ABSTRACT

Title of dissertation: GLOBAL GEOMETRIC CONDITIONS ON SENSING MATRICES FOR THE SUCCESS OF ℓ_1 MINIMIZATION ALGORITHM

Rongrong Wang, Doctor of Philosophy, 2013

Dissertation directed by: Professor John Benedetto, Professor Wojciech Czaja
Department of Mathematics

Compressed Sensing concerns a new class of linear data acquisition protocols that are more efficient than the classical Shannon sampling theorem when targeting at signals with sparse structures.

In this thesis, we study the stability of a Statistical Restricted Isometry Property and show how this property can be further relaxed while maintaining its sufficiency for the Basis Pursuit algorithm to recover sparse signals. We then look at the dictionary extension of Compressed Sensing where signals are sparse under a redundant dictionary and reconstruction is achieved by the ℓ_1 synthesis method. By establishing a necessary and sufficient condition for the stability of ℓ_1 synthesis, we are able to predict this algorithm's performances under different dictionaries. Last, we construct a class of deterministic sensing matrix for the Dirac-Fourier joint dictionary.

GLOBAL GEOMETRIC CONDITIONS ON SENSING MATRICES FOR THE SUCCESS OF ℓ_1 MINIMIZATION ALGORITHM

by

Rongrong Wang

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2013

Advisory Committee:
Professor John Benedetto, Chair/Advisor
Professor Wojciech Czaja, Co-Advisor
Professor Radu Balan
Professor Kasso Okoudjou
Professor Alexander Barg

Acknowledgments

I owe my deep gratitude to my advisor Dr. John Benedetto for his constant help, support, protection and encouragement during my transition from an exam oriented student to, if I may say, a researcher. I also thank him for all the opportunities he gave me, including the opportunity to develop my own research interest and viewpoint.

I own the same amount of gratitude to my co-advisor Dr. Wojciech Czaja, for his numerous good advices, for the countless time he spent helping me modifying the English of my paper, for organizing the FFT and various RIT, and for all other things that are not able to be listed here. He is such a nice person that I can hardly remember he ever turned me down when I ask for help. He is kind of a "yes man" to me.

I deeply thank Dr. Alexander Barg, for shedding lights into my research and for all the kind words during my hardest time of waiting. This thesis can never be done without him.

I thank my committee member Kasso Okoudjou and Radu Balan for being there for my defense and for all the inspiring questions they asked during the seminars.

Special thanks given to my friend Xuemei Chen, for posing a good problem to me and all the following inspiring discussions.

I thank all members of the Norbert Wiener Center.

I thank my good friends Tong Meng, Yu-ru Huang, Minghao Wu, Wei Guo, Wenqing Hu, Changhui Tan and Peng Gao, and many others, for making my graduate life colorful.

I owe my deepest thanks to my family - my mother and father who have always stood by me and guided me through the hard times. Words cannot express the gratitude I owe them.

I would like to acknowledge financial support from the Laboratory of Telecommunications Sciences (LTS).

It is impossible to remember all, and I apologize to those I've inadvertently left out.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Background of Compressed Sensing

Compressed Sensing (CS) concerns the problem of simultaneously sensing and compressing signals that possess special sparse structures. The space of sparse signals affords a succinct representation because it represents a finite union of low-dimensional manifolds all of which are known to be compressible by linear operators. Therefore we are more interested in the following question: for a given sparsity level, to what extent can we compress the sparse signals, or equivalently, at least how many measurements are needed to measure these signal losslessly in the sense that the original signal can be fully recovered from the measurements.

Mathematically, all measurements are stored as rows of a matrix Φ called *sensing matrix*, and each data point is obtained by projecting the sparse signal $\boldsymbol{x} \in \mathbb{R}^N$ on to a row of Φ. Let $\boldsymbol{y}$ be the vector storing all these data points, then it can be written as $\boldsymbol{y} = \Phi\boldsymbol{x}$, with $\boldsymbol{y} \in \mathbb{R}^m$ and $m < N$.

The compressed signal $\boldsymbol{y}$ brings efficacy in data transmission and storage, while at some later point, it needs to be transformed back to the original signal $\boldsymbol{x}$. Solving an underdetermined system is known to be impossible in general but it is no longer ture if we know that $\boldsymbol{x}$ is sparse. The following ℓ_0 minimization algorithm is a straight forward way to exploit sparsity.

$$\min \|\boldsymbol{x}\|_0 \qquad \text{subject to } \boldsymbol{y} = \Phi\boldsymbol{x}. \tag{P_0}$$

(P_0) can recover all sparse signals exactly provided the sensing matrix satisfies some weak condition. However, the algorithm is intractable and has a complexity that grows exponentially in dimension.

It was Candès and Tao [23] who first showed that the following ℓ_1 minimization procedure (also know as Basis Pursuit) can be used as a tractable substitution to (P_0) when Φ is properly chosen.

$$\min \|\boldsymbol{x}\|_1 \qquad \text{subject to } \boldsymbol{y} = \Phi\boldsymbol{x}. \tag{P_1}$$

Specifically, they proved that as long as Φ satisfies the so called (k, δ)-Restricted Isometry Property (RIP) with $\delta_{2k} < \sqrt{2} - 1$, (P_1) is equivalent to (P_0) when recovering k-sparse signals (signals that have at most k nonzero components).

Definition 1. *We say a matrix Φ has the* Restricted isometry property (RIP) *with order k if*

$$(1-\delta)\|\boldsymbol{x}\|_2^2 \leq \|\Phi\boldsymbol{x}\|_2^2 \leq (1+\delta)\|\boldsymbol{x}\|_2^2 \tag{1.1}$$

The nice part about RIP is that it not only guarantees the exact recovery of (P_1), but also its stability [22, 29, 21]. To be more explicit, suppose the measurements are noisy, then the measurements vector becomes $\boldsymbol{y} = \Phi\boldsymbol{x} + \boldsymbol{w}$ with a noise vector $\boldsymbol{w}$ of a known energy level $\|\boldsymbol{w}\|_2 \leq \epsilon$. In this case, either (P_1) or the following denoised version of (P_1) can be used for recovery.

$$\min \|\boldsymbol{x}\|_1 \qquad \text{subject to } \|\boldsymbol{y} - \Phi\boldsymbol{x}\| < \epsilon. \tag{P_2}$$

It has been proved in [21] that, as long as Φ satisfies RIP, small reconstruction error is guaranteed for both (P_1) and (P_2):

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq C_1 \sigma_k(\boldsymbol{x}) + C_2 \epsilon, \tag{1.2}$$

where $\hat{\boldsymbol{x}}$ is the solution to either (P_1) or (P_2), C_1, C_2 are constants depending on k and δ, and $\sigma_k(\boldsymbol{x}) := \min_{x_k \text{ is } k\text{-sparse}} \|\boldsymbol{x} - \boldsymbol{x}_k\|_1$ denotes the ℓ_1 residue of the best k-term approximation to $\boldsymbol{x}$.

A natural question is that why the Basis Pursuit algorithm is chosen for recovery? In fact, since BP is a superlinear algorithm that is barely acceptable in practice, many other algorithms have also been proposed in the literature, such as Orthogonal Matching Pursuit (OMP) [12], CoSaMP [44], One Step Thresholding (OST) [6], Approximate Message Passing Algorithm (AMP) [41], Bregman Iteration [37], etc.. Even though many of these methods are dramatically faster, BP still has its special scientific interest. The most important advantage of BP is perhaps its low requirement for success. In fact, sparsity is the only prior that is required in BP, while in other algorithms this is not true. For example in OST, an additional a priori requirement is that all the nonzero magnitudes $\boldsymbol{x}$ should be comparable to each other; and in AMP, strict analysis is only carried out for certain random matrices combined with Gaussian type of noise.

A necessary and sufficient condition for stable recovery of Basis Pursuit that has brought quite an attention in this community is the Null Space Property. It is formulated as an ℓ_1 condition on the kernel of the sensing matrix.

Definition 2. *A matrix Φ is said to have the* Null space property of order k (k-NSP) *if for* *all $v \in \ker\Phi\backslash\{0\}$ and all index set T with cardinality at most k, we have*

$$\forall v \in \ker\Phi\backslash\{0\},\ \forall|T| \leq k, \quad \|v_T\|_1 < \|v_{T^c}\|_1. \tag{1.3}$$

Despite its equivalence to (P_1), NSP is not as widely used as RIP mainly for two reasons. First RIP is an l_2 criterion and therefore easier for theoretical verification; secondly, since (1.3) has no additivity, examining NSP of a given matrix requires verifying (1.3) for every vector in the kernel of Φ which is more time consuming than examining Φ directly.

In contrast to NSP, RIP is only a sufficient (but not necessary) condition for the success of (P_1) and (P_2). However, this stringency automatically leads to a stronger stability result. In fact, both conditions are equipped with an error guarantee in the form of (1.2), but the C_2 in the error of RIP is smaller than that of NSP by a factor of $\sqrt{N}$.

1.2 Sensing Matrix Analysis

For fixed signal dimensionality N and sparsity level k, among all matrices that satisfies (k, δ_k)-RIP, we are particularly interested in those matrices that have the smallest number of rows, because fewer rows means higher compression rates. It is shown using a Gelfand width based argument that the smallest possible row number is $m = O(k\log(N/k))$. It is also proved [23] that this number is achieved with overwhelming probability if the matrix is random with i.i.d. entries drawn from the standard Gaussian distribution. The possible failure of random matrices and the fact that there is no way to detect them, makes the problem fatal in practice. Therefore certain deterministic matrices have been built and have demonstrated good performances in simulation. Yet so far, none of them has been proved to achieve the optimal compression rate as Gaussian matrices do. The main difficulty lies in the question of how to pass the RIP condition to other global conditions of a matrix that are easier to verify, such as the mutual coherence or the spectral norm. A standard argument to pass RIP to mutual coherence is based on the Gershgorin Circle theorem, but it inevitably leads to a sub-optimal relation $k \le O(\sqrt{m})$, where the square root on m that preventing the order from achieving optimal is known as the square root bottleneck. In the literature, only one matrix constructed by Bourgain et al. [11] has successfully broken this bottleneck. The technique that was used involves the definition of a so-called flat orthogonality constant, which is easier to be verified and yet sufficient for exact recovery. Although the result is significant better than all previous ones, it is still far from satisfactory in the sense that the order on m is only raised from $1/2$ to something slightly larger: $k \le O(m^{1/2+\epsilon})$.

1.3 Compressed sensing in dictionary

A recent direction in CS considers signals that have sparse representation under a redundant dictionary, where the incoming signal $\boldsymbol{x}$ can be expressed as $\boldsymbol{x} = D\boldsymbol{z}$ with $\boldsymbol{z}$ being sparse and D being a fat matrix with more columns than rows. Dictionaries are in general more flexible and representative than orthonormal bases by including more columns (called atoms) into it. Moreover, this model is useful when signals do not naturally have sparse decompositions under orthonormal bases, such as images that are only sparse in curvelet frames (see the numerical experiments in Chapter 4 for what happens if one wrongly assumed such images to be sparse under an orthonormal basis).

Despite all these benefits in using dictionaries, there are surprisingly few results along this direction, especially results related to a well known recovery method called the ℓ_1 synthesis method.

If we denote the measurements by $\boldsymbol{y}$ as before then now it has the representation $\boldsymbol{y} = \Phi\boldsymbol{x} = \Phi D\boldsymbol{z}$. The ℓ_1 synthesis method recovers $\boldsymbol{x}$ from $\boldsymbol{y}$ by solving

$$\begin{aligned} \hat{\boldsymbol{z}} &= \min \|\widetilde{\boldsymbol{z}}\|, \|\Phi D\widetilde{\boldsymbol{z}} - \boldsymbol{y}\|_2 \leq \epsilon, \\ \hat{\boldsymbol{x}} &= D\hat{\boldsymbol{z}}. \end{aligned} \tag{P_3}$$

The only universal condition that is known for (P_3) to converge is that ΦD satisfies NSP, which then requires D to be incoherent. However, the incoherence is sometimes unnecessary if we only care about recovering $\boldsymbol{x}$ but $\boldsymbol{z}$. Therefore, finding a looser condition for the success of (P_3) is considered a major task along this direction.

1.4 Contributions

In Chapter 2, we study a statistical version of RIP that are sufficient for (P_1) to recover nearly all sparse signals except for an ϵ proportion with small ϵ. Moreover, we show how these conditions can be implied by two simpler coherence conditions of a matrix. In this way, we are able to extend the existing theory of deterministic sensing matrices that have near optimal average performances.

In Chapter 3, we study the ℓ_1 synthesis method for the dictionary setting, and prove that ΦD being NSP is indeed necessary when D has full spark. Moreover, we generalize the usual NSP to a dictionary adapted NSP and use it to prove a stability result of the ℓ_1 synthesis method.

1.5 Model Setting

Let $\boldsymbol{x}$ be an N-dimensional real signal that has a sparse representation in a suitably chosen basis. $\boldsymbol{x}$ is said to be k-sparse if it has at most k nonzero coordinates and is said to be approximately k-sparse if it has at most k significant coordinates, i.e., entries of large magnitude compared to the other entries. The observation vector $\boldsymbol{y}$ is formed as a linear transformation of $\boldsymbol{x}$, i.e.,

$$\boldsymbol{y} = \Phi\boldsymbol{x} + \boldsymbol{w},$$

where Φ is an $m \times N$ real matrix, $m \ll N$, and $\boldsymbol{w}$ is a noise vector. We assume that $\boldsymbol{w}$ has bounded energy (i.e., $\|\boldsymbol{w}\|_2 < \varepsilon$).

For the $m \times N$ complex matrix Φ, let $\phi_1, \ldots, \phi_N$ be its columns. Let $[N] = \{1, 2, \ldots, N\}$ and let $I = \{i_1, \ldots, i_k\} \subset [N]$ be a k-subset of the set of coordinates. By $\mathcal{P}_k(N)$ we denote the set of all k-subsets of $[N]$. Below we write Φ_I to refer to the $m \times k$ submatrix of Φ formed of the columns with indices in I. Given a vector $\boldsymbol{x} \in \mathbb{R}^N$, we denote by $\boldsymbol{x}_I$ a k-dimensional vector given by the projection of the vector $\boldsymbol{x}$ on the coordinates in I.

The objective of an estimator is to find a good approximation of the signal $\boldsymbol{x}$ after observing $\boldsymbol{y}$. This is obviously impossible for general signals $\boldsymbol{x}$ but becomes tractable if we seek a sparse approximation $\hat{\boldsymbol{x}}$ which satisfies

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_p \leq C_1 \min_{\boldsymbol{x}' \text{ is } k\text{-sparse}} \|\boldsymbol{x} - \boldsymbol{x}'\|_q + C_2\varepsilon \tag{1.4}$$

for some $p, q \geq 1$ and constants C_1, C_2. Note that if $\boldsymbol{x}$ itself is k-sparse, then (1.4) implies that the recovery error $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$ is at most proportional to the norm of the noise. Moreover it implies that the recovery is stable in the sense that if $\boldsymbol{x}$ is approximately k-sparse then the recovery error is small. If the estimate satisfies an inequality of the type (1.4), we say that the recovery procedure satisfies a (p, q) error guarantee.

The Basis Pursuit algorithm (P_1) we dicussed in the previous section is known to provide both (ℓ_1, ℓ_1) and (ℓ_2, ℓ_1) error guarantees under the condition that Φ satisfies NSP (or RIP).

Another popular estimator for which the recovery guarantees are proved using coherence properties of the sampling matrix Φ is Lasso [50, 24]. Assume the vector $\boldsymbol{w}$ is independent of the signal and formed of independent identically distributed Gaussian random variables with zero mean and variance σ^2. Lasso is a regularization of the ℓ_1 minimization problem written as follows:

$$\hat{\boldsymbol{x}} = \arg\min_{\widetilde{x} \in \mathbb{R}^N} \frac{1}{2} \|\Phi \widetilde{\boldsymbol{x}} - \boldsymbol{y}\|_2^2 + \lambda_N \sigma^2 \|\widetilde{\boldsymbol{x}}\|_1. \tag{1.5}$$

Here λ_N is a regularization parameter which controls the complexity (sparsity) of the optimizer.

We say that Φ satisfies the *coherence property* if the inner product $|\langle \phi_1, \phi_j \rangle|$ is uniformly small, and call $\mu = \max_{i \neq j} |\langle \phi_i, \phi_j \rangle|$ the coherence parameter of the matrix. The importance of incoherent dictionaries has been recognized in a large number of papers on compressed sensing, among them [51, 54, 31, 20, 18, 19, 13]. The coherence condition plays an essential role in proofs of recovery guarantees in these and many other studies.

Chapter 2

A Statistical Restricted Isometry Property and Its Application on Studying Deterministic Sensing Matrices

2.1 Introduction

One of the fundamental problems in compressive sensing concerns constructing efficient deterministic sensing matrices that can universally compress and recover the class of sparse and nearly sparse signals. A sufficient condition for such matrices is given by the restricted isometry property (RIP). It has been shown that sparse signals compressed by an RIP map can be reconstructed using ℓ_1 minimization procedures such as Basis Pursuit and Lasso [22, 21, 17, 13].

While many other conditions such as the Null Space Property (NSP) [30]) and the Sparse Approximation Property (SAP) [49] have also been established, RIP still remains to be the only useful tool in the deterministic setting. However, verifying RIP for a given matrix is by no means an easy task. In fact, direct theoretical verifications have only appeared in the analysis of random sensing matrix, and numerical verification is proved to be NP hard. A usual approach to overcome this difficulty is applying the Gershgorin theorem to reduce the RIP condition to another condition on mutual coherence. Even though the new condition is more convenient to verify, it becomes less effective. For instance, numerical experiments in [6] have shown that the mutual coherence based performance analysis is often too conservative in predicting the sparse recovery results. For these reasons, researchers have started to look for other possible ways of relaxing RIP [54][14].

In this chapter, we shall establish a new useful relaxation of the RIP, prove its sufficiency for stable reconstruction, explore its connection with the matrix coherence properties, and finally use it to study deterministic sensing matrices.

2.1.1 The RIP property

As defined in Chapter 1, a matrix Φ is said to have a (k, δ)-RIP if

$$(1-\delta)\|\boldsymbol{x}\|_2^2 \leq \|\Phi\boldsymbol{x}\|_2^2 \leq (1+\delta)\|\boldsymbol{x}\|_2^2$$

holds for all k-sparse vectors $\boldsymbol{x}$, where $\delta \in (0,1)$ is a parameter. Equivalently, Φ is (k,δ)-RIP if $\|\Phi_I^T\Phi_I - \mathrm{Id}\| \leq \delta$ holds for all $I \in [N], |I| = k$, where $\|\cdot\|$ is the spectral norm and Id is the identity matrix. The RIP property provides a sufficient condition for the solution of (P_2) to satisfy the error guarantees of Basis Pursuit [22, 21, 17, 13]. In particular, by [17], $(2k, \sqrt{2}-1)$-RIP suffices for both (ℓ_1, ℓ_1) and (ℓ_2, ℓ_1) error estimates, while [13] improves this to $(1.75k, \sqrt{2}-1)$-RIP.

As is well known (see [51] [28]), coherence and RIP are related: a matrix with coherence parameter μ is $(k, (k-1)\mu)$-RIP. This connection has served as the starting point in a number of studies on constructing RIP matrices from incoherent dictionaries. To implement this idea one starts with a set of unit vectors $\phi_1, \ldots, \phi_N$ with maximum coherence μ. In other words, we seek a well-separated collection of lines through the origin in $\mathbb{R}^m$, or reformulating again, a good packing of the real projective space $\mathbb{R}P^{m-1}$. One way of constructing such packings begins with taking a set $\mathcal{C}$ of binary m-dimensional vectors whose pairwise Hamming distances are concentrated around $m/2$. Call the maximum deviation from $m/2$ the *width* w of the set $\mathcal{C}$. An incoherent dictionary is obtained by mapping the bits of a small-width code to bipolar signals and normalizing. The resulting coherence and width are related by $w(\mathcal{C}) = \mu m/2$.

One of the first papers to put forward the idea of constructing RIP matrices from binary vectors was the work by DeVore [27]. While [27] did not make a connection to error-correcting codes, a number of later papers pursued both its algorithmic and constructive aspects [8, 14, 15, 26]. Examples of codes with small width are given in [4], where they are studied under the name of small-bias probability spaces. RIP matrices obtained from the constructions in [4] satisfy $m = O(\frac{k \log N}{\log(\log kN)})^2$. Ben-Aroya and Ta-Shma [10] recently improved this to $m = O(\frac{k \log N}{\log k})^{5/4}$ for $(\log N)^{-3/2} \le \mu \le (\log N)^{-1/2}$. The advantage of obtaining RIP matrices from binary or spherical codes is low construction complexity: in many instances it is possible to define the matrix using only $O(\log N)$ columns while the remaining columns can be computed as their linear combinations. We also note a result by Bourgain et al. [11] who gave the first (and the only known) construction of RIP matrices with k on the order of $m^{\frac{1}{2}+\epsilon}$ (i.e., greater than $O(\sqrt{m})$). An overview of the state of the art in the construction of RIP matrices is given in a recent paper [7].

At the same time, in practical problems we still need to write out the entire matrix; so constructions of complexity $O(N)$ are an acceptable choice. Under these assumptions, the best tradeoff between m, k and N for RIP-matrices based on codes and coherence is obtained from Gilbert-Varshamov type code constructions: namely, it is possible to construct (k, δ)-RIP matrices with $m = 4(k/\delta)^2 \log N$. At the same time, already [4] observes that the sketch dimension in RIP matrices constructed from binary codes is at least $m = \Theta((k^2 \log N)/\log k)$.

2.1.2 Statistical incoherence properties

The limitations on incoherent dictionaries discussed in the previous section suggest relaxing the RIP condition. An intuitively appealing idea is to require that condition (1.1) hold for almost all rather than all k-subsets I, replacing RIP with a version of it, in which the near-isometry property holds with high probability with respect to the choice of $I \in \mathcal{P}_k(N)$. The statistical RIP (StRIP) of a matrix is easier to be satisfied, so they have a potential of supporting provable recovery guarantees from shorter sketches compared to the known constructive schemes relying on RIP.

Without loss of generality, we assume all sensing matrices Φ considered in this chapter have unit column norm. Before proceeding to the results, let us introduce a few more notations. Let $[N] := \{1, 2, \ldots, N\}$ and let $\mathcal{P}_k(N)$ denote the set of k-subsets of $[N]$. The usual notation for probability $\Pr$ is used to refer a probability measure when there is no ambiguity. At the same time, we use separate notation for some frequently encountered probability spaces. In particular, we use P_k to denote the uniform probability distribution on $\mathcal{P}_k(N)$. If we need to choose a random k-subset I and a random index in $[N]\backslash I$, we use the notation P_{k+1}. We use P_{R^k} to denote any probability measure on $\mathbb{R}^k$ which assigns equal probability to each of the 2^k orthants (i.e., with uniformly distributed signs).

The following definition is essentially due to Tropp [54, 53], where it is called conditioning of random subdictionaries.

Definition 3. *An $m \times N$ matrix Φ satisfies the statistical RIP property (is (k, δ, ϵ)-StRIP) if*

$$P_k(\{I \in \mathcal{P}_k(N) : \|\Phi_I^T \Phi_I - \mathrm{Id}\| \le \delta\}) \ge 1 - \epsilon.$$

In other words, the inequality

$$(1 - \delta)\|\boldsymbol{x}\|_2^2 \le \|\Phi_I \boldsymbol{x}\|^2 \le (1 + \delta)\|\boldsymbol{x}\|_2^2 \tag{2.1}$$

holds for at least a $(1 - \epsilon)$ proportion of all k-subsets of $[N]$ and for all $\boldsymbol{x} \in \mathbb{R}^k$.

A related but different definition was given later in several papers such as [14, 5, 31] as well as some others. In these works, a matrix is called (k, δ, ϵ)-StRIP if inequality (2.1) holds for at least $(1 - \epsilon)$ proportion of k-sparse unit vectors $\boldsymbol{z} \in \mathbb{R}^N$. While several well-known classes of matrices were shown to have this property, it is not sufficient for sparse recovery procedures. Several additional properties as well as specialized recovery procedures that make signal reconstruction possible were investigated in [14].

In this chapter we focus on the statistical isometry property as given by Def. 3 and mean this definition whenever we mention StRIP matrices. We note that condition (2.1) is scalable, so the restriction to unit vectors is not essential.

Definition 4. *An $m \times N$ matrix Φ satisfies a statistical incoherence condition (is (k, α, ϵ)-SINC) if*

$$P_k(\{I \in \mathcal{P}_k(N) : \max_{i \notin I} \|\Phi_I^T \phi_i\|_2^2 \le \alpha\}) \ge 1 - \epsilon. \tag{2.2}$$

This condition appeared implicitly in [52] and [18]. It has been shown that StRIP and SINC together imply exact recovery of strictly sparse signals. For completeness, we prove in Section 2.2.1 a stability result that may be known but not explicitely established.

Moreover, we note that the SINC property can be further relaxed. In (2.2), we allow a small probability of failure for the random choice of I, but for a fixed I, the coherence between Φ_I and all outside columns should be uniformly small. The condition can thus be relaxed if we can change this uniformity to with large probability (this probability is with respect to i). In other words, we want to build a condition of $\|\Phi_I^T \phi_i\|_2$ that allows a small probability of failure with respect to the random choice of both $I \in \mathcal{P}_k(N)$ and $i \in I^c$.

We let

$$\mathcal{B}(\Phi) = \{\|\Phi_I^T \phi_i\|_2 : I \in \mathcal{P}_k(N),\, i \in I^c\}$$

be the set of values of coherences between a collection of columns of Φ and another column outside this collection. Let us introduce the following definition.

