ABSTRACT

Title of dissertation: QUANTUM NOISE IN OPTICAL PARAMETRIC AMPLIFIERS BASED ON A LOSSY NONLINEAR INTERFEROMETER

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Optical Parametric Amplifiers (OPA) have been of wide interest for the past decades due to their potential for low noise amplification and generation of squeezed light. However, the existing OPAs for fiber applications are based on Kerr effect and require from few centimeters to kilometers of fiber for significant gain.

In this thesis, I review the principles of phase sensitive amplification and derive the expression for gain of a lossless Kerr medium based nonlinear Mach-Zehnder Interferometer (NMZI OPA) using a classical physics model . Using quantum optics, I calculate the noise of a lossless Kerr medium based OPA and show that the noise figure can be close to zero.

Since in real life a Kerr medium is lossy, using quantum electrodynamics, I derive equations for the evolution of a wave propagating in a lossy Kerr medium such as an optical fiber. I integrate these equations in order to obtain the parametric gain, the noise and the noise figure. I demonstrate that the noise figure has a simple expression as a function of loss coefficient and length of the Kerr medium and that the previously published results

by a another research group are incorrect. I also develop a simple expression for the noise figure for high gain parametric amplifiers with distributed loss or gain.

In order to enable construction of compact parametric amplifiers I consider using different nonlinear media, in particular a Saturable Absorber (SA) and a Semiconductor Optical Amplifier (SOA). Using published results on the noise from SOA I conclude that that such device would be prohibitively noisy. Therefore, I perform a detailed analysis of noise properties of a SA based parametric amplifier. Using a quantum mechanical model of an atomic 3 level system and the Heisenberg's equations, I analyze the evolution in time of a single mode coherent optical wave interacting with a saturable absorber. I solve the simultaneous differential equations and find the expression for the noise figure of the SA based NMZI OPA. The results show that noise figure is still undesirably high. The source of the noise is identified. A new approach for low noise parametric amplifier operating with short pulses is proposed.

QUANTUM NOISE IN OPTICAL PARAMETRIC AMPLIFIER BASED ON A LOSSY NONLINEAR INTERFEROMETER

by

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Dedication

To my family

Acknowledgments

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List of Abbreviations

α	alpha
β	beta
NF	Noise Figure
SA	Saturable Absorber
SOA	Semiconductor Optical Amplifier
MZI	Mach-Zehnder Interferometer
NMZI	Nonlinear Mach-Zehnder Interferometer
OSNR	Optical Signal to Noise Ratio
SNR	(Electrical) Signal to Noise Ratio
OPA	Optical Parametric Amplifier
PNF	Photon Number Squared
FAS	Field Amplitude Squared
QFS	Quadrature Field Squared

Chapter 1

Introduction

1.1 Overview

Optical communication plays an important role in the information age we are in. One fiber optic link allows us to reliably send data over thousands of kilometers at very high bandwidth (hundreds of Gb/s). At this time, no other medium can do better. As we are pushing the fiber optic bit rate closer to its channel capacity, it is important to keep the noise generated by the fiber optic devices and the medium low. One of the devices that produces the most noise is the optical amplifier [1]. In chapter one, after a brief introduction to quantum mechanics, I give two established definitions of the Signal to Noise Ratio (SNR), a measure of the quality of a signal, and show why those definitions are not best suited for our problem. I introduce a new definition of the SNR which will be more appropriate. I show that any phase insensitive optical amplifier, such as the commonly used EDFA, degrades Signal to Noise Ratio (SNR) by at least 3 dB. This is particularly important at the optical receiver when the input signals are very weak. There exists a class of amplifier, which is phase sensitive, that does not degrade SNR. The goal of this thesis is to provide a correct quantum mechanical analysis of phase sensitive amplifiers, which include lossy elements. Phase sensitive amplifiers operate on a principle very similar to balanced homodyne detector, which will also be briefly discussed in the introduction.

In chapter 2, I discuss in detail a well known phase sensitive amplifier: The lossless Kerr medium based Nonlinear Mach-Zehnder Interferometer Optical Parametric Amplifier (NMZI-OPA). There are numerous publications which demonstrate the feasibility and applications of this device, which uses nonlinear fiber as the Kerr medium [4, 5, 6, 7, 8, 15, 16]. Phase sensitive gain of this device is calculated using a simple classical model.

In chapter 3, I use a quantum mechanical model to calculate the noise properties of a lossless NMZI OPA and I show that it does not degrade SNR. A possible definition of the Noise Figure (NF) of an amplifier is the ratio of its input to its output SNR. With this definition, the noise figure is zero (0) dB for this ideal NMZI OPA. Unfortunately, there is always loss in a real Kerr medium, which results in a higher noise figure.
This problem was analyzed in detail by Imajuku *et al.* [8]. However, I found that some of their results are incorrect. In order to bracket the noise figure for a lossy NMZI OPA, I consider the effect of lumped loss in front of the Kerr medium and after the Kerr Medium. The true answer for parametric amplifier with distributed loss must fall within those limits. These results, unlike other results which will be discussed later, agree with the calculations of Imajuku *et al.*. In general, calculation of noise for this type of device is tedious[8]. Therefore, I derived a general simple expression for noise figure for an OPA in the limit of large parametric gain, which drastically reduces the complexity of the calculations.

In chapter 4, I calculate the noise figure of a NMZI-OPA with uniformly distributed loss in the medium. This problem was also addressed by Imajuku *et al.* However, they made some wrong assumptions and performed unexplainable mathematical steps, which produced results which I believe to be incorrect. My approach gives a simple and elegant expression for noise figure. I also derive the noise figure for a NMZI-OPA with distributed laser gain instead of loss. I show that even small amount of gain produces 3dB increase in noise figure. In chapter 5, I discuss the implication of the quantum mechanical and the classical noises in the NMZI-OPA. I also find the optimum length of an optic fiber used as a Kerr medium based on its nonlinear property and its loss coefficient.

In order to be able to build a compact OPA, another nonlinear medium besides optical fiber is required. There has been extensive research in optical signal processing, which utilized nonlinear optical properties of semiconductors amplifiers and saturable absorbers (TOAD, wavelength converter, fast optical switch, fast pulse generator ....[9, 10, 11, 12, 13]) . These media are very attractive because of the nonlinearities are very high and the devices are compact. In chapter 6, I use a classical model to show that phase sensitive amplification can be achieved with semiconductor optical amplifiers and saturable absorbers. Recently, there has been some interest in this class of devices [17].

Based on the result in chapter 4, I conclude that an OPA based on SOAs will produce unacceptable noise figure. The calculation of noise figure for an OPA with saturable absorber is non-trivial. Chapter 6 and 7 are dedicated to this problem. In chapter 7, I use full quantum mechanical model to derive basic differential equations for light interaction with saturable absorber. In chapter 8, I solve these equations and evaluate the noise figure for this type of parametric amplifier based on saturable absorber.

In chapter 9, I discuss practical implication of results obtained in the previous chapter.

1.2 The Balanced Homodyne Receiver

The balanced homodyne receiver model has some common characteristics with an optical parametric amplifier. They both have gain sensitive amplification. Therefore, the analysis of an balanced homodyne receiver can give some insight into phase sensitive amplification. Homodyne receivers use interference of a weak signal with a strong local oscillator of the same frequency and with proper phase to produce gain in the receiver [14]. They are the best available receiver for coherent optical signals [3]. However, the local oscillator introduces its own noise, which can be comparable to or greater than the incoming signal [48]. The balanced homodyne receiver overcomes the problem of part of the local oscillator noise, namely its Relative Intensity Noise (RIN), but not its phase noise as we will see in equation (1.8). It works as follows. The message signal and the local oscillator are split in two and mixed using semi-transparent mirror, as shown in figure 1.1. The two signals must have perfect spatial mode overlap, must have the same polarization state and must have the same carrier frequency. The two optical signal are then detected, turned into an electrical signal and subtracted. These operations produce a larger electrical signal than direct detection and cancel RIN from the oscillator. In order to explain the operations of a balance homodyne detector and later the NMZI OPA, the analysis of the optical beam splitter is looked at.

1.2.1 The Lossless Optical Beam Splitter

The optical beam splitter is shown in figure 1.2. It has two inputs, waves with complex amplitude (E_1 and E_2), and two outputs (E_3 and E_4) as shown in the figure.

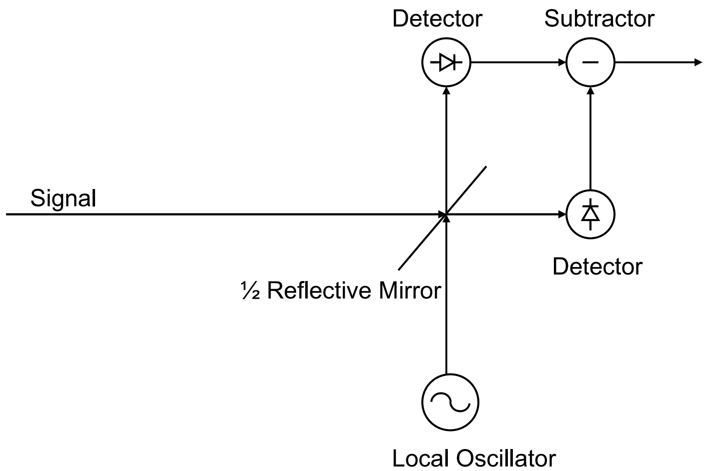

Figure 1.1: Balanced Homodyne Receiver

There is a linear relationship between the input fields and the output fields which are summarized by equation 1.1 and 1.2

$$E_3 = r_{31}E_1 + t_{32}E_2 \tag{1.1}$$

and

$$E_4 = t_{41}E_1 + r_{42}E_2 \tag{1.2}$$

where r_x and t_x are respectively complex reflection and transmission coefficient. The above relations can be written in matrix format:

$$\begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} r_{31} & t_{32} \\ t_{41} & r_{42} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{1.3}$$

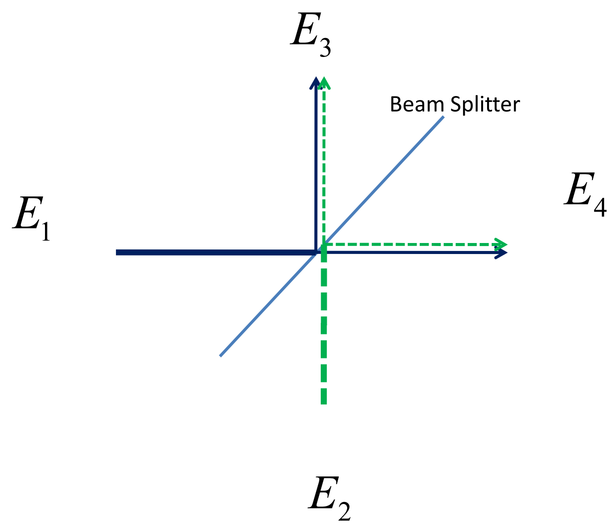

Figure 1.2: Beam Splitter

In order to satisfy conservation of energy and the phase requirement, we can choose a convention where for a 50/50 splitter the matrix is [53]

$$\frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}. \tag{1.4}$$

It is important to note that this solution is not unique. For example, the following matrix could have been chosen

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}. \tag{1.5}$$

1.2.2 Classical Analysis of the Balanced Homodyne Detector

If E_{LO} and E_s are respectively the complex amplitudes of the electric field of the local oscillator and the signal, at the detectors the amplitude of the electric fields are

$$E_1 = \frac{1}{\sqrt{2}}\left(iE_{LO} + E_s\right) \tag{1.6}$$

and

$$E_2 = \frac{1}{\sqrt{2}}\left(E_{LO} + iE_s\right) \tag{1.7}$$

The currents of the two detectors are subtracted, the net output current is

$$\begin{aligned} I &= q\frac{\eta}{\hbar\omega_0}\left(|E_1|^2 - |E_2|^2\right) \\ &= iq\frac{\eta}{\hbar\omega_0}\left(E_{LO}E_s^* - E_{LO}^*E_s\right) \\ &= 2q\frac{\eta}{\hbar\omega_0}|E_{LO}||E_s|\sin\left(\phi + \delta\phi\right), \end{aligned} \tag{1.8}$$

where η is the quantum efficiency, q is the electric charge, ϕ is the average phase of the complex number $E_{LO}^*E_s$, $\delta\phi$ is the phase fluctuation of the complex number $E_{LO}^*E_s$ and ω_0 it the optical frequency. We can see that the currents due to the local oscillator cancel, and therefore, their amplitude fluctuations, their RIN, do not appear in the current output. We can also see that there is a power gain compared to direct detection of the signal, which is

$$\begin{aligned} G &\equiv \frac{|I|^2}{|q\frac{\eta}{\hbar\omega_0}E_s|^2} \\ &= 4|E_{LO}|^2\sin^2\left(\phi + \delta\phi\right). \end{aligned} \tag{1.9}$$

The gain is phase sensitive; it is dependent upon the relative phase between the local oscillator and the signal. However, it is also dependent on phase noise, which leads to RIN noise in the output. I will calculate the noise of the detected signal later in this chapter.

1.3 The Linear MZI

The Mach Zehnder Interferometer (MZI) is a commonly used component in many optical devices [18, 19, 20, 21]. It is often used as a building block of more complex optical devices and functionalities such as optical filters, wavelength demultiplexers, channel interleavers, intensity modulators, switches and optical gates [23]. A schematic diagram of typical MZI based on guided waves in shown on figure 1.3. It consist of two 50:50 optical couplers, which are analogous to beam splitters discussed in the previous section. The two outputs of the first coupler become inputs to the second one. It is assumed that the two outputs of the first coupler travel through similar medium, with same length before arriving at the second coupler. It is straightforward to perform classical calculations of the transmission of signal through this interferometer [24]. The input of one arm of the first coupler is a signal described by its complex electric field amplitude E_s. The input to the second arm is E_p, which is the field of the local oscillator, which I will also call the pump field.

After the first beam coupler, I have

$$E_{\text{out12}} = \frac{1}{\sqrt{2}} \left(E_p + iE_s \right) \tag{1.10}$$

at the output of the first arm and

$$E_{\text{out22}} = \frac{1}{\sqrt{2}} \left(E_s + iE_p \right) \tag{1.11}$$

at the output of the second.

The signals go directly to the second coupler with equal delays and amplitude. I

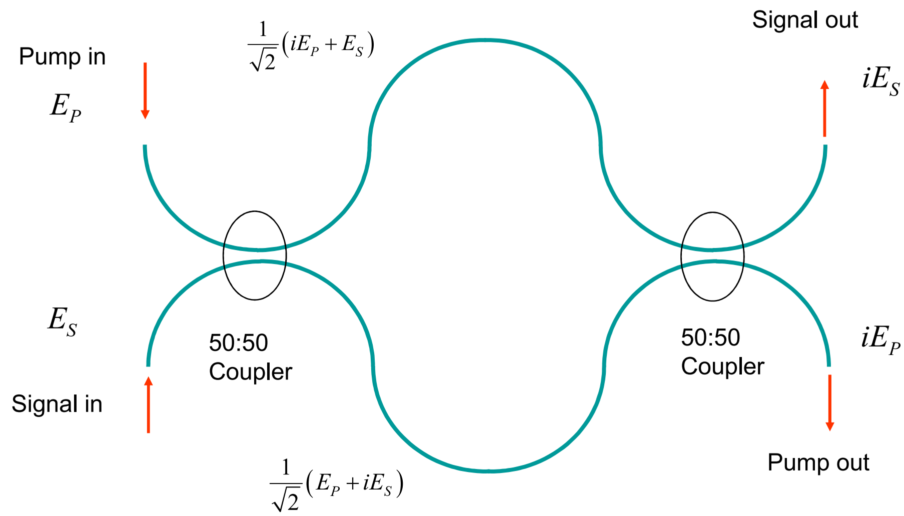

Figure 1.3: Schematic Diagram Linear Mach-Zehnder Interferometer: MZI in linear regime, with proper phase adjustment does not mix the signal and the pump.

obtain at the output of one the arm of the MZI

$$\begin{aligned} E_{\text{out1}} &= \frac{1}{\sqrt{2}}\left(E_{\text{out22}} + iE_{\text{out12}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(E_s + iE_p\right) + i\frac{1}{\sqrt{2}}\left(E_p + iE_s\right)\right) \\ &= \frac{1}{2}E_s - \frac{1}{2}E_s + \frac{i}{2}E_p + \frac{i}{2}E_p. \\ &= iE_p \end{aligned} \tag{1.12}$$

Similarly, at the output of the second arm

$$\begin{aligned}
E_{\text{out2}} &= \frac{1}{\sqrt{2}}\left(E_{\text{out12}} + iE_{\text{out22}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(E_p + iE_s\right) + i\frac{1}{\sqrt{2}}\left(E_s + iE_p\right)\right) \\
&= \frac{1}{2}E_p - \frac{1}{2}E_p + \frac{i}{2}E_s + \frac{i}{2}E_s \\
&= iE_s
\end{aligned} \tag{1.13}$$

Therefore, it can be seen from the balanced MZI that the two signals' amplitude and their phase difference are preserved.

1.4 The Nonlinear MZI Optical Parametric Amplifier

In a NMZI, a nonlinear medium is placed between the two optical couplers as shown in figure 1.4. It is assumed that the nonlinear media have similar properties and the same length. In general, because of interference between the signal and the pump, similar to the balanced homodyne detector, the total intensity is different in the upper and lower arm. In a nonlinear medium, this results in different attenuations, gains and/or phase shift of propagating waves. Unbalancing of MZI redirects some pump power into the signal output port, resulting in effective amplification of the signal. This is a conventional implementation of the OPA [25].

In this thesis, I will first consider a nonlinear optical Kerr effect medium, which produces a phase shift as the function of power intensity in the medium. I will also consider a saturable absorber (SA) as a nonlinear medium in which there is a phase shift and amplitude variation of the field as a function of the intensity. The average properties at the output can be calculated either classically or using a quantum mechanical model. How-

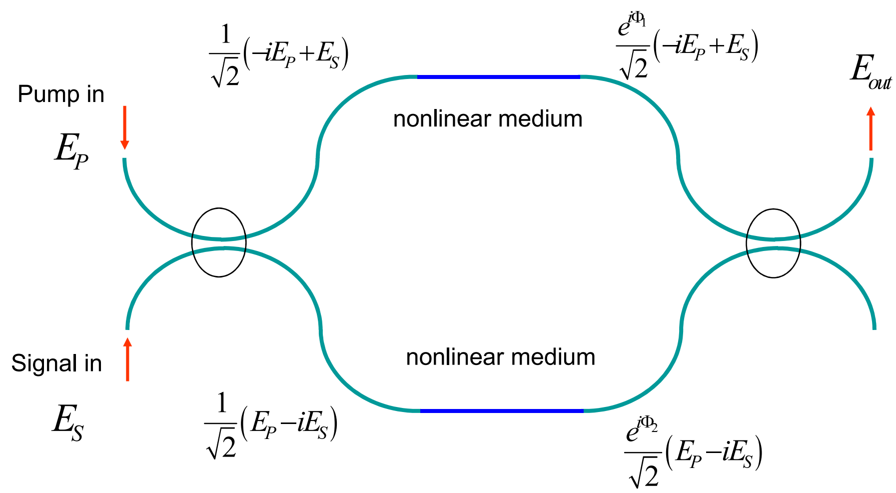

Figure 1.4: Nonlinear Interferometer. It is important to note that Φ is complex

ever, noise properties must be calculated correctly using quantum mechanics. Calculating the noise properties is the main objective of this thesis.

1.5 Brief Introduction to Quantum Mechanics

In quantum mechanics, for any system, there is a state vector $|v\rangle$, containing everything there is to know about that system at a given instant [26]. Any physical quantity is described by an operator $\hat{\mathrm{O}}$, which is Hermitian if the physical quantity is observable. The average value (the expected value) of the physical quantity can be calculated $\left\langle v \left| \hat{\mathrm{O}} \right| v \right\rangle = O$ for the system represented by $|v\rangle$. If the physical quantity is observable then O is real. We are interested in time evolution of physical quantities. There are two approaches for calculation of time evolution of physical quantities. In the Schrödinger picture, the time evolution of a state vector is given by the Schrödinger s equation [56]

$$i\hbar\frac{\mathrm{d}|v(t)\rangle}{\mathrm{d}t} = \hat{\mathbf{H}}|v(t)\rangle \tag{1.14}$$

where $\hat{\mathbf{H}}$ is the Hamiltonian of the system, which is the energy operator. The average of a physical quantity is

$$O(t) = \left\langle v(t) \left| \hat{\mathbf{O}} \right| v(t) \right\rangle . \tag{1.15}$$

and $\hat{\mathbf{O}}$ can be considered constant as a function of time. In Heisenberg picture, state vectors remain constant in time and the operators evolve according to the Heisenberg equation [27]

$$\frac{\mathrm{d}\hat{\mathbf{O}}(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{\mathbf{O}}(t), \hat{\mathbf{H}}\right] . \tag{1.16}$$

The average of a physical quantity is

$$O(t) = \left\langle v \left| \hat{\mathbf{O}}(t) \right| v \right\rangle . \tag{1.17}$$

There is also the interaction picture, which is a combination of the Schrödinger and the Heisenberg picture. In this thesis, I exclusively work in Heisenberg picture because the equations for operators are analogous to classical system [28] and I avoid complications associated with entangled states present in the Schrödinger picture and hidden in the Heisenberg picture[29].

The variance of a physical quantity can be also be calculated. It is

$$(\Delta O)^2 = \left\langle v \left| \hat{\mathbf{O}}^2 \right| v \right\rangle - \left(\left\langle v \left| \hat{\mathbf{O}} \right| v \right\rangle \right)^2 . \tag{1.18}$$

Using the definition of the variance above, the uncertainty involved in the simultaneous measurement of two observable can be calculate using the Heisenberg uncertainty principle, which states [30]

$$\Delta A \Delta B \geq \frac{1}{4} \left| \left\langle \left[\hat{\mathbf{A}}, \hat{\mathbf{B}} \right] \right\rangle \right|, \tag{1.19}$$

where $\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right]$ is the commutator of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ and is defined $\left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] = \hat{\mathbf{A}}\hat{\mathbf{B}} - \hat{\mathbf{B}}\hat{\mathbf{A}}$. In the following sections (1.5.1-1.5.4) are some examples of operator and states I will be using.

1.5.1 Single Mode Field Operators

I will closely follow the formalism of Loudon [31]. Since the thesis is mainly dealing with electric field propagation, it is important to look at the quantum mechanical operator that describes it. Consider an electromagnetic field that has excited a single travelling-wave mode along the z direction with wavevector k . The electric field is expressed as follows

$$\tilde{\mathbf{E}} = -\frac{\partial \tilde{\mathbf{A}}}{\partial t}, \tag{1.20}$$

where $\tilde{\mathbf{A}}$ is the vector potential. In the Heisenberg picture, the scalar electric field operator for a given linear polarization is written

$$\begin{aligned} \hat{\mathbf{E}}(\chi) &= \hat{\mathbf{E}}^{+}(\chi) + \hat{\mathbf{E}}^{-}(\chi) \\ &= \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[\hat{\mathbf{A}} e^{-i\chi} + \hat{\mathbf{A}}^{\dagger} e^{i\chi} \right], \end{aligned} \tag{1.21}$$

where the positive and negative frequency of the field operator correspond respectively to the two terms of the right hand side and V is the volume of the cavity in which the electric field is contained. $\hat{\mathbf{A}}$ is the destruction or the annihilation operator and $\hat{\mathbf{A}}^{\dagger}$ is the raising operator. They are the coefficient of the amplitude of the vector potential. χ is defined as follows

$$\chi = \omega t - kz - \frac{\pi}{2}. \tag{1.22}$$

It is convenient to remove the square root factor from equation (1.21). Therefore, by convention the electric field is measured in units of $2\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$. The operator reduces to

$$\begin{aligned}\hat{\mathbf{E}}(\chi) &= \frac{1}{2}\hat{\mathbf{A}}e^{-i\chi} + \frac{1}{2}\hat{\mathbf{A}}^{\dagger}e^{i\chi} \\ &= \hat{\mathbf{X}}\cos(\chi) + \hat{\mathbf{Y}}\sin(\chi),\end{aligned} \tag{1.23}$$

where $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ are quadrature operators and are defined as follows

$$\hat{\mathbf{X}} = \frac{\hat{\mathbf{A}} + \hat{\mathbf{A}}^{\dagger}}{2} \tag{1.24}$$

and

$$\hat{\mathbf{Y}} = \frac{\hat{\mathbf{A}} - \hat{\mathbf{A}}^{\dagger}}{2i}. \tag{1.25}$$

1.5.2 Number State

The photon number states or Fock state are the eigenstates of the quantum theory of light. They form a complete set for the states of a single mode. They are denoted $|n\rangle$ where n is the number of photon. The action of the destruction operator on the number state is as follows

$$\hat{\mathbf{A}}|n\rangle = n|n-1\rangle \tag{1.26}$$

and the of the creation operator is

$$\hat{\mathbf{A}}^{\dagger}|n\rangle = (n+1)|n+1\rangle. \tag{1.27}$$

From this and using equation (1.23) I can calculate the average value of the electric field in a number state $|n\rangle$, which is

$$\left\langle n \left| \hat{\mathbf{E}}(\chi) \right| n \right\rangle = 0. \tag{1.28}$$

We can also calculate the variance of the electric field for that number state which is

$$\begin{aligned} (\Delta E(\chi))^2 &= \left\langle n \left| \left(\hat{\mathbf{E}}(\chi)\right)^2 \right| n \right\rangle - \left(\left\langle n \left| \hat{\mathbf{E}}(\chi) \right| n \right\rangle\right)^2 \\ &= \left\langle n \left| \left(\hat{\mathbf{E}}(\chi)\right)^2 \right| n \right\rangle \\ &= \frac{1}{2}\left(n + \frac{1}{2}\right). \end{aligned} \tag{1.29}$$

Therefore, for a vacuum state $|0\rangle$ the variance of the electric field is $\frac{1}{4}$. This is also referred to as the vacuum fluctuations.

1.5.3 Number Operator

The number operator is the observable that counts the number of photons. It will be very useful in this thesis as measurements of amplified signals will be needed. It is defined as follows

$$\hat{\mathbf{n}} \equiv \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}}. \tag{1.30}$$

One way to look at this operator is that it counts the photons by removing and replacing them. It acts as follows on a number state

$$\hat{\mathbf{n}}|n\rangle = n|n\rangle. \tag{1.31}$$

It is important to note that for a number state, while the number of photon can be accurately calculated, its phase is undefined. This is consistent with an uncertainty principle, which states that $\Delta n \Delta \phi = \frac{1}{2}$ [32]. Conversely, if there is a state for which the phase is well defined, the pure number would be undefined. Because its phase is undefined, the number state cannot be used as a quantum mechanical model for coherent light.

1.5.4 Coherent States

The coherent state or Glauber state, typically denoted $|\alpha\rangle$ is defined as the following superposition of number states

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{1.32}$$

where α is any complex number. It can be observed from the above equation that the coherent state has a Poissonian number distribution. In other words, the probability of detecting n photons while measuring a coherent state is Poisson distributed and its distribution is

$$p(n) = e^{-|\alpha|^2} \frac{|\alpha|^n}{n!}. \tag{1.33}$$

The action of the destruction operator on a coherent state is as follows

$$\hat{\mathbf{A}}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.34}$$

Therefore $|\alpha\rangle$ is an eigenstate of the destruction operator and α its eigenvalue. Although, $|\alpha\rangle$ is not an eigenstate of the creation operator, the creation operator does satisfy the left-eigenvalue relation conjugate to equation (1.34)

$$\langle\alpha|\hat{\mathbf{A}}^\dagger = \langle\alpha|\alpha^*. \tag{1.35}$$

We can calculate the average number of photon in a coherent state

$$\left\langle \alpha \left| \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \right| \alpha \right\rangle = |\alpha|^2. \tag{1.36}$$

The fluctuation in photon number for a coherent state can also be calculated

$$\begin{aligned}(\Delta n)^2 &= \left\langle \alpha \left| \hat{\mathbf{n}}^2 \right| \alpha \right\rangle - \left(\langle \alpha \left| \hat{\mathbf{n}} \right| \alpha \rangle \right)^2 \\ &= |\alpha|^4 + |\alpha|^2 - |\alpha|^4 \\ &= n \end{aligned} \tag{1.37}$$

The variance of a coherent state is equal to its average number of photons. The coherent state is very often used to model a single mode laser. While it is not the only model used for single mode laser, it is the one for which $\delta n \delta \phi$ is minimum, making a model of choice in this thesis as I will be looking for the minimum noise figure in different OPAs.

1.6 Quantum Noise Added During Amplification and Attenuation

In this section, I go over the process of attenuation and amplification and look at the noises added and some of their properties. These noises will play a very important role when OPA will be analyzed.

1.6.1 Quantum Noise Added During Attenuation

I will first look at the quantum noise added during attenuation. Consider a signal represented at the input by a coherent state $|\alpha\rangle$ (see figure 1.5). Consider also the electric field represented by the lowering operator $\hat{\mathbf{A}}$. The operator $\hat{\mathbf{A}}$ satisfies the following commutator relation [59]:

$$\left[\hat{\mathbf{A}}, \hat{\mathbf{A}}^\dagger \right] = 1. \tag{1.38}$$

Following the formalism of Haus [57], the operator $\hat{\mathbf{A}}$ evolves to $\hat{\mathbf{B}}$ at the output of the attenuator. If L is the fraction of power transmitted through the attenuator, one would

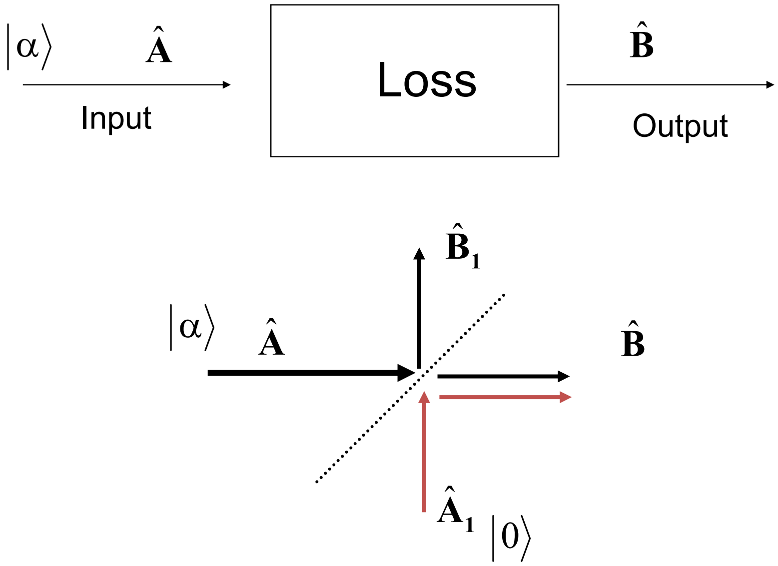

Figure 1.5: Example of Attenuator: Beam coupler. Additional noise from a reservoir channel

expect

$$\hat{\mathrm{B}} = \sqrt{L}\hat{\mathbf{A}}. \tag{1.39}$$

However, the evolution of the electric field inside the attenuator is described by the Heisenberg equation. The Heisenberg equation preserves the commutator relation. Therefore, $\hat{\mathrm{B}}$ must obey the same commutator relation as $\hat{\mathrm{A}}$. Therefore, equation (1.39) is incorrect. What is physically happening is that energy is dissipated. By the

fluctuation-dissipation theorem [44], this results in an addition of noise. Therefore, a Langevin noise operator $\hat{\mathrm{N}}_L$ must be added to the expression of $\hat{\mathrm{B}}$. A specific example of an attenuator is a beam splitter (see figure 1.5), where the field from the second input contributes to the output. The second input of the beam splitter is represented by a vacuum state $|0\rangle$ and its field operator is represented by $\hat{\mathbf{A}}_1$. Therefore, at the output, one would get

$$\hat{\mathbf{B}} = t\hat{\mathbf{A}} + r\hat{\mathbf{A}}_1, \tag{1.40}$$

where t is the transmission coefficient and r the reflection coefficient of the beam splitter. I choose $t = \sqrt{L}$. Therefore, $r = i\sqrt{1-L}$. I calculate the commutator of $\hat{\mathbf{B}}$, which is

$$\left[\hat{\mathbf{B}}, \hat{\mathbf{B}}^\dagger\right] = L\left[\hat{\mathbf{A}}, \hat{\mathbf{A}}^\dagger\right] + (1-L)\left[\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_1^\dagger\right] = 1. \tag{1.41}$$

$\hat{\mathbf{B}}$ obey the same commutator relation as $\hat{\mathbf{A}}$, which is what is expected. It can also be noted that the term $i\left(\sqrt{1-L}\right)\hat{\mathbf{A}}_1$ is an additional noise term proportional to a lowering operator. In general, this noise term is needed to preserve the commutation relation of the evolving field operator. Thus, in general, for the output of an attenuator, it is

$$\hat{\mathbf{B}} = \sqrt{L}\hat{\mathbf{A}} + \hat{\mathbf{N}}_L, \tag{1.42}$$

where $\hat{\mathbf{N}}_L$ is proportional to some annihilation operator[57] operating on the vacuum state $|0\rangle$. Let us find the commutation relation of $\hat{\mathbf{N}}_L$ by computing

$$\left[\hat{\mathbf{B}}, \hat{\mathbf{B}}^\dagger\right] = L\left[\hat{\mathbf{A}}, \hat{\mathbf{A}}^\dagger\right] + \left[\hat{\mathbf{N}}_L, \hat{\mathbf{N}}_L^\dagger\right] = L + \left[\hat{\mathbf{N}}_L, \hat{\mathbf{N}}_L^\dagger\right] = 1. \tag{1.43}$$

Therefore, $\left[\hat{\mathbf{N}}_L, \hat{\mathbf{N}}_L^\dagger\right] = 1 - L$.

In order to characterize the output, let us compute the output number of photons and the photon number fluctuations The number of photons after attenuation is:

$$\begin{aligned}
\langle n_b \rangle &= \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle \\
&= L\left\langle \alpha \left| \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \right| \alpha \right\rangle + \sqrt{L}\left\langle \alpha \left| \hat{\mathbf{A}}^\dagger \right| \alpha \right\rangle \left\langle 0 \left| \hat{\mathbf{N}}_L \right| 0 \right\rangle \\
&\quad + \sqrt{L}\left\langle \alpha \left| \hat{\mathbf{A}} \right| \alpha \right\rangle \left\langle 0 \left| \hat{\mathbf{N}}_L^\dagger \right| 0 \right\rangle + \left\langle 0 \left| \hat{\mathbf{N}}_L^\dagger \hat{\mathbf{N}}_L \right| 0 \right\rangle .
\end{aligned} \tag{1.44}$$

The terms $\left\langle 0 \left| \hat{\mathbf{N}}_L \right| 0 \right\rangle$ and $\left\langle 0 \left| \hat{\mathbf{N}}_L^\dagger \right| 0 \right\rangle$ are zero. The term $\left\langle 0 \left| \hat{\mathbf{N}}_L^\dagger \hat{\mathbf{N}}_L \right| 0 \right\rangle$ is zero since $\hat{\mathbf{N}}_L$ is a lowering operator. Therefore,

$$\begin{aligned} \langle n_b \rangle &= L|\alpha|^2 \\ &= L\langle n_a \rangle \end{aligned} \tag{1.45}$$

as expected, where $\langle n_a \rangle = \left\langle \alpha \left| \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \right| \alpha \right\rangle$. Now, $\langle n_b^2 \rangle$ is calculated

$$\langle n_b^2 \rangle = \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle . \tag{1.46}$$

The operators need to be arranged in normal order, which means having all the lowering operator on the right of the terms and all the raising operator on the left, as I will show. For this, the commutator relation $\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger = 1 + \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}}$ is used. The expression of $\langle n_b^2 \rangle$ becomes

$$\begin{aligned} \langle n_b^2 \rangle &= \left\langle \hat{\mathbf{B}}^\dagger \left(\hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} + 1 \right) \hat{\mathbf{B}} \right\rangle \\ &= \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \hat{\mathbf{B}} \right\rangle + \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle \end{aligned} \tag{1.47}$$

Now, the operators are in normal order. They can easily be evaluated by noting that $\hat{\mathbf{N}}_L$ operates on the state $|0\rangle$ and therefore does not make any contribution. Therefore,

$$\begin{aligned} \langle n_b^2 \rangle &= L^2 \left\langle \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \hat{\mathbf{A}} \right\rangle + L \left\langle \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \right\rangle \\ &= L^2 |\alpha|^4 + L|\alpha|^2 . \end{aligned} \tag{1.48}$$

Therefore,

$$(\Delta n)^2 = L\langle n_a \rangle . \tag{1.49}$$

The output variance is equal to the output average number of photons. This is due to the fact that the output signal is a coherent state. The added noise $\hat{\mathbf{N}}_L$ guaranties that the output field satisfy the Heisenberg uncertainty principle.

1.6.2 Noise in Phase Insensitive Amplification

Closely following Haus [58], an input coherent state $|\alpha\rangle$ is assumed, whose electric field I will represent by a lowering operator $\hat{\mathbf{A}}$. It goes through a phase insensitive amplifier with field gain $\sqrt{G}$. I will denote the output $\hat{\mathbf{B}}$. Just as in the previous section, in order to have $\left[\hat{\mathbf{B}}, \hat{\mathbf{B}}^\dagger\right] = 1$, an extra noise term $\hat{\mathbf{N}}_G$ needs to be added:

$$\hat{\mathbf{B}} = \sqrt{G}\hat{\mathbf{A}} + \hat{\mathbf{N}}_G. \tag{1.50}$$

This noise term $\hat{\mathbf{N}}_G$ satisfies $\left[\hat{\mathbf{N}}_G, \hat{\mathbf{N}}_G^\dagger\right] = 1 - G$. Since $G > 1$, $\left[\hat{\mathbf{N}}_G, \hat{\mathbf{N}}_G^\dagger\right]$ is negative. Therefore, $\hat{\mathbf{N}}_G$ is proportional to a raising operator. Physically, this noise comes from the Heisenberg uncertainty principle. Phase insensitive amplification can be seen as a simultaneous measurement and amplification of the inphase and quadrature phase component, which are two noncommutative observables. Therefore, an associated uncertainty is added. The output number of photons is

$$\begin{aligned} \langle n_b \rangle &= \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle \\ &= \left\langle G\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + \sqrt{G}\left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}_G + \hat{\mathbf{A}}\hat{\mathbf{N}}_G^\dagger\right) + \hat{\mathbf{N}}_G^\dagger\hat{\mathbf{N}}_G \right\rangle \end{aligned} \tag{1.51}$$

since $\hat{\mathbf{N}}_G^\dagger\hat{\mathbf{N}}_G = \hat{\mathbf{N}}_G\hat{\mathbf{N}}_G^\dagger + G - 1$, the expression becomes

$$\langle n_b \rangle = G\langle n_a \rangle + G - 1 \tag{1.52}$$

where $\langle n_a \rangle = \left\langle \alpha \left| \hat{\mathbf{A}}^\dagger \hat{\mathbf{A}} \right| \alpha \right\rangle$ and $G - 1$ is extra noise added by the amplifier. Now, let us calculate $\langle n_b^2 \rangle$

$$\begin{aligned} \langle n_b^2 \rangle &= \left\langle \hat{\mathbf{B}}^\dagger\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger\hat{\mathbf{B}} \right\rangle \\ &= \left\langle \left(\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger - 1\right)\left(\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger - 1\right) \right\rangle \\ &= \left\langle \hat{\mathbf{B}}\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger\hat{\mathbf{B}}^\dagger - 3\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger + 1 \right\rangle. \end{aligned} \tag{1.53}$$

With $\langle n_b^2 \rangle$ expressed in anti normal ordering, the argument can be made that the noise operator $\hat{\mathbf{N}}_G$ does not make any contribution. Therefore,

$$\langle n_b^2 \rangle = \left\langle G^2 \hat{\mathbf{A}}\hat{\mathbf{A}}\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}^\dagger - 3G\hat{\mathbf{A}}\hat{\mathbf{A}}^\dagger + 1 \right\rangle. \tag{1.54}$$

I can now express the relation in normal ordering to evaluate it, which is

$$\begin{aligned}\langle n_b^2 \rangle &= \left\langle G^2 \left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + 1\right)\left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + 1\right) + \left(G^2 - 3G\right)\left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + 1\right) + 1 \right\rangle \\ &= \left\langle G^2 \left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}\hat{\mathbf{A}} + 3\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + 1\right) + \left(G^2 - 3G\right)\left(\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + 1\right) + 1 \right\rangle \\ &= G^2\langle n_a\rangle^2 + 4G^2\langle n_a\rangle - 3G\langle n_a\rangle + 2G^2 - 3G - 1\end{aligned} \tag{1.55}$$

In this fashion, I can compute the variance

$$\begin{aligned}\Delta n_b^2 &= \langle n_b^2 \rangle - \langle n_b \rangle^2 \\ &= G\langle n_a\rangle + 2G(G-1)\langle n_a\rangle + G(G-1).\end{aligned} \tag{1.56}$$

It can be verified that the output signal variance is not equal to the output average number of photons. Therefore, the output signal is not a coherent state.

