ABSTRACT

Title of dissertation:	MOTIVIC DECOMPOSITION OF PROJECTIVE PSEUDO-HOMOGENEOUS VARIETIES
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Let G be a semi-simple algebraic group over a field k. Projective G-homogeneous varieties are projective varieties over which G acts transitively. The stabilizer or the isotropy subgroup at a point on such a variety is a parabolic subgroup which is always smooth when the characteristic of k is zero. However, when k has positive characteristic, we encounter projective varieties with transitive G-action where the isotropy subgroup need not be smooth. We call these varieties *projective pseudo-homogeneous* varieties. To every such variety, we can associate a corresponding projective homogeneous variety. In this thesis, we extensively study the Chow motives (with coefficients from a finite connected ring) of projective pseudo-homogeneous varieties for G inner type over k and compare them to the Chow motives of the corresponding projective homogeneous varieties. This is done by proving a generic criterion for the motive of a variety to be isomorphic to the motive of a projective homogeneous variety which works for any characteristic of k. As a corollary, we give some applications and examples of Chow motives that exhibit an interesting phenomenon. We also show that the motives of projective pseudo-homogeneous varieties satisfy properties such as Rost Nilpotence and Krull-Schmidt.

MOTIVIC DECOMPOSITION OF PROJECTIVE PSEUDO-HOMOGENEOUS VARIETIES

by

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Dedication

To Deepak

Acknowledgments

This thesis is an outcome of curiosity to solve an unknown problem along with persistent encouragement, support and love of some important people. I owe my gratitude to all of those who have been with me through this unforgettable journey.

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Finally, I would like to thank God for being a guiding light all the way along and for blessing me with all such wonderful people around me.

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Chapter 0: Introduction

Let G be a semi-simple algebraic group over a perfect field k. We say that a variety over k is *projective G-homogeneous* if it is projective and if G acts transitively on it over k. Under this action, the stabilizer or the isotropy subgroup of G at a geometric point on the variety is a parabolic subgroup scheme. If the characteristic of the field k is zero, then these parabolic subgroup schemes are always smooth. However, when the characteristic of k is $p \neq 0$, the parabolic subgroup schemes need not necessarily be reduced. We therefore encounter projective varieties with transitive G-action whose isotropy subgroups are non-smooth parabolic subgroup schemes. In this thesis, we study these varieties which we call *projective pseudo-homogeneous varieties* for G inner type over k and establish a connection to a corresponding projective homogeneous variety. We also prove some properties about them and give interesting examples.

0.1 The Problem Setting

Let G be a semi-simple algebraic group of inner type over a perfect field k of characteristic $p > 3$ (See Remark 4.3.4 for why the assumption $p > 3$ is necessary). We follow the terminology of SGA3. So by definition G is smooth and connected with trivial radical. Note that in SGA3, parabolic subgroups are reduced as schemes. Therefore we use the term *parabolic subgroup schemes* to include possibly non-reduced subgroup schemes containing a Borel. Let K denote the algebraic closure of k. For a variety Y over k and an extension $k' \supseteq k$, we write $Y_{k'}$ for $Y \times_{Spec\ k} Spec\ k'$.

Definition 1. A G-variety $\widetilde{X}$ over k is called a *projective pseudo-homogeneous variety*[1] if $\widetilde{X}_K \simeq G_K/\widetilde{P}$ for some parabolic subgroup scheme $\widetilde{P}$ in G_K that is not necessarily reduced.

Such a variety is always smooth since G is smooth (See SGA3, exp $\mathrm{VI_A}$, Theorem 3.2). For detailed construction of the quotient of an algebraic group by a subgroup see Chapter III, §3 of [DG70]. Note that by Proposition 2.1, §3, Chapter III of [DG70], the condition $\widetilde{X}_K \simeq G_K/\widetilde{P}$ is equivalent to saying that the action map $G(\Omega) \times \widetilde{X}(\Omega) \to \widetilde{X}(\Omega) \times \widetilde{X}(\Omega)$ is surjective for every algebraically closed field Ω over K. If $\widetilde{P}$ is a parabolic subgroup scheme over K, we will make slight abuse of notation and write $G/\widetilde{P}$ for $G_K/\widetilde{P}$. Let P denote the underlying reduced scheme of $\widetilde{P}$. Note that since k is perfect, P is a group scheme (See §6 in Chapter VI of [Mil]).

Definition 2. Given $\widetilde{X}$, a projective pseudo-homogeneous variety for G such that $\widetilde{X}_K \simeq G/\widetilde{P}$, let X denote the unique (see Proposition 1.3 in [MPW96]) projective homogeneous variety for G, such that $X_K \simeq G/P$ where P is the underlying reduced subscheme of $\widetilde{P}$. We call X *the projective homogeneous variety corresponding to* $\widetilde{X}$.

By universal property of quotients, there is a canonical G-equivariant finite morphism $\theta : X \to \widetilde{X}$.

[1]The term projective pseudo-homogeneous varieties is coined here to point towards a natural generalization of projective homogeneous varieties. It is not to be confused with the definition used in [Kar16]

Example. Suppose $G = SL_{3,k}$. Let $G/\widetilde{P} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ be given by the equation $\sum_{i=0}^{2} x_i^p y_i = 0$ where the G action is $g.\overrightarrow{x} = g^{p^3}\overrightarrow{x}$ and $g.\overrightarrow{y} = (g^{-t})^{p^4}\overrightarrow{y}$ (Here $g^{-t} = (g^{-1})^t$ is the transpose of the inverse of g. Also by abuse of notation g^{p^n} means taking p^nth power of entries of the matrix g). Then $\widetilde{P} = Stab([1:0:0]\times[0:0:1]) = \{\left(\begin{smallmatrix} * & * & * \\ x & * & * \\ y & z & * \end{smallmatrix}\right)|x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0\}$. The underlying reduced scheme is the standard Borel $P = \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$ and the corresponding homogeneous variety $G/P \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ is given by $\sum_{i=0}^{2} x_i y_i = 0$. This comes with the standard G-action $g.\overrightarrow{x} = g\overrightarrow{x}$ and $g.\overrightarrow{y} = (g^{-t})\overrightarrow{y}$. We have the canonical G-equivariant map

$$G/P \to G/\widetilde{P}$$

$$\overrightarrow{x} \mapsto \overrightarrow{x}^{p^3}$$

$$\overrightarrow{y} \mapsto \overrightarrow{y}^{p^4}$$

We want to emphasize that even over algebraically closed fields, the K-variety $G/\widetilde{P}$ need not be isomorphic to any flag variety (see Theorem 3.3.1). Therefore, X and $\widetilde{X}$ need not be twisted forms of each other.

Projective pseudo-homogeneous varieties are extensively studied in the literature when $k = K$ is algebraically closed. We give a brief survey on what is known so far. In [Wen93a], Wenzel has classified all parabolic subgroup schemes $\widetilde{P}$ and in [Wen93b] he proved that the varieties $G/\widetilde{P}$ are rational. Using this classification, de Salas in [SdS03] has classified all $G/\widetilde{P}$. The varieties of the form $G/\widetilde{P}$ where $\widetilde{P}$ is any parabolic subgroup scheme that may or may not be reduced are known as *parabolic varieties* in [SdS03]. Lauritzen and Haboush answered many interesting questions about the geometry of these varieties including canonical line bundles, vanishing theorems and Frobenius splitting in

[Lau97], [HL93] and [Lau93]. Lauritzen also gave a geometric construction of $G/\widetilde{P}$ in [Lau96] where he realizes these varieties as the G-orbit of a Borel stable line in projective space. They have rich structure and behave quite differently from the analogous *generalized flag varieties* (or simply *flag varieties*) G/P where P is smooth. For example, in [Lau93], Lauritzen has shown that under mild assumptions on G, $G/\widetilde{P}$ is isomorphic to a flag variety if and only if $G/\widetilde{P}$ is Frobenius split. The varieties of the form $G/\widetilde{P}$ that does not admit an isomorphism to a flag variety are known as *varieties of unseparated flags* or simply *VUFs* in [HL93]. In particular, G/P and $G/\widetilde{P}$ are not isomorphic in general. Moreover, in [HL93] one can find explicit examples of VUFs which illustrate that unlike generalized flag varieties, vanishing theorem for ample line bundles and Kodaira's vanishing theorem break down. So over algebraically closed fields, although these varieties exhibit a lot of strange phenomena, they are well understood and it is straightforward to compute their Chow motives (see §4.4).

However, when k is not algebraically closed, nothing much is known about them unlike the analogous projective homogeneous varieties. Projective homogeneous varieties are quite thoroughly studied in the literature ([Art82], [GK], [EKM08] and [KMRT98]) and so are their Chow motives ([Bro05], [CPSZ06], [CGM05], [Kar10] and [Kar13]). Therefore it is natural to study projective pseudo-homogeneous varieties and ask if they exhibit any similarity to projective homogeneous varieties.

In this paper we compute the Chow motives of projective pseudo-homogeneous varieties and prove that Rost nilpotence theorem holds. We also show that their motives are isomorphic to motives of the corresponding projective homogeneous varieties. A crucial ingredient of the proof is Theorem 5.2.1 which gives a characterization of when the motive of a variety is isomorphic to the motive of a projective homogeneous variety. The proof of this theorem is independent of the characteristic of the base field and might be useful for other applications.

0.2 Notations

Throughout this paper k is a perfect field of characteristic $p > 3$ and K denotes the algebraic closure of k. $\mathbb{G}_m$ denotes the usual multiplicative group. G denotes a semi-simple algebraic group of inner type over k unless stated otherwise. The set of vertices of the Dynkin diagram of G (or equivalently the set of conjugacy classes of maximal parabolics in G_K) is denoted by Δ_G. For a field extension E of k, $\tau_E \subseteq \Delta_G$ denotes the subset that contains the classes of those maximal parabolics in G_K defined over E. Given a parabolic subgroup scheme $\widetilde{P}$, P denotes the underlying reduced subscheme. If $\widetilde{X}$ is a projective pseudo-homogeneous variety then X denotes the corresponding projective homogeneous variety.

Λ denotes a connected, finite, associative unital commutative ring. An example to keep in mind is a finite field of some prime characteristic. Let $Chow(k,\Lambda)$ denote the category of Chow motives over k with coefficients in Λ. Detailed exposition of $Chow(k,\Lambda)$ can be found in [EKM08]. For a variety X, $\mathcal{M}(X)$ denotes the Chow motive of X. By $Ch_i(X)$ and $Ch^i(X)$ we mean the i^{th} Chow group of X graded by dimension and codimension respectively. The Tate motive $\mathcal{M}(Spec\ k)\{i\}$ is denoted by $\Lambda\{i\}$ (The notation $\Lambda\{i\}$ is equal to $\Lambda(i)[2i]$ in Voevodsky's category of motives). For a motive M, $M\{i\} := M \otimes \Lambda\{i\}$.

0.3 Statement of Main Results

We say that *Krull-Schmidt principle* holds for an object in an additive category if it is isomorphic uniquely to direct sum of indecomposable summands (up to permutation). Let X be a k-variety. Recall from Karpenko's paper [Kar13] that a summand M of $\mathcal{M}(X)$ is called *upper* if $Ch^0(M) \neq 0$. See Lemma 2.8 in [Kar13] for more details. If the motive of X satisfies Krull-Schmidt principle, let U_X denote the unique upper indecomposable summand of $\mathcal{M}(X)$. It is well known that the motives of projective homogeneous varieties satisfy Krull-Schmidt principle (see Corollary 2.2.2) in $Chow(k, \Lambda)$. If X_τ is projective homogeneous corresponding to the subset $\tau \subseteq \Delta_G$ (see §1.2), we write U_τ for the upper indecomposable summand of $\mathcal{M}(X_\tau)$.

Theorem. *(Rost Nilpotence for Projective Pseudo-Homogeneous Varieties) Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for a semi-simple group G of inner type over k. Then the kernel of the base change map*

$$End(\mathcal{M}(\widetilde{X})) \to End(\mathcal{M}(\widetilde{X}_K))$$

$$f \mapsto f \otimes K$$

consists of nilpotents.