Definition 5. *An $m \times N$ matrix Φ is said to satisfy a weak statistical incoherence condition (to be a $(k, \delta, \alpha, \epsilon)$-WSINC) if*

$$\sum_{t \in \mathcal{B}(\Phi)} P_{k+1}(\{(I, i), I \in A_\alpha(\Phi), i \in I^c \text{ such that } \|\Phi_I^T \phi_i\|_2 = t\}) g(\delta, t) \leq \frac{\epsilon}{N - k}, \quad (2.3)$$

where $g(\delta, t)$ is a positive increasing function of t and

$$A_\alpha(\Phi) = \{I \in \mathcal{P}_k(N) : \exists i \in I^c \text{ such that } \|\Phi_I^T \phi_i\|_2^2 > \alpha\}.$$

We note that this definition is informative if $g(\delta, t)$ is small; otherwise, we will just use the usual SINC condition. Below we use $g(\delta, t) = \exp(-(1-\delta)^2/(8t^2))$. This definition takes account of the distribution of values of the quantity $\|\Phi_I^T \phi_i\|$ and therefore allows the existence of very coherent columns. We will show in Section 2.2.2 that WSINC is enough for BP to find the correct support of $\boldsymbol{x}$.

Definition 6. *We say that a signal $\boldsymbol{x} \in \mathbb{R}^N$ is drawn from a generic random signal model $\mathcal{S}_k$ if*

1) The locations of the k coordinates of $\boldsymbol{x}$ with largest magnitudes are chosen among all k-subsets $I \subset [N]$ with a uniform distribution;

2) Conditional on I, the signs of the coordinates $x_i, i \in I$ are i.i.d. uniform Bernoulli random variables taking values in the set $\{1, -1\}$.

2.2 Statistical Incoherence Properties and Basis Pursuit

In this section we prove approximation error bounds for recovery by Basis Pursuit from linear sketches obtained using deterministic matrices with the StRIP and SINC properties.

2.2.1 StRIP Matrices with incoherence property

It was proved in [54] that random sparse signals sampled using matrices with the StRIP property can be recovered with high probability from low-dimensional sketches using linear programming. In this section we prove a similar result that in addition incorporates stability analysis.

Theorem 2.2.1. *Suppose that $\boldsymbol{x}$ is a generic random signal from the model $\mathcal{S}_k$. Let $\boldsymbol{y} = \Phi\boldsymbol{x}$ and let $\hat{\boldsymbol{x}}$ be the approximation of $\boldsymbol{x}$ by the Basis Pursuit algorithm. Let I be the set of k largest coordinates of $\boldsymbol{x}$. If*

1. *Φ is (k, δ, ϵ)-StRIP;*

2. Φ *is* $(k, \frac{(1-\delta)^2}{8\log(2N/\epsilon)}, \epsilon)$*-SINC,*

then with probability at least $1 - 3\epsilon$

$$\|\boldsymbol{x}_I - \hat{\boldsymbol{x}}_I\|_2 \leq \frac{1}{2\sqrt{2\log(2N/\epsilon)}} \min_{\boldsymbol{x}' \text{is } k \text{ -sparse}} \|\boldsymbol{x} - \boldsymbol{x}'\|_1$$

and

$$\|\boldsymbol{x}_{I^c} - \hat{\boldsymbol{x}}_{I^c}\|_1 \leq 4 \min_{\boldsymbol{x}' \text{is } k \text{ -sparse}} \|\boldsymbol{x} - \boldsymbol{x}'\|_1.$$

This theorem implies that if the signal $\boldsymbol{x}$ itself is k-sparse then the Basis Pursuit algorithm will recover it exactly. Otherwise, its output $\hat{\boldsymbol{x}}$ will be a tight sparse approximation of $\boldsymbol{x}$.

Theorem 2.2.1 will follow from the next three lemmas. Some of the ideas involved in their proofs are close to the techniques used in [23]. Let $\boldsymbol{h} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ be the error in recovery of Basis Pursuit. In the following $I \subset [N]$ refers to the support of the k largest coordinates of $\boldsymbol{x}$.

Lemma 2.2.2. *Let* $s = 8\log(2N/\epsilon)$*. Suppose that* $\|(\Phi_I^T \Phi_I)^{-1}\| \leq \frac{1}{1-\delta}$ *and*

$$\|\Phi_I^T \phi_i\|_2^2 \leq s^{-1}(1-\delta)^2 \quad \textit{for all } i \in I^c := [N] \setminus I.$$

Then

$$\|\boldsymbol{h}_I\|_2 \leq s^{-1/2} \, \|\boldsymbol{h}_{I^c}\|_1.$$

Proof. Clearly, $\Phi\boldsymbol{h} = \Phi\hat{\boldsymbol{x}} - \Phi\boldsymbol{x} = 0$, so $\Phi_I \boldsymbol{h}_I = -\Phi_{I^c}\boldsymbol{h}_{I^c}$ and

$$\boldsymbol{h}_I = -(\Phi_I^T \Phi_I)^{-1} \Phi_I^T \Phi_{I^c} \boldsymbol{h}_{I^c}.$$

We obtain

$$\|\boldsymbol{h}_I\|_2 \le \|(\Phi_I^T\Phi_I)^{-1}\|\|\Phi_I^T\Phi_{I^c}\boldsymbol{h}_{I^c}\|_2 \le \frac{1}{1-\delta}\sum_{i\in I^c}\|\Phi_I^T\phi_i\|_2|h_i|$$

$$\le s^{-1/2}\,\|\boldsymbol{h}_{I^c}\|_1,$$

as required. ∎

Next we show that the error outside I cannot be large. Below $\operatorname{sgn}(\boldsymbol{u})$ is a ±1-vector of signs of the argument vector $\boldsymbol{u}$.

Lemma 2.2.3. *Suppose that there exists a vector $\boldsymbol{v}\in\mathbb{R}^N$ such that*

(i) $\boldsymbol{v}$ *is contained in the row space of* Φ*, say* $\boldsymbol{v}=\Phi^T\boldsymbol{w}$;

(ii) $\boldsymbol{v}_I=\operatorname{sgn}(\boldsymbol{x}_I)$;

(iii) $\|\boldsymbol{v}_{I^c}\|_{\ell_\infty}\le 1/2$.

Then

$$\|\boldsymbol{h}_{I^c}\|_1 \le 4\|\boldsymbol{x}_{I^c}\|_1. \tag{2.4}$$

Proof. By (P_2) we have

$$\|\boldsymbol{x}\|_1 \ge \|\hat{\boldsymbol{x}}\|_1 = \|\boldsymbol{x}+\boldsymbol{h}\|_1 = \|\boldsymbol{x}_I+\boldsymbol{h}_I\|_1 + \|\boldsymbol{x}_{I^c}+\boldsymbol{h}_{I^c}\|_1$$

$$\ge \|\boldsymbol{x}_I\|_1 + \langle\operatorname{sgn}(\boldsymbol{x}_I),\boldsymbol{h}_I\rangle + \|\boldsymbol{h}_{I^c}\|_1 - \|\boldsymbol{x}_{I^c}\|_1.$$

Here we have used the inequality $\|\boldsymbol{a}+\boldsymbol{b}\|_1 \ge \|\boldsymbol{a}\|_1 + \langle\operatorname{sgn}(\boldsymbol{a}),\boldsymbol{b}\rangle$ valid for any two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^N$ and the triangle inequality. From this we obtain

$$\|\boldsymbol{h}_{I^c}\|_1 \le |\langle\operatorname{sgn}(\boldsymbol{x}_I),\boldsymbol{h}_I\rangle| + 2\|\boldsymbol{x}_{I^c}\|_1.$$

Further, using the properties of $\boldsymbol{v}$, we have

$$\begin{aligned}
|\langle \operatorname{sgn}(\boldsymbol{x}_I), \boldsymbol{h}_I\rangle| &= |\langle \boldsymbol{v}_I, \boldsymbol{h}_I\rangle| \\
&= |\langle \boldsymbol{v}, \boldsymbol{h}\rangle - \langle \boldsymbol{v}_{I^c}, \boldsymbol{h}_{I^c}\rangle| \\
&\le |\langle \Phi^T \boldsymbol{w}, \boldsymbol{h}\rangle| + |\langle \boldsymbol{v}_{I^c}, \boldsymbol{h}_{I^c}\rangle| \\
&\le |\langle \boldsymbol{w}, \Phi \boldsymbol{h}\rangle| + \|\boldsymbol{v}_{I^c}\|_{\ell_\infty} \|\boldsymbol{h}_{I^c}\|_1 \\
&\le \frac{1}{2}\|\boldsymbol{h}_{I^c}\|_1.
\end{aligned}$$

The statement of the lemma is now evident. ∎

Now we prove that such a vector $\boldsymbol{v}$ as defined in the last lemma indeed exists.

Lemma 2.2.4. *Let $\boldsymbol{x}$ be a generic random signal from the model $\mathcal{S}_k$. Suppose that the support I of the k largest coordinates of $\boldsymbol{x}$ is fixed. Under the assumptions of Lemma 2.2.2 the vector*

$$\boldsymbol{v} = \Phi^T \Phi_I (\Phi_I^T \Phi_I)^{-1} \operatorname{sgn}(\boldsymbol{x}_I)$$

satisfies (i)-(iii) of Lemma 2.2.3 with probability at least $1 - \epsilon$.

Proof. From the definition of $\boldsymbol{v}$ it is clear that it belongs to the row-space of Φ and $\boldsymbol{v}_I = \operatorname{sgn}(\boldsymbol{x}_I)$. We have $v_i = \phi_i^T \Phi_I (\Phi_I^T \Phi_I)^{-1} \operatorname{sgn}(\boldsymbol{x}_I) = \langle \boldsymbol{s}_i, \operatorname{sgn}(\boldsymbol{x}_I)\rangle$, where

$$\boldsymbol{s}_i = (\Phi_I^T \Phi_I)^{-1} \Phi_I^T \phi_i \in \mathbb{R}^k.$$

We will show that $|v_i| \le \frac{1}{2}$ for all $i \in I^c$ with probability $1 - \epsilon$.

Since the coordinates of $\operatorname{sgn}(\boldsymbol{x}_I)$ are i.i.d. uniform random variables taking values in the set $\{\pm 1\}$, we can use Hoeffding's inequality to claim that

$$P_{R^k}(|v_i| > 1/2) \le 2 \exp\Big(- \frac{1}{8\|\boldsymbol{s}\|_2^2}\Big). \tag{2.5}$$

On the other hand, for all $i \in I^c$,

$$\begin{aligned}
\|\boldsymbol{s}_i\|_2 &= \|(\Phi_I^T\Phi_I)^{-1}\Phi_I^T\phi_i\|_2 \\
&\leq \|(\Phi_I^T\Phi_I)^{-1}\|\|\Phi_I^T\phi_i\|_2 \\
&\leq \frac{1}{1-\delta}\frac{1-\delta}{\sqrt{8\log(2N/\epsilon)}} \\
&= \frac{1}{\sqrt{8\log(2N/\epsilon)}}. \qquad (2.6)
\end{aligned}$$

Equations (2.5) and (2.6) together imply for any $i \in I^c$,

$$P_{R^k}\Big(|v_i| > \frac{1}{2}\Big) \leq 2\exp\Big(-\frac{1}{8(1/\sqrt{8\log(2N/\epsilon)})^2}\Big) = \frac{\epsilon}{N}.$$

Using the union bound, we now obtain the following relation:

$$P_{R^k}\Big(\|\boldsymbol{v}_{I^c}\|_\infty > 1/2\Big) \leq \epsilon. \qquad (2.7)$$

Hence $|v_i| \leq \frac{1}{2}$ for all $i \in I^c$ with probability at least $1-\epsilon$. ∎

Now we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. The matrix Φ is (k, δ, ϵ)-SRIP. Hence, with probability at least $1-\epsilon$, $\|(\Phi_I^T\Phi_I)^{-1}\| \leq \frac{1}{1-\delta}$. At the same time, from the SINC assumption we have, with probability at least $1-\epsilon$ over the choice of I,

$$\|\Phi_I^T\phi_i\|_2^2 \leq \frac{(1-\delta)^2}{8\log(2N/\epsilon)},$$

for all $i \in I^c$. Thus, Φ_I will have these two properties with probability at least $1-2\epsilon$. Then from Lemma 2.2.2 we obtain that

$$\|\boldsymbol{h}_I\|_2 \leq \frac{1}{\sqrt{8\log(2N/\epsilon)}}\|\boldsymbol{h}_{I^c}\|_1,$$

with probability $\geq 1 - 2\epsilon$. Furthermore, from Lemmas 2.2.3, 2.2.4

$$\|\boldsymbol{h}_{I^c}\|_1 \leq 4\|\boldsymbol{x}_{I^c}\|_1,$$

with probability $1 - \epsilon$. This completes the proof. ∎

2.2.2 StRIP Matrices with weak incoherence property

In this section we establish a recovery guarantee of Basis Pursuit under the weak SINC condition defined earlier in this chapter.

Theorem 2.2.5. *Suppose that the sampling matrix Φ is (k, δ, ϵ)-StRIP and $\left(k, \delta, \alpha, \epsilon^2\right)$-WSINC, where $\alpha = (1-\delta)^2/8\log(2N/\epsilon)$ and $g_\delta(t) = \exp(-(1-\delta)^2/8t^2)$. Suppose that the signal $\boldsymbol{x}$ is chosen from the generic random signal model and let $\hat{\boldsymbol{x}}$ be the approximation of $\boldsymbol{x}$ found by Basis Pursuit. Then with probability at least $1 - 4\epsilon$ we have*

$$\|\boldsymbol{x}_{I^c} - \hat{\boldsymbol{x}}_{I^c}\|_1 \leq 4 \min_{\boldsymbol{x}' \text{is } k\text{-sparse}} \|\boldsymbol{x} - \boldsymbol{x}'\|_1.$$

If $\boldsymbol{x}$ is k-sparse and satisfies the condition $\boldsymbol{y} = \Phi\boldsymbol{x}$, then this theorem asserts that Basis Pursuit will find the support of x. If in addition $\boldsymbol{x}$ is the only k-sparse solution to $\boldsymbol{y} = \Phi\boldsymbol{x}$, then we have $\hat{x} = x$. Note that the WSINC property is not sufficient for the (ℓ_2, ℓ_1) error guarantee. However, once the corrected support is detected, the signal $\boldsymbol{x}$ can be found by solving the overcomplete system $\boldsymbol{y} = \Phi_I\boldsymbol{x}$.

To prove Theorem 2.2.5, we refine the ideas used to establish Lemma 2.2.4.

Lemma 2.2.6. *Suppose that the sampling matrix Φ satisfies the conditions of Theorem 2.2.5. For any $\boldsymbol{x} \in \mathbb{R}^k$ and $I \subset [N]$ define $v(\boldsymbol{x}, I) = \Phi^T\Phi_I(\Phi_I^T\Phi_I)^{-1}\operatorname{sgn}(\boldsymbol{x})$. Let*

$$p(I) = P_{R^k}(\|v_{I^c}(\boldsymbol{x}, I)\|_\infty > 1/2),$$

Then

$$P_k(\{I : p(I) > \epsilon\}) < 3\epsilon.$$

Proof. As in the proof of Lemma 2.2.4, we define the vector

$$\boldsymbol{s}_i(I) = (\Phi_I^T \Phi_I)^{-1} \Phi_I^T \phi_i \in \mathbb{R}^k,$$

and let $v_i(\boldsymbol{x}, I)$ be the ith coordinate of the vector $v(\boldsymbol{x}, I)$. From now on we write simply $v_i, \boldsymbol{s}_i$, omitting the dependence on I and $\boldsymbol{x}$. Let $M = M(\Phi) := \{I \in \mathcal{P}_k(N) : \|\Phi_I^T \Phi_I\| \geq 1 - \delta\}$, then the StRIP property of Φ implies that

$$P_k(M) \geq 1 - \epsilon.$$

By definition, for any $I \in M$

$$\|\boldsymbol{s}_i\|_2 = \|(\Phi_I^T \Phi_I)^{-1} \Phi_I^T \phi_i\|_2 \leq \frac{1}{1-\delta} \|\Phi_I^T \phi_i\|_2.$$

Now we split the target probability into three parts:

$$P_k(\{I : p(I) > \epsilon\}) = P_k(\{I \in M \cap A : p(I) > \epsilon\}) + P_k(\{I \in M \cap A^c : p(I) > \epsilon\}) \\ + P_k(\{I \in M^c : p(I) > \epsilon\}),$$

where $A = A_\alpha(\Phi) = \{I : \|\Phi_I^T \phi_i\|_2^2 > \alpha \text{ for some } i \in I^c\}$ is the set of supports appearing in the definition of the WSINC property. If $I \in M \cap A$, i.e., it supports both StRIP and SINC properties, then (2.7) implies that $p(I) \leq \epsilon$, so the first term on the right-hand side equals 0. The third term refers to supports with no SINC property, whose total probability is $\leq \epsilon$. Estimating the second term by the Markov inequality, we have

$$P_k(\{I \in M \cap A^c : p(I) > \epsilon\}) \leq \frac{\mathsf{E}_k[p(I), \mathbf{1}(I \in M \cap A^c)]}{\epsilon}, \qquad (2.8)$$

where $\mathbf{1}(\cdot)$ denotes the indicator random variable. We have

$$\mathsf{E}_k[p(I), I \in M \cap A^c] = \mathsf{E}_k[p(I)\mathbf{1}(I \in M \cap A^c)] = \sum_{I \in M \cap A^c} \frac{1}{\binom{N}{k}} p(I), \tag{2.9}$$

Let us first estimate $p(I)$ for $I \in M \cap A^c$ by invoking Hoeffding's inequality (2.5):

$$\begin{aligned} p(I) &= P_{R^k}(\exists i \in I^c,\ \ |v_i| > 1/2) \leq \sum_{i \in I^c} P_{R^k}(|v_i| > 1/2) \\ &\leq \sum_{i \in I^c} 2\exp\Big(-\frac{1}{8\|\boldsymbol{s}_i\|_2^2}\Big) \\ &\overset{(2.6)}{\leq} \sum_{i \in I^c} 2\exp\Big(-\frac{(1-\delta)^2}{8\|\Phi_I^T \phi_i\|_2^2}\Big) \\ &= 2(N-k)\sum_{t \in \mathcal{B}(\Phi)} \exp\Big(-\frac{(1-\delta)^2}{8t^2}\Big) P_{R_k'}(\|\Phi_I^T \phi_i\|_2 = t \mid I). \end{aligned}$$

Substituting this result into (2.9), we obtain

$$\begin{aligned} \mathsf{E}_k[p(I), \{I \in M \cap A^c\}] &\leq 2(N-k)\sum_{t \in \mathcal{B}(\Phi)} \exp\Big(-\frac{(1-\delta)^2}{8t^2}\Big) \sum_{I \in M \cap A^c} \frac{1}{\binom{N}{k}} P_{R_k'}(\|\Phi_I^T \phi_i\| = t \mid I) \\ &\leq 2(N-k)\sum_{t \in \mathcal{B}(\Phi)} \exp\Big(-\frac{(1-\delta)^2}{8t^2}\Big) P_{R_k'}(I \in A^c, \|\Phi_I^T \phi\|_2 = t) \\ &\leq 2\epsilon^2, \end{aligned}$$

where the last step is on account of (2.8) and the WSINC assumption. ∎

Proof of Theorem 2.2.5: Define the set B by

$$B = \{I \in R_k : P_{R^k}(\|\boldsymbol{v}_{I^c}\|_\infty > 1/2 \mid I) > \epsilon\}.$$

Recall that Theorem 2.2.5 is stated with respect to the random signal $\boldsymbol{x}$. Therefore, let us estimate the probability

$$
\begin{aligned}
&P_{R_k\times R^k}(\{(I,\boldsymbol{x}):\ \|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}) \\
&\qquad = \sum_{I\in\mathcal{P}_k(N)} P_{R_k\times R^k}(\{\boldsymbol{x}:\|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}\mid I)P_{R_k\times R^k}(I) \\
&\qquad = \sum_{I\in B^c} P_{R^k}(\{\boldsymbol{x}:\|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}\mid I)P_k(I) + \sum_{I\in B} P_{R^k}(\{\boldsymbol{x}:\|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}\mid I)P_k(I).
\end{aligned}
$$

We have $P_{R^k}(\{\boldsymbol{x}:\|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}\mid I) < \epsilon$ from Lemma 2.2.4 and $P_k(B)\le 3\epsilon$ from Lemma 2.2.6, so

$$P_{R_k\times R^k}(\{(I,\boldsymbol{x}):\ \|\boldsymbol{v}_{I^c}\|_\infty > 1/2\}) < \epsilon(1+3\epsilon) < 4\epsilon.$$

This implies that with probability $1-4\epsilon$ the signal $\boldsymbol{x}$ chosen from the generic random signal model satisfies the conditions of Lemma 2.2.3, i.e.,

$$\|\boldsymbol{x}_{I^c}-\hat{\boldsymbol{x}}_{I^c}\|_1 \le 4\|\boldsymbol{x}_{I^c}\|_1.$$

This completes the proof. ∎

2.3 Incoherence Properties and Lasso

In this section we prove that sparse signals can be approximately recovered from low-dimensional observations using Lasso if the sampling matrices have statistical incoherence properties. The result is a modification of the methods developed in [18, 54] in that we prove that the conditions used there to bound the error of the Lasso estimate hold with high probability if Φ is has both StRIP and SINC properties. The precise claim is given in the following statement.

Theorem 2.3.1. *Let $\boldsymbol{x}$ be a random k-sparse signal whose support satisfies the two properties of the generic random signal model S_k. Denote by $\hat{\boldsymbol{x}}$ its estimate from $\boldsymbol{y} = \Phi\boldsymbol{x} + \boldsymbol{z}$ via Lasso* (1.5)*, where $\boldsymbol{z}$ is a i.i.d. Gaussian vector with zero mean and variance σ^2 and where $\lambda = 2\sqrt{2\log N}$. Suppose that $k \leq \frac{c_0 N}{\|\Phi\|^2 \log N}$, where c_0 is a positive constant, and that the matrix Φ satisfies the following two properties:*

1. *Φ is $(k, \frac{1}{2}, \epsilon)$-StRIP.*

2. *Φ is $(k, \frac{1}{128\log(N/2\epsilon)}, \epsilon)$-SINC.*

Then we have

$$\|\Phi\boldsymbol{x} - \Phi\hat{\boldsymbol{x}}\|_2^2 \leq C_0 k \log N \sigma^2,$$

with probability at least $1 - 3\epsilon - \frac{1}{N\sqrt{2\pi\log N}} - N^{-a}$, where $C_0 > 0$ is an absolute constant and $a = 0.15\log(2N/\epsilon) - 1$.

The following theorem is implicit in [18], see Theorem 1.2 and Sect 3.2 in that paper.

Theorem 2.3.2. (Candès and Plan) *Suppose that $\boldsymbol{x}$ is a k-sparse signal drawn from the model S_k, $\boldsymbol{y}$, $\boldsymbol{z}$ are the same as in Theorem 2.3.1 and*

$$k \leq \frac{c_0 N}{\|\Phi\|^2 \log N},$$

where $c_0 > 0$ is a constant. Let $I \subset [N]$ be the support of $\boldsymbol{x}$ and suppose the following three conditions are satisfied:

1. *$\|(\Phi_I^T \Phi_I)^{-1}\| \leq 2$.*

2. *$\|\Phi^T \boldsymbol{z}\|_{\ell_\infty} \leq 2\sqrt{\log N}$.*

3. $\|\Phi_{I^c}^T \Phi_I (\Phi_I^T \Phi_I)^{-1} \Phi_I^T \boldsymbol{z}\|_{\ell_\infty} + \sqrt{8 \log N} \|\Phi_{I^c}^T \Phi_I (\Phi_I^T \Phi_I)^{-1} \operatorname{sgn}(\boldsymbol{x}_I)\|_{\ell_\infty} \leq (2-\sqrt{2})\sqrt{2 \log N}$.

Then

$$\|\Phi \boldsymbol{x} - \Phi \hat{\boldsymbol{x}}\|_2^2 \leq C_0 k (\log N) \sigma^2,$$

where C_0 is an absolute constant.

Our aim will be to prove that conditions (1)-(3) of this theorem hold with large probability under the assumptions of Theorem 2.3.1.

First, it is clear that $\|\Phi^T \boldsymbol{z}\|_\infty \leq 2\sqrt{\log N}$ with probability at least $1-(N\sqrt{2\pi \log N})^{-1}$. This follows simply because $\boldsymbol{z}$ is an independent Gaussian vector, and has been discussed in [18] (this is also the reason for selecting the particular value of λ_N). The main part of the argument is contained in the following lemma whose proof uses some ideas of [18].

Lemma 2.3.3. *Suppose that $1/2 \leq \|\Phi_I^T \Phi_I - \mathrm{Id}\| \leq 3/2$ and that for all $i \in I^c$,*

$$\|\Phi_I^T \phi_i\|_2^2 \leq (128 \log(2N/\epsilon))^{-1}.$$

Then Condition (3) of Theorem 2.3.2 holds with probability at least $1 - \epsilon - N^{-a}$ for $a = 0.15 \log(2N/\epsilon) - 1$.

