Technical Research Report

Non-Smooth Robust Stabilization of a Family of Linear Systems in the Plane

by B. Ho-Mock-Qai, W.P. Dayawansa

T.R. 96-10

INSTITUTE FOR SYSTEMS RESEARCH

*Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry*

Non-Smooth Robust Stabilization of a Family of Linear Systems in the Plane

BERTINA HO-MOCK-QAI* WIJESURIYA P. DAYAWANSA*

bertina@src.umd.edu
(301) 405-2942

wpd@src.umd.edu
(301) 405-6621

Abstract

In this paper, we use merely continuous feedback to robustly stabilize a class of parameterized family of linear systems in the plane. We introduce a new interpolation method that enables us to construct a robust stabilizer for the entire family of systems, by using two feedback laws that robustly stabilize two particular sub-families.

Keywords: Continuous feedback, robust stabilization, partition of unity.

1 Introduction

Robust control of linear systems has been a long standing and challenging problem in control theory [3, 4]. In the last twenty five years, research on that matter has evolved along two principal axis: The $\mathcal{H}_\infty$ theory has reached maturity in the design of robust linear controllers [6, 16, 17], while the so-called parametric approach (which originated from Kharitonov theorem [11]) provides criteria to decide whether or not a family of linear plants is robustly stable [1]. [Here by robust stability of a family of systems, we mean that each system of the family is locally asymptotically stable.] Other methods of robust stabilization of linear systems typically rely on obtaining linear controllers [12, 15]. There exist other approaches, e.g., nonlinear $\mathcal{H}_\infty$ theory [14] and Lyapunov methods [5], that use nonlinear feedback laws. However, they do not provide stabilization of each one of the systems of the family.

Because the methods mentioned above are of little help in robustly stabilizing families of linear systems that are *not* robustly stabilized by linear feedback, it is necessary

* Electrical Engineering Department and Institute for Systems Research, University of Maryland, College Park, MD 20742. The work of these authors was supported through NSF Grants: ECS 92-12126 and EEC 94-02384.

to investigate the use of other type of feedback.

Inspired by the surprisingly general results obtained in the context of nonlinear stabilization by continuous feedback [2, 9, 13], we discuss here the use of merely continuous feedback for robustly stabilizing (around $\gamma = 0$) the parameterized family of systems in the plane

$$S(\gamma): \begin{cases} \dot{x}_1 = a(\gamma)x_1 + b(\gamma)x_2 \\ \dot{x}_2 = u \end{cases},$$

where γ is a real parameter, $a(\cdot)$ and $b(\cdot)$ are real-valued functions, u is a scalar control and $b(0) = 0$. Specifically, in case the family $\{S(\gamma)\}$ cannot be robustly stabilized by linear feedback, either we find a continuous robust stabilizer, or we prove that such a stabilizer does not exist. In particular, under some mild assumptions on $a(\cdot)$ and $b(\cdot)$, if the family $\{S(\gamma)\}$ contains the following pair of systems

$$\begin{cases} \dot{x}_1 = \alpha_1 x_1 + \beta_1 x_2 \\ \dot{x}_2 = u \end{cases} \qquad \begin{cases} \dot{x}_1 = \alpha_1 x_1 - \beta_2 x_2 \\ \dot{x}_2 = u \end{cases},$$

for some positive reals α_1, α_2, β_1, and β_2, then, by introducing a new interpolation method between two feedback laws that robustly stabilize two specific sub-families, we find a continuous feedback law that robustly stabilizes the *entire* family of systems.

The paper is organized as follows. We introduce some definitions and state the problem under consideration in Section 2. The robust control problem is solved in the subsequent sections. The situation where the sign of $b(\cdot)$ is constant around $\gamma = 0$ is discussed in Section 3, while that where the sign of $b(\cdot)$ changes is considered in Section 4. The remaining cases are reviewed in Section 5. After some concluding remarks in Section 6, we present in Section 7 the technical lemmas used in the proofs of the main theorems.

2 Statement of the problem

We consider a parameterized family of systems in the plane

$$S(\gamma): \begin{cases} \dot{x}_1 = a(\gamma)x_1 + b(\gamma)x_2 \\ \dot{x}_2 = u \end{cases},$$

where γ is a real parameter, $a(\cdot)$ and $b(\cdot)$ are real-valued functions and u is a scalar control. We assume that the functions $a(\cdot)$ and $b(\cdot)$ are well-defined and smooth on some interval $[-\zeta_0, \zeta_1]$ where ζ_0 and ζ_1 are positive reals. We let Γ denote the set $[-\zeta_0, 0) \cup (0, \zeta_1]$ and we assume that $b(0) = 0$, $a(\gamma) \neq 0$ and $b(\gamma) \neq 0$ for all γ in Γ.

We investigate here the existence of a static feedback law u that robustly stabilizes the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$ in the following sense. For a given system, let $x(\cdot, x_0)$ denote its trajectory that starts from x_0 at time $t = 0$, and let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^n$.

Definition 2.1 *The static feedback law u robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$ if it locally asymptotically stabilizes the system $S(\gamma)$ at the origin for each γ in Γ, and if it is independent of the parameter γ.*

Definition 2.2 *The system $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0) = 0$, is locally asymptotically stable at the origin if the following holds:*

i) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t \geq 0$ and each solution $x(\cdot, x_0)$ of the system starting from x_0, we have $\|x(t, x_0)\| < \varepsilon$ whenever $\|x_0\| < \delta$.*

ii) *There exists $\delta_0 > 0$ such that $x(t, x_0) \to 0$ as $t \to \infty$ whenever $\|x_0\| < \delta_0$.*

We require that u be at least continuous on a neighborhood of the origin and that $u(0) = 0$. In that case continuity of u implies that of the vector-field $[a(\gamma)x_1 + b(\gamma)x_2, u(x)]^t$, which ensures the existence of a solution to the resulting closed-loop system [8, p. 10].

We complete this section with a few words about the notation and terminology used in this paper. A mapping is said to be *almost smooth* if it is smooth on a neighborhood of the origin except at the origin.

For x in $\mathbb{R}^2$, we denote by x_1 and x_2 its coordinates, and we define the mapping $f_\gamma : \mathbb{R}^2 \to \mathbb{R}$ by $f_\gamma(x) = a(\gamma)x_1 + b(\gamma)x_2$. For any subset I of $\mathbb{R}$, we denote respectively by I^- and I^+, the sets $I^- \triangleq \{\rho \in I :\ \rho < 0\}$ and $I^+ \triangleq \{\rho \in I :\ \rho > 0\}$. For a subset Y of $\mathbb{R}^2$, we let $\widehat{Y}$ denote its symmetric with respect to the origin and Y^s its symmetric with respect to the x_1-axis, i.e,

$$\widehat{Y} \triangleq \{-y :\ y \in Y\} \qquad \text{and} \qquad Y^s \triangleq \{(y_1, -y_2) \in \mathbb{R}^2 :\ (y_1, y_2) \in Y\}$$

Finally, for each positive reals α and β, we define

$$\begin{aligned}
\Omega_\alpha &\triangleq \{x \in \mathbb{R}^2 :\ x_1 = (x_2)^{1+\alpha},\ x_2 > 0\} \\
\Delta_\alpha &\triangleq \{x \in \mathbb{R}^2 :\ x_1 = \frac{(x_2)^{1+\alpha}}{2},\ x_2 > 0\} \\
\Psi_\beta &\triangleq \{x \in \mathbb{R}^2 :\ x_2 = x_1 \ln(\frac{x_1}{\beta}),\ x_1 > \beta\}
\end{aligned}$$

In order to discuss the robust stabilization of the family $\{S(\gamma),\ \gamma \in \Gamma\}$, we distinguish several cases based on the sign of $a(\cdot)$ and $b(\cdot)$. Recall that $a(\cdot)$ and $b(\cdot)$ take nonzero values on Γ so that, by continuity, both have a constant sign on Γ^- and Γ^+.

In each section, without further reference, we omit the cases $a(\cdot) < 0$ on Γ, as in that case any feedback law $u(x) = -kx_2$, where k is a positive real, robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

3 Robust stabilization when the sign of $b(\cdot)$ is constant on Γ

In this section, we assume that $b(\cdot)$ is either negative on Γ or positive on Γ. Moreover, we assume that $a(\cdot)$ is either positive on the entire set Γ, or negative on Γ^- and positive on Γ^+. We restrict our discussion to these two cases as the remaining case $a(\cdot)$ positive on Γ^- and negative on Γ^+ is obtained from the latter by replacing Γ^- by Γ^+ and vice versa.