Proof. See §4.5. □

Theorem. *The Krull-Schmidt principle holds for any shift of any summand of the motive of a projective pseudo-homogeneous variety for G.*

Proof. See §4.5. □

The following theorem gives a characterization of when the motive of a variety is isomorphic to the motive a projective homogeneous variety and is independent of the characteristic of the base field k. In particular, it holds for characteristic zero as well. Recall that a k-variety Z is geometrically split if $\mathcal{M}(Z_K)$ is isomorphic to a direct sum of Tate motives.

Theorem. *Let X be a projective G-homogeneous variety over k. Let Z be any geometrically split projective k-variety whose motive satisfies the Rost nilpotence principle such that the following holds in $Chow(k, \Lambda)$:*

1. $U_X \simeq U_Z$
2. $\mathcal{M}(X_L) \simeq \mathcal{M}(Z_L)$ *where* $L = k(X)$

Then $\mathcal{M}(X) \simeq \mathcal{M}(Z)$.

Proof. See §5.2. □

As an application of the above theorem we derive the following main result.

Theorem. *Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for G and let X be the corresponding projective homogeneous variety. Then in the category of motives $Chow(k, \Lambda)$*

$$\mathcal{M}(X) \simeq \mathcal{M}(\widetilde{X})$$

In particular, by Theorem 2.5.2 every indecomposable summand in $\mathcal{M}(\widetilde{X})$ is a shift of some upper motive U_τ satisfying $\tau_{k(X)} \subseteq \tau$.

Proof. See §5.3. □

We also give some examples and applications in §5.4.

0.4 Outline of the Thesis

We start by recalling some background about projective homogeneous varieties and Chow groups in Chapter 1. We also recall the category of Chow motives in this chapter. In Chapter 2, we state some properties of the motives of projective homogeneous varieties such as Rost Nilpotence and Krull-Schmidt principle. This is followed by stating some results from the literature on motivic decompositions of projective homogeneous varieties. In Chapter 3, we recall the definition and classification of the variety of unseparated flags and discuss how they differ from flag varieties. The rest of the chapters are dedicated to proving the main results of this thesis and give some applications.

Chapter 1: Preliminaries

1.1 Linear Algebraic Groups

In this section we recall some facts about algebraic groups and establish the notations we will use in the rest of the thesis. A thorough treatment of linear algebraic groups can be found in [Spr09] and [Bor91].

Recall that G is an *algebraic group* over a field k if the following holds:

- G is an algebraic variety over k.
- There is an operation $\cdot$ on G which makes $(G, \cdot)$ a group
- The maps defining the group structure $\mu : G \times G \to G$ and $i : G \to G$ with $\mu(x, y) = x \cdot y$ and $i(x) = x^{-1}$ are morphism of varieties defined over k.
- The identity element $e \in G$ is a k-rational point.

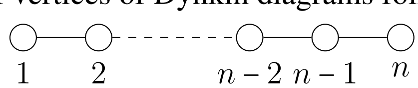
If the underlying variety of G is affine, then it is called an *affine algebraic group* or a *linear algebraic group*.

The *radical* of G denoted by $R(G)$ is the maximal closed, connected, solvable, normal subgroup of G over the algebraic closure K of k. The set of unipotent elements of $R(G)$ is called the *unipotent radical* of G, denoted by $R_u(G)$. The group G is *semi-*

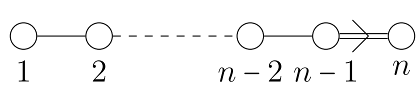
simple (resp. *reductive*) if $R(G)$ (resp.$R_u(G)$) is trivial. A linear algebraic group T over k is called a *torus*, if over K, it becomes isomorphic to a product of several copies of the multiplicative group G_m. If this isomorphism is already defined over k, then T is called a *split torus*. A semi-simple linear algebraic group is called a *split group*, if it contains a split maximal torus.

Let G be a split group. We fix a split maximal torus T in G. Let Φ be the root system of G associated to T. Let $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ denote a basis for Φ i.e, $\Delta \subset \Phi$ is a set of simple roots of G. Then one can associate an oriented graph called the *Dynkin diagram* of Φ whose vertices correspond to the elements of Δ. It is well known that the Dynkin diagram of Φ does not depend on the choice of the basis $\Delta \subset \Phi$ and moreover, uniquely determines Φ. A root system Φ is said to be *irreducible* if it cannot be partitioned into a union of two mutually orthogonal proper subsets. One of the most important results regarding root systems says that all possible Dynkin diagrams of irreducible root systems can be classified into four classical types $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 2)$, $D_n (n \geq 3)$ and five exceptional types E_6, E_7, E_8, F_4 and G_2. Here the subscripts denote the rank of the respective root systems. Listed below are the Dynkin diagrams for various types. The enumeration of vertices of Dynkin diagrams follows Bourbaki [Bou82].

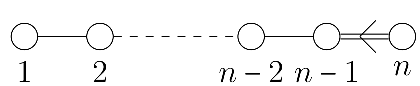
A_n



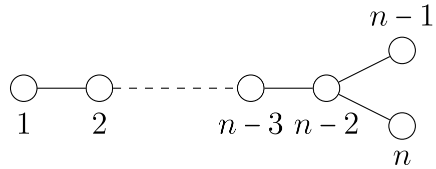
B_n



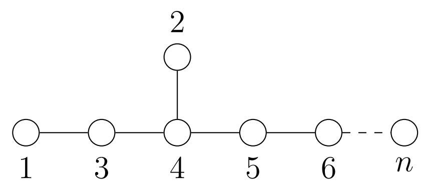
C_n



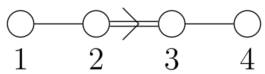
D_n



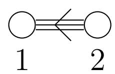
E_n



F_4



G_2



A semi-simple algebraic group G is of inner type if the $*$-action (see [Tit66]) of the absolute Galois group $Gal(k_{sep}/k)$ on the Dynkin diagram of G is trivial. If G is not of inner type, we say that G is of outer type. For any semi-simple algebraic group G there exists a unique (up to an k-isomorphism) minimal finite Galois field extension k_{inn}/k, such that $G_{k_{inn}}$ is of inner type (see Lemma 3.5 in [KR94]).

We say that a subgroup $P \subset G$ is a *parabolic subgroup* if P is smooth and if G/P is a projective variety. Subgroups that are minimal with respect to the above property are called *Borel* subgroups.

1.2 Projective Homogeneous Varieties

Let G be an algebraic group over k. We recall the definition of projective G-homogeneous varieties (§2.3, [Spr09]) and give some of their properties.

A *G-variety* over k is a variety X over k equipped with a G-action, the action being given by morphism of varieties over k. More precisely, there is a morphism of varieties $a : G \times X \to X$, written $a(g,x) = g \cdot x$ defined over k, such that $g \cdot (h \cdot x) = (g \cdot h) \cdot x$, $e \cdot x = x$ where e is the identity element of G.

A *homogeneous variety* X for G or a *G-homogeneous variety* is a G-variety on which G acts transitively, that is, for any $x, y \in X(K)$, there exists $G(K)$ such that $g \cdot x = y$ where K denotes the algebraic closure of k.

A morphism between two G-varieties $\phi : X \to Y$ is called a *G-morphism* or is *G-equivariant* if $\phi(g \cdot x) = g \cdot \phi(x)$ for $g \in G$, $x \in X$.

The *stabilizer* or *isotropy group* of a point $x \in X$ is the closed subgroup $G_x = \{g \in G | g \cdot x = x\}$.

A *projective homogeneous variety for G* or a *projective G-homogeneous variety* is a homogeneous G-variety where the isotropy group of any K-point is a parabolic subgroup (which by definition is smooth). In other words, X is a projective homogeneous variety for G if $X_K \simeq G_K/P$ for some (smooth) parabolic subgroup P.

Let G be a semi-simple algebraic group over k. We fix a maximal torus T, a Borel B containing T and the respective set of simple roots Δ_G of G_K, which can be identified with the nodes of the Dynkin diagram of G. The simple roots also correspond to the conjugacy classes of maximal parabolics in G_K. The subsets of Δ_G are in natural one-to-one correspondence with the set of conjugacy classes of parabolic subgroups in G_K defined as follows: the conjugacy class corresponding $\tau \subseteq \Delta_G$ is the one containing the intersection of all maximal parabolics in τ that contain a given Borel B in G_K. For any subset $\tau \subseteq \Delta_G$, we write X_τ or$X_{\tau,G}$ for the projective homogeneous variety of parabolic subgroups in G of the type τ. For instance, X_{Δ_G} is the variety of the Borel subgroups. Any projective G-homogeneous variety is isomorphic to X_τ for some τ.

1.2.1 Examples

We give examples of projective G-homogeneous varieties over k for various types. Let G_0 be the adjoint split form of G.

- A_n: In this case $G_0 \simeq PGL_{n+1}$, the projective linear group. If G is inner type over k, then $G \simeq PGL_A = Aut_k\ A$, where A is a central simple algebra of degree $n+1$ over k. Any projective G-homogeneous variety X can be identified with variety of flags of (right) ideals in A. For example, the variety of ideals of reduced dimension i in A are the *generalized Severi-Brauer* varieties denoted by $SB_i(A)$. These are twisted forms of Grassmannians and correspond to maximal parabolic subgroups P_i.

  If G is outer over k, then $G \simeq PGU(A,\sigma)$ where A is a central simple algebra with unitary involution σ and any projective G-homogeneous variety is a twisted form of $G_0/P_{i_1,i_2,\cdots,i_m}$ where $i_k = n+1-i_{m-k+1}$ for all $k = 1,2,\cdots,m$.

- B_n: In this case $G_0 \simeq O^+_{2n+1}$. All twisted forms of this group are inner and $G \simeq O^+(q)$ where q is a quadratic form in $2n+1$ variables. Projective G-homogeneous varieties are described as flags of totally q-isotropic subspaces. In particular, the projective quadric given by the equation $q = 0$ is a projective G-homogeneous variety and is a twisted form of G_0/P_1.

- C_n: In this case $G_0 \simeq PGSp(V,h)$ where (V,h) is a non-degenerate alternating form of dimension $2n$. All twisted forms of this group are inner and $G \simeq PGSp(A,\sigma)$ where A is a central simple algebra of degree $2n$ with symplectic involution σ. A projective G-homogeneous variety can be described as the set of flags of (right) ideals

$$X(d_1, d_2, \cdots, d_k) = \{I_1 \subset \cdots \subset I_k \subset A | I_i \subseteq I_i^{\perp}\}$$

  of fixed reduced dimensions $1 \leq d_1 < \cdots < d_k \leq n$, where $I^{\perp} = \{x \in A | \sigma(x) I = 0\}$ is the right ideal of reduced dimension $2n - rdim\ I$ where $rdim\ I$ is the reduced dimension of I. These are twisted forms of $G_0/P_{d_1, d_2, \cdots, d_k}$.

- D_n: In this case $G_0 \simeq PGO^+_{2n}$ and $G \simeq PGO^+(A, \sigma, f)$ where A is a central simple algebra of degree $2n$ and (σ, f) is a quadratic pair (See Definition 5.4, §5 in [KMRT98]). The outer forms of G_0 are characterized by involutions with non-trivial discriminant. Assume that G is inner. If A is split, the projective quadric given by $q = 0$ is a projective homogeneous variety corresponding to the maximal parabolic P_1.

For exceptional groups and for more details, refer to §25, §26 in [KMRT98].

1.3 Algebraic Cycles and Chow groups

Let X be a variety over k. An algebraic cycle of dimension r on X is a finite formal sum $\sum n_Z[Z]$ of integral subvarieties in X of dimension r with integer coefficients. We define the *Chow group* of dimension r to be the additive group of algebraic cycles of dimension r modulo rational equivalence. It is denoted by $CH_r(X)$ or sometimes by $A_r(X)$. The *total Chow group*, $CH(X)$ of X is defined as $CH(X) := \oplus_{i=0}^{dim\ X} CH_i(X)$. Chow groups can also be graded by codimension where $CH^r(X) = Ch_{dim\ X-r}(X)$. One can also replace the integral coefficients with coefficients from a commutative ring Λ to get Chow group with coefficients in Λ denoted by $CH(X;\Lambda)$. Let us recall some operations on Chow groups.

