4.29}
$$

Proposition 4.3 *Let* **(H1)**-**(H5)** *above hold. Then there is an* $\epsilon_0 > 0$ *and for each* $\epsilon \in [0, \epsilon_0]$ *the augmented system (4.29) undergoes a Hopf bifurcation at an equilibrium* $m_0^{\mu_c^\epsilon, \epsilon}$ *near* m_0 *for a critical parameter value* μ_c^ϵ *near* μ_c.

Proof: By virtue of Theorem 2.4 an persistence of Hopf bifurcation under singular perturbation, we only need to show that the reduced system corresponding to (4.29) undergoes a Hopf bifurcation at $(0, 0, z_0)$ at the critical parameter value $\mu = \mu_c$. The reduced system

$$
\begin{aligned}
\dot{x} &= f(x, z, \mu, 0) - cy \\
\dot{y} &= cx \\
0 &= g(x, z, \mu, 0)
\end{aligned}
\tag{4.30}
$$

Since the original reduced system (4.28) undergoes a stationary bifurcation, we can apply Proposition 4.1 to (4.30) to show that the reduced augmented system (4.30) undergoes Hopf bifurcation at the critical parameter value $\mu = \mu_c$. And then, The remainder of proof follows as for the Theorem 2.4.□

Proposition 4.3 is useful because it implies that we only have to augment and feed back slow states in a two-time scale system to transform stationary bifurcation into Hopf bifurcation.

4.3.2 Monitoring system for the system with linearly decoupled Jacobian matrix

In this section, we consider a nonlinear affine control system which has linearly decoupled Jacobian matrix at origin. For those system, we introduce new reduced order monitoring system.

Let this nonlinear affine control system be

$$\dot{x} = f(x, \mu) + \sum_{i=1}^{n} g_i(x) u_i \tag{4.31}$$

where g_i is smooth function of x_i and each u_i is an input which will be used to feedback the augmented states. If the controls $u_i = 0$ for $i = 1, \ldots, m$, the system becomes

$$\dot{x} = f(x, \mu) \tag{4.32}$$

The following assumptions on (4.31) and (4.32) are used in Proposition 4.4 below.

(B1) The origin is an equilibrium point of (4.32) for all μ.

(B2) System (4.32) undergoes stationary bifurcation at $\mu = \mu_c$ (i.e., the uncontrolled system experiences stationary bifurcation).

(B3) Besides the critical zero eigenvalue, all other eigenvalues of the system linearization are in the open left half complex plane for $\mu = \mu_c$.

(B4) The Jacobian matrix of the system at origin for $\mu = \mu_c$ has the form

$$A := \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{4.33}$$

where $A_{11} \in R^{l \times l}$ with $l \leq m$. Moreover, the zero eigenvalue of the system at criticality is an eigenvalue of the A_{11}.

(B5) The subspace $D(x) = span\{g_i(x) : i \in 1, \ldots, m\}$ has constant dimension $k > l$ for all x in a neighborhood of the origin.

Proposition 4.4 *Introduce the augmented system*

$$\begin{aligned} \dot{x} &= f(x, \mu) + \sum_{i=1}^{n} g_i(x) u_i^* \\ \dot{y} &= c \begin{bmatrix} I & 0 \end{bmatrix} x \end{aligned} \tag{4.34}$$

where $x \in R^n$, $y \in R^l$, $c \in R$, and I is $l \times l$ identity matrix. Under assumptions **(B1)**-**(B5)**, *the augmented system (4.34) undergoes Hopf bifurcation at $\mu = \mu_c$ with input $u^* = -cKy$. Here, $c \in R$ and $K \in R^{m \times l}$ satisfies*

$$G(0)K = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.35}$$

Here, $G(0) = [g_1(0), \cdots, g_m(0)]$ and I is $l \times l$ identity matrix. In addition, if origin is asymptotically stable for the original system, then the origin is asymptotically stable for the augmented system.

Proof: Because of assumption (B4), there exists an K satisfies (4.35). Then, the Jacobian matrix of the augmented system with u^* is

$$D = \begin{bmatrix} A_{11} & 0 & -cI \\ A_{21} & A_{22} & 0 \\ cI & 0 & 0 \end{bmatrix} \tag{4.36}$$

Since interchanging rows and columns does not effect the eigenvalues of a matrix, we reformulate D as follows:

$$\tilde{D} = \begin{bmatrix} D_1 & & 0 \\ A_{21} & 0 & A_{22} \end{bmatrix} \tag{4.37}$$

where

$$D_1 = \begin{bmatrix} A_{11} & -cI \\ cI & 0 \end{bmatrix} \tag{4.38}$$

Since the set of eigenvalues of $\tilde{D}$ is the union of the eigenvalues of D_1 and of A_{22}, we can now show that the system undergoes Hopf bifurcation by applying Proposition 4.1 to D_1. It is also clear that the origin is asymptotically stable for the augmented system as long as it is asymptotically stable for the original system from the proof of Proposition 4.1. $\square$

If we have system which is the case with (4.31) under assumption **(B4)** and if system experiences stationary bifurcation, then we can transform the bifurcation into a Hopf bifurcation with reduced order augmented states. In the extreme case, we need only a single augmented state if the dimension of A_{11} is one. It also have advantage over multi-dimension A_{11} which will be illustrated with some example in section 5.2. For those system which is not decoupled as **(B4)**, we could try coordinate transform which render the system into desired form. However, detailed knowledge of the system is needed to find such a transformation.

4.4 Stability of Bifurcated Periodic Solution for the Augmented System

In this section, we consider the stability of bifurcated solutions of the augmented system. We have shown in previous section that the proposed monitoring system renders stationary bifurcation into Hopf bifurcation.

We will investigate in this section the stability of the bifurcated periodic orbit if the original system undergoes either supercritical or subcritical pitchfork bifurcation.

Let the system be

$$\dot{x} = f(x, \mu) \tag{4.39}$$

where $x \in R^n$ is the state vector and $\mu \in R$ is the bifurcation parameter. Suppose that at the critical parameter value $\mu = \mu_c$, the Jacobian matrix of (4.39) evaluated at the equilibrium point $x_0 = 0$ has one zero eigenvalue.

Consider the case in which $n = 1$, that is, let the dimension of the state vector of (4.39) be one. Also, suppose system (4.39) undergoes a pitchfork bifurcation. It is easy to see that left (l) and right (r) eigenvector corresponding to the simple zero eigenvalue at criticality can be taken as any nonzero constants. Set $r = 1$ and $l = 1$ such that r and l satisfy $lr = 1$ and the first component of r is one. Lemma 2.1 then applies. From the assumption that system (4.39) undergoes pitchfork bifurcation, we have

$$\beta_1 = lQ(r, r) = \frac{\partial^2 f}{\partial x^2}(0) = 0 \tag{4.40}$$

Thus, $Q(r, r)$ is zero. Recall that β_2 is given by

$$\beta_2 = 2l\{2Q_0(r, x_2) + C_0(r, r, r)\} \tag{4.41}$$

where x_2 is

$$x_2 = (R^T R)^{-1} R^T(-Q(r,r)) \tag{4.42}$$

with R given by

$$R := \begin{pmatrix} L_0 \\ l \end{pmatrix} \tag{4.43}$$

where L_0 is the Jacobian matrix of (4.39) at criticality. Since $Q(r,r) = 0$, we have $x_2 = 0$. Thus, β_2 becomes

$$\beta_2 = 2lC_0(r,r,r) = \frac{2}{3!}\frac{\partial^3 f}{\partial x^3}(0) \tag{4.44}$$

The augmented system corresponding to the system (4.39) is

$$\begin{aligned} \dot{x} &= f(x,\mu) - cy \\ \dot{y} &= cx \end{aligned} \tag{4.45}$$

where $x, y, \mu \in R$. We have shown that the augmented system undergoes Hopf bifurcation if the original system undergoes pitchfork bifurcation. To check the stability of the bifurcated periodic solution of (4.45), we only have to check the sign of the Hopf bifurcation stability coefficient β_2 (2.50). At criticality, the Jacobian matrix of (4.45) is

$$L_0 := \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \tag{4.46}$$

The matrix L_0 has an eigenvalue cj with corresponding right eigenvector $r = \begin{bmatrix} 1 & -j \end{bmatrix}^T$ and left eigenvector $l = \frac{1}{2}\begin{bmatrix} 1 & j \end{bmatrix}$, respectively. Eigenvalue $-cj$ has right eigenvector $\bar{r}$ and left eigenvector $\bar{l}$, respectively. Note that higher order terms only come from $f(x,\mu)$ and they are not a function of y. From this observation, we have

$$
\begin{aligned}
Q((x,y),(x,y)) &= \begin{pmatrix} \frac{1}{2!}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} \frac{\partial^2 f}{\partial x2}(0) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0)x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4.47)
\end{aligned}
$$

where we used the equation (4.40). Equation (4.47) implies that $Q(r,\bar{r})$ and $Q(r,r)$ are equal to zero. Therefore, $a=0$ and $b=0$ are solution of (2.48) and (2.49), respectively. Hence, β_2 of (4.45) becomes

$$
\beta_2 = 2Re[\frac{3}{4}lC(r,r,\bar{r})] = 2\frac{3}{4}\frac{1}{3!}\frac{\partial^2 f}{\partial x^3}(0) \qquad (4.48)
$$

since higher order terms only come from $f(x,\mu)$ and they are not function of y.

Note that a sign of (4.44) are equal to a sign of (4.48). We have following proposition as a result.

Proposition 4.5 *Consider the system (4.39) is of first order, i.e., n=1. If the original system (4.39) undergoes a supercritical pitchfork bifurcation (respectively a subcritical pitchfork bifurcation), then the transformed system (4.45) undergoes a supercritical Hopf bifurcation (respectively a subcritical Hopf bifurcation).*

Next, we are going to consider the case when $n \geq 2$, that is, dimension of original system is 2 or higher. We use an example to show that a supercritical pitchfork bifurcation (resp. a subcritical pitchfork bifurcation) need not result in a to supercritical Hopf bifurcation (resp. a subcritical Hopf bifurcation) by using the monitoring system proposed in the preceding sections.

To this end, consider the example

$$\begin{aligned}\dot{x_1} &= -\mu x_1 - x_1^3 + x_1 x_2 \\ \dot{x_2} &= -x_2 + k x_1^2\end{aligned} \tag{4.49}$$

where $\mu \in R$ is a bifurcation parameter and $k \in R$ is a constant. It is easy to see that the origin is an equilibrium point for all parameter values μ and that a pitchfork bifurcation occurs for $\mu = 0$. Moreover, a simple calculation shows that β_2 for this pitchfork bifurcation is -1. This implies that system (4.49) undergoes a supercritical pitchfork bifurcation at $\mu = 0$.

The augmented system corresponding to (4.49) is

$$\begin{aligned}\dot{x_1} &= -\mu x_1 - x_1^3 + x_1 x_2 - c y_1 \\ \dot{y_1} &= c x_1 \\ \dot{x_2} &= -x_2 + k x_1^2 - c y_2 \\ \dot{y_2} &= c x_2\end{aligned} \tag{4.50}$$

The Jacobian matrix of (4.50) evaluated at the origin at criticality is

$$\begin{bmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & -1 & -c \\ 0 & 0 & c & 0 \end{bmatrix} \tag{4.51}$$

This matrix has eigenvalue ci with corresponding right eigenvector $r = \begin{bmatrix} 1 & -j & 0 & 0 \end{bmatrix}^T$ and left eigenvector $l = \frac{1}{2}\begin{bmatrix} 1 & j & 0 & 0 \end{bmatrix}$. The eigenvalue $-cj$ has right eigenvector $\bar{r}$ and left eigenvector $\bar{l}$. The Taylor series expansion of the right side of

(4.50) has the following quadratic and cubic terms:

$$
\begin{aligned}
Q((x,y),(x,y)) &= \begin{bmatrix} x_1 x_2 \\ 0 \\ k x_1^2 \\ 0 \end{bmatrix} \\
C((x,y),(x,y),(x,y)) &= \begin{bmatrix} -x_1^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{4.52}
$$

Therefore, we have

$$
\begin{aligned}
Q(r,\bar{r}) &= Q(r,r) = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix} \\
C(r,r,\bar{r}) &= \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{4.53}
$$

By solving (2.48) and (2.49), we have

$$
\begin{aligned}
a &= \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{c} \end{bmatrix}^T \\
b &= \begin{bmatrix} 0 & 0 & \frac{kj}{2j-3c} & \frac{k}{2j-3c} \end{bmatrix}^T
\end{aligned}
\tag{4.54}
$$

Putting these values into (2.50), β_2 for the system (4.50) is

$$\beta_2 = -\frac{3}{4} + \frac{k}{4+9c^2} \tag{4.55}$$

For some large positive k, β_2 becomes positive. Thus, the augmented system (4.50) undergoes a subcritical Hopf bifurcation for such value of k.

Thus, for $n \geq 2$ if the original system undergoes a supercritical pitchfork bifurcation (resp. a subcritical pitchfork bifurcation), then the augmented system need not undergo a supercritical Hopf bifurcation (resp. a subcritical Hopf bifurcation).

4.5 Nonlinear Monitoring System Ensuring Stability

In this section, we modify our monitoring system such that if original system undergoes pitchfork bifurcation for any type with mild assumption which will be specified later, then new nonlinear monitoring system undergoes supercritical Hopf bifurcation, that is, a bifurcated periodic solution is a stable limit cycle.

Consider the following system assumed to undergo a pitchfork bifurcation for $\mu = \mu_c$.

$$\dot{x} = f(x, \mu) \tag{4.56}$$

Here, $x \in R^n$ and $\mu \in R$ is the bifurcation parameter. Denote r_s and l_s as a right eigenvector and left eigenvector corresponding to a simple zero eigenvalue at $\mu = \mu_c$, respectively. Take first component of r_s to be 1 and impose the normalization $l_s r_s = 1$.