Under these assumptions, because the mappings $a(\cdot)$ and $b(\cdot)$ are smooth on $\Gamma \cup \{0\}$ and do not vanish on Γ, it is easily checked by using elementary linear algebra, that the family $\{S(\gamma),\, \gamma \in \Gamma\}$ is robustly stabilizable by smooth (linear) feedback if and only if $\frac{b(\gamma)}{a(\gamma)}$ does not converge to 0 as γ goes to 0. If $\frac{b(\gamma)}{a(\gamma)}$ converges to 0 as γ goes to 0, a linear feedback law with an "infinite gain" would be necessary in order to robustly stabilize the family $\{S(\gamma),\, \gamma \in \Gamma\}$. As we shall see below, it turns out that the family $\{S(\gamma),\, \gamma \in \Gamma\}$ is robustly stabilizable by continuous, almost smooth feedback. Our approach is based on a technique introduced in [2] to establish stabilizability of a class of nonlinear systems in the plane.

In the following theorem we establish robust stabilizability of the family $\{S(\gamma),\, \gamma \in \Gamma\}$: We first construct a feedback law u_{k_0} based on some C^∞ partition of unity. Next, we introduce a base at the origin $\{W_\beta\}_{\beta>0}$ which is independent of the parameter γ. We show that for each parameter value γ in the set Γ, there exists a positive real β_γ such that for each β in $(0, \beta_\gamma]$ the set $\overline{W}_\beta$ is invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$. This enables us to conclude stability of the corresponding closed-loop system. Furthermore, by proving that the only positive limit set in $\overline{W}_\beta$ is the origin, we deduce that u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$.

Theorem 3.1 *Assume that either $a(\gamma)$ is positive on the entire set Γ, or $a(\gamma)$ is negative on Γ^- and positive on Γ^+. Furthermore, assume that $b(\gamma)$ is either negative on Γ, or positive on Γ. Suppose that $\frac{b(\gamma)}{a(\gamma)}$ converges to 0 as γ tends to 0. Then, there exists a continuous and almost smooth feedback law that robustly stabilizes the family of systems $\{S(\gamma),\, \gamma \in \Gamma\}$.*

Proof: We distinguish three cases.

a) $a(\cdot) > 0$ **on** Γ^+**,** $a(\cdot) < 0$ **on** Γ^- **and** $b(\cdot) < 0$ **on** Γ **:**

Recall that $a(\cdot)$ and $b(\cdot)$ are smooth on Γ and that $a(\cdot)$ does not vanish on Γ. Thus, because $\frac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$, there exists $\theta > 0$ such that $|\frac{b(\gamma)}{a(\gamma)}| < \theta$ for all γ in Γ. Therefore, for each γ in Γ^+, the half-line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)} x_2,\, x_2 > 0\}$ (resp. $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)} x_2,\, x_2 < 0\}$) is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 > 0\}$ (resp. below the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 < 0\}$).

Let α be a constant in $(0,1)$, and consider Fig. 1: For each $\beta > 0$, let W_β denote the neighborhood of the origin bounded by the closed curve in bold. Because the curves Ψ_β and Ω_α intersect for each $\beta > 0$ (Lemma 7.1), the neighborhood W_β is well-defined for each $\beta > 0$.

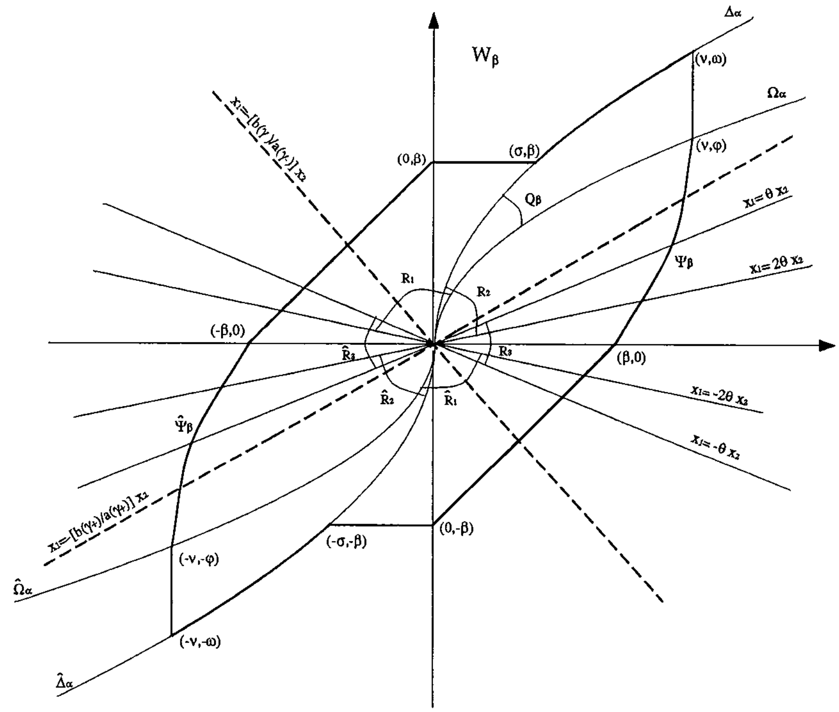

Figure 1: W_β

Besides, as Ω_α is tangent to the x_2-axis at the origin, Ω_α is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 > 0\}$ for x_2 small enough. Moreover, by Lemma 7.1, the unique point $[h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$ at which the sets $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 > 0\}$ and Ψ_β intersect is such that $h(\beta) \to 0$ *as* $\beta \to 0$. Thus, there exists $\bar\beta > 0$ such that for each β in $(0, \bar\beta]$, the point $[h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$ is below Ω_α. Furthermore, as $h(\beta) \to 0$ *as* $\beta \to 0$, it is easily seen from the definition of W_β, that $\{W_\beta\}_{\beta\in(0,\bar\beta]}$ is a base at the origin.

In view of the comments made above, we can now define the following open subsets of $W_{\bar\beta}\backslash\{0\}$:

$$
\begin{aligned}
R_1 &\triangleq \text{ region in } W_{\bar\beta} \text{ between the curves } \{x \in \mathbb{R}^2 :\ x_1 = -2\theta x_2,\ x_2 > 0\} \text{ and } \Omega_\alpha\,,\\
R_2 &\triangleq \text{ region in } W_{\bar\beta} \text{ between the curves } \Delta_\alpha \text{ and } \{x \in \mathbb{R}^2 :\ x_1 = 2\theta x_2,\ x_2 > 0\}\,,\\
R_3 &\triangleq \text{ region in } W_{\bar\beta} \text{ between the half-lines } \{x \in \mathbb{R}^2 :\ x_1 = \theta x_2,\ x_2 > 0\}\\
&\qquad \text{and } \{x \in \mathbb{R}^2 :\ x_1 = -\theta x_2,\ x_2 < 0\}\,,\\
Q_\beta &\triangleq \text{ region delimited by } \Omega_\alpha,\ \Delta_\alpha \text{ and the segment } [(\nu,\varphi),(\nu,\omega)].
\end{aligned}
$$

We note that $\{R_1, R_2, R_3, \widehat{R}_1, \widehat{R}_2, \widehat{R}_3\}$ [where $\widehat{R}_i$ is the symmetric of R_i with respect to the origin for each $i = 1, 2, 3$] is an open cover of $W_{\bar\beta}\backslash\{0\}$, so that there exists a C^∞ partition of unity $\{p_1, p_2, p_3, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3\}$ subordinate to it [7, p. 52]. Without loss of generality, for each $i = 1, 2, 3$, we take p_i (resp. $\widehat{p}_i$) such that its support is included in R_i (resp. $\widehat{R}_i$).

For each $k > 0$, we now define the feedback law $u_k : W_{\bar{\beta}} \to \mathbb{R}$ by setting

$$u_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ k\,\big[-(x_2)^{1-\alpha}p_1(x) + (x_1+x_2)p_2(x) + x_1p_3(x) & \\ \quad +(-x_2)^{1-\alpha}\widehat{p}_1(x) + (x_1+x_2)\widehat{p}_2(x) + x_1\widehat{p}_3(x)\,\big] & \text{otherwise}. \end{cases}$$

We note that the regions R_1 and $\widehat{R}_1$ do not contain any point of the form $(x_1, 0)$, and that the support of p_1 and $\widehat{p}_1$ are included in R_1 and $\widehat{R}_1$, respectively. Therefore, it follows from the smoothness of the mapping p_i and $\widehat{p}_i$ on $W_{\bar{\beta}}\backslash\{0\}$ for each $i = 1, 2, 3$, that u_k is smooth on $W_{\bar{\beta}}\backslash\{0\}$, for each $k > 0$. Furthermore, the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k that

$$|u_k(x)| \;\leq\; k \max\left(|x_2|^{1-\alpha}, |x_1+x_2|, |x_1|\right), \quad x \in W_{\bar{\beta}},$$

and continuity of u_k at the origin follows for each $k > 0$.