- *Proper push-forward:* Let $f : X \to Y$ be a proper morphism. For a cycle V, let $W = f(V)$. Then we define the push-forward of Chow groups as follows:

$$f_* : CH_r(X) \to CH_r(Y)$$
$$f_*[V] = deg(V/W)[W]$$

  where

$$deg(V/W) = \begin{cases} [k(V) : k(W)], \text{if dim V = dim W} \\ 0, \text{else} \end{cases}$$

- *Flat pull-back:* Let $f : X \to Y$ be a flat morphism of relative dimension n. Then we define, pull-back of Chow groups as follows:

$$f^* : CH^r(Y) \to CH^r(X)$$

$$f^*[V] = [f^{-1}V]$$

where $f^{-1}V$ is the scheme-theoretic inverse image of V.

- *Change of field homomorphism:* Let l/k be a field extension. Then the projection morphism $p : X \times_k Spec\ l \to X$ is flat of relative dimension 0 and the induced pull-back

$$res_{l/k} := p^* : CH^r(X) \to CH^r(X_l)$$

is called the change of field homomorphism. Cycles in the image of $res_{l/k}$ are called k-rational.

- *Ring Structure:* One can define intersection product on the total Chow group $CH(X)$ which gives it a graded commutative ring structure. This ring structure is preserved by pull-back homomorphism i.e, f^* as defined above is a ring homomorphism.

- *Projection formula:* Given push-forward f_* and pull-back f^* morphisms, we have the projection formula:

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

where $\alpha \in CH(Y)$ and $\beta \in CH(X)$.

1.4 Chow motives with Coefficients

Let Λ be a commutative ring. To describe the category of Chow motives $Chow(k, \Lambda)$ over a field k with coefficients in Λ, we first start with the definition of the category of correspondences $Corr(k, \Lambda)$. The objects in the category $Corr(k)$ are pairs (X, n) where X is a smooth projective scheme over k and $n \in \mathbb{Z}$.The morphisms are given by

$$Hom_{Corr(k,\Lambda)}((X, n), (Y, m)) = \oplus A_{d_i+n-m}(X_i \times Y) \otimes_{\mathbb{Z}} \Lambda$$

where $X = \coprod X_i$ are the connected components and $d_i = dim\ X_i$.

This category is not idempotent complete, that is, idempotent morphisms need not have kernel or cokernel. The category of Chow motives $Chow(k, \Lambda)$ is obtained by taking the pseudo-abelian envelope of $Corr(k)$. In other words, the objects of $Chow(k, \Lambda)$ are triples (X, n, p) with $p \in End\ (X, n)$ a projector or idempotent, i.e, $p^2 = p$. The morphisms are given by

$$Hom_{Chow(k,\Lambda)}((X, n, p), (Y, m, q)) = q \circ Hom_{Corr(k,\Lambda)}((X, n), (Y, m)) \circ p$$

This category is idempotent complete. By the term *motive of* X, we mean the object $(X, 0, \Delta)$ where Δ is the diagonal in $X \times X$. It is denoted by $\mathcal{M}(X)$.

The category $Chow(k, \Lambda)$ admits tensor structure as follows:

$$(X, n, p) \otimes (Y, m, q) = (X \times Y, n + m, p \times q)$$

A special object in this category is the *trivial Tate motive* $(Spec\, k, 0, \Delta)$, also denoted by Λ. The twists $\Lambda\{n\} := (Spec\, k, n, \Delta)$ are *Tate objects*. For a motive $M \in Obj(Chow(k, \Lambda))$, we define the *Tate twisted* object $M\{n\} := M \otimes \Lambda\{n\}$.

Consider the motive of the projective space $\mathcal{M}(\mathbb{P}^r)$. Then the morphism given by the cycle $\alpha_r = [pt \times \mathbb{P}^r] \in End\ \mathcal{M}(\mathbb{P}^r)$ is a projector where pt is an arbitrary degree 1 closed point in $\mathbb{P}^r$. Then, $Chow(k, \Lambda)$ admits direct sums as follows:

$$(X, n, p) \oplus (Y, m, q) = (X \coprod (Y \times \mathbb{P}^{m-n}), n, p + (q \times \alpha_{m-n}))$$

It is easy to see that if $p \in End\ \mathcal{M}(X)$ is a non-trivial projector, then in the category $Chow(k, \Lambda)$, we have the following decomposition:

$$\mathcal{M}(X) \simeq (X, 0, p) \oplus (X, 0, 1 - p)$$

Chapter 2: Motivic Decomposition of Projective Homogeneous Varieties

2.1 Rost Nilpotence Theorem

In this section we let the coefficient ring Λ to be arbitrary. Consider the category of Chow motives $Chow(F, \Lambda)$ where F is an arbitrary field. Let X be a variety over a field F. We say that *Rost Nilpotence* holds for a variety X if for every field extension E/F the kernel of the base change map

$$End_F(\mathcal{M}(X)) \to End_E(\mathcal{M}(X_E))$$

$$\alpha \mapsto \alpha_E$$

consists of nilpotents. That is , if $\alpha \in End_F(\mathcal{M}(X))$ is such that $\alpha_E = 0$, then $\alpha^{\circ N} = 0$ for some $N > 0$.

Knowing whether the Rost Nilpotence holds for a variety is useful for many reasons. Here are some of the consequences of Rost Nilpotence. Assume that a variety X satisfies Rost Nilpotence in $Chow(F, \Lambda)$. Then the following holds.

- If $p \in End_F(\mathcal{M}(X))$ is idempotent and non-zero, then p_E is non-zero for every field extension E/F.

- If $p \in End_F(\mathcal{M}(X))$ is such that p_E is idempotent for some field extension E, then there is an *idempotent* $q \in End_F(\mathcal{M}(X))$ such that $p_E = q_E$.

- If X and Y are geometrically split varieties over F satisfying Rost Nilpotence such that $\alpha \in Hom(\mathcal{M}(X), \mathcal{M}(Y))$ is an isomorphism over some field extension E/F, then α is already an isomorphism.

We do not yet know if Rost Nilpotence holds for all varieties. But we know that they hold for projective homogeneous varieties as proved by Chernousov, Gille and Merkurjev (Theorem 8.2 in [CGM05]) and Brosnan (Theorem 5.1 in [Bro05]).

Theorem 2.1.1. *(Theorem 5.1 in [Bro05]) Let X be a projective G-homogeneous variety over k. Then X satisfies Rost Nilpotence.*

2.2 Krull-Schmidt Theorem

Recall that K denotes the algebraic closure of a given field k. A variety over k is said to be *geometrically split* if its motive over K is isomorphic to direct sum of Tate motives.

Notation: For the rest of the chapter we assume that Λ is a finite, connected coefficient ring in $Chow(k, \Lambda)$ although some of the results hold for arbitrary Λ.

We say that *Krull-Schmidt principle* holds for an object in an additive category if it is isomorphic uniquely to direct sum of indecomposable summands (up to permutation). A very useful consequence of Rost nilpotence is the following result which can be found in Karpenko's paper [Kar13].

Theorem 2.2.1. *(Corollary 2.6 in [Kar13]) Assume that the coefficient ring Λ is finite. The Krull-Schmidt principle holds for any shift of any summand of the motive of any geometrically split variety satisfying the Rost nilpotence principle. In other words, the Krull-Schmidt principle holds for the objects of the pseudo-abelian Tate subcategory in*

$Chow(k, \Lambda)$ generated by the motives of geometrically split k-varieties satisfying Rost nilpotence.

As a consequence of this theorem, we have the following important result for the motives of projective homogeneous varieties. This is also proved by Chernousov and Merkurjev (Corollary 35 in [CM06]).

Corollary 2.2.2. *The Krull-Schmidt principle holds for any shift of any summand of the motive of projective homogeneous varieties in $Chow(k, \Lambda)$.*

Proof. Observe that any projective homogeneous variety over k is geometrically cellular i.e, has cellular decomposition (see Definition 3.2 in [Kah99]) over the algebraic closure K and therefore by Theorem 2.4.1 is geometrically split i.e, its motive splits into direct sum of Tate motives over K. The result now follows from this fact, Theorem 2.1.1 and Theorem 2.2.1. □

2.3 Upper Indecomposable Motives

The notion of *upper motives* is due to Karpenko ([Kar13]). We recall this from [Kar13] in this section. To a correspondence of degree zero in $Chow(k, \Lambda)$, one can associate an element of Λ called *multiplicity* as follows (see §75 in [EKM08]). For projective varieties X and Y, let $\alpha \in Ch_{dim\ X}(X \times Y) = Hom(\mathcal{M}(X), \mathcal{M}(Y))$ be a correspondence in $Chow(k, \Lambda)$. Then the projection morphism $p : X \times Y \to X$ is proper and hence induces the push-forward homomorphism

$$p_* : Ch_{dim\ X}(X \times Y) \to Ch_{dim\ X}(X) = \Lambda \cdot [X]$$

Then, the element $mult(\alpha) \in \Lambda$ satisfying $p_*(\alpha) = mult(\alpha) \cdot [X]$ is called the *multiplicity* of α. It is easy to see that for any two correspondences $\alpha, \beta \in Hom(\mathcal{M}(X), \mathcal{M}(Y))$, we have $mult(\alpha + \beta) = mult(\alpha) + mult(\beta)$. Moreover, multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (Corollary 1.7 in [Kar00c]). Since the multiplicity of a projector is idempotent, it is either 0 or 1 because the coefficient ring Λ is connected.

For a motive M, let $Ch^i(M)$ denote the group $Hom(M, \Lambda\{i\})$ in the category $Chow(k, \Lambda)$.

Lemma 2.3.1. *(Lemma 2.8 in [Kar13]) Let X be a smooth complete irreducible variety. Let M be a summand of the motive of X and let $\pi \in Ch_{dim\ X}(X \times X)$ be the projector giving M. Then the following are equivalent:*

- $Ch^0(M) \neq 0$
- *the summand $Ch^0(M)$ of the Λ-module $Ch^0(X)$ coincides with the whole $Ch^0(X)$*
- $mult(\pi) \neq 0$
- $mult(\pi) = 1$

Proof. See Lemma 2.8 in [Kar13]. □

Definition 3. (Definition 2.10 in [Kar13]) Let $M \in Chow(k, \Lambda)$ be a summand of the motive of a smooth complete irreducible variety. Then M is called *upper* if it satisfies the four equivalent conditions of the Lemma 2.3.1.

Remark 2.3.2. (Remark 2.13 in [Kar13]) Assume that the coefficient ring Λ is finite. Let X be an irreducible geometrically split variety satisfying the Rost nilpotence principle.

Then the complete motivic decomposition of X contains precisely one upper summand and therefore by Theorem 2.2.1 an upper indecomposable summand of $\mathcal{M}(X)$ is unique upto an isomorphism.

We mention the following results from [Kar13] that will be used later in the proofs.

Theorem 2.3.3. *(Lemma 2.14 in [Kar13]) Assume that the coefficient ring Λ is finite. Let X be an irreducible geometrically split variety satisfying the nilpotence principle. Let M be a motive. Assume that there exist morphisms $\alpha : M(X) \to M$ and $\beta : M \to M(X)$ such that $mult(\beta \circ \alpha) = 1$. Then the indecomposable upper summand of $M(X)$ is isomorphic to a summand of M.*

In the same paper [Kar13], Karpenko gives a necessary and sufficient condition for the upper indecomposable motives of two varieties to be isomorphic.

Theorem 2.3.4. *(Corollary 2.15 in [Kar13]) Let X and Y be irreducible geometrically split varieties satisfying the Rost Nilpotence. The upper indecomposable summands of $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are isomorphic if and only if there exist multiplicity 1 correspondences $\alpha : \mathcal{M}(X) \to \mathcal{M}(Y)$ and $\beta : \mathcal{M}(Y) \to \mathcal{M}(X)$.*

2.4 Useful Techniques in Motivic Decompositions

A very useful technique to decompose a motive is due to Rost ([Ros]) and Karpenko ([Kar00a]). We state this below for convenience of the reader.

Theorem 2.4.1. *([CGM05], [dB01], [Kar00a]) Let X be a smooth, projective variety* *over a field k with a filtration*

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \varnothing$$

where the X_i are closed subvarieties. Assume that, for each integer $i \in [0, n]$, there is a smooth projective variety Z_i and an affine fibration $\phi_i : X_i - X_{i-1} \to Z_i$ of relative dimension a_i. Then, in the category of correspondences,

$$\mathcal{M}(X) = \coprod_{i=0}^{n} \mathcal{M}(Z_i)\{a_i\}$$

A situation where the above theorem can be applied is when X is a smooth projective variety with a $\mathbb{G}_m$-action. The following result is due to Iversen ([Ive72]), Biyałnicki-Birula ([BB73], [BB76]) and Hesselink ([Hes81]). See Theorem 3.3 and Theorem 3.4 in [Bro05] for more details.