Consider the following augmented system

$$\begin{aligned}
\dot{x}_i &= f_i(x, \mu) - c y_i \\
\dot{y}_i &= c x_i - m x_1^2 y_i
\end{aligned} \tag{4.57}$$

where m is a real constant. At criticality, it has Jacobian matrix

$$D := \begin{bmatrix} A & -cI \\ cI & 0 \end{bmatrix} \tag{4.58}$$

where A is the Jacobian matrix of (4.56). Employing Proposition 4.1, it is easy to show that the augmented system (4.57) undergoes Hopf bifurcation and D has eigenvalue $\pm ci$. Moreover, the right and left eigenvectors of D corresponding eigenvalue cj are given by

$$\begin{aligned} r &= \begin{bmatrix} r_s & -jr_s \end{bmatrix}^T \\ l &= \begin{bmatrix} l_s & jl_s \end{bmatrix} \end{aligned} \tag{4.59}$$

Also, the right and left eigenvectors corresponding to the eigenvalue $-cj$ are given by $\bar{r}$ and $\bar{l}$, respectively.

The stability of the bifurcated periodic solution of augmented system is determined by the sign of β_2 (2.50):

$$\beta_2 = 2Re[2lQ_0(r,a) + lQ_0(\bar{r},b) + \frac{3}{4}lC(r,r,\bar{r})] \tag{4.60}$$

Since there are no quadratic terms in y_i in the augmented system (4.57), β_2 of (4.57) simplifies to

$$\beta_2 = 2Re[2l_sQ_f(r,a) + l_sQ_f(\bar{r},b) + \frac{3}{4}lC(r,r,\bar{r})] \tag{4.61}$$

where $Q_f(\cdot,\cdot)$ denotes a quadratic term of $f(x,\mu_c)$. Moreover, we can simplify

$$\begin{aligned} lC(r,r,\bar{r}) &= l_sC_f(r,r,\bar{r}) + jl_sC_y(r,r,\bar{r}) \\ &= l_sC_f(r,r,\bar{r}) - \frac{m}{2}\sum_{i=1}^{n} l_s^i r_s^i (r_s^1)^2 \end{aligned} \tag{4.62}$$

where $C_f(x,x,x)$ are cubic part of $f(x,\mu_c)$ and r_s^i and l_s^i denotes the i-th component of r_s and l_s, respectively. Since the first component of r_s^1 is 1 and $l_sr_s = 1$, (4.62) reduces to

$$lC(r, r, \bar{r}) = l_s C_f(r, r, \bar{r}) - \frac{m}{2} \tag{4.63}$$

Hence, β_2 becomes

$$\beta_2 = 2Re[2l_s Q_f(r, a) + l_s Q_f(\bar{r}, b) + l_s C_f(r, r, \bar{r})] - \frac{m}{2} \tag{4.64}$$

By choosing m large enough and positive, we can make β_2 negative. This will imply that the augmented system (4.57) undergoes a supercritical Hopf bifurcation.

Here, we suggested only one of many possible methods which renders bifurcated periodic orbit stable. Note that we added a nonlinear term only to the dynamics of the augmented states y_i not the physical system states x_i. This implies that we have freedom to choose m to be any value.

4.6 Monitoring System for Equilibrium Point Not at the Origin

We have shown that using the augmented system results in a Hopf bifurcation if the original system undergoes a stationary bifurcation. However, the scheme used places a strict requirement on the system. In the assumptions of Proposition 4.1, note that (A1) demands that the origin be an equilibrium point of the system for all parameter values. This assumption is invoked so that the equilibrium point of the original system is not changed through adding state feedback. In this section, we attempt to remove assumption (A1). It will become clear at the end of the section that removing **(A1)** comes at some expense in terms of simplicity of the results.

Let's assume that equilibrium point of system (4.1) of interest does not lie at the origin, and that it changes as μ varies. Then the augmented system has equilibrium point that differs from the original system. As a matter of fact, $(0, y_0)$ becomes a new equilibrium point for augmented system (4.2), where y_0 solves $f(0, \mu) - cy = 0$. This sudden change in the equilibrium point is highly undesirable. It can result in unstable equilibrium point, or in some cases we simply prefer not to change the nominal operating point. Therefore, an alternative augmented system for monitoring for nearness to instability is presented next. The new system is motivated by washout filters, discussed in Section 2.4.

Consider the augmented system

$$\begin{aligned} \dot{x}_i &= f_i(x, \mu) - cy_i \\ \dot{y}_i &= cx_i + az_i \\ \dot{z}_i &= y_i \end{aligned} \tag{4.65}$$

where $i = 1, 2, \ldots, n$ and $a, c \in R$.

Proposition 4.6 *Assume the original system (4.1) satisfies (A2) and (A3) with an equilibrium point x_0 not necessarily at the origin. Then the augmented system (4.65) undergoes a codimension 2 bifurcation at $\mu = \mu_c$. At criticality, the linearization of (4.65) possesses one simple zero eigenvalue and a pair of pure imaginary eigenvalues.*

Proof: The equilibrium point of the augmented system (4.65) is $(x_0, 0, z_0)$, where z_0 is solution of $cx_i + az_i = 0$. Note that new augmented system keeps x_0 as a component of this equilibrium point. The Jacobian matrix of (4.65) evaluated at this equilibrium point is

$$D = \begin{bmatrix} A & -cI & 0 \\ cI & 0 & aI \\ 0 & I & 0 \end{bmatrix} \tag{4.66}$$

where A is the Jacobian matrix of the original system evaluated at x_0. Let α be any eigenvalue of A and r corresponding eigenvector. Also, assume λ is an eigenvalue of D with eigenvector $v = [v_1^T \quad v_2^T \quad v_3^T]^T$. Then

$$\begin{aligned} \lambda v_1 &= Av_1 - cv_2 & (4.67) \\ \lambda v_2 &= cv_1 + av_3 & (4.68) \\ \lambda v_3 &= v_2 & (4.69) \end{aligned}$$

Attempt a solution v for which $v_1 = r$. Solve (4.68) and (4.69) for v_2 and v_3 in terms of r, we get

$$v_2 = \frac{c\lambda}{\lambda^2 - a} r$$
$$v_3 = \frac{c}{\lambda^2 - a} r$$

Substituting the equation for v_2 into (4.67), gives

$$\lambda^3 - \alpha\lambda^2 + (c^2 - a)\lambda + a\alpha = 0 \tag{4.70}$$

Since one eigenvalue of A becomes 0 at $\mu = \mu_c$, we can set $\alpha = 0$ to get following equation for the expected pair of eigenvalues system at criticality:

$$\lambda^3 + (c^2 - a)\lambda = 0 \tag{4.71}$$

If we choose $a < 0$, then D has eigenvalues $0, \pm\sqrt{c^2 - a}\,j$ which correspond to the zero eigenvalue of the original system criticality.

Next, we check the transversality condition. Equation (4.70) which corresponds to crossing simple real eigenvalue of original system has one real and a pair of complex conjugate as its solution near the critical point. Denote δ as the real eigenvalue and $\beta \pm \gamma j$ as the pair of complex conjugate eigenvalue. Solving this notation in (4.70) and separating real and imaginary parts, we obtain

$$\begin{aligned} \delta + 2 * \beta &= -\alpha \\ \delta(\beta^2 + \gamma^2) &= \alpha a \end{aligned} \tag{4.72}$$

Differentiating these equations with respect to μ, gives

$$\begin{aligned} \frac{d\delta}{d\mu} + 2\frac{d\beta}{d\mu} &= -\frac{d\alpha}{d\mu} \\ \frac{d\delta}{d\mu}(\beta^2 + \gamma^2) + 2\frac{d\beta}{d\mu}\beta\delta + 2\frac{d\gamma}{d\mu}\gamma\delta &= \frac{d\alpha}{d\mu}a \end{aligned} \tag{4.73}$$

At the critical parameter value $\mu = \mu_c$, $\delta = 0$, $\beta = 0$, and $\gamma^2 = c^2 - a$. Thus, at $\mu = \mu_c$,

$$\begin{aligned} \frac{d\delta}{d\mu} + 2\frac{d\beta}{d\mu} &= -\frac{d\alpha}{d\mu} \\ \frac{d\delta}{d\mu}(c^2 - a) &= \frac{d\alpha}{d\mu}a \end{aligned} \tag{4.74}$$

Solving these equations for $\frac{d\beta}{d\mu}$, gives

$$\frac{d\beta}{d\mu} = -\frac{1}{2}\frac{c}{c^2 - a}\frac{d\alpha}{d\mu} \tag{4.75}$$

which is nonzero if $\frac{d\alpha}{d\mu} \neq 0$ and $a < 0$.

As was the case with proposition 4.1, the final step in the proof consists of showing that all other eigenvalues of the matrix D are in the open left half complex plane. There are three eigenvalues of D which correspond to one negative real value eigenvalue of A and these eigenvalues are solutions of the (4.70).

By using the Routh-Hurwitz criterion, we can show that solutions of equation (4.70) in C_- if the corresponding real eigenvalue of A is in C_-. For the complex conjugate pair of eigenvalues of A $(\gamma, \bar{\gamma})$, we have following two equations

$$\lambda^3 - \gamma\lambda^2 + (c^2 - a)\lambda + a\gamma = 0 \quad (4.76)$$

$$\lambda^3 - \bar{\gamma}\lambda^2 + (c^2 - a)\lambda + a\bar{\gamma} = 0 \quad (4.77)$$

Multiply (4.76) and (4.77) to get the sixth order equation

$$\begin{aligned} \lambda^6 &- (\gamma + \bar{\gamma})\lambda^5 + (2(c^2 - a) + \gamma\bar{\gamma})\lambda^4 + (\gamma + \bar{\gamma})(2a - c^2)\lambda^3 \\ &+ ((c^2 - a)^2 - 2a\gamma\bar{\gamma})\lambda^2 + (c^2 - a)a(\gamma + \bar{\gamma})\lambda + a^2\gamma\bar{\gamma} = 0 \end{aligned} \quad (4.78)$$

By applying the Routh-Hurwitz criterion to (4.78), we can show that all solutions of (4.78) are in the open left half complex plane if $Re(\gamma)$ is negative (details are in Appendix A). $\Box$

We have proved that the new augmented system (4.65) with nominal equilibrium not necessarily at the origin replaces a stationary bifurcation with a codimension two bifurcation. Note that the system has the same critical parameter value for the original system and the augmented system. In addition, the crossing eigenvalues at critical point are located 0 and $\pm\sqrt{c^2 - a}\, j$. Also, note that an original simple zero eigenvalue persists under the augmentation. Thus, monitoring system helps by introducing $j\omega$ axis eigenvalue in addition to the zero eigenvalue. Hence, we can expect that the power spectrum peaks near bifurcation to be located at 0 and $\sqrt{c^2 - a}$. By varying c and a (both tunable parameters), we could relocate the peak at $\sqrt{c^2 - a}$ to any desired location. This flexibility gives more assurance to say that peak of power spectrum is caused by closeness to instability rather than other factors such as certain noise burst at that specific frequency. However, this new augmented system (4.65) also comes with some disadvantage compared to the system Proposition 4.1. In Proposition 4.1, we transform a stationary bifurcation into a Hopf bifurcation. In other words, the system has a periodic orbit as its solution instead of a new equilibrium point near of bifurcation. In comparison to the previous augmented system design (4.2), the new augmented system (4.65) shows more complicated bifurcation behavior [31]. The system is no longer guaranteed to have a periodic orbit as a solution near bifurcation. Either a periodic orbit or a new equilibrium point could result at bifurcation. The bifurcation diagram depends strongly on the vector field $f(x, \mu)$. However, it may be possible that augmented system has desired bifurcation diagram by introducing some nonlinear terms into augmented states. Of course, to do that we have detail knowledge on $f(x, \mu)$. Details on codimension two bifurcations can be found in Guckenheimer [31]. However, for the purpose of monitoring, it is enough to have a discernible power spectrum peak when the system approaches instability.

The next proposition asserts that the new augmented system also works for singular perturbation case with fewer states needed to be fed back. Only difference from previous singular perturbation case is that we no longer requires **(H1)** of Section 4.3. Using the same notation in the Section 4.3, we have the following proposition.

Proposition 4.7 *Let (H2)-(H5) in Section 4.3 hold for the system (4.1). Then there is an $\epsilon_0 > 0$ and for each $\epsilon_0 \in [0, \epsilon_0]$ the following extended system undergoes a codimension 2 (one real and a pair of complex eigenvalues crossing) bifurcation at an equilibrium $m_0^{\mu_c^\epsilon, \epsilon}$ near m_0 for a critical parameter value μ_c^ϵ near μ_c.*

$$\dot{x}_i \;=\; f_i(x, z, \mu, \epsilon) - c y_i$$

$$
\begin{aligned}
\dot{y}_i &= cx_i + aw_i \\
\dot{w}_i &= y_i \\
\epsilon \dot{z} &= g(x, z, \mu, \epsilon)
\end{aligned}
\tag{4.79}
$$

where $i = 1, 2, \ldots, n$.

Proof: By direct application of Proposition 4.6 and Theorem 7 of [1]. $\square$

We are going to just state following proposition without detail proof for the nonlinear affine control system since it is direct a result of proposition 4.4 and proposition 4.6. Here, we no longer need assumption **(B1)** which assumes origin remains a equilibrium point for all parameter value μ. The next proposition refers to the following augmented system:

$$
\begin{aligned}
\dot{x} &= f(x, \mu) + \sum_{i=1}^{n} g_i(x) u_i^* \\
\dot{y} &= c \begin{bmatrix} I & 0 \end{bmatrix} x + dz \\
\dot{z} &= y
\end{aligned}
\tag{4.80}
$$

where $x \in R^n$, $y \in R^l$, $c \in R$, and I is $l \times l$ identity matrix. We have new assumption **(B5)'** to replace assumption **(B5)** in Section 4.3.