The following claim is the key argument to establish robust stability.

Claim 1: *There exists $k_0 > 0$, and for each γ in Γ there exists β_γ in $(0, \bar{\beta}]$ such that the sets $\overline{W}_\beta$ and $\overline{Q}_\beta$ are invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$ for each β in $(0, \beta_\gamma]$.*

We note that the invariance of $\overline{W}_\beta$ will be proved if for each x in the boundary ∂W_β, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside the set W_β.

By applying Lemmas 7.3 and 7.4 (with $I = \Gamma$, $\mu = 1$, and θ, $\bar{\beta}$, α as given here), we obtain two positive reals k_1 and k_2. We set $k_0 \triangleq \max(k_1, k_2)$, so that the assertions of both lemmas hold with $k = k_0$.

For each γ in Γ^-, we set $\beta_\gamma \triangleq \bar{\beta}$ and for each γ in Γ^+ we define β_γ through Lemma 7.1: Indeed, because the curves Ω_α and $\widehat{\Omega}_\alpha$ are tangent to the x_2-axis at the origin, Lemma 7.1 yields the existence of β_γ in $(0, \bar{\beta}]$ such that for each β in $(0, \beta_\gamma]$ the segments $[(\nu, \varphi), (\nu, \omega)]$ and $[(-\nu, -\varphi), (-\nu, -\omega)]$ of ∂W_β are respectively above $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)}x_2, x_2 > 0\}$, and below $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)}x_2,\ x_2 < 0\}$.

Next, we fix γ in Γ and β in $(0, \beta_\gamma]$. From the definition of β_γ, it is easily checked that for each x in the segment $[(\nu, \varphi), (\nu, \omega)]$ (resp. $[(-\nu, -\varphi), (-\nu, -\omega)]$) of ∂W_β, we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$), so that $[f_\gamma(x), u_{k_0}(x)]^t$ points into W_β.

Further, recall that for each $i = 1, 2, 3$, the support of the mappings p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) and note that the intersection of more than two sets of the family $\{R_1, R_2, R_3, \widehat{R}_1, \widehat{R}_2, \widehat{R}_3\}$ is empty. Thus, for each x in ∂W_β, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemmas 7.3 and 7.4, and therefore points inside W_β, or is a convex combination of two of them. In the latter case, the result follows either from the fact that we have a convex combination or from the fact that we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) on the segments $[(\nu, \varphi), (\nu, \omega)]$ (resp. $[(-\nu, -\varphi), (-\nu, -\omega)]$) of ∂W_β.

Finally, for each x in $\Omega_\alpha \cap W_\beta$, because $u_{k_0}(x)$ is positive and $f_\gamma(x)$ is negative, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points into Q_β. This, combined with the assertion of Lemma 7.3 **(ii)** and the fact that $\beta \geq \beta_\gamma$, implies that $\overline{Q}_\beta$ is an invariant set with respect to the vector-field $[f_\gamma, u_{k_0}]^t$. The proof of Claim 1 is completed upon noting that the previous results hold for each γ in Γ and each β in $(0, \beta_\gamma]$.

Robust stability:

We now prove that the feedback law u_{k_0}, where k_0 is as given in Claim 1, robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$. Fix γ in Γ and let $\widetilde{S}(\gamma)$ denote the system obtained once u_{k_0} is fed back into $S(\gamma)$. Let β_γ be as defined in Claim 1, let β be in $(0, \beta_\gamma]$, and let x_0 be in $\overline{W}_\beta$. In view of the definition of β_γ, we have $u_{k_0}(x) \neq 0$ for all x in $\overline{W}_\beta \backslash \{0\}$ with $f_\gamma(x) = 0$, so that the origin is the unique equilibrium point of $\widetilde{S}(\gamma)$ in $\overline{W}_\beta$. Thus, by the invariance with respect to $\widetilde{S}(\gamma)$ of the compact set $\overline{W}_\beta$ (Claim 1) and the Poincaré-Bendixson Theorem [8], the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_\beta$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$.

If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [8, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of the set $\overline{Q}_\beta$ and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\widetilde{S}(\gamma)$ starting in $\overline{W}_\beta$ converges to the origin [8, Corollary 1.1 p. 146].

As $\{W_\beta\}_{0<\beta\leq\beta_\gamma}$ is a base at the origin, we easily obtain that the feedback law u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$ for each γ in Γ. In short, u_{k_0} robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

b) $a(\cdot) > 0$ **on** Γ^+**,** $a(\cdot) < 0$ **on** Γ^- **and** $b(\cdot) > 0$ **on** Γ **:**

We consider the family $\{\bar{S}(\gamma),\ \gamma \in \Gamma\}$ of systems

$$\bar{S}(\gamma): \quad \left\{ \begin{array}{lcl} \dot{x}_1 & = & a(\gamma)x_1 - b(\gamma)x_2 \\ \dot{x}_2 & = & u \end{array} \right. .$$

Because $-b(\gamma)$ is negative on Γ, by **(a)**, there exists a feedback laws u_{k_0} which robustly stabilize $\{\bar{S}(\gamma),\ \gamma \in \Gamma\}$. In other words, the system

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & a(\gamma)x_1 - b(\gamma)x_2 \\ \dot{x}_2 & = & u_{k_0}(x_1, x_2) \end{array} \right. \tag{1}$$

is asymptotically stable for each γ in Γ. By the change of variable $(x_1, x_2) \mapsto (x_1, -x_2)$, the system (1) is transformed into the asymptotically stable system

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & a(\gamma)x_1 + b(\gamma)x_2 \\ \dot{x}_2 & = & -u_{k_0}(x_1, -x_2) \end{array} \right.$$

and we conclude that the feedback law v_{k_0} given by $v_{k_0}(x_1, x_2) = -u_{k_0}(x_1, -x_2)$, robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

c) $a(\cdot) > 0$ on Γ, and $b(\cdot)$ is either positive or negative on Γ :

In that case, the result follows easily from the arguments given in **(a)** and **(b)** by replacing Γ^+ by Γ and Γ^- by $\emptyset$. ■

It is easily seen from the proof of Theorem 3.1 that each one of the feedback law of the collection $\{u_k,\ k \in [k_0, \infty)\}$ robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

4 Robust stabilization when $a(\cdot)$ is positive and the sign of $b(\cdot)$ changes

In this section, we only consider the case $b(\cdot)$ negative on Γ^- and positive on Γ^+. The symmetric case $b(\cdot)$ positive on Γ^- and negative on Γ^+ is obtained from the former by replacing Γ^+ by Γ^- and vice versa. By using elementary linear algebra, it is easily checked that there exists no smooth feedback law that simultaneously stabilizes any two systems $S(\gamma_-)$ and $S(\gamma_+)$, with $\gamma_- < 0$ and $\gamma_+ > 0$, so that the family $\{S(\gamma),\ \gamma \in \Gamma\}$ is clearly not robustly stabilizable by smooth feedback. However, as we shall see below, this family is robustly stabilizable by means of *continuous* feedback.

This result is proved in the following theorem. The general line of the proof is to construct two mappings $u^+_{k_0}$ and $u^-_{k_0}$ that robustly stabilize the family $\{S(\gamma),\ \gamma \in \Gamma^+\}$ and $\{S(\gamma),\ \gamma \in \Gamma^-\}$ respectively, by using a first partition of unity similar to that introduced in the proof of Theorem 3.1. In order to obtain a feedback law that robustly stabilizes the entire family $\{S(\gamma),\ \gamma \in \Gamma\}$, we then "piece" together $u^+_{k_0}$ and $u^-_{k_0}$ by using a second partition of unity subordinate to a family of open sets that encircles the origin. Robust stability is shown through an argument similar to that used in the proof of Theorem 3.1.

Theorem 4.1 *Assume that $a(\cdot)$ is positive on Γ, and that $b(\cdot)$ is respectively negative on Γ^- and positive on Γ^+. Then, there exists a continuous and almost smooth feedback law that robustly stabilizes the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$.*

For the sake of clarity we divide the proof of the theorem into several cases.

Lemma 4.1 *Theorem 4.1 holds if* $\dfrac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$.