Theorem 2.4.2. *([BB73], [BB76], [Hes81], [Ive72]) Let X be a smooth projective scheme over k equipped with an action of $\mathbb{G}_m$. Then,*

$$\mathcal{M}(X) = \coprod_{i} \mathcal{M}(Z_i)\{a_i\}$$

where Z_i are connected components of $X^{\mathbb{G}_m}$ and a_i are dimensions of the positive eigenspace of the action of $\mathbb{G}_m$ on the tangent space of X at an arbitrary point in Z_i.

2.5 Motivic Decomposition of Projective Homogeneous Varieties

2.5.1 The Case When G is Isotropic

In [Bro05], Brosnan gave a description about the summands of the motive of projective G-homogeneous varieties for isotropic G. Recall from [Bro05] that a G-scheme X is a *projective quasi-homogeneous scheme* if X is smooth and projective over k and the morphism $G \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is smooth. By Proposition 4.1 in [Bro05], this is equivalent to saying that X_K is a disjoint union of projective homogeneous varieties.

Theorem 2.5.1. *(Corollary 4.1 in [Bro05]) Let X be a projective quasi-homogeneous scheme for an isotropic reductive group G, and let $\lambda : \mathbb{G}_m \to G$ be an embedding of a split torus. Then*

$$\mathcal{M}(X) = \coprod \mathcal{M}(Z_i)\{a_i\}$$

where Z_i are connected components of X^λ. Moreover, Z_i are projective quasi-homogeneous schemes for the centralizer H of λ and the twists a_i are the dimensions of the positive eigenspace of the action of λ on the tangent space of X at an arbitrary point $z \in Z_i$.

2.5.2 The Case When G is Inner

Let G be of inner type over k. Then Karpenko in his paper [Kar13] shows that the complete motivic decomposition of any projective G-homogeneous variety consists of shifts of upper indecomposable motives of other projective G-homogeneous varieties. In other words, the upper indecomposable motives of projective homogeneous varieties are the basic building blocks in the motivic decomposition. Recall from §0.3 that if X_τ is projective homogeneous corresponding to the subset $\tau \subseteq \Delta_G$, U_τ denotes the upper indecomposable summand of $\mathcal{M}(X_\tau)$.

Theorem 2.5.2. *(Theorem 3.5 in [Kar13]) Let X be a projective G-homogeneous variety.*

Then any indecomposable summand of $\mathcal{M}(X)$ is isomorphic to $U_\tau\{i\}$ for some i and some $\tau \subseteq \Delta_G$ satisfying $\tau_{k(X)} \subseteq \tau$.

Theorem 2.5.2 is very useful for motivic decomposition of projective G-homogeneous varieties and has a lot of applications. We refer the reader to [Kar13] and [Zhy12] for some of the applications and examples.

2.5.3 The Case When G is Outer

Let G be of outer type over a field F. Assuming that G becomes inner type over some finite field extension of F of degree a power of a prime p, Karpenko in his paper [Kar10] describes the structure of the Chow motives with coefficients in a finite field of characteristic p of projective G-homogeneous varieties. Any indecomposable direct summand of such a variety is given by a twist of an *upper motive of* G which we define as follows.

Assume that the coefficient Λ associated to the category of Chow motives $Chow(F, \Lambda)$ is a finite field of characteristic p. Let E/F be the minimal field extension (upto F-isomorphism) such that the group G_E is of inner type where the degree of E/F is assumed to be a power of p. For any intermediate field L of the extension E/F let Y be a projective G_L-homogeneous variety. Then we can think of Y as an F-variety via the composition $Y \to Spec\ L \to Spec\ F$. Let U_Y denote the upper motive (see Definition 3) of Y in $Chow(F, \Lambda)$, considered as F-variety. The set of the isomorphism classes of the motives U_Y for all such L and Y is called the set of *upper motives of the algebraic group* G. We will now state Karpenko's motivic decomposition theorem for a projective G-homogeneous variety X.

Theorem 2.5.3. *(Theorem 1.1 in [Kar10]) For F, G, E, and X as earlier, the complete motivic decomposition of X consists of shifts of upper motives of the algebraic group G. More precisely, any indecomposable summand of the motive of X is isomorphic a shift of an upper motive U_Y of G such that the Tits index of G over the function field of the variety Y contains the Tits index of G over the function field of X.*

Remark 2.5.4. The above theorem fails without the hypothesis that the extension E/F is p-primary where p is the characteristic of the coefficient field Λ. See Example 3.3 in [Kar10] for a counterexample.

Chapter 3: Variety of Unseparated Flags (VUFs)

Throughout this chapter k denotes an algebraically closed field of characteristic $p > 0$.

3.1 Parabolic subgroup schemes

Let G denote a reductive linear algebraic group over k. By the term *parabolic subgroup* of G, we mean a *smooth* subgroup of G that contains a Borel subgroup. By definition, these are reduced subgroup schemes of G. When the characteristic of the base field is zero, every subgroup scheme of G is smooth. But for characteristic $p > 0$, we encounter subgroup schemes that are not necessarily reduced.

Example. Suppose $G = SL_{3,k}$. Let

$$\widetilde{P} = \{ \begin{pmatrix} * & * & * \\ x & * & * \\ y & z & * \end{pmatrix} | x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0 \}$$

One can easily verify that $\widetilde{P}$ is a subgroup scheme of G. The underlying reduced scheme of P is the standard Borel $P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$

Definition 4. A subgroup scheme $\widetilde{P}$ of G is said to be a *parabolic subgroup scheme* if it contains a Borel subgroup.

Thus by the above definition, parabolic subgroup schemes need not necessarily be smooth unlike parabolic subgroups. Although the classification of parabolic subgroups of G is well known in the standard literature ([Bor91], [Spr09]), nothing much was known about parabolic subgroup schemes until Wenzel gave a classification in his paper [Wen93a] for arbitrary reductive linear algebraic groups over algebraically closed fields of characteristic $p > 3$.

To state the classification theorem, we first establish some notations as in [Wen93a]. Let G_a denote the 1-dimensional additive linear algebraic group $Spec(K[T])$. For each $n \in \mathbb{N}_0$, let α_n be the subscheme of G_a defined by T^{p^n}. Set $\alpha_{p^\infty} = G_a$. Let B be a Borel subgroup and let U be the unipotent part of B. Let $\phi^+ = \{\beta_1, \beta_2, \cdots \beta_m\}$ be the set of positive roots and let Δ be the set of simple roots. Then there exists morphisms of algebraic groups $x_{\beta_i} : G_a \to U, i \in \{1, 2, \cdots, m\}$ such that

$$G_a^m \to U$$

$$(\xi_1, \cdots, \xi_m) \mapsto \prod x_{\beta_i}(\xi_i)$$

is an isomorphism of varieties. Let $\widetilde{\Delta}$ be the set of maps from Δ to $\mathbb{N}_0 \cup \{\infty\}$ where $\mathbb{N}_0$ is the set of non-negative integers. We will now state the classification theorem.

Theorem 3.1.1. *(Theorem 14 in [Wen93a]) Let G be a reductive linear algebraic group defined over k. There is an injective map from $\widetilde{\Delta}$ to $\mathfrak{B}$, the set of all parabolic subgroup schemes containing B, given by*

$$\widetilde{\Delta} \to \mathfrak{B}$$

$$\phi \mapsto P_\phi$$

where $P_\phi = U_\phi \cdot P_{I(\phi)}$, $I(\phi) = \{\alpha \in \Delta | \phi(\alpha) = \infty\}$, $U_\phi = \prod_{\beta \in \phi^+ - \phi_I} x_{-\beta}(\alpha_{\phi(\beta)})$, ϕ being extended to all of ϕ^+ by $\phi(\beta) = min\{\phi(\gamma) | \gamma \in E(\beta)\}$, $E(\beta) = \{\beta_i \in \Delta | \beta = \sum c_j \cdot$

β_j, with all $c_j \geq 0$ and $c_i \neq 0\}$, ϕ_I the roots generated by $I = I(\phi)$.

If $char\ k > 3$, or if G is simply laced , then this map is also surjective.

Remark 3.1.2. (Remark 15 in [Wen93a]) The map in Theorem 3.1.1 is not surjective in $char(k) = 2, 3$ for certain G; for example for $G = SO_5$ in $char\ k = 2$ and for G with the root system of type G_2 in $char\ k = 3$.

3.2 Construction and Classification of VUFs

In this section we use the following notation. For a parabolic subgroup scheme $\widetilde{P}$ of G, we denote its underlying reduced scheme by P. Recall that a flag variety is a projective homogeneous G- space isomorphic to G/P where P is a (smooth) parabolic subgroup.

Definition 5. We call a projective homogeneous G-space, *variety of unseparated flags* (VUFs in short) if it does not admit an isomorphism to a flag variety.

Lauritzen and Haboush answered many interesting questions about the geometry of these varieties including canonical line bundles, vanishing theorems and Frobenius splitting in [Lau97], [HL93] and [Lau93]. Lauritzen also gave a geometric construction of $G/\widetilde{P}$ in [Lau96] where he realizes these varieties as the G-orbit of a Borel stable line in projective space. Note that since VUFs are projective varieties that are G-homogeneous, they are essentially isomorphic to $G/\widetilde{P}$ where $\widetilde{P}$ is a parabolic subgroup scheme that is not necessarily smooth (or equivalently not reduced).

Example. Suppose $G = SL_{3,k}$. Let $X \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ be given by the equation $\sum_{i=0}^{2} x_i^p y_i = 0$. Then X is a VUF isomorphic to $G/\widetilde{P}$ for a non-reduced parabolic subgroup scheme $\widetilde{P}$.

The construction of VUFs follows from the construction of quotients $G/\widetilde{P}$. For a self contained, complete treatment of construction of quotient of an algebraic group by a subgroup scheme, we refer Jantzen's book (§I.5 in [Jan03]). Note that at geometric level $G/\widetilde{P}$ and G/P have the same underlying topological space with the structure sheaf $\mathcal{O}_{G/\widetilde{P}}$ consisting of $\widetilde{P}$-invariant functions in $\mathcal{O}_G$. Therefore, $\mathcal{O}_{G/\widetilde{P}}$ is a subsheaf of $\mathcal{O}_{G/P}$ and the injection $\mathcal{O}_{G/\widetilde{P}} \hookrightarrow \mathcal{O}_{G/P}$ corresponds to the canonical morphism $G/P \to G/\widetilde{P}$.

Definition 6. Let X be a scheme over k. We define the scheme $X^{[n]}$ as the one with same underlying topological space as X but with k-structure twisted via the ring homomorphism

$$\begin{aligned} k &\to k \\ a &\mapsto \sqrt[p^n]{a} \end{aligned}$$

With the notations as above, the n-th order Frobenius induces a morphism of k-schemes $F^n : X \to X^{[n]}$ where F is the Frobenius morphism. We call $X^{[n]}$, the *n-th Frobenius cover* of X. Note that when X is reduced, $\mathcal{O}_{X^{[n]}}$ can be identified with the k-subalgebra of p^n-th powers of regular functions on X.

Now $G^{[n]}$ is an algebraic group as the same type as G and $F^n : G \to G^{[n]}$ is a homomorphism of algebraic groups. Its kernel denoted by G_n is the *n-th Frobenius kernel* of G. It is easy to see from the above discussion that if $\widetilde{P} = G_nP$, then $G/\widetilde{P}$ is the n-th Frobenius cover of G/P.

Consider all varieties of the form $G/\widetilde{P}$, where $\widetilde{P}$ is a parabolic subgroup scheme that may or may not be reduced. De Salas in his paper [SdS03] calls these varieties as *parabolic G-varieties* or simply *parabolic varieties* if the underlying algebraic group is clear. He also gives a classification of parabolic varieties in [SdS03] for $p = char\ k > 3$.

We will briefly recall them here. The classification is based on the following results:

- The classification of the parabolic subgroups given by Wenzel in [Wen93a] (in characteristic different from 2 and 3).