Proof. Let $i \in I^c$. Define $Z_{0,i} = \langle \boldsymbol{w}_i, \operatorname{sgn}(\boldsymbol{x}_I) \rangle$ and $Z_{1,i} = \langle \boldsymbol{w}'_i, \boldsymbol{z} \rangle$, where

$$\boldsymbol{w}_i = (\Phi_I^T \Phi_I)^{-1} \Phi_I^T \phi_i,$$

$$\boldsymbol{w}'_i = \Phi_I (\Phi_I^T \Phi_I)^{-1} \Phi_I^T \phi_i.$$

Let $Z_0 = \max_{i \in I^c} |Z_{0,i}|$ and $Z_1 = \max_{i \in I^c} |Z_{1,i}|$. We will show that with high probability

$Z_0 \leq 1/4$ and $Z_1 \leq (1.5 - \sqrt{2})\sqrt{2\log N}$ which will imply the lemma. We compute

$$\begin{aligned}\|\boldsymbol{w}_i\|_2 &\leq \|(\Phi_I^T\Phi_I)^{-1}\|\|\Phi_I^T\phi_i\|_2 \leq 2\frac{1}{8\sqrt{2\log(2N/\epsilon)}}\\ &= \frac{1}{4\sqrt{2\log(2N/\epsilon)}},\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{w}_i'\|_2 &\leq \|\Phi_I\|\|(\Phi_I^T\Phi_I)^{-1}\|\|\Phi_I^T\phi_i\|_2 \leq \sqrt{\frac{3}{2}}\frac{2}{8\sqrt{2\log(2N/\epsilon)}}\\ &= \frac{\sqrt{3}}{8\sqrt{\log(2N/\epsilon)}},\end{aligned}$$

for all $i \in I^c$. Let $a_1 = 1.5 - \sqrt{2}$. Since $Z_{1,i} \sim \mathcal{N}(0, \|\boldsymbol{w}_i'\|_2^2)$, we have

$$\begin{aligned}\Pr(Z_1 > a_1\sqrt{2\log N}) &\leq (N-k)\Pr\left(|Z_{1,i}| > a_1\sqrt{2\log N}\right)\\ &\leq \frac{2(N-k)\|\boldsymbol{w}_i'\|_2}{a_1\sqrt{2\pi(2\log N)}}\, e^{-\frac{64}{3}a_1^2\log N\log(2N/\epsilon)}\\ &\leq \frac{2.1}{\sqrt{(2\log N)\log(2N/\epsilon)}} N^{-0.15\log(2N/\epsilon)+1}\\ &\leq N^{-a}.\end{aligned}$$

(the multiplier in front of the exponent is less than 1 for all $N > 4$ and $\epsilon < 1$). Further, since the signs $\mathrm{sgn}(x_i), i \in I$ are uniform i.i.d. random variables, we have

$$\begin{aligned}\Pr(Z_0 > 1/4) &\leq (N-k)\Pr(|\langle\boldsymbol{w}_i, \mathrm{sgn}(\boldsymbol{x}_I)\rangle| > 1/4)\\ &\leq 2(N-k)e^{-1/(32\|w_i\|_2^2)}\\ &< \epsilon.\end{aligned}$$

The proof is complete. ∎

Theorem 2.3.1 is now easily established. Indeed, the assumptions of Lemma 2.3.3 are satisfied with probability at least $1 - 2\epsilon$. The claim of the theorem follows from the above arguments.

2.4 Sufficient conditions for statistical incoherence properties

In this section, we discuss how the StRIP and SINC properties can be controlled by matrix coherence. Upon the completion of this project, we realized another result of Tropp [53] which is better in many cases. However, I feel that this effort is still worth mentioning since it utilizes a different technique and is better than previous results in many special cases.

Specifically, we show in Theorem 2.4.7 that for Φ to satisfies (k, δ)-StRIP, we need its coherence to satisfy $\mu \leq O(k^{-1/4})$. Comparing to Tropp's result which essentially needs $\mu \leq O(\log^{-1} N)$, it is better when $k < \log^4 n$. We comment that $\log^4 n$ is usually not a small number due to the fourth power, so it is quite possible that the sparsity level of the incoming signal falls below this level. For examples of various explicit deterministic constructions on which our theories may apply, we refer the reader to the table in [9].

Let Φ be an $m \times N$ sampling matrix with columns $\phi_i, i = 1, \dots, N$. As above, let $\mu_{ij} = |\langle \phi_i, \phi_j \rangle|$. We also define the mean square coherence and the maximum average square coherence of the dictionary:

$$\bar{\mu}^2 = \frac{1}{N(N-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mu_{ij}^2, \quad \bar{\mu}^2_{\max} = \max_{1 \leq j \leq N} \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} \mu_{ij}^2.$$

Of course, $\bar{\mu}^2 \leq \bar{\mu}^2_{\max}$ with equality if and only if for every j the sum in $\bar{\mu}^2_{\max}$ takes the same value. Dictionaries that satisfy this property will be called *coherence-invariant*. It turns out that a large group of known constructions satisfy the invariance property; see in particular [9]. Our arguments change slightly if the matrix is not coherence-invariant. To deal simultaneously with both cases, define the parameter $\theta = \theta(\Phi)$ as $\theta = \bar{\mu}^2$ if Φ is coherence-invariant and $\theta = \bar{\mu}^2_{\max}$ otherwise.

The next theorem gives sufficient conditions for the SINC property in terms of coherence parameters of Φ.

Theorem 2.4.1. *Let Φ be an $m \times N$ matrix with unit-norm columns, coherence μ and square coherence θ. Suppose that,*

$$\mu^4 \le \frac{(1-a)^2\beta^2}{32k(\log 2N/\epsilon)^3} \quad \text{and} \quad \theta \le \frac{a\beta}{k\log(2N/\epsilon)}, \tag{2.10}$$

where $\beta > 0$ and $0 < a < 1$ are any constants. Then Φ has the (k, α, ϵ)-SINC property with $\alpha = \beta/\log(2N/\epsilon)$.

Before proving this theorem we will introduce some notation. Fix $j \in [N]$ and let $I_j = \{i_1, i_2, \dots, i_k\}$ be a random k-subset such that $j \notin I_j$. The subsets I_j are chosen from the set $[N-1]$ with uniform distribution. Define random variables $Y_{j,l} = \mu^2_{j,i_l}, l = 1, \dots, k$. Next define a sequence of random variables $Z_{j,t}, t = 0, 1, \dots, k$, where

$$Z_{j,0} = \mathsf{E}_{I_j} \sum_{l=1}^{k} Y_{j,l}, \quad Z_{j,t} = \mathsf{E}_{I_j}\Big(\sum_{l=1}^{k} Y_{j,l} \mid Y_{j,1}, Y_{j,2}, \dots, Y_{j,t}\Big),\ t = 1, 2, \dots, k.$$

From the assumption of coherence invariance, the variables $Z_{j,t}$ for different j are stochastically equivalent. Let

$$Z_t = \mathsf{E}_j Z_{j,t} = \mathsf{E}_{R'_k}\Big(\sum_{l=1}^{k} Y_{j,l} \mid Y_{j,1}, Y_{j,2}, \dots, Y_{j,t}\Big), \quad t = 1, \dots, k.$$

The random variables Z_t are defined on the set of $(k+1)$-subsets of $[N]$ with probability distribution P_{k+1}. We will show that they form a Doob martingale. Begin with defining a sequence of σ-algebras $\mathcal{F}_t, t = 0, 1, \ldots, k$, where $\mathcal{F}_0 = \{\emptyset, [N]\}$ and $\mathcal{F}_t, t \geq 1$ is the smallest σ-algebra with respect to which the variables $Y_{j,1}, \ldots, Y_{j,t}$ are measurable (thus, $\mathcal{F}_t$ is formed of all subsets of $[N]$ of size $\leq t+1$). Clearly, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k$, and for each t, Z_t is a bounded random variable that is measurable with respect to $\mathcal{F}_t$. Observe that

$$Z_0 = \mathsf{E}_j Z_{j,0} = \mathsf{E}_{R'_k} \sum_{l=1}^{k} \mu^2_{j,i_l} = \sum_{l=1}^{k} \mathsf{E}_{R'_k} \mu^2_{j,i_l} = k\bar{\mu}^2 \tag{2.11}$$

$$\leq k\bar{\mu}^2_{\max}, \tag{2.12}$$

where (2.11) assumes coherence invariance, and (2.12) is valid independently of that assumption.

Lemma 2.4.2. *The sequence* $(Z_t, \mathcal{F}_t)_{t=0,1,\ldots,k}$ *forms a bounded-differences martingale, namely* $\mathsf{E}_{R'_k}(Z_t \mid Z_0, Z_1, \ldots, Z_{t-1}) = Z_{t-1}$ *and*

$$|Z_t - Z_{t-1}| \leq 2\mu^2\Big(1 + \frac{k}{N-k-2}\Big), \quad t = 1, \ldots, k.$$

Proof. In the proof we write E instead of $\mathsf{E}_{R'_k}$. We have

$$\begin{aligned} Z_t = \mathsf{E}\Big(\sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t\Big) &= \sum_{l=1}^{t} Y_{j,l} + \mathsf{E}\Big(\sum_{l=t+1}^{k} Y_{j,l} \mid \mathcal{F}_t\Big) \\ &= Z_{t-1} + Y_{j,t} + \mathsf{E}\Big(\sum_{l=t+1}^{k} Y_{j,l} \mid \mathcal{F}_t\Big) - \mathsf{E}\Big(\sum_{l=t}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}\Big). \end{aligned}$$

Next,

$$\mathsf{E}(Z_t \mid Z_0, Z_1, \ldots, Z_{t-1}) = Z_{t-1} + \mathsf{E}(Y_{j,t} \mid Z_0, Z_1, \ldots, Z_{t-1})$$

$$
\begin{aligned}
&+ \mathsf{E}\Big(\mathsf{E}\Big(\sum_{l=t+1}^{k} Y_{j,l} \mid \mathcal{F}_t\Big) \mid Z_0, \dots, Z_{t-1}\Big) \\
&\quad - \mathsf{E}\Big(\mathsf{E}\Big(\sum_{l=t}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}\Big) \mid Z_0, \dots, Z_{t-1}\Big) \\
&= Z_{t-1} + \mathsf{E}\Big(Y_{j,t} \mid Z_0, \dots, Z_{t-1}\Big) \\
&\quad + \mathsf{E}\Big(\sum_{l=t+1}^{k} Y_{j,l} \mid Z_0, \dots, Z_{t-1}\Big) - \mathsf{E}\Big(\sum_{l=t}^{k} Y_{j,l} \mid Z_0, \dots, Z_{t-1}\Big) \\
&= Z_{t-1},
\end{aligned}
$$

which is what we claimed.

Next we prove a bound on the random variable $|Z_t - Z_{t-1}|$. We have

$$
\begin{aligned}
|Z_t - Z_{t-1}| &= \Big|\mathsf{E}\Big(\sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t\Big) - \mathsf{E}\Big(\sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}\Big)\Big| \\
&\le \max_{a,b} \Big|\mathsf{E}\Big(\sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a\Big) - \mathsf{E}\Big(\sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b\Big)\Big| \\
&= \max_{a,b} \Big|\sum_{l=1}^{k} \Big(\mathsf{E}\Big(Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a\Big) - \mathsf{E}\Big(Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b\Big)\Big)\Big| \\
&= \max_{a,b} \Big|a - b + \sum_{l=t+1}^{k} \Big(\mathsf{E}\Big(Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a\Big) - \mathsf{E}\Big(Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b\Big)\Big)\Big| \\
&\le \Big|2\mu^2 + \sum_{l=t+1}^{k} \frac{2\mu^2}{N - l - 2}\Big| \\
&= 2\mu^2 \frac{N-2}{N-k-2}.
\end{aligned}
$$

∎

To prove Theorem 2.4.1 we use the Azuma-Hoeffding inequality (see, e.g., [43]).

Proposition 2.4.3. (Azuma-Hoeffding) *Let $X_0, \dots, X_{k-1}$ be a martingale with $|X_i -$*

$X_{i-1}| \leq a_i$ *for each* i*, for suitable constants* a_i*. Then for any* $\nu > 0$,

$$\Pr\Big(\Big|\sum_{t=1}^{k-1}(X_i - X_{i-1})\Big| \geq \nu\Big) \leq 2\exp\frac{-\nu^2}{2\sum a_i^2}.$$

Proof of Theorem 2.4.1: Bounding large deviations for the sum $|\sum_{t=1}^{k}(Z_t - Z_{t-1})| = |Z_k - Z_0|$, we obtain

$$\Pr(|Z_k - Z_0| > \nu) \leq 2\exp\Big(-\frac{\nu^2}{8\mu^4 k(\frac{N-2}{N-k-2})^2}\Big), \tag{2.13}$$

where the probability is computed with respect to the choice of *ordered* $(k+1)$-tuples in $[N]$ and $\nu > 0$ is any constant. Assume coherence invariance. Using (2.11) and the inequality $(N-2)/(N-k-2) < 2$ valid for all $k < \frac{N}{2} - 1$, we obtain

$$\Pr(Z_k \geq \nu + k\bar{\mu}^2) \leq \Pr(|Z_k - k\bar{\mu}^2| \geq \nu) \leq 2\exp\Big(-\frac{\nu^2}{32\mu^2 k}\Big).$$

Now take $\beta > 0$ and $\nu = \frac{\beta}{\log(2N/\epsilon)} - k\bar{\mu}^2$. Suppose that for some $a \in (0,1)$

$$k\mu^4 \leq \frac{((1-a)\beta)^2}{32}\Big(\log\frac{2N}{\epsilon}\Big)^{-3}, \quad k\bar{\mu}^2 \leq \frac{a\beta}{\log(2N/\epsilon)}, \tag{2.14}$$

then we obtain

$$\Pr\Big(\|\Phi_{I_j}^T\phi_j\|_2^2 \geq \frac{\beta}{\log(2N/\epsilon)}\Big) \leq 2\exp\Big(-\frac{\nu^4}{32\mu^4 k}\Big) \leq \frac{\epsilon}{N}. \tag{2.15}$$

Now the first claim of Theorem 2.4.1 follows by the union bound with respect to the choice of the index j.

Assume that Φ does not satisfy the invariance condition. Then we rely on (2.12) and repeat the above argument with respect to $\bar{\mu}^2_{\max}$. ∎

The above proof contains the following statement.

Corollary 2.4.4. *Let Φ be an $m \times N$ matrix with coherence μ and $\theta = \bar{\mu}^2$ or $\bar{\mu}^2_{\max}$, as appropriate. Let $a \in (0,1)$ and $\beta > 0$ be any constants. Suppose that for $\alpha < \beta \log_2 e$,*

$$\mu^4 \leq \frac{(1-a)^2\alpha^3}{32\beta k}, \quad k\theta \leq a\alpha.$$

Then $P_{k+1}(\sum_{l=1}^{k} \mu^2_{i_l,j} \geq \alpha) \leq 2e^{-\beta/\alpha}$.

Proof. Denote $\alpha = \beta/(\log(2N/\epsilon))$, then $\epsilon/N = 2e^{-\beta/\alpha}$. The claim is obtained by substituting α in (2.14)-(2.15). ∎

We note that this corollary follows directly from the SINC property under our assumptions on coherence and mean square coherence. We observe that the SINC property naturally implies some StRIP condition as given in the following theorem.

Theorem 2.4.5. *Let Φ be an $m \times N$ matrix. Let $I \subset [N]$ be a random ordered k-subset and suppose that for all $j \in I$, $\Pr(\sum_{m=1}^{k-1} \mu^2_{j,i_m} > \delta^2/k) < \epsilon_1/k$. Then Φ is a (k, δ, ϵ_1)-StRIP matrix.*

Proof. Given I let $H(I) = \Phi_I^T \Phi_I - \mathrm{Id}$ be the "hollow Gram matrix". Let $B = \{I : \|H(I)\| > \delta\} \subset \mathcal{P}_k(N)$. We need to prove that $P_k(B) \leq \epsilon$. Let $(e_1, \ldots, e_k)$ be the standard basis of $\mathbb{R}^k$. Define a subset $C \subset \mathcal{P}_k(N)$ as follows:

$$C = \{I : \exists i \in I \text{ s.t. } \|H(I)e_i\|_2 \geq \delta/\sqrt{k}\}.$$

Let us show that $B \subseteq C$ by proving $C^c \subseteq B^c$. Indeed, if $I \in C^c$, then we have

$$\begin{aligned}\|H(I)\| &= \max_{|\boldsymbol{x}\|_2=1} \|H(I)\boldsymbol{x}\|_2 = \max_{|\boldsymbol{x}\|_2=1} \|H(I)(x_1e_1 + x_2e_2 + \cdots + x_ke_k)\| \\ &\leq \max_{|\boldsymbol{x}\|_2=1} \sum_l |x_l| \, \|H(I)e_l\|_2\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{|\boldsymbol{x}\|_2=1} \|\boldsymbol{x}\|_1 \max_{1\leq l\leq k} \|H(I)e_l\|_2 \\
&\leq \sqrt{k} \max_{1\leq l\leq k} \|H(I)e_l\|_2. \\
&\leq \delta,
\end{aligned}
$$

which implies $I \in B^c$. Now since $B \subseteq C$, we only need to show that $P_k(C) \leq \epsilon$.

Careful readers may have already noticed that the target quantity $P_k(C)$ uses a different probability measure from that in theorem's assumption. We note that a change of measure is actually inevitable since the probability measure in Azuma-Hoeffding's inequality we used in Proposition 2.4.3 is with respect to ordered k-tuples while that in the definition of StRIP is with respect to unordered ones. In the following, we provide a rigorous calculation that supports this measure transformation.

For any $I \in C$, by definition, there exists at least one $l \in I$ such that $\|H_I e_l\| \geq \delta/\sqrt{k}$. Among such l, let $i(I)$ be the smallest one $i(I) = \min\{l \in I : \|H_I e_l\|_2 \geq \delta/\sqrt{k}\}$. Now we define a map from an unordered k-tuple $I \in C \subseteq \mathcal{P}_k(N)$ to a set of ordered k-tuples $Q(I) = \{(i_1, \ldots, i_{k-1}, i(I)) : (i_1, \ldots, i_{k-1}) = \sigma(I \backslash i(I)), \sigma \in S_{k-1}\}$, where S_{k-1} denotes the set of all permutations of $k-1$ elements. Obviously, $|Q(I)| = (k-1)!$ for all I, and $Q(I_1) \cap Q(I_2) = \emptyset$ for distinct k-subsets I_1, I_2. Moreover, if $(i_1, \ldots, i_k) \in Q(I)$, then $\|H(I)e_k\|_2 \geq \delta/\sqrt{k}$ or $\sum_{l=1}^{k-1} \mu_{i_l,i_k}^2 > \delta^2/k$. Therefore

$$
\bigcup_{I\in C} Q(I) \subseteq \Big\{(i_1, \ldots, i_k) \subset [N] : \sum_{l=1}^{k-1} \mu_{i_l,i_k}^2 > \delta^2/k\Big\}.
$$

Now compute

$$
\begin{aligned}
P_k(B) = \frac{|B|}{\binom{N}{k}} &\leq \frac{|C|(k-1)!}{\binom{N}{k}(k-1)!} = \frac{\sum_{I\in C} |Q(I)|}{\binom{N}{k}(k-1)!} \\
&= \frac{\big|\bigcup_{I\in C} Q(I)\big|}{\binom{N}{k}(k-1)!}
\end{aligned}
$$

$$\leq \frac{k}{k!\binom{N}{k}}\Big|\Big\{(i_1,\dots,i_k)\subset[N]:\sum_{l=1}^{k-1}\mu_{i_l,i_k}^2>\delta^2/k\Big\}\Big|$$
$$= k\,\Pr(\sum_{m=1}^{k-1}\mu_{j,i_m}^2>\delta^2/k).$$

By the assumption of the theorem the last expression is at most ϵ which proves our claim.

∎

Theorem 2.4.12 implies the following

Corollary 2.4.6. *Let Φ be an $m\times N$ matrix. If*

$$\theta\leq\frac{a\delta^2}{k^2},\quad \textit{and}\quad \mu^4\leq\frac{(1-a)^2\delta^4}{32k^3\log(2k/\epsilon_1)},$$

where $0<a<1$, then Φ is (k,δ,ϵ_1)-StRIP.

Proof. Take $\epsilon_1=2ke^{-\beta/\alpha}$, then $\beta=\frac{\delta^2}{k}\log(2k/\epsilon_1)$. The claim is obtained by substituting this value into the conditions of Corollary 2.4.4. ∎

Observe that the sufficient condition for the (k,δ)-RIP property from the Gershgorin theorem is $\mu<\delta/k$, so the result of Corollary 2.4.6 gives a better result, namely $\mu=O(k^{-3/4})$. At the same time, Tropp's result in [54, Thm. B] implies that the matrix Φ is (k,δ,ϵ)-StRIP under a weaker (i.e., more inclusive) condition. Below we improve upon these results by analyzing the StRIP property directly rather than relying on the SINC condition.

Theorem 2.4.7. *Let Φ by an $m\times N$ matrix and let $\theta=\bar{\mu}^2$ or $\theta=\bar{\mu}^2_{\max}$, depending on whether Φ is coherence-invariant or not. Let $\epsilon<\min\{1/k,e^{1-1/\log 2}\}$ and suppose that Φ satisfies*

$$k\mu^4\leq\frac{1}{\log^2(1/\epsilon)}\min\Big(\frac{(1-a)^2b^2}{32\log(2k)\log(e/\epsilon)},c^2\Big)\quad\textit{and}\quad k\theta\leq\frac{ab}{\log(1/\epsilon)},\qquad(2.16)$$

where $a, b, c \in (0, 1)$ are constants such that

$$\sqrt{b} + \sqrt{2ab} + \sqrt{c} + \frac{2k}{N}\|\Phi\|^2 \leq e^{-1/4}\delta/6\sqrt{2}. \tag{2.17}$$

Then Φ is (k, δ, ϵ)-StRIP.

The proof relies on several results from [54]. The following theorem is a modification of Theorem 25 in that paper. Below R denotes a linear operator that performs a restriction to k coordinates chosen according to some rule (e.g., randomly). Its domain is determined by the context. Its adjoint R^* acts on $\mathbb{R}^k$ by padding the k-vector with the appropriate number of zeros.

Theorem 2.4.8. (Decoupling of the spectral norm) *Let A be a $2N \times 2N$ symmetric matrix with zero diagonal. Let $\eta \in \{0, 1\}^{2N}$ be a random vector with N components equal to one. Define the index sets $T_1(\eta) = \{i : \eta_i = 0\}, T_2(\eta) = \{i : \eta_i = 1\}$. Let R be a random restriction to k coordinates. For any $q \geq 1$ we have*

$$(\mathsf{E}\|RAR^*\|^q)^{1/q} \leq 2 \max_{k_1+k_2=k} \mathsf{E}_\eta(\mathsf{E}\|R_1 A_{T_1(\eta)\times T_2(\eta)} R_2^*\|^q)^{1/q}, \tag{2.18}$$

where $A_{T_1(\eta)\times T_2(\eta)}$ denotes the submatrix of A indexed by $T_1(\eta) \times T_2(\eta)$ and the matrices R_i are independent restrictions to k_i coordinates from $T_i, i = 1, 2$.

When A has order $(2N+1)\times(2N+1)$, then an analogous result holds for partitions into blocks of size N and $N + 1$.

Inequality (2.18) is implicitly proved in the proof of the decoupling theorem (Theorem 9) [54]. The ideas behind it are due to [38].

The next lemma is due to Tropp [53] and Rudelson and Vershinin [48].

Lemma 2.4.9. *Suppose that A is a matrix with N columns and let R be a random restriction to k coordinates. Let $q \geq 2, p = \max(2, 2\log(\operatorname{rk} AR^*), q/2)$. Then*

$$(\mathsf{E}\|AR^*\|^q)^{1/q} \leq 3\sqrt{p}(E\|AR^*\|_{1\to 2}^q)^{1/q} + \sqrt{\frac{k}{N}}\|A\|,$$

where $\|\cdot\|_{1\to 2}$ is the maximum column norm.

The following lemma is a simple application of Markov's inequality, a similar result can be found in [38], Lemma 4.10; see also [54].

Lemma 2.4.10. *Let $q, \lambda > 0$ and let ξ_q be a positive function of q. Suppose that Z is a positive random variable whose qth moment satisfies the bound*

$$(\mathsf{E}Z^q)^{1/q} \leq \xi_q\sqrt{q} + \lambda.$$

Then

$$P(Z \geq e^{1/4}(\xi_q\sqrt{q} + \lambda)) \leq e^{-q/4}.$$

Proof: By the Markov inequality,

$$P\left(Z \geq e^{1/4}(\xi_q\sqrt{q} + \lambda)\right) \leq \frac{\mathsf{E}Z^q}{(e^{1/4}(\xi_q\sqrt{q} + \lambda))^q} \leq \left(\frac{\xi_q\sqrt{q} + \lambda}{e^{1/4}(\xi_q\sqrt{q} + \lambda)}\right)^q = e^{-q/4}. \qquad \blacksquare$$

The main part of the proof of Theorem 2.4.7 is contained in the following lemma.