(B5)' The subspace $D(x) = span\{g_i(x) : i \in 1, \ldots, m\}$ has constant dimension $k > l$ for all$x$ in R^n.

Proposition 4.8 *Assumptions* **(B2)**-**(B4)** *in Section 4.3 and* **(B5)'** *hold for the system (4.1), then the augmented system (4.80) experiences a codimension 2 bifurcation at* $\mu = \mu_c$ *with special input* u^*. *In addition, if the nominal equilibrium of the original system is asymptotically stable, then it is also asymptotically stable for the augmented system (4.80).*

Proof: Let's set $u^* = -ch(x)y$ where $h(x) \in R^{m\times l}$ and $c \in R$. Let $G(x) = [g_1(x), \cdots, g_m(x)]$. For any given x, $G(x)$ can be changed into singular value decomposition form as follows.

$$
\begin{aligned}
G(x) &= U(x)\Sigma(x)V(x) \\
&= \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix} \begin{bmatrix} \begin{matrix} \sigma_1(x) & 0 & \cdots & 0 \\ 0 & \sigma_2(x) & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_l \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(x) \\ \vdots \\ v_m(x) \end{bmatrix}
\end{aligned} \tag{4.81}
$$

where $U(x)$ and $V(x)$ are unitary matrix, i.e., $U^H(x)U(x) = I$ and $V^H(x)V(x) = I$, with dimension $n \times n$ and $m \times m$, respectively. Here, A^H denotes complex conjugate transpose of A. Because of assumption **(B5)'** and smoothness of g_i, $U(x)$, $V(x)$, and $\Sigma(x)$ are smooth function of x. Set $h(x)$ to

$$
\begin{aligned}
h(x) &= V^H(x)M(x) \\
\Sigma(x)M(x) &= \begin{bmatrix} u_1^H & \cdots & u_l^H & 0 & \cdots & 0 \end{bmatrix}
\end{aligned} \tag{4.82}
$$

Then,

$$
G(x)h(x) = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.83}
$$

where I is $l \times l$ identity matrix. Rest of the proof is direct application of Proposition 4.4. $\square$

Chapter 5

Application of Monitoring System to Axial Flow Compression System

There have recently been several important developments in analysis and control of axial flow compressor both in analysis of instability phenomena and their control. These developments make possible the increased performance of the axial flow compressor by operating near the maximum pressure rise. However, the increased performance comes at the price of significantly reduced stability margin. Loss of stability is of course unacceptable. It results decreased performance of the compressor and to mechanical damage of the compression system. Axial flow compressor are known to be susceptible to two basic types of instability, [29]. One of these is surge which is a low-frequency, large-amplitude oscillation of the mean mass flow rate and pressure rise. The other is rotating stall which corresponds to a traveling wave of gas around the annulus of the compressor. These rich nonlinear characteristics of the axial flow compressor are well suited to apply our suggested monitoring system.

Several control laws have been introduced to permit operation near peak pressure rise while maintaining stability (e.g. [35], [40], [51], [56], etc.). Many of those suggested control laws ([50] and [51]) are employ linear control for avoiding or delaying the occurrence of stall. There are also nonlinear controls ([11], [56], [39] and [40]) that aims to ensure results in only stable rotating stall. Thus, even though the nominal equilibrium is not stabilized, it may be possible to stabilize a neighborhood of the nominal solution for a range of parameter values including the stall value of the throttle opening parameter, to finite perturbations. However, there is no known control law which totally eliminates the bifurcation and maintains stable nominal equilibrium point for all parameter values. Since bifurcation will occur for any controller, the axial flow compressor is a good application for our monitoring systems.

5.1 Modeling and Bifurcation Analysis

Compressor dynamic modeling is a subject that has attracted significant attention (see, e.g., [20], [24], [14] and [33]). Greitzer [30] developed a nondimensional fourth order compression system model and introduced a nondimensional parameter, B, which he found to be a determinant of the nature of post-instability behavior. A global bifurcation of periodic solutions and other bifurcations were found for this model [41], and were used to explain the observed dependence of the dynamical behavior on the B parameter.

Moore and Greitzer [48] introduced a refined model to describe stall phenomena in axial compressor. This model includes the dynamics of nonaxisymmetric flow patterns, which was not present in the Greitzer model [30].

For this dissertation, we employ a Moore and Greitzer's model in [48] with slight modification. Moore and Greitzer derived following basic partial differen-

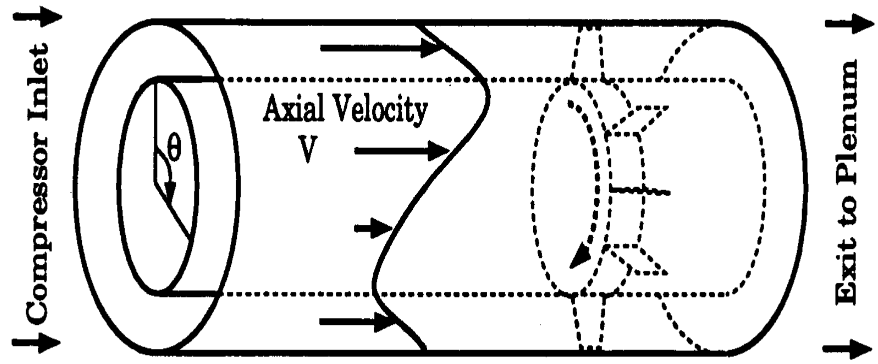


Figure 5.1: Schematic of an axial compressor

tial equation representing a local momentum balance in the compressor and its associated ducting:

$$\Delta_P = C_{ss}(V + v_0) - l_C \frac{dV}{dt} - m \frac{\partial}{\partial t} \int_{-\infty}^{0} v d\eta - \frac{1}{2\alpha}[2\frac{\partial v_0}{\partial t} + \frac{\partial v_0}{\partial \theta}] \qquad (5.1)$$

where V denotes the annulus-averaged (mean) gas axial velocity; v_0 is the axial velocity perturbation evaluated at $\eta = 0$ (the inlet face of the compressor); Δ_P is the plenum to atmosphere pressure rise; η, θ are the axial and angular coordinates, respectively; and α, l_c, m are internal compressor lag, overall compressor length, and exit duct length factor, respectively.

The compressor characteristic C_{ss} is particular to each compressor. Moore and Greitzer [48] used following cubic function as a compressor characteristic C_{ss}.

$$C_{ss}(V_{loc}) = f_0 + H[1 + \frac{3}{2}(\frac{V_{loc}}{\omega} - 1) - \frac{1}{2}(\frac{V_{loc}}{\omega} - 1)^3] \qquad (5.2)$$

where $V_{loc} = V + v_0$ (the total local axial flow); f_0 is shut-off head; ω is a compressor characteristic width factor; and H is a is a compressor characteristic height factor. Unlike Moore and Greitzer's model, we only assume here that C_{ss} be a smooth function of $V+v_0$ as in Liaw and Abed [40]. If there are no spatial variation of gas density and pressure in the plenum, an overall material balance on the gas over plenum gives:

$$\frac{d\Delta P}{dt}=\frac{1}{4B^2}[V-F(\gamma,\Delta P)] \tag{5.3}$$

where $F(\gamma,\Delta P)$ is a inverse of throttle characteristic and γ is a parameter proportional to the throttle opening. Note that parameter B which was first introduced by Greitzer [30] appears in (5.3).

Moreover, Moore and Greitzer retain only the first harmonic of the axial velocity perturbation and assume that a first harmonic axial velocity perturbation is a sine wave with time varying phase and amplitude (i.e. $v_0 = WA(t)sin(\theta+p(t))$). By employing Galerkin procedure to one mode truncation approach, the model is reduced to the following 3 dimension nonlinear ODE:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(V+WA\sin\theta)\sin\theta d\theta && (5.4)\\
\frac{dV}{dt} &= -\Delta P+\frac{1}{2\pi}\int_0^{2\pi} C_{ss}(V+WA\sin\theta)d\theta && (5.5)\\
\frac{d\Delta P}{dt} &= \frac{1}{4B^2}[V-F(\gamma,\Delta P)] && (5.6)
\end{aligned}$$

Stability analysis has been done for compressor without employing bifurcation theory ([19],[44] and [49]). McCaughan [43] performs bifurcation analysis on model of Moore and Greitzer [48] and shows that a stationary bifurcation from nominal equilibrium point occurs as the throttle opening parameter γ is varied. Liaw and Abed [40] extended the work of McCaughan [43] to the case in which the axisymmetric compressor characteristic is taken to be a general smooth function of axial velocity. Since Liaw and Abed [40] did bifurcation analysis of the axial compressor model (5.4)-(5.6), we follow the bifurcation analysis therein.

We assume that $B > 0$ and F is strictly increasing function with respect to each of the variables γ and ΔP in (5.6). Seeking equilibrium point of (5.4)-(5.6), we note that $A = 0$ always results in $\frac{dA}{dt} = 0$. There may also be equilibrium points for which $A \neq 0$. Consider a nominal equilibrium point for which there is no asymmetry flow , i.e. let $A = 0$. Denote this nominal equilibrium point by

$$x^0(\gamma) = \begin{bmatrix} 0 & V^0(\gamma) & \Delta P^0(\gamma) \end{bmatrix}^T \tag{5.7}$$

where $V^0(\gamma)$ and $\Delta P^0(\gamma)$ satisfy $V^0 = F(\gamma, \Delta P^0)$ and $\Delta P^0 = C_{ss}(V^0)$. At this equilibrium point, the system Jacobian matrix is

$$L_0 = \begin{bmatrix} \alpha C_{ss}^{'}(V^0(\gamma)) & 0 & 0 \\ 0 & C_{ss}^{'}(V^0(\gamma)) & -1 \\ 0 & \frac{1}{4B^2} & -\frac{1}{4B^2} D_{\Delta P} F(\gamma, \Delta P^0) \end{bmatrix} \tag{5.8}$$

At the parameter value $\gamma = \gamma^0$ for which $C_{ss}^{'}(V^0(\gamma)) = 0$, L_0 has a zero eigenvalue and two eigenvalue with negative real part since we assume that $B > 0$ and F is strictly increasing function with respect to each of variables γ and Δ_P. This suggests that a static bifurcation occurs at $\gamma = \gamma^0$. Moreover, it is not difficult to see that for the matrix L_0 to have a pair of pure imaginary eigenvalues, $C_{ss}^{'}$ must be positive. Because the trace of the lower right 2×2 submatrix of L_0 should be zero. However, if this were the case then the matrix L_0 would have $\alpha C_{ss}^{'}(V^0(\gamma))$ as a positive real eigenvalue, and the equilibrium would therefore unstable. This means that a stationary bifurcation must be occur before an Hopf bifurcation for the nominal equilibrium of x^0. In the remainder of this section, we focus on the stationary bifurcation that occurs for $\gamma = \gamma^0$.

To analyze bifurcation behavior of the model, we employ bifurcation formulae in Section 2.3. Let x^0 be equilibrium point at which $C'_{ss}(V^0(\gamma)) = 0$ for $\gamma = \gamma^0$. The Taylor series expansion of (5.4)-(5.6) for (x, γ) near (x^0, γ^0) is given by

$$\frac{dx}{dt} = L_0 x + Q_0(x, x) + C_0(x, x, x) + (\gamma - \gamma^0) L_1 x + \cdots \tag{5.9}$$

where x now denotes the deviation $\begin{bmatrix} A & V & \Delta_P \end{bmatrix}^T - x^0$. Here, L_0 is as in (5.8), and

$$Q_0(x, x) = \begin{pmatrix} \alpha C''_{ss}(V^0) x_1 x_2 \\ \frac{1}{4} C''_{ss}(V^0)(W^2 x_1^2 + 2x_2^2) \\ -\frac{1}{8B^2} D_{(\Delta P)^2} F(\gamma^0, \Delta P^0) x_3^2 \end{pmatrix} \tag{5.10}$$

$$C_0(x, x, x) = \begin{pmatrix} \frac{1}{8} \alpha C'''_{ss}(V^0)(w^2 x_1^3 + 4x_1 x_2^2) \\ C'''_{ss}(V^0)(\frac{1}{4} W^2 x_1^2 x_2 + \frac{1}{6} x_2^3) \\ -\frac{1}{24B^2} D_{(\Delta P)^3} F(\gamma^0, \Delta P^0) x_3^3 \end{pmatrix} \tag{5.11}$$

$$L_1 = \begin{pmatrix} \alpha C''_{ss}(V^0)\xi_1 & 0 & 0 \\ 0 & C''_{ss}(V^0)\xi_1 & 0 \\ 0 & 0 & -\frac{1}{4B^2}\xi_2 \end{pmatrix} \tag{5.12}$$

where $\xi_1 = D_\gamma F(\gamma^0, \Delta P^0)$ and $\xi_2 = D_{\Delta P \gamma} F(\gamma^0, \Delta P^0)$. Set $l = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $r = l^T$, the left and right eigenvectors respectively corresponding to the zero eigenvalues of L_0. Check the transversality condition by using (2.43)

$$l L_1 r = \alpha C''_{ss}(V^0) D_\gamma F(\gamma^0, \Delta P^0) \tag{5.13}$$

We calculate the bifurcation stability coefficients using (2.44) and (2.45):

$$\beta_1 = l Q_0(r, r) = 0 \tag{5.14}$$

$$\begin{aligned} \beta_2 &= 2l\{2Q_0(r, x_2) + C_0(r, r, r)\} \\ &= \frac{1}{4} \alpha W^2 \{2 D_{\Delta P} F(\gamma, \Delta P^0) [C''_{ss}(V^0)]^2 + C'''_{ss}(V^0)\} \end{aligned} \tag{5.15}$$

The next theorem follows from the foregoing discussion.

Theorem 5.1 *Suppose that $C''_{ss} \neq 0$ and that F is strictly increasing in each of its variables. Then the axial compression system model (5.4)-(5.6) exhibits a pitchfork bifurcation with respect to small variation of γ at the point (x^0, γ^0) for which $C'_{ss}(V^0(\gamma)) = 0$.*

Note that the formula (5.15) shows that stability of bifurcation equilibria depends on derivatives of the axisymmetric compressor characteristic at the bifurcation point.