- The determination given by Demazure in [Dem77] of the pairs $P \subset G$, where G is a simple group of adjoint type and $P \subset G$ is a reduced parabolic subgroup such that $G = Aut^0(G/P)$. Here $Aut^0(G/P)$ is the identity component of automorphism group of G/P. The pairs satisfying this condition are called *non-exceptional*. Demazure shows that the only exceptional pairs are the following ones:

  - $G = SO_{2l+1}(k)$ and G/P the variety that parameterizes the totally isotropic subspaces $V_l \subset k_{2l+1}$ (with $2l+1 \geq 5$). In this case $Aut^0(G/P) \simeq PSO_{2l+2}$.

  - $G = Sp_{2l}(k)$ and $G/P = \mathbb{P}^{2l-1}$ the variety that parameterizes the lines of k^{2l}. In this case $Aut^0(\mathbb{P}^{2l-1}) \simeq PGl_{2l}(k)$.

  - G is the simple group of adjoint type with semi-simple rank 2 and type G_2; that is, it is the group of automorphisms of an algebra of octonions Ω. Let $\widetilde{\Omega} \subset \Omega$ be the hyperplane of the pure octonions and G/P the variety of isotropic lines of $\widetilde{\Omega}$. Then G/P is isomorphic to a projective quadric of dimension 5, and hence, $Aut^0(G/P) = PSO_6(k)$.

Let us fix a Borel B in G. Let $P_1, P_2, \cdots P_s$ be the maximal (reduced) parabolic subgroups containing B and let $\mathcal{P}_1 = G/P_1, \mathcal{P}_2 = G/P_2, \cdots, \mathcal{P}_s = G/P_s$ denote the associated parabolic varieties. Similarly, $\mathcal{B} = G/B$ denotes the associated variety of Borels.

Given two parabolic G-varieties $\mathcal{P}$and $\mathcal{P}'$ together with the canonical morphisms

$\pi : \mathcal{B} \to \mathcal{P}$ and $\pi' : \mathcal{B} \to \mathcal{P}'$, let $\mathcal{P} * \mathcal{P}'$ denote the parabolic G-variety given by the image of the G-morphism $\pi \times \pi' : \mathcal{B} \to \mathcal{P} \times \mathcal{P}'$. Note that if $\mathcal{P} = G/\widetilde{P}$ and $\mathcal{P}' = G/\widetilde{P}'$ with $\widetilde{P}, \widetilde{P}' \subset G$ parabolic subgroups schemes containing B, then $\mathcal{P} * \mathcal{P}' = G/\widetilde{P} \cap \widetilde{P}'$.

Recall that for a scheme X and a non-negative integer $n \in \mathbb{N}_0$, we have a natural morphism over k given by $F^n : X \to X^{[n]}$ where F is the Frobenius. We now provide the classification of parabolic varieties from [SdS03].

Theorem 3.2.1. *(Theorem 6.8 in [SdS03]) Assume that $p = char\ k > 3$. For each parabolic G-variety $\mathcal{P}$, there exist unique indices $1 \le i_1 \le \cdots \le i_r \le s$ and exponents $n_1, \cdots, n_r \in \mathbb{N}_0$ such that*

$$\mathcal{P} = \mathcal{P}_{i_1}^{[n_1]} * \cdots * \mathcal{P}_{i_s}^{[n_r]}$$

Moreover, $Aut^0(\mathcal{P}) = G$ if and only if $n_h = 0$ for some $1 \le h \le r$ and $\mathcal{P}_{i_j}$ is non-exceptional for some $1 \le j \le r$. That is, $G \to Aut^0(\mathcal{P})$ is not an isomorphism if and only if either $n_1, \cdots, n_r > 0$ or P is maximal and $P \subset G$ is an exceptional pair.

3.3 Flag varieties vs VUFs

Let X be a parabolic variety, i.e, $X \simeq G/\widetilde{P}$ where $\widetilde{P}$ is a parabolic subgroup scheme that may or may not be reduced. A natural question that arises in this case is when $G/\widetilde{P}$ is isomorphic to a flag variety. This is answered by Lauritzen in his paper [Lau93] using a property call *Frobenius splitting* which we now recall.

Definition 7. We call a scheme Y, (X, f)-split, where $f : X \to Y$ is a finite morphism, if $\mathcal{O}_Y \to f_*\mathcal{O}_X$ splits as a morphism of $\mathcal{O}_Y$-modules, that is, if there is a morphism $f_*\mathcal{O}_X \to \mathcal{O}_Y$ such that $\mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{O}_Y$ is the identity morphism.

Theorem 3.3.1. *(Theorem 5.2 in [Lau93]) Let G be an algebraic group of simple type of Coxeter number h over an algebraically closed field k of characteristic $p > 0$. Suppose $G/\widetilde{P}$ is a complete homogeneous G-space and let P be the reduced part of $\widetilde{P}$. If $p > h$ then the following conditions are equivalent*

(1) $G/\widetilde{P}$ is Frobenius split

(2) $G/\widetilde{P}$ is $(G/P, \pi)$-split, where $\pi : G/P \to G/\widetilde{P}$ is the canonical map

(3) $G/\widetilde{P}$ is a Frobenius cover of G/P

Since G/P can be obtained by base change of a $\mathbb{Z}$-scheme, G/P is isomorphic to all its Frobenius covers. Therefore (3) states that $G/\widetilde{P}$ is isomorphic to G/P as varieties (but not as G-spaces).

VUFs have rich structure and behave quite differently from the *generalized flag varieties* G/P where P is smooth. One can find explicit examples of VUFs in [HL93] which illustrate that unlike generalized flag varieties, vanishing theorem for ample line bundles and Kodaira's vanishing theorem break down.

Chapter 4: Motives of Projective Pseudo-Homogeneous Varieties- I

For the rest of this thesis, we assume that G is a semi-simple algebraic group of inner type over a perfect field k of characteristic $p > 3$ (See Remark 4.3.4 for why the assumption $p > 3$ is necessary). For a variety Y over k and an extension $k' \supseteq k$, we write $Y_{k'}$ for $Y \times_{Spec\ k} Spec\ k'$. As before K denotes the algebraic closure of k. We start by recalling §0.1 here for convenience of the reader.

4.1 Projective Pseudo- Homogeneous Varieties

Definition 8. A G-variety $\widetilde{X}$ over k is called a *projective pseudo-homogeneous variety* if $\widetilde{X}_K \simeq G_K/\widetilde{P}$ for some parabolic subgroup scheme $\widetilde{P}$ in G_K that is not necessarily reduced.

Such a variety is always smooth since G is smooth (See SGA3, exp $\mathrm{VI_A}$, Theorem 3.2). For detailed construction of the quotient of an algebraic group by a subgroup see Chapter III, §3 of [DG70]. Note that by Proposition 2.1, §3, Chapter III of [DG70], the condition $\widetilde{X}_K \simeq G_K/\widetilde{P}$ is equivalent to saying that the action map $G(\Omega) \times \widetilde{X}(\Omega) \to \widetilde{X}(\Omega) \times \widetilde{X}(\Omega)$ is surjective for every algebraically closed field Ω over K. If $\widetilde{P}$ is a parabolic subgroup scheme over K, we will make slight abuse of notation and write $G/\widetilde{P}$ for $G_K/\widetilde{P}$. As before P denotes the underlying reduced scheme of $\widetilde{P}$. Note that since k

is perfect, P is a group scheme (See §6 in Chapter VI of [Mil]).

4.2 Projective Homogeneous Variety Corresponding to a Projective Pseudo-Homogeneous Variety

Definition 9. Given $\widetilde{X}$, a projective pseudo-homogeneous variety for G such that $\widetilde{X}_K \simeq G/\widetilde{P}$, let X denote the unique (see Proposition 1.3 in [MPW96]) projective homogeneous variety for G, such that $X_K \simeq G/P$ where P is the underlying reduced subscheme of $\widetilde{P}$. We call X *the projective homogeneous variety corresponding to* $\widetilde{X}$.

By universal property of quotients, there is a canonical G-equivariant finite morphism $\theta : X \to \widetilde{X}$.

Example. Suppose $G = SL_{3,k}$. Let $G/\widetilde{P} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ be given by the equation $\sum_{i=0}^{2} x_i^p y_i = 0$ where the G action is $g.\overrightarrow{x} = g^{p^3}\overrightarrow{x}$ and $g.\overrightarrow{y} = (g^{-t})^{p^4}\overrightarrow{y}$ (Here $g^{-t} = (g^{-1})^t$ is the transpose of the inverse of g. Also by abuse of notation g^{p^n} means taking p^nth power of entries of the matrix g). Then $\widetilde{P} = Stab([1:0:0] \times [0:0:1]) = \{\left(\begin{smallmatrix} * & * & * \\ x & * & * \\ y & z & * \end{smallmatrix}\right) | x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0\}$. The underlying reduced scheme is the standard Borel $P = \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$ and the corresponding homogeneous variety $G/P \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ is given by $\sum_{i=0}^{2} x_i y_i = 0$. This comes with the standard G-action $g.\overrightarrow{x} = g\overrightarrow{x}$ and $g.\overrightarrow{y} = (g^{-t})\overrightarrow{y}$. We have the canonical G-equivariant map

$$G/P \to G/\widetilde{P}$$

$$\overrightarrow{x} \mapsto \overrightarrow{x}^{p^3}$$

$$\overrightarrow{y} \mapsto \overrightarrow{y}^{p^4}$$

We want to emphasize that by Theorem 3.3.1, the K-varieties $G/\widetilde{P}$ and G/P are not in general isomorphic. Therefore, X and $\widetilde{X}$ need not be twisted forms of each other.

4.3 A Motivic Decomposition Theorem for Isotropic G

In this section we assume that G is an isotropic, semi-simple group of inner type over k. We fix an embedding $\lambda : \mathbb{G}_m \to G$ of a k-split torus. Let H denote the centralizer of λ in G. Then by Theorem 6.4.7 in [Spr09], H is connected and reductive. It is defined over k by Proposition 13.3.1 of [Spr09]. Recall that if $X_K \simeq G/P$ and $\widetilde{X}_K \simeq G/\widetilde{P}$, we have a canonical G-equivariant finite morphism $\theta : X \to \widetilde{X}$.

Recall from [Ive72] that for a smooth projective variety X equipped with an action of $\mathbb{G}_m$, the fixed point locus $X^{\mathbb{G}_m}$ is a smooth closed subscheme of X.

Proposition 4.3.1. *Let X and Y be smooth projective varieties equipped with an action of $\mathbb{G}_m$. Let $\theta : X \to Y$ be a finite surjective $\mathbb{G}_m$-equivariant morphism. Then the restriction morphism $\theta|_{X^{\mathbb{G}_m}} : X^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$ is surjective.*

Proof. Pick a point $y \in Y^{\mathbb{G}_m}$. Clearly $\mathbb{G}_m$ acts on the fiber $X_y = X \times_Y \mathit{Spec}\ k(y)$. Since θ is finite, X_y is finite. Therefore $\mathbb{G}_m$ fixes the underlying reduced subschemes of each point in X_y. □

A morphism $X \to Y$ of finite type is surjective if and only if the induced map $X(\Omega) \to Y(\Omega)$ is surjective for every algebraically closed field Ω (EGA IV, Chapter 1, §6, Proposition 6.3.10). Using this we get an easy corollary of the above proposition.