1.7 Definition of Signal to Noise Ratio and Noise Figure

In this thesis, I will use the Signal to Noise Ratio (SNR) as a measure of quality of an optical signal. There are more than one way to define SNR. In general, for an electrical signal the definition used is [45]

$$\text{SNR} = \frac{\text{Average Power of Electrical Signal}}{\text{Average Power of Electric Noise}}. \tag{1.57}$$

My problems will involve the characterization of optical signals. This can be done by characterizing the signal and the noise of an ideal receiver illuminated by an optical

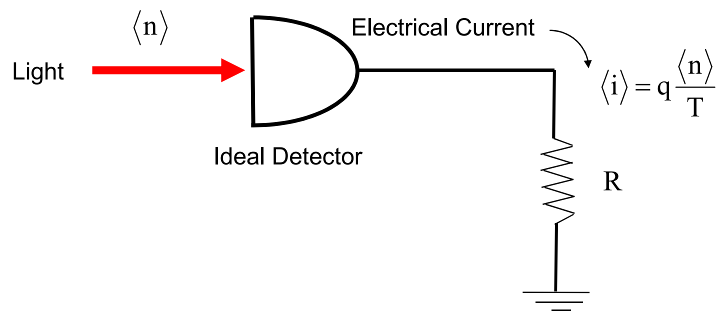

Figure 1.6: Schematic of an ideal receiver

signal (see figure 1.6). In an ideal receiver, for each photon received, the detector emits one electron in a circuit. To characterize the electric current generated by the receiver, I consider a repeated experiment where the statistical average number of photons received $\langle n \rangle$ is defined as the signal. The generated current flows for a time T equal to the duration of the optical signal in a circuit with a resistance R, where R is large enough not to allow any oscillation in the circuit. The average current in this circuit is

$$\langle i \rangle = \frac{q\langle n \rangle}{T}. \tag{1.58}$$

Therefore, the average electric signal power is

$$P_S = \frac{q^2\langle n \rangle^2}{\tau^2 R}. \tag{1.59}$$

The average total electric power is

$$P_{\text{Tot}} = \frac{\langle i^2 \rangle}{R} = \frac{q^2 \langle n^2 \rangle}{\tau^2 R}. \tag{1.60}$$

I can define the noise power as

$$P_N = P_{\text{Tot}} - P_S = \frac{q^2 \langle n^2 \rangle}{T^2 R} - \frac{q^2 \langle n \rangle^2}{\tau^2 R} = \frac{q^2}{T^2 R} \left(\langle n^2 \rangle - \langle n \rangle^2 \right), \tag{1.61}$$

which is proportional to the variance of the number of photons. Therefore, the SNR of the detected optical signal is

$$\text{SNR}_{\text{PNF}} = \frac{P_S}{P_N} = \frac{\langle n \rangle^2}{\langle n^2 \rangle - \langle n \rangle^2} = \frac{\langle n \rangle^2}{(\Delta n)^2}. \tag{1.62}$$

This definition of SNR is called the photon number fluctuations SNR [41]. This SNR is in principle measurable in an experiment and can be readily calculated for some common types of signals. For example, for a coherent state $|\alpha\rangle$ with $\langle n \rangle = |\alpha|^2$ and $(\Delta n)^2 = |\alpha|^2$ it is

$$\text{SNR}_{\text{PNF}} = \frac{|\alpha|^4}{|\alpha|^2} = |\alpha|^2. \tag{1.63}$$

When an optical signal goes through a device, its SNR changes. To compare the noise performance of different devices, a quantity called Noise Figure (NF) is introduced. There are more than one definition of NF based on which SNR is used. In general, the noise figure is [43]

$$\text{NF} = -10 \log \left[\frac{\text{SNR}_{\text{out}}}{\text{SNR}_{\text{in}}} \right]. \tag{1.64}$$

where SNR_{in} is the Signal to Noise Ratio of the signal before the device and SNR_{out} is the Signal to Noise Ratio of the signal after the device.

1.7.1 Definition of the Photon Number Fluctuation Noise Figure

I can use the definition in equation (1.64) to calculate the Photon-Number-Fluctuation NF ($\mathrm{NF}_{\mathrm{PNF}}$) of an attenuator with field loss $\sqrt{L}$. It has been seen in section 1.6.1 that for an input coherent state $|\alpha\rangle$, the output average number of photons is $L|\alpha|^2$ and the photon number fluctuations is $L|\alpha|^2$. Therefore, the output SNR is

$$\mathrm{SNR}_{\mathrm{PNF}} = L\langle n_a \rangle \tag{1.65}$$

Since the input $\mathrm{SNR}_{\mathrm{PNF}} = |\alpha|^2$, the noise figure based on the photon number fluctuation defition is

$$\begin{aligned} \mathrm{NF}_{\mathrm{PNF}} =& \; -10\log\left[\frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{in}}}\right] \\ =& \; -10\log(L) \end{aligned} \tag{1.66}$$

(see figure 1.7). Let us now calculate the noise figure for a phase insensitive amplifier with field gain $\sqrt{G}$. It has been seen in section 1.6.2 that for an input coherent state $|\alpha\rangle$, the output average number of photons is $G|\alpha|^2+G-1$ and the photon number fluctuations is $G|\alpha|^2 + 2G(G-1)|\alpha|^2 + G(G-1)$. Therefore, the output SNR is

$$\mathrm{SNR}_{\mathrm{PNF}} = \frac{G^2|\alpha|^4}{G|\alpha|^2 + 2G(G-1)|\alpha|^2 + G(G-1)}. \tag{1.67}$$

Since the input $\mathrm{SNR}_{\mathrm{PNF}} = |\alpha|^2$, the noise figure is

$$\mathrm{NF}_{\mathrm{PNF}} = \; -10\log\left[\frac{G^2|\alpha|^2}{G|\alpha|^2 + 2G(G-1)|\alpha|^2 + G(G-1)}\right], \tag{1.68}$$

which is plotted in figure 1.8. With higher amplification, there is a higher deterioration of the SNR with the $\mathrm{NF}_{\mathrm{PNF}}$ converging to 3 dB. It can also be seen that the $\mathrm{NF}_{\mathrm{PNF}}$ for phase insensitive amplifiers is signal dependent for weak signal, which can be a problem.

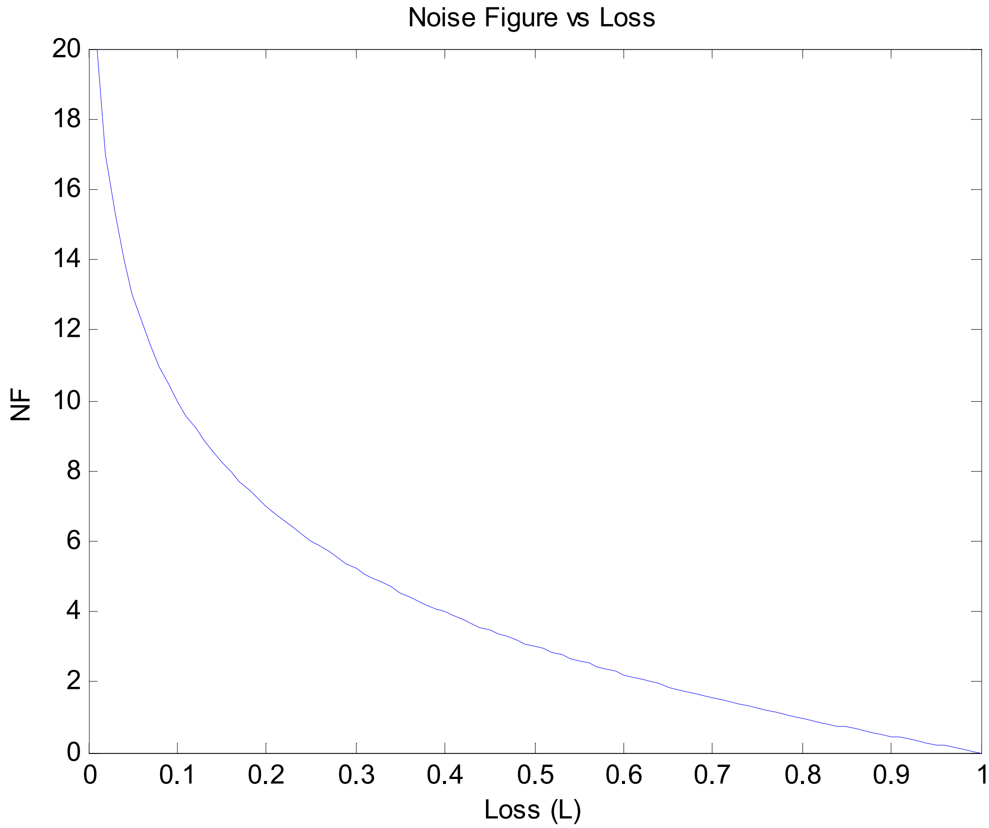

Figure 1.7: Noise Figure in dB as a function of attenuation

In section 5, I will show that this signal dependence of the $\mathrm{NF_{PNF}}$ is also true for $\mathrm{NF_{PNF}}$ of Kerr based NMZI OPA.

However, if a large signal is assumed ($n \gg \frac{1}{2}$), then the $\mathrm{NF_{PNF}}$ for phase insensitive amplifiers becomes

$$\begin{aligned} \mathrm{NF_{PNF}} \approx& -10\log\left[\frac{G^2|\alpha|^2}{G|\alpha|^2+2G(G-1)|\alpha|^2}\right] \\ =& -10\log\left[\frac{G^2}{G+2G(G-1)}\right]. \end{aligned} \tag{1.69}$$

The noise figure is no longer signal dependent. I will show in section 3.4, assuming a large signal, that the $\mathrm{NF_{PNF}}$ is signal independent.

For large values of G,

$$G|\alpha|^2 + 2G(G-1)|\alpha|^2 + G(G-1) \approx 2G^2|\alpha|^2 \tag{1.70}$$

Therefore,

$$\begin{aligned} \mathrm{SNR}_{\mathrm{PNF}} &\approx \frac{G^2\langle n_a\rangle^2}{2G^2\langle n_a\rangle} \\ &= \frac{\langle n_a\rangle}{2}. \end{aligned} \tag{1.71}$$

By equation (1.68) $\mathrm{NF}_{\mathrm{PNF}} = 10\log(2) \approx 3$ dB. That 3 dB is the minimum in consequence of the uncertainty principle. A simultaneous measurement of two noncommuting variables must double the Heisenberg uncertainty [41]. For other reasons, a practical amplifier will have a larger noise figure.

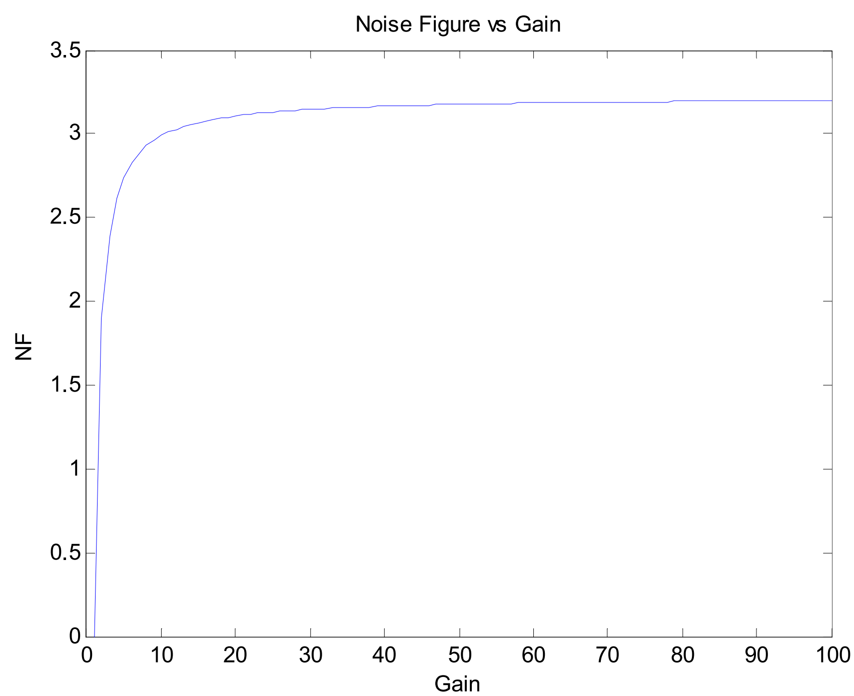

Figure 1.8: Noise Figure in dB as a function of gain (equation (1.67))

1.7.2 Definition of the Field Amplitude Squared Noise Figure

Haus proposed another definition of SNR called the field amplitude squared SNR whose noise figure is not signal dependent. It is defined as follows

$$\mathrm{SNR}_{\mathrm{FAS}} = \frac{\langle E\rangle^2}{(\Delta E)^2}. \tag{1.72}$$

For a coherent state,

$$\mathrm{SNR}_{\mathrm{FAS}} = \frac{|\alpha|^2}{\frac{1}{2}} = 2|\alpha|^2 = 2\langle n\rangle. \tag{1.73}$$

For an attenuator with field loss $\sqrt{L}$ with an input coherent state $|\alpha\rangle$, it was said in section 1.6.1 that the output is also a coherent state with average number of photon $L\langle n\rangle$. Therefore, its output SNR is

$$\mathrm{SNR}_{\mathrm{FAS}} = \frac{|\alpha|^2}{\frac{1}{2}} = 2|\alpha|^2 = 2L\langle n\rangle. \tag{1.74}$$

Therefore, its noise figure is

$$\mathrm{NF}_{\mathrm{FAS}} = -10\log(L) = \mathrm{NF}_{\mathrm{PNF}}. \tag{1.75}$$

1.7.3 $\mathrm{NF}_{\mathrm{FAS}}$ of an Phase Insensitive Amplifier

Let us calculate the field amplitude squared noise figure of the phase insensitive amplifier with field gain $\sqrt{G}$ and with a coherent state $|\alpha\rangle$ as an input. Using equation (1.50), I calculate the variance of the output field, which is

$$\begin{aligned}\Delta\hat{\mathbf{B}}^2 &= \Delta\hat{\mathbf{X}}^2 + \Delta\hat{\mathbf{Y}}^2 \\ &= G - \frac{1}{2}.\end{aligned} \tag{1.76}$$

Since the average number of photons at the output is $G|\alpha^2|$, the output SNR is

$$\mathrm{SNR}_{\mathrm{FAS}_{\mathrm{out}}} = \frac{G|\alpha|^2}{G - \frac{1}{2}}. \tag{1.77}$$

Since the input $\mathrm{SNR}_{\mathrm{FAS}_{\mathrm{in}}} = 2|\alpha|^2$, the noise figure is

$$\mathrm{NF}_{\mathrm{FAS}} = 10 \log \left[\frac{2G}{G - \frac{1}{2}} \right]. \tag{1.78}$$

It can be seen that this noise figure in independent of the signal without any assumption on the value of the signal. Also, for large values of G the field amplitude squared noise figure goes to 3 dB, which is roughly equal to the photon number fluctuations noise figure. However, this definition also has its problems. The electric field is not a measurable quantity. Therefore, for the calculations of noise figure, the signal to noise ratio has to be estimated. Moreover, this definition of the SNR assumes that the inphase component of the noise and the quadrature phase component of the noise are have the same power, which makes it unsuitable for phase sensitive optical parametric amplifiers as I will show.

1.7.4 Definition of the Quadrature Field Squared Noise Figure

A new definition of SNR and its associated noise figure that is more suitable to OPAs is introduced. It is the quadrature field squared SNR and is defined as follows

$$\mathrm{SNR}_{\mathrm{QFS}} = \frac{\langle n \rangle}{\left\langle \Delta \hat{\mathbf{X}}^2 \right\rangle}. \tag{1.79}$$

If this definition is applied to a coherent state $|\alpha\rangle$, its signal to noise ratio is

$$\mathrm{SNR}_{\mathrm{QFS}} = 4|\alpha|^2. \tag{1.80}$$

For an attenuator with field loss $\sqrt{L}$ with an input coherent state $|\alpha\rangle$, the output SNR is

$$\mathrm{SNR}_{\mathrm{QFS}_{\mathrm{out}}} = 4L|\alpha|^2. \tag{1.81}$$

Since $\mathrm{SNR}_{\mathrm{QFS}_{\mathrm{in}}} = 4|\alpha|^2$, the attenuator quadrature field squared noise figure is

$$\mathrm{NF}_{\mathrm{QFS}} = -10\log(L) = \mathrm{NF}_{\mathrm{PNF}} = \mathrm{NF}_{\mathrm{FAS}}. \tag{1.82}$$

To calculate the quadrature phase squared noise figure for a phase insensitive amplifier with field gain $\sqrt{G}$, I first calculate the quadrature phase noise, which is

$$\left\langle \delta\hat{\mathbf{X}}^2 \right\rangle = \frac{G - \frac{1}{2}}{2}. \tag{1.83}$$

Since the average number of photons at the output is $G|\alpha^2|$, the output SNR is

$$\mathrm{SNR}_{\mathrm{QFS}_{\mathrm{out}}} = \frac{2G|\alpha|^2}{G - \frac{1}{2}}. \tag{1.84}$$

Since the input $\mathrm{SNR}_{\mathrm{QFS}_{\mathrm{in}}} = 4|\alpha|^2$, the noise figure is

$$\mathrm{NF}_{\mathrm{QFS}} = -10\log\left[\frac{2G}{G - \frac{1}{2}}\right] = \mathrm{NF}_{\mathrm{FAS}}. \tag{1.85}$$

For large values of G, it can be verified that $\mathrm{NF}_{\mathrm{QFS}}$ for the phase insensitive amplifier goes to 3 dB, which is roughly equal to the $\mathrm{NF}_{\mathrm{PNS}}$. The added advantage of the $\mathrm{NF}_{\mathrm{QFS}}$ over the $\mathrm{NF}_{\mathrm{FAS}}$ is that it is in principle measurable.

1.7.5 Definition of the Quality Factor

Another measure of a signal quality is called Quality Factor or Q. It is to similar SNR and is very applicable to On-Off Keying (OOK). It is defined as follows [42]

$$\mathrm{Q} = \frac{\langle n_1\rangle - \langle n_0\rangle}{\Delta n_1 + \Delta n_0}, \tag{1.86}$$

where $\langle n_1 \rangle$ and $\langle n_0 \rangle$ are respectively the average number of photons when bit one (1) is sent and bit zero (0) is sent, and Δn_1 and Δn_0 are respectively the standard deviation of the signal when bit on (1) and bit zero (0) is sent. The Q factor of an Pulse-Amplitude-Modulated (PAM) OOK coherent state signal can be calculated. $\alpha = 0$ is chosen for the bit zero (0) and $\alpha \neq 0$ is chosen for the bit one (1). Since for a coherent state $|\alpha\rangle$ the variance $\Delta n = |\alpha|^2$, $\Delta n_1 = |\alpha|^2$ and $\Delta n_0 = 0$. Therefore,

$$\mathrm{Q} = |\alpha| = \sqrt{\langle n \rangle}. \tag{1.87}$$

The Q factor is very convenient for calculating the Bit Error Rate (BER) [42], which is.

$$\mathrm{BER} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\mathrm{Q}}{\sqrt{2}}\right). \tag{1.88}$$

In this thesis, I will be analyzing the output signal of OPA and using the different noise figures defined above.

1.8 Balanced Homodyne Detection of a Coherent State

Because of similarities to Phase-Sensitive-Amplified detection, I will first analyzed the Balanced Homodyne Detector (BHD), which is more widely known. For this, I assume that all dark and thermal noise is negligible in comparison to the photon noise. Balanced Homodyne Detectors can be readily described in Quantum Mechanics using the Heisenberg picture [2]. I denote $\hat{\mathbf{A}}_{LO}$, $\hat{\mathbf{A}}_s$, $\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2$ respectively, the operators of the complex amplitude of the electric field of the local oscillator, the incoming signal, and the signal incident on the photodetectors after an ideal splitter.

$$\hat{\mathbf{B}}_1 = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{A}}_{LO} + i\hat{\mathbf{A}}_s\right), \tag{1.89}$$

$$\hat{\mathbf{B}}_2 = \frac{1}{\sqrt{2}} \left(\hat{\mathbf{A}}_s - i \hat{\mathbf{A}}_{LO} \right). \tag{1.90}$$

The difference between the charges collected by the two detectors is

$$\begin{aligned} \delta \hat{\mathbf{q}} &= q \left(\hat{\mathbf{B}}_1^\dagger \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2^\dagger \hat{\mathbf{B}}_2 \right) \\ &= - iq \left(\hat{\mathbf{A}}_{LO}^\dagger \hat{\mathbf{A}}_s - \hat{\mathbf{A}}_s^\dagger \hat{\mathbf{A}}_{LO} \right). \end{aligned} \tag{1.91}$$

I assume that the frequency of the local oscillator is the same as the incoming signal frequency. Using the coherent states $|\alpha_{LO}\rangle$ and $|\alpha_s\rangle$ respectively as the wave vectors of the local oscillator signal and the incoming signal, I have

$$\begin{aligned} \langle \delta \hat{\mathbf{q}} \rangle =& - \left\langle \alpha_s \left| \left\langle \alpha_{LO} \left| iq \left(\hat{\mathbf{A}}_{LO}^\dagger \hat{\mathbf{A}}_s - \hat{\mathbf{A}}_s^\dagger \hat{\mathbf{A}}_{LO} \right) \right| \alpha_{LO} \right\rangle \right| \alpha_s \right\rangle \\ &= 2q|\alpha_{LO}\alpha_s| \sin(\phi), \end{aligned} \tag{1.92}$$

where ϕ is the phase of $\alpha_{LO}^* \alpha_s$ Under optimum phase,

$$\langle \delta \hat{\mathbf{q}} \rangle = 2q|\alpha_{LO}\alpha_s|. \tag{1.93}$$

The mean square fluctuation is

$$\begin{aligned} \Delta(\delta \hat{\mathbf{q}})^2 =& \left\langle (\delta \hat{\mathbf{q}})^2 \right\rangle - \langle \delta \hat{\mathbf{q}} \rangle^2 \\ &= q^2 \left(|\alpha_{LO}|^2 + |\alpha_s|^2 \right) \\ &= q^2 \left(\langle n_{LO} \rangle + \langle n_s \rangle \right), \end{aligned} \tag{1.94}$$

where

$$\langle n_{LO} \rangle = \left\langle \hat{\mathbf{A}}_{LO}^\dagger \hat{\mathbf{A}}_{LO} \right\rangle \tag{1.95}$$

and

$$\langle n_s \rangle = \left\langle \hat{\mathbf{A}}_s^\dagger \hat{\mathbf{A}}_s \right\rangle. \tag{1.96}$$

For large relative values of the local oscillator signal, $\langle n_s \rangle$ can be neglected

$$\Delta \hat{\mathbf{q}}^2 = q^2 \langle n_{LO} \rangle. \tag{1.97}$$

Therefore,

$$\text{SNR}_{\text{PNS}} = 4\langle n_s \rangle, \tag{1.98}$$

which is equal to twice field amplitude squared SNR of the optical signal and four times its photon number square SNR. However, the SNR is not the whole story when it comes to determining sensitivity of a receiver in digital communications. Let us calculate its quality factor. To calculate Q for this receiver, $\alpha_s = 0$ is chosen for the bit zero (0) and $\alpha_s \neq 0$ is chosen for the bit one (1). It is

$$\text{Q} = \frac{2q|\alpha_{LO}\alpha_s|}{\sqrt{q^2\langle n_{LO}\rangle} + \sqrt{q^2\langle n_{LO}\rangle}} = \sqrt{\langle n_s \rangle}. \tag{1.99}$$

The quality factor of a ideal detector and a balanced homodyne detector are the same.

1.9 Nonlinear Mach-Zehnder Interferometer OPA Noise

In this section, I derive the expression for the noise at the output of a NMZI OPA without input signal using quantum mechanics, and I compare the results with the one from classical analysis. Assuming that a field operator $\hat{\mathbf{A}}$ can be expanded as a perturbation which is small around the average value, the field can be represented as

$$\hat{\mathbf{A}} = A + \delta\hat{\mathbf{A}} \tag{1.100}$$

where $A \equiv \left\langle \hat{\mathbf{A}} \right\rangle$. This is known as Quazi Linearization. This assumes that $A \gg \delta\hat{\mathbf{A}}$ .Also, the fluctuations can be broken up into a inphase and a quadrature phase as follows

$$\hat{\mathbf{A}} = A + \left(\delta\hat{\mathbf{X}} + i\delta\hat{\mathbf{Y}}\right). \tag{1.101}$$

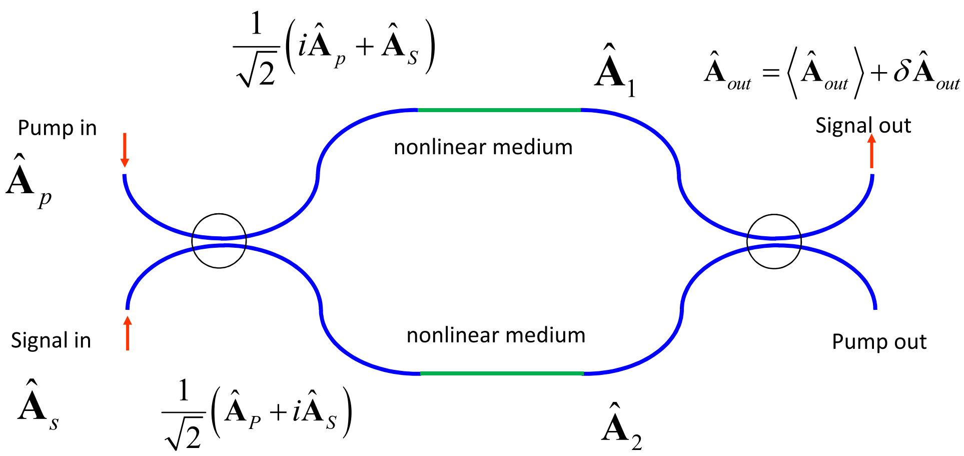

Figure 1.9: Schematic of NMZI OPA. Appropriate operators as detailed in the text are shown.

Now, let us consider a NMZI OPA as shown in figure 1.9, with an input pump field denoted with the operator $\hat{\mathbf{A}}_p$ and a signal field denoted with the operator $\hat{\mathbf{A}}_s$. In the following chapters, I will show how the evolution of the field operator is calculated in a nonlinear medium. Below is the computation of the NMZI output noise assuming that field operators at the output of the two nonlinear medium are known.

I denote respectively $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ the field operators at the inputs to the output coupler. Under normal conditions, the signal comes out in one arm and the pump in the other. On the signal arm, I have

$$\begin{aligned}\hat{\mathbf{A}}_{\text{out}} &= \frac{1}{\sqrt{2}}\left(\hat{\mathbf{A}}_1 + i\hat{\mathbf{A}}_2\right) \\ &= \frac{1}{\sqrt{2}}\left(A_1 + iA_2 + \left[\delta\hat{\mathbf{X}}_1 + i\delta\hat{\mathbf{Y}}_1\right] + i\left[\delta\hat{\mathbf{X}}_2 + i\delta\hat{\mathbf{Y}}_2\right]\right).\end{aligned} \tag{1.102}$$

Assuming that the input signal power is zero, the average field at the input of the nonlinear medium a the upper arm of the NMZI is $\frac{A_p}{\sqrt{2}}$, while the one at the lower arm of the nonlinear medium is $\frac{iA_p}{\sqrt{2}}$. Since the power of the field going through the two nonlinear medium are the same, their amplitude and phase response will be the same. Therefore, at the output $iA_1 = A_2$, so that equation (1.102) becomes

$$\hat{\mathbf{A}}_{\text{out}} = \frac{1}{\sqrt{2}} \left[\left(\delta\hat{\mathbf{X}}_1 - \delta\hat{\mathbf{Y}}_2 \right) + i \left(\delta\hat{\mathbf{Y}}_1 + \delta\hat{\mathbf{X}}_2 \right) \right]. \tag{1.103}$$

The noise in the upper arm of the NMZI OPA described by $\delta\hat{\mathbf{X}}_1$ and $\delta\hat{\mathbf{Y}}_1$ is uncorrelated to the noise in the lower arm described by $\delta\hat{\mathbf{X}}_2$ and $\delta\hat{\mathbf{Y}}_2$)(see appendix A.4). Therefore, the Amplified Spontaneous Emission (ASE) is

$$\begin{aligned} \text{ASE} &= \left\langle \hat{\mathbf{A}}^{\dagger}_{\text{out}} \hat{\mathbf{A}}_{\text{out}} \right\rangle \\ &= \frac{1}{2} \Bigg[\left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle + \left\langle \delta\hat{\mathbf{Y}}_1^2 \right\rangle + i \left\langle \left[\delta\hat{\mathbf{X}}_1, \delta\hat{\mathbf{Y}}_1 \right] \right\rangle + \left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle \\ &\quad + \left\langle \delta\hat{\mathbf{Y}}_2^2 \right\rangle + i \left\langle \left[\delta\hat{\mathbf{X}}_2, \delta\hat{\mathbf{Y}}_2 \right] \right\rangle \Bigg]. \end{aligned} \tag{1.104}$$

Since the noise at the output of the nonlinear medium have the same statistics

$$\text{ASE} = \left[\left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle + \left\langle \delta\hat{\mathbf{Y}}_1^2 \right\rangle + i \left\langle \left[\delta\hat{\mathbf{X}}, \delta\hat{\mathbf{Y}} \right] \right\rangle \right]. \tag{1.105}$$

It is well known that $\delta\hat{\mathbf{X}}$ and $\delta\hat{\mathbf{Y}}$ do not commute. Using equations (1.24) and (1.25) I can calculate their commutator, which is[55]

$$\left[\delta\hat{\mathbf{X}}, \delta\hat{\mathbf{Y}} \right] = \frac{i}{2}, \tag{1.106}$$

and obtain

$$\text{ASE} = \left[\left\langle \delta\hat{\mathbf{Y}}_1^2 \right\rangle + \left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle \right] - \frac{1}{2}. \tag{1.107}$$

Chapter 2

Classical Treatment of the Kerr Medium Based NMZI

2.1 Kerr Effect

Discovered in 1875 by John Kerr, the DC Kerr effect or the quadratic electro-optic effect (QEO effect)is a change in the refractive index of a material in response to the power of an electric field. The optical Kerr effect or the AC Kerr effect is the case in which the change is due to light. The relationship for Optical Kerr effect is described as follows [52, 33]. Let us assume an electric vector field $\mathbf{E} = \mathbf{E_s}\cos(\omega t)$. For a nonlinear material, in general the electric polarization field $\mathbf{P}$ will depend on the electric field $\mathbf{E}$ such that

$$P_i = \epsilon_0 \sum_{j=1}^{3} \chi_{ij}^{(1)} E_j + \epsilon_0 \sum_{j=1}^{3}\sum_{k=1}^{3} \chi_{ijk}^{(2)} E_j E_k + \epsilon_0 \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \chi_{ijkl}^{(3)} E_j E_k E_l + \cdots + HOT, \tag{2.1}$$

where ϵ_0 is the vacuum permittivity and $\chi^{(n)}$ is the n-th order component of the electric susceptibility of the medium, and where $i = 1, 2, 3$. It is assumed that 1 represents x, 2 y and 3 z. For materials exhibiting a non-negligible Kerr effect, the third, $\chi^{(3)}$ term is significant, with the even-order terms typically cancelling due to inversion symmetry of the nonlinear medium (a change in sign in $\mathbf{E}$ means a changing sign in $\mathbf{P}$). From that, the following equation can be obtained [33]:

$$\mathbf{P} \approx \epsilon_0 \left(\chi^{(1)} + \frac{3}{4}\chi^{(3)}|\mathbf{E_s}|^2 \right) \mathbf{E_s}\cos(\omega t). \tag{2.2}$$

This equation looks like the polarization equation for linear material

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E_s} \cos(\omega t). \tag{2.3}$$

The difference is that in (2.2), the linear susceptibility has an extra nonlinear term:

$$\begin{aligned} \chi &= \chi_{Lin} + \chi_{NL} \\ &= \chi^{(1)} + \frac{3}{4}\chi^{(3)}|\mathbf{E_s}|^2. \end{aligned} \tag{2.4}$$

Since the index of refraction $n = (1+\chi)^{\frac{1}{2}}$, we define $n_0 = (1+\chi_{Lin})^{\frac{1}{2}}$. The Kerr medium's index of refraction can be written

$$\begin{aligned} n &= (1 + \chi_{Lin} + \chi_{NL})^{\frac{1}{2}} \\ &= (1+\chi_{Lin})^{\frac{1}{2}}(1 + \frac{1}{1+\chi_{Lin}}\chi_{NL})^{\frac{1}{2}} \\ &= n_0\left(1 + \frac{1}{n_0^2}\chi_{NL}\right)^{\frac{1}{2}} \\ &\approx n_0\left(1 + \frac{1}{2n_0^2}\chi_{NL}\right) \\ &= n_0\left(1 + \frac{3}{4n_0^2}\chi^{(3)}|\mathbf{E_s}|^2\right) \\ &= n_0 + n_2 I. \end{aligned} \tag{2.5}$$

There are many phenomena such as four wave mixing, polarization rotation and cross phase modulation built into the nonlinear index n_2. However, the principal phenomenon is a phase shift due to the retardation of propagation in proportion to the intensity, called self phase modulation. The Kerr medium of choice for many applications is Highly Nonlinear Fiber (HNLF)[4].

2.2 Nonlinear MZI with Kerr Media

I will assume a noiseless pump and a negligeable thermal noise. Let us consider an experimental setup as in figure 1.4. At one input, a strong pump signal is assumed represented by the complex phasor E_p, which I will pick to be real without loss of generality. At the second input of the NMZI, a weak signal E_s is assumed. After the first coupler, the expression of the two outputs are

$$\bar{\mathrm{E}}_{out11} = \frac{1}{\sqrt{2}}\left(E_p + iE_s\right) \tag{2.6}$$

and

$$\bar{\mathrm{E}}_{out21} = \frac{1}{\sqrt{2}}\left(E_s + iE_p\right). \tag{2.7}$$

As discussed earlier, when those two signals go through the Kerr media in both arms of the NMZI, they self phase modulate and pick up a phase shift. It is assumed that the Kerr media in both arms have the same length. The expression of the electric field at the output of the two Kerr media can be expressed as follows

$$\bar{\mathrm{E}}_{out12} = \frac{e^{i\Phi_1}}{\sqrt{2}}\left(E_p + iE_s\right) \tag{2.8}$$

and

$$\bar{\mathrm{E}}_{out22} = \frac{e^{i\Phi_2}}{\sqrt{2}}\left(E_s + iE_p\right). \tag{2.9}$$

the phase of shifts are calculated as follows [52].

$$\begin{aligned}\Phi_1 &= \frac{2\pi}{\lambda}n_2L\left(\frac{\left(E_s - iE_p\right)\left(E_s^* + iE_p\right)}{2}\right) + \frac{2\pi}{\lambda}n_0L \\ &\approx \frac{\pi}{\lambda}n_2L\left(E_p^2 + iE_p\left(E_s - E_s^*\right)\right) + \frac{2\pi}{\lambda}n_0L\end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
\Phi_2 &= \frac{2\pi}{\lambda} n_2 L \left(\frac{(E_p - iE_s)\,(iE_s^* + E_p)}{2} \right) + \frac{2\pi}{\lambda} n_0 L \\
&\approx \frac{\pi}{\lambda} n_2 L \left(E_p^2 - iE_p\,(E_s - E_s^*) \right) + \frac{2\pi}{\lambda} n_0 L.
\end{aligned}
\tag{2.11}
$$

The approximations are valid because E_s is weak compared to E_p.

I can also write $\Phi_1 = \Phi_{10} + \Phi_{11}$ and $\Phi_2 = \Phi_{10} - \Phi_{11}$, where the common phase is

$$
\Phi_{10} = \frac{\pi}{\lambda} n_2 L E_p^2 + \frac{2\pi}{\lambda} n_0 L, \tag{2.12}
$$

and the phase difference is,

$$
\Phi_{11} = -2\frac{\pi}{\lambda} n_2 L E_p \Im(E_s), \tag{2.13}
$$

where $\Im(\cdot)$ means imaginary part. After the two signals propagate through the second coupler, I get at one of the outputs of the NMZI

$$
E_{\text{out}} = \frac{E_p}{2}\left(e^{i\Phi_2} - e^{i\Phi_1}\right) - \frac{iE_s}{2}\left(e^{i\Phi_2} + e^{i\Phi_1}\right). \tag{2.14}
$$

I can rewrite $e^{i\Phi_1} \approx e^{i\Phi_{10}}(1 + i\Phi_{11})$ and $e^{i\Phi_2} \approx e^{i\Phi_{10}}(1 - i\Phi_{11})$ since Φ_{11} is small. This allows a further reduction of the electric field output equation to.

$$
\begin{aligned}
E_{\text{out}} &= -iE_p e^{i\Phi_{10}} \Phi_{11} - ie^{i\Phi_{10}} \\
&= -ie^{i\Phi_{10}} (E_p \Phi_{11} + E_s) \\
&= -ie^{i\Phi_{10}} \left(i\Phi_{10}\,(E_s - E_s^*) + E_s \right).
\end{aligned}
\tag{2.15}
$$

If I rewrite $E_s = |E_s| e^{i\phi}$, then

$$
E_{\text{out}} = -ie^{i\Phi_{10}} |E_s| \left(2\Phi_{10} \sin(\phi) + e^{i\phi} \right). \tag{2.16}
$$

Therefore, the gain of the NMZI is

$$
\begin{aligned}
G &= \left| \frac{E_{\text{out}}}{E_s} \right|^2 \\
&= \left| \frac{2\pi}{\lambda} n_2 L E_p^2 \sin(\phi) + e^{i\phi} \right|^2,
\end{aligned}
\tag{2.17}
$$

which is a phase sensitive gain.