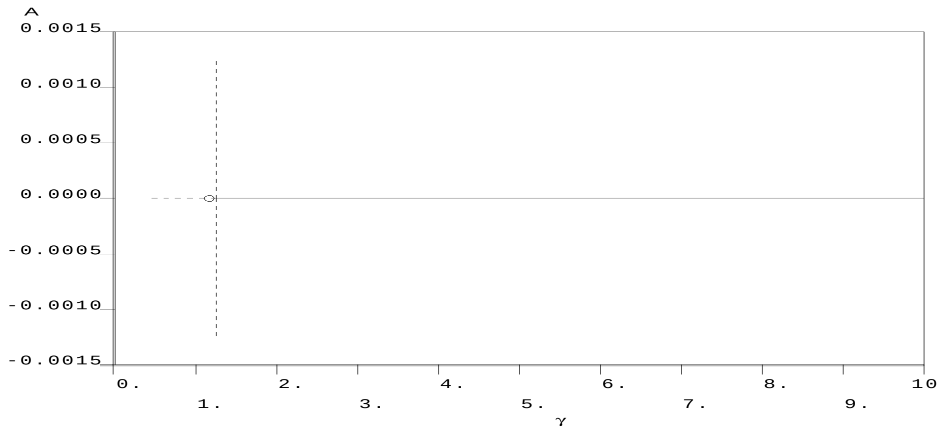
To illustrate Theorem 5.1 numerically, we consider an example in which the compressor characteristic is cubic. Let the axisymmetric compressor characteristic $C_{ss}(V)$ and the throttle characteristic F be

$$C_{ss}(V) = 1.56 + 1.5(V-1) - 0.5(V-1)^3 \tag{5.16}$$

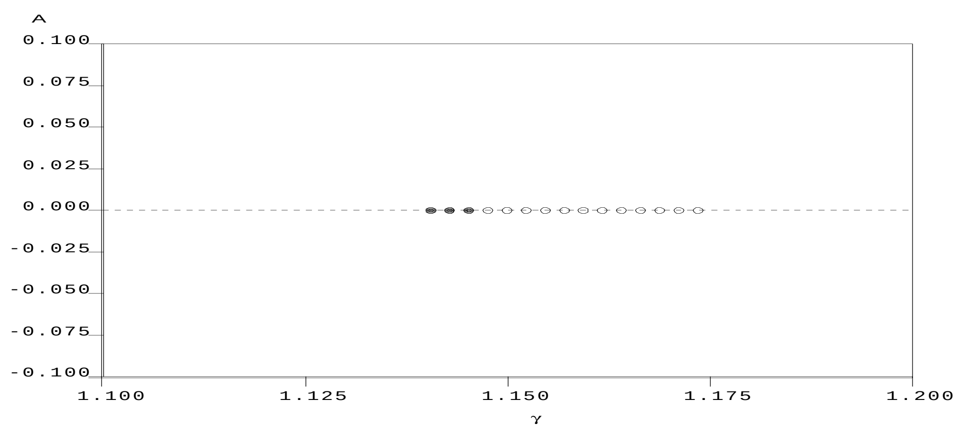
$$F(\gamma, \Delta P) = \gamma\sqrt{\Delta P} \tag{5.17}$$

Note that (5.16) is the same cubic function which used in Moore and Greitzer [48] (see (5.2)). Also, throttle characteristic (5.17) is same throttle characteristic given in Moore and Greitzer [48] and satisfies strictly increasing function condition.

Select parameter values $\alpha = 3, W = 1.0$, and $B = 0.5$. By using the bifurcation analysis package AUTO [22], we generate Figure 5.2. Figure 5.2 shows the first bifurcation occurring at $\gamma = 1.25$ and second bifurcation occurring at $\gamma = 1.17325$. First one is a pitchfork bifurcation corresponding to rotating stall and second one is Hopf bifurcation. In the Figure 5.2, solid line, dotted line, and circle correspond to stable equilibrium point, unstable equilibrium point, and periodic orbit, respectively. Figure 5.2 (b) is the blown up figure of Figure 5.2



(a)



(b)

Figure 5.2: Bifurcation Diagram for Axial Compressor

(a) for γ value between and 1.3 where fill up circle corresponds to stable periodic solution and empty circle corresponds to unstable periodic solution. This figures are drawn for variable A.

5.2 Precursor for Rotating Stall

In the previous section, we have shown that the axial compression system model (5.4)-(5.6) experiences a pitchfork bifurcation as the parameter γ varies. Based on this information, we will apply our monitoring system (4.2) in Section 4.1 to the axial flow compressor.

The Jacobian matrix of the compressor model (5.8) shows that the linearized system is decoupled as assumption **(B3)** of Proposition 4.4 between A and V, ΔP. Thus, we need only one augmented state corresponding to A. However, to apply our monitoring system, we should have monitoring input to state A. But, our compressor model ((5.4)-(5.6)) does not have input to amplitude of first harmonic of asymmetric flow (A). Therefore, we are going to adopt new compressor model which has control input to state A.

It is known that there are many ways to generate asymmetric flow in an axial compressor, such as oscillating the inlet guide vanes, vanes with oscillating flaps, jet flaps, tip bleed above the rotor, etc. In Paduano et al. [51], control was implemented using a circumferential array of hot wires to sensor propagating waves of axial velocity upstream of the compressor. Using this information, additional circumferential traveling waves were then generated with appropriate phase and amplitude by *wiggling* inlet guide vanes driven by individual actuator. Their modified compressor schematics is shown in Fig 5.3.

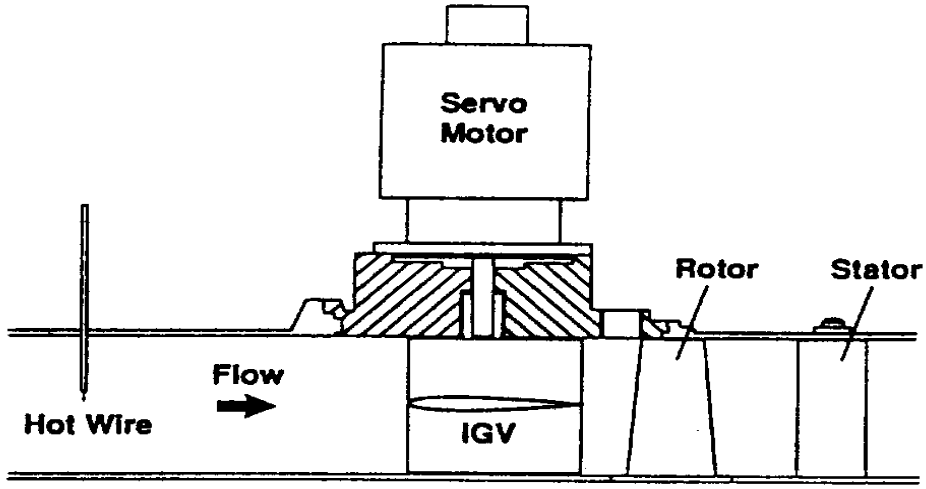


Figure 5.3: Schematic of controlled IGV axial compressor

In another paper [50], Paduano et al. obtained a linear state space model which includes the effect of the input (moving the inlet guide vanes). In [50], experiments were used to identify the effect of input. Until now, no low order nonlinear model is available that includes the effect of the dithering the inlet guide vanes. Therefore, we settle for using the following general nonlinear model in this section. Let the compression system with controlled inlet guide vanes be described by

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(V + WA\sin\theta)\sin\theta d\theta + g_1(A, V, \Delta P)u \\
\frac{dV}{dt} &= -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(V + WA\sin\theta)d\theta + g_2(A, V, \Delta P)u \\
\frac{d\Delta P}{dt} &= \frac{1}{4B^2}[\dot{m}_C - F(\gamma, \Delta P)] + g_3(A, V, \Delta P)u
\end{aligned}
\tag{5.18}
$$

Here, we assume that the effect inputs to the system can be modeled as affine. However, since dynamics of Δ_P is derived from balance of entering, leaving, and stored mass of the plenum, changing the phase and amplitude of traveling wave

A is most likely not affect the dynamics of Δ_P.

We have the following proposition for the model above.

Proposition 5.1 *Assume that g_1,g_2, and g_3 are general smooth function with respect to each of their variables and g_1 does not depend on the bifurcation parameter. In addition, suppose that g_1 does not change sign along the equilibrium path as the parameter γ varies and its sign is known to us but not the exact function itself. Then following augmented system with $u = -sgn(g_1)cy$ undergoes a Hopf bifurcation at γ^0 which corresponds to a pitchfork bifurcation point from nominal equilibrium point x^0 (5.7) for the original system:*

$$\begin{aligned}
\frac{dA}{dt} &= \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(V + WA\sin\theta)\sin\theta d\theta + g_1(A, V, \Delta P)u \\
\frac{dy}{dt} &= cA \\
\frac{dV}{dt} &= -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(V + WA\sin\theta)d\theta + g_2(A, V, \Delta P)u \\
\frac{d\Delta P}{dt} &= \frac{1}{4B^2}[V - F(\gamma, \Delta P)] + g_3(A, V, \Delta P)u
\end{aligned} \tag{5.19}$$

Proof: Linearize the equation (5.19) at the equilibrium point. The equilibrium point for system (5.19) corresponding to a nominal equilibrium point x^0 is given by

$$\begin{bmatrix} 0 & 0 & V^0(\gamma) & \Delta P^0(\gamma) \end{bmatrix}^T \tag{5.20}$$

Note that augmentation does not change the nominal equilibrium point x^0. Then the Jacobian matrix

$$\begin{bmatrix}
\alpha C_{ss}^{'}(V^0(\gamma)) & -c \mid g_1 \mid & 0 & 0 \\
c & 0 & 0 & 0 \\
0 & \tilde{g}_2 & C_{ss}^{'}(V^0(\gamma)) & -1 \\
0 & \tilde{g}_3 & \frac{1}{4B^2} & -\frac{1}{4B^2}D_{\Delta P}F(\gamma, \Delta P^0)
\end{bmatrix} \tag{5.21}$$

where $\tilde{g}_2 := g_2 sgn(g_1)$ and $\tilde{g}_3 := g_3 sgn(g_1)$. Since the Jacobian matrix is in block lower triangular form, we can apply Proposition 4.1 to the upper left 2×2 submatrix to verify occurrence of Hopf bifurcation. At the critical point, we have a pair of imaginary eigenvalues

$$\lambda = \frac{\pm c\sqrt{\mid g_1(\cdot) \mid} j}{2} \tag{5.22}$$

□

From equation (5.22), without knowing g_1 we cannot determine exact location of crossing pair of imaginary eigenvalues. However, this equation does show that the values of the eigenvalues at crossing are proportional to the value of c, which we can control.

Now consider the case in which g_1 is exactly known. We have the following Proposition.

Proposition 5.2 *Consider the augmented system (5.19). Assume that g_1^{-1} exists and is independent of the bifurcation parameter γ. Then the system (5.19) with $u = -g_1^{-1} cy$ undergoes a Hopf bifurcation for $\gamma = \gamma^0$ which corresponds to a pitchfork bifurcation point for the original system. Also, a crossing pair of imaginary eigenvalues has value $\pm ci$ at the critical point.*

Note that if g_1^{-1} fails to exist in the region of interest, then we can use Proposition 5.1. However, we can determine the exact location of crossing pair of imaginary eigenvalues unlike Proposition 5.1 since we have exact knowledge of g_1.

In the rest of this section, we try to verify the propositions above numerically. All of the bifurcation diagram shown here are obtained using numerical bifurcation analysis package AUTO. We assume that g_1 and g_2 are given as

$$\begin{aligned} g_1(V) &= V^2 + 1 \\ g_2(\Delta P) &= \Delta P^2 \\ g_3 &= 0 \end{aligned}$$

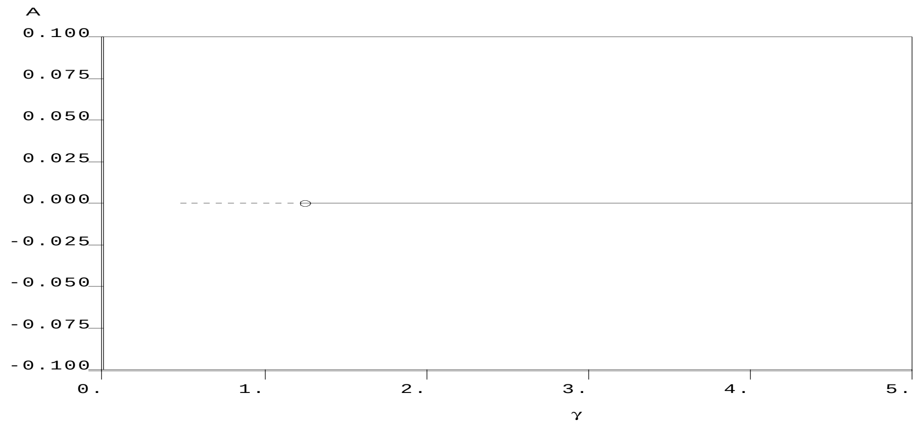
We also take the same compressor characteristic (5.16), inverse function of throttle pressure rise (5.17), and parameters which are given in Section 5.1. Moreover, small white Gaussian noise ($n_1(t)$) is added to $\frac{dA}{dt}$ in equation (5.18) either naturally or artificially such that

$$\frac{dA}{dt} = \frac{\alpha}{\pi W} \int_0^{2\pi} C_{ss}(V + WA\sin\theta)\sin\theta d\theta + g_1(A, V, \Delta P)u + n_1(t) \tag{5.23}$$

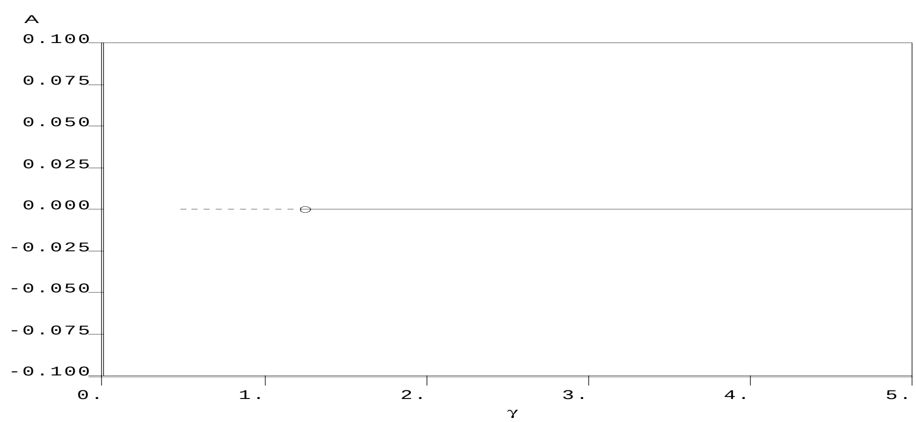
This assumes that there exist noise disturbance in the natural environment. We have shown that the compressor with these specifications undergoes pitchfork bifurcation at $\gamma = 1.25$ without input.