Corollary 4.3.2. *With notations as in Proposition 4.3.1, let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^m$ denote the connected components of $X^{\mathbb{G}_m}$ and $Y^{\mathbb{G}_m}$ respectively. Suppose $\theta : X(\Omega) \to Y(\Omega)$ is* *bijective for every algebraically closed field Ω. Then $n = m$ and after permuting indices, $\theta|_{X_i} : X_i(\Omega) \to Y_i(\Omega)$ is also bijective.*

Theorem 4.3.3. *Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for G and let X be the corresponding projective homogeneous variety. Then each connected component of the fixed point locus $\widetilde{X}^\lambda$ is projective pseudo-homogeneous for H. Moreover if $\widetilde{X}^\lambda = \coprod \widetilde{Z}_i$, then $X^\lambda = \coprod Z_i$ where Z_i are the projective H-homogeneous varieties corresponding to $\widetilde{Z}_i$*

Proof. First note that H acts on $\widetilde{X}^\lambda$ because $\lambda(t) \cdot h \cdot x = h \cdot \lambda(t) \cdot x = h \cdot x \ \forall h \in H, t \in \mathbb{G}_m, x \in \widetilde{X}^\lambda$. Let Y be a connected component of $\widetilde{X}^\lambda$. It suffices to show that the action map $H \times Y \to Y \times Y$ is surjective on Ω-points for every algebraically closed field Ω over K. By III, §1, 1.15 of [DG70], the G-equivariant morphism $\theta(\Omega) : X(\Omega) \to \widetilde{X}(\Omega)$ is bijective. Therefore, by Corollary 4.3.2, $X^\lambda(\Omega) \to \widetilde{X}^\lambda(\Omega)$ is also bijective. So there exists a connected component Z of X^λ such that $\theta : Z(\Omega) \to Y(\Omega)$ is a bijection. By Theorem 7.1 in [Bro05], Z is projective homogeneous for H. Therefore the action map $H \times Z \to Z \times Z$ is surjective on Ω-points. We have the following commutative diagram:

$$\begin{tikzcd}
H \times Z \arrow[r] \arrow[d, "{(id, \theta)}"'] & Z \times Z \arrow[d, "{(\theta, \theta)}"] \\
H \times Y \arrow[r] & Y \times Y
\end{tikzcd}$$

The morphisms given by the top arrow and (θ, θ) are surjective on Ω-points as argued before. Hence we conclude that the bottom arrow is surjective on Ω-points. This proves that each Y is projective pseudo-homogeneous for H.

For the second part of the claim note that if $x \in Z(K)$, then $Stab_H(x) \subseteq Stab_H(\theta(x))$. This together with the bijectivity of $\theta : Z(K) \to Y(K)$ shows that Z is the projective homogeneous variety corresponding to Y. □

We now analyze the action of λ on the tangent space at any point in the fixed point locus $\widetilde{X}^{\lambda}$. As before $X_K \simeq G/P$ and $\widetilde{X}_K \simeq G/\widetilde{P}$. Let $b \in (G/P)^{\lambda}$. Let $a \in G/P$ be the unique point whose stabilizer in G_K is P and let $b = g \cdot a$ for some $g \in G(K)$. Then $g^{-1}\lambda g \subseteq T \subseteq P$ for some maximal torus T. Let $T' = gTg^{-1}$. Let $\beta_1, \beta_2, \cdots, \beta_n$ be the negative roots of G_K with respect to T and a Borel B such that $T \subseteq B \subseteq P$. Recall from Theorem 6 in [HL93] that to every parabolic subscheme, one can associate a W-function defined as follows.

Definition 10. (Definition 5 in [HL93]) Write $\mathbb{N}^*$ to signify the set of non-negative integers together with ∞. Let ϕ^+ denote the set of positive roots of G. A *W-function* on ϕ^+ is a function, f, on ϕ^+ with values in $\mathbb{N}^*$ satisfying the condition,

$$f(\beta) = \inf_{\alpha \in supp(\beta)} f(\alpha)$$

where $supp(\beta) = \{\gamma \in \phi^+ | \beta = \gamma + \delta, \text{ for some } \delta \in \phi^+\}$.

Remark 4.3.4. In order to associate a W-function to a parabolic subgroup scheme as in Theorem 6 in [HL93], the authors of the paper assume that $char\ K > 3$. This assumption is necessary by Remark 3.1.2.

Let f be the W-function associated to $\widetilde{P}$ and let $n_i = f(-\beta_i)$. Without loss of generality, assume that $\beta_1, \beta_2, \cdots, \beta_m$ are the negative roots such that $f(-\beta_i) < \infty$.

Lemma 4.3.5. *With the notations above, there exists a T'-stable affine open neighborhood of $\theta(b)$ in $(G/\widetilde{P})^\lambda$ parametrized by T' - eigen functions with weights $p^{n_i}\alpha_i$ where α_i are characters of T'. In other words, one can find an open set $V = Spec\ K[X_1, X_2, \cdots, X_m]$ containing $\theta(b)$ such that*

$$t' \cdot X_i = \alpha_i^{p^{n_i}}(t')\ X_i \qquad \forall t' \in T'$$

Proof. Let U_P^0 denote the opposite of the unipotent radical of P. By Theorem 1 in [HL93], $U = U_P^0 \cdot \theta(a) = Spec\ K[Y_1, Y_2, \cdots, Y_m]$ is an affine open neighborhood of $\theta(a)$ invariant under T, where

$$t \cdot Y_i = \beta_i^{p^{n_i}}(t)\ Y_i \qquad \forall t \in T$$

Consider the affine open neighborhood $V = gU_P^0 \cdot \theta(a)$ of $\theta(b)$. Then

$$T' \cdot V = T' g U_P^0 \cdot \theta(a) = gTU_P^0 \cdot \theta(a) = gU_P^0 \cdot \theta(a) = V$$

So V is T'-invariant. Moreover $V = Spec\ K[X_1, X_2, \cdots, X_m]$ where $X_i = g^{-1} \cdot Y_i$. Let α_i be the character of T' defined by $\alpha_i(t') = \beta_i(g^{-1}t'g)\ \forall t' \in T'$. For any point $x \in V$, write $x = gy$ where $y \in U$. Then

$$\begin{aligned} t' \cdot X_i(x) &= t' \cdot (g^{-1} \cdot Y_i)(gy) = Y_i(g^{-1}t'gy) \\ &= \beta_i^{p^{n_i}}(g^{-1}t'g) Y_i(y) = \alpha_i^{p^{n_i}}(t')\ X_i(x) \qquad \forall t' \in T' \end{aligned}$$

□

Lemma 4.3.6. *For any point $b \in X^\lambda$, the dimension of positive eigenspaces of the λ-action on the tangent spaces at b and $\theta(b)$ are equal.*

Proof. It suffices to prove the lemma over the algebraic closure K where $X_K \simeq G/P$ and $\widetilde{X}_K \simeq G/\widetilde{P}$. So assume that $k = K$. By Lemma 4.3.5, there exists an affine open cover $U = Spec\, K[Y_1, Y_2, \cdots, Y_m]$ of b and an affine open cover $V = Spec\, K[X_1, X_2, \cdots, X_m]$ of $\theta(b)$ parametrized by λ-eigen functions with weights $\{\alpha_i\}$ and $\{p^{n_i}\alpha_i\}$ respectively. Let $\overline{Y_i} \in \mathfrak{m}_b/\mathfrak{m}_b^2$ and $\overline{X_i} \in \mathfrak{m}_{\theta(b)}/\mathfrak{m}_{\theta(b)}^2$ denote the cosets of Y_i and X_i respectively. Note that $\{\overline{Y_i}\}$ and $\{\overline{X_i}\}$ form a basis for $\mathfrak{m}_b/\mathfrak{m}_b^2$ and $\mathfrak{m}_{\theta(b)}/\mathfrak{m}_{\theta(b)}^2$respectively . It is now easy to see that the span of $\overline{Y_i}$ is a positive eigenspace for λ if and only if the span of $\overline{X_i}$ is so. By taking the dual, we are done.

□

By Theorem 2.4.2, Theorem 4.3.3 and Lemma 4.3.6, we get the following motivic decomposition for $\widetilde{X}$.

Corollary 4.3.7. *With the notations as in Theorem 4.3.3,*

$$\mathcal{M}(\widetilde{X}) = \coprod_i \mathcal{M}(\widetilde{Z}_i)\{a_i\}$$

and

$$\mathcal{M}(X) = \coprod_i \mathcal{M}(Z_i)\{a_i\}$$

where $\widetilde{Z}_i$ is projective pseudo-homogeneous for H and Z_i is the corresponding projective homogeneous variety. The twists a_i are dimensions of the positive eigenspace of the action of λ on the tangent space of X at an arbitrary point $z \in Z_i$.

Applying the above result inductively, we see that each of the components in the decomposition are projective (pseudo-) homogeneous for the centralizer $Z(S)$ of a maximal k-split torus S. By Proposition 2.2 in [BT72], we have an almost direct product decomposition $Z(S) = DZ(S) \cdot Z$ where Z is the center of $Z(S)$ and $DZ(S)$ is the semi-simple anisotropic kernel. Since the center of a group is contained in every parabolic subscheme, it acts trivially on any projective pseudo-homogeneous variety. Hence, each of the $\widetilde{Z}_i$ (respectively Z_i) are projective pseudo-homogeneous (respectively homogeneous) for the adjoint group of the semi-simple anisotropic kernel. Therefore we conclude:

Corollary 4.3.8. *With the notations as in Theorem 4.3.3,*

$$\mathcal{M}(\widetilde{X}) = \coprod_i \mathcal{M}(\widetilde{Z}_i)\{a_i\}$$

and

$$\mathcal{M}(X) = \coprod_i \mathcal{M}(Z_i)\{a_i\}$$

where each $\widetilde{Z}_i$ (respectively Z_i) is either $Spec\, k$ or anisotropic projective pseudo-homogeneous (respectively homogeneous) variety for the semi-simple anisotropic kernel of G.

Proof. From Corollary 4.3.7, each $\widetilde{Z}_i$ is projective pseudo-homogeneous variety for H. Let $(\widetilde{Z}_i)_K \simeq H/\widetilde{Q}$, for a parabolic subgroup scheme $\widetilde{Q}$ of H_K. If $\widetilde{Z}_i$ is anisotropic we are done. Suppose $\widetilde{Z}_i$ is isotropic, i.e., $\widetilde{Z}_i$ has a k-point. Then its stabilizer is defined over k by Proposition 12.1.2 in [Spr09]. Without loss of generality we can assume that $\widetilde{Q}$ is defined over k. Since k is perfect, the underlying reduced scheme Q is also defined over k and hence is isomorphic to $Q(\lambda)$ for some co-character λ of H defined over k (Lemma 15.1.2 in [Spr09]). So H is isotropic. If λ is a central torus, $Q(\lambda) = H$ and $\widetilde{Z}_i \simeq Spec\, k$. If λ is non-central, then we can inductively use Corollary 4.3.7 to get the result. □

4.4 Motivic Decomposition when G is split

In this section we assume that G is split, so that $\widetilde{X} \simeq G/\widetilde{P}$ and $X \simeq G/P$. The goal of this section is to understand the cellular structure of $G/\widetilde{P}$ and compute its motive.

Lemma 4.4.1. *$\widetilde{X}$ is a cellular variety i.e., it has decomposition into affine cells. Moreover, the affine cells can be obtained by the image of the Schubert cells in G/P under $\theta : G/P \to G/\widetilde{P}$.*

Proof. We follow the proof of §2.2 in [Lau97]. We know that $X = G/P$ is cellular because G/P is a disjoint union of Schubert cells $C(w) = UwP/P$ where U is the unipotent radical of B. Let $X(w) = \overline{C(w)}$ be the corresponding Schubert variety. Let $\widetilde{X}(w)$ be the scheme theoretic image of $X(w)$ in $\widetilde{X} = G/\widetilde{P}$ under the canonical map $\theta : G/P \to G/\widetilde{P}$. Call it a Schubert variety in $\widetilde{X}$. We get a filtration $\widetilde{X} = \widetilde{X}_0 \supseteq \widetilde{X}_1 \supseteq \widetilde{X}_2 \supseteq \dots$ where $\widetilde{X}_i$ is the union of codimension i Schubert varieties in $\widetilde{X}$ and $\widetilde{X}_i - \widetilde{X}_{i+1} = \coprod \theta(C(w))$. Here $\theta(C(w))$ are disjoint because θ is bijective. Moreover θ is U-equivariant and U acts transitively on $\theta(C(w))$. Therefore by IV.3.16 in [DG70], $\theta(C(w))$ is affine. So $\widetilde{X}$ is a disjoint union of affine cells $\theta(C(w))$. □

Lemma 4.4.2. *With the notations in the proof of Lemma 4.4.1, the classes of Schubert varieties $[\widetilde{X}(w)]$ form a basis for the Chow group of $G/\widetilde{P}$. As a consequence $Ch_i(G/\widetilde{P}) \simeq Ch_i(G/P)$.*

Proof. By Example 1.9.1 in [Ful98], it is clear that the classes of Schubert varieties $[\widetilde{X}(w)]$ form a basis for $Ch_*(G/\widetilde{P})$ and we get an isomorphism

$$Ch_*(G/P) \to Ch_*(G/\widetilde{P})$$

$$[X(w)] \mapsto [\widetilde{X}(w)]$$

□

Theorem 4.4.3. *The motive $\mathcal{M}(\widetilde{X})$ is split i.e., it decomposes into direct sum of Tate motives. Moreover, $\mathcal{M}(X) \simeq \mathcal{M}(\widetilde{X})$.*

Proof. This follows directly from Corollary 4.3.8. Alternatively, one can also argue as follows. The fact that $\mathcal{M}(\widetilde{X})$ splits into Tate motives follows by Lemma 4.4.1, and Theorem 2.4.1. Now observe that for any variety whose motive splits into Tate motives, the rank of the i^{th} Chow group is equal to the number of summands isomorphic to $\Lambda\{i\}$. Therefore by Lemma 4.4.2, $\mathcal{M}(X) \simeq \mathcal{M}(\widetilde{X})$. □

4.5 Rost Nilpotence and Krull- Schmidt for Pseudo-Homogeneous Varieties

In this section we prove that Rost nilpotence and Krull-Schmidt holds for projective pseudo-homogeneous varieties.