Lemma 2.4.11. *Let Φ be an $m \times N$ matrix with coherence parameter μ. Suppose that for some $0 < \epsilon_1, \epsilon_2 < 1$*

$$P_{k+1}(\{(I, i) : \|\Phi_I^T \phi_i\|^2 \geq \epsilon_1\} \mid i) \leq \epsilon_2. \tag{2.19}$$

Let R be a random restriction to k coordinates and $H = \Phi^T\Phi - Id$. For any $q \geq 2, p =$ $\max(2, 2\log(\operatorname{rk} RHR^*), q/2)$ we have

$$(\mathsf{E}\|RHR^*\|^q)^{1/q} \leq 6\sqrt{p}(\sqrt{\epsilon_1} + (k\epsilon_2)^{1/q}\mu\sqrt{k} + \sqrt{2k\theta}\,) + \frac{2k}{N}\|\Phi\|^2. \tag{2.20}$$

Proof. We begin with setting the stage to apply Theorem 2.4.8. Let $\eta \in \{0,1\}^N$ be a random vector with $N/2$ ones and let R_1, R_2 be random restrictions to k_i coordinates in the sets $T_i(\eta), i = 1,2$, respectively. Denote by $\mathrm{supp}(R_i), i = 1,2$ the set of indices selected by R_i and let $H(\eta) := H_{T_1(\eta)\times T_2(\eta)}$. Let $q \geq 1$ and let us bound the term $\mathsf{E}_\eta(\mathsf{E}\|R_1 H(\eta) R_2\|^q)^{1/q}$ that appears on the right side of (2.18). The expectation in the q-norm is computed for two random restrictions R_1 and R_2 that are conditionally independent given η. Let E_i be the expectation with respect to $R_i, i = 1,2$. Given η we can evaluate these expectations in succession and apply Lemma 2.4.9 to E_2 :

$$\begin{aligned}
\mathsf{E}_\eta(\mathsf{E}\|R_1 H(\eta) R_2^*\|^q)^{1/q} &= \mathsf{E}_\eta\Big[\mathsf{E}_1(\mathsf{E}_2\|R_1 H(\eta) R_2^*\|^q)^{q/q}\Big]^{1/q} \\
&\leq \mathsf{E}_\eta\Big\{\mathsf{E}_1\Big[3\sqrt{p}\,(\mathsf{E}_2\|R_1 H(\eta) R_2^*\|_{1\to 2}^q)^{1/q} + \sqrt{\frac{2k_2}{N}}\|R_1 H(\eta)\|\Big]^q\Big\}^{1/q} \\
&\leq \mathsf{E}_\eta\Big\{3\sqrt{p}\,\Big[\mathsf{E}_1\big(\mathsf{E}_2\|R_1 H(\eta) R_2^*\|_{1\to 2}^q\big)\Big]^{1/q} + \sqrt{\frac{2k_2}{N}}\Big[\mathsf{E}_1\|R_1 H(\eta)\|^q\Big]^{1/q}\Big\},
\end{aligned}$$

where on the last line we used the Minkowski inequality (recall that the random variables involved are finite). Now use Lemma 2.4.9 again to obtain

$$\begin{aligned}
\mathsf{E}_\eta(\mathsf{E}\|R_1 H(\eta) R_2^*\|^q)^{1/q} \leq 3\sqrt{p}\,\mathsf{E}_\eta\Big[\mathsf{E}_1\mathsf{E}_2\|R_1 H(\eta) R_2^*\|_{1\to 2}^q\Big]^{1/q} + 3\sqrt{\frac{2k_2 p}{N}}\mathsf{E}_\eta\big(\mathsf{E}_1\|H(\eta)^* R_1^*\|_{1\to 2}^q\big)^{1/q} \\
+ \sqrt{\frac{4k_1k_2}{N^2}}\mathsf{E}_\eta\|H(\eta)^*\|. \qquad (2.21)
\end{aligned}$$

Let us examine the three terms on the right-hand side of the last expression. Let $\eta(R_2)$ be the random vector conditional on the choice of k_2 coordinates. The sample space for $\eta(R_2)$ is formed of all the vectors $\eta \in \{0,1\}^N$ such that $\mathrm{supp}(R_2) \subset T_2(\eta)$. In other words, this is a subset of the sample space $\{0,1\}^N$ that is compatible with a given R_2. The random restriction R_1 is still chosen out of $T_1(\eta)$ independently of R_2. Denote by

$\widetilde{R}$ a random restriction to k_1 indices in the set $(\mathrm{supp}(R_2))^c$ and let $\widetilde{\mathsf{E}}$ be the expectation computed with respect to it. We can write

$$\begin{aligned}\mathsf{E}_\eta(\mathsf{E}_1\mathsf{E}_2\|R_1H(\eta)R_2^*\|_{1\to2}^q)^{1/q} &\le (\mathsf{E}_\eta\mathsf{E}_1\mathsf{E}_2\|R_1H(\eta)R_2^*\|_{1\to2}^q)^{1/q}\\ &= (\mathsf{E}_2\widetilde{\mathsf{E}}\|\widetilde{R}H(\eta)R_2^*\|_{1\to2}^q)^{1/q}.\end{aligned}$$

Recall that $H_{ij} = \mu_{ij}\mathbf{1}_{\{i\neq j\}}$ and that $\widetilde{R}$ and R_2 are 0-1 matrices. Using this in the last equation, we obtain

$$\mathsf{E}_2\widetilde{\mathsf{E}}\|\widetilde{R}H(\eta)R_2^*\|_{1\to2}^q \le \mathsf{E}_2\widetilde{\mathsf{E}}\max_{j\in\mathrm{supp}(R_2)}\Big(\sum_{i\in\mathrm{supp}(\widetilde{R})}\mu_{ij}^2\Big)^{q/2}. \tag{2.22}$$

Now let us invoke assumption (2.19). Recalling that $k_1<k$, we have

$$P_{R_2,\widetilde{R}}\Big(\max_{j\in\mathrm{supp}(R_2)}\sum_{i\in\mathrm{supp}(\widetilde{R})}\mu_{ij}^2\ge\epsilon_1\Big)\le k_2\epsilon_2.$$

Thus with probability $1-k_2\epsilon_2$ the sum in (2.22) is bounded above by ϵ_1. For the other instances we use the trivial bound $k_1\mu^2$. We obtain

$$\begin{aligned}3\sqrt{p}\,\mathsf{E}_\eta\mathsf{E}_1(\mathsf{E}_2\|R_1H(\eta)R_2^*\|_{1\to2}^q)^{1/q} &\le 3\sqrt{p}((1-k_2\epsilon_2)\epsilon_1^{q/2}+k_2\epsilon_2(k_1\mu^2)^{q/2})^{1/q}\\ &\le 3\sqrt{p}(\epsilon_1^{q/2}+k_2\epsilon_2(k_1\mu^2)^{q/2})^{1/q}\\ &\le 3\sqrt{p}(\sqrt{\epsilon_1}+(k\epsilon_2)^{1/q}\sqrt{k_1\mu^2}),\end{aligned}$$

where in the last step we used the inequality $a^q+b^q\le(a+b)^q$ valid for all $q\ge1$ and positive a,b. Let us turn to the second term on the right-hand side of (2.21). Assuming coherence invariance, we observe that

$$\|H(\eta)^*R_1^*\|_{1\to2}=\max_{j\in T_1(\eta)}\|H_{j,T_2(\eta)}\|_2\le\max_{j\in[N]}\|H_{j,\cdot}\|_2=\sqrt{N\bar{\mu}^2},$$

where $H_{j,\cdot}$ denotes the jth row of H and $H_{j,T_2(\eta)}$ is a restriction of the jth row to the indices in $T_2(\eta)$. At the same time, if the dictionary is not coherence-invariant, then in the last step we estimate the maximum norm from above by $\sqrt{N\bar{\mu}^2_{\max}}$, so overall the second term is not greater than $\sqrt{N\theta}$,

Finally, the third term in (2.21) can be bounded as follows:

$$\sqrt{\frac{4k_1k_2}{N^2}}\mathsf{E}_\eta\|H(\eta)\| \le \sqrt{\frac{(k_1+k_2)^2}{N^2}}\|H\| = \frac{k}{N}\|\Phi^T\Phi - I_N\|$$
$$\le \frac{k}{N}\max(1, \|\Phi\|^2 - 1) \le \frac{k}{N}\|\Phi\|^2,$$

where the last step uses the fact that the columns of Φ have unit norm, and so $\Phi^2 \ge N/m > 1$.

Combining all the information accumulated up to this point in (2.21), we obtain

$$\mathsf{E}_\eta(\mathsf{E}\|R_1H(\eta)R_2^*\|^q)^{1/q} \le 3\sqrt{p}(\sqrt{\epsilon_1} + (k\epsilon_2)^{1/q}\mu\sqrt{k} + \sqrt{2k_2\theta}\,) + \frac{k}{N}\|\Phi\|^2.$$

Finally, use this estimate in (2.18) to obtain the claim of the lemma. ∎

Proof of Theorem 2.4.7:

Proof. The strategy is to fix a triple $a, b, c \in (0,1)$ that satisfies (2.17) and to prove that (2.16) implies (k,δ,ϵ)-StRIP. Let $\epsilon_1 = \frac{b}{\log 1/\epsilon}$ and $\epsilon_2 = k^{-1+\log\epsilon}$. In Corollary 2.4.4 set $\alpha = \epsilon_1$ and $\beta = \alpha\log(2/\epsilon_2)$. Under the assumptions in (2.16) this corollary implies that

$$P_{R'}\Big(\sum_{m=1}^{k}\mu^2_{i_m,j} > \epsilon_1\Big) < \epsilon_2.$$

Invoking Lemma 2.4.11, we conclude that (2.20) holds with the current values of ϵ_1, ϵ_2. For any $q \ge 4\log k$ we have $p = q/2$, and thus (2.20) becomes

$$(\mathsf{E}\|RHR^*\|^q)^{1/q} \le 3\sqrt{2q}(\sqrt{\epsilon}_1 + (k\epsilon_2)^{1/q}\mu\sqrt{k} + \sqrt{2k\theta}) + 2\frac{k}{N}\|\Phi\|^2. \qquad (2.23)$$

Introduce the following quantities:

$$\xi_q = 3\sqrt{2}(\sqrt{\epsilon_1} + (k\epsilon_2)^{1/q}\mu\sqrt{k} + \sqrt{2k\theta}) \ \text{ and } \ \lambda = \frac{2k}{N}\|\Phi\|^2.$$

Now (2.23) matches the assumption of Lemma 2.4.10, and we obtain

$$P_k(\|RHR^*\| \geq e^{1/4}(\xi_q\sqrt{q}+\lambda)) \leq e^{-q/4}. \tag{2.24}$$

Choose $q = 4\log(1/\epsilon)$, which is consistent with our earlier assumptions on k, q, and ϵ. With this, we obtain

$$P_k\big(\|RHR^*\| \geq e^{1/4}(\xi_q\sqrt{q}+\lambda)\big) \leq \epsilon. \tag{2.25}$$

Now observe that $\|RHR^*\| \leq \delta$ is precisely the RIP property for the support identified by the matrix R. Let us verify that the inequality

$$6\sqrt{2}\big(\sqrt{\epsilon_1}+(k\epsilon_2)^{1/q}\sqrt{k\mu^2}+\sqrt{2k\theta}\big)\sqrt{\log(1/\epsilon)}+\frac{2k}{N}\|\Phi\|^2 < e^{-1/4}\delta$$

is equivalent to (2.17). This is shown by substituting ϵ_1 and ϵ_2 with their definitions, and μ and θ with their bounds in statement of the theorem. Thus, $P_k(\|RHR^*\| \geq \delta) \leq \epsilon$, which establishes the StRIP property of Φ. ∎

Let $f(k,\epsilon)$ be the $(1-\epsilon)$'th percentile of the random variable $\|RHR^*\|$ at sparsity level k (recall that R is a function of k). Then equation (2.25) essentially says that the quantity $e^{1/4}(\xi_q\sqrt{q}+\lambda)$, as a function of k and ϵ, is an upper bound on f. We denote this quantity by g, i.e., $g(k,\epsilon) = e^{1/4}(\xi_q\sqrt{q}+\lambda)$. In fact, other upper bounds of f can be similarly constructed by modifying the assignment to ϵ_2 (see e.g. Theorem 2.4.12). In the above proof, the particular upper bound is chosen because of the specific purpose of that Theorem. To be clearer, recall that the goal of Theorem 2.4.7 is finding the largest k such that the (k,δ,ϵ)-StRIP of a given matrix holds; and the larger the k is, the more likely StRIP is to fail. Therefore, if we are not able to find an upper bound that is uniformly tight for every k, we should at least require it to be tight for large ks, which is exactly the way the above g is defined.

In the following theorem, we derive another upper bound on f which is required to be tight for small ks. This upper bound, though not quantitatively optimal due to the large constants which could arise as an artifact of our technique, is useful for qualitative analysis, such as predicting the order of growth of f as a function of k.

Theorem 2.4.12. *Let Φ, μ and θ be defined as in the assumption of Theorem 2.4.7. Let $\epsilon < \min\{1/k, e^{1-1/\log 2}\}$ and suppose that Φ satisfies*

$$k\mu^4 \leq \frac{(1-a)^2 b^2}{32\log^2(1/\epsilon)\big(\log(2k)\log(e/\epsilon) + 4\log(\epsilon)\log(c)\big)} \quad \textit{and} \quad k\theta \leq \frac{ab}{\log(1/\epsilon)}, \tag{2.26}$$

where $a, b, c \in (0,1)$ are constants such that

$$\sqrt{b} + \sqrt{2ab} + c\mu + \frac{2k}{N}\|\Phi\|^2 \leq e^{-1/4}\delta/6\sqrt{2}.$$

Let R and H be the same as those in the proof of Lemma 2.4.11, then with probability exceeding $1-\epsilon$, we have

$$\|RHR^*\| \leq 6\sqrt{2}e^{1/4}\big(-\log\epsilon\sqrt{k\theta} + \sqrt{b} - c\mu\log\epsilon\big) + \frac{2k}{N}\|\Phi\|^2. \tag{2.27}$$

In particular, when Φ is a tight frame and let $f(k,\epsilon)$ be the $(1-\epsilon)$'th percentile of the random variable $\|RHR^*\|$, then (2.27) becomes

$$f(k,\epsilon) \leq 6\sqrt{2}e^{1/4}\big(-\log\epsilon\sqrt{k\theta} + \sqrt{b} - c\mu\log\epsilon\big) + \frac{2k}{m}. \tag{2.28}$$

Proof. In the proof of Theorem 2.4.7, change the assignment of ϵ_2 to $\epsilon_2 = c^{-1/(4\log\epsilon)}k^{-1+2\log\epsilon}$ and keep everything else the same. ∎

Remark: The upper bound in equation (2.28) grows in the order of $k^{1/2}$ when k is small enough to satisfy (2.26). If this upper bound is tight, it is reasonable to expect the left hand side of (2.28) to have the same order of growth. This conjecture is supported by our numerical experiment in the next section.

2.4.1 StRIP matrices from orthogonal arrays

Let us briefly consider another way of constructing StRIP matrices based on elementary arguments. Let $\mathcal{C} = \{\phi_1, \dots, \phi_N\}$ be a collection of binary m-vectors. We assume that the entries of the vectors are of the form $\pm 1/\sqrt{m}$ and denote the correlation of ϕ_i and ϕ_j by $\mu_{ij} = |\langle \phi_i, \phi_j \rangle|$.

The set $\mathcal{C}$ is called an orthogonal array of strength t if every subset of $r \le t$ coordinates of the vectors of $\mathcal{C}$ supports a uniformly random binary r-vector. A good reference for orthogonal arrays is the book by Hedayat et al. [33]. An orthogonal array has the property that any t coordinates of a randomly chosen vector behave as independent random variables (therefore, of course, t is much smaller than m). In particular, the first t moments of the distance distribution of $\mathcal{C}$ are equal to the moments of the binomial distribution. Let $d_{ij} = \frac{m}{2}(1 - \phi_i^T \phi_j)$ be the Hamming distance between ϕ_i and ϕ_j.

Lemma 2.4.13. (Pless identities, e.g. [40, p.132]) *Let $\mathcal{C}$ be an orthogonal array of strength t. Let $B_w = (1/N)|\{(\phi_i, \phi_j) \in \mathcal{C}^2 \mid d_{ij} = w\}|$ be the number of pairs vectors in $\mathcal{C}$ at distance w. For all $l = 1, 2, \dots, t$*

$$\sum_{w=0}^{m} \frac{B_w}{N}\left(w - \frac{m}{2}\right)^l = \frac{1}{2^m} \sum_{w=0}^{m} \binom{m}{w}\left(w - \frac{m}{2}\right)^l. \tag{2.29}$$

We will need a manageable estimate of the right-hand side of (2.29). We quote from [40, p.288]: let $l \geq 2$ be even, then

$$\frac{1}{2^m} \sum_{w=0}^{m} \binom{m}{w} \Big(w - \frac{m}{2}\Big)^l \leq \Big(\frac{ml}{4e}\Big)^{l/2} \sqrt{l}\, e^{1/6}. \tag{2.30}$$

The main result of this section is given by the following theorem.

Theorem 2.4.14. *Let $\mathcal{C}$ be an orthogonal array of strength t and cardinality N and let $l \leq t$ be even. If $m \geq (3/4)\, l(k/\delta)^2(k/\epsilon)^{2/l}$ then Φ is (k, δ, ϵ)-StRIP.*

Proof. Let $I \subset [N]$ be a uniformly random k-subset. We clearly have

$$\lambda_{\min}(\Phi_I^T \Phi_I)\|\boldsymbol{x}\|_2^2 \leq \|\Phi_I \boldsymbol{x}\|_2^2 \leq \lambda_{\max}(\Phi_I^T \Phi_I)\|\boldsymbol{x}\|_2^2,$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues of the argument.

By the Gershgorin theorem, any eigenvalue λ of the Gram matrix $\Phi_I^T \Phi$ satisfies

$$|\lambda - 1| \leq \sum_{j \in I_i} \mu_{ij},$$

for some $i \in [N]$, where we used the notation $I_i := I \backslash \{i\}$. Now consider the probability that for some $i \in I$ the sum $\sum_{j \in I_i} \mu_{ij} > \delta$. The proof will be finished if we show that this probability is less than ϵ. Let $I = \{i_1, \dots, i_k\}$. We have

$$\begin{aligned}
P_k\Big(\exists i \in I : \sum_{j \in I_i} \mu_{ij} > \delta\Big) &\leq k P_k\Big(\sum_{j \in I_{i_1}} \mu_{i_1,j} > \delta\Big) \leq k \frac{1}{\delta^l} \mathsf{E}_k\Big(\sum_{j \in I_{i_1}} \mu_{i_1,j}\Big)^l \\
&= k \frac{(k-1)^l}{\delta^l} \mathsf{E}_k\Big(\frac{1}{k-1} \sum_{j \in I_{i_1}} \mu_{i_1,j}\Big)^l \\
&\leq \frac{k(k-1)^{l-1}}{\delta^l} \mathsf{E}_k \sum_{j \in I_{i_1}} \mu_{i_1,j}^l,
\end{aligned}$$

where the last step uses convexity of the function $z \mapsto z^l$. The trick is to show that the expectation on the last line, presently computed over the choice of I, can be also found with respect to a pair of random uniform elements of $\mathcal{C}$ chosen without replacement. This is established in the next calculation:

$$\begin{aligned}
\mathsf{E}_k \sum_{j\in I_{i_1}} \mu_{i_1,j}^l &= \sum_{i_1<i_2<\cdots<i_k} \frac{1}{\binom{N}{k}} \sum_{j=2}^{k} \mu_{i_1,i_j}^l = \frac{1}{k!\binom{N}{k}} \sum_{i_1\neq i_2\neq\cdots\neq i_k} \sum_{j=2}^{k} \mu_{i_1,i_j}^l \\
&= \frac{1}{N(N-1)} \sum_{j=2}^{k} \sum_{i_1=1}^{N} \sum_{i_j\neq i_1} \mu_{i_1,i_j}^l \\
&= (k-1)\mathsf{E}\mu_{ij}^l, \qquad (2.31)
\end{aligned}$$

where the expectation on the last line (and below in the proof) is computed with respect to a pair of uniformly chosen distinct random vectors from $\mathcal{C}$. Next using (2.29) and switching to the variable $w = (m/2)(1-\mu)$, we obtain

$$\begin{aligned}
\mathsf{E}\mu_{ij}^l &= \Big(\frac{2}{m}\Big)^l \sum_{w=1}^{m} \frac{B_w}{N-1}\Big(w-\frac{m}{2}\Big)^l \\
&= \Big(\frac{2}{m}\Big)^l \frac{N}{N-1}\Big[\sum_{w=0}^{m} \frac{B_w}{N}\Big(w-\frac{m}{2}\Big)^l - \frac{1}{N}\Big(\frac{m}{2}\Big)^l\Big] \\
&= \Big(\frac{2}{m}\Big)^l \frac{N}{N-1}\Big[\frac{1}{2^m}\sum_{w=0}^{m} \binom{m}{w}\Big(w-\frac{m}{2}\Big)^l - \frac{1}{N}\Big(\frac{m}{2}\Big)^l\Big],
\end{aligned}$$

Now we can use (2.30) and $l < m$ to write

$$\mathsf{E}\mu_{ij}^l \le \Big(\frac{l}{em}\Big)^{l/2} \frac{N}{N-1}\sqrt{le^{1/3}} - \frac{1}{N-1} \le e^{1/6} l^{(l+1)/2} (em)^{-l/2}.$$

Conclude using the condition on m :

$$P_k\Big(\exists i \in I : \sum_{j\in I_i} \mu_{ij} > \delta\Big) \le k^{l+1}\delta^{-l} e^{1/6} l^{(l+1)/2} (em)^{-l/2} < \epsilon.$$

∎

Observe that the condition of this theorem is nonasymptotic, and is satisfied by a number of known constructions of orthogonal arrays.

Example: Consider sampling matrices obtained from the binary Delsarte-Goethals *codes already mentioned above; see Eq.*(**??**)*. It is known that the underlying code forms an orthogonal array of strength* $t = 7$*, so taking* $l = 6$ *we obtain a family of* (k, δ, ϵ)*-StRIP matrices of dimensions* $m \times N$ *for sparsity*

$$k \le 0.52\,(\delta^6 \epsilon\, m^3)^{1/7} = 0.52(\delta^6 \epsilon)^{1/7}(2^r N)^{3/(7(r+2))}.$$

The case $r = 0$ *was considered in [15] where these matrices were analyzed based on the detailed properties of this particular case of the construction. Our computation, while somewhat crude, permits a uniform estimate for the entire family of matrices. The estimate can be improved if the expectation* $\mathsf{E}\mu_{ij}^l$ *can be computed explicitly from the known distribution of correlations. For instance, taking* $r = 1$ *and using the distribution given in [40, p.477] we obtain that* $\mathsf{E}\mu^6 \approx (4/3)m^{-3}$*. With this, the condition on sparsity that emerges has the form* $k < 0.95(\delta^6 \epsilon m^3)^{1/7}$*, with a better constant compared to the general estimate. For instance, we obtain* $m \times (m^3/2)$ *matrices with the* $(k, \delta, 0.001)$ *StRIP property for all* $k \le 0.35\delta^{6/7}m^{3/7}$*.*

Another similar possibility arises if $\mathcal{C}$ *is taken to be a binary dual BCH code with* $m = 2^s - 1, N = m^r, \mu = 2(r-1)m^{-1/2}, r = 1, 2, 3, \dots$ *Many more such constructions can be obtained from other algebraic codes such as the Kerdock codes, Gold codes, etc. [34]. This lends further support to earlier studies of sampling matrices constructed from the BCH codes [1], Delsarte-Goethals codes, and other binary codes related to the second-order Reed-Muller codes [14, 15].*

It would be desirable to show that orthogonal arrays also suffice for the SINC property; however, the technique introduced above results in parameters that contradict the Rao bound on the number of rows in an array [33]. Thus, we are unable to show that this construction results in matrices that are good for linear estimators.

2.4.2 Further constructions from binary codes

We remark that it is easy to show existence of matrices with low coherence. The following observation is a rephrasing of the result known in coding theory as the Gilbert-Varshamov existence bound for binary linear codes.

Proposition 2.4.15. *Let* $l = \log_2 N, l < m$ *and let* $G = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_l)$ *be an* $m \times l$ *binary matrix whose rows are chosen independently and uniformly from* $\mathbb{F}_2^l$. *Let* $m = 4 \log N/\mu^2$, *where* $0 < \mu < 1$. *Form the matrix* Φ *by constructing an* $\mathbb{F}_2$*-linear span of the columns of* G *and using the map* $\{0,1\} \to \{\frac{1}{\sqrt{m}}, \frac{-1}{\sqrt{m}}\}$. *Then* Φ *has coherence* μ *with probability at least* $1 - 2/N$ *and mean square coherence* $\bar{\mu}^2 < 1/m$ *with probability at least* $(1 - (m/N))^m$.

Proof. Note that the Hamming distance d between any two columns of a matrix with coherence μ satisfies $\mu \geq |1 - 2d/m|$. The set of columns of C forms a linear space, so it suffices to argue about Hamming weights rather than pairwise correlations. Let $\boldsymbol{u} \in \{0,1\}^l$ be a nonzero vector, then the probability that the vector $\boldsymbol{v} = G\boldsymbol{u}$ has weight w equals $\binom{m}{w} 2^{-m}$. Let X be the random number of columns with weight $|w - m/2| \geq m\mu/2$. We have

$$\mathsf{E}X \leq 2\frac{N-1}{2^m} \sum_{w=0}^{m(\frac{1}{2}-\frac{\mu}{2})} \binom{m}{w} \leq N 2^{1-m(1-h(\frac{1}{2}-\frac{\mu}{2}))}, \tag{2.32}$$

where $h(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. Using the inequality

$$1 - h(1/2 - x) \geq 2x^2/\log 2, \quad 0 \leq x < 1/2$$

and the condition for μ, we obtain $\mathsf{E}X \leq 2/N$. Since $P(X > 0) \leq \mathsf{E}X$, this implies the first claim. The second part follows because there are $\prod_{i=1}^{m}(N-i)$ matrices G with distinct nonzero rows. ∎

The derandomizing of Gilbert-Varshamov codes was recently addressed by Porat and Rothschild [45]. They presented a $O(mN)$ deterministic algorithm that constructs codes with large minimum distance. To construct incoherent dictionaries, we need a bit more, namely that all the pairwise distances are in a narrow segment around $m/2$. The algorithm in [45] can be easily tailored to do this. A simplified version of this procedure which results in the algorithm of complexity $O(mN^2)$ (i.e., not as good as in [45]), was given in [42]. In a nutshell it is as follows. Instead of constructing the $m \times N$ matrix, $N = 2^l$, we aim at constructing a basis of the space of columns, i.e., an $m \times l$ matrix G. The rows of G are selected recursively. Before any rows are selected, the expected number of codewords of weight far from $m/2$ is given by (2.32). The algorithm selects rows one by one so that the expectation of the number of outlying vectors *conditional* on the rows already chosen is the smallest possible.