A common constant used to describe the characteristics of a Kerr medium is γ in units of $\mathrm{W}^{-1}\mathrm{km}^{-1}$ [34] which is defined as follows:

$$\gamma = \frac{2\pi n_2}{\lambda A_{\text{eff}}}, \tag{2.18}$$

where A_{eff} is the effective area. In this case $A_{\text{eff}} = 1$ For 1 km of nonlinear fiber with $\gamma = 10$ and a pump signal of 1 W and a signal of 1 mW, a phase sensitive gain is shown in figure 2.1

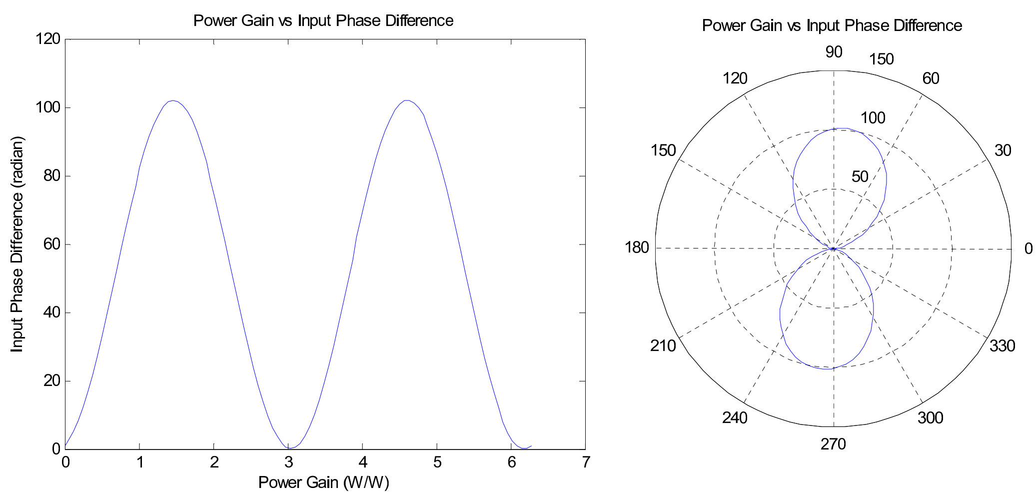

Figure 2.1: Plots of gain as a function of input phase difference for 1 km of Kerr medium with $\gamma = 10$ and pump power of 1 W

2.3 Frequency Response of a NMZI

Even for linear MZI, unbalancing the interferometer by making one arm shorter than the other can introduce bandwidth limitations. For example, let us consider a MZI with the same parameters as the one in section 2.2 but without nonlinear effects and with one arm longer by ΔL. It can be shown following the same procedure as the one in that section that the output field is

$$E_{\text{out}} = ie^{i\frac{2\pi}{\lambda}n_0\left(L+\frac{\Delta L}{2}\right)}\left[E_p \sin\left(\frac{\pi}{\lambda}n_0\Delta L\right) - E_s \cos\left(\frac{\pi}{\lambda}n_0\Delta L\right)\right]. \tag{2.19}$$

Looking only at the field signal output, I assume that its maximum is reached at frequency ω_0. Therefore,

$$\cos\left(\frac{\omega_0}{2c}n_0\Delta L\right) = 1. \tag{2.20}$$

The minimum is reached at frequency $\omega = \omega_0 + \Delta\omega$, where

$$\Delta\omega = \frac{\pi c}{n_0\Delta L}. \tag{2.21}$$

$\Delta\omega$ is the estimated bandwidth. For $\Delta L = 1$mm, the bandwidth is about 100 Ghz. To avoid this bandwidth limitation, an interferometer called Sagnac is used in practice (see figure 2.2). The Sagnac is a MZI folded on itself so that one arm is used instead of two. It guaranties that the interferometer is always balanced, while its behavior is similar to a regular MZI. Therefore, for an easier analysis, the MZI is used. But in practice, the Sagnac is used.

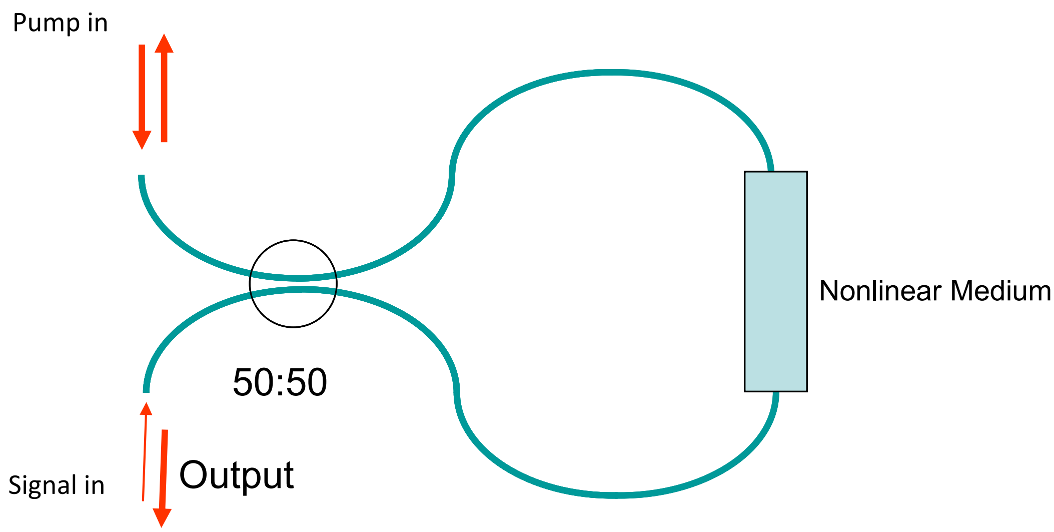

Figure 2.2: Schematic of a Sagnac

2.3.1 Phase Insensitive Gain

Here I am considering a case where the signal and the pump have a different carrier frequency and there is no dispersion in the Kerr medium. If Ω is the difference between their carrier frequencies, then $E_s = A_s(t)e^{i[\Omega t+\phi]}$ is the input signal and $E_p = A_p$ is the pump. Based on equation (2.15), the total output electric field of the NMZI is:

$$\begin{aligned} E^T_{\text{out}} &= ie^{i\omega_0 t}A_s e^{i\Phi_{10}}\left(\Phi_{10}(2\sin(\Omega t+\phi)) + e^{i[\Omega t+\phi]}\right) \\ &= ie^{i\omega_0 t}A_s e^{i\Phi_{10}}\left(\Phi_{10}(4i\left(e^{(\Omega t+\phi)} - e^{-i(\Omega t+\phi)}\right)) + e^{i[\Omega t+\phi]}\right). \end{aligned} \tag{2.22}$$

Note that the output electric field includes the idler signal $A_s(t)e^{-i[(\Omega)t+\phi]}$, which is not part of the input signal. To look at the amplification of the signal, we keep only the terms that includes the input signal and obtain:

$$E_{\text{out}}^S = ie^{i\omega_0 t}A_s e^{i\Phi_{10}}\left(-i\Phi_{10}e^{i(\Omega t+\phi)} + e^{i[\Omega t+\phi]}\right). \tag{2.23}$$

From this I get an expression for the gain, which is:

$$\begin{aligned} G &= \frac{|E_{\text{out}}^S|^2}{|E_s|^2} \\ &= \left|-i\Phi_{10} + 1\right|^2 \\ &= \Phi_{10}^2 + 1, \end{aligned} \tag{2.24}$$

with $\Phi_{10} = \frac{\pi}{\lambda}n_2 L|A_p|^2$. It can be seen that a phase insensitive amplification is obtained and that it is independent of frequency detuning Ω. In the real world, the bandwidth is determined by the finite response time of $\chi^{(3)}$, which can be almost instantaneous.

2.3.2 Phase Sensitive Gain

Consider the signal and the pump to be at the same carrier frequency. However, the signal is amplitude modulated. Therefore, I can write $E_s = A_s(t)e^{i\phi}$ and $E_p = A_p$ where $A_s(t)$ real. Then, based on equation (2.15), the electric field output of the NMZI is:

$$E_{\text{out}} = iA_s(t)e^{i\Phi_{10}}e^{i\omega_0 t}\left(\Phi_{10}(2\sin(\phi)) + e^{i\phi}\right). \tag{2.25}$$

I can then get an expression for the gain, which is:

$$\begin{aligned} G &= \frac{|E_{\text{out}}|^2}{|E_s|^2} \\ &= \left[4\Phi_{10}^2\sin^2(\phi) + 2\Phi_{10}\sin(2\phi) + 1\right], \end{aligned} \tag{2.26}$$

where $\Phi_{10} = \frac{\pi}{\lambda} n_2 L |A_p|^2$. It can be seen that the gain is sensitive to phase but is bandwidth independent.

In conclusion, we can say that the Kerr medium is an important amplifier with large gain and wide bandwidth.

Chapter 3

Quantum Mechanical Analysis of the Lossless Kerr Medium Based NMZI

3.1 Calculation of Average Quantities Using QM

Using notation from section 1.6.2 I can write the output field operator $\hat{\mathbf{B}}$ in terms of the input field operator $\hat{\mathbf{A}}$ by replacing in equation (2.15). Further, by replacing E_s with $\hat{\mathbf{A}}$, E_s^* with $\hat{\mathbf{A}}^\dagger$, and E_{out} with $\hat{\mathbf{B}}$. I obtain:

$$\begin{aligned}\hat{\mathbf{B}} &= -ie^{i\Phi_{10}}\left(i\Phi_{10}\left(\hat{\mathbf{A}} - \hat{\mathbf{A}}^\dagger\right) + \hat{\mathbf{A}}\right) \\ &= \mu\hat{\mathbf{A}} + \nu\hat{\mathbf{A}}^\dagger\end{aligned} \tag{3.1}$$

where $\mu \equiv -ie^{i\Phi_{10}}\left(i\Phi_{10} + 1\right)$ and $\nu \equiv -e^{i\Phi_{10}}\Phi_{10}$. As a cross check, using equation (3.1), I compute the commutator of $\hat{\mathbf{B}}$ and obtain

$$\left[\hat{\mathbf{B}}, \hat{\mathbf{B}}^\dagger\right] = \left(|\mu|^2 - |\nu|^2\right) = 1. \tag{3.2}$$

This time, unlike in section 1.6.2, the commutator has been preserved by the device without the need of the addition of an extra noise term operator in the expression of $\hat{\mathbf{B}}$. This is interesting because there is gain. To explain what happens physically consider a phasor representing the electric field of a coherent state as shown in figure 3.1 and a disk representing uncertainties in amplitude and phase of the electric field. After amplification, the inphase component of the noise with the gain gets amplified while the out of phase component of the noise gets attenuated, in a such way that the product of photon number and phase uncertainties is kept $\Delta n \Delta\phi = \frac{1}{2}$. This noise is called squeezed noise.

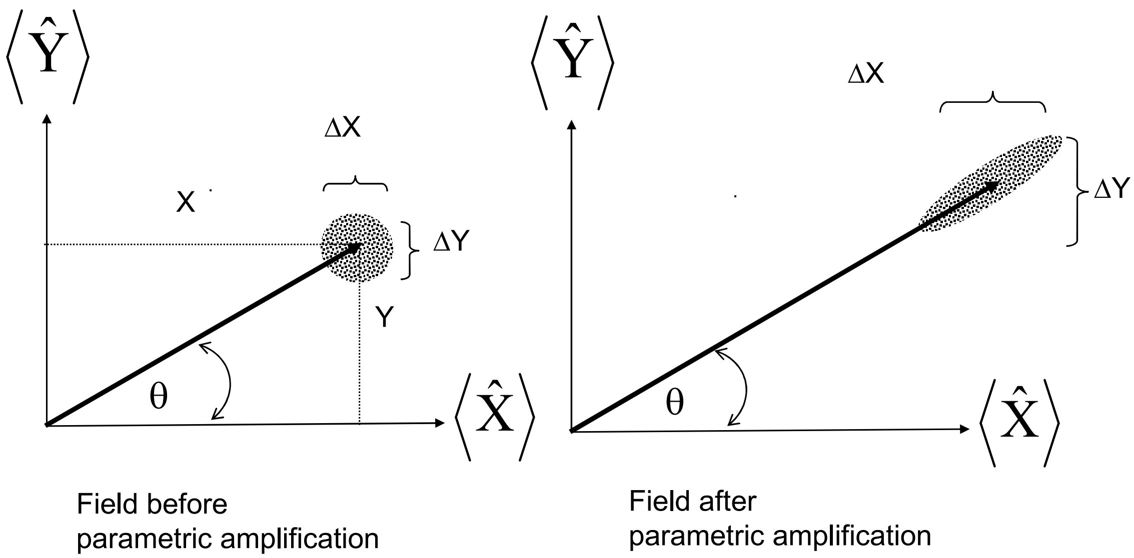

Figure 3.1: Representation of the electric-field properties of the coherent before and after parametric amplification

For an input signal modeled by a coherent state $|\alpha\rangle$, I can compute the number of photons at the output:

$$\begin{aligned} \left\langle \alpha \left| \hat{\mathbf{B}}^{\dagger}\hat{\mathbf{B}} \right| \alpha \right\rangle &= \left\langle \alpha \left| \left(\mu^{*}\hat{\mathbf{A}}^{\dagger} + \nu^{*}\hat{\mathbf{A}}\right) \left(\mu\hat{\mathbf{A}} + \nu\hat{\mathbf{A}}^{\dagger}\right) \right| \alpha \right\rangle \\ &= |\mu|^2|\alpha|^2 + 2\Re\left(\nu^{*}\mu\alpha^2\right) + |\nu|^2\left(|\alpha|^2 + 1\right), \end{aligned} \tag{3.3}$$

where $\Re(\cdot)$ means real part. From the above equation, it can be seen that $(|\mu|^2+|\nu|^2)|\alpha|^2+2\Re\left(\nu^{*}\mu\alpha^2\right)$ is the amplified input signal and ν^2 is the ASE. The maximum signal output happens when $\Re\left(\nu^{*}\mu\alpha^2\right)$ is maximum, which happens when all the term $\nu^{*}\mu\alpha^2$ is real.

Therefore, the maximum signal output relative to signal input phase occurs when

$$\Re\left(\nu^{*}\mu\alpha^{2}\right) = |\mu||\nu||\alpha|^{2}. \tag{3.4}$$

The output is then

$$\begin{aligned}\left\langle \alpha \left| \hat{\mathbf{B}}^{\dagger}\hat{\mathbf{B}} \right| \alpha \right\rangle_{\max} &= |\mu|^{2}|\alpha|^{2} + 2|\mu||\nu||\alpha|^{2} + |\nu|^{2}\left(|\alpha|^{2}+1\right) \\ &= \left(|\mu|+|\nu|\right)^{2}|\alpha|^{2} + |\nu|^{2} \\ &= G|\alpha|^{2} + |\nu|^{2},\end{aligned} \tag{3.5}$$

where

$$G = \left(|\mu|+|\nu|\right)^{2} \tag{3.6}$$

and is the parametric gain. Some useful equations involving the parametric gain are:

$$|\nu|^{2} = \frac{G+G^{-1}-2}{4} \tag{3.7}$$

and

$$|\mu|^{2} = \frac{G+G^{-1}+2}{4}. \tag{3.8}$$

Using these equation, the maximum signal output relative to signal input phase, which occurs when α is real can be obtained. It is

$$\begin{aligned}\langle n_b\rangle_{\max} &= \left(|\mu|+|\nu|\right)^{2}|\alpha|^{2} + \frac{G+G^{-1}-2}{4} \\ &= G\langle n_a\rangle + \frac{G+G^{-1}-2}{4} \\ &\approx G\langle n_a\rangle + \frac{G}{4},\end{aligned} \tag{3.9}$$

where $\langle n_b\rangle \equiv \left\langle \alpha \left| \hat{\mathbf{B}}^{\dagger}\hat{\mathbf{B}} \right| \alpha \right\rangle$ and $\langle n_a\rangle \equiv |\alpha|^{2}$. Since G is assumed to be large, in the order of 20 dB, the following approximation can be made

$$\langle n_b\rangle_{\max} \approx G\langle n_a\rangle + \frac{G}{4}. \tag{3.10}$$

Similarly, the minimum output, which occurs when α is purely imaginary, is:

$$\begin{aligned} \langle n_b \rangle_{\min} =& \ (|\mu| - |\nu|)^2 |\alpha|^2 + \frac{G + G^{-1} - 2}{4} \\ =& \ \frac{\langle n_a \rangle}{G} + \frac{G + G^{-1} + 2}{4} \\ \approx& \ \frac{\langle n_a \rangle}{G} + \frac{G}{4}. \end{aligned} \tag{3.11}$$

3.2 Quantum Mechanical Noise With Loss in Linear Elements

To compute the noise, I calculate the variance of power signal Δn. In order to do so, I compute $\left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle$ using (3.1).

$$\begin{aligned} \left\langle \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \hat{\mathbf{B}}^\dagger \hat{\mathbf{B}} \right\rangle =& \ |\mu|^4 \left(|\alpha|^2 (1 + |\alpha|^2) \right) + 2\Re \left(\mu |\mu|^2 \nu^* \alpha^2 (2 + |\alpha|^2) \right) \\ & + 2\Re \left(\mu |\mu|^2 \nu^* \alpha^2 |\alpha|^2 \right) + |\mu|^2 |\nu|^2 \left(4|\alpha|^4 + 8|\alpha|^2 + 2 \right) \\ & + 2\Re \left(\mu^2 \nu^{*2} \alpha^4 \right) + 2\Re \left(\nu^* \mu |\nu|^2 \alpha^2 \left(4 + 2|\alpha|^2 \right) \right) \\ & + |\nu|^4 \left(1 + 3|\alpha|^2 + |\alpha|^4 \right). \end{aligned} \tag{3.12}$$

I then find $\langle n \rangle^2$ using (3.3) , which is

$$\begin{aligned} \langle n \rangle^2 =& \ |\mu|^4 |\alpha|^4 + 2\Re \left(\nu^{*2} \mu^2 \alpha^4 \right) + |\nu|^4 \left(1 + 2|\alpha|^2 + |\alpha|^4 \right) \\ & + 4\Re \left(|\mu|^2 |\alpha|^2 \nu^* \mu \alpha^2 \right) + 4\Re \left(\nu^* \mu \alpha^2 |\nu|^2 \left(|\alpha|^2 + 1 \right) \right) \\ & + 2|\mu|^2 |\alpha|^2 |\nu|^2 \left(|\alpha|^2 + 1 \right) + 2|\mu|^2 |\nu|^2 |\alpha|^4. \end{aligned} \tag{3.13}$$

The variance is:

$$\begin{aligned} (\Delta n)^2 =& \ \left(|\mu|^4 + |\nu|^4 \right) |\alpha|^2 + 4\Re \left(\mu |\mu|^2 \nu^* \alpha^2 \right) + 4\Re \left(\nu^* \mu |\nu|^2 \alpha^2 \right) \\ & + |\mu|^2 |\nu|^2 \left(6|\alpha|^2 + 2 \right). \end{aligned} \tag{3.14}$$

when the relative phase between the signal and the pump is adjusted to give maximum gain, I use (3.4)

$$(\Delta n)^2 = \left(|\mu|^4 + |\nu|^4 \right) |\alpha|^2 + 4 \left(\mu|^2 + |\nu|^2 \right) |\mu||\nu||\alpha|^2 + |\mu|^2 |\nu|^2 \left(6|\alpha|^2 + 2 \right). \tag{3.15}$$

Substituting equation (3.7) and (3.8) in the above equation and using the approximation $|\mu|^2 \approx |\nu|^2$, I get the variance for the maximum output to be

$$(\Delta n)^2 \approx G^2 \left(\langle n_a \rangle + \frac{1}{8} \right). \tag{3.16}$$

This implies that the output photon number squared SNR is

$$\mathrm{SNR}_{\mathrm{PNS}_{\mathrm{out}}} = \frac{G^2 \langle n_a \rangle^2}{G^2 \left(\langle n_a \rangle + \frac{1}{8} \right)} \tag{3.17}$$

Since $\mathrm{SNR}_{\mathrm{PNS}_{\mathrm{in}}} = \langle n_a \rangle$, the photon number squared NF is

$$\mathrm{NF}_{\mathrm{PNS}} = 10 \log \left[\frac{\langle n_a \rangle}{\left(\langle n_a \rangle + \frac{1}{8} \right)} \right]. \tag{3.18}$$

Even for high parametric gain, the photon number squared NF of this OPA is signal dependent for weak signal. For large signal ($\langle n_a \rangle \gg \frac{1}{8}$), the photon number squared NF approaches 0 dB. Also, I have

$$(\Delta E)^2 = \frac{G + G^{-1}}{4}. \tag{3.19}$$

Therefore, the output field amplitude squared SNR is

$$\mathrm{SNR}_{\mathrm{FAS}_{\mathrm{out}}} = \frac{4G \langle n_a \rangle}{G + G^{-1}}. \tag{3.20}$$

Since $\mathrm{SNR}_{\mathrm{FAS}_{\mathrm{in}}} = 2\langle n_a \rangle$, the field amplitude squared NF is

$$\mathrm{NF}_{\mathrm{FAS}} = -10 \log \left[\frac{2G}{G + G^{-1}} \right]. \tag{3.21}$$

For large parametric gain, $\mathrm{NF}_{\mathrm{FAS}}$ approaches -3 dB, which shows that $\mathrm{NF}_{\mathrm{FAS}}$ is not appropriate for this OPA. The quadrature field noise variance is

$$\left\langle \Delta \hat{\mathbf{X}}^2 \right\rangle = \frac{G + G^{-1} - 1}{4}. \tag{3.22}$$

Therefore, the output quadrature field squared SNR is

$$\mathrm{SNR}_{\mathrm{FAS_{out}}} = \frac{4G\langle n_a\rangle}{G + G^{-1} - 1}. \tag{3.23}$$

Since $\mathrm{SNR}_{\mathrm{QFS_{in}}} = 4\langle n_a\rangle$, the field amplitude squared NF is

$$\mathrm{NF}_{\mathrm{QFS}} = -10\log\left[\frac{G}{G + G^{-1} - 1}\right]. \tag{3.24}$$

For a large parametric gain G, $\mathrm{NF}_{\mathrm{QFS}}$ converges to 0 dB.

3.3 Quantum Mechanical Noise of NMZI with Lumped Loss

Real couplers are usually the largest practical loss and contribute heavily to total noise. To get an estimate of their impact in the NMZI OPA's noise figure, I will consider a lumped loss before the Kerr medium and one after.

3.3.1 Quantum Mechanical Noise of Kerr Based NMZI with all the loss at the Input

If there is a loss that is placed before each Kerr Medium, it is equivalent to placing the losses at the inputs of the NMZI's. I have shown that for an attenuator with loss L in section (1.6.1) $\mathrm{NF}_{\mathrm{QFS}} = \mathrm{NF}_{\mathrm{FAS}} = \mathrm{NF}_{\mathrm{PNS}} = -10\log(L)$. The lossless parametric amplifier has a noise figure of $\mathrm{NF}_{\mathrm{QFS}} = \mathrm{NF}_{\mathrm{PNS}} = 0$ dB and $\mathrm{NF}_{\mathrm{FAS}} = -3$ dB for large parametric gain. Therefore, for a NMZI OPA with lumped loss placed before the Kerr medium, the noise figures are

$$\mathrm{NF}_{\mathrm{PNS}} = \mathrm{NF}_{\mathrm{QFS}} = -10\log(L) \tag{3.25}$$

and

$$\mathrm{NF}_{\mathrm{FAS}} = -10\log(L) - 3 \tag{3.26}$$

Let us now look for noise figures expressions with the lumped loss after the Kerr medium.

3.3.2 Quantum Mechanical Noise of Lossy Kerr Based NMZI with all the loss at the Output

If the losses are placed after the Kerr Medium, it is equivalent to placing them after the NMZI's. The output field operator of the OPA is as given by equation (3.1). The output after the loss is

$$\hat{\mathbf{C}} = \sqrt{L}\hat{\mathbf{B}} + \hat{\mathbf{N}}, \tag{3.27}$$

where $\hat{\mathbf{N}}$ is Langevin noise operator proportional to the annihilation operator and that $\left[\hat{\mathbf{N}}^\dagger, \hat{\mathbf{N}}\right] = 1 - L$. In order to characterize this output signal, I need to calculate its average and its variance. The average signal output is

$$\left\langle \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \right\rangle = L\left[|\mu|^2|\alpha|^2 + 2\Re\left(\nu^*\mu\alpha^2\right) + |\nu|^2\left(|\alpha|^2+1\right)\right] \tag{3.28}$$

To calculate its variance, I first calculate $\left\langle \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \right\rangle$

$$\begin{aligned}\left\langle \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \right\rangle = L^2\Big\{ & |\mu|^4\left(|\alpha|^2(1+|\alpha|^2)\right) + 2\Re\left(\mu|\mu|^2\nu^*\alpha^2(2+|\alpha|^2)\right) \\ & + 2\Re\left(\mu|\mu|^2\nu^*\alpha^2|\alpha|^2\right) + |\mu|^2|\nu|^2\left(4|\alpha|^4 + 8|\alpha|^2 + 2\right) \\ & + 2\Re\left(\mu^2\nu^{*2}\alpha^4\right) + 2\Re\left(\nu^*\mu|\nu|^2\alpha^2\left(4+2|\alpha|^2\right)\right) \\ & + |\nu|^4\left(1+3|\alpha|^2+|\alpha|^4\right)\Big\} + L\Big[6\Re\left(\nu^*\mu\alpha^2\right) \\ & + |\mu|^2\left(1+3|\alpha|^2\right) + |\nu|^2\left(2+3|\alpha|^2\right) + L\Big].\end{aligned} \tag{3.29}$$

I then find $\langle n\rangle^2 = \left\langle \hat{\mathbf{C}}^\dagger \hat{\mathbf{C}} \right\rangle^2$ using (3.28) , which is

$$\begin{aligned}\langle n\rangle^2 = L^2 \Big\{ & |\mu|^4|\alpha|^4 + 2\Re\left(\nu^{*2}\mu^2\alpha^4\right) + |\nu|^4\left(1+2|\alpha|^2+|\alpha|^4\right) \\ & + 4\Re\left(|\mu|^2|\alpha|^2\nu^*\mu\alpha^2\right) + 4\Re\left(\nu^*\mu\alpha^2|\nu|^2\left(|\alpha|^2+1\right)\right) \\ & + 2|\mu|^2|\alpha|^2|\nu|^2\left(|\alpha|^2+1\right) + 2|\mu|^2|\nu|^2|\alpha|^4 \Big\} + L\Big[2|\mu|^2|\alpha|^2 \\ & + 4\Re\left(\nu^*\mu\alpha^2\right) + 2|\nu|^2\left(|\alpha|^2+1\right) + L\Big].\end{aligned} \tag{3.30}$$

Therefore, the variance is

$$\begin{aligned}(\Delta n)^2 = L^2 \Big\{ & \left(|\mu|^4+|\nu|^4\right)|\alpha|^2 + 4\Re\left(\mu|\mu|^2\nu^*\alpha^2\right) + 4\Re\left(\nu^*\mu|\nu|^2\alpha^2\right) \\ & + |\mu|^2|\nu|^2\left(6|\alpha|^2+2\right) \Big\} \\ & + L\Big[2\Re\left(\nu^*\mu\alpha^2\right) + |\mu|^2\left(1+|\alpha|^2\right) + |\nu|^2|\alpha|^2\Big].\end{aligned} \tag{3.31}$$

When the relative phase between the signal and the pump is adjusted to give maximum gain, I use equation (3.4) to get

$$\begin{aligned}(\Delta n)^2 = L^2 \Big\{ & \left(|\mu|^4+|\nu|^4\right)|\alpha|^2 + 4\left(|\mu|^2+|\nu|^2\right)|\mu||\nu||\alpha|^2 + |\mu|^2|\nu|^2\left(6|\alpha|^2+2\right) \Big\} \\ & + L\Big[2|\nu||\mu||\alpha|^2 + |\mu|^2\left(1+|\alpha|^2\right) + |\nu|^2|\alpha|^2\Big].\end{aligned} \tag{3.32}$$

For high parametric gain, using equations (3.7) and (3.8), the approximation $|\mu|^2 \approx |\nu|^2$ can be made. Substituting equation (3.7) and (3.8) in the above equation and using the approximation that $|\mu|^2 \approx |\nu|^2$, I get the variance for the maximum output to be

$$(\Delta n)^2 \approx L^2G^2\left(\langle n_a\rangle + \frac{1}{8}\right) + LG\left(\langle n_a\rangle + \frac{1}{4}\right). \tag{3.33}$$

For high parametric gain G much larger than the loss L, I have

$$(\Delta n)^2 \approx L^2G^2\left(\langle n_a\rangle + \frac{1}{8}\right). \tag{3.34}$$

With the output signal being $GL\langle n_a\rangle$, the output photon number squared SNR is

$$\begin{aligned}\mathrm{SNR}_{\mathrm{PNS_{out}}} &= \frac{G^2L^2\langle n_a\rangle^2}{L^2G^2\left(\langle n_a\rangle + \frac{1}{8}\right)} \\ &\approx \langle n_a\rangle + \frac{1}{8}.\end{aligned} \tag{3.35}$$

Since $\mathrm{SNR}_{\mathrm{PNS_{in}}} = \langle n_a\rangle$,

$$\mathrm{NF}_{\mathrm{PNS}} = -10\log\left[\frac{\langle n_a\rangle}{\langle n_a\rangle + \frac{1}{8}}\right]. \tag{3.36}$$

For large signals $\langle n_a\rangle \gg \frac{1}{8}$, $\mathrm{NF}_{\mathrm{PNS}}$ is roughly 0 dB.

To calculate the field amplitude squared NF, I compute $(\Delta E)^2$, which is

$$(\Delta E)^2 = L\frac{G + G^{-1} - 2}{4} + \frac{1}{2}. \tag{3.37}$$

From this, it is then easy to show, using equations (1.64) and (1.72) , that

$$\mathrm{NF}_{\mathrm{FAS}} = -10\log\left[\frac{2GL}{LG + LG^{-1} - 2L + 2}\right]. \tag{3.38}$$

For a large parametric gain G, $\mathrm{NF}_{\mathrm{FAS}}$ converges to 0 dB.

To calculate the quadrature field squared NF, I compute $\left(\Delta\hat{\mathbf{X}}\right)^2$, which is

$$\left(\Delta\hat{\mathbf{X}}\right)^2 = L\frac{G + G^{-1} - 2}{4} + \frac{1}{4}. \tag{3.39}$$

It is then easy to show that

$$\mathrm{NF}_{\mathrm{QFS}} = -10\log\left[\frac{2GL}{LG + LG^{-1} - 2L + 1}\right]. \tag{3.40}$$

For a large parametric gain G, $\mathrm{NF}_{\mathrm{QFS}}$ converges to 0 dB.

Let us find an expression that will reduce the amount of necessary calculations for the NF.

3.4 Simple Expression of Noise Figure for High Gain Parametric Amplifier

In this section, I derive an expression for the noise for a general high gain parametric amplifier, which will simplify the necessary calculations in the following chapters. For important applications, the amplifier gain will be at least 10 dB. Unlike in the section 1.9, I assume that there is an input signal that is much greater than the noise. I calculate the output photon number variance of the OPA. I denote $\hat{\mathbf{A}}_{\mathrm{out}}$, the field operator at the output of the parametric amplifier. Assuming that perturbation is valid, i.e. $\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle \gg \delta\hat{\mathbf{A}}_{\mathrm{out}}$, I can write

$$\hat{\mathbf{A}}_{\mathrm{out}} = \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle + \delta\hat{\mathbf{A}}_{\mathrm{out}}, \tag{3.41}$$

where $\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle$ is the amplified signal field and $\delta\hat{\mathbf{A}}_{\mathrm{out}}$ is the noise field operator. From this equation above, I find the variance of photon number, which is (see appendix B.1)

$$\begin{aligned}\Delta\left(\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\hat{\mathbf{A}}_{\mathrm{out}}\right)^2 = &\left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle + \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger} \right\rangle \\ &+ \left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}\delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger} \right\rangle + \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^{*2} \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle \\ &+ \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}}\delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle - \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^2 .\end{aligned} \tag{3.42}$$

I assume that the signal power is much greater than the power of the noise. Therefore, I only keep the beat terms between noise and signal since these are the source of the noise in the electrical domain. Therefore, I have

$$\begin{aligned}\Delta\left(\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\hat{\mathbf{A}}_{\mathrm{out}}\right)^2 \approx &\left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle + \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger} \right\rangle \\ &+ \left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}\delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger} \right\rangle + \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^{*2} \left\langle \delta\hat{\mathbf{A}}_{\mathrm{out}}\delta\hat{\mathbf{A}}_{\mathrm{out}} \right\rangle \\ = &\left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle \delta\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger} + \left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle^{*} \delta\hat{\mathbf{A}}_{\mathrm{out}}\right|^2 .\end{aligned} \tag{3.43}$$

The average number of photon at the output is $\left|\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle\right|^2$. At the input, I denote the average input number of photon $|\alpha_{\mathrm{in}}|^2$. I denote by G_p, the parametric gain of the OPA. Therefore,

$$\left\langle \hat{\mathbf{A}}_{\mathrm{out}} \right\rangle = \sqrt{G_p}|\alpha_{\mathrm{in}}|e^{i\phi}, \tag{3.44}$$

where $|\phi|$ is some phase.

$$\Delta \left(\hat{\mathbf{A}}^{\dagger}_{\mathrm{out}} \hat{\mathbf{A}}_{\mathrm{out}} \right)^2 = G_p |\alpha_{\mathrm{in}}|^2 \left| \delta \hat{\mathbf{A}}^{\dagger}_{\mathrm{out}} e^{i\phi} + \delta \hat{\mathbf{A}}_{\mathrm{out}} e^{-i\phi} \right|^2 \tag{3.45}$$

I define $\delta\hat{\mathbf{X}}$ and $\delta\hat{\mathbf{Y}}$, such that

$$\delta\hat{\mathbf{X}} \equiv \frac{\delta \hat{\mathbf{A}}^{\dagger}_{\mathrm{out}} e^{i\phi} + \delta \hat{\mathbf{A}}_{\mathrm{out}} e^{-i\phi}}{2} \tag{3.46}$$

and

$$\delta\hat{\mathbf{Y}} \equiv \frac{\delta \hat{\mathbf{A}}^{\dagger}_{\mathrm{out}} e^{i\phi} - \delta \hat{\mathbf{A}}_{\mathrm{out}} e^{-i\phi}}{2i}. \tag{3.47}$$

This is equivalent to the following expression

$$\delta \hat{\mathbf{A}}_{\mathrm{out}} = \left(\delta\hat{\mathbf{X}} + i\delta\hat{\mathbf{Y}} \right) e^{i\phi}. \tag{3.48}$$

Substituting this expression and equation (3.44) into equation (3.43), I get

$$\Delta \left(\hat{\mathbf{A}}^{\dagger}_{\mathrm{out}} \hat{\mathbf{A}}_{\mathrm{out}} \right)^2 \approx 4G_p |\alpha_{\mathrm{in}}|^2 \left\langle \delta\hat{\mathbf{X}}^2 \right\rangle. \tag{3.49}$$

In equation (1.107), we saw that

$$\mathrm{ASE} = \left[\left\langle \delta\hat{\mathbf{Y}}^2 \right\rangle + \left\langle \delta\hat{\mathbf{X}}^2 \right\rangle \right] - \frac{1}{2}. \tag{3.50}$$

Under the condition that $\left\langle \delta\hat{\mathbf{X}}^2 \right\rangle \gg \left\langle \delta\hat{\mathbf{Y}}^2 \right\rangle$

$$\mathrm{ASE} = \left\langle \delta\hat{\mathbf{X}}^2 \right\rangle. \tag{3.51}$$

This condition is expected since the component of the noise that is in phase with the signal is expected to be considerably amplified compared to the component of the noise in the quadrature phase. Therefore,

$$\Delta\left(\hat{\mathbf{A}}_{\mathrm{out}}^{\dagger}\hat{\mathbf{A}}_{\mathrm{out}}\right)^{2}=4G|\alpha_{\mathrm{in}}|^{2}\mathrm{ASE}. \tag{3.52}$$

I can compute using equation (1.62) the output photon number squared SNR, using equation (3.44) for the signal and (3.52) for the noise. I get

$$\mathrm{SNR}_{\mathrm{PNS}_{\mathrm{out}}}=\frac{G|\alpha_{\mathrm{in}}|^{2}}{4\mathrm{ASE}}. \tag{3.53}$$

For a coherent state input, the input SNR is $|\alpha_{\mathrm{in}}|^2$. Therefore, the photon number squared noise figure is

$$\mathrm{NF}_{\mathrm{PNS}}=-10\log\left[\frac{G}{4\mathrm{ASE}}\right]. \tag{3.54}$$

It is important to note that this result is only valid for large parametric gain (greater than 10dB) and for large signal power (greater than 10 photons).

To compute the quadrature field squared noise figure, I assume an input coherent state $|\alpha_{\mathrm{in}}\rangle$ for an OPA. The output quadrature phase squared SNR is

$$\mathrm{SNR}_{\mathrm{QAS}_{\mathrm{out}}}\approx\frac{G|\alpha_{\mathrm{in}}|^{2}}{\left\langle\delta\hat{\mathbf{X}}^{2}\right\rangle}. \tag{3.55}$$

For large gain, assuming the out of phase component of the noise is negligible relative to the inphase component, the SNR becomes

$$\mathrm{SNR}_{\mathrm{QAS}_{\mathrm{out}}}\approx\frac{G|\alpha_{\mathrm{in}}|^{2}}{\mathrm{ASE}}. \tag{3.56}$$

The input quadrature field squared SNR of an coherent state $|\alpha_{\text{in}}\rangle$ is $4|\alpha_{\text{in}}|^2$, the field amplitude squared noise figure is

$$\text{NF}_{\text{QFS}} = -10\log\left[\frac{G}{4\text{ASE}}\right]. \tag{3.57}$$

This means that for large signals and large parametric gain,

$$\text{NF}_{\text{QFS}} = \text{NF}_{\text{PNS}}. \tag{3.58}$$

Now, I compute NF_{FAS}. I first calculate the field amplitude squared output SNR for an OPA with input coherent state $|\alpha_{\text{in}}\rangle$, which is

$$\begin{aligned}\text{SNR}_{\text{FAS}_{\text{out}}} &= \frac{G|\alpha_{\text{in}}|^2}{(\Delta E)^2} \\ &= \frac{G|\alpha_{\text{in}}|^2}{\left\langle \delta\hat{\mathbf{X}}^2 \right\rangle + \left\langle \delta\hat{\mathbf{Y}}^2 \right\rangle} \\ &= \frac{G|\alpha_{\text{in}}|^2}{\text{ASE} + \frac{1}{2}}.\end{aligned} \tag{3.59}$$

For large gain, I have

$$\text{SNR}_{\text{FAS}_{\text{out}}} \approx \frac{G|\alpha_{\text{in}}|^2}{\text{ASE}}. \tag{3.60}$$

Since the input field amplitude squared SNR of an coherent state $|\alpha_{\text{in}}\rangle$ is $2|\alpha_{\text{in}}|^2$, the field amplitude squared noise figure is

$$\text{NF}_{\text{FAS}} = -10\log\left[\frac{G}{2\text{ASE}}\right], \tag{3.61}$$

which shows that for large signal and large parametric gain,

$$\text{NF}_{\text{FAS}} = \text{NF}_{\text{PNS}_{\text{out}}} - 3 = \text{NF}_{\text{QFS}} - 3. \tag{3.62}$$

This result shows one more time that the $\mathrm{NF}_{\mathrm{FAS}}$ is inappropriate for OPAs. Therefore, I will stop using it. The expression for $\mathrm{NF}_{\mathrm{QFS}}$ and $\mathrm{NF}_{\mathrm{NPS}}$ will be used to significantly reduce the necessary calculations in the following chapters of this thesis.

Chapter 4

Quantum Mechanical Noise of a NMZI with Lossy Kerr Media

4.1 Overview

In the previous chapter, we have looked at the NMZI noise in which loss is either at the input or at the output of the Kerr medium . In this chapter, we look at another case in which the loss is uniformly distributed throughout the Kerr medium. In section 3.3, I have shown that the signal is least degraded when all the loss is located after the Kerr medium, in which case $\mathrm{NF}_{\mathrm{QFS}} \approx \mathrm{NF}_{\mathrm{PNS}} \approx 0$. I have also shown that the degradation is the strongest when all the loss is placed before the Kerr medium, in which case $\mathrm{NF}_{\mathrm{QFS}} \approx \mathrm{NF}_{\mathrm{PNS}} \approx -10\log(L)$. Therefore, I can expect the noise figure for the distributed loss in the Kerr medium to be bracketed by the previous two extreme cases, which is to be between $-10\log(L)$ and 0 dB.

While, this problem has been considered before by Imajuku *et al.* [8], their derivation seems to have errors. In this chapter I show a proper method for solving the problem. In order to solve this problem, I derive the differential equation involving the quantum operators in the lossy Kerr Medium. I then use first order perturbation theory to linearize and solve the differential equation.