Consider problem setting of Proposition 5.2. Since g_1 is known, we can cancel out g_1 in (5.23) and result $g_1 u = -cy$ by setting input $u = -\frac{1}{V^2+1}cy$. Figure 5.4 (a) shows that augmented system undergoes Hopf bifurcation and the bifurcation point is the same as original system parameter value. In Figure 5.5 where we set $c = 5$, note that the clear pronounced spectrum peak is at 5 as γ approaches critical value. Another Figure 5.6 shows that the power spectrum peak is moved to 10 when c is changed to 10.

Let's consider the case such that there is no way to add artificial disturbance to A and also no natural disturbance in A. Since existence of a small white Gaussian noise is crucial to see a growing peak in the power spectrum as a system closes to a bifurcation, we have to add some noise to the system. Here,

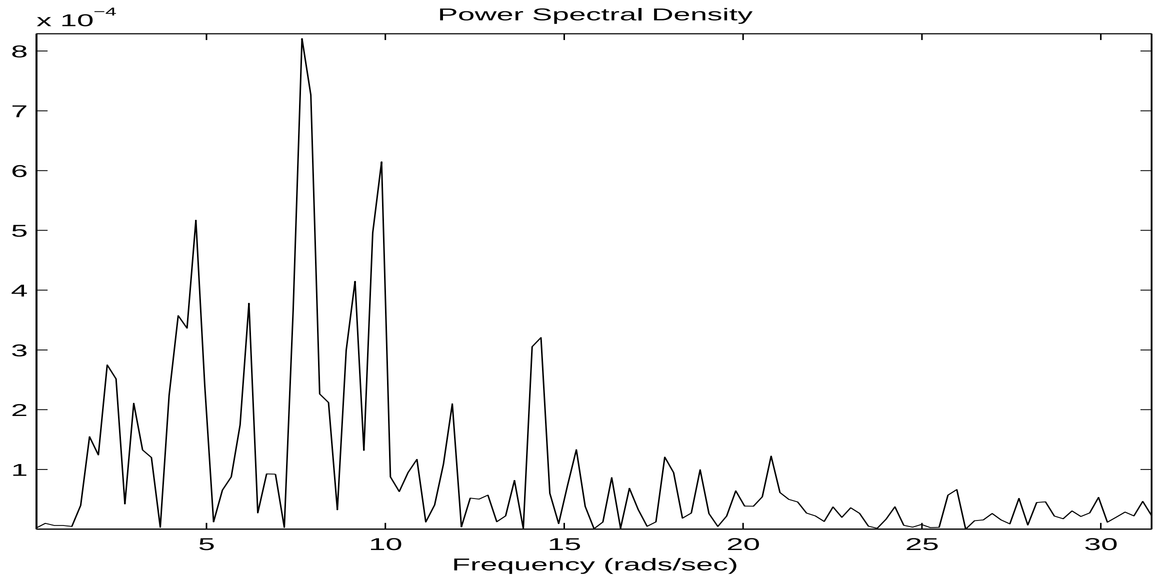


(a) Exact cancelation of $g_1(\cdot)$

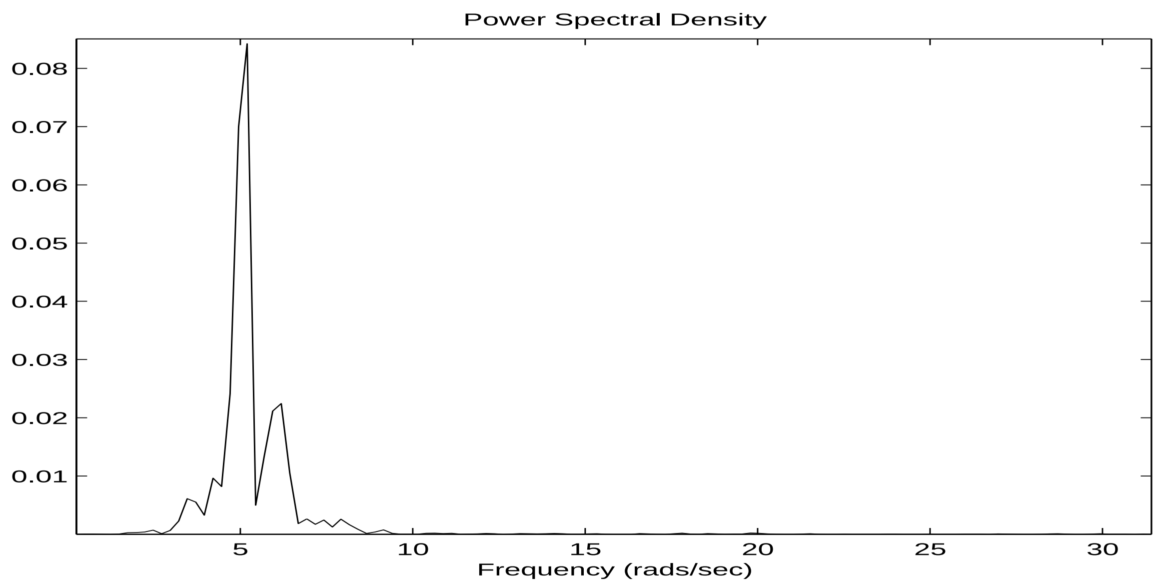


(b) Partially known $g_1(\cdot)$ or non-invertible $g_1(\cdot)$

Figure 5.4: Bifurcation Diagram of Axial Compressor for Augmented System

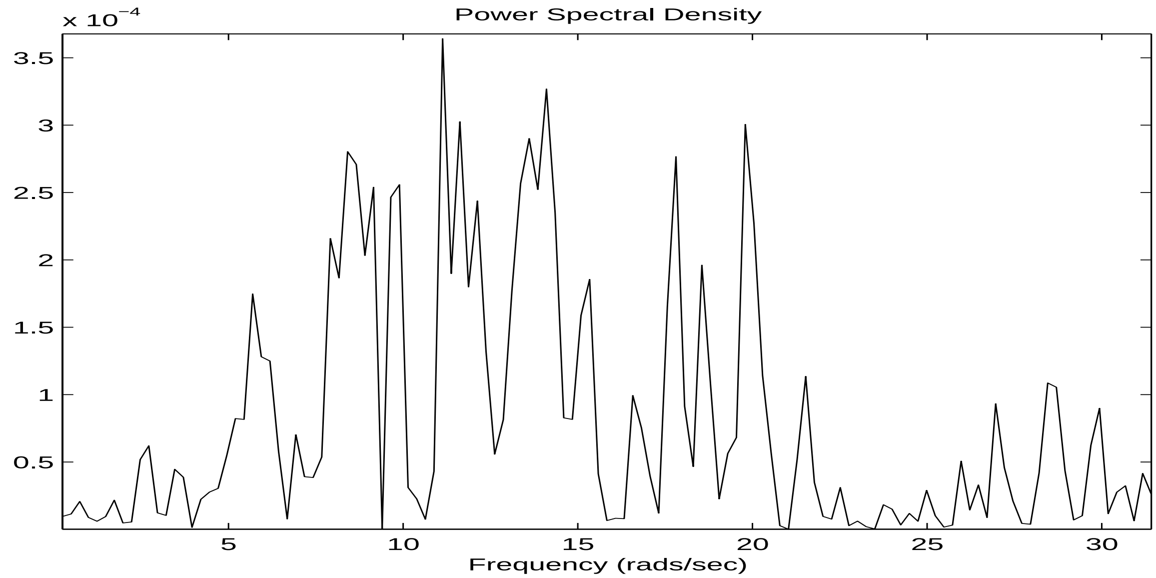


a. $\gamma = 10$

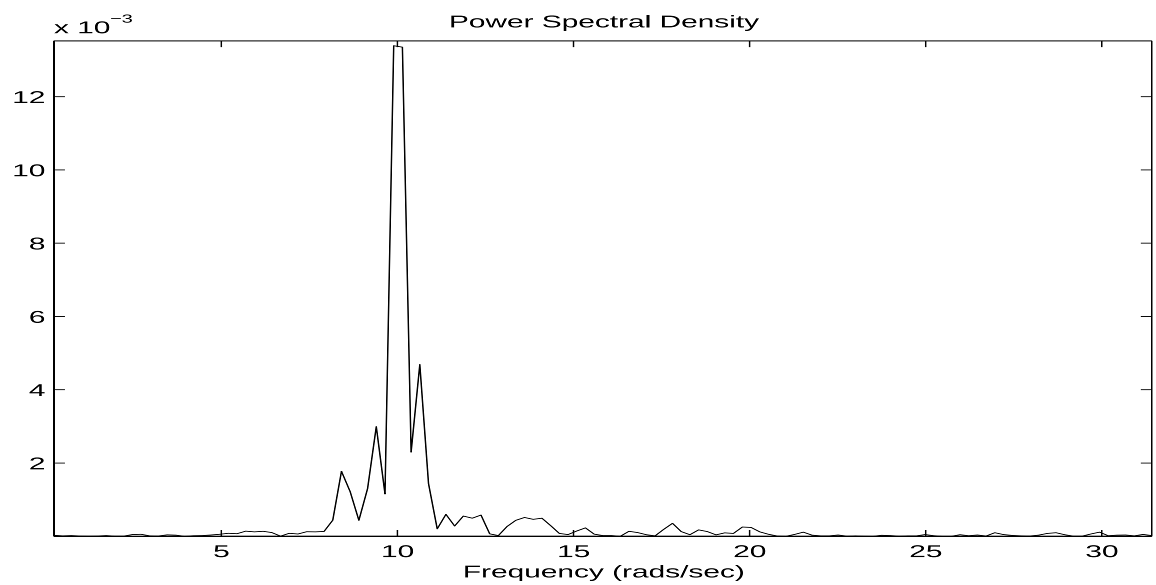


b. γ= 1.3

Figure 5.5: Power spectrum of A when c=5



a. $\gamma = 10$



b. $\gamma = 1.3$

Figure 5.6: Power spectrum of A when c=10

we are going to add a white Gaussian noise ($n_2(t)$) to the augmented state y such that

$$\frac{dy}{dt} = cA + n_2(t) \tag{5.24}$$

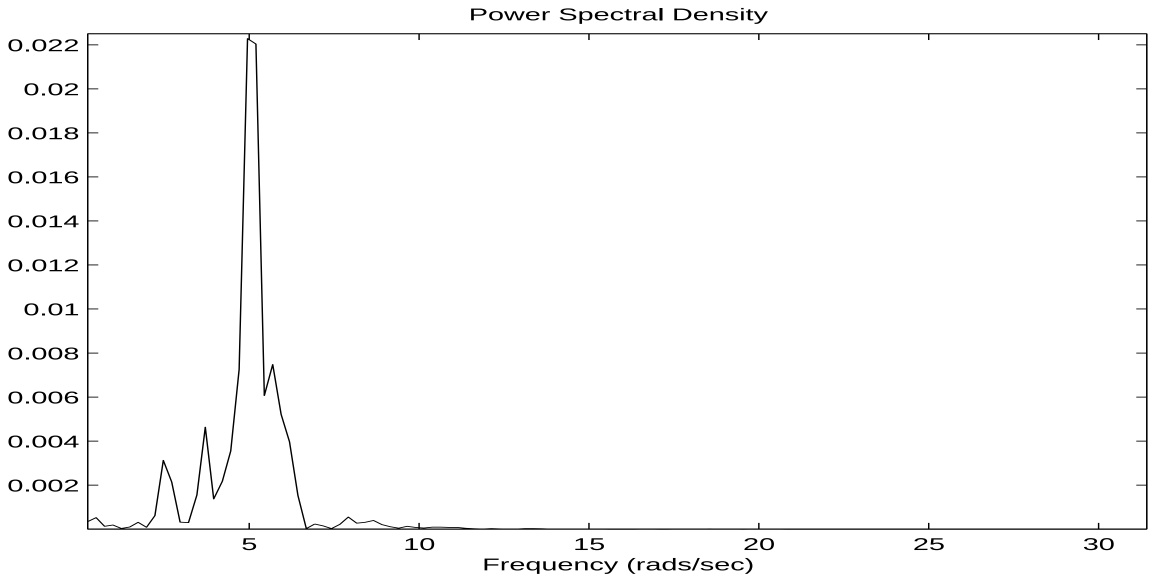
Since we have total control over the augmented state y, it is easy to add noise to state variable y.

With noise injected into the y dynamics as in (5.24), Figure 5.7 shows that we have a peak at $\omega = 5$ for power spectrum measurement of both A and y. This shows that even if noise does not enter naturally or cannot be injected into the physical system dynamics, it may still be possible to obtain a precursor by injecting noise into the augmented states.