Proof:

Construction of u^-_k and u^+_k :

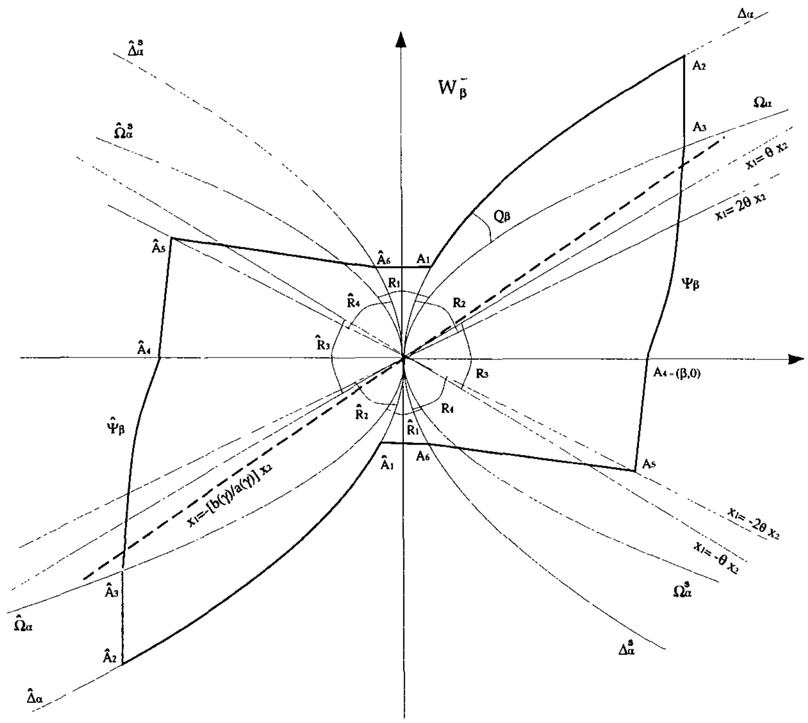

Figure 2: W_β^- in the case $\dfrac{b(\gamma)}{a(\gamma)} \to 0 \;\; as \;\; \gamma \to 0$

In that case, the assumptions on $a(\cdot)$ and $b(\cdot)$ yield the existence of $\theta > 0$ such that $|\dfrac{b(\gamma)}{a(\gamma)}| < \theta$ for all γ in Γ. Let α be a constant in $(0, 1)$ and consider Fig. 2 and Fig. 3: For each $\beta > 0$, we let W_β^- and W_β^+ be the open subsets of $\mathbb{R}^2$ bounded by the closed curves in bold, in Fig. 2 and Fig. 3 respectively. The neighborhood W_β^+ is obtained by rotating W_β^- around the x_1-axis by 180 degrees. In Fig. 2, the segments $[\widehat{A}_6, A_1]$ and $[A_2, A_3]$ are respectively horizontal and vertical, while the segments $[A_6, A_5]$ and $[A_4, A_5]$ have respective slopes $\dfrac{dx_1}{dx_2} = -\delta$ and $\dfrac{dx_1}{dx_2} = \mu$ where μ and δ are fixed positive reals such that $\delta > 2\theta$. Combining this last inequality with the fact that the curves Ψ_β and Ω_α intersect for each $\beta > 0$ (Lemma 7.1), we obtain that the neighborhoods W_β^- and W_β^+ are well-defined for each $\beta > 0$.

Besides, because the curve Ω_α is tangent to the x_2-axis at the origin, it is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \, x_2 > 0\}$ for x_2 small enough. Furthermore, Lemma 7.1 yields the existence of $\bar{\beta} > 0$ such that for each β in $(0, \bar{\beta}]$, the intersection of $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \, x_2 > 0\}$ with Ψ_β is below Ω_α. Finally, it is easily checked that $\bar{\beta}$ can be chosen such that for each β in $(0, \bar{\beta}]$, both A_5 and the intersection of $[A_6, A_5]$ with $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2, \, x_2 < 0\}$ are above Ω_α^s. We now define the set W by

$$W \triangleq W_{\bar{\beta}}^- \cup W_{\bar{\beta}}^+.$$

In view of the comments made above and the symmetry of the neighborhoods W_β^- and

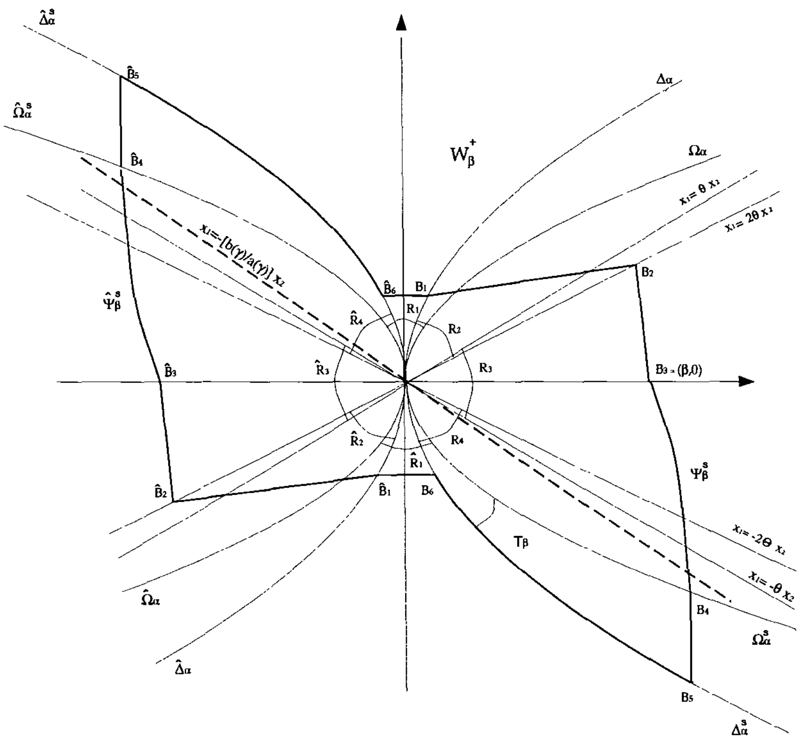

Figure 3: W_β^+ in the case $\dfrac{b(\gamma)}{a(\gamma)} \to 0 \ as \ \gamma \to 0$

W_β^+, we can define the following open subsets of $W \backslash \{0\}$:

$R_1 \triangleq$ region in W between the curves $\widehat{\Omega}_\alpha^s$ and Ω_α,
$R_2 \triangleq$ region in W between the curves Δ_α and $\{x : \ x_1 = 2\theta x_2, \ x_2 > 0\}$,
$R_3 \triangleq$ region in W between the half-lines $\{x : \ x_1 = \theta x_2, \ x_2 > 0\}$ and $\{x : \ x_1 = -\theta x_2, \ x_2 < 0\}$,
$R_4 \triangleq$ region in W between the half-line $\{x : \ x_1 = -2\theta x_2, \ x_2 < 0\}$ and the curve Δ_α^s,
$Q_\beta \triangleq$ region delimited by Δ_α, Ω_α and the segment $[A_2, A_3]$,
$T_\beta \triangleq$ region delimited by Ω_α^s, Δ_α^s and the segment $[B_4, B_5]$,

Because $\{R_1, .., R_4, \widehat{R}_1, .., \widehat{R}_4\}$ is an open cover of $W \backslash \{0\}$, there exists a partition of unity $\{p_1, .., p_4, \widehat{p}_1, .., \widehat{p}_4\}$ subordinate to it [7, p. 52]. Without loss of generality, we choose this partition of unity such that for each $i = 1, \ldots, 4$, the support of p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$).

For each $k > 0$, we now define the mappings $u_k^-, u_k^+ : W \to \mathbb{R}$, by setting

$$u_k^-(x) = \begin{cases} 0 & \text{if } x = 0 \\ -k(x_2)^{1-\alpha}p_1(x) + k(x_1+x_2)p_2(x) + kx_1\,p_3(x) - (x_1)^2\,p_4(x) & \\ +k(-x_2)^{1-\alpha}\widehat{p}_1(x) + k(x_1+x_2)\widehat{p}_2(x) + kx_1\,\widehat{p}_3(x) + (x_1)^2\,\widehat{p}_4(x) & \text{otherwise}, \end{cases}$$

and

$$u_k^+(x) = \begin{cases} 0 & \text{if } x = 0 \\ -k(x_2)^{1-\alpha}p_1(x) + (x_1)^2\,p_2(x) - kx_1\,p_3(x) + k(-x_1+x_2)p_4(x) & \\ +k(-x_2)^{1-\alpha}\widehat{p}_1(x) - (x_1)^2\,\widehat{p}_2(x) - kx_1\,\widehat{p}_3(x) + k(-x_1+x_2)\widehat{p}_4(x) & \text{otherwise}. \end{cases}$$

The argument given in the proof of Theorem 3.1 to show the smoothness of u_k transposes easily here, and for each $k > 0$, both mappings u_k^- and u_k^+ are smooth on $W\backslash\{0\}$ and continuous at the origin.

Using u_k^- and u_k^+, we now construct the desired stabilizing feedback law u_k.