4.2 Noise Figure of a Lossy Kerr Based NMZI OPA

4.2.1 Differential Equation of Field Operator in Lossy Kerr Medium

For this analysis, I chose a small segment on the Kerr medium. I can choose a model for that segment from a variety of possibilities. For example, it can be a lossless propagation followed by loss, or loss followed by lossless propagation, or half of loss followed by lossless propagation and then followed by half of loss. These different models for the small segments adds to the same Kerr medium. Therefore, they are all expected to yield the same results. I chose the model in which there is lossless propagation followed by loss because I expect the calculations to be simpler. It is important to note that the results in the previous chapter in which it was said that loss in front of a lossless Kerr medium yield different results from loss after the Kerr medium, assumes a high gain. For infinitesimal length, the gain produced by the Kerr medium is infinitesimal. Therefore, those results do not apply.

During the lossless propagation, the field goes through self phase modulation which can be described by the following equation [54]

$$\frac{\mathrm{d}\hat{\mathbf{A}}(z)}{\mathrm{d}z} = i\gamma\hat{\mathbf{A}}^{\dagger}(z)\hat{\mathbf{A}}(z)\hat{\mathbf{A}}(z). \tag{4.1}$$

Since $\hat{\mathbf{A}}^{\dagger}(z)\hat{\mathbf{A}}(z)$ is invariant of motion within the lossless Kerr medium after propagation $\mathrm{d}z$, I have

$$\begin{aligned}\hat{\mathbf{A}}'(z) &= \exp\left(i\gamma\hat{\mathbf{A}}^{\dagger}(z)\hat{\mathbf{A}}(z)\mathrm{d}z\right)\hat{\mathbf{A}}(z)\\ &\approx \left(1 + i\gamma\hat{\mathbf{A}}^{\dagger}(z)\hat{\mathbf{A}}(z)\mathrm{d}z\right)\hat{\mathbf{A}}(z).\end{aligned} \tag{4.2}$$

Then the signal goes through loss. The signal field operator is expressed as follows:

$$\begin{aligned}\hat{\mathbf{A}}(z+\mathrm{d}z) &= \hat{\mathbf{A}}'(z)e^{-\beta \mathrm{d}z} + \mathrm{d}\hat{\mathbf{f}} \\ &= \hat{\mathbf{A}}(z) + \left(i\gamma\hat{\mathbf{A}}^\dagger(z)\hat{\mathbf{A}}(z) - \beta\right)\hat{\mathbf{A}}(z)\mathrm{d}z + \mathrm{d}\hat{\mathbf{f}},\end{aligned} \tag{4.3}$$

where $\mathrm{d}\hat{\mathbf{f}}$ is Langevinian noise such that

$$\begin{aligned}\left[\mathrm{d}\hat{\mathbf{f}}, \mathrm{d}\hat{\mathbf{f}}^\dagger\right] &= 1 - \left(e^{-\beta \mathrm{d}z}\right)^2 \\ &\approx 2\beta \mathrm{d}z.\end{aligned} \tag{4.4}$$

Therefore, I have

$$\frac{\mathrm{d}\hat{\mathbf{A}}(z)}{\mathrm{d}z} = \left(i\gamma\hat{\mathbf{A}}^\dagger(z)\hat{\mathbf{A}}(z) - \beta\right)\hat{\mathbf{A}}(z) + \hat{\mathbf{N}} \tag{4.5}$$

,

where

$$\left[\hat{\mathbf{N}}(z), \hat{\mathbf{N}}^\dagger(z')\right] = 2\beta\delta(z-z'), \tag{4.6}$$

where $\delta(\cdot)$ is a Dirac delta function.

4.2.2 Using First Order Perturbation Theory

To solve equation (4.5), I introduce $\delta\hat{\mathbf{A}}(z)$ such that

$$\delta\hat{\mathbf{A}}(z) \equiv \hat{\mathbf{A}}(z) - \left\langle \hat{\mathbf{A}}(z) \right\rangle. \tag{4.7}$$

I substitute this in the differential equation and separate fluctuating terms from steady state terms, keep the linear terms and throw away the higher order terms, I get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left(\left\langle \hat{\mathbf{A}}(z)\right\rangle + \delta\hat{\mathbf{A}}(z)\right) = &\left[i\gamma\left(\left|\left\langle \hat{\mathbf{A}}(z)\right\rangle\right|^2 + \left\langle \hat{\mathbf{A}}(z)\right\rangle \delta\hat{\mathbf{A}}^\dagger(z) + \left\langle \hat{\mathbf{A}}(z)\right\rangle^* \delta\hat{\mathbf{A}}(z)\right)\right. \\ &\left. - \beta\right]\left(\left\langle \hat{\mathbf{A}}(z)\right\rangle + \delta\hat{\mathbf{A}}(z)\right) + \hat{\mathbf{N}}.\end{aligned} \tag{4.8}$$

Therefore

$$\frac{\mathrm{d}\delta\hat{\mathbf{A}}(z)}{\mathrm{d}z} = \Bigg[i\gamma\left(\left|\left\langle\hat{\mathbf{A}}(z)\right\rangle\right|^2\delta\hat{\mathbf{A}}(z) + \left\langle\hat{\mathbf{A}}(z)\right\rangle^2\delta\hat{\mathbf{A}}^\dagger(z) + \left|\left\langle\hat{\mathbf{A}}(z)\right\rangle\right|^2\delta\hat{\mathbf{A}}(z)\right) \\ - \beta\delta\hat{\mathbf{A}}(z)\Bigg] + \hat{\mathbf{N}}. \tag{4.9}$$

For the average value equation, I get:

$$\frac{\mathrm{d}\left\langle\hat{\mathbf{A}}(z)\right\rangle}{\mathrm{d}z} = \left(i\gamma\left|\left\langle\hat{\mathbf{A}}(z)\right\rangle\right|^2 - \beta\right)\left\langle\hat{\mathbf{A}}(z)\right\rangle. \tag{4.10}$$

4.2.3 Solving the Differential Equation for the Average Value

To solve (4.10), I take an integral on both side of the equation as follows

$$\int_0^x \frac{\mathrm{d}\left\langle\hat{\mathbf{A}}(z)\right\rangle}{\left\langle\hat{\mathbf{A}}(z)\right\rangle} = \int_0^x \left(i\gamma\left|\left\langle\hat{\mathbf{A}}(z)\right\rangle\right|^2 - \beta\right)\mathrm{d}z. \tag{4.11}$$

Since input state is $|\alpha\rangle$, $\left\langle\hat{\mathbf{A}}(0)\right\rangle \equiv \alpha$. Since this is a propagation of a field through a lossy medium with loss coefficient β,I have

$$\left\langle\hat{\mathbf{A}}(z)\right\rangle = \alpha e^{-\beta z}. \tag{4.12}$$

I substitute this result in the equation then evaluate the integral to get an expression for $\left\langle\hat{\mathbf{A}}(z)\right\rangle$ (see appendix A.1)

$$\left\langle\hat{\mathbf{A}}(z)\right\rangle = \alpha e^{-\beta z}\exp\left(\frac{i\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta z}\right)\right). \tag{4.13}$$

4.2.4 Solving the Differential Equation for the Fluctuating Terms

Finding a solution for equation (4.9) is a bit more involved. I start by simplifying the equation with a series of substitution. I begin with the following for short hand notation:

$$\phi(z) \equiv \frac{\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta z}\right) + \theta_{\mathrm{in}}. \tag{4.14}$$

These equation are then substituted in equation (4.9)

$$\frac{\mathrm{d}\delta\hat{\mathbf{A}}(z)}{\mathrm{d}z} = \left(i2\gamma|\alpha|^2 e^{-2\beta z} - \beta\right)\delta\hat{\mathbf{A}}(z) + \left(i\gamma|\alpha|^2 e^{-2\beta z} e^{2i\phi(z)}\right)\delta\hat{\mathbf{A}}^\dagger(z) + \hat{\mathbf{N}}. \tag{4.15}$$

I make a series of change of variable starting with $\delta\hat{\mathbf{A}}(z) \equiv \delta\hat{\mathbf{B}}e^{i\phi(z)}$, which gives

$$\frac{\mathrm{d}\delta\hat{\mathbf{B}}(z)}{\mathrm{d}z} = \left(i\gamma|\alpha|^2 e^{-2\beta z} - \beta\right)\delta\hat{\mathbf{B}}(z) + i\gamma|\alpha|^2 e^{-2\beta z}\delta\hat{\mathbf{B}}^\dagger(z) + \hat{\mathbf{N}}e^{-i\phi(z)}. \tag{4.16}$$

Then, substituting $\delta\hat{\mathbf{B}}(z) = \delta\hat{\mathbf{C}}(z)e^{-\beta z}$ in equation (4.16) gives

$$\frac{\mathrm{d}\delta\hat{\mathbf{C}}(z)}{\mathrm{d}z} = i\gamma|\alpha|^2 e^{-2\beta z}\left(\delta\hat{\mathbf{C}}(z) + \delta\hat{\mathbf{C}}^\dagger(z)\right) + \hat{\mathbf{N}}e^{-i\phi(z)}e^{\beta z}. \tag{4.17}$$

Then, I introduce the variables

$$\delta\hat{\mathbf{X}} \equiv \frac{\delta\hat{\mathbf{C}}(z) + \delta\hat{\mathbf{C}}^\dagger(z)}{2} \tag{4.18}$$

and

$$\delta\hat{\mathbf{Y}} \equiv \frac{\delta\hat{\mathbf{C}}(z) - \delta\hat{\mathbf{C}}^\dagger(z)}{2i}. \tag{4.19}$$

$\delta\hat{\mathbf{X}}$ and $\delta\hat{\mathbf{Y}}$ are Hermitian and

$$\delta\hat{\mathbf{C}}(z) = \delta\hat{\mathbf{X}} + i\delta\hat{\mathbf{Y}} \tag{4.20}$$

Therefore, if I substitute this relation in equation (4.17), I get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\delta\hat{\mathbf{X}} + i\delta\hat{\mathbf{Y}}\right) = i\gamma|\alpha|^2 e^{-2\beta z}\left(\delta\hat{\mathbf{C}}(z) + \delta\hat{\mathbf{C}}^\dagger(z)\right) + \hat{\mathbf{N}}e^{-i\phi(z)}e^{\beta z}. \tag{4.21}$$

The Hermitian conjugate of the equation is

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\delta\hat{\mathbf{X}} - i\delta\hat{\mathbf{Y}}\right) = -i\gamma|\alpha|^2 e^{-2\beta z}\left(\delta\hat{\mathbf{C}}(z) + \delta\hat{\mathbf{C}}^\dagger(z)\right) + \hat{\mathbf{N}}^\dagger e^{i\phi(z)}e^{\beta z}. \tag{4.22}$$

Summing equations (4.21) and (4.22), I get

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta\hat{\mathbf{X}} = \frac{e^{\beta z}}{2}\left(\hat{\mathbf{N}}^{\dagger}e^{i\phi(z)} + \hat{\mathbf{N}}e^{-i\phi(z)}\right). \tag{4.23}$$

Subtracting equations (4.21) and (4.22), I get

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta\hat{\mathbf{Y}} = 2\gamma|\alpha|^2 e^{-2\beta z}\delta\hat{\mathbf{X}} + \frac{e^{\beta z}}{2i}\left(\hat{\mathbf{N}}e^{-i\phi(z)} - \hat{\mathbf{N}}^{\dagger}e^{i\phi(z)}\right). \tag{4.24}$$

From equation (4.23), I get

$$\delta\hat{\mathbf{X}}(L) = \delta\hat{\mathbf{X}}(0) + \int_0^L \frac{e^{\beta z}}{2}\left(\hat{\mathbf{N}}^{\dagger}e^{i\phi(z)} + \hat{\mathbf{N}}e^{-i\phi(z)}\right)\mathrm{d}z. \tag{4.25}$$

From equation (4.24), I get

$$\begin{aligned}
\delta\hat{\mathbf{Y}}(L) =& \int_0^L\left[2\gamma|\alpha|^2 e^{-2\beta z}\delta\hat{\mathbf{X}}(z) + \frac{e^{\beta z}}{2i}\left(\hat{\mathbf{N}}e^{-i\phi(z)} - \hat{\mathbf{N}}^{\dagger}e^{i\phi(z)}\right)\right]\mathrm{d}z \\
&= \delta\hat{\mathbf{Y}}(0) + \frac{\gamma}{\beta}|\alpha|^2\left(1 - e^{-2\beta L}\right)\delta\hat{\mathbf{X}}(0) \\
&+ \int_0^L 2\gamma|\alpha|^2 e^{-2\beta z}\int_0^z \frac{e^{\beta x}}{2}\left(\hat{\mathbf{N}}^{\dagger}e^{i\phi(x)} + \hat{\mathbf{N}}e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}z \\
&+ \int_0^L \frac{e^{\beta z}}{2i}\left(\hat{\mathbf{N}}e^{-i\phi(z)} - \hat{\mathbf{N}}^{\dagger}e^{i\phi(z)}\right)\mathrm{d}z,
\end{aligned} \tag{4.26}$$

where x is a dummy variable and not a physical dimension. Therefore,

$$\begin{aligned}
\left\langle\delta\hat{\mathbf{X}}^2(L)\right\rangle =& \left\langle\delta\hat{\mathbf{X}}^2(0)\right\rangle \\
&+ \frac{1}{4}\left\langle\int_0^L\int_0^L e^{\beta(z+z')}\left(\hat{\mathbf{N}}^{\dagger}e^{i\phi(z)} + \hat{\mathbf{N}}e^{-i\phi(z)}\right)\left(\hat{\mathbf{N}}^{\dagger}e^{i\phi(z')} + \hat{\mathbf{N}}e^{-i\phi(z')}\right)\mathrm{d}z\mathrm{d}z'\right\rangle \\
&= \left\langle\delta\hat{\mathbf{X}}^2(0)\right\rangle + \frac{1}{4}\int_0^L\int_0^L e^{\beta(z+z')}\left\langle\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^{\dagger}(z')\right\rangle\mathrm{d}z\mathrm{d}z' \\
&= \left\langle\delta\hat{\mathbf{X}}^2(0)\right\rangle + \frac{e^{2\beta L} - 1}{4} \\
&= \frac{e^{2\beta L}}{4},
\end{aligned} \tag{4.27}$$

where I used $\left\langle \delta\hat{\mathbf{X}}^2(0)\right\rangle = \frac{1}{4}$ since our input signal is a coherent state. I will use $|\alpha|^2 = \frac{|\alpha_p|^2}{2}$, where $|\alpha_p|^2$ is the pump power (only half of the pump power goes into each Kerr medium). Therefore,

$$\begin{aligned}
\left\langle \delta\hat{\mathbf{Y}}^2(L)\right\rangle = & \left\langle \delta\hat{\mathbf{Y}}^2(0)\right\rangle + \frac{\gamma^2}{4\beta^2}|\alpha_p|^4\left(1-e^{-2\beta L}\right)^2\left\langle \delta\hat{\mathbf{X}}^2(0)\right\rangle \\
& + \frac{1}{4}\left\langle \int_0^L\int_0^L e^{\beta(z+z')}\left(\hat{\mathbf{N}}^\dagger e^{i\phi(z)} - \hat{\mathbf{N}}e^{-i\phi(z)}\right)\left(\hat{\mathbf{N}}e^{-i\phi(z')} - \hat{\mathbf{N}}^\dagger e^{i\phi(z')}\right)\mathrm{d}z\mathrm{d}z'\right\rangle \\
& + \frac{\gamma^2}{4}|\alpha_p|^4\int_0^L\int_0^L\int_0^z\int_0^{z'} e^{\beta(x-2z+y-2z')+i(\phi(x)-\phi(y))}\left\langle \hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\right\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}z'\mathrm{d}z \\
= & \frac{e^{2\beta L}}{4} + \frac{\gamma^2}{16\beta^2}|\alpha_p|^4\left(1-e^{-2\beta L}\right)^2 + \frac{1}{8}\frac{\gamma^2}{\beta}|\alpha_p|^4\left(1-e^{-2\beta L}\right)\left(L - \frac{1-e^{-2\beta L}}{2\beta}\right) \\
= & \frac{e^{2\beta L}}{4} + \frac{1}{2}\frac{\gamma^2}{4\beta}|\alpha_p|^4\left(1-e^{-2\beta L}\right)L
\end{aligned} \tag{4.28}$$

It is conventional to define

$$L_{\text{eff}} \equiv \frac{1-e^{-2\beta L}}{2\beta}. \tag{4.29}$$

Therefore,

$$\left\langle \delta\hat{\mathbf{Y}}^2(L)\right\rangle = \frac{e^{2\beta L}}{4} + \frac{\gamma^2}{4}|\alpha_p|^4 L_{\text{eff}}L. \tag{4.30}$$

Substituting back the changes of variables, it is easy to show that

$$\delta\hat{\mathbf{A}}(L) = \left(\delta\hat{\mathbf{X}}(L) + i\delta\hat{\mathbf{Y}}(L)\right)e^{i\phi(L)-\beta L}. \tag{4.31}$$

I define $\delta\hat{\mathbf{X}}_B \equiv \delta\hat{\mathbf{X}}e^{\frac{1}{2}(i\phi(L)-\beta L)}$ and $\delta\hat{\mathbf{Y}}_B \equiv \delta\hat{\mathbf{Y}}e^{\frac{1}{2}(i\phi(L)-\beta L)}$. Therefore,

$$\left\langle \delta\hat{\mathbf{X}}_B^2(L)\right\rangle = \frac{1}{4} \tag{4.32}$$

and

$$\left\langle \delta\hat{\mathbf{Y}}_B^2(L)\right\rangle = \frac{1}{4} + \frac{\gamma^2}{4}|\alpha_p|^4 L_{\text{eff}}Le^{-2\beta L} \tag{4.33}$$

Using equation (1.107), I get the output ASE of the parametric amplifier, which is

$$\begin{aligned}\text{ASE} &= \left\langle \delta\hat{\mathbf{X}}_B^2(L)\right\rangle + \left\langle \delta\hat{\mathbf{Y}}_B^2(L)\right\rangle - \frac{1}{2} \\ &= \frac{\gamma^2}{4}|\alpha_p|^4 L_{\text{eff}} L e^{-2\beta L}\end{aligned} \tag{4.34}$$

The large gain expression of the parametric amplifier is derived in appendix A.3 and is

$$G = \gamma^2 e^{-2\beta L}|\alpha_p|^4 L_{\text{eff}}^2. \tag{4.35}$$

The observation of equation (A.24) shows that the field gain phase is $ie^{i\phi(L)}$ relative to that of the input field, while that of the noise is $e^{i\phi(L)}$ (see equation (4.31)). Therefore, at the output, the noise field is $\frac{\pi}{2}$ out of phase relative to our phase frame of reference. Therefore, the in phase component of noise with the output signal is $\delta\hat{\mathbf{Y}}_B$ and the quadrature phase component is $\delta\hat{\mathbf{X}}_B$. $\left\langle \delta\hat{\mathbf{Y}}_B^2(L)\right\rangle$ is much greater than $\left\langle \delta\hat{\mathbf{X}}_B^2(L)\right\rangle$. Therefore, I can use equation (3.54) to get the noise figure since it requires that the in phase component of the noise be much larger than its quadrature phase. I calculate the noise figure and get

$$\text{NF}_{\text{QFS}} = \text{NF}_{\text{PNS}} = -10\log\left[\frac{1-e^{-2\beta L}}{2\beta L}\right]. \tag{4.36}$$

See figure 4.1. This expression was obtained with the approximate expression (3.54) valid only for high gain and large signal. In Appendix B, I derive the photon number squared noise figure of a lossy Kerr Medium based NMZI assuming only that the parametric gain is high. In other words, the input signal can be weak, the expression of the noise figure is as follows

“

$$\text{NF}_{\text{PNS}} = -10\log\left[\frac{\alpha_s^2}{\frac{L}{L_{\text{eff}}}\alpha_s^2 + \frac{1}{8}\left(\frac{L^2}{L_{\text{eff}}^2} - \frac{L}{L_{\text{eff}}} + 1\right)}\right]. \tag{4.37}$$

The noise figure shows that Haus [41] is right since it is dependent on the input signal, which is a problem for this choice of definition of the noise figure.

If I assume that the input signal is large, I can ignore in the noise figure expression any term not containing the input signal power. I get

$$\mathrm{NF}_{\mathrm{PNS}} \approx\ -10\log\left[\frac{\alpha_s^2}{\frac{L}{L_{\mathrm{eff}}}\alpha_s^2}\right]. \tag{4.38}$$

Therefore,

$$\mathrm{NF}_{\mathrm{PNS}} = -10\log\left[\frac{1-e^{-2\beta L}}{2\beta L}\right], \tag{4.39}$$

which is in aggreement with equation (4.36).

This result is different from the result given by Imajuku *et al.* [8], which is

$$\mathrm{NF}_{\mathrm{PNS}} = -10\log\left[1+\frac{2}{(2\beta L)^2}\left(e^{2\beta L}-1-2\beta L-\frac{1}{2}(2\beta L)^2\right)\right] \tag{4.40}$$

4.3 Detailed Comparison with Imajuku's Calculations

In this section, I want to show the steps in Imajuku *et al.*'s calculations that led to the differences in our results. Imajuku *et al.* [8] write that the output field operator of the NMZI OPA is

$$\hat{\mathbf{A}}(L) = \hat{\mathbf{A}}_0(L)e^{-\beta L} + \sqrt{1-e^{-2\beta L}}\hat{\mathbf{\Gamma}}_s(L), \tag{4.41}$$

where

$$\hat{\mathbf{A}}_0(L) = e^{i(\psi+\theta')}\left(\sqrt{1+\left(\frac{\gamma|\alpha_{\mathrm{out}}|^2L}{2}\right)^2}\hat{\mathbf{A}}(0)e^{-i\theta'}+\frac{\gamma|\alpha_{\mathrm{out}}|^2L}{2}\hat{\mathbf{A}}^\dagger(0)e^{i\theta'}\right), \tag{4.42}$$

$$|\alpha_{\mathrm{out}}|^2 = |\alpha|^2\frac{1-e^{-2\beta L}}{2\beta L}, \tag{4.43}$$

$|\alpha|^2$ is the averaged photon number of the input pump,

$$\theta' = \phi_b + \frac{\frac{\pi}{2}-\phi_0}{2}, \tag{4.44}$$

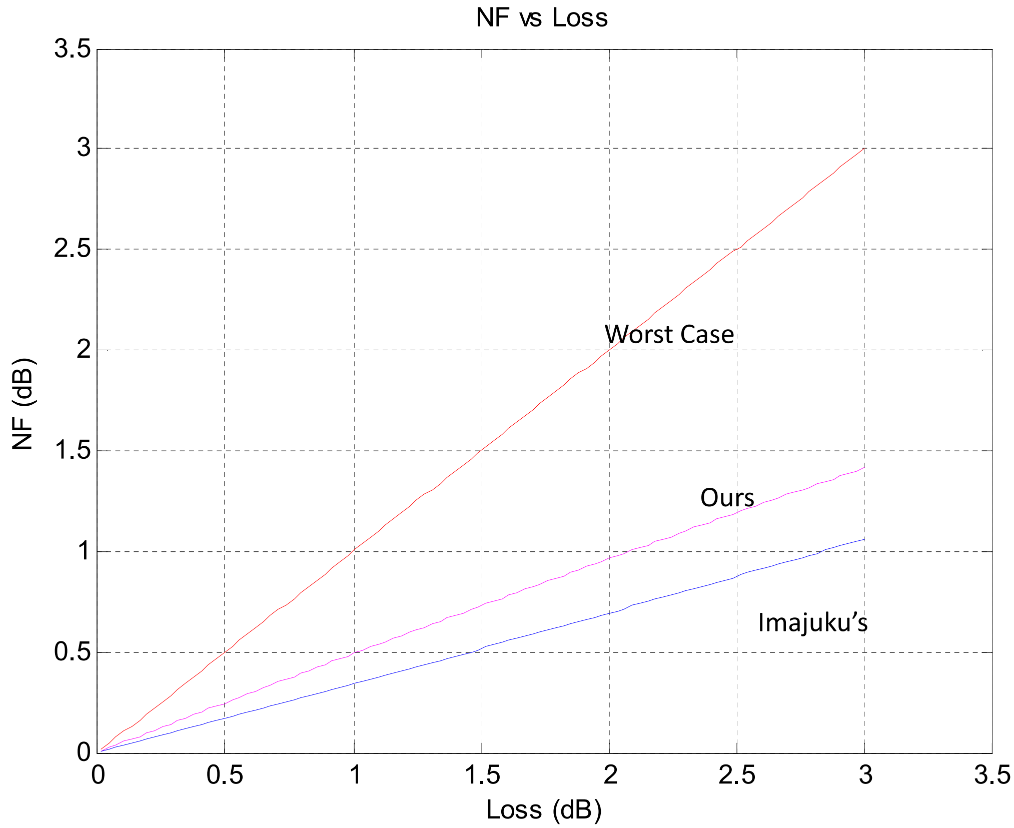

Figure 4.1: Plots of NF vs loss for a Lossy Kerr medium based NMZI-OPA

$$\phi_0 = \arctan\left(\frac{\gamma|\alpha_{\text{out}}|^2 L}{2}\right), \tag{4.45}$$

$$\psi = \phi_0 + \frac{\gamma|\alpha_{\text{out}}|^2 L}{2}, \tag{4.46}$$

ϕ_b is the pump field phase at the output of the NMZI,

$$\begin{aligned}\hat{\mathbf{\Gamma}}_s(L) = e^{i(\psi+\theta')}\sqrt{\frac{2\beta e^{-2\beta L}}{1-e^{-2\beta L}}}\Bigg(&e^{-i\phi_0}\int_0^L e^{\beta z}\hat{\mathbf{c}}_0(z)\mathrm{d}z \\ &+ \frac{\gamma|\alpha_{\text{out}}|^2}{2}\int_0^L\int_0^z \left[ie^{-i\phi_0}e^{\beta z}\hat{\mathbf{c}}_0(z') + e^{\beta z}\hat{\mathbf{c}}_0^\dagger(z')\right]\mathrm{d}z'\mathrm{d}z\Bigg),\end{aligned} \tag{4.47}$$

$$\hat{\mathbf{c}}_0(z) = e^{-i\theta' - i\frac{\gamma|\alpha_{\text{out}}|^2 z}{2}}\hat{\mathbf{c}}(z), \tag{4.48}$$

$$\hat{\mathbf{c}}(z) = \frac{\hat{\mathbf{c}}_{m,1}(z) + \hat{\mathbf{c}}_{m,2}(z)}{\sqrt{2}}, \tag{4.49}$$

$\hat{\mathbf{c}}_{m,1}(z)$ and $\hat{\mathbf{c}}_{m,2}(z)$ are the vacuum operator in each nonlinear medium of the NMZI. They obey the following commutation relation

$$\left[\hat{\mathbf{c}}_{m,i}(z) + \hat{\mathbf{c}}^{\dagger}_{m,i}(z')\right] = \delta(z - z'), \tag{4.50}$$

where $i = 1, 2$. At this point, this result is similar to the one I obtained in equation (A.8). Without any explanation, they use the following mathematical identity (possibly it is an approximation)

$$\int_0^z \hat{\mathbf{c}}_0(z')\mathrm{d}z' = (L - z)\hat{\mathbf{c}}_0(z) \tag{4.51}$$

Consequently, in equation (53) of their publication, they write

$$\begin{aligned}\hat{\mathbf{\Gamma}}_s(L) = e^{i(\psi+\theta')}\sqrt{\frac{2\beta e^{-2\beta L}}{1 - e^{-2\beta L}}}\Bigg(& e^{-i\phi_0}\int_0^L e^{\beta z}\hat{\mathbf{c}}_0(z)\mathrm{d}z \\ &+ \frac{\gamma|\alpha_{\text{out}}|^2}{2}\int_0^L (L - z)\left[ie^{-i\phi_0}e^{\beta z}\hat{\mathbf{c}}_0(z) + e^{\beta z}\hat{\mathbf{c}}^{\dagger}_0(z)\right]\mathrm{d}z\Bigg).\end{aligned} \tag{4.52}$$

From then on, our results diverge. See figure 4.2 and 4.3

4.4 Quantum Mechanical Noise of a NMZI with Gain in the Kerr Medium

It has been shown that the loss in the nonlinear medium of NMZI OPA deteriorates its noise figure. This gives me an idea on what to expect if the nonlinear medium is a saturable absorber. However, it does not give much insight of what to expect in the case where the nonlinear medium is an SOA. For this, I am going to consider a NMZI OPA with gain in the Kerr medium instead of loss.

With a process similar to the previous section (see appendix C), I get the noise figure of a NMZI with gain in the Kerr medium instead of loss. If I choose

$$L_{\text{eff}} \equiv \frac{e^{2g_0 L} - 1}{2g_0}, \tag{4.53}$$

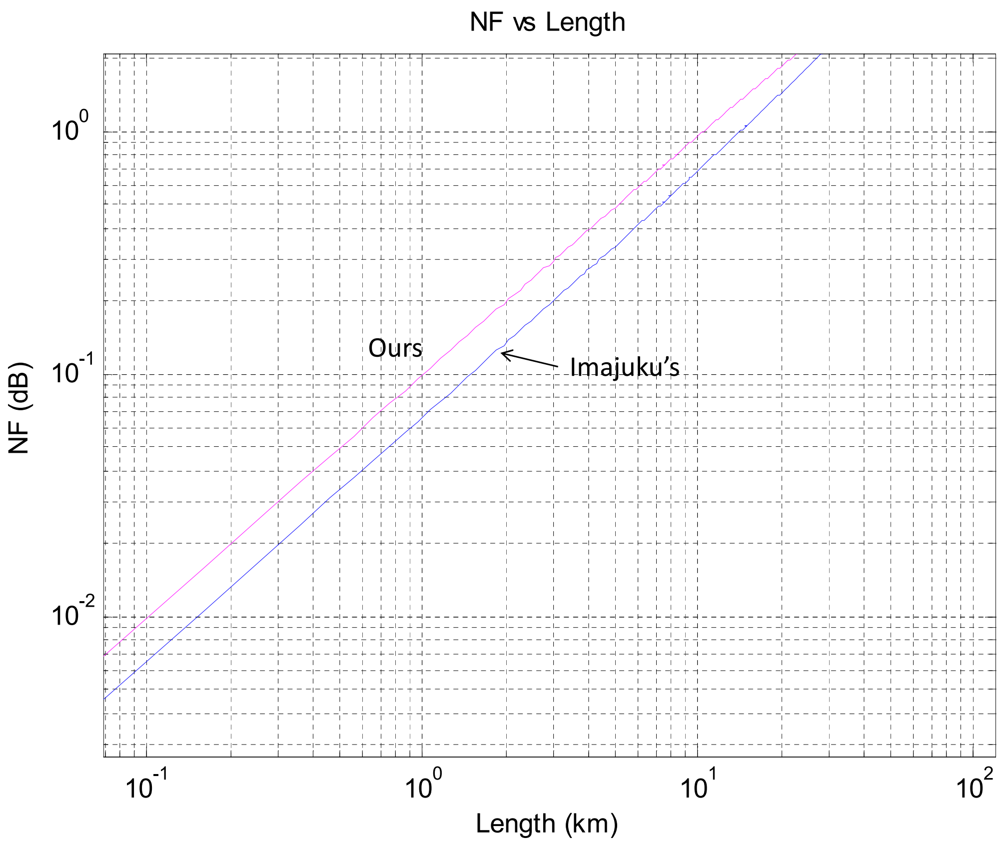

Figure 4.2: Plots of NF_{NPS} vs Length for a Lossy Kerr medium based NMZI-OPA as a comparison to Imajuku's *et al.* figure 6 [8]

then I have

$$\left\langle \delta \hat{\mathbf{E}}_{\text{out}}^{\dagger} \delta \hat{\mathbf{E}}_{\text{out}} \right\rangle \approx \gamma^2 e^{2g_0 L} \alpha_p^4 L_{\text{eff}}^2 \left[\frac{1}{2} - \frac{L g_0}{2\left(e^{2g_0 L} - 1\right)} \right]. \tag{4.54}$$

Under those condition, the expression of the parametric gain is

$$G \approx \gamma^2 e^{g_0 L} \alpha_p^4 L_{\text{eff}}^2. \tag{4.55}$$

I use the noise figure formula I derived earlier

$$\begin{aligned} \text{NF}_{\text{NPS}} \approx & -10 \log \left[\frac{G}{4\text{ASE}} \right] \\ &= 10 \log \left[2 - \frac{L}{L_{\text{eff}}} \right]. \end{aligned} \tag{4.56}$$

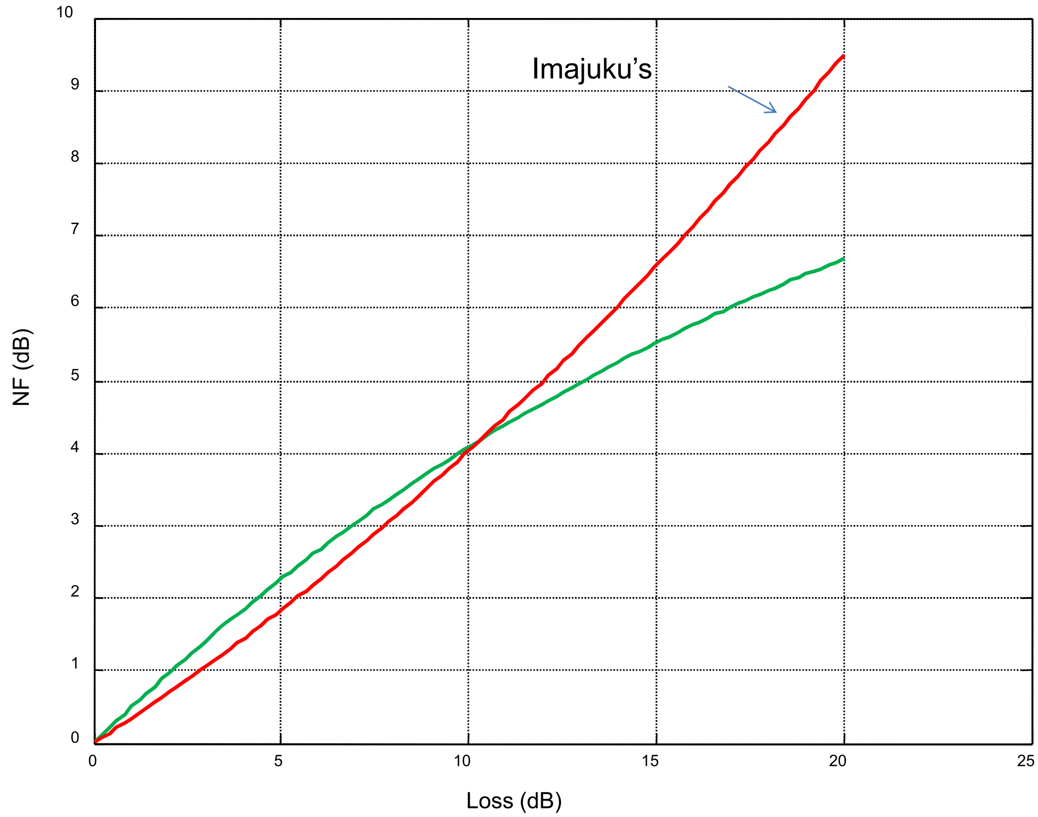

Figure 4.3: Plots of NF_{NPS} vs Loss for a Lossy Kerr medium based NMZI-OPA as a comparison to Imajuku's *et al.*

It can be seen that the noise figure goes very quickly to 3 dB(See figure 4.4).

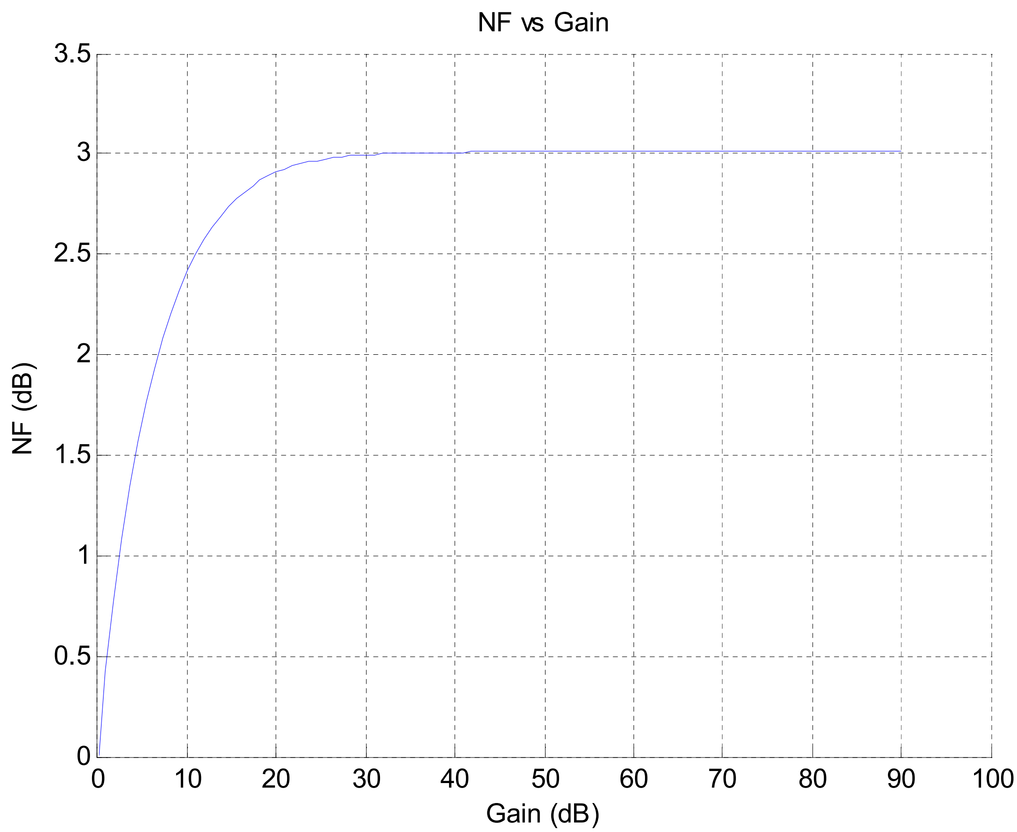

Figure 4.4: Plots of NF_{NPS} vs Gain for Kerr Medium with gain based NMZI-OPA

Chapter 5

Discussions of the Results from the Lossy Kerr Based NMZI OPA

In the previous chapters, I have looked the affects of the loss in Kerr based NMZI OPA on its noise figure. In order to calculate its noise, I calculated quantum mechanically the fluctuations of the field after the nonlinear medium. From that result, I calculated the output ASE of the OPA. From those results interesting things can be noticed.

First, in classical physics, when a noisy signal goes through a lossy medium, both the signal and the noise decay the same way. Therefore, the SNR does not degrade. We have seen that in section 4 that in quantum mechanics this is not true. It was seen that the in phase component of the noise remaining constant (see equation (4.32)) while the quadrature phase component grew (see equation (4.33)). Quantum mechanics guaranteed that the noise did not go below the minimum threshold imposed by the Heisenberg uncertainty principle. Since the signal decays, the SNR degrades. Furthermore, it was seen that the inphase component of the noise, while remaining constant, made the quadrature phase component grow (see equation (4.26)). This difference between classical physics and quantum mechanics justifies the use of quantum mechanics in that section. Moreover, The ASE at the output is the amplified vacuum fluctuations. Let us get a typical value of the ASE for a gain of 30 dB. From equation (4.34) and (4.35), I have

$$\mathrm{ASE} = \frac{G}{4}\left(\frac{2\beta L}{1-e^{-2\beta L}}\right). \tag{5.1}$$

It can be seen in the absence of gain, there is no ASE. Let us choose $2\beta L = \ln(2)$ (this choice will be justified later in this chapter). Then the expression of the ASE becomes

$$\text{ASE} = \frac{G}{2}\ln(2). \tag{5.2}$$

Therefore, for a 30 dB gain, the ASE is about 347 photons. Note that in the absence of loss, it is 250 photons. It is also important to note that classical OPA has classical noise, for example from pump fluctuation (RIN and phase noise) and thermal noise. However, in the absence of photon quantization there is no quantum noise due to the uncertainty of $\langle n \rangle$ and no vacuum input giving rise to ASE. In actuality, for both the classical and quantum OPA, the pump noise is the dominant noise. Loss in classical OPA does however lead to noise. Thermal heat causes the index of refraction to have fluctuations, introducing phase noise, and the thermal Rayleigh scattering causes RIN. These are thoroughly classical noises.