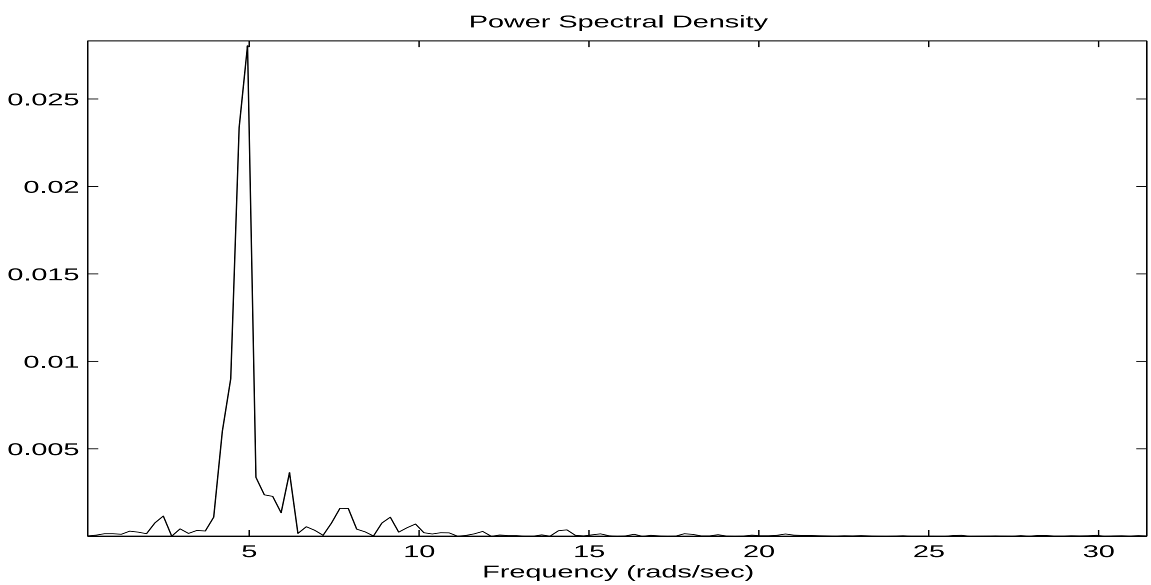
Next, we consider Proposition 5.1 case. This case we cannot cancel out g_1 since we assume no knowledge on g_1 except its sign which is locally non varying. Figure 5.4 (b) shows that augmented system still undergoes Hopf bifurcation at the same critical value despite of partially known g_1 assumption. Let our monitoring input $u = -cy$ where $c > 0$ because $g_1(V) = V^2 + 1 > 0$ for all $\dot{m}_c$ In the Figure 5.8 where we set $c = 5$, we see clear power spectrum peak around 7 which we could predict from (5.22) as γ closes its bifurcation point. Note that power spectrum peak moves to around 14.14 by changing c to 10 in the Figure 5.9.

5.3 Precursor for Hopf Bifurcation of the Axisymmetric Solution

From previous sections, we know that axial compressor undergoes a pitchfork bifurcation as parameter varies. Here, we are going to consider the case of after

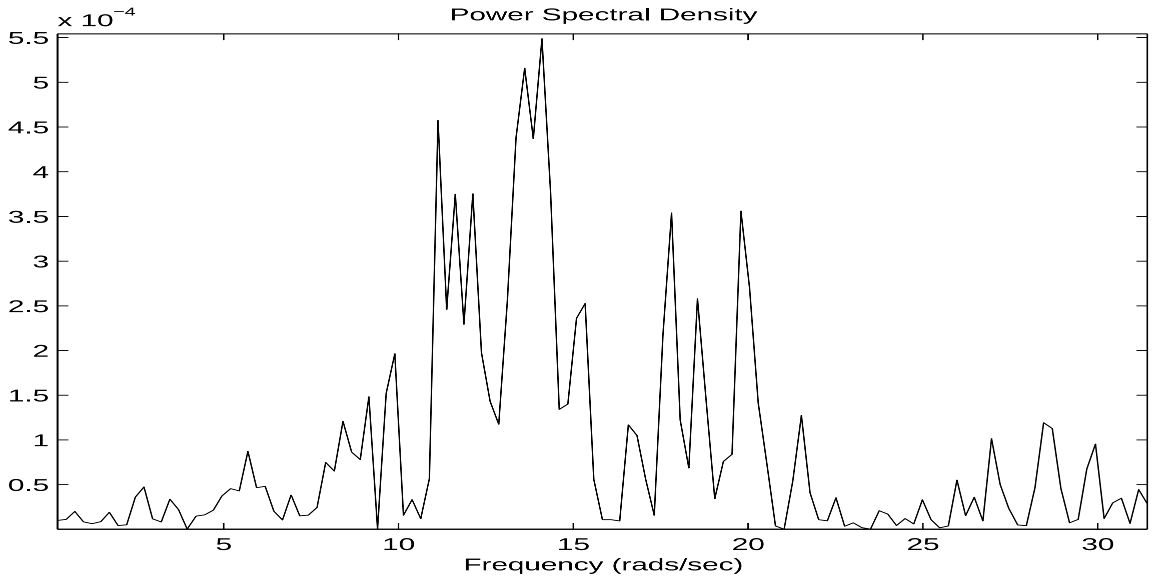


a. $\gamma = 1.3$ and power spectrum of A

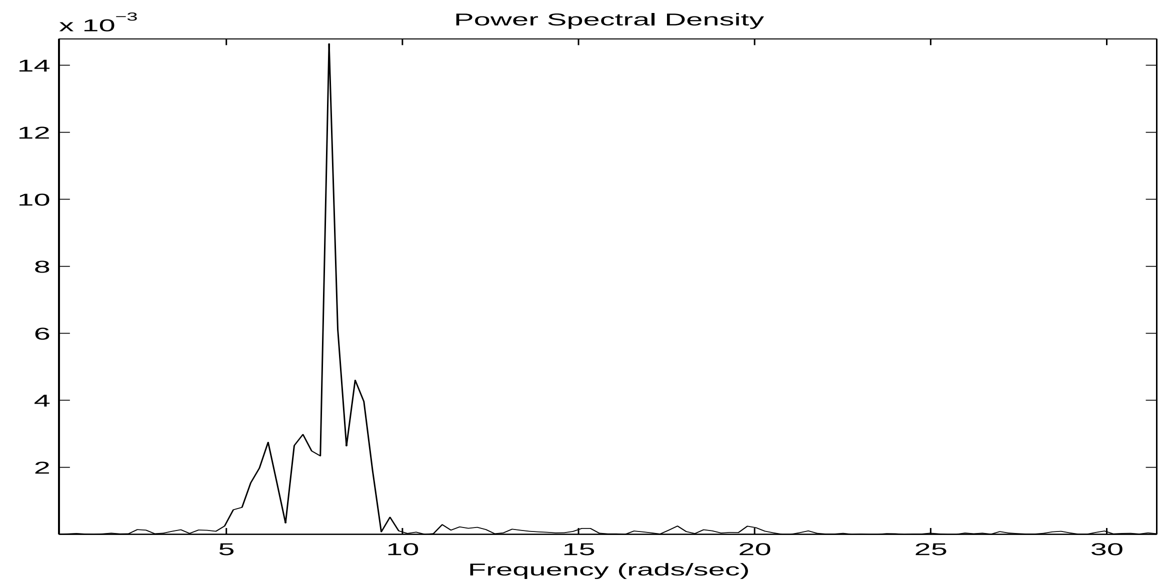


b. γ= 1.3 and power spectrum of y

Figure 5.7: Power spectrum with noise driving only the augmented state y

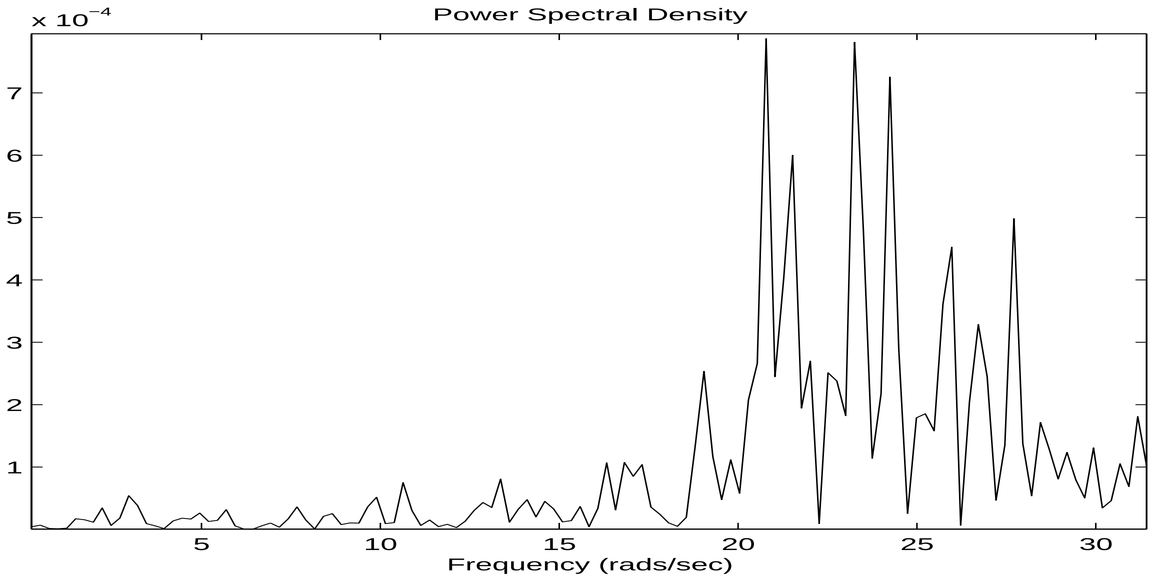


a. $\gamma = 10$

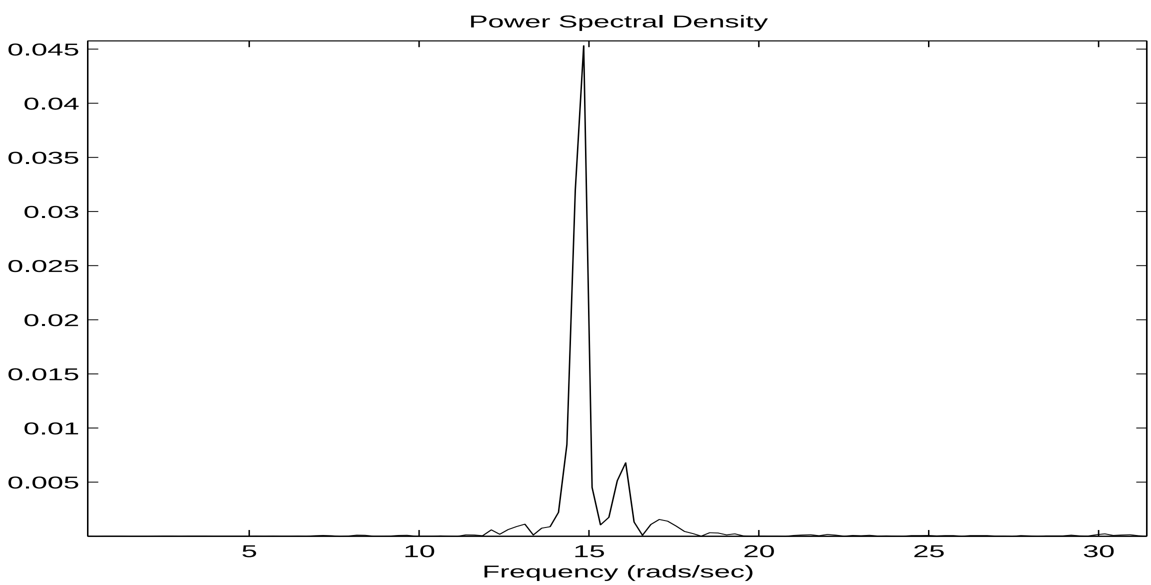


b. γ= 1.3

Figure 5.8: Power spectrum of A when c=5



a. $\gamma = 10$



b. γ= 1.3

Figure 5.9: Power spectrum of A when c=10

pitchfork bifurcation occurred. For this section, we limit our case to a cubic compressor characteristic which is given by equation (5.16). Also, we employ throttle characteristic given by equation (5.17).

After the pitchfork bifurcation occurs, there are two new equilibrium points for the amplitude of the first harmonic of asymmetric flow (A). Moreover, the equilibrium point with $A = 0$ is no longer stable. However, if initially $A = 0$ and there is no perturbation to dynamics of A, then A remains zero for all the time (see equation (5.4)). This corresponds to an axisymmetric flow condition and it is stable for given cubic compressor characteristic and throttle characteristic. For this *ideal* compressor which A being fixed to 0 even after the pitchfork bifurcation, the first bifurcation will be a Hopf bifurcation.

Since we assume that A remains zero all the time, we can drop equation (5.4) and the following two dimensional ODE remains:

$$\begin{aligned} \frac{dV}{dt} &= -\Delta P + C_{ss}(V) \\ \frac{d\Delta P}{dt} &= \frac{1}{4B^2}[V - F(\gamma, \Delta P)] \end{aligned} \tag{5.25}$$

Figure 5.2 shows the Hopf bifurcation occurring at $\gamma = 1.17325$ for system (5.25).

Before we apply our monitoring system, there are two things to note. First, the equilibrium point varies as the parameter γ varies. Hence, we are going to use our monitoring system with wash out filter (4.65). Second, we have to have monitoring signal input to both states of the system (5.25). However, our axial compressor has one input as form of $\gamma = \gamma^n + u$. From this, it is clear that we cannot transform axial compressor model system directly into suggested augmented state form. Therefore, we suggest a modified method as follows.