Construction of u_k :
It is not hard to see from Lemma 7.2 that both families $\{W_\beta^-\}_{\beta\in(0,\bar\beta]}$ and $\{W_\beta^+\}_{\beta\in(0,\bar\beta]}$ are bases at the origin with

$$W_\beta^- \subset W_{\beta'}^- \quad \text{and} \quad W_\beta^+ \subset W_{\beta'}^+ \quad \text{whenever} \quad \beta < \beta'. \tag{2}$$

Thus, there exists a sequence of positive reals $\{\beta_j\}_{j=0}^\infty$ included in $(0,\bar\beta]$ such that

$$\overline{W}_{j+1} \subset W_j, \quad j = 0, 1, 2, \ldots. \tag{3}$$

and

$$\beta_j \to 0 \;\; as \;\; j \to \infty \tag{4}$$

where we have set

$$\begin{aligned} W_{2n} &\triangleq W_{\beta_{2n}}^+, \quad n = 0, 1, 2, \ldots, \\ W_{2n+1} &\triangleq W_{\beta_{2n+1}}^-, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Combining the inclusions (3) with the fact that $\{W_j\}_{j=0}^\infty$ is a base at the origin [which follows from (4)], it is not hard to check that $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j=1}^\infty$ is an open cover of $W_0\backslash\{0\}$. Let $\{q_j\}_{j=1}^\infty$ be a partition of unity subordinate to the cover $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j=1}^\infty$ such that the support of q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$, for each $j = 1, 2, \ldots$ [8].

For each $k > 0$, we now define the feedback law $u_k : W_0 \to \mathbb{R}$ by setting

$$u_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ u_k^+(x)\displaystyle\sum_{n=1}^\infty q_{2n}(x) + u_k^-(x)\sum_{n=0}^\infty q_{2n+1}(x) & \text{otherwise}. \end{cases}$$

Next, we fix $k > 0$ and we show that u_k is smooth on $W_0\backslash\{0\}$ and continuous at the origin. Let x be in $W_0\backslash\{0\}$. It is easily checked that there exists a neighborhood U_x of

x such that U_x intersects with at most three sets of the collection $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j=1}^{\infty}$. Because the support of each mapping q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$, the infinite sums in the expressions of u_k reduce to the sum of at most three fixed terms on U_x. Therefore, the smoothness of u_k on $W_0\backslash\{0\}$ follows from that of the mappings u_k^-, u_k^+ and q_j, $j = 1, 2, \ldots$.

Furthermore the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k that

$$|u_k(x)| \;\leq\; \max(\,|u_k^+(x)|, |u_k^-(x)|\,), \quad x \in W_0\backslash\{0\},$$

and for each $k > 0$, continuity of u_k at the origin follows from that of u_k^- and u_k^+.

The key argument for proving robust stabilizability lies in the following claim.

Invariance of the sets W_j :

Claim 1: *There exists $k_0 > 0$, and for each γ in Γ^- (resp. in Γ^+) there exists an integer n_γ such that the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ (resp. $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$) are invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$ for each $n = n_\gamma, n_\gamma + 1, \ldots$.*

The invariance of $\overline{W}_j$ will be proved if for each x in the boundary ∂W_j, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside the set W_j.

By definition of the partition of unity $\{q_j\}_{j=1}^{\infty}$, we have

$$q_m(x) \;=\; 1 \quad \text{and} \quad q_j(x) = 0, \quad j \neq m, \tag{5}$$

for each x in some set $W_{m-1}\backslash\overline{W}_{m+1}$ which does not belong to any other set of the family $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j=1}^{\infty}$. Therefore, because for each $m = 1, 2, \ldots$, the boundary ∂W_m of the neighborhood W_m is included in $W_{m-1}\backslash\overline{W}_{m+1}$ and does not intersect with any other set $W_{j-1}\backslash\overline{W}_{j+1}, j \neq m$, the definition of u_k yields

$$u_k(x) \;=\; u_k^+(x), \quad x \in \partial W_{2n}, \quad n = 1, 2, \ldots, \tag{6}$$

and

$$u_k(x) \;=\; u_k^-(x), \quad x \in \partial W_{2n+1}, \quad n = 0, 1, \ldots\ . \tag{7}$$

Recall that for each $j = 1, 2, \ldots$ we have $\overline{W}_j \subset W_0$. Because $u_k^-(x)$ and $u_k^+(x)$ are both equal to $-k(x_2)^{1-\alpha}$ if x is in $\Delta_\alpha \cap W_0$ and to $k(-x_2)^{1-\alpha}$ if x is in $\Delta_\alpha^s \cap W_0$, we get

$$u_k(x) = -k(x_2)^{1-\alpha}, \quad x \in \Delta_\alpha \cap \overline{W}_j, \quad j = 1, 2, \ldots \tag{8}$$

and

$$u_k(x) = k(-x_2)^{1-\alpha}, \quad x \in \Delta_\alpha^s \cap \overline{W}_j, \quad j = 1, 2, \ldots\ . \tag{9}$$

Furthermore, by definition of u_k we have

$$u_k(x) > 0, \quad x \in \Omega_\alpha \cap \overline{W}_j, \quad j = 1, 2, \ldots \tag{10}$$

and

$$u_k(x) < 0, \quad x \in \Omega_\alpha^s \cap \overline{W}_j, \quad j = 1, 2, \dots . \tag{11}$$

From Lemmas 7.6 (with $I = \Gamma^-$, and θ, δ as given here) and 7.7 (with $I = \Gamma^+$, and θ, δ as given here), we obtain for each γ in Γ^- (resp. in Γ^+), a neighborhood U_γ such that the assertions of Lemma 7.6 (resp. Lemma 7.7) hold. Fix γ in Γ^- (resp. in Γ^+). Because the curves Ω_α and $\widehat{\Omega}_\alpha$ (resp. Ω_α^s and $\widehat{\Omega}_\alpha^s$) are tangent to the x_2-axis at the origin, Lemma 7.1 yields the existence of β_γ in $(0, \bar{\beta}]$ such that for each β in $(0, \beta_\gamma]$, the vertical segments of the boundary ∂W_β^- (resp. ∂W_β^+) are as follows: $[A_2, A_3]$ (resp. $[\widehat{B}_4, \widehat{B}_5]$) is above the half-line $\{x \in \mathbb{R}^2 : x_1 = -\dfrac{b(\gamma)}{a(\gamma)}\, x_2,\ x_2 > 0\}$ and the segment $[\widehat{A}_2, \widehat{A}_3]$ (resp. $[B_4, B_5]$) is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -\dfrac{b(\gamma)}{a(\gamma)}\, x_2,\ x_2 < 0\}$, as shown in Fig. 2 (resp. Fig. 3).

Because $\beta_j \to 0$ *as* $j \to \infty$, the family $\{W_j\}_{j=1}^\infty$ is a base at the origin, and for each γ in Γ, there exists an integer n_γ such that

$$\beta_j \ \le\ \beta_\gamma, \quad j = 2n_\gamma,\, 2n_\gamma + 1, \dots \tag{12}$$

and

$$W_j \ \subset\ U_\gamma, \quad j = 2n_\gamma,\, 2n_\gamma + 1, \dots . \tag{13}$$

It follows from (12) and the inclusions (2) that for each $n = n_\gamma, n_\gamma + 1, \dots$, the neighborhood $W_{\beta_{2n+1}}^-$ (resp. $W_{\beta_{2n}}^+$) is included in $W_{\beta_\gamma}^-$ (resp. $W_{\beta_\gamma}^+$). In other words, for each $n = n_\gamma, n_\gamma + 1, \dots$, the neighborhood W_{2n+1} (resp. W_{2n}) is included in $W_{\beta_\gamma}^-$ (resp. $W_{\beta_\gamma}^+$). Thus, the definition of β_γ implies that for each x on the vertical segments of ∂W_{2n+1} (resp. ∂W_{2n}), we have $f_\gamma(x) < 0$ for x in $[A_2, A_3]$ (resp. $[B_4, B_5]$) while $f_\gamma(x) > 0$ for x in $[\widehat{A}_2, \widehat{A}_3]$ (resp. $[\widehat{B}_4, \widehat{B}_5]$).

We now apply Lemmas 7.3 and 7.4 (with $I = \Gamma^-$, and θ, μ, $\bar{\beta}$ and α as given here) and Lemma 7.5 (with $I = \Gamma^+$, and θ, μ, α as given here): we obtain positive reals k_1, k_2 and k_3. We set $k_0 \triangleq \max(k_1, k_2, k_3)$, so that the assertions of those three lemmas hold with $k = k_0$. Fix γ in Γ^- and n in $\{n_\gamma, n_\gamma + 1, \dots\}$. Recall that for each $i = 1, 2, 3, 4$, the support of the mappings p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$). Further, note that the intersection of more than two sets of the family $\{R_1, \dots, R_4, \widehat{R}_1, \dots, \widehat{R}_4\}$ is empty. Thus, for each x in ∂W_{2n+1}, the vector $[f_\gamma(x), u_{k_0}^-(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemmas 7.3, 7.4 and 7.6, and therefore points inside W_{2n+1}, or is a convex combination of two of them. In the latter case, $[f_\gamma(x), u_{k_0}^-(x)]^t$ points inside W_{2n+1} either because we have a convex combination, or because we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) on the vertical segments $[A_2, A_3]$ (resp. $[\widehat{A}_2, \widehat{A}_3]$) of ∂W_{2n+1}.