Theorem 4.5.1. *(Rost Nilpotence for projective pseudo-homogeneous varieties) Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for a semi-simple group G of inner type over k. Then the kernel of the base change map*

$$End(\mathcal{M}(\widetilde{X})) \to End(\mathcal{M}(\widetilde{X}_K))$$

$$f \mapsto f \otimes K$$

consists of nilpotents.

Proof. The proof is similar to the one in [Bro05]. For a field extension L/k, let n_L denote the number of terms appearing in the decomposition of Corollary 4.3.8 for the the motive of the G_L-variety $\widetilde{X}_L$. Clearly, $L \subset M \Rightarrow n_M \geq n_L$ and the maximal number of terms in the coproduct occurs precisely when each $\widetilde{Z}_i$ is $Spec\ L$. In particular, this happens when $L = K$.

Claim: Set $N(d,n) = (d+1)^{n_K - n}$ where d is the dimension of $\widetilde{X}$. Then, for any morphism $f \in End(M(\widetilde{X}))$ with $f \otimes K = 0$, $f^{N(d,n_k)} = 0$.

The claim obviously implies the theorem. Note that when $n_k = n_K$, $\mathcal{M}(\widetilde{X})$ completely splits into Tate motives and $End(\mathcal{M}(\widetilde{X})) = Ch_0(Spec\ k)^{\oplus r}$ for some r. Therefore the claim is valid for $n_k = n_K$. Now we use descending induction on $n = n_k$. Let $f \in End(M(\widetilde{X}))$ be an endomorphism in the kernel of the base change map. If all components $\widetilde{Z}_i$ appearing in the motivic decomposition of Corollary 4.3.8 are isotropic, n is maximal and the claim is already proved. If not, pick a point z in one of the anisotropic components Z_i and set $L = k(z)$. Over L, $\widetilde{Z}_i$ is isotropic. Therefore, the number $n_i = n_L$ of terms appearing in the motivic decomposition of $\widetilde{X}_L$ is strictly greater than n. Thus the claim holds for $\mathcal{M}(\widetilde{X}_L)$ and $f_L^{N(d,n_i)} = 0$. Since $N(d,n_i) \leq N(d,n+1)$, it follows that $f_L^{N(d,n+1)} = 0$. Now we use Theorem 3.1 in [Bro03] to conclude that the composition

$$\mathcal{M}(\widetilde{Z}_i)\{a_i\} \xrightarrow{j_1} \mathcal{M}(\widetilde{X}) \xrightarrow{f^{(d+1)N(d,n+1)}} \mathcal{M}(\widetilde{X})$$

vanishes where the first arrow comes from the coproduct decomposition. Since for each summand the composition is zero, we are done. □

Theorem 4.5.2. *The Krull-Schmidt principle holds for any shift of any summand of the motive of projective pseudo-homogeneous variety for G.*

Proof. This follows from Theorem 4.5.1, Theorem 2.2.1 and Theorem 4.4.3. □

Chapter 5: Motives of Projective Pseudo-Homogeneous Varieties- II

As before G is assumed to be inner over k. Therefore by results of §1.1, the $*$-action is trivial.

5.1 Upper Motives of Projective Pseudo-Homogeneous Varieties

Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for G and let X be the corresponding homogeneous variety. We show that the upper indecomposable motives of $\mathcal{M}(\widetilde{X})$ and $\mathcal{M}(X)$ are isomorphic. Recall the following well-known fact about parabolic subgroups ([Tit66]).

Fact 5.1.1. *Let G be a semi-simple algebraic group over a field k. Let P be a parabolic subgroup corresponding to subset τ of nodes of the Dynkin diagram (See §1.2). Let $\mathcal{P}$ denote the conjugacy class of P. Then $\mathcal{P}$ contains a parabolic subgroup defined over k if and only if the nodes in τ are circled in the Tits index of G over k and τ is invariant under the $*$-action of $Gal(K/k)$.*

In our case, since G is assumed to be inner over k, the $*$-action is trivial. Let X and $\widetilde{X}$ be as before.

Lemma 5.1.2. *Let F be any field extension of k. Then X has an F-point iff $\widetilde{X}$ has an F-point.*

Proof. Clearly if X has an F-point, its image via the canonical map $X \to \widetilde{X}$ gives an F-point on $\widetilde{X}$. Now assume that $\widetilde{X}$ has an F-point. Let F' be the perfect closure of F. Then by Proposition 12.1.2 of [Spr09] the stabilizer in G of this F-point is defined over F'. Without loss of generality we can assume that $\widetilde{P}$ is defined over F'. Since F' is perfect the underlying reduced subscheme P is also defined over F'. Let τ be the subset of nodes of Dynkin diagram corresponding to P. Since G is inner over k, it is inner over F. Therefore the $*$-action is trivial over F. Moreover, by Exercise 13.2.5 (4) in [Spr09], the Tits index of F' and F are the same. Therefore by Fact 5.1.1, the conjugacy class $\mathcal{P}$ of P contains an F-defined parabolic and therefore X has an F-point. $\square$

Note that by Theorem 4.5.2, the motive $\mathcal{M}(\widetilde{X})$ satisfies the Krull-Schmidt principle. Therefore we can talk about *the unique* upper summand $U_{\widetilde{X}}$ of $\mathcal{M}(\widetilde{X})$.

Corollary 5.1.3. *Let X and $\widetilde{X}$ be as above. Then in $Chow(k, \Lambda)$, $U_X \simeq U_{\widetilde{X}}$.*

Proof. By Theorem 2.3.4, it suffices to show multiplicity one correspondences $\alpha : \mathcal{M}(X) \to \mathcal{M}(\widetilde{X})$ and $\beta : \mathcal{M}(\widetilde{X}) \to \mathcal{M}(X)$. Take α to be the correspondence induced from the canonical map $X \to \widetilde{X}$. For β, first observe that $\widetilde{X}$ has an $k(\widetilde{X})$-point. Then by Lemma 5.1.2, so does X. Now take β to be the correspondence induced from the rational map $\widetilde{X} \dashrightarrow X$. $\square$

5.2 A General Criterion for Isomorphic motives

In this section, we give a characterization of when the motive of a variety is isomorphic to the motive a projective homogeneous variety.

Notation: For a variety X, $A^i(X,\Lambda)$ denotes the i^{th} Chow group of X with coefficients in Λ graded by codimension. We simply write A^i if X and Λ are clear from the context. The symbol $A^{\geq i}$ denotes $\bigoplus_{j\geq i} A^j$. Similarly define $A^{>i}$, $A^{\leq i}$ and $A^{<i}$.

Let $A_i(X,\Lambda)$ denote the i^{th} Chow group of X with coefficients in Λ graded by dimension. We make similar definitions for $A_{\geq i}$, $A_{>i}$, $A_{\leq i}$ and $A_{<i}$.

Recall that for a motive M, $Ch^i(M)$ is defined as $Hom(M,\Lambda\{i\})$ in the category $Chow(k,\Lambda)$.

Definition: Let ϵ be the function on the objects of $Chow(k,\Lambda)$ defined as follows:

$$\begin{aligned} \epsilon : Ob(Chow(k,\Lambda)) \longrightarrow \quad & \mathbb{Z} \bigcup \{-\infty\} \\ M \quad \longmapsto & \, min\{i \,|Ch^i(M_K) \neq 0\} \end{aligned}$$

Theorem 5.2.1. *Let X be a projective G-homogeneous variety over k. Let Z be any geometrically split projective k-variety whose motive satisfies the Rost nilpotence principle such that the following holds in $Chow(k,\Lambda)$:*

1. $U_X \simeq U_Z$
2. $\mathcal{M}(X_L) \simeq \mathcal{M}(Z_L)$ *where* $L = k(X)$

Then $\mathcal{M}(X) \simeq \mathcal{M}(Z)$.

Remark 5.2.2. In the above theorem, $\mathcal{M}(Z)$ satisfies Krull-Schmidt by Theorem 2.2.1 and hence the upper motive U_Z of Z is well-defined.

Proof. Since X is projective homogeneous variety for G, by Theorem 1.1 of [Kar10], every indecomposable summand M of $\mathcal{M}(X)$ is isomorphic to $U_Y\{i\}$ for some projective homogeneous variety Y corresponding to τ such that $\tau \supseteq \tau_L$. By condition (2), $U_{Y_L}\{i\}$ comes from an indecomposable summand $\widetilde{M}$ of $\mathcal{M}(Z)$ (Here U_{Y_L} denotes the upper motive of Y_L. It is not the same as $(U_Y)_L$. But is the upper motive of $(U_Y)_L$). We claim that $M \simeq \widetilde{M}$. It is clear that if M and N are distinct (they may or may not be isomorphic) indecomposable summands of $\mathcal{M}(X)$, $\widetilde{M}$ and $\widetilde{N}$ are distinct indecomposable summands of $\mathcal{M}(\widetilde{X})$. This together with condition (2) implies that it suffices to prove the claim to complete the proof.

The proof of the claim is by induction on $\epsilon(M)$. For the base case $\epsilon(M) = 0$, the claim clearly holds by condition (1). Now let $M \simeq U_Y\{i\}$ be a summand of $\mathcal{M}(X)$ as above. Then $\epsilon(M) = i$ and assume that for all indecomposable summands N with $\epsilon(N) < i$, $N \simeq \widetilde{N}$. Write $\mathcal{M}(X) = P \bigoplus Q$ where $\epsilon(P') < i$ for every indecomposable summand P' of P and $\epsilon(Q) \geq i$. Then by induction hypothesis, $\mathcal{M}(Z) \simeq P \bigoplus R$. By Theorem 2.2.1, $Q_L \simeq R_L$. By assumption M is a summand of Q and so $\widetilde{M}$ is a summand of R. Observe that $\epsilon(\widetilde{M_L}) = i$ as $\epsilon(Q_L) \geq i$. Therefore if $\pi \in End(\mathcal{M}(Z))$ is the projector giving rise to the summand $\widetilde{M}$, then $\pi_{\overline{L}} = \sum b_k \times a_k \in \sum_I A^r \times A_r$ for a multiset I such that $r \geq i$ for every $r \in I$ and $a_k \cdot b_j = \delta_{kj}$ (Here δ_{kj} is the Kronecker delta function).

To complete the proof, it suffices to find $\alpha : \mathcal{M}(Y)\{i\} \longrightarrow \widetilde{M}$ and $\beta : \widetilde{M} \longrightarrow \mathcal{M}(Y)\{i\}$ such that $mult(\beta \circ \alpha) = 1$ (See Theorem 2.3.3).

For a motive N over k, let $\overline{N}$ denote the motive base changed to $\overline{L}$ and for a variety V over k, $\overline{V}$ denotes $V \times_{Spec\ k} \overline{L}$.

First note that we have $a \in Hom(\Lambda\{i\}, \mathcal{M}(\overline{Z})) = A_i(\overline{Z})$ given by $\Lambda\{i\} \hookrightarrow \overline{U}_{Y_L}\{i\} \hookrightarrow \mathcal{M}(\overline{Z})$ and $b \in Hom(\mathcal{M}(\overline{Z}), \Lambda\{i\}) = A^i(\overline{Z})$ given by $\mathcal{M}(\overline{Z}) \to \overline{U}_{Y_L}\{i\} \to \Lambda\{i\}$ such that $mult(b \circ a) = 1$ i.e., $a \cdot b = 1$. Observe that with this notation, $\overline{\pi} = b \times a + \sum_k b_k \times a_k$

where $b_k \times a_k \in A^{\geq i} \times A_{\geq i}$, $a \cdot b_k = 0 \ \forall b_k$ and $a_k \cdot b = 0 \ \forall a_k$.