We note that in the context of sparse recovery, the dependence between N and m is likely to be polynomial. In this range of parameters the above complexity is acceptable and is in fact comparable with the size of the matrix Φ which needs to be stored for sampling and processing.

Chapter 3

Compressive sensing with dictionary

3.1 Introduction

A recent direction of interest in compressed sensing concerns problems where signals are sparse in an overcomplete dictionary D instead of a basis, see [16, 47, 30, 39, 2]. This is motivated by the widespread use of overcomplete dictionaries in signal processing and data analysis. Many signals naturally possess sparse frame coefficients, such as images consisted of curves (curvelet frame). In addition, the greater flexibility and stability of frames make them preferable for practical purposes in order to compensate the imperfectness of measurements.

In this setting, the signal $\boldsymbol{x} \in \mathbb{C}^d$ can be represented as $\boldsymbol{x} = D\boldsymbol{z}$, where $\boldsymbol{z}$ is k-sparse and D is a $N \times d$ matrix with $d \geq N$. The columns of D may be thought of as an overcomplete frame or dictionary for $\mathbb{C}^N$. The linear measurements are $\boldsymbol{y} = \Phi\boldsymbol{z}$, with $\Phi \in \mathbb{C}^{m,N}$.

A natural way to recover $\boldsymbol{x}$ from $\boldsymbol{y}$ is first solving

$$\hat{\boldsymbol{z}} = \arg\min_{\boldsymbol{z} \in R^d} \|\boldsymbol{z}\|_1, \text{ subject to } \boldsymbol{y} = \Phi D\boldsymbol{z}. \tag{3.1}$$

for the sparse coefficients $\hat{\boldsymbol{z}}$, then synthesizing it to obtain $\hat{\boldsymbol{x}}$, i.e., $\hat{x} = D\hat{\boldsymbol{z}}$. The resulting method is therefore called ℓ^1-synthesis or synthesis based method [39, 47]. In the case when the measurements are perturbed, we naturally solve the following problem:

$$\hat{\boldsymbol{z}} = \arg\min_{\boldsymbol{z}\in R^d} \|\boldsymbol{z}\|_1, \text{ subject to } \|\boldsymbol{y} - \Phi D\boldsymbol{z}\| \leq \epsilon. \tag{3.2}$$

The work in [47] established conditions on Φ and D to make the compound ΦD satisfy RIP. However, as pointed in [16, 39], forcing ΦD to satisfy RIP or even the weaker NSP (defined in Section 1.1) implies exact recovery of both $\boldsymbol{z}$ and $\boldsymbol{x}$, which is unnecessary if we only care about obtaining a good estimate of $\boldsymbol{x}$. In particular, it is argued in [16, 39] that if D is perfectly correlated (has two identical columns), then there are infinitely many minimizers of (3.1) that may be assigned to $\hat{\boldsymbol{z}}$, but all of them lead to the true signal $\boldsymbol{x}$. It seems reasonable to expect that a similar result may hold in the case of highly correlated dictionaries, since they are only a small perturbation away from the perfectly correlated ones.

3.2 Overview and main results

In the following, we will generalize the ordinary NSP to the dictionary case (D-NSP), and prove (in Theorem 3.3.1) that this new condition is equivalent to the successful recovery of signals in $D\Sigma_k$ via ℓ^1-synthesis, where $D\Sigma_k = \{x : \exists\ z,$ such that $x = Dz, \|z\|_0 \leq k\}$ is the set of signals that have k-sparse representations in D. Moreover, a stability result is given in Theorem 3.4.2. To the best of our knowledge, these results are the first characterization of compressed sensing with dictionaries via ℓ^1-synthesis approach.

Section 3.5 studies further properties of D-NSPand shows that the condition Φ being D-NSP is equivalent to ΦD being NSP as long as D is "full spark" (every d columns of D are linearly independent). As a consequence, under the full spark assumption, the ℓ^1-synthesis method cannot accurately recover the signals without accurate recoveries of their sparse representations, therefore an incoherent dictionary is needed under these circumstances. Further analysis on D-NSP can be found in [25].

3.3 A sufficient and necessary condition for noiseless sparse recovery

In this section, we develop a necessary and sufficient condition for the ℓ^1-synthesis method (3.1) to achieve accurate reconstruction of sparse signals with noiseless measurements. We say the ℓ^1-synthesis method (3.1) is *successful* in recovering $\boldsymbol{x}$ when every minimizer $\hat{\boldsymbol{z}}$ of (3.1) satisfies $D\hat{\boldsymbol{z}} = \boldsymbol{x}$. We show that the following property on Φ is a necessary and sufficient condition for successfully recovering all signals in $D\Sigma_s$ via (3.1).

Definition 7 (Null Space Property of the dictionary D (D-NSP))**.** *Fix a dictionary $D \in \mathbb{C}^{N,d}$, a matrix $\Phi \in \mathbb{C}^{m,N}$ is said to satisfy the D-NSP of order k (k-D-NSP) if for any index set T with $|T| \leq k$, and any $v \in D^{-1}(\ker\Phi\backslash\{0\})$, there exists $u \in \ker D$, such that*

$$\|v_T + u\|_1 < \|v_{T^c}\|_1. \tag{3.3}$$

Theorem 3.3.1. *D-NSP is a necessary and sufficient condition for the success of ℓ^1-synthesis for all signals in the set $D\Sigma_k$.*

Proof. Suppose that ℓ^1-synthesis is successful for all the signals in $D\Sigma_k$. Take any support T and $v \in D^{-1}(\ker\Phi/\{0\})$, Let $\boldsymbol{x} = Dv_T$ be the signal that we are trying to recover, then by assumption, the minimizer must be $v_T + u$ with some $u \in \ker D$.

$v_T - v$ is another feasible representation, however, it cannot be a minimizer since $D(v_T - v) \neq Dv_T$, therefore

$$\|v_T + u\|_1 < \|v_T - v\|_1 = \|v_{T^c}\|_1.$$

On the other hand, assuming that D-NSP is satisfied, suppose that $D\hat{\boldsymbol{z}} \neq Dz_0$, then $v = \boldsymbol{z}_0 - \hat{\boldsymbol{z}} \in D^{-1}(\ker \Phi/\{0\})$, Let T be the support of $\boldsymbol{z}_0$, therefore there exists $u \in \ker D$, $\|v_T + u\|_1 < \|v_{T^c}\|_1$, i.e. $\|\boldsymbol{z}_0 - \hat{\boldsymbol{z}}_T + u\|_1 < \|\hat{\boldsymbol{z}}_{T^c}\|_1$, so

$$\|\boldsymbol{z}_0 + u\|_1 \leq \|\boldsymbol{z}_0 - \hat{\boldsymbol{z}}_T + u\|_1 + \|\hat{\boldsymbol{z}}_T\|_1 < \|\hat{\boldsymbol{z}}_{T^c}\|_1 + \|\hat{\boldsymbol{z}}_T\|_1 = \|\hat{\boldsymbol{z}}\|_1.$$

Since $\hat{\boldsymbol{z}}$ is the minimizer, this is a contradiction.∎

Notice that when D is the canonical basis of $\mathbb{C}^d$, D-NSP is reduced to the normal NSP with the same order. In other words, D-NSP is a generalization of NSP for the dictionary case.

The intuition for D-NSP rises from the fact that we are only interested in recovering $\boldsymbol{x}$ instead of the representation $\boldsymbol{z}_0$. As long as the minimizer $\hat{\boldsymbol{z}}$ lies in the affine plane $\boldsymbol{z}_0 + \ker D$, our reconstruction is a success.

3.4 D-NSP based stability analysis

It is known that NSP is a necessary and sufficient condition not only for the sparse and noiseless recovery, but also for compressible signals with noisy measurement [2, 49]. However, the stability analysis of NSP [2] cannot be easily generalized to our case because essentially we need the function $f(v) = (\|v_{T^c}\|_1 - \|v_T + u\|_1)/\|Dv\|_2$ to be bounded away from zero. In the basis case, we have knowledge of $f(v)$ on a compact set, and consequently the extreme value theorem can be applied to prove the exisitence of a positive lower bound. In our case we do not have a compact set, therefore another approach to overcome this difficulty is necessary.

Definition 8 (Strong Null Space Property of the dictionary D (D-SNSP)). *A sensing matrix Φ is said to have the strong null space property with respect to D of order k (k-D-SNSP) if for any index set T with $|T| \leq k$, and any $v \in \ker(\Phi D)$, there exists $u \in \ker D$ such that*

$$\|v_{T^c}\|_1 - \|v_T + u\|_1 \geq c\|Dv\|_2. \tag{3.4}$$

D-SNSP seems to be a stronger assumption than D-NSP by definition. However, in the real case, we are able to show that it is actually equivalent to the D-NSP.

Theorem 3.4.1. *If $D \in \mathbb{R}^{N,d}$ and $\Phi \in \mathbb{R}^{m,N}$, then D-NSP is equivalent to D-SNSP.*

Since the proof is tedious, we postpone it to Section 3.7. First we prove under D-SNSP, the ℓ^1-synthesis recovery is stable with respect to perturbations of the measurement vector y.

Theorem 3.4.2. *If Φ is k-D-NSP, then any solution $\hat{\boldsymbol{z}}$ of problem* (3.2) *satisfies*

$$\|D\hat{\boldsymbol{z}} - \boldsymbol{x}\|_2 \leq C_1\sigma_k(\boldsymbol{x}) + C_2\epsilon,$$

where $\sigma_k(\boldsymbol{x})$ denotes the ℓ^1 residue of the best k-term approximation to $\boldsymbol{x}$, C_1, C_2 are constant dependent on n, the c in (3.4)*, the minimum singular values of Φ and D, but not on $\boldsymbol{x}$.*

Proof. Let $\boldsymbol{x} = D\boldsymbol{z}_0$ be the true sparse representation. Let $h = D(\hat{\boldsymbol{z}} - \boldsymbol{z}_0)$, and we can decompose h as $h = Dw + \eta$ where $Dw \in \ker \Phi$, $\eta \in \ker \Phi^{\perp}$, and $\|\eta\|_2 \leq \frac{1}{\nu_\Phi}\|\Phi h\|_2 \leq \frac{2\epsilon}{\nu_\Phi}$

with ν_Φ being the smallest singular value of Φ.

Let $\xi = D^T(DD^T)^{-1}\eta$, then $\eta = D\xi$, and

$$\|\xi\|_2 \leq \frac{1}{\nu_D}\|\eta\|_2 \leq \frac{2}{\nu_\Phi \nu_D}\epsilon. \tag{3.5}$$

Since $D(\hat{\boldsymbol{z}} - \boldsymbol{z}_0) = h = D(w+\xi)$, then $\hat{\boldsymbol{z}} - \boldsymbol{z}_0 = w+\xi+u_1$ with some $u_1 \in \ker D$.

Let $v = w + u_1$, then we have $Dv \in \ker \Phi$ and $\hat{\boldsymbol{z}} - \boldsymbol{z}_0 = v + \xi$.

So there exists $u \in \ker D$ such that (3.4) holds, therefore

$$\begin{aligned}
&\|v + \boldsymbol{z}_{0,T}\|_1 - \| - u + \boldsymbol{z}_{0,T}\|_1 \\
\geq &\|v_{T^c}\|_1 + \|v_T + \boldsymbol{z}_{0,T}\|_1 - \| - u_T + \boldsymbol{z}_{0,T}\|_1 - \|u_{T^c}\|_1 \\
\geq &\|v_{T^c}\|_1 - \|v_T + u_T\| - \|u_{T^c}\|_1 \\
= &\|v_{T^c}\|_1 - \|v_T + u\|_1 \geq c\|Dv\|_2.
\end{aligned} \tag{3.6}$$

On the other hand, from the fact that $\hat{\boldsymbol{z}}$ is a minimizer, we get

$$\begin{aligned}
&\| - u + \boldsymbol{z}_{0,T}\|_1 + \|\boldsymbol{z}_{0,T^c}\|_1 \geq \| - u + \boldsymbol{z}_0\|_1 \geq \|v + \boldsymbol{z}_0 + \xi\|_1 \\
&\geq \|v + \boldsymbol{z}_0\|_1 - \|\xi\|_1 \geq \|v + \boldsymbol{z}_{0,T}\|_1 - \|\boldsymbol{z}_{0,T^c}\|_1 - \|\xi\|_1.
\end{aligned}$$

It follows that

$$\|v + \boldsymbol{z}_{0,T}\|_1 - \| - u + \boldsymbol{z}_{0,T}\|_1 \leq 2\|\boldsymbol{z}_{0,T^c}\|_1 + \|\xi\|_1. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\|Dv\|_2 \leq \frac{2}{c}\|\boldsymbol{z}_{0,T^c}\|_1 + \frac{1}{c}\|\xi\|_1 \leq \frac{2}{c}\|\boldsymbol{z}_{0,T^c}\|_1 + \frac{\sqrt{n}}{c}\|\xi\|_2. \tag{3.8}$$

In the end, using (3.8) and (3.5),

$$\begin{aligned}\|h\|_2 = \|Dv + D\xi\|_2 = \|Dv + \eta\|_2 &\leq \|Dv\|_2 + \|\eta\|_2 \\ &\leq \frac{2}{c}\|\boldsymbol{z}_{0,T^c}\|_1 + \frac{\sqrt{n}}{c}\|\xi\|_2 + \frac{1}{\nu_\Phi}2\epsilon \\ &\leq \frac{2}{c}\|\boldsymbol{z}_{0,T^c}\|_1 + \frac{2\sqrt{n}}{c\nu_\Phi\nu_D}\epsilon + \frac{1}{\nu_\Phi}2\epsilon.\end{aligned}$$

∎

It is natural to ask how much stronger this new assumption is than D-NSP. We address this question partially in the next section.

3.5 A further study of D-NSP and admissible dictionaries

This section further explores the two assumptions D-NSP and D-SNSP for the purpose of answering the following important questions: What kind of dictionaries allow sensing matrices Φ with few measurements to satisfy D-NSP? How to find those sensing matrices given a dictionary?

We call an $N \times d$ dictionary D *k-admissible* if there exists a measurement matrix $\Phi \in \mathbb{C}^{m,N}$ with $m < N$ such that Φ is k-D-NSP. We call D *inadmissible* if D is not k-admissible for any $k \geq 2$.

The following proposition shows that adding repeated columns to the dictionary D will not affect admissibility. This is quite intuitive since we do not change the set $D\Sigma_k$ during this procedure, and we only care about recovering the signal $\boldsymbol{x}$ rather than the representation $\boldsymbol{z}_0$.

Proposition 3.5.1. *Let $D \in \mathbb{C}^{N,d}$, and let I be any index set $I \subset \{1, ..., n\}$. Define*

$\widetilde{D} = [D, D_I]$, *then for any sensing matrix* $\Phi \in \mathbb{C}^{m,N}$, *we have* Φ *is* D*-NSP if and only if* Φ *is* $\widetilde{D}$*-NSP.*

Proposition 3.5.1 states that a perfectly correlated dictionary D does not preclude the reconstruction of signals. It is natural to ask whether this is still the case for a highly coherent dictionary. We answer this question partially by showing that a class of highly correlated dictionaries is inadmissible. Moreover, easily verifiable conditions that are equivalent to D-NSP are given in Section 3.6 under the assumption that D is full spark.

3.5.1 A Class of inadmissible matrices

The following theorem constructs a class of inadmissible matrices with a dimension 1 kernel.

Theorem 3.5.2. *Given an orthonormal basis* $\Phi = [\phi_1, ..., \phi_N]$. *Let* $H = \bigcup_{I_i = \{1,...,N\}\setminus i} span(\Phi_{I_i})$ *be a union of hyperplanes spanned by every combination of* $N-1$ *columns of* Φ. *Then there exists a small constant* r_0 *such that for every* $v \in B(\phi_1, r_0)\backslash H$, $D = [\Phi, v] \in \mathbb{C}^{N,N+1}$ *is not admissible.*

We need the following lemma for the proof of this Theorem.

Lemma 3.5.3. *Suppose* D *is a* $N \times (N+1)$ *dictionary. If there exists* $T \subset \{1, ..., N+1\}$ *with* $|T| \geq 2$ *such that the normalized vector* $u \in \ker D$ *satisfies*

1. $\|u_T\|_1 > \|u_{T^c}\|_1$, *and*

2. $T^c \subset supp(u)$.

Then D *cannot be* $|T|$*-admissible.*

Proof. Assume that D is a dictionary which satisfies the assumptions of Lemma 3.5.3 and is $|T|$-admissible at the same time. We shall prove this leads to a contradiction.

For a vector $w \in \mathbb{C}^N$, we define $\|w\|_{\min} = \min\{|w_i|,\ i = 1, ..., N\}$ to be the minimum magnitude in w. Assumption 2 then implies $\|u\|_{\min} > 0$. Suppose Φ is D-NSP and fix a $v_0 \in D^{-1}(\ker(\Phi)\backslash\{0\})$. We define $\alpha = 2\|v_0\|_\infty / \|u\|_{\min}$. Now that $v_0 + \alpha u, -v_0 + \alpha u \in D^{-1}(\ker(\Phi)\backslash\{0\})$, we can use the definition of D-NSP to derive: there exist $c_1, c_2 \in \mathbb{C}$ such that

$$\|v_T + \alpha u_T - c_1 u\|_1 < \|v_{T^c} + \alpha u_{T^c}\|_1,$$

and

$$\| - v_T + \alpha u_T - c_2 u\|_1 < \| - v_{T^c} + \alpha u_{T^c}\|_1,$$

Adding up the two equations, we get

$$\begin{aligned} & \|v_T + \alpha u_T - c_1 u\|_1 + \| - v_T + \alpha u_T - c_2 u\|_1 \\ < \ & \|v_{T^c} + \alpha u_{T^c}\|_1 + \| - v_{T^c} + \alpha u_{T^c}\|_1 \\ = \ & 2\alpha \|u_{T^c}\|_1. \end{aligned} \tag{3.9}$$

The equality in (3.9) follows from our definition of α. On the other hand,

$$\begin{aligned} & \|v_T + \alpha u_T - c_1 u\|_1 + \| - v_T + \alpha u_T - c_2 u\|_1 \\ = \ & \|v_T + (\alpha - c_1) u_T\|_1 + |c_1| \|u_{T^c}\|_1 \\ + \ & \| - v_T + (\alpha - c_2) u_T\|_1 + |c_2| \|u_{T^c}\|_1 \\ \geq \ & |2\alpha - c_1 - c_2| \|u_T\|_1 + (|c_1| + |c_2|) \|u_{T^c}\|_1. \end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) together imply

$$|2\alpha - c_1 - c_2| \|u_T\|_1 + (|c_1| + |c_2|) \|u_{T^c}\|_1 < 2\alpha \|u_{T^c}\|_1,$$

which can be simplified to

$$\|u_T\|_1 < \|u_{T^c}\|_1.$$

This is a contradiction to Assumption 1 of Lemma 3.5.3. ∎

Proof of Theorem 3.5.2: Notice that ker(D) is one dimensional and set its basis to be $u = (a^T, -1)$. Pick an index set T with $|T| \geq 2$ such that $\{1, N+1\} \in T$. First if $v \notin H$, then $\langle v, \phi_i \rangle \neq 0$ for $i = 1, ..., N$. This means that all coordinates of u are nonzero. Second, we can pick r_0 small enough such that whenever $v \in B(\phi_1, r)$, we have $\|u_T\|_1 > \|u_{T^c}\|_1$.

Therefore picking $v \in B(\phi_1, r_0) \backslash H$ fulfills the two assumptions of Lemma 3.5.3. This completes the proof. ∎

Proposition 3.5.4. *If $D = [B, v]$ where B is a full rank $N \times (d-1)$ matrix and $v = B\alpha$ with $\|\alpha\|_1 \leq 1$, then Φ has D-NSP implies that Φ has B-NSP with the same order k.*

With this proposition, we can add more columns to the inadmissible dictionaries constructed in Theorem 3.5.2 to obtain inadmissible dictionaries with arbitrary dimension.

3.6 Relation between D-NSP and NSP

It is obvious that ΦD being NSP implies Φ being D-NSP, which explains why imposing RIP or incoherence conditions on ΦD could be too strong and unnecessary. Quantifying the gap between these two conditions can possibly answer the question whether we can allow highly coherent dictionaries or not, since ΦD being NSP will inevitably lead to the incoherence of D. Surprisingly enough, we show that whenever D is full spark, these two conditions are equivalent.

Theorem 3.6.1. *The following conditions are equivalent under the assumption that D is full spark,*

- Φ *is k-D-NSP;*
- ΦD *is k-NSP;*
- Φ *is k-D-SNSP;*
- *For any $v \in \ker \Phi D$, there exists a u such that*

$$\|v_T + u\|_1 < \|v_{T^c}\|_1.$$

Remark 3.6.1. *Theorem 3.6.1 implies that for a given full spark dictionary D and a given sensing matrix Φ, if Φ satisfies D-NSP, then all signals x will be recovered by synthesizing the already correctly recovered representations $\boldsymbol{z}$. If Φ does not satisfy D-NSP, although certain signals cannot be recovered accurately, there might be signals that are recovered from a "wrong" representation.*

In the beginning of Section 3.4, we mentioned the difficulty of proving stability result for D-NSP is due to the non-compactness of the set $D^{-1}(\ker \Phi \backslash \{0\})$. However, in the full spark case, Theorem 3.6.1 guarantees that we can extend this set to its closure, and then the result of Theorem 3.4.2 will trivially hold under the necessary assumption Φ being D-NSP.

We remark that full spark is not a restrictive assumption on matrices. In fact, full spark matrices are dense in the space of matrices, and a large class of full spark Harmonic frames are constructed in [3]. This means that for "most" dictionaries, we need to study the composite ΦD, and ΦD being NSP is the equivalent condition for successful recovery of $\boldsymbol{x} \in D\Sigma_k$. Hence for "most" dictionaries, D is not allowed to be very coherent, which is somewhat unexpected.

Remark 3.6.2. *Given an admissible dictionary D that is perfectly correlated, we can always find a full spark and highly coherent dictionary D' that is arbitrarily close to D, therefore we cannot find a sensing matrix Φ such that $\Phi D'$ satisfies k-NSP for any $k \geq 2$. By Theorem 3.6.1, D' is inadmissible, indicating that a small perturbation on the dictionary cannot preserve admissibility.*

3.7 Proofs of the main theorems

Lemma 3.7.1. *Assume Φ satisfies D-NSP. If in addition, for any $u \in \ker D$, there exists a $\widetilde{u} \in \ker D$, such that*

$$\|u_T + \widetilde{u}\|_1 < \|u_{T^c}\|_1, \tag{3.11}$$

then ΦD satisfies NSP.

Proof. Let

$$f(u) = \min_{\widetilde{u} \in \ker D} \frac{\|u_{T^c}\|_1 - \|u_T + \widetilde{u}\|_1}{\|u\|_2}.$$

Continuity of $f(u)$ together with (3.11) implies that it attains minimum on the closed set $\{u : u \in \ker D),\ \|u\|_2 = 1\}$, i.e. $f(u) \geq c > 0$. Then we have for any $u \in \ker D$, there exists a $\widetilde{u}$ such that

$$\|u\|_2 \leq \frac{1}{c}(\|u_{T^c}\|_1 - \|u_T + \widetilde{u}\|_1). \tag{3.12}$$

Now suppose that $\boldsymbol{x}$ is a signal that has sparse representation under D, i.e.$\boldsymbol{x} = D\boldsymbol{z}$ for some $\boldsymbol{z} \in \Sigma_k$ and $\hat{\boldsymbol{z}}$ the solution to (3.2). Since Φ is assumed to be D-NSP, we must have $D\hat{\boldsymbol{z}} = D\boldsymbol{z}$, which implies $h := \hat{\boldsymbol{z}} - \boldsymbol{z} \in \ker D$. Hence there exists a $\widetilde{u}$ such that (3.12) holds for h. Since $\hat{\boldsymbol{z}}$ is the minimizer, we have,

$$\begin{aligned} 0 &\geq \|h + \boldsymbol{z}\|_1 - \|\boldsymbol{z} - \widetilde{u}\| \\ &\geq \|h_{T^c}\|_1 + \|h_T + \boldsymbol{z}\|_1 - \|\boldsymbol{z} - \widetilde{u}_T\|_1 - \|\widetilde{u}_{T^c}\|_1 \\ &\geq \|h_{T^c}\|_1 - \|h_T + \widetilde{u}_T\|_1 - \|\widetilde{u}_{T^c}\|_1 \\ &= \|h_{T^c}\_1 - \|h_T + \widetilde{u}\|_1 \\ &\geq c\|h\|_2, \end{aligned}$$

which implies $\hat{\boldsymbol{z}} = \boldsymbol{z}$, the sparse coefficients are accurately recovered. Since our choice of $\boldsymbol{z}$ is arbitrary, and for all sparse coefficient to be recovered universally, ΦD must satisfy NSP. ∎

Proof of of Theorem 3.6.1 . Here we only prove Φ is D-NSP implies ΦD is NSP. Other equivalences are either trivial or similar to this proof.

To rule out the trivial case, suppose that $\ker \Phi \neq \emptyset$. According to Lemma 3.7.1, we only need to show that (3.11) holds.

Step 1. Fix a T with $|T| < N$, we will show that for any $u \in \ker D$, there exists $v \in D^{-1}(\ker \Phi \backslash \{0\})$, such that $\operatorname{supp} u_{T^c} \subset \operatorname{supp} v_{T^c}$.