By (7), $u_{k_0} = u_{k_0}^-$ on ∂W_{2n+1} and it follows that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside W_{2n+1}, for each x in ∂W_{2n+1}.

Because we have $u_{k_0}(x) > 0$ [by (10)] and $f_\gamma(x) < 0$ for each x in $\Omega_\alpha \cap W_{2n+1}$, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$. Further, (8) combined with the assertions of Lemma 7.3 **(ii)** imply that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$ for each x in $\Delta_\alpha \cap W_{2n+1}$. Thus, as $\beta_{2n+1} \leq \beta_\gamma$, it follows from the definition of β_γ that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$ for each x in $\partial Q_{\beta_{2n+1}}$.

Therefore, for each n in $\{n_\gamma, n_\gamma + 1, \ldots\}$, the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ are invariant with respect to the vector-field $[f_\gamma, u_{k_0}]^t$.

Similarly, (6), (9), (11), (12) and (13), together with the assertions of Lemmas 7.5 and 7.7 yield the invariance with respect to the vector-field $[f_\gamma, u_{k_0}]^t$ of the sets $\overline{W}_{2n}$ and $\overline{Q}_{\beta_{2n}}$, for each γ in Γ^+ and for each n in $\{n_\gamma, n_\gamma + 1, \ldots\}$. The proof of Claim 1 is now complete.

Robust stability :

Let k_0 be as defined in Claim 1. Fix γ in Γ^-. Let n_γ be as given in Claim 1 and let $n = n_\gamma, n_\gamma + 1, \ldots$. In view of (12) and the definition of β_γ, we have $u_{k_0}(x) \neq 0$ for all x in $\overline{W}_{2n+1} \backslash \{0\}$ with $f_\gamma(x) = 0$, so that the origin is the unique equilibrium point in $\overline{W}_{2n+1}$ of the system $\widetilde{S}(\gamma)$ obtained once u_{k_0} is fed back into $S(\gamma)$. Thus, by the invariance with respect to $\widetilde{S}(\gamma)$ of the compact set $\overline{W}_{2n+1}$ (Claim 1) and the Poincaré-Bendixson Theorem [8], the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_{2n+1}$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$.

If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [8, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of the set $\overline{Q}_{\beta_{2n+1}}$ and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\widetilde{S}(\gamma)$ starting in $\overline{W}_{2n+1}$ remains in $\overline{W}_{2n+1}$ and converges to the origin [8, Corollary 1.1 p. 146].

As $\{W_{2n+1}\}_{n=n_\gamma}^{\infty}$ is a base at the origin, we obtain that the feedback law u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$ for each γ in Γ^-.

Similarly, by using the invariance of the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$, we get that u_{k_0} locally asymptotically stabilizes $S(\gamma)$ for each γ in Γ^+. Hence the lemma. ■

Using this lemma, we now prove Theorem 4.1.

Proof of Theorem 4.1:

If $\dfrac{b(\gamma)}{a(\gamma)} \to 0 \;\; as \;\; \gamma \to 0$, the claim of the theorem follows from Lemma 4.1.

Thus, we now assume that $\dfrac{b(\gamma)}{a(\gamma)}$ does not converge to 0 as γ tends to 0. In this case, because $a(\cdot)$ and $b(\cdot)$ are smooth and do not vanish on Γ, we have $|\dfrac{b(\gamma)}{a(\gamma)}| \to +\infty \;\; as \;\; \gamma \to 0$. Therefore, there exists $\theta > 0$ such that $|\dfrac{b(\gamma)}{a(\gamma)}| > \theta, \;\; \gamma \in \Gamma$. The result is now obtained through the same arguments as those in the proof of Lemma 4.1 with θ as defined above. ■

We note that each one of the feedback law of the collection $\{u_k, \, k \in [k_0, \infty)\}$ robustly stabilizes the family $\{S(\gamma), \, \gamma \in \Gamma\}$. To complete our study, we now investigate the case when the sign of both $a(\cdot)$ and $b(\cdot)$ change as γ takes the value 0.

5 Robust stabilization when the sign of $a(\cdot)$ and $b(\cdot)$ changes

The next theorem yields a necessary and sufficient condition for robust stabilizability of the family $\{S(\gamma),\ \gamma \in \Gamma\}$, in case $a(\cdot)$ is negative on Γ^- and positive on Γ^+. The companion case $a(\cdot)$ positive on Γ^- and negative on Γ^+ is easily deduced by replacing Γ^- by Γ^+ and vice versa.

Theorem 5.1 *Assume that $a(\cdot)$ is negative on Γ^- and positive on Γ^+, and that $b(\cdot)$ is negative (resp. positive) on Γ^- and positive (resp. negative) on Γ^+. Then the family $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly stabilizable using continuous (smooth) feedback if and only if*

$$\frac{b(\gamma_-)}{a(\gamma_-)} < \frac{b(\gamma_+)}{a(\gamma_+)} \qquad \textit{(resp.} \quad \frac{b(\gamma_+)}{a(\gamma_+)} < \frac{b(\gamma_-)}{a(\gamma_-)}\textit{)}\,, \tag{14}$$

for all γ_- in Γ^- and all γ_+ in Γ^+.

Proof: We only consider the case $b(\cdot)$ negative on Γ^- and positive on Γ^+, as the arguments presented below carry over to the case $b(\cdot)$ positive on Γ^- and negative on Γ^+.

We first show that under (14), the family $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly stabilizable by smooth feedback. Under the assumptions made on $a(\cdot)$ and $b(\cdot)$, $\frac{b(\gamma)}{a(\gamma)}$ converges either to 0, to $+\infty$, or to a positive real τ. Because $\frac{b(\cdot)}{a(\cdot)}$ is positive on Γ, it is easily checked that $\frac{b(\gamma)}{a(\gamma)}$ converges to some positive real τ whenever (14) holds.

Note that under (14), there does not exist any γ_- in Γ^- and γ_+ in Γ^+ such that $\frac{b(\gamma_-)}{a(\gamma_-)} = \frac{b(\gamma_+)}{a(\gamma_+)} = \tau$. Let $k < -\sup\{a(\gamma) : \gamma \in \Gamma\}$ and define the feedback law $u : \mathbb{R}^2 \to \mathbb{R}$ as follows: If $\frac{b(\gamma)}{a(\gamma)} = \tau$ for some γ in Γ^-, set $u(x) = k[\frac{1}{\tau}x_1 + x_2] + x_1^3$. If $\frac{b(\gamma)}{a(\gamma)} = \tau$ for some γ in Γ^+, set $u(x) = k[\frac{1}{\tau}x_1 + x_2] - x_1^3$. Finally, if $\frac{b(\gamma)}{a(\gamma)} \neq \tau$ for all γ in Γ, set $u(x) = k[\frac{1}{\tau}x_1 + x_2]$. By adapting Example 3.8 in [10, p. 118] to our setup, it is not hard to check that u robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

Conversely, we prove by contradiction that if $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly stabilizable by means of continuous feedback, then (14) holds.

To this end, assume that there exists a continuous feedback law u that robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$ and that there exist γ_- in Γ^- and γ_+ in Γ^+ such that (14) is violated. Define the sets Σ_{γ_-} and Σ_{γ_+} by setting

$$\Sigma_{\gamma_-} \triangleq \{x \in \mathbb{R}^2 :\ x_1 = -\frac{b(\gamma_-)}{a(\gamma_-)}x_2,\ x_2 > 0\}$$

and

$$\Sigma_{\gamma_+} \triangleq \{x \in \mathbb{R}^2 : \ x_1 = -\frac{b(\gamma_+)}{a(\gamma_+)} x_2, \ x_2 > 0\}.$$

Because the feedback law u stabilizes $S(\gamma_-)$ and $S(\gamma_+)$, the origin is an isolated equilibrium of the corresponding closed-loop systems. Note that $f_{\gamma_-}(x)$ is negative for all x in the region above Σ_{γ_-} that is between Σ_{γ_-} and $\{(0, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$. Furthermore, $f_{\gamma_+}(x)$ is negative of all x in the region below Σ_{γ_+} that is between Σ_{γ_+} and $\{(0, x_2) \in \mathbb{R}^2 : \ x_2 < 0\}$. Thus, there exists a ball $B_\varepsilon(0)$ of radius ε centered at the origin such that

$$u(x) < 0, \quad x \in \Sigma_{\gamma_-} \cap B_\varepsilon(0) \qquad \text{and} \qquad u(x) > 0, \quad x \in \Sigma_{\gamma_+} \cap B_\varepsilon(0). \tag{15}$$

In view of the assumption that (14) is violated, we have either $\dfrac{b(\gamma_-)}{a(\gamma_-)} = \dfrac{b(\gamma_+)}{a(\gamma_+)}$, or $\dfrac{b(\gamma_-)}{a(\gamma_-)} > \dfrac{b(\gamma_+)}{a(\gamma_+)}$. In the former case, we obtain $\Sigma(\gamma_-) = \Sigma(\gamma_+)$, a contradiction with (15).