Second, in section 1.6.2, I calculated an expression for the ASE of a NMZI OPA using quantum mechanics. From equation (1.107), the expression of the ASE is

$$\text{ASE} = \left[\left\langle \delta\hat{\mathbf{Y}}_1^2 \right\rangle + \left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle\right] - \frac{1}{2}. \tag{5.3}$$

This expression tells us that if the output from the nonlinear medium are coherent beams, for which $\left\langle \delta\hat{\mathbf{Y}}_1^2 \right\rangle = \left\langle \delta\hat{\mathbf{X}}_1^2 \right\rangle = \frac{1}{4}$ there will be no ASE. We have also seen that the output of a NMZI OPA amplifying a coherent state always contains ASE. Therefore, in the absence of ASE, there is no gain. ASE as derived here is in number of photons. Typically for high gain parametric amplification, ASE is in the order of 10 to a 100 photons. Therefore, the $-\frac{1}{2}$ in the expression of the ASE is not important and can be neglected. The expression becomes the same as the one given by classical physics. Therefore, in that scenario quantum mechanics is not needed.

Table 5.1: Different fiber that could be used in a Kerr based NMZI based OPA and their optimum length and their maximum gain.

Fiber Type	γ (/km/W)	β (dB)	Opt. Length (km)	Max. Gain (dB)
SF	2.2	-0.1	15	20.2
HNL Fiber	15.8	-0.35	4.3	31.9
BOBF	460	$-1.9 \cdot 10^3$	$0.8 \cdot 10^{-3}$	23.8
LSF	640	$-1.3 \cdot 10^3$	$1.2 \cdot 10^{-3}$	28.4

Now, for practical devices Kerr based NMZI OPA, let us calculate the best performance we can expect if different types of lossy fiber were used as Kerr medium. In table 5.1, I show the loss coefficient and the nonlinear coefficient of Standard Fiber (SF)[8], Highly Nonlinear Fiber (HNLF)[8], Bismuth-Oxide Based Fiber (BOBF) [39], Lead Silicate Fiber (LSF) [40]. Let us calculate their optimum gain. The expression of the parametric gain from (4.35) is

$$G = \gamma^2 |\alpha_p|^4 e^{-2\beta L} \left(\frac{1 - e^{-2\beta L}}{2\beta} \right). \tag{5.4}$$

The parametric gain is length dependent (See figure 5.1) . Its maximum as a function of L is reached when

$$L_{\text{opt}} = \frac{\ln(2)}{2\beta}. \tag{5.5}$$

The above expression justifies the earlier choice of $2\beta L = \ln(2)$. The maximum gain is

$$G_{\max} = \frac{\gamma^2}{8\beta} |\alpha_p|^4 \tag{5.6}$$

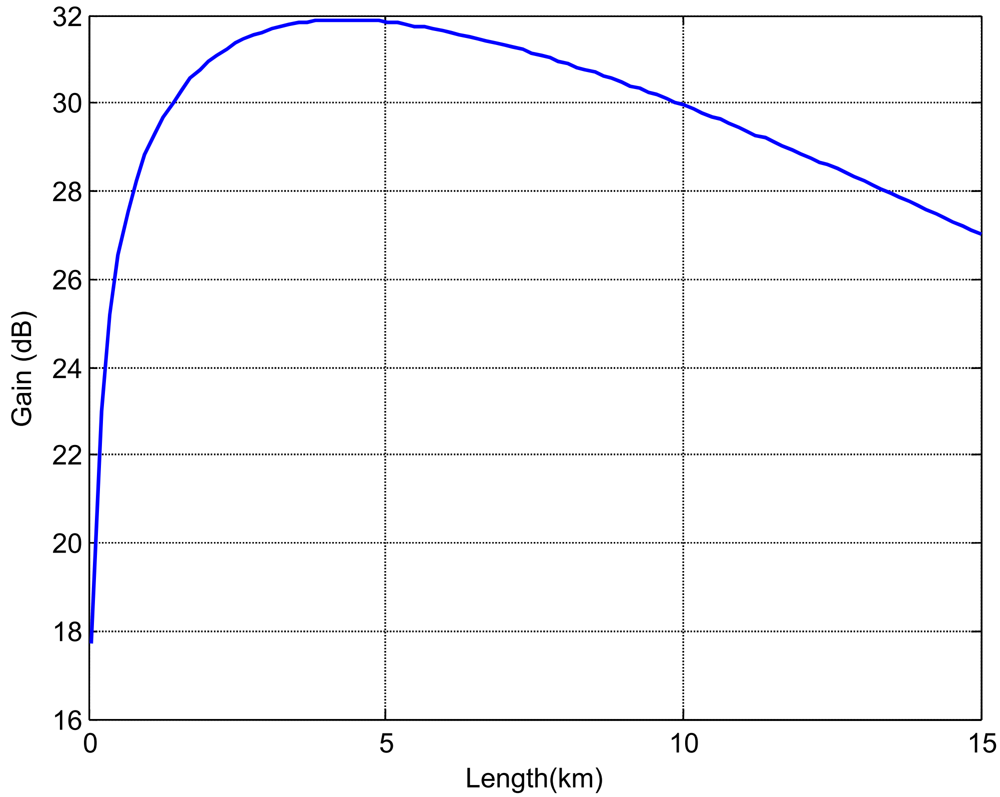

Figure 5.1: Plot of gain versus length for a HNLF ($\gamma = 15.8$)/km/W, $\beta = -0.7$dB/km)

At maximum gain, the noise figure is

$$\mathrm{NF}_{\mathrm{QFS}} = \mathrm{NF}_{\mathrm{PNS}} = -10 \log \left[\frac{1}{2 \ln(2)} \right] \approx 1.4186. \tag{5.7}$$

To illustrate this, I compare the maximum gain for OPAs based on the fibers in table 5.1 for an input pump of 2 Watt. It shows that HNLF fiber would yield the highest gain.

In the next chapter, I consider a different nonlinear medium.

Chapter 6

Semiclassical Treatment of Optical Parametric Amplifier Based on Saturable Absorber and Amplifier

In this chapter, I show that a Saturable Absorber (SA) or Semiconductor Optical Amplifier (SOA) can be used as a non linear medium in the NMZI OPA [65]. The steady state parametric amplifier gain is calculated from a simple classical model. I assume that variation of gain or loss is usually accompanied by large phase modulation as generally described by the Kramers-Kronig relation, which is typical of semiconductor media. The calculations are done for saturable absorber. The results are identical for an amplifier except for the sign of the absorption coefficient.

6.1 Saturable absorber/Amplifier Overview

6.1.1 Loss Characteristics

For a cross section σ , the absorption or gain of a travelling wave is [51]

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = -\sigma N(z)P(z), \tag{6.1}$$

where $P(z)$ is the field amplitude squared, z is the axis of propagation and $N(z)$ is the number of atoms per unit length. For a SA/SOA, the absorption/ stimulated emission of a photon changes N. Assuming one atom is removed for each photon and also assuming that atoms relaxes and return with some time constant τ, the equation of the number of atoms is

$$\frac{\mathrm{d}N(z)}{\mathrm{d}z} = -\sigma\frac{N(z)P(z)}{\hbar\omega} + \frac{N_0 - N(z)}{\tau}, \tag{6.2}$$

where N_0 is the total number of atoms per unit length. In steady state

$$N(z) = \frac{N_0}{1 + \frac{P(z)}{P_{\mathrm{sat}}}}, \tag{6.3}$$

where $P_{\mathrm{sat}} = \frac{\hbar\omega}{\sigma\tau}$. P_{sat} is the saturation power. Substituting the equation above in (6.1), I get

$$\frac{1}{P(z)}\frac{\mathrm{d}I(z)}{\mathrm{d}z} = -\frac{\sigma N_0}{1 + \frac{P(z)}{P_{\mathrm{sat}}}}, \tag{6.4}$$

The absorption/gain coefficient is $\beta(z) = \sigma N(z)$. Multiplying equation (6.2) by σ I get

$$\frac{\mathrm{d}\beta(z)}{\mathrm{d}z} = \frac{\beta(z)P(z)}{\tau P_{\mathrm{sat}}} + \frac{\beta_0 - \beta(z)}{\tau}, \tag{6.5}$$

where $\beta_0 = \sigma N_0$. β_0 is the loss/gain per unit length in the material. Also,[49]

$$\frac{1}{P(z)}\frac{\mathrm{d}P(z)}{\mathrm{d}z} = \frac{-\beta_0}{1 + \frac{P(z)}{P_{\mathrm{sat}}}}. \tag{6.6}$$

The same equation may represent the behavior of a SOA if the sign of β is inverted. β_0 is positive if the device is a SA and is negative if it is an SOA. The solution of this equation in implicit algebraic form for a length L of the device is

$$\ln\left(\frac{P(L)}{P(0)}\right) + \left(\frac{P(L) - P(0)}{P_{\mathrm{sat}}}\right) = -\beta_0 L. \tag{6.7}$$

The solution to this equation is

$$P(L) = P_{\mathrm{sat}} \cdot W\left(\frac{P(0)}{P_{\mathrm{sat}}} e^{\frac{P(0)}{P_{\mathrm{sat}}} - \beta_0 L}\right), \tag{6.8}$$

where $W(y)$ is the Lambert W function defined as the inverse of the function.

$$y = We^W. \tag{6.9}$$

The loss ratio is then [65]

$$\Gamma = \frac{P(L)}{P(0)} = \frac{I_{\text{out}}}{I_{\text{in}}} \tag{6.10}$$

I define

$$\beta = -\ln(\Gamma). \tag{6.11}$$

Therefore,

$$e^{\beta(z)} = \frac{P_{\text{sat}}}{P(0)} \cdot W\left(\frac{P(0)}{P_{\text{sat}}} e^{\frac{P(0)}{P_{\text{sat}}} - \beta_0 z}\right), \tag{6.12}$$

6.1.2 Phase Variation

The SA also shifts the phase of the electric field as it propagates through it. This change in phase depends on the Henry-alpha factor, α_H, of the medium which is the ratio between the real and imaginary parts of a complex loss. Typical values for the Henry-alpha factor ranges between three and five. For an input $E(0)$ in an SA, the output electric field after the SA is [65]

$$E(L) = E(0)e^{-\frac{\beta}{2}(i\alpha_H + 1)}. \tag{6.13}$$

Therefore,

$$\theta_{\text{out}} = -\frac{\beta(z)}{2}\alpha_H + \theta_{\text{in}}, \tag{6.14}$$

where θ_{in} is the phase at the input to the medium.

6.2 Parametric Gain of SA based NMZI-OPA

Here I consider an NMZI as shown in figure 1.4 in which the nonlinear medium is a saturable absorber. At the input of one arm of the NMZI, I have the electric field of the pump E_p, which I will chose to be real, and on the other arm, I have the CW input signal $E_s e^{i\theta_{\text{in}}}$, where θ_{in} is the relative phase between E_s and E_p. As previously, after the first beam coupler, I have

$$E_{\text{out11}} = \frac{1}{\sqrt{2}} \left(E_p + iE_s e^{i\theta_{\text{in}}} \right), \tag{6.15}$$

at the output of the first arm and

$$E_{\text{out11}} = \frac{1}{\sqrt{2}} \left(E_s e^{i\theta_{\text{in}}} + iE_p \right) \tag{6.16}$$

in the other arm. I denote by $P_a(z)$ and $P_b(z)$ respectively the power in the first SA and second SA. From equations (6.15), I obtain

$$P_1(0) = \frac{1}{2} \left(E_s^2 + E_p^2 - 2E_p E_s \sin(\theta_{\text{in}}) \right) \tag{6.17}$$

and from equation (6.16)

$$P_2(0) = \frac{1}{2} \left(E_s^2 + E_p^2 + 2E_p E_s \sin(\theta_{\text{in}}) \right). \tag{6.18}$$

Assuming that E_s is weak, then to the first order I have

$$P_1(0) \approx \frac{1}{2} \left(E_p^2 - 2E_p E_s \sin(\theta_{\text{in}}) \right) \tag{6.19}$$

and

$$P_2(0) \approx \frac{1}{2} \left(E_p^2 + 2E_p E_s \sin(\theta_{\text{in}}) \right). \tag{6.20}$$

I will define the total relative loss of each SA using (6.8)

$$\begin{aligned} \Gamma_1 &\equiv \frac{P_1(L)}{P_1(0)} \\ &= \frac{P_{\text{sat}}}{P_1(0)} W \left(\frac{P_1(0)}{P_{\text{sat}}} e^{\frac{P_1(0)}{P_{\text{sat}}} - \beta_0 L} \right) \end{aligned} \tag{6.21}$$

and

$$\begin{aligned}\Gamma_2 &\equiv \frac{P_2(L)}{P_2(0)} \\ &= \frac{P_{\text{sat}}}{P_2(0)} W\left(\frac{P_2(0)}{P_{\text{sat}}} e^{\frac{P_2(0)}{P_{\text{sat}}}-\beta_0 L}\right).\end{aligned} \tag{6.22}$$

I write the exponential loss $\beta_1 \equiv -\ln(\Gamma_1)$ and $\beta_2 \equiv -\ln(\Gamma_2)$. Since at the input of each SA I had

$$E_{\text{out11}}(0) = \frac{1}{\sqrt{2}}\left(E_p + ie^{i\theta_{\text{in}}}E_s\right) \tag{6.23}$$

$$E_{\text{out12}}(0) = \frac{1}{\sqrt{2}}\left(e^{i\theta_{\text{in}}}E_s + iE_p\right). \tag{6.24}$$

At the output of each SA I have

$$E_{\text{out11}}(L) = \frac{e^{-\frac{1}{2}(i\alpha_H+1)\beta_1 L}}{\sqrt{2}}\left(E_p + iE_s\right) \tag{6.25}$$

and

$$E_{\text{out12}}(L) = \frac{e^{-\frac{1}{2}(i\alpha_H+1)\beta_2 L}}{\sqrt{2}}\left(E_s + iE_p\right). \tag{6.26}$$

The output fields of the NMZI are

$$\begin{aligned}E_{\text{out1}} = \frac{1}{2}\Big[&\left(e^{-\frac{1}{2}(i\alpha_H+1)\beta_1 L} - e^{-\frac{1}{2}(i\alpha_H+1)\beta_2 L}\right)E_p \\ &+ ie^{i\theta_{\text{in}}}\left(e^{-\frac{1}{2}(i\alpha_H+1)\beta_1 L} + e^{-\frac{1}{2}(i\alpha_H+1)\beta_2 L}\right)E_s\Big]\end{aligned} \tag{6.27}$$

and

$$\begin{aligned}E_{\text{out2}} = \frac{1}{2}\Big[&i\left(e^{-\frac{1}{2}(i\alpha_H+1)\beta_1 L} + e^{-\frac{1}{2}(i\alpha_H+1)\beta_2 L}\right)E_p \\ &+ e^{i\theta_{\text{in}}}\left(e^{-\frac{1}{2}(i\alpha_H+1)\beta_2 L} - e^{-\frac{1}{2}(i\alpha_H+1)\beta_1 L}\right)E_s\Big].\end{aligned} \tag{6.28}$$

I am only interested in output field one since it is the signal output, which can be rewritten as

$$\begin{aligned}E_{\text{out1}} = &\frac{e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1+\beta_2)L}}{2}\Big[\left(e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)L} - e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)L}\right)E_p \\ &+ ie^{i\theta_{\text{in}}}\left(e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)L} + e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)L}\right)E_s\Big].\end{aligned} \tag{6.29}$$

Next, I simplify the above expression by defining the differential absorption $\beta_1 - \beta_2$ and calculating its approximate expression.

6.2.1 Approximate Expression for the Differential Absorption

I substitute (6.19) and (6.20) in (6.21) and (6.22) and I obtain

$$\begin{aligned}\Gamma_1 = & \frac{P_{\text{sat}}}{E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})} \\ & \times W\left(\frac{\left(E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} e^{\frac{\left(E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} - \beta_0}\right)\end{aligned} \tag{6.30}$$

and

$$\begin{aligned}\Gamma_2 = & \frac{P_{\text{sat}}}{E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})} \\ & \times W\left(\frac{\left(E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} e^{\frac{\left(E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} - \beta_0}\right).\end{aligned} \tag{6.31}$$

For any differentiable function $f(x)$, to the first order, I have

$$f(x + \Delta) - f(x - \Delta) \approx 2f'(x)\Delta. \tag{6.32}$$

I choose $\Delta = 2E_pE_s\sin(\theta_{\text{in}})$. Therefore (see appendix D.1),

$$\Gamma_2 - \Gamma_1 = \Gamma(\Delta) - \Gamma(-\Delta) \approx 2\Delta\Gamma'(E_p^2), \tag{6.33}$$

where

$$\Gamma(x) = \frac{P_{\text{sat}}}{x} \cdot W\left(\frac{x}{P_{\text{sat}}} e^{\frac{x}{P_{\text{sat}}} - \beta_0}\right). \tag{6.34}$$

It can be shown (see appendix D.2)

$$\Gamma'(E_p^2) = -\frac{\Gamma(\Gamma - 1)}{P_{\text{sat}} + \Gamma E_p^2}. \tag{6.35}$$

Therefore,

$$\beta_2 - \beta_1 \approx = 2E_pE_s\frac{(\Gamma - 1)}{P_{\text{sat}} + \Gamma p}\sin(\theta_{\text{in}}), \tag{6.36}$$

where $p \equiv \frac{1}{2}E_p^2$ (see appendix D.3).

6.2.2 Parametric Gain Calculation

Since Δ is small, to the first order $\beta_2 + \beta_1 \approx 2\beta$ where $\beta \equiv \ln(\Gamma(E_p^2))$. Therefore, equation (6.29) can be written

$$\begin{aligned} E_{\text{out1}} \approx & \frac{e^{-\frac{\beta}{2}(i\alpha_H+1)}}{2}\Big[\left(e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)} - e^{-\frac{1}{4}(i\alpha_H+1)(\beta_2-\beta_1)}\right)E_p \\ & + ie^{i\theta_{\text{in}}}\left(e^{-\frac{1}{4}(i\alpha_H+1)(\beta_1-\beta_2)} + e^{-\frac{1}{4}(i\alpha_H+1)(\beta_2-\beta_1)}\right)E_s\Big]. \end{aligned} \tag{6.37}$$

Substituting for the approximation of $\beta_1 - \beta_2$ from equation (6.36) and introducing $p \equiv \frac{1}{2}E_p^2$, I get

$$E_{\text{out1}} \approx e^{-\frac{\beta}{2}(i\alpha_H+1)}\left[ie^{i\theta_{\text{in}}} - (i\alpha_H+1)\frac{(1-\Gamma)\sin(\theta_{\text{in}})}{\Gamma p + P_{\text{sat}}}p\right]E_s. \tag{6.38}$$

I define

$$E_{\text{in}} = E_s e^{i\theta_{\text{in}}}. \tag{6.39}$$

Substituting for E_{in}, I get

$$E_{\text{out1}} = e^{-\frac{\beta}{2}(i\alpha_H+1)}\left[iE_{\text{in}} - \frac{(i\alpha_H+1)}{2i}\frac{(1-\Gamma)}{E_p^2\Gamma + P_{\text{sat}}}p\left(E_{\text{in}} - E_{\text{in}}^*\right)\right]. \tag{6.40}$$

I define μ and ν such that

$$E_{\text{out1}} = \mu E_{\text{in}} + \nu E_{\text{in}}^*, \tag{6.41}$$

then I have

$$\mu \equiv e^{-\frac{\beta}{2}(i\alpha_H+1)}\left[i + \frac{i}{2}(i\alpha_H+1)\frac{(1-\Gamma)}{\Gamma p + P_{\text{sat}}}p\right] \tag{6.42}$$

and

$$\nu \equiv e^{-\frac{\beta}{2}(i\alpha_H+1)}\left[\frac{i}{2}(i\alpha_H+1)\frac{(1-\Gamma)}{\Gamma p+P_{\text{sat}}}p\right]. \tag{6.43}$$

Therefore, the parametric gain is

$$G_{par} = \left|\mu e^{i\theta_{\text{in}}} + \nu e^{-i\theta_{\text{in}}}\right|^2. \tag{6.44}$$

The maximum parametric gain that can be attained by adjusting the relative phase between the pump and the signal is

$$G_{par} = |\mu + \nu|^2. \tag{6.45}$$

The minimum parametric gain that can be attained by adjusting the relative phase between the pump and the signal is

$$G_{par} = |\mu - \nu|^2. \tag{6.46}$$

See figures 6.1 and 6.2

It can be seen that a phase sensitive amplifier is obtained with a possibility of more than 10 dB gain difference between the minimum and the maximum depending on the relative phase.

6.3 Bandwidth of the SA Based NMZI

So far, I have considered a CW signal. In this section, I will look for the bandwidth and noise of the SA based NMZI. As before, I assume that the arms are normally balanced so that that the low power bandwidth is very large.

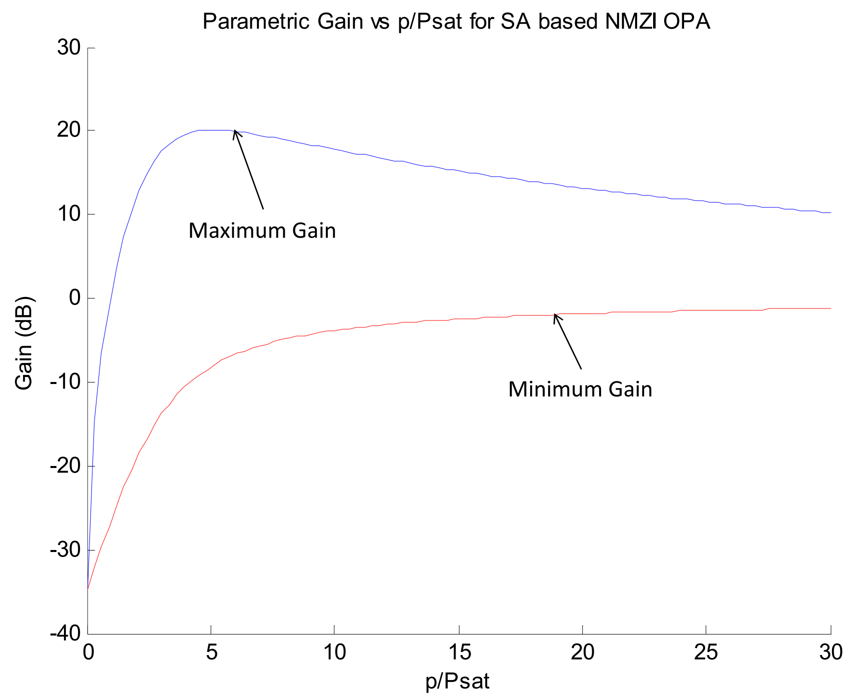

Figure 6.1: Minimum and maximum parametric gain vs p/P_{sat} for a SA based NMZI OPA with $\alpha_H = 25$, $\beta_0 L = 4$.

6.3.1 Response of a SA to a Modulated Signal

To find the response of a SA to a modulated signal, I will assume a weak modulated signal at the input and a strong CW pump. I will write for our signal $E_s = A_s(t)e^{i\theta_{\text{in}}}$ where $A_s(t)$ a real signal and and $E_p = \sqrt{2}A_p$ for the pump. From equation (6.5), the expression of the absoption coefficient is

$$\frac{\partial\beta(z)}{\partial t} = \frac{\beta_0 - \beta(z)}{\tau} - \frac{\beta(z)|E(z)|^2}{P_{\text{sat}}\tau} \tag{6.47}$$

with

$$|E(z)|^2 \approx A_p^2(z) - 2A_p(z)A_s(z,t)\sin(\theta_{\text{in}}) \tag{6.48}$$

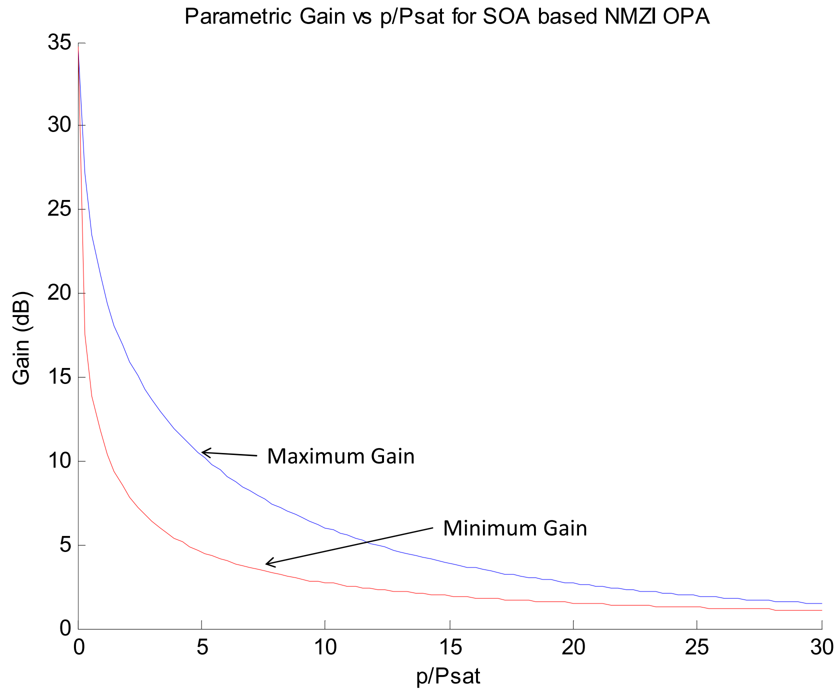

Figure 6.2: Minimum and maximum parametric gain vs. p/P_{sat} for a SOA based NMZI OPA with $\alpha_H = 5$, $\beta_0 L = -4$.

and τ is the carrier lifetime. To simplify the expressions, I will set $P_{\text{sat}} = 1$, which is equivalent to saying that powers are normalized to P_{sat}. I will reintroduce P_{sat} at the end by dividing all the powers by P_{sat}. Using a perturbation technique , I write $\beta(z) = \beta_s(z) + \delta\beta(z)$, where $\beta_s(z)$ is the steady state loss and $\delta\beta(z)$ is the perturbation in loss. $-2A_p(z)A_s(z,t)\sin(\theta_{\text{in}})$ is the driving term of the perturbation. I substitute these relations in equation (6.47) and separate the constant terms from the pertubation terms and obtain the two equations:

$$\tau \frac{\partial \delta\beta(z)}{\partial t} = 2\beta_s(z)A_p(z)A_s(z,t)\sin(\theta_{\text{in}}) - \delta\beta(z)(1 + A_p^2(z)) \tag{6.49}$$

$$0 = \frac{\beta_0 - \beta}{\tau} - \frac{\beta A_p^2(z)}{\tau}. \tag{6.50}$$

Equation (6.50) gives me the steady state solution, which is

$$\beta_s(z) = \frac{\beta_0}{1+P(z)}, \tag{6.51}$$

where $P(z) = A_p^2(z)$. Taking the Fourier transform of equation (6.49), I get

$$i\Omega\tau\delta\tilde{\beta}_\Omega(z) = 2\beta_s(z)A_p(z)\tilde{A}_s(z,\Omega)\sin(\theta_{\rm in}) - \delta\tilde{\beta}_\Omega(z)(1+A_p^2(z)), \tag{6.52}$$

where Ω is the signal envelope frequency, $\delta\tilde{\beta}_\Omega(z)$ is the Fourier transform of $\delta\beta(z)$ and $\tilde{A}_s(z,\Omega)$ is the Fourier transform of $A_s(z,t)$. I shall here after represent the Fourier transform of any variable by a over the variable symbol. Solving for $\delta\tilde{\beta}_\Omega$, I get

$$\delta\tilde{\beta}_\Omega(z) = \frac{-\beta_s(z)\Delta P_s(z,\Omega)}{1+P(z)+i\Omega\tau} \tag{6.53}$$

The equation of the power inside each SA is governed by [52]

$$\frac{\mathrm{d}P_t(z)}{\mathrm{d}z} = -\beta(z)P_t(z). \tag{6.54}$$

Taking the Fourier transform of the equation, I get

$$\frac{\mathrm{d}\tilde{P}_t(z,\Omega)}{\mathrm{d}z} = -\tilde{\beta}_\Omega(z,\Omega)\tilde{P}_t(z,\Omega). \tag{6.55}$$

Using a perturbation technique , I will write $\tilde{P}_t(z,\Omega) = P(z) + \Delta\tilde{P}_s(z,\Omega)$ where $P(z)$ is the constant portion and $\Delta\tilde{P}_s(z,\Omega)$ is the perturbation of the power. Obviously,

$$P(z) \equiv A_p^2(z) \tag{6.56}$$

and

$$\Delta\tilde{P}_s(z,\Omega) \equiv -2A_p(z)\tilde{A}_s(z,\Omega)\sin(\theta_{\rm in}). \tag{6.57}$$

I substitute this perturbation equation in equation (6.55) and obtain

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} + \frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\mathrm{d}z} = -\beta_s(z)P(z) - \delta\tilde{g}_\Omega(z)P(z) - \beta_s(z)\Delta\tilde{P}_s(z,\Omega). \tag{6.58}$$

I separate this equation into a steady state equation and a perturbation equation. For the steady state equation, I have

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = -\beta_s(z)P(z), \tag{6.59}$$

where $\beta_s(z)$ is given by equation (6.51). Therefore,

$$\int_0^L \left(\frac{1}{P(z)} + 1\right) \mathrm{d}P(z) = -\int_0^L \beta_0 \, \mathrm{d}z. \tag{6.60}$$

It has been already seen that the solution to this equation is

$$P(z) = P(0)W\left(P(0)e^{P(0)-\beta_0 z}\right). \tag{6.61}$$

For the perturbation equation, I have

$$\frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\mathrm{d}z} = -\delta\tilde{\beta}_\Omega(z)P(z) - \beta_s(z)\Delta\tilde{P}_s(z,\Omega). \tag{6.62}$$

If I define the total steady state gain as $\Gamma_s = \frac{P(L)}{P(0)}$, then I can show that the solution of this equation is (see appendix E.1)

$$\Delta P_s(L,\Omega) = \Delta P_s(0,\Omega)\Gamma_s\left(\frac{1 + P(0) + i\Omega\tau}{1 + P(L) + i\Omega\tau}\right). \tag{6.63}$$

I will define

$$\Delta\beta(\Omega) = \int_0^L \delta\tilde{\beta}_\Omega(z)\mathrm{d}z. \tag{6.64}$$

Therefore, I can prove that (see appendix E.1)

$$\Delta\beta(\Omega) = \frac{-(1-\Gamma_s)}{1 + \Gamma_s P(0) + i\Omega\tau}\Delta P_s(0,\Omega). \tag{6.65}$$

This results is consistent with the previous results since for $\Omega = 0$, I obtain equation (6.36).

6.3.2 Overall System Output

In this section, I will follow a procedure identical to the one used in section 6.2.2 (see appendix E.2). I define

$$\tilde{E}_{\text{in}}(\Omega) \equiv \tilde{A}_s(\Omega) e^{i\theta_{\text{in}}}. \tag{6.66}$$

The output field is then

$$\tilde{E}_{\text{out}} = \mu \tilde{E}_{\text{in}}(\Omega) + \nu \tilde{E}^*_{\text{in}}(\Omega), \tag{6.67}$$

where

$$\mu \equiv i e^{-\frac{\beta_s}{2}(i\alpha_H + 1)} \left[1 - \frac{1}{2}(i\alpha_H + 1) \frac{(\Gamma - 1)}{\Gamma_s p + P_{\text{sat}} + i\Omega\tau P_{\text{sat}}} p\right] \tag{6.68}$$

and

$$\nu \equiv i e^{-\frac{\beta_s}{2}(i\alpha_H + 1)} \left[\frac{1}{2}(i\alpha_H + 1) \frac{(\Gamma - 1)}{\Gamma p + P_{\text{sat}} + i\Omega\tau P_{\text{sat}}} p\right], \tag{6.69}$$

where P is the power of the input pump on each arm of the NMZI: $p = A_p^2$. Reintroducing P_{sat}, I see that for $\Omega = 0$, equation (6.68) and (6.69) are equivalent to equation (6.42) and (6.43). I can now find the parametric gain, which is

$$\begin{aligned} G(\Omega) &= \frac{|\tilde{E}_{\text{out}}|^2}{|\tilde{A}_s(\Omega)|^2} \\ &= |\mu(\Omega) e^{i\theta_{\text{in}}} + \nu(\Omega)| e^{-i\theta_{\text{in}}}|^2 \end{aligned} \tag{6.70}$$

See figure 6.3.

To calculate the noise figure, I need to use quantum mechanics to calculate the noise generated by this device.

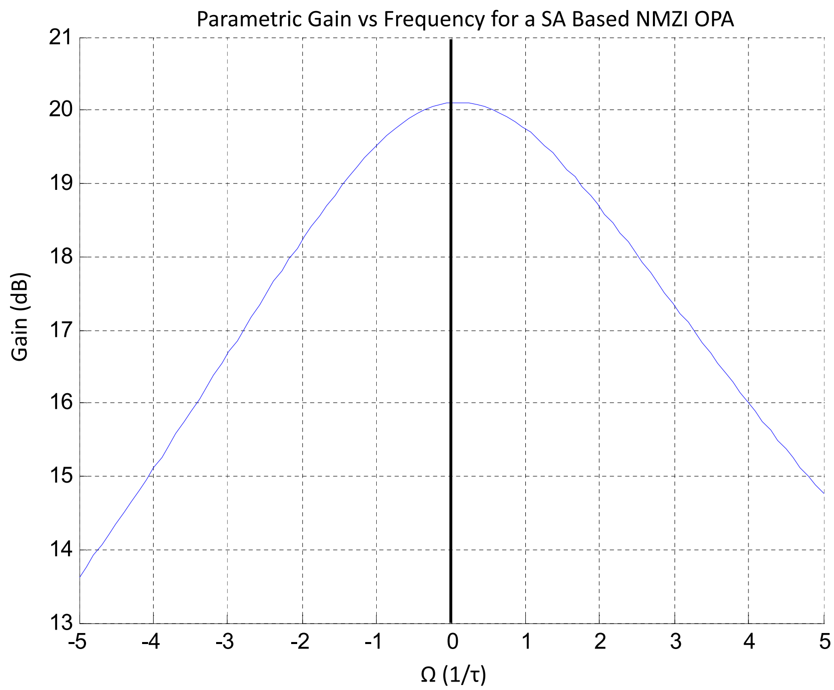

Figure 6.3: Parametric gain in dB vs Ω in $1/\tau$ for a SA based NMZI OPA with $p/P_{\mathrm{sat}} = 25$, $\alpha_H = 5$and $\beta_0 L = 4$.

Chapter 7

Quantum Mechanical Model for Interaction of Light with a Saturable Absorber

7.1 Overview

I considered two possible candidates for the new type of phase sensitive amplifier, based on either SOA or on SA. Based on results from Shtaif *et al.* [66], I concluded that the SOA produces too much ASE noise to be useful. Therefore, I decided to investigate the noise properties of non linear media based saturable absorbers. The question which I am trying to answer in the next several chapters is what is the minimum noise in a SA based NMZI OPA. In this chapter, I will follow closely the formalism of Professor Rana [61] to derive the equations for time evolution of operators describing the field and the medium.

7.2 Quantum Mechanical Model

Figure 7.1 shows schematically a saturable absorber interacting with a single quantized mode of electromagnetic field. I will use Jaynes-Cummings formalism to describe the electromagnetic field interacting with atoms [46, 62]. I will start with the Hamiltonian, which gives the energy of the atom, the energy of the electric field and their coupling.

$$\hat{\mathbf{H}} = E_1\hat{\mathbf{N}}_1 + E_2\hat{\mathbf{N}}_2 + \hbar\omega_0\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + \kappa\hat{\mathbf{A}}\hat{\sigma}_+ + \kappa^*\hat{\mathbf{A}}^\dagger\hat{\sigma}_- + \frac{1}{2}, \tag{7.1}$$

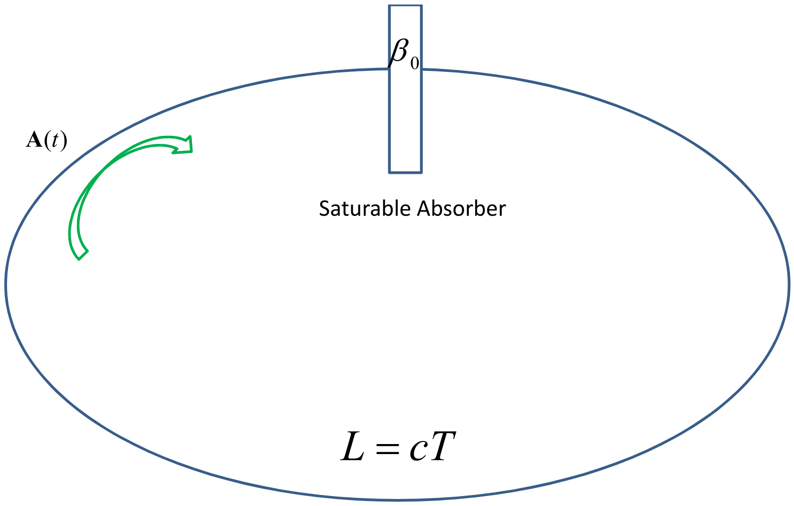

Figure 7.1: Single Mode Quantum Mechanical Model

where κ is a complex constant that I will discuss later, E_1 and E_2 are the energy levels of the atom $\hat{\sigma}_-$, $\hat{\sigma}_+$, $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$ are atomic state operators. For two atomic energy levels $|e_1\rangle$ and $|e_2\rangle$, the operator $\hat{\sigma}_+$ raises the atomic state from level 1 to level 2, while $\hat{\sigma}_-$ lowers it. Therefore,

$$\hat{\sigma}_+ = |e_2\rangle\langle e_1|, \tag{7.2}$$

$$\hat{\sigma}_- = |e_1\rangle\langle e_2|, \tag{7.3}$$

thus

$$\hat{\sigma}_- = \hat{\sigma}_+^\dagger. \tag{7.4}$$

Further,

$$\hat{\mathbf{N}}_1 = |e_1\rangle\langle e_1| \tag{7.5}$$

and

$$\hat{\mathbf{N}}_2 = |e_2\rangle\langle e_2|. \tag{7.6}$$

From this, it follows that

$$\hat{\mathbf{N}}_1 = \hat{\sigma}_-\hat{\sigma}_+ \tag{7.7}$$

and

$$\hat{\mathbf{N}}_2 = \hat{\sigma}_+\hat{\sigma}_-. \tag{7.8}$$

The Hamiltonian for the electric field interacting with N atoms becomes

$$\hat{\mathbf{H}} = \sum_{k=1}^{N}\left(E_1\hat{\mathbf{N}}_{1\ k} + E_2\hat{\mathbf{N}}_{2\ k} + \kappa\hat{\mathbf{A}}\hat{\sigma}_{+\ k} + \kappa^*\hat{\mathbf{A}}^\dagger\hat{\sigma}_{-\ k}\right) + \hbar\omega_0\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}} + \frac{1}{2}, \tag{7.9}$$

where $\hat{\mathbf{N}}_{1\ k}$ and $\hat{\mathbf{N}}_{2\ k}$, $k \in \{1, 2, .., N\}$ are the raising operator and lowering operator of the k^{th} atom. I define

$$\hat{\mathbf{N}}_1 \equiv \sum_{k=1}^{N}\hat{\mathbf{N}}_{1\ k}, \tag{7.10}$$

$$\hat{\mathbf{N}}_2 \equiv \sum_{k=1}^{N}\hat{\mathbf{N}}_{2\ k}, \tag{7.11}$$

$$\hat{\sigma}_- \equiv \sum_{k=1}^{N}\hat{\sigma}_{-\ k} \tag{7.12}$$

and

$$\hat{\sigma}_+ \equiv \sum_{k=1}^{N}\hat{\sigma}_{+\ k}. \tag{7.13}$$

The Heisenberg equation for a general operator $\hat{\mathbf{O}}(t)$ is given by

$$\frac{\mathrm{d}\hat{\mathbf{O}}(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{\mathbf{O}}(t), \hat{\mathbf{H}}\right]. \tag{7.14}$$

Applying this equation to each of the operators, and using the relations between the operators, I get

$$\frac{\mathrm{d}\hat{\mathbf{N}}_2(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\kappa\hat{\sigma}_+(t)\hat{\mathbf{A}}(t) - \kappa^*\hat{\sigma}_-(t)\hat{\mathbf{A}}^\dagger(t)\right], \tag{7.15}$$

$$\frac{d\hat{\mathbf{N}}_1(t)}{dt} = -\frac{d\hat{\mathbf{N}}_2(t)}{dt}, \tag{7.16}$$

$$\frac{d\hat{\mathbf{N}}_1(t)}{dt} = -\frac{1}{i\hbar}\left[\kappa\hat{\sigma}_+(t)\hat{\mathbf{A}}(t) - \kappa^*\hat{\sigma}_-(t)\hat{\mathbf{A}}^\dagger(t)\right], \tag{7.17}$$

$$\frac{d\hat{\mathbf{A}}(t)}{dt} = i\omega_0\hat{\mathbf{A}}(t) - \frac{i\kappa^*}{\hbar}\hat{\sigma}_-, \tag{7.18}$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = i\omega_{21}\hat{\sigma}_+(t) - \frac{i\kappa^*}{\hbar}\hat{\mathbf{A}}^\dagger\left[\hat{\mathbf{N}}_2(t) - \hat{\mathbf{N}}_1(t)\right]. \tag{7.19}$$

where

$$\omega_{21} \equiv \frac{E_2 - E_1}{\hbar}. \tag{7.20}$$

Any population in level 2 causes spontaneous emission, which is very detrimental. Because, I am trying to consider only saturable absorbers introducing minimum noise into the parametric amplifier, I will consider only a system in which the upper level is quickly depopulated by fast transitions to other levels lumped as level 3. As illustrated in figure 7.2 this level then slowly relaxes back to ground state. This is typical of what occurs in many semiconductor absorbers.