Since $spark(D) = N + 1$ and $u \in \ker D$, then we have $|\operatorname{supp} u| \geq N + 1$, and thus $|\operatorname{supp} u_{T^c}| \geq N + 1 - |T|$. Therefore there exists a $G \in \operatorname{supp} u_{T^c}$ with $|G| = N - |T|$ and $|G \cup T| = N$. On the other hand, $\ker \Phi \neq \emptyset$ implies $D^{-1}(\ker \Phi \backslash \{0\}) \neq \emptyset$. Assume

v_0 is an element in this nonempty set. Let $D_{G\cup T}$ be the submatrix of D corresponding to the index set $G \cup T$. Then $D_{G\cup T}$ is full rank by assumption. Let v be the vector defined by $v_{(G\cup T)^c} = 0$ and $v_{G\cup T} = D^{-1}_{G\cup T} D v_0$. Then obviously we have $Dv = Dv_0$, $\operatorname{supp} v_{T^c} \subset \operatorname{supp} u_{T^c}$ and $v \in D^{-1}(\ker \Phi \backslash \{0\}$. This finishes Step 1.

Step 2: Consider the same T as in Step 1. Given a $u \in \ker D$, find the vector v with $\operatorname{supp} v_{T^c} \in \operatorname{supp} u_{T^c}$ using Step 1. Choose α large enough such that $\alpha \|u\|_{\min} > \|v\|_\infty$. Then by the assumption that Φ is D-NSP, there exist $u_1, u_2 \in \ker D$, such that

$$\|(v + \alpha u)_T + u_1\|_1 < \|(v + \alpha u)_{T^c}\|_1$$

and

$$\|(-v + \alpha u)_T + u_2\|_1 < \|(-v + \alpha u)_{T^c}\|_1$$

hold. Adding the above two equations, and using convexity of the l_1 norm, we get

$$\|2\alpha u_T + (u_1 + u_2)\|_1 < 2\alpha \|u_{T^c}\|_1.$$

The proof is completed by recalling that T is arbitrary and by invoking Lemma 3.7.1. ∎

In order to prove Theorem 3.4.1, we need the following two lemmas.

Lemma 3.7.2. *Define*

$$h(w) = \sup_{\widetilde{u} \in \ker D} \frac{\|w_{T^c}\|_1 - \|w_T + \widetilde{u}\|_1}{\|Dw\|},$$

then $h(w)$ is positive and bounded away from zeros. Set $W = \{w : w \in D^{-1}(\ker \Phi \backslash \{0\}), C_1 \leq \|w\| \leq C_2 \|Dw\|\}$. In addition, this bound is independent of C_1.

Proof. First, $h(w) > 0$, and it is a continuous function because

$$\sup_{\widetilde{u} \in \ker D} -\|w_T + \widetilde{u}\|_1 = -\inf_{\widetilde{u} \in \ker D} \|w_T + \widetilde{u}\|_1 = dist(w_T, \ker D)$$

is continuous.

Secondly, $W \cap B(0, C_1) = \ker(\Phi D) \cap B(0, C_1) \cap \{\|w\| \leq C_2\|Dw\|\}$ is a compact set, so there exists a $C_3 > 0$ such that $h(w) \geq C_3$ on $W \cap B(0, 1)$.

Thirdly, take any $w \in W$ in general, since $h(C_1 w/\|w\|) \geq C_3$, there exists $\widetilde{u} \in \ker D$ such that $\frac{\|C_1 w_{T^C}/\|w\|_2\|_1 - \|C_1 w_T/\|w\|_2 + \widetilde{u}\|_1}{\|C_1 Dw\|_2/\|w\|_2} > C_3/2$, i.e. $\frac{\|w_{T^C}\|_1 - \|w_T + \widetilde{u}\cdot\|w\|_2/C_1\|_1}{\|Dw\|_2} > C_3/2$, which implies $h(w) > C_3/2$. ∎

Fix a support T and a vector $v \in D^{-1}(\ker \Phi \backslash \{0\})$ and define for all $u \in \ker D$, and all $t > 0$ the functions

$$g_v(u, t) = \sup_{\widetilde{u} \in \ker D} \|(tv + u)_{T^c}\|_1 - \|(tv + u)_T + \widetilde{u}\|_1, \text{ and } f_v(u, t) = g_v(u, t)/t.$$

The D-NSP then implies that $g_v(u, t) > 0, f_v(u, t) > 0$.

Lemma 3.7.3. *For any fixed* v, $\inf_{u \in \ker D, t>0} f_v(u, t) > 0$.

Proof. **Step 1.** It is sufficient to prove $\inf_{\|u\|=1, u \in \ker D, t>0} f_v(u, t) > 0$. This is due to the fact that when $u \neq 0$, $f_v(u, t) = f_v(\frac{u}{\|u\|}, \frac{t}{\|u\|})$; when $u = 0$, $f_v(0, t) = f_v(0, 1) > 0$, so

$$\inf_{u \in \ker D, t>0} f_v(u, t) = \min\{\inf_{\|u\|=1, u \in \ker D, t>0} f_v(u, t), f_v(0, 1)\}.$$

Suppose $\inf_{\|u\|=1, u \in \ker D, t>0} f_v(u, t) = 0$, then there exists (u_i, t_i) such that $\lim_{i \to \infty} f_v(u_i, t_i) = 0$.

Step 2. Here we prove that $\{t_i\}$ has a subsequence converging to 0. Otherwise, $t_i \geq t_0 > 0$, which result $t_i v + u_i \in W$ with $W = \{w : w \in D^{-1}(\ker \Phi \backslash \{0\}), C_1 \leq \|w\| \leq C_2\|Dw\|\}$ some constants C_1, C_2 (depending on v). Indeed,

$\|t_i v + u_i\| \geq \|P_{\ker D^\perp}(t_i v + u_i)\| = \|P_{\ker D^\perp}(t_i v)\| \geq t_0\|P_{\ker D^\perp}(v)\| \neq 0$, and

$$\|t_iv+u_i\| \leq \|t_iv\|+1 \leq \begin{cases} \|v\|+1 \leq \frac{\|v\|+1}{t_0\|Dv\|}\|D(t_iv+u_i)\|, & \text{if } t_i \leq 1 \\ t_i(\|v\|+1) = \frac{\|v\|+1}{\|Dv\|}\|D(t_iv+u_i)\|, & \text{if } t_i > 1 \end{cases}.$$

Therefore by Lemma 3.7.2, $f_v(u_i,t_i) = h(t_iv+u_i)\|Dv\| \geq C_3\|Dv\|$ which is a contradiction.

Now we assume $(u_i,t_i) \to (u_0,0)$. There must be infinitely many of $\{u_i - u_0\}$ falling into one orthant (closed) of $\mathbb{R}^d$, say O. Without loss of generality, we assume $x_i := u_i - u_0 \in O$.

Let $\{w_j\}_{j=1}^m$ be the unit vectors on each extremal ray of the polyhedral cone $\ker D \cap O$, i.e., any vector in $\ker D \cap O$ can be expressed as a nonnegative linear combination of $\{w_j\}_{j=1}^m$.

We write $x_i = u_i - u_0 = \sum_{j=1}^m \beta_i(j)w_j$, where $\beta_i(j) \geq 0$. Again, without loss of generality, we assume $\frac{\beta_i(j)}{t_i}$ has a limit for every j as $i \to \infty$. There are only three possibilities of the limits: 0, constants, ∞.

Step 3. We can assume $\frac{\beta_i(j)}{t_i} \to \infty$ for every j.

If $\frac{\beta_i(j_0)}{t_i} \to 0$, for some j_0, then

$$g_v(u_i,t_i) \leq o(t_i) + g_v(u_i - \beta_i(j_0)w_{j_0}, t_i) \leq o(t_i) + g_v(u_i,t_i).$$

Divide all sides by t_i and take the limit, to get

$$\lim_{i\to\infty} f_v(u_i - \beta_i(j_0)w_{j_0}, t_i) = \lim_{i\to\infty} f_v(u_i,t_i) = 0.$$

If $\frac{\beta_i(j_0)}{t_i} \to a_{j_0} \neq 0$, for some j_0, then similarly,

$$\begin{aligned} g_v(u_i,t_i) &= \sup_{\widetilde{u}\in\ker D} \|(t_iv + a_{j_0}t_iw_{j_0} + \sum_{j\neq j_0}\beta_i(j)w_j + u_0 + (\beta_i(j_0) - a_{j_0}t_i)w_{j_0})_{T^c}\|_1 \\ &- \|(t_iv + a_{j_0}t_iw_{j_0} + \sum_{j\neq j_0}\beta_i(j)w_j + u_0 + (\beta_i(j_0) - a_{j_0}t_i)w_{j_0})_T + \widetilde{u}\|_1 \end{aligned}$$

$$\leq \quad o(t_i) + g_{v+a_{j_0}w_{j_0}}(u_i - \beta_i(j_0)w_{j_0}, t_i)$$

$$\leq \quad o(t_i) + g_v(u_i, t_i),$$

which leads to

$$\lim_{i\to\infty} f_{v+a_{j_0}w_{j_0}}(u_i - \beta_i(j_0)w_{j_0}, t_i) = \lim_{i\to\infty} f_v(u_i, t_i) = 0.$$

In summary, take $J_1 = \{j : \frac{\beta_i(j)}{t_i} \to 0\}$, $J_2 = \{j : \frac{\beta_i(j)}{t_i} \to a_j \neq 0\}$, we get

$$\lim_{i\to 0} f_{v'}(u_i', t_i) = 0,$$

where $v' = v + \sum_{j\in J_2} a_j w_j$, $u_i' = u_i - \sum_{j\in J_1\cup J_2} \beta_i(j)w_j$.

Notice that the coefficients β_i of $u_i' - u_0$ in the expansion of w_j will all have the property that $\frac{\beta_i(j)}{t_i} \to \infty$.

Step 4. Final contradiction.

Choose K large enough (the choice of K will be specified later)

Let $x_i - \frac{t_i}{t_K}x_K = \sum c_i(j)w_j$, so we can find an I_0 such that for all $i > I_0$, we have

$$c_i(j) = \beta_i(j) - \frac{t_i}{t_K}\beta_K(j) > 0.$$

Consider

$$\sum c_i(j) g_v(w_j + u_0, 0) + \frac{t_i}{t_K} g_v(u_K, t_K) + (1 - \sum c_i(j) + \frac{t_i}{t_K}) g_{u_0}(0)$$
$$\leq \sum c_i(j) \left[\|(w_j + u_0)_{T^c}\|_1 - \|(w_j + u_0)_T + \widetilde{u}_1\|_1\right] + \epsilon$$
$$+ \frac{t_i}{t_K} \left[\|(t_K v + x_K + u_0)_{T^c}\|_1 - \|(t_K + x_K + u_0)_T + \widetilde{u}_2\|_1\right] + \epsilon$$
$$+ (1 - \sum c_i(j) + \frac{t_i}{t_K})\left[\|(u_0)_{T^c}\|_1 - \|(u_0)_T + \widetilde{u}_3\|_1\right] + \epsilon$$

$$=\|[\sum c_i(j)(w_j+u_0)+\frac{t_i}{t_K}(t_Kv+x_K+u_0)+(1-\sum c_i(j)+\frac{t_i}{t_K})u_0]_{T^c}\|_1+3\epsilon \tag{3.13}$$

$$-\sum c_i(j)\|(w_j+u_0)_T+\widetilde{u}_1\|_1-\frac{t_i}{t_K}\|(t_K+x_K+u_0)_T+\widetilde{u}_2\|_1-\|(u_0)_T$$

$$+(1-\sum c_i(j)+\frac{t_i}{t_K})\widetilde{u}_3\|_1$$

$$\leq g_v(u_i,t_i)+3\epsilon. \tag{3.14}$$

In order for (3.13) to hold, due to the fact that $c_i(j)>0$, $\frac{t_i}{t_K}>0$, and $1-\sum c_i(j)+\frac{t_i}{t_K}>0$ (if $i>I_0$), a sufficient condition is that for each $k\in T^c$, the sign of $w_j(k)+u_0(k), t_Kv(k)+x_K(k)+u_0(k),$ and $u_0(k)$ are all the same. This indeed holds because we can choose K such that

$$\frac{\beta_K(j)}{t_K}>\frac{|v(k)|}{\max_j|w_j(k)|},\ \text{for all index } k\in T^c.$$

With such choice of K, we get $|v(k)|<\sum_{j=1}^m|w_j(k)|\frac{\beta_K(j)}{t_K}=|\sum_{j=1}^m w_j(k)\frac{\beta_K(j)}{t_K}|$ (equality holds since all w_j are in the same orthant), hence

$$sgn(t_Kv(k)+\sum_{j=1}^m\beta_K(j)w_j(k))=sgn(\sum_{j=1}^m\beta_K(j)w_j(k))=sgn(w_j(k)).$$

So if $u_0(k)=0$, we have $sgn(w_j(k)+u_0(k))=sgn(w_j(k))$ and

$sgn(t_Kv(k)+x_K(k)+u_0(k))=sgn(t_Kv(k)+\sum_{j=1}^m\beta_K(j)w_j(k))=sgn(w_j(k))$.

If $u_0(k)\neq 0$, we have $sgn(w_j(k)+u_0(k))=sgn(u_0(k))$ and

$sgn(t_Kv(k)+x_K(k)+u_0(k))=sgn(u_0(k))$ with a big enough choice of K since $t_i\to 0, x_i\to 0$.

Now that (3.13) is justified, let $\epsilon\to 0$ in (3.14), we get

$$g_v(u_i,t_i)\geq\frac{t_i}{t_K}g_v(u_K,t_K)\Rightarrow f_v(u_i,t_i)\geq f_v(u_K,t_K),$$

which is a contradiction.∎

Proof of Theorem 3.4.1. Suppose Φ has D-NSP, we need to show the function

$$F(w) = \sup_{\widetilde{u}\in\ker D} \frac{\|w_{T^c}\|_1 - \|w_T + \widetilde{u}\|_1}{\|Dw\|_2}$$

has a positive lower bound on $D^{-1}(\ker\setminus\{0\})$.

Decompose w as $w = tv + u$ where $u = P_{\ker D}w, tv = P_{\ker D^\perp}w$, $\|v\| = 1$, and $t > 0$. Therefore

$$\inf_{w\in D^{-1}(\ker\setminus\{0\})} F(w) = \inf_{v\in\ker D^\perp, \|v\|=1} \inf_{u\in\ker D, t>0} f_v(u,t)/\|Dv\|. \tag{3.15}$$

By Lemma 3.7.3, the function $\inf_{u\in\ker D,t>0} f_v(u,t)$ is always positive. Since the set $\ker D^\perp \cap B(0,1)$ is compact, it is sufficient to prove that the function $\inf_{u\in\ker D,t>0} f_v(u,t)$ is lower-semicontinuous with respect to v.

$$\begin{aligned} f_{v+e}(u,t) &= \sup_{\widetilde{u}\in\ker D} \frac{\|(tv+te+u)_{T^c}\|_1 - \|(tv+te+u)_T + \widetilde{u}\|_1}{t} \\ &\geq \sup_{\widetilde{u}\in\ker D} \frac{\|(tv+u)_{T^c}\|_1 - \|(tv+u)_T + \widetilde{u}\|_1 - \|te\|_1}{t}. \end{aligned}$$

Taking the infimum over u, t on both sides, we obtain

$$\inf_{u\in\ker D,t>0} f_{v+e}(u,t) \geq \inf_{u\in\ker D,t>0} f_v(u,t) - \|e\|_1,$$

which shows this function is lower-semicontinuous.∎

Chapter 4

Deterministic Sensing Matrices for Dictionaries

A natural question arising from Theorem 3.6.1 is as follows: given a full spark dictionary D, how to actually construct a sensing matrix Φ such that the composition ΦD is NSP. Note that D itself should satisfy NSP for this question to be well-posed. This problem was addressed by Rauhut et al. in [47], but only random sensing matrices were considered. In particular, they proved if D has a small restricted isometry constant δ_D and the random $m \times N$ sensing matrix Φ satisfies the concentration inequality

$$P\Big(|\|\Phi v\| - \|v\|| \geq \epsilon \|v\| \Big) \leq 2e^{-cm\epsilon^2/2}, \quad \epsilon \in (0, 1/3),$$

for all v and some constant c, then with large probablity, the restricted isometry constant of ΦD is small and linearly depends on δ_D. It has been shown that many usual random families satisfy the above concentration inequality. Among them are the Gaussian and Bernoulli ensembles as well as the so-called isotropic subgaussian ensembles, which are constructed by stacking independent copies of a random vector Y as rows of Φ, where Y is such that $\mathbb{E}|\langle Y, v\rangle|^2 = \|v\|^2$ for all $v \in \mathbb{R}^n$.

If we want Φ to be deterministic, then it can no longer be universal in the sense that no single Φ can make ΦD to be RIP (or NSP) for all D. In the following subsections, we construct two classes of deterministic sensing matrices that are compatible with the Dirac-Fourier dictionary $D = [I, F]$, where I is the identity matrix and F the discrete Fourier matrix. While the second class might not be as useful as the first one, it has a very interesting mathematical structure.

4.1 A Class of Deterministic Matrices For the Dirac-Fourier Joint Dictionary

In this section, we construct a class of matrices that are compatible with the Dirac-Fourier joint dictionary. To the best of our knowledge, this is the first class of deterministic sensing matrices for dictionaries constructed in the literature. The matrices are formed by stacking shifted versions of a single chirp sequence, and thus are constant magnitude and quasi-circulant. Verifying why the resulting composition of the sensing matrix and the dictionary satisfies RIP is essentially the same as why the matrix itself is RIP, which is quite straightforward as soon as we know the property of a very similar construction in [32] and our result in Chapter 2. The next three theorems include both the construction of the matrices and the charaterizations of their RIP properties.

Theorem 4.1.1. *Let $p > 2$ be a prime and A be a chirp matrix defined by*

$$A_{j,k} = e^{2\pi i \frac{(j+k)^2}{p}}.$$

Let $f(n)$ be a polynomial of degree $d \geq 2$ with integer coefficients. Choose m to be an integer satisfying

$$p^{1/d} \leq m \leq p.$$

Let $\Omega = \{f(n) \mod p : n = 1, 2, ..., m\}$ and fix any $\eta > 0$. Then for any $\delta_s \in (0, 1)$, the matrix $m^{-1/2} A_\Omega$ satisfies RIP(k, δ_k) whenever the following conditions on k are satisfied:

$$\begin{cases} k \le c_1 m^{2^{1-d}-\eta}, & \text{if } p^{1/(d-1)} \le m \le p, \\ k \le c_2 m^{(\frac{\ln p}{\ln m}-d)2^{1-d}-\eta}, & \text{if } p^{1/d} \le m \le p^{1/(d-1)}. \end{cases} \tag{4.1}$$

where c_1 c_2 are constants that only depend on d and η, and δ_k.

As made clear earlier, any matrix will satisfy an equal or higher order of StRIP than RIP.

Theorem 4.1.2. *Fix any $\eta > 0$, and suppose p, m, Ω, and A_Ω are the same as those in Theorem 4.1.1, then the matrix $m^{-1/2}A_\Omega$ satisfies the (k, δ_k, ϵ)-StRIP if*

$$\begin{cases} k \le \max\left\{\alpha_1 m, \alpha_2 m^{2^{3-d}-4\eta}\right\}, & \text{if } p^{1/(d-1)} \le m \le p, \\ k \le \max\left\{\alpha_1 m, \alpha_3 m^{d2^{3-d}(\ln p/\ln m-1)-4\eta}\right\}, & \text{if } p^{1/d} \le m \le p^{1/(d-1)}. \end{cases} \tag{4.2}$$

where $\alpha_1 - \alpha_3$ only depend on d, δ_k, η and ϵ.

Remark 4.1.1. *The required relations between k, m, and p for the matrix to satisfy SINC are essentially the same up to some logarithmic factor. Since the proof is also similar, we omit it here.*

Theorem 4.1.3. *Suppose p, m, Ω, and A_Ω are the same as those in Theorem 4.1.1. Let F be the $p \times p$ DFT matrix and $D = [I, F]$, then $m^{-1/2}A_\Omega F$ has the same order of RIP and StRIP as those in Theorem 4.1.1 and Theorem 4.1.2. In addition, $m^{-1/2}A_\Omega D$ satisfies RIP if*

$$\begin{cases} k \le c_1 m^{2^{1-2d}-\eta}, & \text{if } p^{1/(d-1)} \le m \le p, \\ k \le c_2 m^{(\frac{\ln p}{\ln m}-2d)2^{1-2d}-\eta}, & \text{if } p^{1/d} \le m \le p^{1/(d-1)}, \end{cases}$$

and satisfies the (k, δ_k, ϵ)-StRIP if

$$\begin{cases} k \le \max\left\{\alpha_1 m, \alpha_2 m^{2^{4-2d}-4\eta}\right\}, & \text{if } p^{1/(d-1)} \le m \le p, \\ k \le \max\left\{\alpha_1 m, \alpha_3 m^{d2^{4-2d}(\ln p/\ln m-1)-4\eta}\right\}, & \text{if } p^{1/d} \le m \le p^{1/(d-1)}. \end{cases}$$

Proofs of these results essentially rely on the following theorem of Weil and Theorem 2.4.7.

Theorem 4.1.4. *([55]). Let m, a, q be integers such that $(a, q) = 1$ and $q > 0$. If f is a real polynomial of degree $k \geq 1$ with leading coefficient α such that $|\alpha - \frac{a}{q}| \leq tq^{-2}$ for some $t \leq 1$ then for any $\eta > 0$ we have*

$$\sum_{x=1}^{m} e^{2\pi i f(x)} = O\left(m^{1+\eta}\left(\frac{t}{q} + \frac{1}{m} + \frac{t}{m^{k-1}} + \frac{q}{m^k}\right)^{2^{1-k}}\right).$$

Proof of Theorem 4.1.1. For the matrix $m^{-1/2}A_\Omega$ defined in the theorem, we can calculate its mutual coherence,

$$\mu_{j,l} = m^{-1}\left|\sum_{x=1}^{m} e^{2\pi i \frac{j-l}{p} f(x)}\right|, \quad \forall j \in [p], l \in [p]\backslash j \tag{4.3}$$

Let $g(x) = \frac{j-l}{p} f(x)$, then g satisfies the assumption in Theorem 4.1.4. Hence we have

$$\left|\sum_{x=1}^{m} e^{2\pi i g(x)}\right| \leq c m^{1+\eta}(\frac{1}{m} + \frac{p}{m^d})^{2^{1-d}},$$

where c is some constant. By the definition of mutual coherence, we have

$$\mu \leq c m^{\eta}\left(\frac{1}{m} + \frac{p}{m^d}\right)^{2^{1-d}}. \tag{4.4}$$

Applying the Gershgorin Theorem, we obtain the following condition on k for the matrix $m^{-1/2}A_\Omega$ to satisfy RIP,

$$k < c\delta_k m^{-\eta}\left(\frac{1}{m} + \frac{p}{m^d}\right)^{-2^{1-d}}.$$

When $p^{1/(d-1)} \leq m \leq p$, $\frac{1}{m}$ is the leading term in the above parentheses, thus we can use $\frac{2}{m}$ to bound the whole brackets. Rearranging the inequality, we get the first constraint in (3). Similarly, when $p^{1/(d)} \leq m \leq p^{1/(d-1)}$, we obtain the second. ∎

Proof of Theorem 4.1.2. In order to apply Theorem 2.4.7, let us first calculate the quantity $\bar{\mu}^2(m^{-1/2}A_\Omega)$.

$$\mathsf{E}_{j,l:j\neq l}(\mu_{j,l}^2) = \frac{1}{p(p-1)} \sum_j \sum_{l:l\neq j} \frac{1}{m^2} \left| \sum_{x=1}^{m} e^{2\pi i \frac{j-l}{p} f(x)} \right|^2.$$

Expanding the square and interchanging the summations, we obtain

$$\begin{aligned} \mathsf{E}_{j,l:j\neq l}(\mu_{j,l}^2) &= \frac{1}{p(p-1)m^2} \sum_j \sum_{l:l\neq j} \sum_{x_1,x_2} e^{2\pi i \frac{j-l}{p}\left(f(x_1)-f(x_2)\right)} \\ &= \frac{1}{p(p-1)m^2} \sum_{x_1,x_2} \sum_j \sum_{l:l\neq j} e^{2\pi i \frac{j-l}{p}\left(f(x_1)-f(x_2)\right)}. \end{aligned} \tag{4.5}$$

Since the sum of roots of unity is 1, the value that comes out of the first summation will be -1 if $f(x_1) - f(x_2) \neq 0$, and $p-1$ otherwise. This observation implies the right hand side of 4.5 has the following equivalent form:

$$\begin{aligned} & \frac{1}{p(p-1)m^2} \sum_{x_1,x_2} \sum_j [-(1-\delta(f(x_1)-f(x_2)) + (p-1)\delta(f(x_1)-f(x_2))] \\ = & \frac{1}{p(p-1)m^2} \sum_{x_1,x_2} [-p(1-\delta(f(x_1)-f(x_2)) + (p-1)p\delta(f(x_1)-f(x_2))] \\ = & \frac{1}{p(p-1)m^2} \left[-p\big(m^2 - |\{(x_1,x_2) : f(x_1) = f(x_2)\}|\big) + (p-1)p\,|\{(x_1,x_2) : f(x_1) = f(x_2)\}|\right. \end{aligned}$$

If there exist $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, then we must have chosen two identical rows, which contradicts our definition of A_Ω. Therefore $|\{(x_1,x_2) : f(x_1) = f(x_2)\}| = m$, which, together with the previous equation leads to

$$\bar{\mu}^2 < \frac{1}{m}. \tag{4.6}$$

Plugging (4.1) and (4.6) into Theorem 2.4.7 completes the proof. ∎

Proof of Theorem 4.1.3. By direct calculation, we obtain $A_\Omega F = F_\Omega \Lambda$ where Λ is a diagonal matrix whose diagonal vector λ is given by $\lambda = FA_{1,\cdot}^T$ with $A_{1,\cdot}$ being the first row of A. A well known property of the chirp sequences are that both they and their Fourier transforms belong to the class of constant magnitude and zero auto-correlation sequences. Therefore, λ has constant magnitude 1, implying $\mu(F_\Omega\Lambda) = \mu(F_\Omega)$ and $\bar{\mu}^2(F_\Omega\Lambda) = \bar{\mu}^2(F_\Omega)$. The following calculations are the same as those in Theorem 4.1.1 and Theorem 4.1.2.