The modified augmented model is

$$\begin{aligned}
\frac{dV}{dt} &= -\Delta P + \frac{1}{2\pi}C_{ss}(V) \\
\frac{d\Delta P}{dt} &= \frac{1}{4B^2}[V - \gamma^n\sqrt{\Delta P}] - cy \\
\frac{dy}{dt} &= c\Delta P + dz \\
\frac{dz}{dt} &= y
\end{aligned} \tag{5.26}$$

We set $\gamma = \gamma^n + \frac{4B^2cy}{\sqrt{\Delta P}}cy$ to get above equations where γ^n denotes original parameter setting. Note that equilibrium point of (5.25) is still the equilibrium point of (5.26).

The linearization of (5.26) at equilibrium point $(V^0, \Delta P^, 0, -\frac{c}{d}\Delta P^0)$ where V^0 and ΔP^0 satisfy $V^0 = F(\gamma, \Delta P^0)$ and $\Delta P^0 = C_{ss}(V^0)$. gives Jacobian matrix

$$D = \begin{pmatrix} C'_{ss}(V^0) & -1 & 0 & 0 \\ \frac{1}{4B^2} & -\frac{\gamma^n}{8B^2\sqrt{\Delta P^0}} & -c & 0 \\ 0 & c & 0 & d \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.27}$$

The characteristic polynomial of D is

$$\lambda^4 - (\alpha+\beta)\lambda^3 + (\alpha\beta + \frac{1}{4B^2} + c^2 - f)\lambda^2 + ((\alpha+\beta)d - \alpha c^2)\lambda - d(\alpha\beta + \frac{1}{4B^2}) \tag{5.28}$$

where $\alpha = C'_{ss}(V^0)$ and $\beta = -\frac{\gamma^n}{8B^2\sqrt{\Delta P^0}}$. To check the stability of augmented system if all the eigenvalues of Jacobian matrix of the system (5.25) is in the open left half complex plane, we to use the Routh-Hurwitz criterion. The Routh array is

$$\begin{array}{llll}
s^4 & 1 & \alpha\beta + \frac{1}{4B^2} + c^2 - d & -d(\alpha\beta + \frac{1}{4B^2}) \\
s^3 & -(\alpha+\beta) & ((\alpha+\beta)d - \alpha c^2) & 0 \\
s^2 & \frac{(\alpha+\beta)(\alpha\beta+\frac{1}{4B^2})+\beta c^2}{\alpha+\beta} & -d(\alpha\beta + \frac{1}{4B^2}) & 0 \\
s^1 & \frac{c^2(-\alpha(\alpha\beta+\frac{1}{4B^2})(\alpha+\beta)-\alpha\beta(c^2-d)+\beta^2 d)}{(\alpha+\beta)(\alpha\beta+\frac{1}{4B^2})+\beta c^2} & 0 & 0 \\
s^0 & -d(\alpha\beta + \frac{1}{4B^2}) & 0 & 0
\end{array} \tag{5.29}$$

Note that $(\alpha + \beta) < 0$ since original system has eigenvalues in open left half complex plane and $\alpha\beta + \frac{1}{4B^2} > 0$ since determinant of Jacobian matrix of (5.25) is positive if (5.25) has all the eigenvalues in the open left half complex plane. From these, we have following sufficient conditions to guarantee asymptotic stability of augmented system.

$$\beta < 0 \text{ and } d < -|K| \tag{5.30}$$

where d is constant which we can choose and should be chosen big enough to guarantee the stability (see s^1 row). Since $\beta = -\frac{\gamma^n}{8B^2\sqrt{\Delta P^0}}$, β is always less than zero if the equilibrium point of (5.26) is real and γ^n is positive. These are the case for our axial compressor set-up.

By using the fact that at criticality $\alpha + \beta = 0$, the characteristic polynomial at the criticality simplifies to

$$\lambda^4 + (\omega^2 + c^2 - d)\lambda^2 - \alpha c^2 \lambda + d\omega^2 = 0 \tag{5.31}$$

where $\omega = \pm\sqrt{\alpha\beta + \frac{1}{4B^2}}$ is a crossing pair of imaginary eigenvalues of (5.25). For the compressor with cubic characteristic function, α is close to zero near the critical point. Hence, we can expect equation (5.31) has solutions close to

$$\lambda = \pm\frac{\sqrt{d - (c^2 + \omega^2) \pm \sqrt{d^2 - 2dc + c^4 + 2d\omega^2 + 2c^2\omega^2 + \omega^4}}}{\sqrt{2}} \tag{5.32}$$

It is easy to see that inside of outer square root of above equation is always less than zero. Therefore, augmented system has two pair of complex conjugate eigenvalues near critical point.

As we have pointed out before, the augmented system has two pair of complex conjugate eigenvalues with real part guaranteed to be less than zero but very close to imaginary axis near bifurcation point. Actual bifurcation diagram is not shown here since until now there are no numerical package which detects codimension 2 bifurcation. However, Figure 5.3 which is generated by AUTO shows stability change in equilibrium point at same critical value of γ as original system.

Despite of this unclear post bifurcation behavior, we can still use augmented system to get more assurance for closeness to bifurcation point as follows. First, we observe the power spectrum without augmentation. Once we saw power spectrum peak at ω, then apply feedback with state augmentation. If the power spectrum peak moves the point where it supposed to be, then we have more assurance that system is really close to its instability.

Note that original system has a complex conjugate pair of eigenvalues $\pm i$ at bifurcation point for our chosen parameter. From the calculation of section 3.2, we can expect power spectrum peak at $\omega = 1$ and we can verify that from Figure 5.11.

For the augmented system, we see two peaks as system close to its bifurcation point. In Figure 5.12 where we have chosen $c = 5$ and $d = -100$, we see to peaks around at $\omega = 1$ and $\omega = 11$. We have moved these power spectrum peak to $\omega = 1$ and $\omega = 15$ by varying d to -200 in Figure 5.13. Those location of power spectrum peak could be predicted by solving (5.31). We can say that

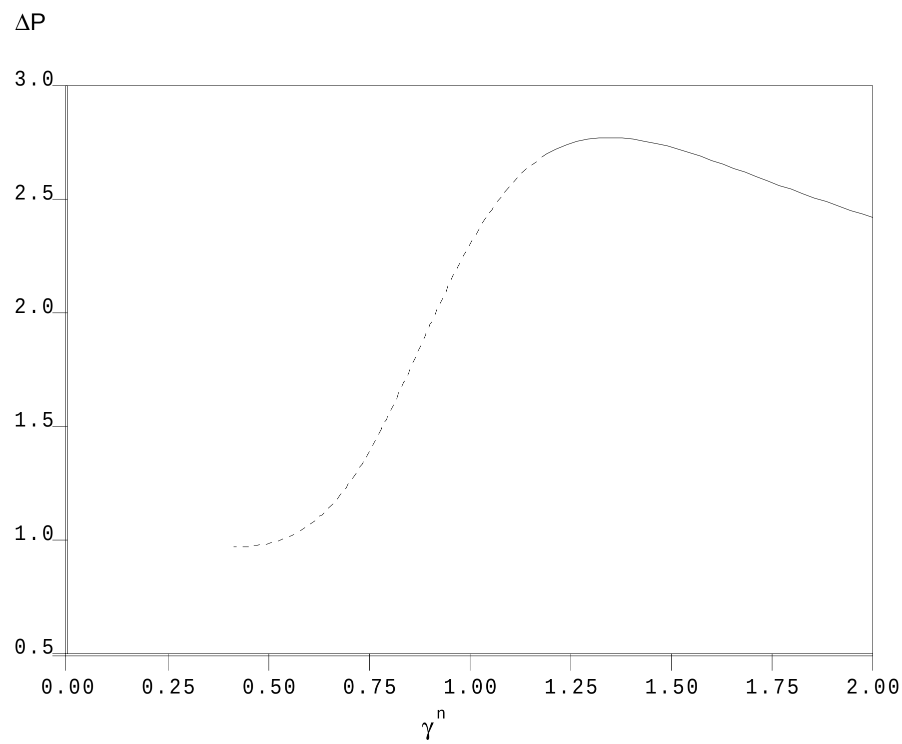
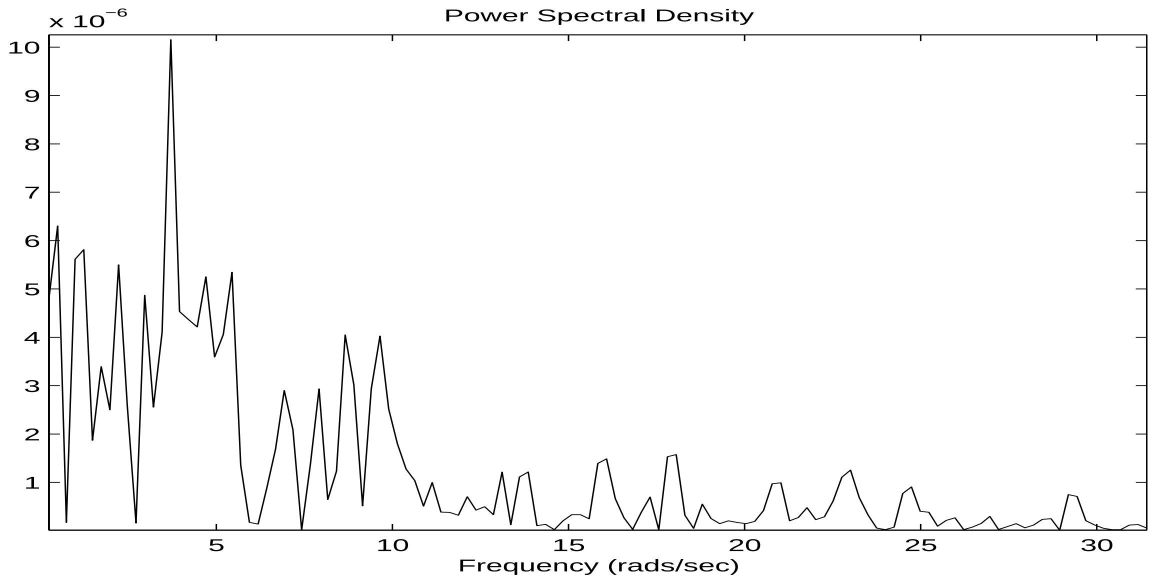
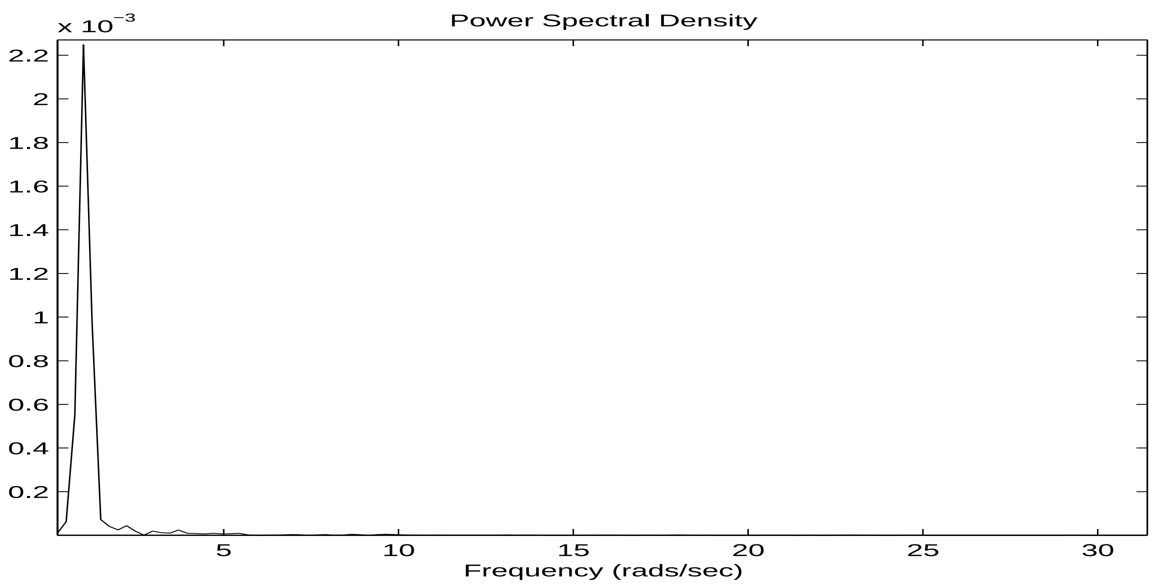


Figure 5.10: Showing change of stability of equilibrium corresponding to codimension 2 bifurcation