Therefore, I am going to add fast relaxation and dephasing to equation (7.19) [50]

$$\frac{d\hat{\sigma}_+(t)}{dt} = (i\omega_{21} - \gamma)\,\hat{\sigma}_+(t) + \frac{i\kappa^*}{\hbar}\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}_1(t) + \hat{\mathbf{F}}_+(t), \tag{7.21}$$

where γ is the relaxation and dephasing rate and $\hat{\mathbf{F}}_+(t)$ is the noise associated with it. I also add an empirical term for repopulation of level 1 to equation (7.17), which results in

$$\frac{d\hat{\mathbf{N}}_1(t)}{dt} = \frac{N - \hat{\mathbf{N}}_1(t)}{\tau} + \frac{i}{\hbar}\left[\kappa\hat{\sigma}_+(t)\hat{\mathbf{A}}(t) - \kappa^*\hat{\sigma}_-(t)\hat{\mathbf{A}}^\dagger(t)\right] + \hat{\mathbf{F}}_N(t), \tag{7.22}$$

where τ is the rate of relaxation and $\hat{\mathbf{F}}_N(t)$ is the noise due to relaxation. I will see later in this chapter why $\hat{\mathbf{F}}_+(t)$ and $\hat{\mathbf{F}}_N(t)$ had to be added and I will evaluate their properties. Now I have a complete system of differential equations. To solve them, I will first find an expression for $\hat{\sigma}_+$, which I will later substitute in the differential equation in $\hat{\mathbf{A}}(t)$.

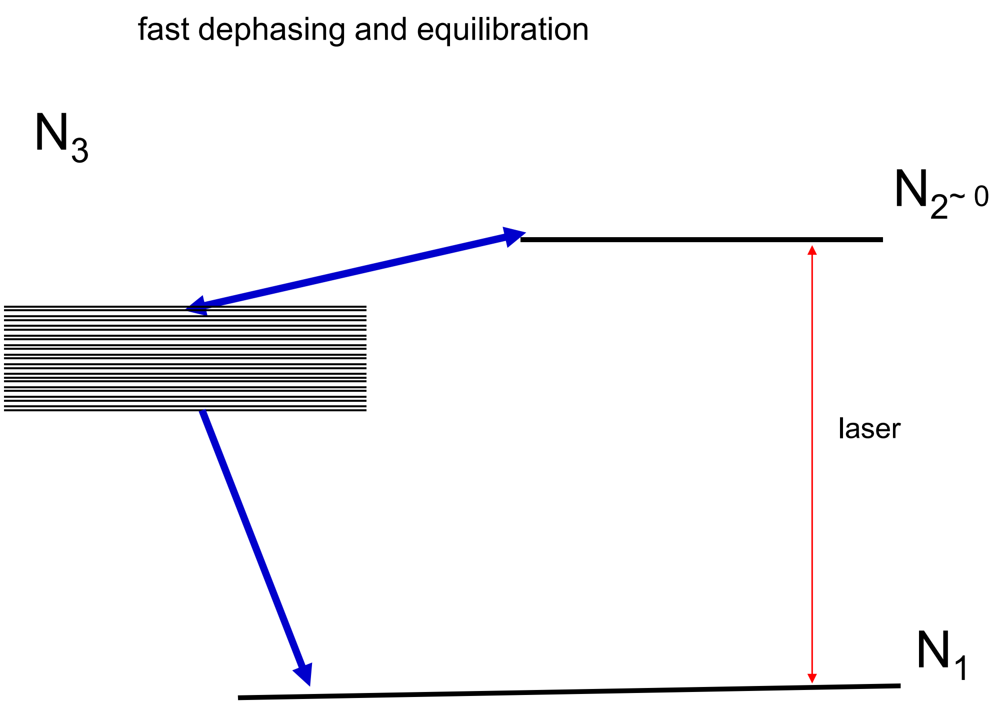

Figure 7.2: 3 Level System

From equation (7.21), it can be shown that (see appendix F.1),

$$\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{(i\omega_{21}-\gamma)t} + \frac{i\kappa^*}{\hbar}e^{(i\omega_{21}-\gamma)t}\int_0^t \hat{\mathbf{A}}^\dagger(t')\hat{\mathbf{N}}_1(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t' + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}, \tag{7.23}$$

where

$$\hat{\mathbf{F}}^\dagger_{\mathrm{ab}} \equiv \frac{i\kappa}{\hbar}e^{-i\omega_0 t}\int_0^t \hat{\mathbf{F}}_+(t')e^{-(i\omega_{21}-\gamma)(t-t')}\mathrm{d}t'. \tag{7.24}$$

For the semiconductor I am considering, γ is large. As a result, the first term of the above equation can be neglected and the initial conditions are quickly forgotten by the system.

Therefore,

$$\hat{\sigma}_+(t) = \frac{i\kappa^*}{\hbar} e^{(i\omega_{21}-\gamma)t} \int_0^t \hat{\mathbf{A}}^\dagger(t')\hat{\mathbf{N}}_1(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t' + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{7.25}$$

Let $\hat{\mathbf{A}}(t) \equiv \hat{\mathbf{B}}(t)e^{-i\omega_0 t}$. $\hat{\mathbf{N}}_1(t)$ and $\hat{\mathbf{B}}(t)$ are varying much slower than the term $e^{(i\omega_{21}-\gamma)t}$. Therefore, they can be moved in front of the integral

$$\hat{\sigma}_+(t) = \frac{i\kappa^*}{\hbar} e^{(i\omega_{21}-\gamma)t}\hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{N}}_1(t) \int_0^t e^{-(i(\omega_{21}-\omega_0)-\gamma)t'}\mathrm{d}t' + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{7.26}$$

After evaluating the integral, I have

$$\hat{\sigma}_+(t) = \frac{\kappa^*}{i\hbar}\frac{\hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{N}}_1(t)}{(i(\omega_{21}-\omega_0)-\gamma)}\left(e^{i\omega_0 t} - e^{(i\omega_{21}-\gamma)t}\right) + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{7.27}$$

When γ is very large, I can again neglect $e^{(i\omega_{21}-\gamma)t}$. I then obtain the expression for $\hat{\sigma}_+$, which is

$$\hat{\sigma}_+(t) = \frac{\kappa^*}{\hbar}\frac{\hat{\mathbf{A}}^\dagger(t)\hat{\mathbf{N}}_1(t)}{((\omega_0-\omega_{21})-i\gamma)} + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{7.28}$$

Now, I will substitute this expression of $\hat{\sigma}_+$ into the differential equation of $\hat{\mathbf{A}}(t)$.

Equation (7.18) is a differential equation for $A(t)$ in terms of $\hat{\sigma}_-$

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = i\omega_0\hat{\mathbf{A}}(t) - \frac{i\kappa^*}{\hbar}\hat{\sigma}_- \tag{7.29}$$

I can use the expression for $\hat{\sigma}_+$ from equation (7.28) to get an expression of $\hat{\sigma}_-$, since $\hat{\sigma}_+ = \hat{\sigma}_-^\dagger$, which can be substituted in the above equation. I get

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = \frac{|\kappa|^2}{\hbar^2}\hat{\mathbf{A}}(t)\hat{\mathbf{N}}_1(t)\left(-\frac{\gamma}{((\omega_0-\omega_{21})^2+\gamma^2)} - i\frac{(\omega_0-\omega_{21})}{((\omega_0-\omega_{21})^2+\gamma^2)}\right) \\ + \hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t} + i\omega_0\hat{\mathbf{A}}(t),\end{aligned} \tag{7.30}$$

I define $\mho$ such that

$$\mho \equiv \frac{|\kappa|^2}{\hbar^2}\left(-\frac{\gamma}{((\omega_0-\omega_{21})^2+\gamma^2)} - i\frac{(\omega_0-\omega_{21})}{((\omega_0-\omega_{21})^2+\gamma^2)}\right) \tag{7.31}$$

The first term corresponds to absorption of the field with coefficient

$$\mho_r \equiv -\frac{|\kappa|^2}{\hbar^2}\frac{\gamma}{((\omega_0-\omega_{21})^2+\gamma^2)}, \tag{7.32}$$

and the second corresponds to index of refraction change

$$\mho_i \equiv \frac{|\kappa|^2}{\hbar^2}\frac{(\omega_0-\omega_{21})}{((\omega_0-\omega_{21})^2+\gamma^2)}. \tag{7.33}$$

Now, I will calculate the properties of $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$, required when I solve the differential equation in $\hat{\mathbf{A}}(t)$.

In equation (7.21), I added an empirical term for the noise associated with relaxation and dephasing. This noise lead to the expression for $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$ in equation (7.24). The noise term is necessary so that so that $\left[\hat{\mathbf{A}}(t), \hat{\mathbf{A}}^{\dagger}(t)\right] = 1$ for all values of t. Without $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$, I would have

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\mho\hat{\mathbf{A}}(t)\hat{\mathbf{N}}_1(t) + i\omega_0\hat{\mathbf{A}}(t). \tag{7.34}$$

I define $\hat{\mathbf{C}} \equiv \mho\hat{\mathbf{N}}_1(t) + i\omega_0$

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\hat{\mathbf{C}}\hat{\mathbf{A}}(t). \tag{7.35}$$

I then have

$$\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(0)e^{-\hat{\mathbf{C}}t}. \tag{7.36}$$

Therefore

$$\begin{aligned}\left[\hat{\mathbf{A}}(t), \hat{\mathbf{A}}^{\dagger}(t)\right] &= \left[\hat{\mathbf{A}}(0), \hat{\mathbf{A}}^{\dagger}(0)\right] e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t} \\ &= e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t},\end{aligned} \tag{7.37}$$

which is obviously wrong as the answer should be one (1) . To fix this, I must add the following noise term

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\hat{\mathbf{C}}\hat{\mathbf{A}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t} \tag{7.38}$$

I can assume that $\left[\hat{\mathbf{F}}_{\mathrm{ab}}(t), \hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t')\right] = \hat{\mathbf{C}}_0\delta(t-t')$, since the noise fluctuates much faster that the other variables(on a time scale $1/\gamma$) [63]. The solution of this equation is

$$\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(0)e^{-\hat{\mathbf{C}}t} + \int_0^t e^{-\hat{\mathbf{C}}(t-t')-i\omega_0 t'}\hat{\mathbf{F}}_{\mathrm{ab}}(t')\mathrm{d}t'. \tag{7.39}$$

Computing the commutator of $\hat{\mathbf{A}}(t)$, I can show that (see appendix F.1.1)

$$\begin{aligned}\hat{\mathbf{C}}_0 &= \hat{\mathbf{C}} + \hat{\mathbf{C}}^{\dagger} \\ &= 2\mho_r\hat{\mathbf{N}}_1(t).\end{aligned} \tag{7.40}$$

Therefore,

$$\left[\hat{\mathbf{F}}_{\mathrm{ab}}(t), \hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t')\right] = 2\mho_r\hat{\mathbf{N}}_1(t)\delta(t-t'). \tag{7.41}$$

In general, starting with equation (7.21), I would obtain

$$\left[\hat{\mathbf{F}}_{\mathrm{ab}}(t), \hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t')\right] = -2\mho_r\left(\hat{\mathbf{N}}_2(t) - \hat{\mathbf{N}}_1(t)\right)\delta(t-t'), \tag{7.42}$$

and

$$\left\langle\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t')\right\rangle = 2\mho_r\left\langle\hat{\mathbf{N}}_2(t)\right\rangle\delta(t-t') \tag{7.43}$$

and [64]

$$\left\langle\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t')\right\rangle = 2\mho_r\left\langle\hat{\mathbf{N}}_1(t)\right\rangle\delta(t-t'). \tag{7.44}$$

Because I am neglecting $\hat{\mathbf{N}}_2(t)$, I can set

$$\left\langle \hat{\mathbf{F}}^{\dagger}_{\rm ab}(t)\hat{\mathbf{F}}_{\rm ab}(t')\right\rangle = 0. \tag{7.45}$$

$\hat{\mathbf{N}}_1(t)$ cannot be neglected due to its importance. Now, I will derive a differential equation for $\hat{\mathbf{n}}(t)$, the number of photons in the mode operator, which will later help us solve the system of differential equations. By definition $\hat{\mathbf{n}}(t) = \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{A}}(t)$. Taking the derivative of this equation, I get

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = \hat{\mathbf{A}}^{\dagger}(t)\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t}\hat{\mathbf{A}}(t). \tag{7.46}$$

After substituting equation (7.38) I get

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = -2\mho_r\hat{\mathbf{N}}_1(t)\hat{\mathbf{n}}(t) + \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{F}}_{\rm ab}(t)e^{-i\omega_0 t} + \hat{\mathbf{F}}^{\dagger}_{\rm ab}\hat{\mathbf{A}}(t)e^{i\omega_0 t}. \tag{7.47}$$

The term $-2\mho_r\hat{\mathbf{N}}_1(t)\hat{\mathbf{n}}(t)$ represent absorption. Using equation (7.39), I can compute $\left\langle \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{F}}_{\rm ab}(t)e^{-i\omega_0 t}\right\rangle$ and $\left\langle \hat{\mathbf{F}}^{\dagger}_{\rm ab}(t)\hat{\mathbf{A}}(t)e^{i\omega_0 t}\right\rangle$. I can obtain (see appendix F.1.2)

$$\left\langle \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{F}}_{\rm ab}(t)e^{-i\omega_0 t}\right\rangle = 0. \tag{7.48}$$

Similarly

$$\left\langle \hat{\mathbf{F}}^{\dagger}_{\rm ab}(t)\hat{\mathbf{A}}(t)e^{-i\omega_0 t}\right\rangle = 0. \tag{7.49}$$

Using perturbation theory in equation (7.47), it can easily be shown that

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -2\mho_r N_1 n(t), \tag{7.50}$$

where $N_1 \equiv \left\langle \hat{\mathbf{N}}_1(t)\right\rangle$ and is assumed to be constant, and $n(t) \equiv \langle\hat{\mathbf{n}}(t)\rangle$. Solving this differential equation gives us

$$\ln\left[\frac{n(T)}{n(0)}\right] = -2\mho_r N_1 T. \tag{7.51}$$

Referring back to equation (7.32) for the definition of $\mho_r$, we can see that $\frac{\kappa}{\hbar}$ is the Rabi frequency [37]. It can also be expressed as follows [38]

$$\frac{|\kappa|^2}{\hbar^2} = \frac{\omega_0}{2\epsilon_0 \hbar V}|\vec{e} \cdot \vec{D}_{12}|^2 \tag{7.52}$$

,

where V is the volume, D_{12} is the dipole moment. Therefore,

$$\frac{|\kappa|^2}{\hbar^2} = \frac{\omega_0}{2\epsilon \hbar S L}|\vec{e} \cdot \vec{D}_{12}|^2, \tag{7.53}$$

where S is the cross sectional area of the optical beam and L the length of the cavity. Since $T = \frac{L}{c}$, I get

$$\ln\left[\frac{n(T)}{n(0)}\right] = -\frac{\omega_0}{\epsilon_0 c \hbar S}|\vec{e} \cdot \vec{D}_{12}|^2 \frac{\gamma}{((\omega_0 - \omega_{21})^2 + \gamma^2)} N_1. \tag{7.54}$$

Since this expression is the absorption coefficient, I define

$$\hat{\beta}(t) \equiv \frac{\omega_0}{\epsilon_0 c \hbar S}|\vec{e} \cdot \vec{D}_{12}|^2 \frac{\gamma}{((\omega_0 - \omega_{21})^2 + \gamma^2)} \hat{\mathbf{N}}_1(t). \tag{7.55}$$

Returning to the differential equation of $\hat{\mathbf{A}}(t)$, I simplify equation (7.30) by substituting for $\hat{\mathbf{A}}(t) = \hat{\mathbf{B}}(t)e^{-i\omega_0 t}$. I get

$$\frac{\mathrm{d}\hat{\mathbf{B}}(t)}{\mathrm{d}t} = -\mho \hat{\mathbf{N}}_1(t)\hat{\mathbf{B}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t). \tag{7.56}$$

Equation (7.56) can be rewritten

$$\frac{\mathrm{d}\hat{\mathbf{B}}(t)}{\mathrm{d}t} = -\mho_r\left(1 + i\frac{\mho_i}{\mho_r}\right)\hat{\mathbf{N}}_1(t)\hat{\mathbf{B}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t). \tag{7.57}$$

Then using equation (7.55), I have

$$\frac{\mathrm{d}\hat{\mathbf{B}}(t)}{\mathrm{d}t} = -\frac{\hat{\beta}(t)}{2T}\left(1 + i\frac{\mho_i}{\mho_r}\right)\hat{\mathbf{B}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t). \tag{7.58}$$

Let

$$\alpha_H \equiv \frac{\mho_i}{\mho_r}, \tag{7.59}$$

the Henry alpha factor. Finally, the equation becomes

$$\frac{\mathrm{d}\hat{\mathbf{B}}(t)}{\mathrm{d}t} = -\frac{\hat{\beta}(t)}{2T}\left(1 + i\alpha_H\right)\hat{\mathbf{B}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t). \tag{7.60}$$

As a check on this equation, I consider the case of unsaturable loss. For unsaturable loss $\hat{\beta}(t) = \beta$ is constant. After the interaction, I have

$$\hat{\mathbf{B}}_{\mathrm{out}} = \hat{\mathbf{B}}_{\mathrm{in}} e^{-\frac{\beta}{2}(1+i\alpha_H)} + \hat{\mathbf{F}}_{loss}, \tag{7.61}$$

where $\hat{\mathbf{B}}_{\mathrm{in}} \equiv \hat{\mathbf{B}}(0)$ is the input wave, before interaction, $\hat{\mathbf{B}}_{\mathrm{out}} \equiv \hat{\mathbf{B}}(T)$ is the output wave and

$$\hat{\mathbf{F}}_{loss} \equiv e^{-\frac{\beta}{2}(1+i\alpha_H)} \int_0^T \hat{\mathbf{F}}_{\mathrm{ab}}(t) e^{\frac{c\beta}{2L}(1+i\alpha_H)t} \mathrm{d}t. \tag{7.62}$$

I use equation (7.45) to get

$$\left\langle \hat{\mathbf{F}}_{loss}^{\dagger} \hat{\mathbf{F}}_{loss} \right\rangle = 0. \tag{7.63}$$

I then use equation (7.44)to get

$$\left\langle \hat{\mathbf{F}}_{loss} \hat{\mathbf{F}}_{loss}^{\dagger} \right\rangle = 1 - e^{-\beta}, \tag{7.64}$$

which is consistent with what I said in section 1.6.1 and α_H has no effect on the loss induced noise.

Now, I will obtain a differential equation for $\hat{\mathbf{N}}_1(t)$. I substitute the expression of $\hat{\sigma}_-$ and $\hat{\sigma}_+$ from equation (7.28) in equation (7.22), I get

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{N}}_1(t)}{\mathrm{d}t} = & -2\mho_r \hat{\mathbf{n}}(t)\hat{\mathbf{N}}_1(t) + \hat{\mathbf{F}}_N(t) + \frac{N - \hat{\mathbf{N}}_1(t)}{\tau} \\ & - \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t} - \hat{\mathbf{F}}_{\mathrm{ab}}^{\dagger}\hat{\mathbf{A}}(t)e^{i\omega_0 t}. \end{aligned} \tag{7.65}$$

To solve the differential equation above, I will need to find the properties of $\hat{\mathbf{F}}_N(t)$. I use an argument similar to the one I used for $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$ to justify its addition in equation (7.17) and to prove that for a single atom (see appendix F.2)

$$\left\langle \hat{\mathbf{F}}_{Nj}(t)\hat{\mathbf{F}}_{Nj}(t') \right\rangle = \frac{1-\left\langle \hat{\mathbf{N}}_{1j}(t) \right\rangle}{\tau}\delta(t-t'). \tag{7.66}$$

For a collection of N atoms, it is (see appendix F.3)

$$\left\langle \hat{\mathbf{F}}_{N}(t)\hat{\mathbf{F}}_{N}(t') \right\rangle = \frac{N-\left\langle \hat{\mathbf{N}}_{1}(t) \right\rangle}{\tau}\delta(t-t'). \tag{7.67}$$

I can now simplify equation (7.65) by substituting $\hat{\mathbf{A}}(t) = \hat{\mathbf{B}}(t)e^{i\omega_0 t}$. I get

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{N}}_1(t)}{\mathrm{d}t} = & -2\mho_r\hat{\mathbf{n}}(t)\hat{\mathbf{N}}_1(t) + \hat{\mathbf{F}}_N(t) + \frac{N-\hat{\mathbf{N}}_1(t)}{\tau} \\ & - \hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t) - \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}\hat{\mathbf{B}}(t). \end{aligned} \tag{7.68}$$

I substitute in the above relation equation (7.55) and after some algebraic manipulations, I get

$$\frac{\mathrm{d}\hat{\beta}(t)}{\mathrm{d}t} = \frac{\beta_0 - \hat{\beta}(t)}{\tau} - 2\mho_r\hat{\mathbf{n}}(t)\hat{\beta}(t) + \hat{\mathbf{F}}_L(t), \tag{7.69}$$

where

$$\beta_0 \equiv 2\frac{L}{c}\mho_r N \tag{7.70}$$

and

$$\hat{\mathbf{F}}_L(t) \equiv -2\frac{L}{c}\mho_r\left(\hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t) + \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}\hat{\mathbf{B}}(t)\right) + 2\frac{L}{c}\mho_r\hat{\mathbf{F}}_N(t). \tag{7.71}$$

$\hat{\mathbf{F}}_L(t)$ is Hermitian. I can compute its properties by computing $\left\langle \hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t) \right\rangle$. For this, I first substitute equation (7.55) in equation (7.44) to get

$$\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t') \right\rangle = \frac{1}{T}\left\langle \hat{\beta}(t) \right\rangle \delta(t-t'). \tag{7.72}$$

I then use this result to get

$$\left\langle \hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t')\right\rangle = 2\mho_r T\left[\frac{\beta_0 + (2\tau\mho_r \left\langle \hat{\mathbf{n}}(t)\right\rangle - 1)\left\langle \hat{\beta}(t)\right\rangle}{\tau}\right]\delta(t-t'). \tag{7.73}$$

I multiply equation (7.71) by $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$ and take the average to get

$$\left\langle \hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\right\rangle = -2\mho_r\left\langle \hat{\mathbf{B}}^\dagger(t)\right\rangle\left\langle \hat{\beta}(t)\right\rangle\delta(t-t'). \tag{7.74}$$

I multiply equation (7.71) by $\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)$ and take the average to get

$$\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_L(t')\right\rangle = -2\mho_r\left\langle \hat{\mathbf{B}}(t)\right\rangle\left\langle \hat{\beta}(t)\right\rangle\delta(t-t'). \tag{7.75}$$

In conclusion, I obtained from this chapter the following results

$$\frac{\mathrm{d}\hat{\beta}(t)}{\mathrm{d}t} = \frac{\beta_0 - \hat{\beta}(t)}{\tau} - 2\mho_r\hat{\mathbf{n}}(t)\hat{\beta}(t) + \hat{\mathbf{F}}_L(t), \tag{7.76}$$

$$\frac{\mathrm{d}\hat{\mathbf{B}}(t)}{\mathrm{d}t} = -\frac{1}{2T}\hat{\beta}(t)\left(1 + i\alpha_H\right)\hat{\mathbf{B}}(t) + \hat{\mathbf{F}}_{\mathrm{ab}}(t), \tag{7.77}$$

and I can rewrite equation (7.47) as follows

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = -\frac{1}{T}\hat{\beta}(t)\hat{\mathbf{n}}(t) + \hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t) + \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}\hat{\mathbf{B}}(t). \tag{7.78}$$

Chapter 8

Solving Differential Equation of SA

8.1 Overview

In the previous chapter, I derived equation (7.76) and equation (7.77), which are the simultaneous differential equation describing the interaction between the field operator and a saturable absorber. In this chapter, I will find a solution to those differential equation. In these solutions, I assumed that the noise terms are small compared to the average values of the parameters. Therefore, I will use first order perturbation theory.

8.2 Deriving Simultaneous Differential Equations of the Inphase and Quadrature Phase component of Noise

It is important to note that the average $\langle \cdot \rangle$ is a statistical average, or ensemble average and not a time average. I use perturbation theory to solve the differential equation of $\hat{\mathbf{n}}(t)$ and $\hat{\beta}(t)$. For that, I make the following substitution

$$\hat{\mathbf{B}}(t) = \delta\hat{\mathbf{B}}(t) + \left\langle \hat{\mathbf{B}}(t) \right\rangle . \tag{8.1}$$

To simplify notation, I will use $B(t) \equiv \left\langle \hat{\mathbf{B}}(t) \right\rangle$. Similarly

$$\hat{\beta}(t) = \delta\hat{\beta}(t) + \left\langle \hat{\beta}(t) \right\rangle . \tag{8.2}$$

To simplify notation, I will use $\beta(t) \equiv \left\langle \hat{\beta}(t) \right\rangle$.

$$\hat{\mathbf{n}}(t) = \delta\hat{\mathbf{n}}(t) + \langle \hat{\mathbf{n}}(t) \rangle \,. \tag{8.3}$$

To simplify notation, I will use $n(t) \equiv \langle \hat{\mathbf{n}}(t) \rangle$. I had

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = -\frac{1}{T}\hat{\beta}(t)\hat{\mathbf{n}}(t) + \hat{\mathbf{B}}^{\dagger}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t) + \hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t)\hat{\mathbf{B}}(t). \tag{8.4}$$

I substitute equation (8.3) in equation (7.78) and looking at the averaged equation, I get

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -\frac{1}{T}\beta(t)n(t). \tag{8.5}$$

Substituting equation (8.2) in equation (7.76) gives

$$\begin{aligned} \frac{\mathrm{d}\delta\hat{\beta}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\beta(t)}{\mathrm{d}t} = \frac{\beta_0 - \beta(t) - \delta\hat{\beta}(t)}{\tau} + \hat{\mathbf{F}}_L(t) \\ - 2\mho_r\left(n(t) + \delta\hat{\mathbf{n}}(t)\right)\left(\delta\hat{\beta}(t) + \beta(t)\right). \end{aligned} \tag{8.6}$$

Separating the averaged terms to the fluctuating terms, I get the following equation for the fluctuating terms

$$\tau\frac{\mathrm{d}\delta\hat{\beta}(t)}{\mathrm{d}t} = -\delta\hat{\beta}(t) + \tau\hat{\mathbf{F}}_L(t) - 2\mho_r\tau\left(\delta\hat{\beta}(t)n(t) + \beta(t)\delta\hat{\mathbf{n}}(t)\right) \tag{8.7}$$

and the following equation for the averaged terms

$$\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} = \frac{\beta_0 - \beta(t)}{\tau} - 2\mho_r\beta(t)n(t). \tag{8.8}$$

$n(t)$ is the driving term in the above equation. I will assume that $n(t)$ changes slowly enough for the atom to follow without oscillation. In other words that $\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \approx 0$. Therefore,

$$\beta(t) = \frac{\beta_0}{1 + 2\tau\mho_r n(t)}. \tag{8.9}$$

Substituting this result in equation (8.5), I get

$$\frac{1}{n(t)}\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -\frac{1}{T}\frac{\beta_0}{1+2\tau\mho_r n(t)}. \tag{8.10}$$

I define

$$n_{\text{sat}} \equiv \frac{1}{2\tau\mho_r}, \tag{8.11}$$

and use this definition in equation (8.10) to get

$$\frac{1}{n(t)}\frac{\mathrm{d}n(t)}{\mathrm{d}t} = -\frac{\frac{1}{T}\beta_0}{1+\frac{n(t)}{n_{\text{sat}}}}. \tag{8.12}$$

which is a differential equation which agrees with the semiclassical result I had in equation (6.6), where the differential equation was

$$\frac{1}{P(z)}\frac{\mathrm{d}P(z)}{\mathrm{d}z} = \frac{-\beta_1}{1+\frac{P(z)}{P_{\text{sat}}}}, \tag{8.13}$$

where β_1 is the absorption coefficient per unit length. The solution of equation (8.12) is

$$n(t) = n(0)\cdot W\left(\frac{n(0)}{n_{\text{sat}}}e^{\frac{n(0}{n_{\text{sat}}}-\frac{1}{T}\beta_0 T}\right) \tag{8.14}$$

$n(0)$ is the input number of photons and $n(T)$ is the output number of photons, I have

$$n_{\text{out}} = n_{\text{in}}\cdot W\left(\frac{n_{\text{in}}}{n_{\text{sat}}}e^{\frac{n_{\text{out}}}{n_{\text{sat}}}-\beta_0}\right). \tag{8.15}$$

Again, this agrees with semiclassical results, where the solution to equation (6.6) is

$$P_{\text{out}} = P_{\text{in}}\cdot W\left(\frac{P_{\text{in}}}{P_{\text{sat}}}e^{\frac{P_{\text{in}}}{P_{\text{sat}}}-\beta_1 l}\right). \tag{8.16}$$

I use (8.11) to get

$$\tau = \frac{1}{2\mho_r n_{\text{sat}}}. \tag{8.17}$$

Using the above equation, equation (7.76) can be rewritten

$$\frac{\mathrm{d}\hat{\beta}(t)}{\mathrm{d}t} = \frac{\beta_0 - \hat{\beta}(t)}{\tau} - \frac{\hat{\mathbf{n}}(t)\hat{\beta}(t)}{n_{\mathrm{sat}}\tau} + \hat{\mathbf{F}}_L(t), \tag{8.18}$$

equation (8.7) can be written

$$\tau\frac{\mathrm{d}\delta\hat{\beta}(t)}{\mathrm{d}t} = -\delta\hat{\beta}(t) + \tau\hat{\mathbf{F}}_L(t) - \frac{1}{n_{\mathrm{sat}}}\left(\delta\hat{\beta}(t)n(t) + \beta(t)\delta\hat{\mathbf{n}}(t)\right), \tag{8.19}$$

equation (7.73) can be written

$$\left\langle\hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t')\right\rangle = \frac{T}{n_{\mathrm{sat}}}\left[\frac{\beta_0 + \left(\frac{n(t)}{n_{\mathrm{sat}}} - 1\right)\left\langle\hat{\beta}(t)\right\rangle}{\tau^2}\right]\delta(t-t'), \tag{8.20}$$

equation (8.9) can be written

$$\beta(t) = \frac{\beta_0}{1 + \frac{n(t)}{n_{\mathrm{sat}}}}, \tag{8.21}$$

and equation (7.74) and (7.75) can be written consecutively

$$\left\langle\hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\right\rangle = -\frac{1}{\tau n_{\mathrm{sat}}}\left\langle\hat{\mathbf{B}}^\dagger(t)\right\rangle\left\langle\hat{\beta}(t)\right\rangle\delta(t-t'), \tag{8.22}$$

and

$$\left\langle\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_L(t')\right\rangle = -\frac{1}{\tau n_{\mathrm{sat}}}\left\langle\hat{\mathbf{B}}(t)\right\rangle\left\langle\hat{\beta}(t)\right\rangle\delta(t-t'). \tag{8.23}$$

It can be verified that if $n_{\mathrm{sat}} \to \infty$, I am back to the non saturable case. In that case equation (8.20) becomes

$$\left\langle\hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t')\right\rangle = 0. \tag{8.24}$$

Since $\hat{\mathbf{F}}_L(t)$ is Hermitian, in that case $\hat{\mathbf{F}}_L(t) = 0$.

$\hat{\mathbf{B}}(t)$ can be written

$$\hat{\mathbf{B}}(t) = B(t) + \delta\hat{\mathbf{X}}(t) + i\delta\hat{\mathbf{Y}}(t), \tag{8.25}$$

where $\delta\hat{\mathbf{X}}(t)$ and $\delta\hat{\mathbf{Y}}(t)$ are the in phase and the quadrature phase component of $\delta\hat{\mathbf{B}}(t)(t)$. $\hat{\mathbf{B}}(t)$ can also be written

$$\begin{aligned}\hat{\mathbf{B}}(t) &= B(t)\left(1 + \frac{\delta\hat{\mathbf{X}}(t)}{B(t)} + i\frac{\delta\hat{\mathbf{Y}}(t)}{B(y)}\right) \\ &= B(t)\left(1 + \hat{\mathbf{x}}(t) + i\hat{\mathbf{y}}(t)\right),\end{aligned} \tag{8.26}$$

where

$$\hat{\mathbf{x}}(t) \equiv \frac{\delta\hat{\mathbf{X}}(t)}{B(t)}, \tag{8.27}$$

and

$$\hat{\mathbf{y}}(t) \equiv \frac{\delta\hat{\mathbf{Y}}(t)}{B(t)}. \tag{8.28}$$

Substituting equation (8.26) in equation (7.77) and after some algebraic manipulation (see appendix G.1) I get the following equations

$$\frac{\mathrm{d}\hat{\mathbf{y}}(t)}{\mathrm{d}t} = \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} - B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2i} - \frac{\alpha_H}{2T}\delta\hat{\beta}(t), \tag{8.29}$$

and

$$\frac{\partial\hat{\mathbf{x}}(t)}{\mathrm{d}t} = -\frac{1}{2T}\delta\hat{\beta}(t) + \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} + B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2}. \tag{8.30}$$

As a check, I assume that $n_{\mathrm{sat}} \to \infty$, which is the case for unsaturable loss. In that case equations (8.30)and (8.19) can be easily solved and gives

$$\langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle = \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle + \frac{e^{\beta_0} - 1}{4n(0)}. \tag{8.31}$$

Multiplying the whole equation by $n(T) = n(0)e^{-\beta_0}$, I get

$$\left\langle\delta\hat{\mathbf{X}}^2(T)\right\rangle = e^{-\beta_0}\left\langle\delta\hat{\mathbf{X}}^2(0)\right\rangle + \frac{1 - e^{-\beta_0}}{4}, \tag{8.32}$$

which is what is expected.

8.3 Solving the Simultaneous Differential Equations

The solution of (8.19) is

$$\delta\hat{\beta}(t) = \left[\int_0^t H_1(t')\left(\hat{\mathbf{F}}_L(t') - \frac{2n(t')\beta(t')}{\tau n_{\mathrm{sat}}}\hat{\mathbf{x}}(t')\right)\mathrm{d}t'\right] H_1^{-1}(t), \tag{8.33}$$

since $\delta\hat{\beta}(0) = 0$ and where

$$H_1(t) \equiv \exp\left(\int_0^t \frac{n_{\mathrm{sat}} + n(t')}{\tau n_{\mathrm{sat}}}\mathrm{d}t'\right). \tag{8.34}$$

It is important to note from equation (8.19) that for time long compared to τ $\delta\hat{\beta}(t)$ can be considered memoryless and can be determined by event that occur at time very close to t. Therefore,

$$H_1(t) \approx \exp\left(\left(1 + \frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{t}{\tau}\right). \tag{8.35}$$

This approximation can be used because $\delta\hat{\beta}(t)$ is memoryless. The equation can further be reduced as follows

$$\begin{aligned}
\delta\hat{\beta}(t) &\approx \left[\int_0^t H_1(t')\hat{\mathbf{F}}_L(t')\mathrm{d}t' - \frac{2n(t)\beta(t)}{\tau n_{\mathrm{sat}}}\hat{\mathbf{x}}(t)\int_0^t H_1(t')\mathrm{d}t'\right] H_1^{-1}(t) \\
&\approx \left[\int_0^t H_1(t')\hat{\mathbf{F}}_L(t')\mathrm{d}t' - \frac{2n(t)\beta(t)}{\tau n_{\mathrm{sat}}}\hat{\mathbf{x}}(t)\int_0^t e^{\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{t'}{\tau}}\mathrm{d}t'\right] H_1^{-1}(t) \\
&\approx \left[\int_0^t H_1(t')\hat{\mathbf{F}}_L(t')\mathrm{d}t' - \frac{2\frac{n(t)}{n_{\mathrm{sat}}}\beta(t)}{1 + \frac{n(t)}{n_{\mathrm{sat}}}}\hat{\mathbf{x}}(t)H_1(t)\right] H_1^{-1}(t),
\end{aligned} \tag{8.36}$$

where the lower values of the integral has been ignored, since $\delta\hat{\beta}(t)$ forgets and only remembers what happened at time t and $\hat{\mathbf{x}}(t)$, $n(t)$ and $\beta(t)$ are considered slowly varying and can be taken outside of the integral. The equation gives

$$\delta\hat{\beta}(t) \approx H_1^{-1}(t)\int_0^t H_1(t')\hat{\mathbf{F}}_L(t')\mathrm{d}t' - \frac{2\frac{n(t)}{n_{\mathrm{sat}}}\beta(t)}{1 + \frac{n(t)}{n_{\mathrm{sat}}}}\hat{\mathbf{x}}(t), \tag{8.37}$$

since everything is slowly varying except for $\hat{\mathbf{F}}_L(t)$. Substituting this result in equation (8.30) gives us.

$$\frac{\partial \hat{\mathbf{x}}(t)}{\mathrm{d}t} = \frac{1}{T}\frac{\frac{n(t)}{n_{\mathrm{sat}}}\beta(t)\hat{\mathbf{x}}(t)}{\frac{n(t)}{n_{\mathrm{sat}}}+1} + \hat{\mathbf{N}}_x, \tag{8.38}$$

where

$$\hat{\mathbf{N}}_x \equiv \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} + B^{*-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}}^{\dagger}}{2} + \hat{\mathbf{F}}_{\beta}(t), \tag{8.39}$$

where

$$\hat{\mathbf{F}}_{\beta}(t) \equiv -\frac{1}{2T}\int_0^t e^{\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{t'-t}{\tau}}\hat{\mathbf{F}}_L(t')\mathrm{d}t'. \tag{8.40}$$

The solution of equation (8.38) is then

$$\hat{\mathbf{x}}(t) = H_2^{-1}(t)\left[H_2(0)\hat{\mathbf{x}}(0) + \int_0^t H_2(t')\hat{\mathbf{N}}_x\mathrm{d}t'\right], \tag{8.41}$$

where

$$\begin{aligned} H_2(t) =\ & \exp\left(\frac{1}{T}\int_t^T \frac{\frac{n(t')}{n_{\mathrm{sat}}}\beta(t')}{\frac{n(t')}{n_{\mathrm{sat}}}+1}\mathrm{d}t'\right) \\ =\ & \left(\frac{n(t)+n_{\mathrm{sat}}}{n(T)+n_{\mathrm{sat}}}\right). \end{aligned} \tag{8.42}$$

I use this result to get the expression of the power of the inphase noise

$$\begin{aligned} \langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle =& \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle H_2(0)^2 \\ &+ \int_0^T\int_0^T H_2(t')H_2(t'')\left\langle\hat{\mathbf{N}}_x(t')\hat{\mathbf{N}}_x(t'')\right\rangle \mathrm{d}t'\mathrm{d}t'', \end{aligned} \tag{8.43}$$

where $\left\langle\hat{\mathbf{x}}(0)\hat{\mathbf{N}}_x(t')\right\rangle = 0$ and $\left\langle\hat{\mathbf{N}}_x(t')\hat{\mathbf{x}}(0)\right\rangle = 0$.