Since $A_\Omega D = [A_\Omega, F_\Omega\Lambda]$, then $\mu(A_\Omega D) = \max\{\mu(A_\Omega), \mu(F_\Omega), \mu(A_\Omega, F_\Omega)\}$, where $\mu(\Phi, \Psi) := \max_{j,l} |\langle \phi_j, \psi_l \rangle|$ denotes the maximum coherence between dictionaries Φ and Ψ. Let $\mu_{j,l}$ denote the magnitude of the inner product of the j'th column of A_Ω and the l'th column of F_Ω. Then

$$\mu_{j,l} = m^{-1} \Big| \sum_{x=1}^{m} e^{2\pi i \frac{(f(x)+j)^2 - lf(x)}{p}} \Big| = m^{-1} \Big| \sum_{x=1}^{m} e^{2\pi i \frac{g_{j,l}(x)}{p}} \Big|,$$

where $g_{j,l}(x) = (f(x) + j)^2 - lf(x)$. By definition $g_{j,l}(x)$ is a $2d$'th order polynomial that satisfies the condition in Theorem 4.1.4 with $k = 2d$. Calculations stating from here are all the same as in the previous two theorems. ∎

4.2 Another Statistical Restricted Isometry Property

The next class of matrices that we construct does not satisfy the strict RIP nor the previously defined StRIP, but it satisfies another Statistical RIP proposed by Calderbank et al. ([14]) as a guarantee for the Quadratic Reconstruction Algorithm they established in an earlier paper [35].

Definition 4.2.1. *(Statistical Restricted Isometry Property STRIP) An $m \times N$ sensing matrix Φ is said to be a (k, ϵ, δ_k)-Statistical Restricted Isometry Property matrix if, for* *any k-sparse vectors $x \in R^n$, the inequalities*

$$(1 - \delta_k)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_k)\|x\|_2^2 \quad (4.7)$$

hold with probability exceeding $1 - \epsilon$ (with respect to a uniform distribution of the vector x among all k-sparse vectors in R^n with fixed magnitudes).

Unlike the StRIP defined in Chapter 2 which depends only on the distribution of active locations, this definition relies on both the locations and the magnitudes of active components of x, and therefore is a weaker condition. Again, STRIP does not automatically imply unique reconstruction itself, which brings about the definition of a second property called Uniqueness-guaranteed Statistical RIP.

Definition 4.2.2. *(Uniqueness-guaranteed Statistical RIP) ((k, ϵ, δ_k)-UStRIP Matrix): An $m \times N$ sensing matrix Φ is said to be a (k, ϵ, δ_k)-Uniqueness-guaranteed Statistical Restricted Isometry Property matrix if Φ is a (k, ϵ, δ_k)-StRIP matrix, and*

$$\{y \in R^N, y \; is \; k - sparse; \; \Phi x = \Phi y\} = \{x\} \quad (4.8)$$

with probability exceeding $1 - \epsilon$ (with respect to a uniform distribution of the vector x among all k-sparse vectors in R^N)

4.3 Another Class of Deterministic Sensing Matrix for Dictionaries

In this section, we construct a class of matrices that satisfy the STRIP and UStRIP defined in the previous section. These matrices are structured as a repetitive stack of a group of smaller orthogonal matrices in the most redundant way, in the sense that further repetition of the small matrices will lead to the apperance of identical columns. Because of this redundancy, the matrix is computationally efficient.

Definition 4.3.1. *Let Φ be a $N \times N$ chirp matrix (i.e.,$\Phi_{k,j} = e^{2\pi i \frac{(k+j)^2}{N}}$) with $N = p_1 \times p_2 \times ... \times p_r$ is a product of prime numbers and $r = p_1^{\alpha}$ with $0 < \alpha < 1$. We modify this matrix as follows:*

- *If $k \neq N$ and $kp_j \mid N$ for some j, multiply the kth row by $\sqrt{\frac{\ln(p_j)}{p_j \ln(N)}}$;*
- *If $kp_j \nmid N$ for all j=1,..,r, remove this row from Φ;*
- *Multiply the last row by $\sqrt{\sum_{j=1}^{r} \frac{\ln(p_j)}{p_j \ln(N)}}$.*

Theorem 4.3.1. *The matrix Φ defined above obeys the (k, ϵ, δ_k) statistical RIP, for all $\delta_k < 1$ and $k < \max\{\frac{\delta_k(1-\alpha)\ln(N)}{\ln(2k/\epsilon)}, \sqrt{\frac{\epsilon p_1}{r}}\}$.*

Theorem 4.3.2. *The matrix Φ satisfies the (k, ϵ, δ_k)-UStRIP whenever it satisfies the $(k, \epsilon, \min\{\delta_k, 1/3\})$ StRIP.*

Theorem 4.3.3. *Let F be the DFT matrix, then ΦF satisfies the same order of StRIP and UStRIP.*

Example: Let $r = 2$, $p_1 = 2$, $p_2 = 3$, the matrix is structurally similar to that in Figure 4.1 except for some extra rotation and scaling of the sub matrices. This special structure will make the matrix-vector multiplication operation (which is the most time consuming step in nearly all reconstruction algorithms) more efficient.

Proposition 4.3.1. *The matrix-vector multiplication cost of this matrix is $rN + \sum_{i=1}^{r} p_i \log(p_i)$.*

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

Figure 4.1: Structure of the matrix

Proof. By construction, the first p_1 rows are simply $\frac{N}{p_1}$ repetitions of the $p \times p$ DFT matrix with some scaling. The matrix-vector multiplication is then equivalent to taking $\frac{N}{p_1}$ Fourier transforms and then adding them together. Instead, we add the vectors first and do the Fourier transform only once. The total number of addition operations is N and that of the Fast Fourier transform of a prime order matrix is $p_i \log(p_i)$. Applying this analysis r times results in the quantity in the statement of the proposition. ∎

Remark 4.3.1. *If r is fixed and we let p_i go to infinity, the cost is only $O(N)$.*

Remark 4.3.2. *We note that if we take only the first few columns of Φ, then we get a deterministic RIP matrix which has the same property as the matrix constructed in* [*36*].

Before proceeding to the proof of Theorem 3.6, we first introduce some special notation.

- Let Φ^j be the submatrix of Φ containing only the rows with magnitude $\sqrt{\frac{log(p_j)}{p_j \ln(N)}}$. Thus $\Phi = [\Phi^1; \Phi^2; ...; \Phi^r]$.

- Let ϕ_l denote the lth column of Φ and ϕ_l^j the lth column of Φ^j.

- Use (p) to denote the set $\{1, 2, ..., p\}$ and $[x]$ to denote the largest integer less than or equal to x.

- Use $\underline{x}^n = (x_1, x_2, .., x_n),\ with\ x_i \in \{0, 1\}$ to denote the binary code of length n, i.e. $\underline{x}^n \in \mathbb{F}_2^n$.

- $d(\underline{x}^n - \underline{y}^n)$ denotes the hamming distance between two binary codes. And thus $d(\underline{x}^n) \equiv d(\underline{x}^n - (0, 0, ..., 0))$ is the number of 1s in $\underline{x}^n$.

- We say $\underline{x}^n \leq \underline{y}^n$ if $x_i \leq y_i$ for every i. And $\underline{x}^n < \underline{y}^n$ if $\underline{x}^n \leq \underline{y}^n$ and $\underline{x}^n \neq \underline{y}^n$.

4.3.0.1 Proof of the Theorem 4.3.1

Lemma 4.3.2. *For fixed l_1, l_2 with $l_1 \neq l_2$, $|\langle \phi_{l_1}, \phi_{l_2} \rangle| = \frac{1}{\log N} \sum_{j=1}^{r} \log(p_j) x_j$, where $x_j = 1$ if $p_j | (l_1 - l_2)$ and 0 otherwise.*

Proof.

$$
\begin{aligned}
\langle \phi_{l_1}, \phi_{l_2} \rangle &= \sum_{j=1}^{r} (\phi_{l_1}^j)^T (\phi_{l_2}^j) \\
&= \sum_{j=1}^{r} \sum_{m=1}^{p_j} \frac{\log(p_j)}{p_j \log(N)} e^{-2\pi i \frac{(\frac{mN}{p_j} + l_1)^2}{N}} \cdot e^{2\pi i \frac{(\frac{mN}{p_j} + l_2)^2}{N}} \\
&= e^{2\pi i \frac{l_2^2 - l_1^2}{N}} \sum_{j=1}^{r} \frac{\log(p_j)}{p_j \log(N)} \sum_{m=1}^{p_j} e^{4\pi i \frac{m(l_2 - l_1)}{p_j}} \\
&= e^{2\pi i \frac{l_2^2 - l_1^2}{N}} \sum_{j=1}^{r} \frac{\log(p_j)}{p_j \log(N)} x_j p_j \\
&= e^{2\pi i \frac{l_2^2 - l_1^2}{N}} \frac{\sum_{j=1}^{r} \log(p_j) x_j}{\log(N)}.
\end{aligned}
$$

∎

Lemma 4.3.3. *$P(|\langle\phi_{l_1},\phi_{l_2}\rangle| > \varepsilon) < \max\{\frac{\log(N)}{\log(p_1)}N^{-\varepsilon(1-\alpha)}, 1-\frac{r}{p_1}\}$, where the probability is with respect to the random choices of l_1 and l_2.*

Proof. WLOG, assume $l_1 = 1$. We discuss the cases $\varepsilon < \frac{\log p_1}{\log N}$ and $\varepsilon \geq \frac{\log p_1}{\log N}$ separately.

Case I ($\varepsilon < \frac{\log p_1}{\log N}$). In this case, we can assert that

$$P(|\langle\phi_{l_1},\phi_{l_2}\rangle| > \varepsilon) = P(|\langle\phi_{l_1},\phi_{l_2}\rangle| > 0). \tag{4.9}$$

This is because the set $\{|\langle\phi_{l_1},\phi_{l_2}\rangle| : l_1 \neq l_2\}$ is finite, and the smallest positive value in this set is $\frac{\log p_1}{\log N}$. Since we assumed ε to be less than this value, it has to equal 0. We proceed to calculate the probability on the right hand side of 4.9 using union bound:

$$P(|\langle\phi_{l_1},\phi_{l_2}\rangle| > 0) = 1 - P(|\langle\phi_{l_1},\phi_{l_2}\rangle| = 0) \leq 1 - \frac{1}{p_1} - \frac{1}{p_2}, ..., \frac{1}{p_r} \leq 1 - \frac{r}{p_1}.$$

Case II ($\varepsilon > \frac{\log p_1}{\log N}$). Define

$$\begin{aligned}
A_\varepsilon &= \left\{\underline{x}^r : \ \frac{1}{\log N}\sum_{j=1}^{r}\log(p_j)x_j > \varepsilon\right\}, \\
B_{\underline{x}^r} &= \left\{l : \ \ |\langle\phi_1,\phi_l\rangle| = \frac{1}{\log N}\sum_{j=1}^{r}\log(p_j)x_j\right\}, \\
\widetilde{A}_\varepsilon &= \left\{\underline{x}^r : \ \ \underline{x}^r \in A_\varepsilon \ and \ \underline{y}^r \notin A_\varepsilon, \ for \ all \ \underline{y}^r < \underline{x}^r\right\}, \\
\widetilde{B}_{\underline{x}^r} &= \left\{k : \ \ k \in B_{\underline{y}^r} \ for \ some \ \underline{y}^r \geq \underline{x}^r\right\}.
\end{aligned}$$

We will prove later in Lemma 4.3.4 and Lemma 4.3.5 that $|\widetilde{B}_{\underline{x}^r}| = \frac{N}{\prod_{j=1}^{r} p_j^{x_j}}$ and $|\widetilde{A}_\varepsilon| \leq \frac{\log(N)}{\log(p_1)}N^{\alpha\varepsilon}$. Here we first use these results to prove the current lemma. By Lemma 4.3.2:

$$P\left(|\langle\phi_{l_1},\phi_{l_2}\rangle| > \varepsilon\right) = P\left(\frac{\sum_{j=1}^{r}\log(p_j)x_j}{\log(N)} > \varepsilon\right) \leq \sum_{\underline{x}\in\widetilde{A}}\frac{|\widetilde{B}_{\underline{x}^r}|}{N}$$

$$
\begin{aligned}
&= \sum_{\underline{x}\in\widetilde{A}} \frac{1}{\prod_{j=1}^{r} p_j^{x_j}} = \sum_{\underline{x}\in\widetilde{A}} \frac{1}{e^{\log(\prod_{j=1}^{r} p_j^{x_j})}} = \sum_{\underline{x}\in\widetilde{A}} \frac{1}{e^{\sum_{j=1}^{r}\log(p_j)x_j}} \leq \sum_{\underline{x}\in\widetilde{A}} \frac{1}{e^{\varepsilon\log(N)}} \\
&= \sum_{\underline{x}\in\widetilde{A}} N^{-\varepsilon} = |\widetilde{A}|N^{-\varepsilon} = \frac{\log(N)}{\log(p_1)} N^{-(1-\alpha)\varepsilon},
\end{aligned}
$$

where the second to last inequality made use of the fact that $\underline{x}^r \in \widetilde{A} \subseteq A$. ∎

Lemma 4.3.4. $|\widetilde{B}_{\underline{x}^r}| = \frac{N}{\prod_{j=1}^{r} p_j^{x_j}}$.

Proof. From the definition of x_j in Lemma 4.3.2, if $l \in \widetilde{B}$, then $p_j^{x_j} \mid (l-1)$. Since this is true for all j and the p_js are relatively prime, we get $(\prod_{j=1}^{r} p_j^{x_j})|(l-1)$ meaning $l-1$ is a multiple of $\prod_{j=1}^{r} p_j^{x_j}$. The number of such multiples is $\frac{N}{\prod_{j=1}^{r} p_j^{x_j}}$, and so is $|\widetilde{B}_{\underline{x}^r}|$. ∎

Lemma 4.3.5. $|\widetilde{A}_\varepsilon| \leq N^{2\alpha\varepsilon}$.

Proof. For any $\underline{x}^r \in \widetilde{A}$, let $\underline{y}^r$ be the element obtained by changing a "1" element in $\underline{x}^r$ to "0" and keeping other elements the same. So we have $d(\underline{y}^r) = d(\underline{x}^r) - 1$. By the definition of $\widetilde{A}$, if $\underline{y}^r < \underline{x}^r$, then $\underline{y}^r \notin A$ and $\frac{\sum_{j=1}^{r}\log(p_j)y_j}{\log(N)} < \varepsilon$. We use this inequality to obtain an upper bound on $d(\underline{y}^r)$:

$$
\varepsilon \; > \; \frac{\sum_{j=1}^{r}\log(p_j)y_j}{\log(N)} > \frac{d(\underline{y}^r)\log(p_1)}{\log(N)}.
$$

Therefore,

$$
d(\underline{x}^r) = d(\underline{y}^r) + 1 < \frac{\log(N)}{\log(p_1)}\varepsilon + 1.
$$

We use this result to estimate the cardinality of $\widetilde{A}$ as follows,

$$\begin{aligned}
|\widetilde{A}| &\leq \left|\{\underline{x}^r : \quad d(\underline{x}^r) = [\varepsilon\frac{\log(N)}{\log(p_1)} + 1]\}\right| \\
&= \binom{r}{[\varepsilon\frac{\log(N)}{\log(p_1)} + 1]} \\
&\leq r^{\varepsilon\frac{\log(N)}{\log(p_1)}+1} \\
&\leq \frac{\log(N)}{\log(p_1)} r^{\frac{\log(N)}{\log(p_1)}\varepsilon} \\
&= \frac{\log(N)}{\log(p_1)} N^{\frac{\log(r)}{\log(p_1)}\varepsilon} \\
&= \frac{\log(N)}{\log(p_1)} N^{\alpha\varepsilon}.
\end{aligned}$$

∎

Proof of Theorem 4.3.1. Recall we use δ_k to denote the RIP constant of order k, and Ω to denote the index of the nonzero components of a k-sparse vector x. First we define a set C as follows,

$$C = \{\Omega|\ |\langle\phi_i, \phi_j\rangle| < \frac{\delta_k}{k-1},\ for\ all\ i, j \in \Omega, and\ j \neq i\}. \tag{4.10}$$

Then by the Gershgorin Circle Theorem, for any $\Omega \in C$, we have

$$\begin{aligned}
|1 - \lambda(\Phi_\Omega^T \Phi_\Omega)| &\leq \max_{i\in\Omega} \sum_{j\in\Omega, j\neq i} | < \phi_j, \phi_i > | \\
&\leq (k-1)\max_{j\neq i} | < \phi_i, \phi_j > | \\
&\leq \delta_k.
\end{aligned}$$

Thus the set C is where the (k, δ_k)-RIP holds. We are about to bound the probability of the complement of C:

$$\begin{aligned} 1 - P(C) &= P\left(|\langle \phi_{l_i}, \phi_{l_j} \rangle| > \frac{\delta_k}{k-1},\ for\ some\ i \in [k], j \in [k], i \neq j \right) \\ &\leq k(k-1) P\left(|\langle \phi_{l_1}, \phi_{l_2} \rangle| > \frac{\delta_k}{k-1} \right) \\ &\leq k(k-1) \max\left\{ \frac{r}{p_1}, \frac{\log(N)}{\log(p_1)} N^{-\frac{\delta_k}{k-1}(1-\alpha)} \right\}. \end{aligned}$$

For the matrix to satisfy the (k, ϵ, δ_k)-StRIP, we only need to impose the above probability to be bounded by ϵ, i.e.

$$k(k-1) \max\{\frac{r}{p_1}, \frac{\log(N)}{\log(p_1)} N^{-\frac{\delta_k}{k-1}(1-\alpha)}\} < \epsilon.$$

Solving this inequality gives us the condition on k in the statement of the theorem. ∎

4.3.0.2 Proof of Theorem 4.3.2

To prove this Theorem, we need to prove the following lemma.

Lemma 4.3.6. *Φ satisfies the unique recovery property (i.e. $P(x : \nexists\ k\text{-sparse}\ y,\ st\ \Phi x = \Phi y) > 1 - \epsilon$) if and only if*

$$P(\Omega : \exists\, \Omega',\ with |\Omega'| = k\ and\ span(\Phi_\Omega) = span(\Phi_{\Omega'})) < \epsilon.$$

Here the first probability is with respect to the uniform distribution of all k-sparse vectors and the second is with respect to the uniform distribution of all sets Ω with cardinality k.

Proof. For a fixed index set Ω ($|\Omega| = k$), the following two statements are equivalent:

1. There is no Ω' with $|\Omega'| = k$, which is different from Ω but have the same span: $\text{span}(\Omega') = \text{span}(\Omega)$.

2. $P(x \in R^k : \exists y \in R^k,\ st.\ \Phi_\Omega x = \Phi_{\Omega'} y) = 0.$

We use $\mathcal{R}$ to denotes the set of Ωs which satisfies 1 and 2. Now we can calculate the probability in the definition of UStRIP:

$$\begin{aligned} & P(x \in R^N : x \text{ is } k-sparse,\ \nexists\, k-sparse\, y,\ st\ \Phi x = \Phi y) \\ = \ & 1 - P(x \in R^N : \operatorname{supp}(x) \in \mathcal{R}) \\ \geq \ & 1 - \epsilon. \end{aligned}$$

∎

Proof of Theorem 4.3.2. Proof by contradiction. For any $\Omega \in C$, assume there exists an Ω' such that $|\Omega'| = k$ and $span(\Phi_\Omega) = span(\Phi_{\Omega'})$. Then any column $\phi_{\omega'}$ of $\Phi_{\Omega'}$ can be expressed as a linear combination of the vectors in Φ_Ω: $\phi_{\omega'} = \sum_{i=1}^{k} a_k \phi_{\omega_i}$ (recall we assumed that the coefficients are all real). Since $\Omega \in C$, we have $\langle \phi_{\omega_i}, \phi_{\omega_j} \rangle < \frac{\delta_k}{k}, \forall\, i \neq j$. We define $x^l_{i,j}$ and x^l_j as follows:

$$|\langle \phi_{\omega_i}, \phi_{\omega_j} \rangle| = \frac{\sum_{l=1}^{r} x^l_{i,j} \log(p_l)}{\log(N)}, \quad |\langle \phi_{\omega'}, \phi_{\omega_j} \rangle| = \frac{\sum_{l=1}^{r} x^l_{j} \log(p_l)}{\log(N)}.$$

So $x^l_{i,j}$ denotes whether the lth block of ϕ_{ω_i} and ϕ_{ω_j} are collinear or orthogonal, respectively. If they are collinear, then $x^l_{i,j} = 1$, orthogonal $x^l_{i,j} = 0$. The same explanation holds for x^l_j. Fix an i such that $a_i \neq 0$ and sum the magnitude of the inner product over all the indices $j \neq i$, we obtain

$$\begin{aligned} \delta_k \ & \geq \ \sum_{j, j\neq i} |\langle \phi_{\omega_i}, \phi_{\omega_j} \rangle| \\ & = \ \sum_{j, j\neq i} \frac{\sum_{l=1}^{r} x^l_{i,j} \log(p_l)}{\log(N)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{r} \frac{(\sum_{j,j\neq i} x_{i,j}^{l}) \log(p_l)}{\log(N)} \\
&\geq \sum_{l=1}^{r} \frac{(\max_{j,j\neq i} x_{i,j}^{l}) \log(p_l)}{\log(N)}. \qquad (4.11)
\end{aligned}
$$

Let $y_i^l = 1 - \max_{j,j\neq i}(x_{i,j}^l)$, and insert it into 4.11 to have

$$
\begin{aligned}
\delta_k &\geq \sum_{l=1}^{r} \frac{(1-y_i^l)\log(p_l)}{\log(N)} = 1 - \sum_{l=1}^{r} \frac{y_i^l \log(p_l)}{\log(N)} \\
&\Rightarrow \sum_{l=1}^{r} \frac{y_i^l \log(p_l)}{\log(N)} \geq 1 - \delta_k.
\end{aligned}
$$

Note that $y_i^l = 1$ means $\max_{j,j\neq i} x_{i,j}^l = 0$, which indicates all the ϕ_j are orthogonal to ϕ_i at the lth block. For all the l such that $y_i^l = 1$, we have

$$
\langle \phi_{\omega'}^l, \phi_{\omega_i}^l \rangle =< \sum_{j=1}^{k} a_j \phi_{\omega_j}^l, \phi_{\omega_i}^l >= \frac{a_i \log(p_l)}{\log(N)} \neq 0. \qquad (4.12)
$$

Since the lth blocks of two columns of Φ are either collinear or orthogonal, 4.12 implies that $\phi_{\omega'}^l$ is collinear with $\phi_{\omega_i}^l$ so it must be orthogonal to other $\phi_{\omega_j}^l$ for $j \neq i$. This leads to

$$
\phi_{\omega'}^l = a_i \phi_{\omega_i}^l.
$$

Moreover, since $\phi_{\omega'}^l$ and $\phi_{\omega_i}^l$ have the same magnitude $\frac{\log p_l}{\log N}$, thus $|a_i| = 1$. Now for any $i \in \{1, ..., k\}$, we have

$$
\begin{aligned}
|\langle \phi_{\omega_i}, \phi_{\omega'} \rangle| &= |\langle \phi_{\omega_i}, \sum_{j=1}^{k} a_j \phi_{\omega_j} \rangle| \\
&\geq \sum_{l: y_i^l \neq 0} |\langle \phi_{\omega_i}^l, \sum_{j=1}^{k} a_j \phi_{\omega_j}^l \rangle| \\
&= \sum_{l: y_i^l \neq 0} |\langle \phi_{\omega_i}^l, \phi_{\omega_i}^l \rangle| \\
&= \sum_{l=1}^{r} \frac{y_i^l \log(p_l)}{\log(N)} \geq 1 - \delta_k. \qquad (4.13)
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{k}|\langle\phi_{\omega'},\phi_{\omega_i}\rangle| &= \sum_{i=1}^{k}\frac{\sum_{l=1}^{r}x_i^l\log p_l}{\log N}\\ &= \frac{1}{\log N}\sum_{l=1}^{r}(\sum_{i=1}^{k}x_i^l\log p_l). \qquad (4.14)\end{aligned}$$

Let $i_l \in \{1,...,k\}$ be an index such that $x_{i_l}^l = 1$. The existence of such indices is guaranteed by the linearly dependence of $\phi_{\omega'}$ on $\phi_{\omega_i}, i = 1,...,k$. Then

$$\begin{aligned}\frac{1}{\log N}\sum_{l=1}^{r}(\sum_{i=1}^{k}x_i^l\log p_l) &= \frac{1}{\log N}\sum_{l=1}^{r}(x_{i_l}^l+\sum_{i:i\neq i_l}x_i^l)\log p_l\\ &= \frac{1}{\log N}\sum_{l=1}^{r}(1+\sum_{i\neq i_l}x_{i,i_l}^l)\log p_l\\ &= 1+\frac{1}{\log N}\sum_{l=1}^{r}\sum_{i\neq i_l}x_{i,i_l}^l\log p_l\\ &\leq 1+\frac{1}{\log N}\sum_{l=1}^{r}\sum_{i,j:i\neq j}x_{i,j}^l\log p_l\\ &= 1+\sum_{i,j}\sum_{l=1}^{r}\frac{\log p_l}{\log N}\\ &= 1+\sum_{i,j:i\neq j}|\langle\phi_{\omega_i},\phi_{\omega_j}\rangle|\\ &\leq 1+k\delta_k. \qquad (4.15)\end{aligned}$$

Combine 4.13-4.15, to get

$$1+k\delta_k \geq \sum_{i=1}^{k}|<\phi_{\omega'},\phi_{\omega_i}>| \geq k(1-\delta_k),$$

which implies

$$\delta_k \geq \frac{k-1}{2k} \geq \frac{1}{3}.$$

This contradicts the assumption that $\delta_k < 1/3$. ∎

Proof of Theorem 4.3.3. The Fourier transform of this matrix is equivalent to applying the truncating and rescaling operations in Definition 4.1.1 on another chirp matrix given by:

$$\widetilde{\Phi}_{j,k} = e^{2\pi i \frac{2bj-b^2}{N}}.$$

with $b \equiv 2k \; mod(N)$. Therefore the same arguments follow. ∎

4.4 Numerical Results

Experiment 1 (Standard RIP): In this experiment, we compare the matrix A constructed in Section 4.1 to a Guassian matrix $\mathcal{N}$ with the same dimension in their performances of sparse recoveries by basis pursuit. In particular, we set $m = 100$, $N = 1031$, and let k vary. Signals are generated by first choosing the k nonzero locations uniformly at random, and then assigning values to these locations from the standard normal distribution. A recovery $\hat{x}$ is deemed as successful if $\|\boldsymbol{x} - \hat{x}\|_{l_2}/\|\boldsymbol{x}\|_{l_2} \leq 0.01$, where $\boldsymbol{x}$ denotes the original signals as before. Figure 4.2 plots the average success rate taken over 100 independent draws of x. The result shows that on average, the two matrices act very similarly to each other.