Construction of α:

Let $\alpha_1 \in Hom(\mathcal{M}(Y_L)\{i\}, \mathcal{M}(Z_L)) = A^{dim\ Z-i}(Y_L \times Z_L)$ be given by $\mathcal{M}(Y_L)\{i\} \to U_{Y_L}\{i\} \hookrightarrow \mathcal{M}(X_L) \xrightarrow{\simeq} \mathcal{M}(Z_L)$. Then,

$$\overline{\alpha}_1 \in 1 \times a + A^{>0} \times A_{>i}$$

Let α_2 be the image of α_1 under the pull back of Chow groups

$$A^{dim\ Z-i}(Y_L \times_L Z_L) \longrightarrow A^{dim\ Z-i}(Spec\ L(Y) \times_L Z_L)$$

induced by $Spec\ L(Y) \times_L \times Z_L \to Y_L \times_L Z_L \simeq (Y \times Z)_L$. Then

$$\overline{\alpha}_2 = Spec\ \overline{L}(Y) \times a.$$

Since $\tau_L \subseteq \tau$, X has an $k(Y)$-point. So $k(Y)(X)/k(Y) = L(Y)/k(Y)$ is purely transcendental. Therefore α_2 is $k(Y)$ rational. So $\alpha_2 \in A^{dim\ Z-i}(Spec\ k(Y) \times Z)$. Let α' be any preimage of α_2 under the surjective map of Chow groups

$$A^{dim\ Z-i}(Y \times Z) \twoheadrightarrow A^{dim\ Z-i}(Spec\ k(Y) \times Z)$$

induced by $Spec\ k(Y) \times Z \to Y \times Z$. Then

$$\overline{\alpha'} \in 1 \times a + A^{>0} \times A_{>i}$$

Let $p : \mathcal{M}(Z) \to \widetilde{M}$ be the projection from our decomposition. Define

$$\alpha = p \circ \alpha'$$

Construction of β:

Let $\beta_1 \in Hom(\mathcal{M}(Z_L), \mathcal{M}(Y_L)\{i\})$ be given by $\mathcal{M}(Z_L) \xrightarrow{\simeq} \mathcal{M}(X_L) \to U_{Y_L}\{i\} \to$

$\mathcal{M}(Y_L)\{i\}$. Then,

$$\overline{\beta}_1 \in b \times y + A^{>i} \times A_{>0}$$

where y is the class of a point in $\overline{Y}$. Let β_2 be an element in the inverse image of β_1 under the surjective map of Chow groups

$$A^{dim\ Y+i}(Z \times X \times Y) \twoheadrightarrow A^{dim\ Y+i}(Z_L \times Y_L)$$

induced by $Z_L \times_L Y_L \simeq (Z \times_k Y) \times Spec\ k(X) \to Z \times Y \times X \to Z \times X \times Y$ where the last map is obtained by switching second and third factors. Then

$$\overline{\beta}_2 \in b \times 1 \times y + A^{>i} \times 1 \times A_{>0} + A^* \times A^{>0} \times A^*$$

Recall that $\pi \in End(\mathcal{M}(Z))$ is the projector giving the summand $\widetilde{M}$. Let $\beta_3 = \beta_2 \circ \pi$ where β_2 is thought of as an element in $Hom(\mathcal{M}(Z), \mathcal{M}(X \times Y)\{i - dim\ X\})$. Then

$$\overline{\beta}_3 \in p_{134*}[(b \times a \times 1 \times 1 + \sum_k b_k \times a_k \times 1 \times 1) \cdot (1 \times b \times 1 \times y + 1 \times A^{>i} \times 1 \times A_{>0} + 1 \times A^* \times A^{>0} \times A^*)]$$

and hence,

$$\overline{\beta}_3 \in b \times 1 \times y + A^i \times A^{>0} \times A^* + A^{>i} \times 1 \times A_{>0} + A^{>i} \times A^{>0} \times A^*$$

By condition (1) in the hypothesis of the theorem, $U_X \simeq U_Z$. This implies by Theorem 2.3.4 that we have a multiplicity 1 correspondence $\Gamma \in A_{dim\ Z}(Z \times X)$. Then $\overline{\Gamma} = 1 \times x + A^{>0} \times A_{>0}$ where x refers to the class of a point in $\overline{X}$.

Now $\Gamma \times 1 \in A_{dim\ Z + dim\ Y}(Z \times X \times Y)$. Define $\beta' = p_{13*}[(\Gamma \times 1) \cdot \beta_3] \in A^{dim\ Y+i}(Z \times Y) = Hom(\mathcal{M}(Z), \mathcal{M}(Y)\{i\})$. Then,

$$\overline{\beta'} \in p_{13*}[(1 \times x \times 1 + A^{>0} \times A_{>0} \times 1) \cdot (b \times 1 \times y + A^i \times A^{>0} \times A^* + A^{>i} \times 1 \times A_{>0} + A^{>i} \times A^{>0} \times A^*)]$$

Therefore,

$$\overline{\beta'} \in b \times y + A^{>i} \times A_{>0}$$

Now define $\beta = \beta' \circ q$ where $q : \widetilde{M} \hookrightarrow \mathcal{M}(Z)$ is inclusion map from our decomposition.

We now observe that $\beta \circ \alpha = \beta' \circ q \circ p \circ \alpha' = \beta' \circ \pi \circ \alpha'$. Note that

$$\overline{\pi} \circ \overline{\alpha'} \in p_{13*}[(1 \times a \times 1 + A^{>0} \times A_{>i} \times 1) \cdot (1 \times b \times a + \sum_k 1 \times b_k \times a_k)]$$

and hence,

$$\overline{\pi} \circ \overline{\alpha'} \in 1 \times a + A^{>0} \times A_{>i}$$

Finally we see that

$$\overline{\beta} \circ \overline{\alpha} \in p_{13*}[(1 \times a \times 1 + A^{>0} \times A_{>i} \times 1) \cdot (1 \times b \times y + 1 \times A^{>i} \times A_{>0})]$$

This imples,

$$\overline{\beta} \circ \overline{\alpha} \in 1 \times y + A^{>0} \times A_{>0}$$

Therefore, $mult(\beta \circ \alpha) = 1$. □

5.3 Motives of Projective Pseudo-Homogeneous vs Homogeneous Varieties

As an application of Theorem 5.2.1, we derive the following main result.

Theorem 5.3.1. *Let $\widetilde{X}$ be a projective pseudo-homogeneous variety for G and let X be the corresponding projective homogeneous variety. Then in the category of motives*

$Chow(k, \Lambda)$

$$\mathcal{M}(X) \simeq \mathcal{M}(\widetilde{X})$$

In particular, by Theorem 2.5.2 every indecomposable summand in $\mathcal{M}(\widetilde{X})$ is a shift of some upper motive U_τ satisfying $\tau_{k(X)} \subseteq \tau$.

Proof. We will prove by induction on $n = \mathrm{rank}(G)$. The claim is trivially true for $n = 0$. Assume that the claim is true for all groups with rank less than n. Let $\mathrm{rank}(G) = n$. We can assume that $X \neq Spec(k)$ (otherwise there is nothing to prove). Let $L = k(X)$ and G' the anisotropic kernel of G_L. Then $\mathrm{rank}(G') < \mathrm{rank}(G)$. Now by Corollary 4.3.8, $\mathcal{M}(\widetilde{X}_L) = \coprod_i \mathcal{M}(\widetilde{Z}_i)\{a_i\}$ and $\mathcal{M}(X_L) = \coprod_i \mathcal{M}(Z_i)\{a_i\}$ where $\widetilde{Z}_i$ is projective pseudo-homogeneous for G' and Z_i the corresponding projective homogeneous variety. By induction hypothesis, we have $\mathcal{M}(\widetilde{Z}_i) \simeq \mathcal{M}(Z_i)$ and thus $\mathcal{M}(\widetilde{X}_L) \simeq \mathcal{M}(X_L)$. Moreover by Corollary 5.1.3 $U_X \simeq U_{\widetilde{X}}$. Therefore, by Theorem 5.2.1, we are done.

□

5.4 Examples and Applications

Let A be a central simple algebra of degree n over k. Let $X = X(d_1, d_2, \cdots, d_m, A)$ be the variety of right ideals of reduced dimensions $1 \leq d_1 < d_2 < \cdots < d_m \leq n$. Note that X is projective homogeneous for $G = PGL(A)$. Write $X_K \simeq G/P$ for some parabolic subgroup P. Let $A^{(p)} = A \otimes_{Fr} k$ and $X^{(p)} = X \times_{Fr} Spec\ k$ where $Fr : k \to k$ is the Frobenius morphism. Then it is easy to see that $X^{(p)}{}_K \simeq G/\widetilde{P}$ where $\widetilde{P} = G_p P$ and G_p is the kernel of the Frobenius morphism $Fr : G \to G^{(p)}$. Moreover, X is the projective homogeneous variety corresponding to $X^{(p)}$.

Recall the following fact from [Flo13] (See also Theorem 3.9 in [KOS76]).

Lemma 5.4.1. *(Proposition 3.2 in [Flo13]): Let A be a central simple algebra of degree n over k. Then $A^{(p)}$ is Brauer equivalent to $A^{\otimes p}$.*

An easy consequence of Theorem 5.3.1 is the following.

Corollary 5.4.2. *For a central simple algebra A over k of degree n, let B denote the central simple algebra of degree n that is Brauer equivalent to $A^{\otimes p}$. Then in the category $Chow(k, \Lambda)$, the motives of twisted flag varieties $X(d_1, d_2, \cdots, d_m, A)$ and $X(d_1, d_2, \cdots, d_m, B)$ are isomorphic. That is,*

$$\mathcal{M}(X(d_1, d_2, \cdots, d_m, A)) \simeq \mathcal{M}(X(d_1, d_2, \cdots, d_m, B))$$

Taking $m = 1$, we get $\mathcal{M}(SB_d(A)) \simeq \mathcal{M}(SB_d(B))$ for twisted Grassmannians. In particular, for the case of Severi-Brauer varieties we have $\mathcal{M}(SB(A)) \simeq \mathcal{M}(SB(B))$.

Proof. Note that $B = A^{(p)}$ by Lemma 5.4.1. Therefore,

$$\begin{aligned}
\mathcal{M}(X(d_1, d_2, \cdots, d_m, B)) &\simeq \mathcal{M}(X(d_1, d_2, \cdots, d_m, A^{(p)})) \\
&\simeq \mathcal{M}(X(d_1, d_2, \cdots, d_m, A)^{(p)}) \qquad \text{(by functoriality of the Frobenius)} \\
&\simeq \mathcal{M}(X(d_1, d_2, \cdots, d_m, A)) \qquad \text{(by Theorem 5.3.1)}
\end{aligned}$$

The rest follows easily. □

Remark 5.4.3. Let A be a central simple algebra over k with exponent (i.e., the order of its Brauer class as an element in the Brauer group) not dividing $p^2 - 1$. Let $X = SB(A)$ be the Severi-Brauer variety associated with A and let $X^{(p)} = SB(A)^{(p)} \simeq SB(A^{(p)})$.

Then by Corollary 5.4.2, $\mathcal{M}(X)$ and $\mathcal{M}(X^{(p)})$ are isomorphic in $Chow(k, \Lambda)$ for all coefficient rings Λ that are finite fields (of any characteristic). But they are not isomorphic in the integral Chow motive category $Chow(k, \mathbb{Z})$. Indeed, if they were isomorphic in $Chow(k, \mathbb{Z})$, Criterion 7.1 in [Kar00b] would imply that $A^{(p)}$ is isomorphic either to A or its opposite A^{op}. Since $A^{(p)}$ is Brauer equivalent to $A^{\otimes p}$ by Lemma 5.4.1, this contradicts our assumption on the exponent of A. Therefore we get examples of varieties whose motives are isomorphic over all finite field coefficients but not over integral coefficients.

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