It is clear that in order to get an expression for $\langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle$ I will need an expression for $\left\langle\hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$, $\left\langle\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$ and $\left\langle\hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t')\right\rangle$. Let us first find an expression for $\left\langle\hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$.

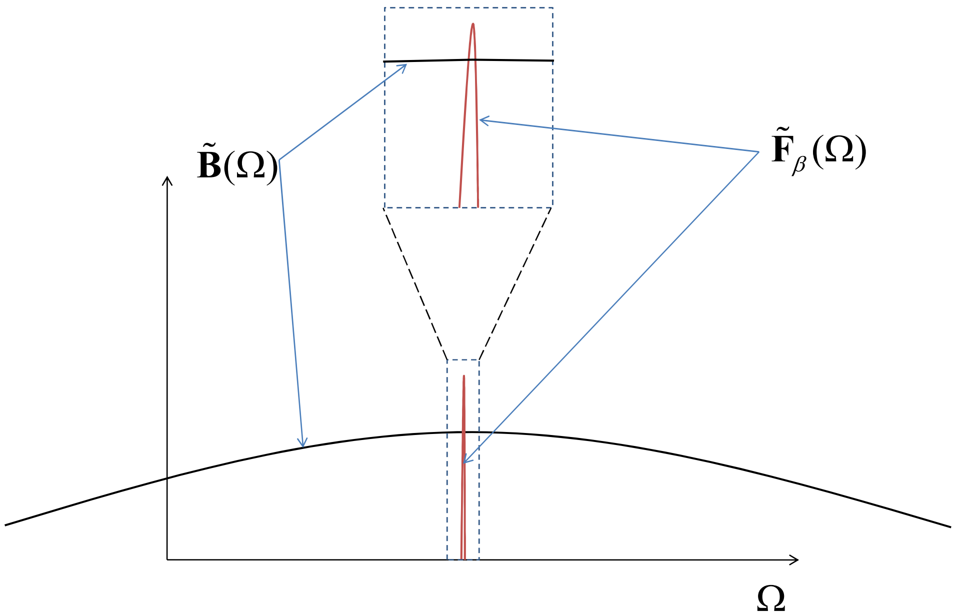

Figure 8.1: Illustration of spectrum of $\hat{\mathbf{F}}_\beta(t)$ relative to the spectrum of $\hat{\mathbf{B}}(t)$. $\tilde{\mathbf{F}}_\beta(\Omega)$ is the he Fourier Transform of $\hat{\mathbf{F}}_\beta(t)$ and $\tilde{\mathbf{B}}(\Omega)$ is the Fourier transform of $\hat{\mathbf{B}}(t)$. The spectrum of $\hat{\mathbf{F}}_\beta(t)$ looks flat relative to the spectrum of $\hat{\mathbf{B}}(t)$

On the scale at which I look at this problem, $\hat{\mathbf{F}}_\beta(t)$ is roughly a delta correlated noise. This can be understood by observing that $\hat{\mathbf{F}}_\beta(t)$ oscillates much faster than the signal and can be approximated by a delta correlated function. Yet, another way to look at it (see figure 8.1) is to see that the spectrum of $\hat{\mathbf{F}}_\beta(t)$ relative to the spectrum of the signal looks constant, similar to the one of a delta correlated function.

I will compute the integral of $\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t') \right\rangle$ and rewrite it as a delta correlated function (see appendix G.1.1). I can rewrite equation (8.20) as such

$$\left\langle \hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t') \right\rangle = K\delta(t-t'), \tag{8.44}$$

where

$$K \equiv \frac{T}{n_{\mathrm{sat}}}\left[\frac{\beta_0+\left(\frac{n(t)}{n_{\mathrm{sat}}}-1\right)\left\langle\hat{\beta}(t)\right\rangle}{\tau^2}\right]. \tag{8.45}$$

Equation (8.40) can be rewritten

$$\hat{\mathbf{F}}_\beta(t)=\int_0^t G(\tau_1)\hat{\mathbf{F}}_L(t-\tau_1)\mathrm{d}\tau_1, \tag{8.46}$$

where

$$G(\tau_1)=\frac{1}{2T}e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}. \tag{8.47}$$

Therefore,

$$\left\langle\hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t-\tau_1)\right\rangle \approx \frac{\tau K}{8T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}. \tag{8.48}$$

I now compute the integral

$$\int_{-\infty}^{\infty}\left\langle\hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t-\tau_1)\right\rangle \mathrm{d}\tau_1 \approx \frac{\tau^2 K}{4T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-2}. \tag{8.49}$$

Therefore,

$$\left\langle\hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t')\right\rangle \approx \frac{\tau^2 K}{4T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-2}\delta(t-t'). \tag{8.50}$$

I now look for an expression for $\left\langle\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_\beta(t')\right\rangle$. Equation (8.23) can be rewritten

$$\left\langle\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_L(t')\right\rangle = K_1\delta(t-t'), \tag{8.51}$$

where

$$K_1 \equiv -\frac{B(t)}{n_{\mathrm{sat}}\tau}\beta(t). \tag{8.52}$$

Following a procedure identical to the one in the previous section (see appendix G.1.2), I get

$$\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle \approx \frac{\tau K_1}{T}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}\delta(t-t'). \tag{8.53}$$

Similarly

$$\left\langle \hat{\mathbf{F}}_{\beta}(t')\hat{\mathbf{F}}_{\mathrm{ab}}(t)\right\rangle \approx \frac{\tau K_1}{T}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}\delta(t-t'). \tag{8.54}$$

I can now use the expression for $\left\langle \hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$, $\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$ and $\left\langle \hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t')\right\rangle$ to get an expression for the power of the inphase noise, $\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle$. I take equation (8.43) and use equation (8.39) to substitute the expression of $\hat{\mathbf{N}}_x(t)$, with the results in equations (8.53),(8.54), (8.50) and (7.72), I get (see appendix G.2)

$$\begin{aligned}\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle =&\langle \hat{\mathbf{x}}^2(0)\rangle H^2(0)\\ &+\frac{1}{4T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n(t')}+\frac{\beta_0+\left(\frac{n(t')}{n_{\mathrm{sat}}}-1\right)\beta(t)}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}}+1\right)^2}\right)\mathrm{d}t'\\ &+\frac{1}{T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}}+1\right)}\right)\mathrm{d}t'.\end{aligned} \tag{8.55}$$

The inphase noise can be broken into four noises: the amplified initial noise, the relaxation noise, the absorption noise and the beat noise between the the absorption noise and the relaxation noise. The amplified initial noise is noise due to the amplification of the incoming signal's noise $\langle \hat{\mathbf{x}}^2(0)\rangle$. The relaxation noise is the noise due to $\left\langle \hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$. The absorption noise is noise due to $\hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t')$. The beat noise is noise due to $\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle$ and $\left\langle \hat{\mathbf{F}}_{\beta}(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t')\right\rangle$.

The amplified initial amplitude noise is

$$\mathrm{Init}_x = H_2^2(0)\langle \hat{\mathbf{x}}^2(0)\rangle. \tag{8.56}$$

Using equation (8.42), I have

$$\text{Init}_x = \frac{1}{4}\left(\frac{n(0)+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2 \frac{1}{n(0)}. \tag{8.57}$$

I compute the first integral of equation (8.55)(see appendix G.2.2), which is noise due to absorption

$$\text{Noise}_{abs_x} = \frac{1}{4}\frac{2n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\text{sat}}^2}{n(T)} - \frac{n_{\text{sat}}^2}{n(0)} + n(0) - n(T)}{(n(T)+n_{\text{sat}})^2}, \tag{8.58}$$

which, as $n_{\text{sat}} \to \infty$ converges to

$$\text{Noise}_{abs_x} = \frac{1}{4}\frac{e^{\beta_0}-1}{n(0)}, \tag{8.59}$$

which is what was expected. I compute the second integral of equation (8.55) (see appendix G.2.3), which is noise due to relaxation

$$\text{Noise}_{rel_x} = \frac{1}{2}\frac{n(0)-n(T)}{(n(T)+n_{\text{sat}})^2} \tag{8.60}$$

I compute the third integral of equation (8.55)(see appendix G.2.4), which is the beat noise between the relaxation noise and the absorption noise

$$\text{Noise}_{beat_x} = \frac{n(0)-n(T)+n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right)}{(n(T)+n_{\text{sat}})^2}. \tag{8.61}$$

The power of the amplitude noise is the sum of all the noises and is

$$\begin{aligned}\langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle = {} & \frac{1}{4}\left(\frac{n(0)+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2\frac{1}{n(0)} + \frac{1}{4}\frac{6n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right)+\frac{n_{\text{sat}}^2}{n(T)}-\frac{n_{\text{sat}}^2}{n(0)}}{(n(T)+n_{\text{sat}})^2} \\ & + \frac{3}{2}\frac{(n(0)-n(T))}{(n(T)+n_{\text{sat}})^2}.\end{aligned} \tag{8.62}$$

Now, I calculate the quadrature phase noise $\langle \hat{\mathbf{y}}^2(T) \rangle$ Using equation (8.29) and equation (8.30), It can be shown that (see appendix (G.3))

$$\left\langle \hat{\mathbf{y}}^2(T) \right\rangle = \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(T)}\right] + \alpha_H^2 \left\langle \hat{\mathbf{x}}^2(T) \right\rangle - \frac{\alpha_H^2}{2}\frac{1}{n(0)}\left(\frac{n(0)+n_{\mathrm{sat}}}{n(T)+n_{\mathrm{sat}}}\right) - \frac{\alpha_H^2}{2}\left[\frac{3\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\mathrm{sat}}}{n(T)} - \frac{n_{\mathrm{sat}}}{n(0)}}{n(T)+n_{\mathrm{sat}}}\right] \tag{8.63}$$

For verification, I set $n_{\mathrm{sat}} \to \infty$. Then using equation (8.31), I can verify that

$$\left\langle \hat{\mathbf{y}}^2(T) \right\rangle = \frac{1}{4}\left[\frac{1}{n(T)}\right], \tag{8.64}$$

which is what was expected. The incoming signal's noise is amplified. At the output, it becomes

$$\mathrm{Init}_y = \frac{1}{4}\left[(1+\alpha_H^2) + \alpha_H^2\left(\frac{n(0)+n_{\mathrm{sat}}}{n(T)+n_{\mathrm{sat}}}\right)^2 - 2\alpha_H^2\left(\frac{n(0)+n_{\mathrm{sat}}}{n(T)+n_{\mathrm{sat}}}\right)\right]\left[\frac{1}{n(0)}\right]. \tag{8.65}$$

From equation (8.63), the expression of the absorption noise is

$$\mathrm{Noise}_{abs_y} = \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(T)} - \frac{1}{n(0)}\right] + \alpha_H^2 \mathrm{Noise}_{abs_x} - \frac{\alpha_H^2}{2}\left[\frac{\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\mathrm{sat}}}{n(T)} - \frac{n_{\mathrm{sat}}}{n(0)}}{n(T)+n_{\mathrm{sat}}}\right] \tag{8.66}$$

From equation (8.63), the expression of the relaxation noise is

$$\mathrm{Noise}_{rel_y} = \alpha_H^2 \mathrm{Noise}_{rel_x}. \tag{8.67}$$

From equation (8.63), the expression of the beat noise is

$$\mathrm{Noise}_{beat_y} = \alpha_H^2 \mathrm{Noise}_{beat_x} - \alpha_H^2\left[\frac{\ln\left(\frac{n(0)}{n(T)}\right)}{n(T)+n_{\mathrm{sat}}}\right]. \tag{8.68}$$

Now that I have the inphase and quadrature phase noise of the saturable absorber, I use it to get the noise figure of the NMZI OPA.

8.4 Noise Figure

From equation (3.54), assuming a large parametric gain and a large signal, a good estimate of the noise figure is

$$\mathrm{NF}_{\mathrm{QFS}} = \mathrm{NF}_{\mathrm{FAS}} + 3 = \mathrm{NF}_{\mathrm{PNS}} \approx -10 \log \left[\frac{G_{par}}{4\mathrm{ASE}} \right], \tag{8.69}$$

with

$$\mathrm{ASE} = n(T) \left[\left\langle \hat{\mathbf{y}}^2(T) \right\rangle + \left\langle \hat{\mathbf{x}}^2(T) \right\rangle \right] - \frac{1}{2}. \tag{8.70}$$

It is easy to prove that the parametric gain is

$$G_{par} = \left| \mu e^{i\theta_{\mathrm{in}}} + \nu e^{-i\theta_{\mathrm{in}}} \right|^2. \tag{8.71}$$

where

$$\mu \equiv e^{-\frac{\beta}{2}(i\alpha_H+1)} \left[i + \frac{i}{2}(i\alpha_H + 1) \frac{(1-\Gamma)}{\Gamma n(0) + n_{\mathrm{sat}}} n(0) \right] \tag{8.72}$$

and

$$\nu \equiv e^{-\frac{\beta}{2}(i\alpha_H+1)} \left[\frac{i}{2}(i\alpha_H + 1) \frac{(1-\Gamma)}{\Gamma n(0) + n_{\mathrm{sat}}} n(0) \right]. \tag{8.73}$$

I calculated the noise figure for this device for a large range of parameters and found that it is always above 3 dB. I plot the noise figure of a SA based NMZI OPA with a Henry alpha factor 5, $\beta_0 = 2$ and $n_{\mathrm{sat}} = 1$ (see figure 8.2). It can be seen that the noise figure remains high, more than 8 dB. I plot the gain and the noise figure for a saturable absorber of $\alpha_H = 25$, $\beta_0 = 2$ and $n_{\mathrm{sat}} = 1$ on the same graph (see figure 8.3).

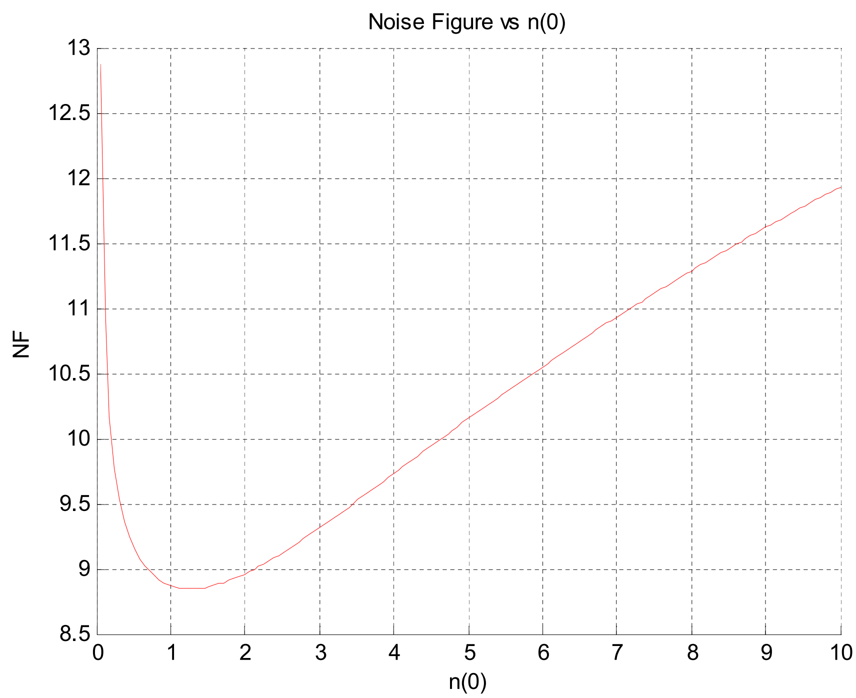

Figure 8.2: Noise Figure as a function of n(0) for $\alpha_H = 5$, $\beta_0 = 2$ and $n_{\text{sat}} = 1$

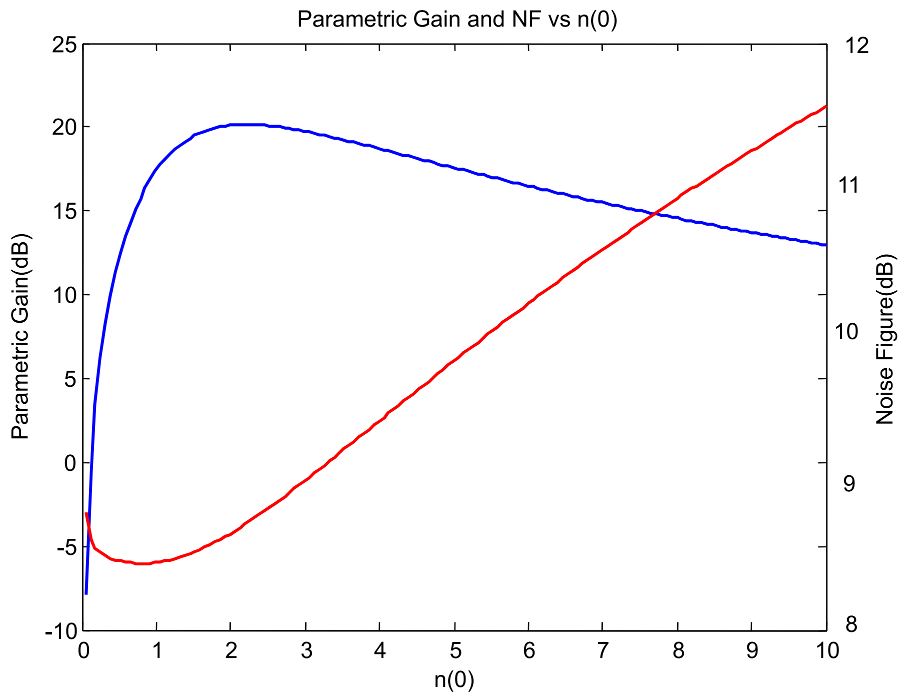

Figure 8.3: Parametric gain (blue) and noise figure (red) as a function of n(0) for $\alpha_H = 25$, $\beta_0 = 2$ and $n_{\text{sat}} = 1$

Chapter 9

Discussions

In this thesis, I have done the following:

- I went over two definitions of noise figure (NF_{NPS} and NF_{FAS}) and shown that the NF_{NPS} is signal dependent for weak signals in OPAs. I have also shown that the NF_{FAS} is inappropriate for OPAs. Therefore, a new definition of the noise figure was introduced, namely the quadrature phase noise figure (NF_{QFS}), which works very well for OPAs and phase insensitive amplifiers. In other words, it is not signal dependent and is much easier to calculate.

- I have a derived a simple expression to get the noise figure for OPA in high gain regime and large signal based on their parametric gain and ASE.

- I have derived the proper expression of the noise figure of a lossy Kerr based NMZI OPA. I have also found the expression for the optimum length for the length of a fiber used as a Kerr medium based on its nonlinear properties and its loss coefficient.

- I have shown that Kerr based NMZI OPA with gain instead of loss under high gain have a minimum 3dB noise figure.

- I have demonstrated the feasibility of a SOA based and SA based NMZI OPA as an alternative to the Kerr based NMZI OPA. I have

  - I have calculated their steady state parametric gain.

- I have calculated their steady state bandwidth.
- I have used Quantum Mechanics to show that the noise figure of SA Based NMZI OPA is very high, in the order of 9 dB.

While at the beginning, the unexpectedly high noise figure of the SA based NMZI OPA came as a little surprise, because the expected noise figure was in the order of 1 dB, after a closer look at the expression something interesting became apparent. I expected most of the noise to come from the absorption noise $\hat{\mathbf{F}}_{\mathrm{ab}}$, the noise due to the signal absorption. However, most of the noise comes from relaxation noise $\hat{\mathbf{F}}_{\beta}(t)$, due to the relaxation of the electrons. This noise is well known and was by Yamamoto *et al.* [67]. The magnitude of the relaxation noise could be due to an overestimate because of the way that noise was calculated. Indeed, the noise average value was calculated in appendix F.2 based on the assumption that

$$\hat{\mathbf{N}}_3\hat{\mathbf{N}}_3 = \hat{\mathbf{N}}_3 \tag{9.1}$$

This is actually only true if the atom is isolated, which is not really the case. While I could get a better estimate of this noise, it is better to completely cancel that noise or make it not interfere with the signal. There are two ways this could be achieved.

The first way is to not let the electrons relax back to level one. Instead, the electrons and their associated holes can be removed by applying a very-low-noise current. A possible technique to produce very low fluctuations current is discussed by Yamamoto *et al.* [67]. Yamamoto *et al.* explain that if such current is used to pump a laser oscillator, it is no longer subject to the standard quantum limit. It can then produce amplitude squeezed states.

The second way, is to send very short pulses as the signal and the pump, much shorter that the relaxation time τ but still much longer than the decoherence time $1/\gamma$. In this fashion, by the time the relaxation occur, the experiment is over.

Assuming these techniques are applied, then the noise due to relaxation $\hat{\mathbf{F}}_{\beta}(t)$ can be neglected. This significantly improves the noise figure and places it in the order of 1 to 0.5 dB (see figure 9.1). However, the application of these techniques are left to future research.

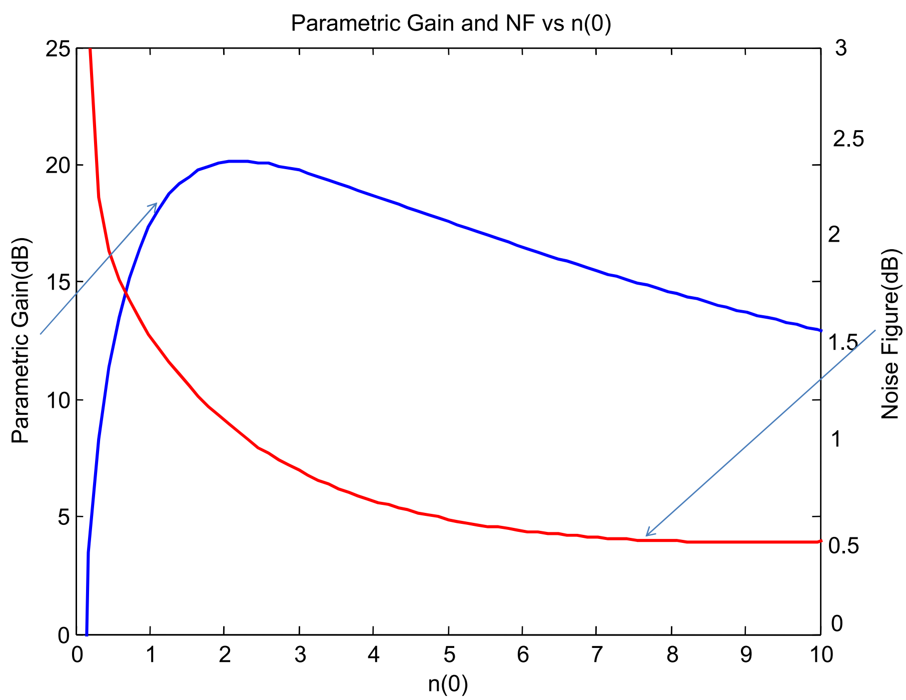

Figure 9.1: Parametric gain (blue) and noise figure (red) as a function of n(0) for $\alpha_H = 25$, $\beta_0 = 2$ and $n_{\text{sat}} = 1$

Appendix A

Detailed Derivation of Noise Field Operator in Lossy Kerr Medium

A.1 Solving the Differential Equation for the Average Terms

To solve (4.10), I take an integral on both side of the equation as follows

$$\int_0^x \frac{\mathrm{d}\left\langle \hat{\mathbf{A}}(z) \right\rangle}{\left\langle \hat{\mathbf{A}}(z) \right\rangle} = \int_0^x \left(i\gamma \left| \left\langle \hat{\mathbf{A}}(z) \right\rangle \right|^2 - \beta \right) \mathrm{d}z. \tag{A.1}$$

To solve it, I define $\left\langle \hat{\mathbf{A}}(0) \right\rangle \equiv \alpha$. Since this is a propagation of a field through a lossy medium with loss coefficient β, I have

$$\left\langle \hat{\mathbf{A}}(z) \right\rangle = \alpha e^{-\beta z}. \tag{A.2}$$

I substitute this result in our equation. I get

$$\int_0^x \frac{\mathrm{d}\left\langle \hat{\mathbf{A}}(z) \right\rangle}{\left\langle \hat{\mathbf{A}}(z) \right\rangle} = \int_0^x \left(i\gamma |\alpha|^2 e^{-2\beta z} - \beta \right) \mathrm{d}z. \tag{A.3}$$

I evaluate the right left hand side of the equation. I get

$$\ln \left(\frac{\left\langle \hat{\mathbf{A}}(x) \right\rangle}{\left\langle \hat{\mathbf{A}}(0) \right\rangle} \right) = \int_0^x \left(i\gamma |\alpha|^2 e^{-2\beta z} - \beta \right) \mathrm{d}z. \tag{A.4}$$

After some more algebraic manipulation, I get

$$\ln \left(\frac{\left\langle \hat{\mathbf{A}}(x) \right\rangle}{\alpha} \right) = \left(-\frac{i\gamma}{2\beta} |\alpha|^2 \left(1 - e^{-2\beta x} \right) - \beta x \right). \tag{A.5}$$

I set $x = z$ and obtain the following solution of the differential equation

$$\begin{aligned} \left\langle \hat{\mathbf{A}}(z) \right\rangle &= \alpha \exp \left(\frac{i\gamma}{2\beta} |\alpha|^2 \left(1 - e^{-2\beta z} \right) - \beta z \right) \\ &= \alpha e^{-\beta z} \exp \left(\frac{i\gamma}{2\beta} |\alpha|^2 \left(1 - e^{-2\beta z} \right) \right). \end{aligned} \tag{A.6}$$

A.2 Derivation of the Output Noise Field Operator

The definition of $\delta\hat{\mathbf{A}}(L)$ is

$$\delta\hat{\mathbf{A}}(L) = \left(\delta\hat{\mathbf{X}}(L) + i\delta\hat{\mathbf{Y}}(L)\right) e^{i\phi(L)-\beta L}. \tag{A.7}$$

Substituting for the values of $\delta\hat{\mathbf{X}}(L)$ and $\delta\hat{\mathbf{Y}}(L)$ from equation (4.25) and (4.26), I get

$$\delta\hat{\mathbf{A}}(L) = \mu(\alpha)\delta\hat{\mathbf{A}}(0)e^{i\theta_{\text{in}}} + \nu(\alpha)\delta\hat{\mathbf{A}}^{\dagger}(0)e^{-i\theta_{\text{in}}} + \hat{\mathbf{N}}, \tag{A.8}$$

where

$$\mu(\alpha) \equiv e^{i\phi(L)-\beta L}\left(1 + i\frac{\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta L}\right)\right), \tag{A.9}$$

$$\nu(\alpha) \equiv e^{i\phi(L)-\beta L}\left(i\frac{\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta L}\right)\right) \tag{A.10}$$

and

$$\begin{aligned}\hat{\mathbf{N}} \equiv \gamma|\alpha|^2 &\int_0^L\int_0^z e^{\beta(x-2z)}\left(\hat{\mathbf{N}}^{\dagger}(x)e^{i\phi(x)} + \hat{\mathbf{N}}(x)e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}z \\ &+ \int_0^L e^{\beta z}\hat{\mathbf{N}}(z)e^{-i\phi(z)}\mathrm{d}z.\end{aligned} \tag{A.11}$$

A.3 Parametric Gain Derivation

I consider Kerr based NMZI with a strong pump in the first input with average field intensity α_p. Similarly, at the second input of the NMZI I assume a weak signal with average field intensity α_s. After the first coupler, I have the two outputs

$$E_{\text{out11}} = \frac{1}{\sqrt{2}}\left(\alpha_p + i\alpha_s\right) \tag{A.12}$$

at the output of the first arm and

$$E_{\text{out21}} = \frac{1}{\sqrt{2}}\left(\alpha_s + i\alpha_p\right). \tag{A.13}$$

To get an expression for the output of the first Kerr medium, I use the previous results, namely equation (A.6) and (A.8), which gives

$$E_{\text{out12}} = \alpha_1 e^{-\beta L} \exp\left(\frac{i\gamma}{2\beta}|\alpha_1|^2\left(1-e^{-2\beta L}\right)\right) \tag{A.14}$$

where

$$\alpha_1 = \frac{1}{\sqrt{2}}\left(\alpha_p + i\alpha_s\right) \tag{A.15}$$

Similarly, the output of the second Kerr medium is

$$E_{\text{out22}} = \alpha_2 e^{-\beta L} \exp\left(\frac{i\gamma}{2\beta}|\alpha_2|^2\left(1-e^{-2\beta L}\right)\right) \tag{A.16}$$

where

$$\alpha_2 = \frac{1}{\sqrt{2}}\left(\alpha_s + i\alpha_p\right) \tag{A.17}$$

At the output of the NMZI, I have

$$E_{\text{out}} = \frac{\alpha_p e^{-\beta L}}{2}\left(e^{i\Phi_2} - e^{i\Phi_1}\right) - \frac{i\alpha_s e^{-\beta L}}{2}\left(e^{i\Phi_2} + e^{i\Phi_1}\right), \tag{A.18}$$

where

$$\Phi_1 = \frac{\gamma}{2\beta}|\alpha_1|^2\left(1-e^{-2\beta L}\right) \tag{A.19}$$

and

$$\Phi_2 = \frac{\gamma}{2\beta}|\alpha_2|^2\left(1-e^{-2\beta L}\right). \tag{A.20}$$

I define $\Phi_1 \equiv \Phi_{10} + \Phi_{11}$, $\Phi_2 \equiv \Phi_{10} - \Phi_{11}$,

$$\Phi_{10} \equiv \frac{\gamma}{4\beta}|\alpha_p|^2\left(1-e^{-2\beta L}\right), \tag{A.21}$$

and

$$\Phi_{11} \equiv \frac{\gamma}{4\beta}\left(1-e^{-2\beta L}\right)\left(\alpha_s\alpha_p^* - \alpha_s^*\alpha_p\right). \tag{A.22}$$

Using the above definitions, we can simplify equation (A.18) further and get

$$E_{\text{out}} = -ie^{-\beta L}e^{i\Phi_{10}}\left(\alpha_p\Phi_{11} + \alpha_s\right). \tag{A.23}$$

Substituting Φ_{11} into equation (A.23), we get

$$E_{\text{out}} = -ie^{-\beta L}e^{i\Phi_{10}}\left(\frac{\gamma}{2\beta}\left(1 - e^{-2\beta L}\right)\sin(\theta_{\text{in}}) + e^{i\theta_{\text{in}}}\right)|\alpha_s|. \tag{A.24}$$

This equation can be rewritten as follows

$$E_{\text{out}} = \mu(\alpha)|\alpha_s|e^{i\theta_{\text{in}}} + \nu(\alpha)|\alpha_s|e^{-i\theta_{\text{in}}}, \tag{A.25}$$

where

$$\mu(\alpha) \equiv e^{i\phi(L)-\beta L}\left(1 + i\frac{\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta L}\right)\right), \tag{A.26}$$

$$\nu(\alpha) \equiv e^{i\phi(L)-\beta L}\left(i\frac{\gamma}{2\beta}|\alpha|^2\left(1 - e^{-2\beta L}\right)\right). \tag{A.27}$$

Therefore, the maximum parametric gain is

$$G = \gamma^2 e^{-2\beta L}|\alpha_p|^4 L_{\text{eff}}^2. \tag{A.28}$$

A.4 Correlation of the Vacuum Noise in the two Arms of the NMZI

It is most important to note that $\delta\hat{\mathbf{A}}_2(0)$ and $\delta\hat{\mathbf{A}}_1(0)$ are standard vacuum fluctuations that are uncorrelated. The proof is as follows:

$$\begin{aligned}
\left\langle \delta\hat{\mathbf{A}}_1(0)\hat{\mathbf{A}}_2^\dagger(0)\right\rangle &= \frac{1}{2}\left\langle \left(\delta\hat{\mathbf{A}} + i\delta\hat{\mathbf{A}}_{lo}\right)\left(i\delta\hat{\mathbf{A}} + \delta\hat{\mathbf{A}}_{lo}\right)^\dagger\right\rangle \\
&= \frac{1}{2}\left\langle -i\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger + \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}_{lo}^\dagger + \delta\hat{\mathbf{A}}_{lo}\delta\hat{\mathbf{A}}^\dagger + i\delta\hat{\mathbf{A}}_{lo}\delta\hat{\mathbf{A}}_{lo}^\dagger\right\rangle \\
&= \frac{i - i}{2} \\
&= 0
\end{aligned} \tag{A.29}$$

and

$$\left\langle \delta\hat{\mathbf{A}}_1^\dagger(0)\hat{\mathbf{A}}_2(0)\right\rangle = \frac{1}{2}\left\langle \left(\delta\hat{\mathbf{A}} + i\delta\hat{\mathbf{A}}_{lo}\right)^\dagger \left(i\delta\hat{\mathbf{A}} + \delta\hat{\mathbf{A}}_{lo}\right)\right\rangle$$
$$= 0. \tag{A.30}$$

It is very easy to see that $\left\langle \delta\hat{\mathbf{A}}_1(0)\hat{\mathbf{A}}_2(0)\right\rangle = \left\langle \delta\hat{\mathbf{A}}_1^\dagger(0)\hat{\mathbf{A}}_2^\dagger(0)\right\rangle = 0$.