Experiment 2 (RIP in the joint dictionary): We take the same matrices as in the previous experiment, but test their performances on signals that are sparse under the Dirac-Fourier joint dictionary $D = [I, F]$. In particular, let $\boldsymbol{x}$ be such that $\boldsymbol{x} = D\boldsymbol{z}$ for some $\boldsymbol{z} \in \Sigma_k$, and let $y_A = A\boldsymbol{x}$ and $y_{\mathcal{N}} = \mathcal{N}\boldsymbol{x}$ be the measurements taken from the two sensing schemes. If the reconstruction from the ℓ_1 synthesis approach is recorded in $\hat{x}$, then we

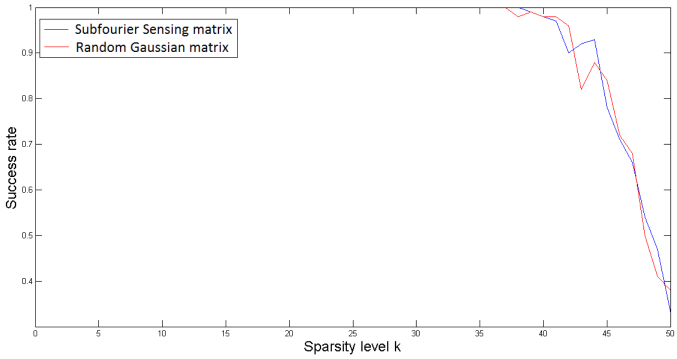
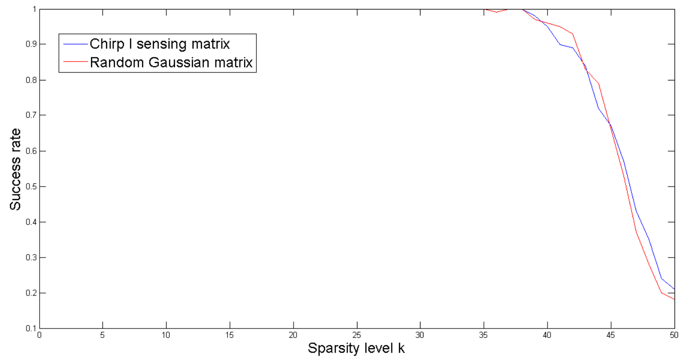


Figure 4.2: Success rate of sparse signal sensed by Chirp matrix vs Gaussian matrix

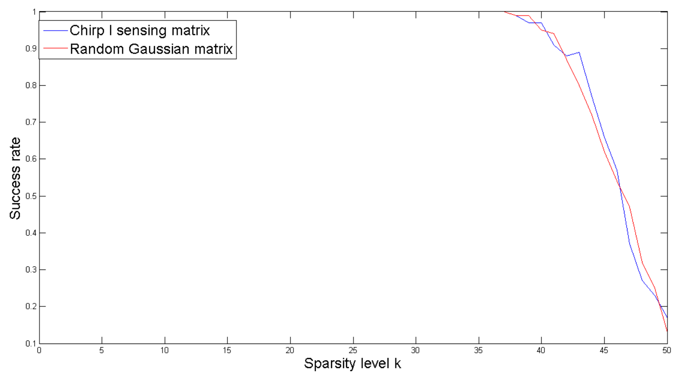
deem the recovery to be successful if $\|\boldsymbol{x} - \hat{x}\|_{l_2}/\|\boldsymbol{x}\|_{l_2} \leq 0.01$. Figure 4.3(a) plots the average success rate taken over 100 independent draws of x. Figure 4.3(b) shows similar result but for the Dirac-Haar joint dictionary. Again, the performances of these two matrices are nearly indistinguishable.

Experiment 3: In this experiment we compare reconstructions of real scene images based on difference sparsity assumptions, that is either assuming images being sparse in canonical basis I, or in the Dirac-Fourier joint dictionary $D_1 = [I, F]$, or the Dirac-Haar joint dictionary $D_2 = [I, H]$.

For a given vectorized image X, let $\nabla_x X$, $\nabla_y X$ be the horizontal and vertical (both are directions in the original image) gradients of X. Then there exist finite difference matrices P_1 and P_2 independent of X such that $\nabla_x X = P_1 X$, $\nabla_y X = P_2 X$. Suppose A is the same matrix as in previous experiments, and the measurements $Y = [Y1; Y2]$ are obtained from projections $Y_i = AP_i X$ for $i = 1, 2$. Notice that now the composition $[AP_1; AP_2]$ is the actual underlying sensing matrix. First assuming both gradients are



(a) Success recovery rate for signals which are sparse under the Dirac-Fourier joint dictionary



(b) Success recovery rate for signals which are sparse under the Dirac-Haar joint dictionary

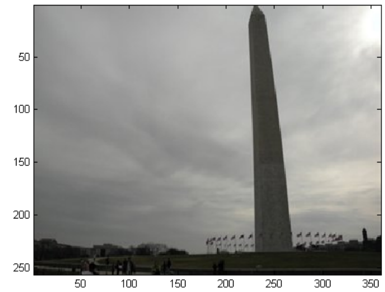
Figure 4.3: Success recovery rate of sparse signals under different dictionaries

sparse, we reconstruct $\nabla_x X$ and $\nabla_y X$ from Y by solving

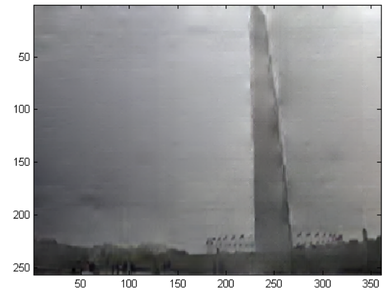
$$\arg\min \|Z_i\|_{l_1} \text{ subject to } Y_i = AZ_i, \quad i = 1, 2. \tag{4.16}$$

As soon as $\widehat{\nabla_x X}$ and $\widehat{\nabla_y X}$ are obtained as solutions to (4.16), they can be used to construct $\widehat{X}$ by applying the Frankot-Chellappa algorithm [46].

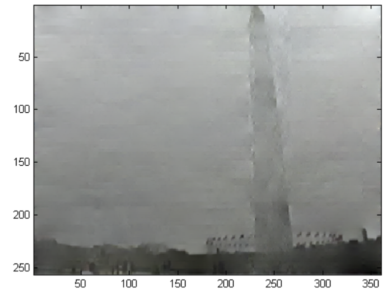
We test the above method using a 256×363 photo of the monument. In order to speed



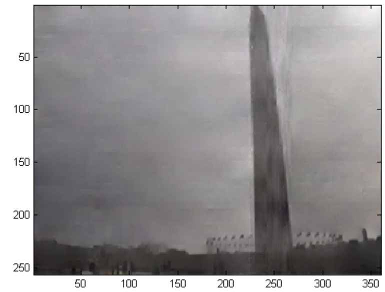
(a) Original Image



(b) Reconstructed image if assuming the gradients are sparse under the Dirac-Fourier joint dictionary



(c) Reconstructed image if assuming the gradients are sparse



(d) Reconstructed image if assuming the gradients are sparse under the Dirac-Haar joint dictionary

Figure 4.4: Reconstruction under various dictionaries using 25% measurements

up the reconstruction, the image is broken into subimages each containing four columns of the original image. The subimages are compressed and reconstructed separately and then pieced together. Reconstruction result using 25% of the total measurements is shown in Figure 4.4(c). Secondly we assume the image is sparse under D_1. Since the sensing technique should be universal, we keep Y_1 and Y_2 the same as above, and only change the

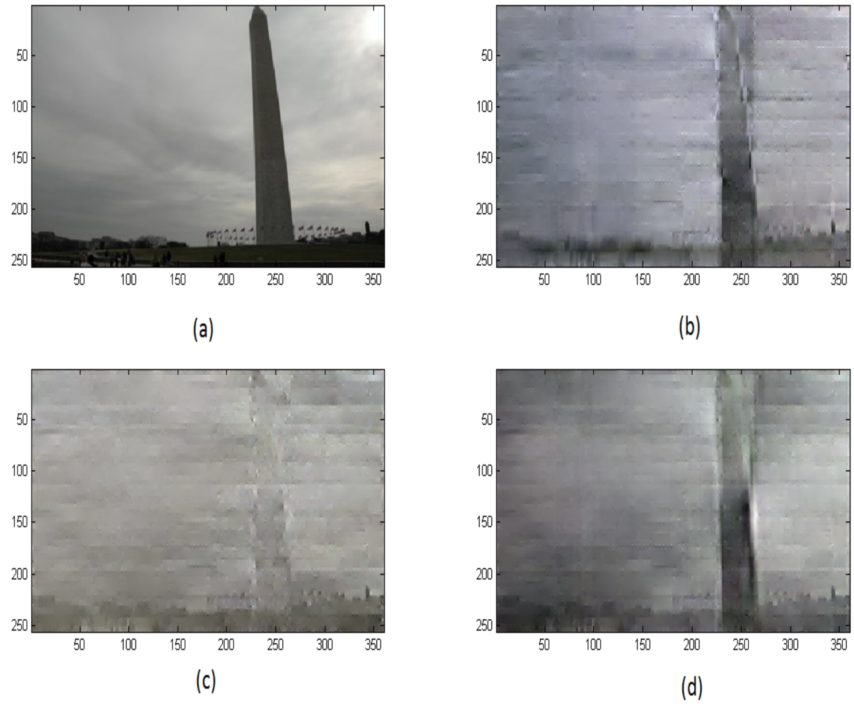


Figure 4.5: Compression rate 10:1, subfigures' order: (a) original image (b) Dirac-Fourier dictionary (c) orthonormal basis (d) Dirac-Haar dictionary

recovery algorithm to the ℓ_1 synthesis algorithm

$$\arg\min \|Z_i\|_{l_1} \text{ subject to } Y_i = AD_1 Z_i, \;\; i = 1, 2,$$

and $\widehat{\nabla_x X} = D_1 Z_1$, $\widehat{\nabla_y X} = D_1 Z_2$. Exactly the same procedure is used to reconstruct image based on the Dirac-Haar dictioinary D_2. Results on different images are shown in Figure 4.4-4.7. As expected, when an image does have sparse gradients, the joint dictionaries seems to work similar to orthonormal bases, otherwise the increased redundancy in dictionaries guarantees a more stable recovery in general.

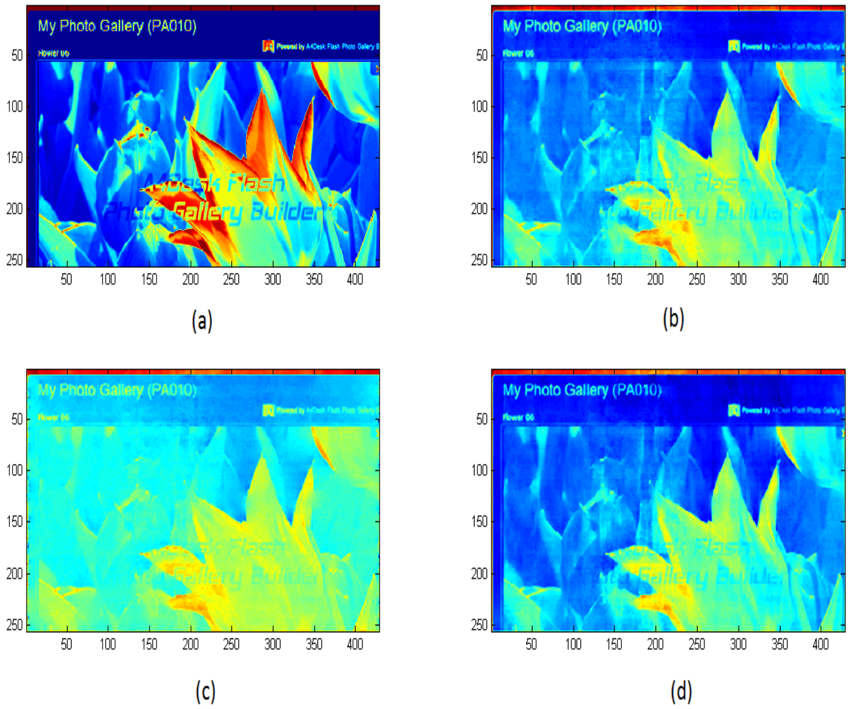


Figure 4.6: Compression rate 2:1, subfigures' order: (a) original image (b) Dirac-Fourier dictionary (c) orthonormal basis (d) Dirac-Haar dictionary

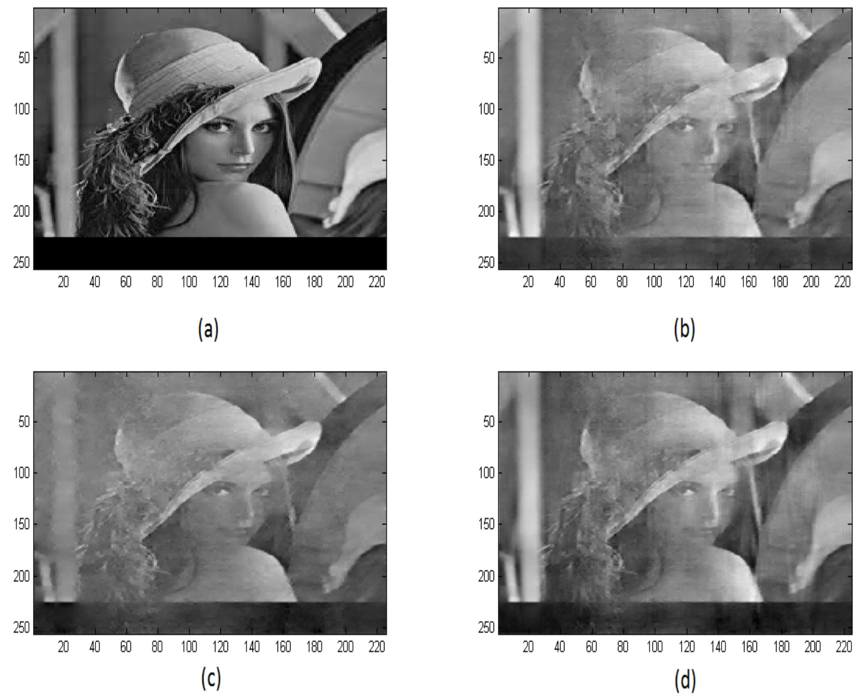


Figure 4.7: Compression rate 2:1, subfigures' order: (a) original image(b) Dirac-Fourier dictionary (c) orthonormal basis (d) Dirac-Haar dictionary

Bibliography

[1] N. Ailon and E. Liberty. Fast dimension reduction using Rademacher series on dual BCH codes. *Discrete Comput. Geom.*, 42(4):615–630, 2009.

[2] A. Aldroubi, X Chen, and A. M. Powell. Perturbations of measurement matrices and dictionaries in compressed sensing. *Appl. Comput. Harmon. Anal.*, 33(2):282–291, 2012.

[3] B. Alexeev, J Cahill, and D. G. Mixon. Full Spark Frames. *J. Fourier Anal. Appl.*, 18(6):1167–1194, 2012.

[4] N. Alon, O. Goldreich, J. Hastad, and R. Peralta. Simple constructions of almost k-wise independent random variables. *Random Structures and Algorithms*, 3:289–304, 1992.

[5] W. U. Bajwa, R. Calderbank, and S. Jafarpour. Why Gabor frames? Two fundamental measures of coherence and their role in model selection. *J. Commun. Networks*, 12:289–307, 2010.

[6] W. U. Bajwa, R. Calderbank, and D. G. Mixon. Two are better than one: fundamental parameters of frame coherence. *Appl. Comput. Harmon. Anal.*, 33(1):58–78, 2012.

[7] A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong. The road to deterministic matrices with the restricted isometry property. arXiv:1202.1234.

[8] A. Barg and A. Mazumdar. Small ensembles of sampling matrices constructed from coding theory. In *Proc. IEEE International Symposium on Information Theory, Austin, TX, June 2010*, pages 1963–1967.

[9] A. Barg, A. Mazumdar, and R. Wang. Random subdictionaries and coherence conditions for sparse signal recovery. In *http://arxiv.org/pdf/1303.1847.pdf*.

[10] A. Ben-Aroya and A. Ta-Shma. Constructing small-bias sets from algebraic-geometric codes. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, pages 191–197. IEEE Computer Soc., Los Alamitos, CA, 2009.

[11] J. Bourgain, S. J. Dilworth, K. Ford, S. Konyagin, and D. Kutzarova. Explicit constrcuctions of RIP matrices and related problems. arXiv:1008:4535.

[12] T. Cai and L. Wang. *IEEE Trans Infrom Theory*, (7).

[13] T. T. Cai, G. Xu, and J. Zhang. On recovery of sparse signals via ℓ_1 minimization. *IEEE Trans. Inform. Theory*, 55(1):3388–3397, 2009.

[14] R. Calderbank, S. Howard, and S. Jafarpour. Construction of a large class of deterministic sensing matrices that satisfy a statistical restricted isometry property. *IEEE J. Selected Topics Signal Proc.*, 4(2):358–374, 2010.

[15] R. Calderbank and S. Jafarpour. Reed-Muller sensing matrices and the LASSO. In C. Carlet and A. Pott, editors, *Sequences and Their Applications (SETA2010), Lect. Notes Comput. Science, vol. 6338*, pages 442–463, 2010.

[16] E. Candes, Y. C. Eldar, D. Needell, and P. Randall. Compressed sensing with coherent and redundant dictionaries. *Applied and Computational Harmonic Analysis*, 31(1):59–73, 2010.

[17] E. J. Candès. The restricted isometry property and its implications for compressed sensing. *C. R. Math. Acad. Sci. Paris*, 346(9-10):589–592, 2008.

[18] E. J. Candès and Y. Plan. Near-ideal model selection by ℓ_1 minimization. *Ann. Statist.*, 37(5A):2145–2177, 2009.

[19] E. J. Candes and Y. Plan. A probabilistic and RIPless theory of compressed sensing. *IEEE Trans. Inform. Theory*, 57(11):7235–7254, 2011.

[20] E. J. Candès and J. Romberg. Sparsity and incoherence in compressive sampling. *Inverse Problems*, 23:969–985, 2007.

[21] E. J. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inform. Theory*, 52(2):489–509, 2006.

[22] E. J. Candès and T. Tao. Decoding by linear programming. *IEEE Trans. Inform. Theory*, 51(12):4203–4215, 2005.

[23] E. J. Candès and T. Tao. Near-optimal signal recovery from random projections: universal encoding strategies? *IEEE Trans. Inform. Theory*, 52(12):5406–5425, 2006.

[24] S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursiut. *SIAM J. Sci. Comput.*, 20(1):33–61, 1998.

[25] X. Chen, H. Wang, and R. Wang. Compressed sensing in dictionaries with null space property. *in preperation*, 2013.

[26] W. Dai and O. Milenkovic. Weighted superimposed codes and constrained integer compressed sensing. *IEEE Trans. Inform. Theory*, 55(5):2215–2229, 2009.

[27] R. A. DeVore. Deterministic constructions of compressed sensing matrices. *J. Complexity*, 23(4-6):918–925, 2007.

[28] D. L. Donoho and M. Elad. Optimally sparse representations in general (nonorthogonal) dictionaries via ℓ^1 minimization. *Proc. Natl. Acad. Sci.*, 100:2197–2202, 2003.

[29] S. Foucart and M. Lai. Sparsest solutions of underdetermined linear systems via l_q-minimization for $0 < q \leq 1$. *Applied and Computational Harmonic Analysis*, 26(3):395–407, 2009.

[30] R. Gribonval and M. Nielsen. Highly sparse representations from dictionaries are unique and independent of the sparseness measure. *Applied and Computational Harmonic Analysis*, 22(3):335–355, May 2007.

[31] A. Gurevich and R. Hadani. Statistical RIP and semi-circle distribution of incoherent dictionaries, 2009. arXiv:0903.3627.

[32] J. Haupt, L. Applebaum, and R. Nowak. On the restricted isometry of deterministically subsampled Fourier matrices. In *Proc. 44th Annual Conf. Information Sciences and Systems (CISS)*, pages 1–6, 2010.

[33] A. S. Hedayat, N. J. A. Sloane, and J. Stufken. *Orthogonal arrays*. Springer-Verlag, New York, 1999.

[34] T. Helleseth and P. V. Kumar. Sequences with low correlation. In V. S. Pless and W. C. Huffman, editors, *Handbook of Coding Theory*, volume II, pages 1768–1853. Elsevier Science, 1998.

[35] Calderbank R. Howard, S and S. Searle. A fast reconstruction algorithm for deterministic compressive sensing using second order reed- muller codes. *Proc. Conf. Inf. Sci. Syst. (CISS)*, pages 11–15, 2008.

[36] M. A. Iwen. Simple deterministically constructible rip matrices with sublinear fourier sampling requirements,. In *Proc. CISS*, pages 870 – 875, 2008.

[37] S. Osher J. Cai and Z. Shen. Linearied gregman iterations for compressed sensing. 2008.

[38] M. Ledoux and M. Talagrand. *Probability in Banach spaces: Isoperimetry and Processes*. Springer, 1991.

[39] Shidong Li, Tiebin Mi, and Yulong Liu. Performance analysis of ℓ_1-synthesis with coherent frames. *http://arxiv.org/abs/1202.2223*, 2012.

[40] F. J. MacWilliams and N. J. A. Sloane. *The theory of error-correcting codes*. North-Holland, Amsterdam, 1991.

[41] A. Maleki and A. Montanari. Analysis of approximate message passing algorithm. 2010.

[42] A. Mazumdar. *Combinatorial methods in coding theory*. PhD thesis, University of Maryland, 2011. http://hdl.handle.net/1903/11547.

[43] C. McDiarmid. On the method of bounded differences. In *Surveys in combinatorics, 1989 (Norwich, 1989)*, volume 141 of *London Math. Soc. Lecture Note Ser.*, pages 148–188. Cambridge Univ. Press, Cambridge, 1989.

[44] D. Needell and J. A. Tropp. Cosamp: Iterative signal recovery from incomplete and inaccurate samples. *Applied and Computational Harmonic Analysis*, 26(3):301–321.

[45] E. Porat and A. Rothschild. Explicit non-adaptive combinatorial group testing schemes. In *Automata, languages and programming. Part I*, volume 5125 of *Lecture Notes in Comput. Sci.*, pages 748–759. Springer, Berlin, 2008.

[46] R. Chellappa R. T. Frandkot. A method for enforcing integrability in shape from shading algorithms. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 10(4):439–351, July 1988.

[47] H. Rauhut, K. Schnass, and P. Vandergheynst. Compressed sensing and redundant dictionaries. *IEEE Transactions on Information Theory*, 54(5):2210–2219, 2008.

[48] M. Rudelson and R. Vershynin. Sampling from large matrices: An approach through geometric functional analysis. *J. Assoc. Comput. Mach.*, 54(4):1–19, 2007.

[49] Q. Sun. Sparse approximation property and stable recovery of sparse signals from noisy measurements. *IEEE Trans. Signal Process.*, 59(10):5086–5090, 2011.

[50] R. Tibshirani. Regression shrinkage and selection via the lasso. *J. Roy. Statist. Soc. Ser. B*, 58:267–288, 1996.

[51] J. A. Tropp. Greed is good: Algorithmic results for sparse approximation. *IEEE Trans. Inform. Theory*, 50(10):2231–2242, 2004.

[52] J. A. Tropp. Recovery of short, complex linear combinations via l_1 minimization. *IEEE Trans. Inform. Theory*, 51(4):1568–1570, 2005.

[53] J. A. Tropp. Norms of random submatrices and sparse approximation. *C.R. Acad. Sci. Paris, Ser. I*, 346:1271–1274, 2008.

[54] J. A. Tropp. On the conditioning of random subdictionaries. *Appl. Comput. Harmon. Anal.*, 25(1):1–24, 2008.

[55] A. Weil. On some exponential sums. *Proc. Nat. Acad. Sci. U.S.A*, 34:204–207, 1948.