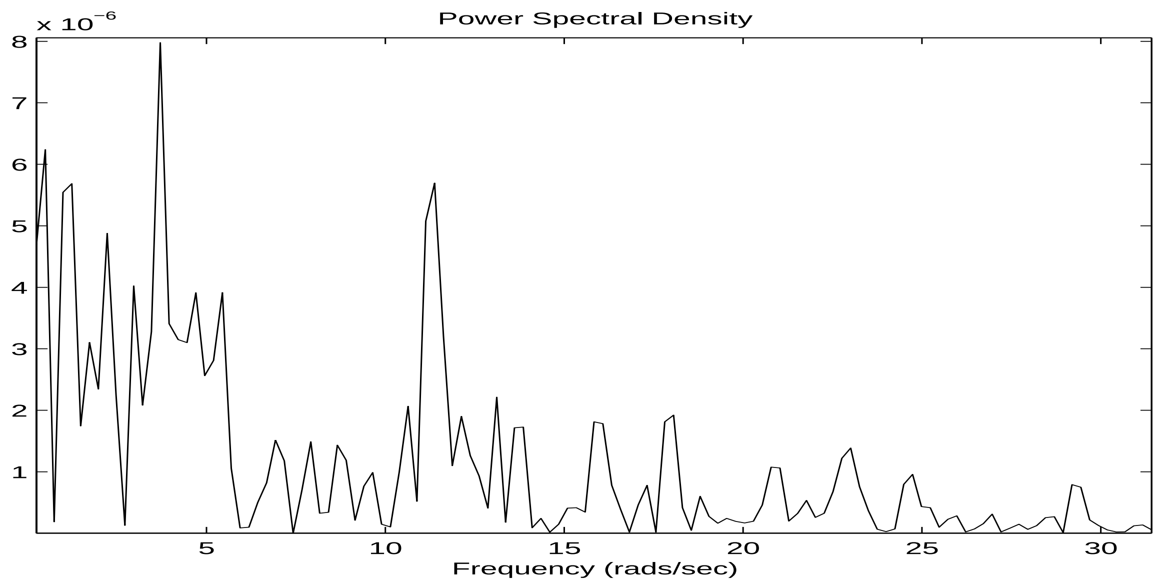
(a) $\gamma^n = 5$



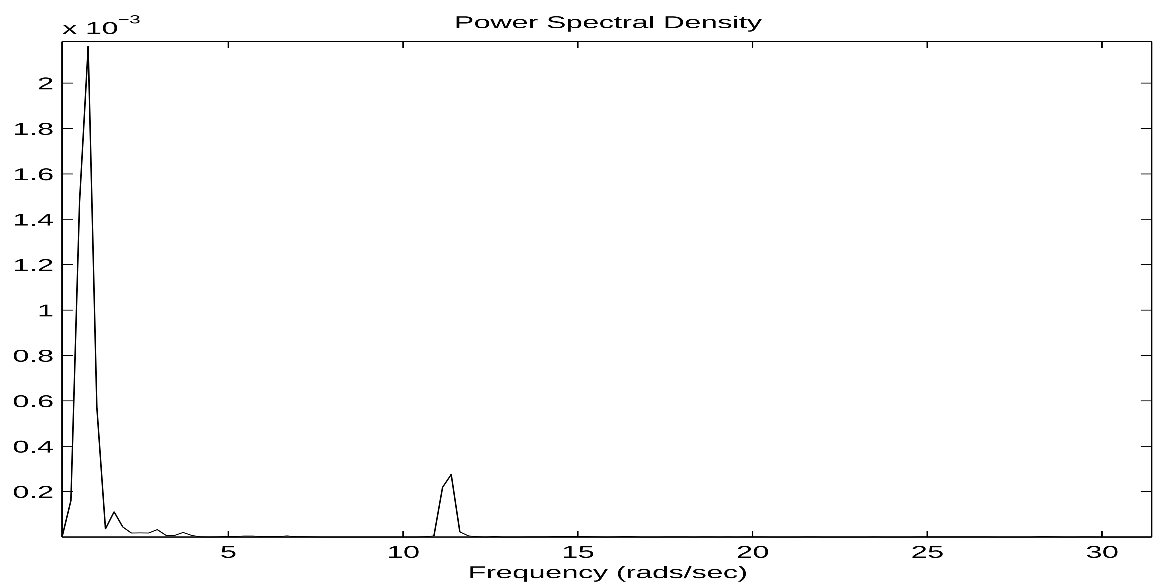
(b) γ^n= 1.215

Figure 5.11: Power spectrum of ΔP

our augmentation does not have much effect on original power spectrum peak location rather we create new power spectrum peak which is more influenced by changing d than original peak.

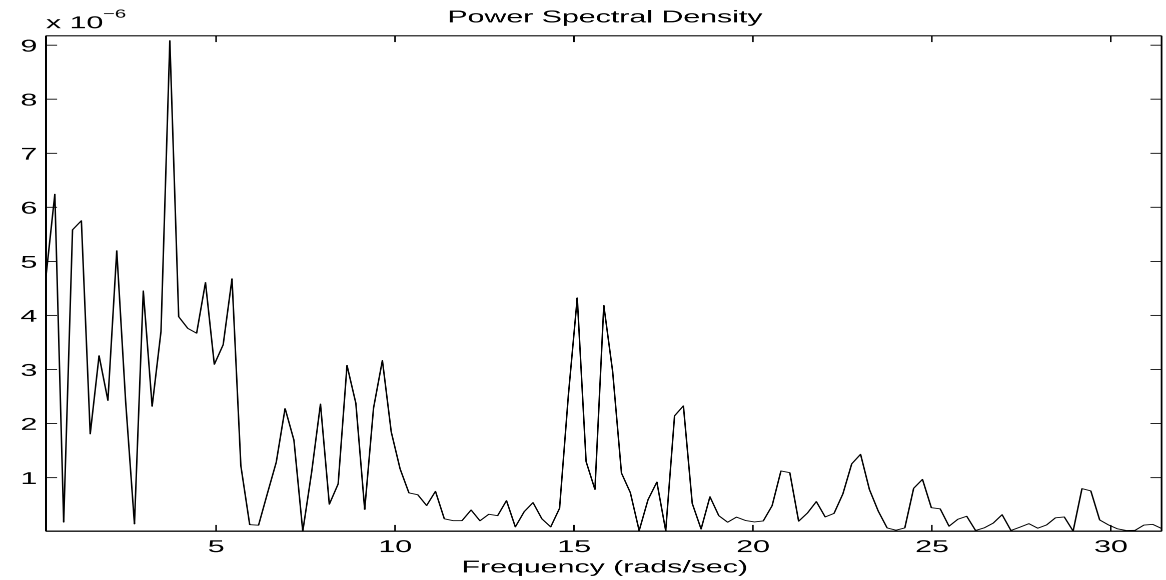


(a) $\gamma^n = 5$

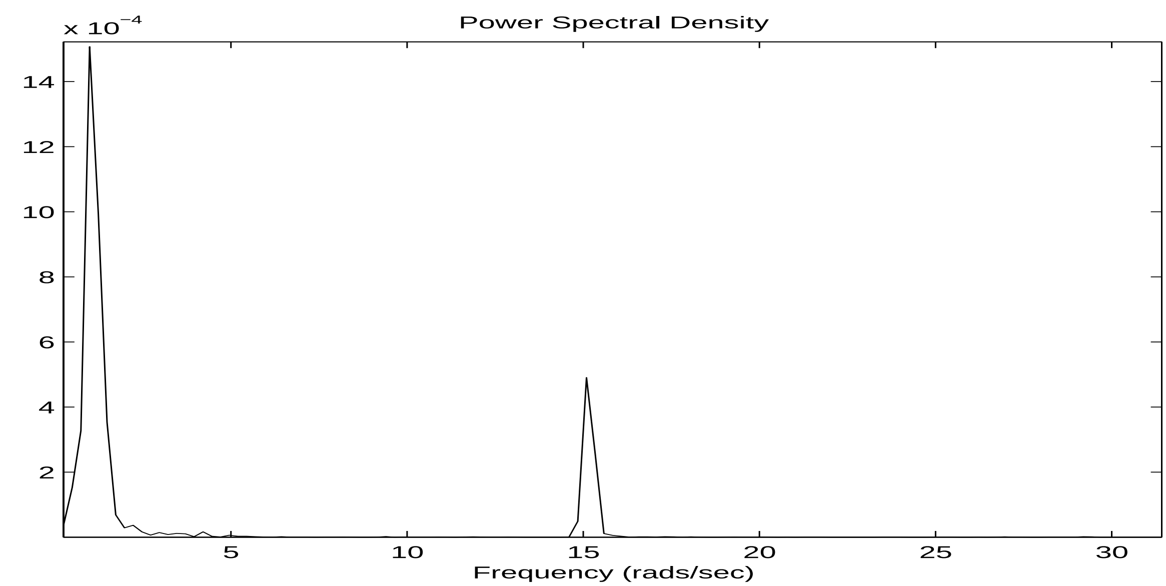


(b) γ^n= 1.215

Figure 5.12: Power spectrum of ΔP with $d = -100$



(a) $\gamma^n = 5$



(b) γ^n= 1.215

Figure 5.13: Power spectrum of ΔP with $d = -200$

Chapter 6

Conclusions and Suggestions for Future Research

We have discussed that a power spectrum precursor can be a useful measurement to detect nearness to instability in a system. It has been shown that a power spectrum precursor is a useful signal of closeness to bifurcation in the case of Hopf bifurcation. Unlike Hopf bifurcation, employing a power spectrum precursor to stationary bifurcation does not result in satisfactory performance. To circumvent this we have suggested an augmented monitoring system.

The augmented system has been shown to keep most of the original nonlinear characteristics of the system, except that it replaces a stationary bifurcation by a Hopf bifurcation. The transformed system undergoes bifurcation at the same critical paremeter value as the original system. Moreover, it has been shown that we can transform either supercritical and subcritical pitchfork bifurcation to a supercritical Hopf bifurcation under mild assumptions. This achieves both precursor for bifurcation and stabilizing the bifurcated solution. Moreover, should the system undergo a Hopf bifurcation, the augmented system undergoes a codimension two bifurcation. Thus the approach allows detection of both stationary and Hopf bifurcations.

One advantage of the augmented system is that it does not require detailed knowledge of the system being monitored. Mere knowledge that the system undergoes stationary bifurcation is enough to rendering stationary bifurcation to Hopf bifurcation. Another advantage is that we have to total control on the frequency of bifurcated periodic solution. This means we can only measure certain range of frequency rather than measuring the entire frequency domain.

These advantages are achieved under some restrictive assumptions such as the origin being an equilibrium point as the parameter varies and having independent inputs into all original system states. We have discovered that in some cases we can transform stationary bifurcation to Hopf bifurcation with restricted input. Moreover, for singularly perturbed system case we need inputs only to the slow varying states. For the case of non origin equilibrium point, we used wash out type filter. However, it renders stationary bifurcation to codimension 2 bifurcation.

We have successfully applied a precursor to axial compressor. Numerical simulation on compressor have shown that we have a clear precursor as a system closes a bifurcation. We have examined a precursor for both of a stationary bifurcation corresponding to rotating stall and a Hopf bifurcation corresponding surge. It should be noted that with some detail knowledge of the system we can do more. This is certainly the case for axial flow compressors, especially for surge detection.

Several directions for extension of this work are summarized as follows.

1. In the derivation of a power spectrum precursor, we assumed that all other eigenvalues have relatively large negative real part compared to imaginary axis crossing critical eigenvalue. Of course some systems might have two or more eigenvalues near the imaginary axis. Moreover, in some cases system could have imaginary axis approaching eigenvalues without ever crossing imaginary axis as parameter varies. A power spectrum precursor could not discern these cases. Research to address these issues is recommended.

2. We assumed that white Gaussian noise enters the dynamical model in the form of additive forcing. This noise may come from the natural environment of the system or can be added to the system artificially. In the former case (i.e., a given noise structure), determining the best states to measure for obtaining the best precursor is an important problem for future work. In the latter setting, we can consider finding the optimal location for injection of noise to result in the clearest possible precursor.

3. In some systems, the naturally occurring perturbations cannot be modeled as white noise, but rather as time periodic disturbances. Precursors and monitoring systems for such settings deserve investigation.

4. Experimental testing of the use of the monitoring systems presented here on an axial flow compression system rig would be worthwhile.

Appendix A

Routh-Hurwitz Calculation for Proposition 4.6

Use the same notation as in Proposition 4.6 and letting $\gamma = \sigma + j\omega$ and $\bar{\gamma} = \sigma - j\omega$. Equation (4.78) becomes

$$\begin{aligned} \lambda^6 - 2\sigma\lambda^5 + (2(c^2 - a) + \sigma^2 + \omega^2)\lambda^4 + 2\sigma(2a - c^2)\lambda^3 &\quad + \\ ((c^2 - a)^2 - 2a(\sigma^2 + \omega^2))\lambda^2 + 2(c^2 - a)a\sigma\lambda + a^2(\sigma^2 + \omega^2) &= 0 \end{aligned}$$

Applying the Routh-Hurwitz criterion to the equation above, we obtain the Routh array

$$\begin{array}{lcccc} s^6 & 1 & 2(c-a) + \sigma & (c-a)^2 - 2a(\sigma^2 + \omega^2) & a^2(\sigma^2 + \omega^2) \\ s^5 & -2\sigma & 2\sigma(2a - c) & 2\sigma a(c - a) & 0 \\ s^4 & c + \sigma^2 + \omega^2 & c^2 - ac - 2a(\sigma^2 + \omega^2) & a^2(\sigma^2 + \omega^2) & 0 \\ s^3 & \frac{2\sigma c(a-(\sigma^2+\omega^2))}{c+\sigma^2+\omega^2} & \frac{2\sigma ac(c-a+\sigma^2+\omega^2)}{c+\sigma^2+\omega^2} & 0 & 0 \\ s^2 & \Lambda & a^2(\sigma^2 + \omega^2) & 0 & 0 \\ s^1 & \frac{-2ac^3\sigma(c+\sigma^2+\omega^2)}{a(\sigma^2+\omega^2+c)-(a+c)^2} & 0 & 0 & 0 \\ s^0 & a^2(\sigma^2 + \omega^2) & 0 & 0 & 0 \end{array}$$

where $\Lambda = \frac{a(\sigma^2+\omega^2)(c+\sigma^2+\omega^2)-(a+c)^2(\sigma^2+\omega^2)}{a-(\sigma^2+\omega^2)}$. If $a < 0$ and $\sigma < 0$, then all entries in the first column of this array are positive. Therefore, under this condition all solutions of (4.78) have negative real part.

Appendix B

Frequently Used Notation

$D_x f$, $\frac{\partial f}{\partial x}$	partial derivatives with respect to x
C^k	functions k times differentiable
$\dot{x}$, $\frac{dx}{dt}$	time derivative of x
R^n	real n dimensional space
C_-	open left half complex plane
μ	parameters
I	$n \times n$ identity matrix
e^{At}	exponential of matrix At
$\det(A)$	determinant of $n \times n$ matrix A
λ	eigenvalue
$Re(\alpha)$	real part of α
$\bar{\gamma}$	complex conjugate of γ
$\|\alpha\|$	absolute value of α
$N(t)$, $n(t)$	white Gaussian noise
$< X >$	expected value of X
$\overline{X(t)}^t$	time average over all time

$sgn(\alpha)$	sign of real number α
A^H	complex conjugate transpose of matrix A
$g^{-1}(x)$	reciprocal of function $g(x)$

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