Assume now that $\dfrac{b(\gamma_-)}{a(\gamma_-)} > \dfrac{b(\gamma_+)}{a(\gamma_+)}$ so that Σ_{γ_-} is below Σ_{γ_+}. Let S be the region of $B_\varepsilon(0)$ below Σ_{γ_-} that is between Σ_{γ_-} and $\{(0, x_2) \in \mathbb{R}^2 : \ x_2 < 0\}$. Because, f_{γ_+} is negative on S, the stability of the system associated with the vector-field $[f_{\gamma_+}, u]^t$ implies that each trajectory $x(\cdot, x_0)$ of this system starting in S leaves S. Hence, it follows from the negativeness of f_{γ_+} and u on Σ_{γ_-}, together with that of f_{γ_+} on $\{(0, x_2) \in \mathbb{R}^2 : \ x_2 < 0\}$, that $x(\cdot, x_0)$ cannot leave S, neither through Σ_{γ_-}, nor through the x_2-axis. We therefore conclude that $x(\cdot, x_0)$ leaves S through the boundary of $B_\varepsilon(0)$. In short $x(\cdot, x_0)$ leaves $B_\varepsilon(0)$ whenever x_0 lies in S, a contradiction with the fact that u stabilizes $S(\gamma_+)$. The proof of the theorem is now complete. ∎

6 Concluding Remarks

We have addressed and solved the robust stabilization problem of a general class of parameterized families of systems in the plane that are not robustly stabilized by smooth feedback. Our solution encompasses the study of two main cases. We approach the first one by adapting a technique introduced in [2], in the context of nonlinear stabilization. The second one is solved through a new method that enables us to construct a robust stabilizer for the entire family of systems, using two feedback laws that robustly stabilize two particular sub-families.

We have completed a first step towards the understanding of non-smooth robust stabilization and we believe that the techniques developed in this paper will be useful for solving more involved robust stabilization problems.

7 Appendix

We present here several technical lemmas that were used in the proofs of Theorems 3.1 and 4.1.

The first one can be easily derived using Lemma 3.1 of [2, p. 1328] and the second one is elementary.

Lemma 7.1 *Let $\eta > 0$ and α in $[0,1)$. Then, for each $\beta > 0$, the intersection of the sets $\{x \in \mathbb{R}^2 : x_1 = \eta\,(x_2)^{1+\alpha},\, x_2 > 0\}$ and Ψ_β contains a unique point $[h(\beta), h(\beta)log(\frac{h(\beta)}{\beta})]$. Moreover, we have $h(\beta) \to 0$ as $\beta \to 0$.*

Lemma 7.2 *For each positive reals β and β', the curve Ψ_β is on the left of $\Psi_{\beta'}$ whenever $\beta < \beta'$.*

The next two lemmas were needed in establishing Claim 1 in the proof of Theorem 3.1.

Lemma 7.3 *Let I be some subset of $\mathbb{R}$. Assume that $b(\cdot)$ is negative on I and $a(\cdot)$ is either positive on I or negative on I. Suppose that $a(\cdot)$ and $b(\cdot)$ are bounded on I. Let θ, μ and $\bar{\beta}$ be positive reals and let α be in $(0,1)$. Then, there exists $k_1 > 0$ such that for each k in $[k_1, +\infty)$ and each γ in I the following holds:*

- **i) a)** *For each $\beta > 0$ and each x in Ψ_β, the vector $[f_\gamma(x), k(x_1 + x_2)]^t$ points towards the left of Ψ_β,*
  - **b)** *For each $\beta > 0$ and each x in Ψ_β, the vector $[f_\gamma(x), kx_1]^t$ points towards the left of Ψ_β if x is below the half-line $\{x \in \mathbb{R}^2 :\ x_1 = \theta x_2,\ x_2 > 0\}$.*
- **ii)** *For each x in Δ_α, the vector $[f_\gamma(x), -k(x_2)^{1-\alpha}]^t$ points into the region below Δ_α if x is above the half-line $\{x \in \mathbb{R}^2 :\ x_1 = \theta x_2,\, x_2 > 0\}$.*
- **iii)** *For each β in $(0, \bar{\beta}]$ and each x in the segment $D_\beta \triangleq \{x \in \mathbb{R}^2 : \mu x_2 - x_1 = -\beta,\, x_1 \in [0, \beta]\,\}$, we have:*
  - **a)** *If x is above the line $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2,\, x_2 < 0\}$, then the vector $[f_\gamma(x), kx_1]^t$ points into the region above D_β.*
  - **b)** *If x is below the line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2,\, x_2 < 0\}$, then the vector $[f_\gamma(x), k(-x_2)^{1-\alpha}]^t$ points into the region above D_β.*

Proof:

(i) Let $\beta > 0$ and x in Ψ_β. The tangent to Ψ_β at x is given by $\dfrac{dx_1}{dx_2} = \dfrac{x_1}{x_1 + x_2}$. If $f_\gamma(x) \le 0$, **(a)** and **(b)** are immediate. Assume now that $f_\gamma(x) > 0$. As $b(\gamma)x_2$ is negative, we get

$$\frac{f_\gamma(x)}{k(x_1 + x_2)} \le \frac{a(\gamma)}{k}\frac{x_1}{x_1 + x_2} \qquad \text{and} \qquad \frac{f_\gamma(x)}{kx_1} \le \frac{a(\gamma)}{k}.$$

The first inequality combined with the boundedness of $a(\cdot)$ on I, yields claim **(a)**. Furthermore, if x is below the line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 > 0\}$, then the tangent to Ψ_β at x is greater than $\dfrac{\theta}{1+\theta}$ and claim **(b)** follows from the second inequality upon recalling that $a(\cdot)$ is bounded on I.

(ii) Let x be in Δ_α. If $f_\gamma(x) \geq 0$, then the claim clearly holds. On the other hand, if $f_\gamma(x) < 0$, it is then easily checked that

$$\frac{-f_\gamma(x)}{k(x_2)^{1-\alpha}} \leq \max\{\frac{-b(\gamma)}{k}, \frac{-\theta a(\gamma) - b(\gamma)}{k}\}\,(x_2)^\alpha, \tag{16}$$

and the desired result follows, once it is seen that for k large enough, the right-hand side of (16) is smaller than $(1+\alpha)\,(x_2)^\alpha$.

(iii) Let x be in D_β: Because both claims clearly hold if $f_\gamma(x) \leq 0$, we assume that $f_\gamma(x) > 0$. If x is above the line $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2,\, x_2 < 0\}$, then $-x_2 < \frac{x_1}{\theta}$ and we get

$$\frac{f_\gamma(x)}{kx_1} \leq \frac{\theta a(\gamma) - b(\gamma)}{\theta}\frac{1}{k}. \tag{17}$$

Also, if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2,\, x_2 < 0\}$, then it is easily checked that

$$\frac{-f_\gamma(x)}{k(-x_2)^{1-\alpha}} \leq (\frac{\bar{\beta}}{\mu})^\alpha \max\{\frac{-b(\gamma)}{k}, \frac{2\theta a(\gamma) - b(\gamma)}{k}\}, \tag{18}$$

so that for k large enough, by boundedness of $a(\cdot)$ and $b(\cdot)$ on I, the right-hand sides of (17) and (18) are smaller than μ. Hence claims **(a)** and **(b)**. ■

The proof of the following Lemma is similar to that of Lemma 7.3 and is therefore omitted.

Lemma 7.4 *Let I be some subset of $\mathbb{R}$. Assume that $b(\cdot)$ is negative on I and $a(\cdot)$ is either positive on I or negative on I. Suppose that $a(\cdot)$ and $b(\cdot)$ are bounded on I. Let θ, μ and $\bar{\beta}$ be positive reals and let α be in $(0,1)$. Then, there exists $k_2 > 0$ such that for each k in $[k_2, +\infty)$ and each γ in I the following holds:*

- **i) a)** *For each $\beta > 0$ and each x in $\widehat{\Psi}_\beta$, the vector $[f_\gamma(x), k(x_1 + x_2)]^t$ points towards the right of $\widehat{\Psi}_\beta$.*
  - **b)** *For each $\beta > 0$ and each x in $\widehat{\Psi}_\beta$, the vector $[f_\gamma(x), kx_1]^t$ points towards the right of $\widehat{\Psi}_\beta$ if x is above the line $\{x \in \mathbb{R}^2 :\ x_1 = \theta x_2,\ x_2 < 0\}$.*
- **ii)** *For each x in $\widehat{\Delta}_\alpha$, the vector $[f_\gamma(x), k(-x_2)^{1-\alpha}]^t$ points into the region above $\widehat{\Delta}_\alpha$ if x is below the half-line $\{x \in \mathbb{R}^2 :\ x_1 = \theta x_2, x_2 < 0\}$.*
- **iii)** *For each β in $(0, \bar{\beta}]$ and each x in the segment $\widehat{D}_\beta \triangleq \{x \in \mathbb{R}^2 : \mu x_2 - x_1 = \beta,\, x_1 \in [-\beta, 0]\,\}$, we have:*
  - **a)** *If x is below the line $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2, x_2 > 0\}$, then the vector $[f_\gamma(x), kx_1]^t$ points into the region below $\widehat{D}_\beta$.*
  - **b)** *If x is above the line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2, x_2 > 0\}$, then the vector $[f_\gamma(x), -k(-x_2)^{1-\alpha}]^t$ points into the region below $\widehat{D}_\beta$.*

When $b(\cdot)$ is positive on some subset I of $\mathbb{R}$, the assertions **(i)**, **(ii)** and **(iii) (a)** of Lemmas 7.3 and 7.4 translate to the following lemma.

Lemma 7.5 *Let I be some subset of $\mathbb{R}$. Assume that $b(\cdot)$ is positive on I and $a(\cdot)$ is either positive on I or negative on I. Suppose that $a(\cdot)$ and $b(\cdot)$ are bounded on I. Let θ and μ are positive reals let α be in $(0,1)$. Then, there exists $k_3 > 0$ such that for each k in $[k_3, +\infty)$ and each γ in I the following holds.*

i) a) *For each $\beta > 0$ and each x in Ψ^s_β, the vector $[f_\gamma(x), k(-x_1+x_2)]^t$ points towards the left of Ψ^s_β.*

b) *For each $\beta > 0$ and each x in Ψ^s_β, the vector $[f_\gamma(x), -kx_1]^t$ points towards the left of Ψ^s_β if x is above the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 < 0\}$.*

ii) *For each x in Δ^s_α, the vector $[f_\gamma(x), k(-x_2)^{1-\alpha}]^t$ points into the region above Δ^s_α if x is below the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, x_2 < 0\}$.*

iii) *For each β in $(0, \bar\beta]$ and each x in the segment $D_\beta \triangleq \{x \in \mathbb{R}^2 : \mu x_2 + x_1 = \beta, \, x_1 \in [0, \beta]\}$, the vector $[f_\gamma(x), -kx_1]^t$ points into the region below D_β if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \, x_2 > 0\}$.*

iv) a) *For each $\beta > 0$ and each x in $\widehat\Psi^s_\beta$, the vector $[f_\gamma(x), k(-x_1 + x_2)]^t$ points towards the right of $\widehat\Psi^s_\beta$.*

b) *For each $\beta > 0$ and each x in $\widehat\Psi^s_\beta$, the vector $[f_\gamma(x), -kx_1]^t$ points towards the right of $\widehat\Psi^s_\beta$ if x is below the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 > 0\}$.*

v) *For each x in $\widehat\Delta^s_\alpha$, the vector $[f_\gamma(x), -k(-x_2)^{1-\alpha}]^t$ points into the region below $\widehat\Delta^s_\alpha$ if x is above the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, x_2 > 0\}$.*

vi) *For each β in $(0, \bar\beta]$ and each x in the segment $\widehat D_\beta \triangleq \{x \in \mathbb{R}^2 : \mu x_2 + x_1 = -\beta, \, x_1 \in [-\beta, 0]\}$, the vector $[f_\gamma(x), -kx_1]^t$ points into the region above $\widehat D_\beta$ if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \, x_2 < 0\}$.*

Finally the last two lemmas are used in the proof of Theorem 4.1.

Lemma 7.6 *Assume that $a(\cdot)$ is positive on some subset I of $\mathbb{R}$ and that $b(\cdot)$ is negative on I. Let θ and δ be fixed positive reals with $2\theta < \delta$. Then, for each γ in I, there exists a neighborhood U_γ of the origin such that for each $\tau > 0$ the following holds:*

i) *For each x in U_γ and in the half-line $D_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 - \tau, \, x_1 > 0\}$ the vector $[f_\gamma(x), -(x_1)^2]^t$ points into the region above D_τ, if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2, \, x_2 < 0\}$.*

ii) *For each x in U_γ and in the half-line $\widehat D_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 + \tau, \, x_1 < 0\}$ the vector $[f_\gamma(x), (x_1)^2]^t$ points into the region below $\widehat D_\tau$, if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2, \, x_2 > 0\}$.*

Proof: We prove only **(i)** as the proof of **(ii)** is similar. Fix γ in I, and let $\tau > 0$ and x in D_τ. Under the assumptions of the lemma, $f_\gamma(x)$ is positive and $-x_2 > \frac{x_1}{2\theta}$, so that

$$\frac{(x_1)^2}{f_\gamma(x)} \le \frac{2\theta}{2\theta a(\gamma) - b(\gamma)} x_1 .$$

Thus, for $x_1 > 0$ small enough, we have $\frac{(x_1)^2}{f_\gamma(x)} \leq \frac{1}{\delta}$, for all $\tau > 0$ and the claim follows. ∎

Lemma 7.7 *Assume that $a(\cdot)$ is positive on some subset I of $\mathbb{R}$ and that $b(\cdot)$ is positive on I. Let θ and δ be positive reals with $2\theta < \delta$. Then, for each γ in I, there exists a neighborhood U_γ of the origin such that for each $\tau > 0$ the following holds.*

i) *For each x in U_γ and in the half-line $D_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 - \tau, x_1 > 0\}$, the vector $[f_\gamma(x), (x_1)^2]^t$ points into the region below D_τ, if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = 2\theta x_2, x_2 > 0\}$,.*

ii) *For each x in U_γ and in the half-line $\widehat{D}_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 + \tau, x_1 < 0\}$, the vector $[f_\gamma(x), -(x_1)^2]^t$ points into the region above $\widehat{D}_\tau$ if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = 2\theta x_2, x_2 < 0\}$,.*

References

[1] B. R. Barmish. *New Tools for Robustness of Linear Systems.* Macmillan Publishing Company, 1994.

[2] W. Dayawansa, C. Martin, and G. Knowles. Asymptotic stabilization of a class of smooth two-dimensional systems. *SIAM Journal of Control and Optimization*, 28(6):1321–1349, 1990.

[3] P. Dorato. A historical review of robust control. *IEEE Control Systems Magazine*, 7:44–47, 1987.

[4] P. Dorato, R. Tempo, and G. Muscato. Bibliography on robust control. *Automatica*, 29(1):201–213, 1993.

[5] J. Doyle, A. Packard, and K. Zhou. Review of LFTs, LMIs, and μ. In *Proceedings of the 30^{th} Conference on Decision and Control.* IEEE, 1991.

[6] B. Francis. *A course on H_∞ control theory.* Springer-Verlag, Berlin, 1987.

[7] V. Guillemin and A. Pollack. *Differential Topology.* Prentice Hall, Inc., Englewood Cliffs, NJ, 1974.

[8] P. Hartman. *Ordinary Differential Equations.* John Wiley and Sons, Inc., 1964.

[9] M. Kawski. Stabilization of nonlinear systems in the plane. *Systems and Control Letters*, 12:169–175, 1989.

[10] H. K. Khalil. *Nonlinear Systems.* Macmillan Publishing Company, 1992.

[11] V. Kharitonov. Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differentsial'nye Uravneniya*, 14:2086–2088, 1978.

[12] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[13] E. Sontag and H. Sussmann. Remarks on continuous feedback. In *Proceedings of the Conference on Decision and Control*, pages 916–921. IEEE, 1980.

[14] A. van der Schaft. Nonlinear state space H_∞ control theory. In *Essays on Control: Perspectives in the Theory and its Applications.* Birkhauser, Boston, MA, 1993.

[15] O. Yaniv and I. Horowitz. A quantitative design method for MIMO linear feedback systems having uncertain plants. *International Journal of Control*, 43:401–421, 1986.

[16] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26:301–320, 1981.

[17] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, Englewood Cliffs, NJ, 1996.