Appendix B

Detailed Derivation of NF for the Lossy Kerr Medium Based NMZI OPA

B.1 Expression for $\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}\right)^2$

To calculate the variance of $\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}$, I use the definition

$$\hat{\mathbf{E}}_{\mathrm{out}} \equiv \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle + \delta\hat{\mathbf{E}}_{\mathrm{out}}. \tag{B.1}$$

I can then compute $\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}$, which is

$$\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}} = \left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}} + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^{*} \delta\hat{\mathbf{E}}_{\mathrm{out}} + \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} \tag{B.2}$$

I can then compute $\left(\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}\right)^2$, which is

$$\begin{aligned}
\left(\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}\right)^2 = {} & \left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^4 + 2\left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}} + 2\left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^{*} \delta\hat{\mathbf{E}}_{\mathrm{out}} \\
& + 3\left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^2 \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}} \\
& + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} + \left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 \delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}} \\
& + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^{*2} \delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^{*} \delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} \\
& + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}} + \left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle^{*} \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} \\
& + \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}
\end{aligned} \tag{B.3}$$

Taking the average of equation (B.2), I get

$$\left\langle \hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}} \right\rangle = \left|\left\langle \hat{\mathbf{E}}_{\mathrm{out}} \right\rangle\right|^2 + \left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}} \right\rangle. \tag{B.4}$$

From the above equation, I get

$$\left\langle \hat{\mathbf{E}}_{\text{out}}^{\dagger}\hat{\mathbf{E}}_{\text{out}} \right\rangle^2 = \left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^4 + \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle^2 \\ + 2\left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle \tag{B.5}$$

Taking the average of equation (B.3), I get

$$\begin{aligned} \left\langle \hat{\mathbf{E}}_{\text{out}}^{\dagger}\hat{\mathbf{E}}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}^{\dagger}\hat{\mathbf{E}}_{\text{out}} \right\rangle = &\left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^4 \\ &+ 3\left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle + \left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger} \right\rangle \\ &+ \left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger} \right\rangle + \left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle^{*2} \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle \\ &+ \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle . \end{aligned} \tag{B.6}$$

Therefore,

$$\begin{aligned} \Delta\left(\hat{\mathbf{E}}_{\text{out}}^{\dagger}\hat{\mathbf{E}}_{\text{out}}\right)^2 = &\left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle + \left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger} \right\rangle \\ &+ \left|\left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle\right|^2 \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger} \right\rangle + \left\langle \hat{\mathbf{E}}_{\text{out}} \right\rangle^{*2} \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle \\ &+ \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle - \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle^2 . \end{aligned} \tag{B.7}$$

B.2 Calculation of $\left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}} \right\rangle$

To calculate the noise commutator, I need to compute the following averages:

B.2.1 Calculation of $\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^{\dagger}(y)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^{\dagger}(x) \right\rangle$

From equation (4.6)

$$\left[\hat{\mathbf{N}}(z), \hat{\mathbf{N}}^{\dagger}(z')\right] = 2\beta\delta(z - z'). \tag{B.8}$$

where it is a Dirac delta function. Therefore,

$$\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^{\dagger}(x) - \hat{\mathbf{N}}^{\dagger}(x)\hat{\mathbf{N}}(z) = 2\beta\delta(z - x). \tag{B.9}$$

Therefore,

$$\left\langle \hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x) \right\rangle = 2\beta\delta(z-x). \tag{B.10}$$

Therefore,

$$\hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x) - \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z) = 2\beta\delta(z-x)\hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y). \tag{B.11}$$

Since

$$\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z) \right\rangle = 0, \tag{B.12}$$

I have

$$\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x) - \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z) \right\rangle = 2\beta\delta(z-x)\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y) \right\rangle. \tag{B.13}$$

Therefore,

$$\begin{aligned} \left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x) \right\rangle &= 2\beta\delta(z-x)\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y) \right\rangle \\ &= 4\beta^2\delta(z-x)\delta(w-y). \end{aligned} \tag{B.14}$$

B.2.2 Calculation of $\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) \right\rangle$

From the commutator relationship, I have

$$\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x) - \hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z) = 2\beta\delta(z-x). \tag{B.15}$$

Therefore,

$$\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) - \hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(y) = 2\beta\delta(z-x)\hat{\mathbf{N}}^\dagger(y). \tag{B.16}$$

Therefore,

$$\hat{\mathbf{N}}(w)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) - \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(y) = 2\beta\delta(z-x)\hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(y). \tag{B.17}$$

Therefore,

$$\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) - \hat{\mathbf{N}}(w)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(y) \right\rangle = 4\beta^2\delta(z-x)\delta(w-y). \quad \text{(B.18)}$$

Using the result of the previous section, I have

$$\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) \right\rangle - 4\beta^2\delta(w-x)\delta(z-y) = 4\beta^2\delta(z-x)\delta(w-y). \quad \text{(B.19)}$$

Therefore,

$$\left\langle \hat{\mathbf{N}}(w)\hat{\mathbf{N}}(z)\hat{\mathbf{N}}^\dagger(x)\hat{\mathbf{N}}^\dagger(y) \right\rangle = 4\beta^2\Big[\delta(z-x)\delta(w-y) + \delta(w-x)\delta(z-y)\Big]. \quad \text{(B.20)}$$

B.3 Simplified Expression for $\Delta\left(\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\hat{\mathbf{E}}_{\mathrm{out}}\right)^2$

It was seen that the noise from the noise from the upper arm of the NMZI and the lower arm are uncorrelated. Therefore, the noise at the output of the NMZI is of the same form as the ones in each of the arm of the NMZI given in equation (A.8), which is

$$\delta\hat{\mathbf{E}}_{\mathrm{out}} = \mu(\alpha_p)\delta\hat{\mathbf{A}}(0)e^{i\theta_{\mathrm{in}}} + \nu(\alpha_p)\delta\hat{\mathbf{A}}^\dagger(0)e^{-i\theta_{\mathrm{in}}} + \hat{\mathbf{N}}. \quad \text{(B.21)}$$

I make the following approximation

$$\begin{aligned}\hat{\mathbf{N}} &= \gamma|\alpha_p|^2\int_0^L\int_0^z e^{\beta(x-2z)}\left(\hat{\mathbf{N}}^\dagger(x)e^{i\phi(x)} + \hat{\mathbf{N}}(x)e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}z \\ &\quad + \int_0^L e^{\beta z}\hat{\mathbf{N}}(z)e^{-i\phi(z)}\mathrm{d}z \\ &\approx \gamma|\alpha_p|^2\int_0^L\int_0^z e^{\beta(x-2z)}\left(\hat{\mathbf{N}}^\dagger(x)e^{i\phi(x)} + \hat{\mathbf{N}}(x)e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}z\end{aligned} \quad \text{(B.22)}$$

The term is neglected since it will small relative to the pump power $|\alpha_p|^2$. In general, I am only going to keep terms with factor of $|\alpha_p|^8$ in variance of the output . Also, I will use the high gain approximation

$$|\mu|^2 \approx |\nu|^2. \quad \text{(B.23)}$$

I compute $\left\langle \delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\right\rangle$, which is

$$\begin{aligned}\left\langle \delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\right\rangle \approx{}& |\mu|^4\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\right\rangle \\ &+|\mu|^4\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\delta\hat{\mathbf{A}}^\dagger\right\rangle \\ &+|\mu|^2\left\langle \hat{\mathbf{N}}^\dagger\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}\right\rangle+|\mu|^2\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\right\rangle \\ &+|\mu|^2\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle+|\mu|^2\left\langle \delta\hat{\mathbf{A}}\hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\delta\hat{\mathbf{A}}^\dagger\right\rangle \\ &+\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle.\end{aligned} \tag{B.24}$$

I use the following expressions

$$\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\right\rangle = 1, \tag{B.25}$$

$$\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\delta\hat{\mathbf{A}}^\dagger\right\rangle = 2, \tag{B.26}$$

$$\left\langle \hat{\mathbf{N}}^\dagger\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}\right\rangle = \left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle, \tag{B.27}$$

$$\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\right\rangle = \left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle, \tag{B.28}$$

$$\left\langle \delta\hat{\mathbf{A}}\delta\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle = \left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle, \tag{B.29}$$

$$\left\langle \delta\hat{\mathbf{A}}\hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\delta\hat{\mathbf{A}}^\dagger\right\rangle = \left\langle \hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\right\rangle, \tag{B.30}$$

and the fact that

$$\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle = \left\langle \hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\right\rangle. \tag{B.31}$$

I substitute these expressions into the equation, I get

$$\left\langle \delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\delta\hat{\mathbf{E}}^\dagger_{\mathrm{out}}\delta\hat{\mathbf{E}}_{\mathrm{out}}\right\rangle = 3|\mu|^4+4|\mu|^2\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle+\left\langle \hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle. \tag{B.32}$$

Also, it can easily be shown that

$$\begin{aligned}\left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle &= \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\right\rangle \\ &= \left\langle \delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle \\ &= \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\right\rangle .\end{aligned} \tag{B.33}$$

Therefore,

$$\begin{aligned}\Delta\left(\hat{\mathbf{E}}_{\text{out}}^{\dagger}\hat{\mathbf{E}}_{\text{out}}\right)^{2} = &\left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle \left[2\left|\left\langle \hat{\mathbf{E}}_{\text{out}}\right\rangle\right|^{2} + 2\Re\left(\left\langle \hat{\mathbf{E}}_{\text{out}}\right\rangle^{2}\right)\right] \\ &+ \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle - \left\langle \delta\hat{\mathbf{E}}_{\text{out}}^{\dagger}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle^{2}\end{aligned} \tag{B.34}$$

B.4 Computing $\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle$

$$\begin{aligned}\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle = \gamma^{4}\frac{e^{-4\beta L}|\alpha_p|^{8}}{16}\Bigg\langle &\int_0^L\int_0^{x'} e^{\beta(x-2x')}\left(\hat{\mathbf{N}}^{\dagger}(x)e^{i\phi(x)} + \hat{\mathbf{N}}(x)e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}x' \\ &\times\int_0^L\int_0^{w'} e^{\beta(w-2w')}\left(\hat{\mathbf{N}}^{\dagger}(w)e^{i\phi(w)} + \hat{\mathbf{N}}(w)e^{-i\phi(w)}\right)\mathrm{d}w\mathrm{d}w' \\ &\times\int_0^L\int_0^{y'} e^{\beta(y-2y')}\left(\hat{\mathbf{N}}^{\dagger}(y)e^{i\phi(y)} + \hat{\mathbf{N}}(y)e^{-i\phi(y)}\right)\mathrm{d}y\mathrm{d}y' \\ &\times\int_0^L\int_0^{z'} e^{\beta(z-2z')}\left(\hat{\mathbf{N}}^{\dagger}(z)e^{i\phi(z)} + \hat{\mathbf{N}}(z)e^{-i\phi(z)}\right)\mathrm{d}z\mathrm{d}z'\Bigg\rangle.\end{aligned} \tag{B.35}$$

Keeping only the terms that are not going to average to zero, I have

$$\begin{aligned}\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{16} &\int_0^L \int_0^{x'} \int_0^L \int_0^{w'} \int_0^L \int_0^{y'} \int_0^L \int_0^{z'} \\ &\times e^{\beta(x+w+y+z-2(x'+w'+y'+z'))} \\ &\times \left[\left\langle \hat{\mathbf{N}}(x) \hat{\mathbf{N}}^\dagger(w) \hat{\mathbf{N}}(y) \hat{\mathbf{N}}^\dagger(z) \right\rangle \right. \\ &\left. + \left\langle \hat{\mathbf{N}}(x) \hat{\mathbf{N}}(w) \hat{\mathbf{N}}^\dagger(y) \hat{\mathbf{N}}^\dagger(z) \right\rangle \right] \\ &\times \mathrm{d}x\mathrm{d}x'\mathrm{d}w\mathrm{d}w'\mathrm{d}y\mathrm{d}y'\mathrm{d}z\mathrm{d}z'.\end{aligned} \tag{B.36}$$

I now use the results in section B.2, I get

$$\begin{aligned}\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \beta^2\gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{4} &\int_0^L \int_0^{x'} \int_0^L \int_0^{w'} \int_0^L \int_0^{y'} \int_0^L \int_0^{z'} \\ &\times e^{\beta(x+w+y+z-2(x'+w'+y'+z'))} \\ &\times \left[\delta(x-w)\delta(y-z) + \delta(w-y)\delta(x-z) \right. \\ &\left. + \delta(x-y)\delta(w-z) \right] \\ &\times \mathrm{d}x\mathrm{d}x'\mathrm{d}w\mathrm{d}w'\mathrm{d}y\mathrm{d}y'\mathrm{d}z\mathrm{d}z'.\end{aligned} \tag{B.37}$$

From this, I get

$$\begin{aligned}\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \beta^2\gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{4} &\int_0^L \int_0^L \int_0^{w'} \int_0^L \int_0^{y'} \int_0^L \int_0^{z'} \\ &\times \left[e^{\beta(2w+y+z-2(x'+w'+y'+z'))}\delta(y-z) \right. \\ &+ e^{\beta(w+y+2z-2(x'+w'+y'+z'))}\delta(w-y) \\ &\left. + e^{\beta(w+2y+z-2(x'+w'+y'+z'))}\delta(w-z) \right] \\ &\times \mathrm{d}x'\mathrm{d}w\mathrm{d}w'\mathrm{d}y\mathrm{d}y'\mathrm{d}z\mathrm{d}z'.\end{aligned} \tag{B.38}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \beta^{2}\gamma^{4}\frac{e^{-4\beta L}|\alpha_{p}|^{8}}{4}\Bigg\{ \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{w'}\int_{0}^{y'} \times \left[e^{\beta(2w+2y-2(x'+w'+y'+z'))}\right]\mathrm{d}y\mathrm{d}w\mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z' + \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{y'}\int_{0}^{z'}\Big[e^{\beta(2y+2z-2(x'+w'+y'+z'))} + e^{\beta(2y+2z-2(x'+w'+y'+z'))}\Big]\mathrm{d}z\mathrm{d}y\mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z'\Bigg\}. \tag{B.39}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \beta\gamma^{4}\frac{e^{-4\beta L}|\alpha_{p}|^{8}}{8}\Bigg\{ \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{w'} \times \left[e^{\beta(2w-2(x'+w'+z'))} - e^{\beta(2w-2(x'+w'+y'+z'))}\right] \times \mathrm{d}w\mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z' + \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{y'} \times 2\left[e^{\beta(2y-2(x'+w'+y'))} - e^{\beta(2y-2(x'+w'+y'+z'))}\right] \times \mathrm{d}y\mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z'\Bigg\}. \tag{B.40}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \gamma^{4}\frac{e^{-4\beta L}|\alpha_{p}|^{8}}{16}\Bigg\{ \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L} \times \Big[e^{-2\beta(x'+z')} - e^{-2\beta(x'+w'+z')} + e^{-2\beta(x'+w'+y'+z')} - e^{-2\beta(x'+y'+z')}\Big] \times \mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z' + \int_{0}^{L}\int_{0}^{L}\int_{0}^{L}\int_{0}^{L} \times 2\Big[e^{-2\beta(x'+w')} - e^{-2\beta(x'+w'+y')} + e^{-2\beta(x'+w'+y'+z')} - e^{-2\beta(x'+w'+z')}\Big] \times \mathrm{d}x'\mathrm{d}w'\mathrm{d}y'\mathrm{d}z'\Bigg\}. \tag{B.41}$$

Rewriting the integrals, I have

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{16} \Bigg\{ \int_0^L \int_0^L \int_0^L \int_0^L \times \Big[e^{-2\beta(x+z)} - e^{-2\beta(x+w+z)} + 3e^{-2\beta(x+w+y+z)} - 3e^{-2\beta(x+y+z)} \Big] + 2\Big[e^{-2\beta(x+w)} - e^{-2\beta(x+w+y)} \Big] \mathrm{d}x\mathrm{d}w\mathrm{d}y\mathrm{d}z \Bigg\}. \tag{B.42}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{32\beta} \Bigg\{ \int_0^L \int_0^L \int_0^L \times \Big[e^{-2\beta z} - e^{-2\beta(L+z)} + e^{-2\beta(L+w+z)} - e^{-2\beta(w+z)} + 3e^{-2\beta(w+y+z)} - 3e^{-2\beta(L+w+y+z)} + 3e^{-2\beta(L+y+z)} - 3e^{-2\beta(y+z)} + 2\Big[e^{-2\beta w} - e^{-2\beta(L+w)} \Big] + 2\Big[e^{-2\beta(L+w+y)} - e^{-2\beta(w+y)} \Big] \mathrm{d}w\mathrm{d}y\mathrm{d}z \Bigg\}. \tag{B.43}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{32\beta} \Bigg\{ \int_0^L \int_0^L \Big[Le^{-2\beta z} - Le^{-2\beta(L+z)} + 3Le^{-2\beta(L+y+z)} - 3Le^{-2\beta(y+z)} + \frac{1}{2\beta} \Bigg(e^{-2\beta(L+z)} + e^{-2\beta(L+z)} - e^{-2\beta z} + 3e^{-2\beta(y+z)} - 3e^{-2\beta(L+y+z)} - e^{-2\beta(2L+z)} + 3e^{-2\beta(2L+y+z)} - 3e^{-2\beta(L+y+z)} + 2\Big[1 - e^{-2\beta L} \Big] + 2\Big[e^{-2\beta(2L)} - e^{-2\beta L} \Big] + 2\Big[e^{-2\beta(L+y)} - e^{-2\beta(2L+y)} \Big] + 2\Big[e^{-2\beta(L+y)} - e^{-2\beta y} \Big] \Bigg) \mathrm{d}y\mathrm{d}z \Bigg\}. \tag{B.44}$$

Rewriting the equation, I have

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{32\beta} \Bigg\{ \int_0^L \int_0^L \Big[L e^{-2\beta z} - L e^{-2\beta(L+z)} + 3L e^{-2\beta(L+y+z)} - 3L e^{-2\beta(y+z)} + \frac{1}{2\beta} \Bigg(2e^{-2\beta(L+z)} - e^{-2\beta(2L+z)} - e^{-2\beta z} + 3e^{-2\beta(y+z)} - 6e^{-2\beta(L+y+z)} + 3e^{-2\beta(2L+y+z)} + 2\Big[1 - 2e^{-2\beta L} + e^{-2\beta(2L)}\Big] + 2\Big[2e^{-2\beta(L+y)} - e^{-2\beta(2L+y)} - e^{-2\beta y}\Big]\Bigg) \mathrm{d}y \mathrm{d}z \Bigg\}. \tag{B.45}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{32\beta} \Bigg\{ \int_0^L \Big[L^2 e^{-2\beta z} - L^2 e^{-2\beta(L+z)} + \frac{3L}{2\beta}\Big[e^{-2\beta(L+z)} - e^{-2\beta(2L+z)} + e^{-2\beta(L+z)} - e^{-2\beta z}\Big] + \frac{1}{2\beta}\Bigg(2L e^{-2\beta(L+z)} - L e^{-2\beta(2L+z)} - L e^{-2\beta z} + \frac{3}{2\beta}\Big[e^{-2\beta z} - e^{-2\beta(L+z)} + 2e^{-2\beta(2L+z)} - 2e^{-2\beta(L+z)} + e^{-2\beta(2L+z)} - e^{-2\beta(3L+z)}\Big] + 2L\Big[1 - 2e^{-2\beta L} + e^{-2\beta(2L)}\Big] + \frac{1}{\beta}\Big[2e^{-2\beta L} - 2e^{-4\beta L} + e^{-6\beta L} - e^{-4\beta L} + e^{-2\beta L} - 1\Big]\Bigg) \mathrm{d}z \Bigg\}. \tag{B.46}$$

Rewritting the equation, I have

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = \gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{32\beta} \Bigg\{ \Bigg[\frac{L^2}{2\beta}\Big[1 - 2e^{-2\beta L} + e^{-4\beta L}\Big] \\ + \frac{3L}{4\beta^2}\Big[3e^{-2\beta L} - 3e^{-4\beta L} + e^{-6\beta L} - 1\Big] \\ + \frac{1}{2\beta}\Bigg(\frac{L}{2\beta}\Big[3e^{-2\beta L} - 3e^{-4\beta L} + e^{-6\beta L} - 1\Big] \\ + \frac{3}{4\beta^2}\Big[1 - e^{-2\beta L} + 3e^{-4\beta L} - 3e^{-2\beta L} \\ + 3e^{-4\beta L} - 3e^{-6\beta L} + e^{-8\beta L} - e^{-6\beta L}\Big] \\ + 2L^2\Big[1 - 2e^{-2\beta L} + e^{-2\beta(2L)}\Big] \\ + \frac{L}{\beta}\Big[3e^{-2\beta L} - 3e^{-4\beta L} + e^{-6\beta L} - 1\Big]\Bigg)\Bigg]\Bigg\}. \tag{B.47}$$

From this, I get

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = 3\gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{64\beta^2} \Bigg\{ \frac{L}{\beta}\Big[3e^{-2\beta L} - 3e^{-4\beta L} + e^{-6\beta L} - 1\Big] \\ + \frac{1}{4\beta^2}\Big[1 - 4e^{-2\beta L} + 6e^{-4\beta L} + e^{-8\beta L} - 4e^{-6\beta L}\Big] \\ + L^2\Big[1 - 2e^{-2\beta L} + e^{-4\beta L}\Big]\Bigg\}. \tag{B.48}$$

Which reduces to

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = 3\gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{64\beta^2} \Bigg\{ \frac{-L}{\beta}\left(1 - e^{-2\beta L}\right)^3 \\ + \frac{1}{4\beta^2}\left(1 - e^{-2\beta L}\right)^4 + L^2\left(1 - e^{-2\beta L}\right)^2 \Bigg\}. \tag{B.49}$$

I define

$$L_{\text{eff}} \equiv \frac{1 - e^{-2\beta L}}{2\beta}. \tag{B.50}$$

Substituting L_{eff} in the equation, I get

$$\left\langle \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \hat{\mathbf{N}}^\dagger \hat{\mathbf{N}} \right\rangle = 3\gamma^4 \frac{e^{-4\beta L}|\alpha_p|^8}{8} \left\{ \frac{1}{2}L_{\text{eff}}^4 + \frac{1}{2}L^2 L_{\text{eff}}^2 - L L_{\text{eff}}^3 \right\} \tag{B.51}$$

B.5 Computing $\left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle$

If I rewrite equation (A.10) using L_{eff}, I get

$$\begin{aligned}|\mu|^4 &\approx e^{-4\beta L}\frac{\gamma^4}{16}|\alpha_p|^8 L_{\text{eff}}^4 \\ &\approx |\nu|^4.\end{aligned} \tag{B.52}$$

Using equation (B.22), I find an expression for $\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle$, which is

$$\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle = \gamma^2\frac{e^{-2\beta L}}{4}\alpha_p^4\left(L - L_{\text{eff}}\right)L_{\text{eff}}. \tag{B.53}$$

Using equation (B.21), I find an expression for $\left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle$, which is

$$\left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle = \frac{\gamma^2 e^{-2\beta L}}{4}\alpha_p^4 L L_{\text{eff}}. \tag{B.54}$$

I use these results in equation (B.32) to get

$$\begin{aligned}\left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle &\approx 3|\mu|^4 + 4|\mu|^2\left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle + \left\langle \hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\hat{\mathbf{N}}^{\dagger}\hat{\mathbf{N}}\right\rangle \\ &\approx \gamma^4 e^{-4\beta L}|\alpha_p|^8\left\{\frac{3L_{\text{eff}}^4}{16} + \frac{\left(L - L_{\text{eff}}\right)L_{\text{eff}}^3}{4}\right. \\ &\quad \left. + \frac{3}{8}\left(\frac{1}{2}L_{\text{eff}}^4 + \frac{1}{2}L^2L_{\text{eff}}^2 - LL_{\text{eff}}^3\right)\right\} \\ &= \frac{\gamma^4}{16}e^{-4\beta L}|\alpha_p|^8\left(3L^2L_{\text{eff}}^2 - 2LL_{\text{eff}}^3 + 2L_{\text{eff}}^4\right).\end{aligned} \tag{B.55}$$

Putting everything together, I have

$$\begin{aligned}\Delta\left(\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right)^2 &= \frac{\gamma^4}{16}e^{-4\beta L}|\alpha_p|^8\left(2L^2L_{\text{eff}}^2 - 2LL_{\text{eff}}^3 + 2L_{\text{eff}}^4\right) \\ &= \frac{\gamma^4}{8}e^{-4\beta L}|\alpha_p|^8\left(L^2L_{\text{eff}}^2 - LL_{\text{eff}}^3 + L_{\text{eff}}^4\right)\end{aligned} \tag{B.56}$$

B.6 Computing $\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}\right)^2$

Since

$$\begin{aligned}\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}\right)^2 = &\left\langle \delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle \left[2\left|\left\langle\hat{\mathbf{E}}_{\text{out}}\right\rangle\right|^2 + 2\Re\left(\left\langle\hat{\mathbf{E}}_{\text{out}}\right\rangle^2\right)\right] \\ &+ \left\langle\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle - \left\langle\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right\rangle^2,\end{aligned} \tag{B.57}$$

I have

$$\begin{aligned}\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}\right)^2 = &\frac{\gamma^2 e^{-2\beta L}}{4}\alpha_p^4 L L_{\text{eff}}\left[2G\alpha_s^2 + 2G\alpha_s^2\cos(2\Phi_{10})\right] \\ &+ \Delta\left(\delta\hat{\mathbf{E}}^{\dagger}_{\text{out}}\delta\hat{\mathbf{E}}_{\text{out}}\right)^2,\end{aligned} \tag{B.58}$$

where the expression of Φ_{10} is given by (A.21). The maximum noise in then when $\cos(2\Phi_{10}) = 1$. After some algebraic manipulation, I get

$$\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}\right)^2 = G^2\left[\frac{L}{L_{\text{eff}}}\alpha_s^2 + \frac{1}{8}\left(\frac{L^2}{L_{\text{eff}}^2} - \frac{L}{L_{\text{eff}}} + 1\right)\right]. \tag{B.59}$$

It can be verified that if $\beta \longrightarrow 0$

$$\Delta\left(\hat{\mathbf{E}}^{\dagger}_{\text{out}}\hat{\mathbf{E}}_{\text{out}}\right)^2 = G^2\left[\alpha_s^2 + \frac{1}{8}\right], \tag{B.60}$$

which is the result I obtain in equation (3.16). Therefore, the output SNR is

$$\text{SNR}_{\text{out}} = \frac{\alpha_s^4}{\frac{L}{L_{\text{eff}}}\alpha_s^2 + \frac{1}{8}\left(\frac{L^2}{L_{\text{eff}}^2} - \frac{L}{L_{\text{eff}}} + 1\right)}. \tag{B.61}$$

B.7 Noise Figure

With $\text{SNR}_{\text{in}} = \alpha_s^2$, I get the following expression of the noise figure

$$\text{NF} = -10\log\left[\frac{\alpha_s^2}{\frac{L}{L_{\text{eff}}}\alpha_s^2 + \frac{1}{8}\left(\frac{L^2}{L_{\text{eff}}^2} - \frac{L}{L_{\text{eff}}} + 1\right)}\right]. \tag{B.62}$$

Appendix C

Derivation of Noise for NMZI Based Kerr Medium with Gain

C.1 ASE of Kerr Medium with Gain

To get the field output of Kerr Medium with gain instead of loss, I follow the same procedure as in section 4.2 The differences in the equation are that instead of β, I have $-g_0$, g_0 being the gain coefficient, and instead of $\hat{\mathbf{N}}$ being a lowering operator, it is a raising operator. I get

$$\left\langle \hat{\mathbf{A}}(z) \right\rangle = \alpha e^{g_0 z} \exp\left(\frac{i\gamma}{2g_0} |\alpha|^2 \left(e^{2g_0 z} - 1 \right) \right), \tag{C.1}$$

$$\delta\hat{\mathbf{X}}(L) = \delta\hat{\mathbf{X}}(0) + \int_0^L \frac{e^{-g_0 z}}{2} \left(\hat{\mathbf{N}}^\dagger e^{i\phi(z)} + \hat{\mathbf{N}} e^{-i\phi(z)} \right) \mathrm{d}z \tag{C.2}$$

and

$$\begin{aligned} \delta\hat{\mathbf{Y}}(L) = \delta\hat{\mathbf{Y}}(0) &+ \frac{\gamma}{g_0}|\alpha|^2 \left(e^{2g_0 L} - 1 \right) \delta\hat{\mathbf{X}}(0) \\ &+ \int_0^L 2\gamma|\alpha|^2 e^{2g_0 z} \int_0^z \frac{e^{-g_0 x}}{2} \left(\hat{\mathbf{N}}^\dagger e^{i\phi(x)} + \hat{\mathbf{N}} e^{-i\phi(x)} \right) \mathrm{d}x\mathrm{d}z \\ &+ \int_0^L \frac{e^{-g_0 z}}{2i} \left(\hat{\mathbf{N}} e^{-i\phi(z)} - \hat{\mathbf{N}}^\dagger e^{i\phi(z)} \right) \mathrm{d}z. \end{aligned} \tag{C.3}$$

Therefore,

$$\begin{aligned} \left\langle \delta\hat{\mathbf{X}}^2(L) \right\rangle &= \left\langle \delta\hat{\mathbf{X}}^2(0) \right\rangle + \frac{1}{4} \int_0^L \int_0^L e^{-g_0(z+z')} \left\langle \hat{\mathbf{N}}^\dagger(z) \hat{\mathbf{N}}(z') \right\rangle \mathrm{d}z\mathrm{d}z' \\ &= \frac{1}{4} + \frac{1 - e^{-2g_0 L}}{4}, \end{aligned} \tag{C.4}$$

where I used $\left\langle \delta\hat{\mathbf{X}}^2(0) \right\rangle = \frac{1}{4}$ since our input signal is a coherent state. To calculate $\left\langle \delta\hat{\mathbf{Y}}^2(L) \right\rangle$, I first make the following approximation based on the fact that $|\alpha|^2$, the pump power is very large

$$\delta\hat{\mathbf{Y}}(L) = \frac{\gamma}{g_0}|\alpha|^2\left(e^{2g_0L}-1\right)\delta\hat{\mathbf{X}}(0) + \int_0^L 2\gamma|\alpha|^2 e^{2g_0z}\int_0^z \frac{e^{-g_0x}}{2}\left(\hat{\mathbf{N}}^\dagger e^{i\phi(x)} + \hat{\mathbf{N}}e^{-i\phi(x)}\right)\mathrm{d}x\mathrm{d}z. \tag{C.5}$$

I will use $= |\alpha|^2 = \frac{|\alpha_p|^2}{2}$, where $|\alpha_p|^2$ is the pump power (only half of the pump power goes into each Kerr medium). Therefore,

$$\begin{aligned}\left\langle\delta\hat{\mathbf{Y}}^2(L)\right\rangle &= \frac{\gamma^2}{4g_0^2}|\alpha|^4\left(e^{2g_0L}-1\right)^2\left\langle\delta\hat{\mathbf{X}}^2(0)\right\rangle \\ &\quad + \frac{\gamma^2}{4}|\alpha|^4\int_0^L\int_0^L\int_0^z\int_0^{z'} e^{g_0(2z-x+2z'-y)+i(\phi(x)-\phi(y))}\left\langle\hat{\mathbf{N}}^\dagger\hat{\mathbf{N}}\right\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}z'\mathrm{d}z \\ &= \frac{\gamma^2}{16g_0^2}|\alpha|^4\left(e^{2g_0L}-1\right)^2 + \frac{1}{8}\frac{\gamma^2}{g_0}|\alpha|^4\left(1-e^{-2g_0L}\right)\left(\frac{e^{2g_0L}-1}{2g_0} - L\right). \\ \\ &= \frac{\gamma^2 e^{2g_0L}}{4g_0}|\alpha|^4\left(e^{2g_0L}-1\right)\left[\frac{e^{2g_0L}-1}{2g_0} - \frac{L}{2}\right] \\ &= \frac{\gamma^2 e^{2g_0L}}{4g_0^2}|\alpha|^4\left(e^{2g_0L}-1\right)^2\left[\frac{1}{2} - \frac{Lg_0}{2\left(e^{2g_0L}-1\right)}\right].\end{aligned} \tag{C.6}$$

I define

$$L_{\text{eff}} \equiv \frac{e^{2g_0L}-1}{2g_0}. \tag{C.7}$$

Therefore, I have

$$\left\langle\delta\hat{\mathbf{Y}}^2(L)\right\rangle = \gamma^2 e^{2g_0L}|\alpha|^4 L_{\text{eff}}^2\left[\frac{1}{2} - \frac{Lg_0}{2\left(e^{2g_0L}-1\right)}\right]. \tag{C.8}$$

C.2 Noise Figure

The gain of this OPA is calculated using the same steps as in appendix A.3, but with $-g_0$ instead of β. I obtain

$$G = \gamma^2 e^{g_0 L} |\alpha|^4 L_{\text{eff}}^2. \tag{C.9}$$

using the same argument as in section 4.2, it can be shown that $\delta\hat{\mathbf{Y}}$ is the component of noise in phase with the output signal. Therefore, I can use the equation (3.54) to geet the noise figure, which is

$$\begin{aligned} \text{NF} \approx & -10\log\left[\frac{G}{4\text{ASE}}\right] \\ &= 10\log\left[2 - \frac{2Lg_0}{(e^{2g_0 L} - 1)}\right] \\ &= 10\log\left[2 - \frac{L}{L_{\text{eff}}}\right]. \end{aligned} \tag{C.10}$$

Appendix D

Derivation of SA based NMZI-OPA Parametric Gain

D.1 Approximation of $\Gamma_2 - \Gamma_1$

I substitute (6.19) and (6.20) in (6.21) and (6.22) and I obtain

$$\Gamma_1 = \frac{P_{\text{sat}}}{E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})} \times W\left(\frac{\left(E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} e^{\frac{\left(E_p^2 - 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} - \beta_0}\right) \tag{D.1}$$

and

$$\Gamma_2 = \frac{P_{\text{sat}}}{E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})} \times W\left(\frac{\left(E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} e^{\frac{\left(E_p^2 + 2E_pE_s\sin(\theta_{\text{in}})\right)}{P_{\text{sat}}} - \beta_0}\right). \tag{D.2}$$

For any differentiable function f at x, to the first order I have

$$f(x+\Delta) - f(x-\Delta) \approx 2f'(x)\Delta. \tag{D.3}$$

I choose $\Delta = 2E_pE_s\sin(\theta_{\text{in}})$. Therefore,

$$\Gamma_2 - \Gamma_1 = \Gamma(\Delta) - \Gamma(-\Delta) \approx 2\Delta\Gamma'(E_p^2), \tag{D.4}$$

D.2 Derivative of the Saturated Loss

I set $p \equiv \frac{1}{2}E_p^2$. Therefore,

$$\Gamma(p) = \frac{P_{\text{sat}}}{p} W\left(\frac{p}{P_{\text{sat}}} e^{\frac{p}{P_{\text{sat}}} - \beta_0 L}\right). \tag{D.5}$$

Taking the derivative with respect to p

$$\Gamma'(p) = -\frac{P_{\mathrm{sat}}}{p^2} W\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right) + \frac{P_{\mathrm{sat}}}{p} \frac{\mathrm{d}}{\mathrm{d}p} W\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right) \tag{D.6}$$

For short hands, I will write W for $W\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right)$ and W' for $W'\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right)$. Then the derivative of the gain can be written

$$\Gamma'(p) = -\frac{P_{\mathrm{sat}}}{p^2} W + \frac{1}{p}\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L} + e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right) W'. \tag{D.7}$$

Substituting for the derivative of the Lambert W function which is

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{W(x)}{x\left(1 + W(x)\right)} \tag{D.8}$$

I get

$$\begin{aligned}
\Gamma'(p) &= -\frac{P_{\mathrm{sat}}}{p^2} W + \frac{1}{p}\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L} + e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right) \frac{W}{(1+W)\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right)} \\
&= -\frac{P_{\mathrm{sat}}}{p^2} W + \frac{1}{p}\left(\frac{p}{P_{\mathrm{sat}}} + 1\right) \frac{W}{(1+W)\left(\frac{p}{P_{\mathrm{sat}}}\right)} \\
&= -\frac{P_{\mathrm{sat}}}{p^2} W + \frac{1}{p}\left(\frac{P_{\mathrm{sat}}}{p} + 1\right) \frac{W}{(1+W)}.
\end{aligned} \tag{D.9}$$

Using Γ as short for $\Gamma(p)$ and substituting for

$$\Gamma(p) = \frac{P_{\mathrm{sat}}}{p} W\left(\frac{p}{P_{\mathrm{sat}}} e^{\frac{p}{P_{\mathrm{sat}}} - \beta_0 L}\right), \tag{D.10}$$

I get

$$\begin{aligned}
\Gamma'(p) &= -\frac{\Gamma}{p} + \frac{\Gamma}{p} \frac{1}{1 + \frac{\Gamma p}{P_{\mathrm{sat}}}} \left(\frac{p}{P_{\mathrm{sat}}} + 1\right) \\
&= -\frac{\Gamma(\Gamma - 1)}{P_{\mathrm{sat}} + \Gamma p}
\end{aligned} \tag{D.11}$$

D.3 Approximation of $\beta_2 - \beta_1$

I denote $\beta(p) \equiv -\ln(\Gamma(p))$. Therefore,

$$\begin{aligned} \beta(p+\Delta) &\approx -\ln(\Gamma(p)) - \Delta\Gamma'(p) \\ &= -\ln(\Gamma(p)) - \Delta\frac{\Gamma'}{\Gamma} \end{aligned} \tag{D.12}$$

Therefore,

$$\begin{aligned} \beta'(p) &= -\frac{\Gamma'}{\Gamma} \\ &= \frac{(\Gamma-1)}{P_{\text{sat}} + \Gamma p}. \end{aligned} \tag{D.13}$$

Since $\beta_2 = \beta(p+\Delta)$ and $\beta_1 = \beta(p-\Delta)$,

$$\begin{aligned} \beta_2 - \beta_1 &\approx 2\Delta\frac{(\Gamma-1)}{P_{\text{sat}} + \Gamma p} \\ &= 2E_pE_s\frac{(\Gamma-1)}{P_{\text{sat}} + \Gamma p}\sin(\theta_{\text{in}}). \end{aligned} \tag{D.14}$$

Appendix E

Bandwidth of the Saturable Absorber Based NMZI

E.1 Response of a Saturable absorber to non CW signal

I substitute for $\delta\beta_\Omega(z)$ using (6.53) in (6.62) and obtain

$$\frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\mathrm{d}z} = \frac{\beta_s(z)\Delta\tilde{P}_s(z,\Omega)}{1+P(z)+i\Omega\tau}P(z) - \beta_s(z)\Delta\tilde{P}_s(z,\Omega). \tag{E.1}$$

I then factor for $\beta_s(z)$ and $\Delta\tilde{P}_s(z,\Omega)$, I get

$$\frac{1}{\Delta\tilde{P}_s(z,\Omega)}\frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\mathrm{d}z} = \beta_s(z)\left(\frac{P(z)}{1+P(z)+i\Omega\tau} - 1\right). \tag{E.2}$$

I take the integrals. I get

$$\int_0^L \frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\Delta P_s(z,\Omega)} = \int_0^L \beta_s(z)\left(\frac{P(z)}{1+P(z)+i\Omega\tau} - 1\right)\mathrm{d}z. \tag{E.3}$$

We introduce a change of variable in right hand side of the equation $X = P(z)$. From equation (6.59), we have

$$\frac{\mathrm{d}X}{\mathrm{d}z} = -\beta_s(z)P(z). \tag{E.4}$$

Therefore,

$$\mathrm{d}X = -\beta_s(z)P(z)\mathrm{d}z. \tag{E.5}$$

After substitution into the integral, I obtain

$$\int_0^L \frac{\mathrm{d}\Delta\tilde{P}_s(z,\Omega)}{\Delta\tilde{P}_s(z,\Omega)} = \int_{P(0)}^{P(L)} \left(\frac{1}{X} - \frac{1}{1+X+i\Omega\tau}\right)\mathrm{d}X. \tag{E.6}$$

The evaluation of the integrals leads to

$$\left[\ln\left(\Delta\tilde{P}_s(z,\Omega)\right)\right]_0^L = \left[\ln\left(\frac{X}{1+X+i\Omega\tau}\right)\right]_{P(0)}^{P(L)}, \tag{E.7}$$

with

$$P(z) = P(0)W\left(P(0)e^{P(0)-\beta_0 z}\right). \tag{E.8}$$

I can then solve for $\Delta\tilde{P}_s(L,\Omega)$ as follows

$$\ln\left(\frac{\Delta P_s(L,\Omega)}{\Delta P_s(0,\Omega)}\right) = \ln\left(\frac{P(L)(1+P(0)+i\Omega\tau)}{P(0)(1+P(L)+i\Omega\tau)}\right). \tag{E.9}$$

If I define the total steady state gain as $\Gamma_s = \frac{P(L)}{P(0)}$, I then have

$$\Delta P_s(L,\Omega) = \Delta P_s(0,\Omega)\Gamma_s\left(\frac{1+P(0)+i\Omega\tau}{1+P(L)+i\Omega\tau}\right). \tag{E.10}$$

I define the total loss of an SA as

$$\Gamma(\Omega) = e^{-\int_0^L \beta_s(z)\mathrm{d}z - \int_0^L \delta\beta_\Omega(z)\mathrm{d}z}. \tag{E.11}$$

I know that

$$\Gamma_s = e^{-\int_0^L \beta_s(z)\mathrm{d}z}. \tag{E.12}$$

I define

$$\Delta\beta(\Omega) = \int_0^L \delta\beta_\Omega(z)\mathrm{d}z. \tag{E.13}$$

Therefore,

$$\Gamma(\Omega) = \Gamma_s(1-\Delta\beta(\Omega)). \tag{E.14}$$

I then have

$$\Gamma_s(1-\Delta\beta(\Omega))(P(0)+\Delta P_s(0,\Omega)) = P(L)+\Delta P_s(L,\Omega). \tag{E.15}$$

I expand the equation, dropping the higher order terms. I get

$$\Gamma_s P(0) + \Gamma_s \Delta P_s(0,\Omega) - \Gamma_s \Delta\beta(\Omega) P(0) = P(L) + \Delta P_s(L,\Omega). \quad \text{(E.16)}$$

Since $\Gamma_s P(0) = P(L)$, I have

$$\Gamma_s \Delta P_s(0,\Omega) - \Gamma_s \Delta\beta(\Omega) P(0) = \Delta P_s(L,\Omega). \quad \text{(E.17)}$$

Substituting for $\Delta P_s(L,\Omega)$, I have

$$\Gamma_s \Delta P_s(0,\Omega) - \Gamma_s \Delta\beta(\Omega) P(0) = \Delta P_s(0,\Omega)\Gamma_s \left(\frac{1 + P(0) + i\Omega\tau}{1 + P(L) + i\Omega\tau}\right). \quad \text{(E.18)}$$

Solving for $\Delta\beta(\Omega)$, I get

$$\Delta\beta(\Omega) = \frac{\Delta P_s(0,\Omega)}{P(0)} \left(1 - \frac{1 + P(0) + i\Omega\tau}{1 + P(L) + i\Omega\tau}\right). \quad \text{(E.19)}$$

Therefore,

$$\Delta\beta(\Omega) = \frac{\Delta P_s(0,\Omega)}{P(0)} \left(\frac{P(L) - P(0)}{1 + P(L) + i\Omega\tau}\right), \quad \text{(E.20)}$$

which can be rewritten

$$\Delta\beta(\Omega) = \frac{-(1-\Gamma_s)}{1 + \Gamma_s P(0) + i\Omega\tau} \Delta P_s(0,\Omega). \quad \text{(E.21)}$$

This results is consistent with our previous results since for $\Omega = 0$ I obtain equation (D.14).

E.2 Overall System Output

At the output of each SA, I have

$$E_1(L,\Omega) = \frac{1}{\sqrt{2}} \left(E_1(0,\Omega) e^{-\frac{\beta_s + \Delta\beta(\Omega)}{2}(i\alpha_H + 1)}\right) \quad \text{(E.22)}$$

and

$$E_2(L,\Omega) = \frac{1}{\sqrt{2}} \left(E_2(0,\Omega) e^{-\frac{\beta_s - \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} \right), \tag{E.23}$$

where $\beta_s = -\ln(\Gamma_s)$,

$$E_1(0,\Omega) = \frac{1}{\sqrt{2}} \left(iA_p - \tilde{A}_s(\Omega) e^{i\theta_{\text{in}}} \right) \tag{E.24}$$

and

$$E_2(0,\Omega) = \frac{1}{\sqrt{2}} \left(-A_p + i\tilde{A}_s(\Omega) e^{i\theta_{\text{in}}} \right). \tag{E.25}$$

Therefore, the NMZI field output at one of its arms is:

$$\begin{aligned} E_{\text{out}} &= \frac{-1}{2} \Bigg(-i(-iA_p + \tilde{A}_s(\Omega) e^{i\theta_{\text{in}}}) e^{-\frac{\beta_s + \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} \\ &\qquad + (A_p - i\tilde{A}_s(\Omega) e^{i\theta_{\text{in}}}) e^{-\frac{\beta_s - \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} \Bigg) \\ &= \frac{-1}{2} \Bigg(A_p \left(e^{-\frac{\beta_s - \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} - e^{-\frac{\beta_s + \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} \right) \\ &\qquad - i\tilde{A}_s(\Omega) e^{i\theta_{\text{in}}} \left(e^{-\frac{\beta_s - \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} + e^{-\frac{\beta_s + \Delta\beta(\Omega)}{2}(i\alpha_H + 1)} \right) \Bigg). \end{aligned} \tag{E.26}$$

factoring $e^{-\frac{\beta_s}{2}(i\alpha_H+1)}$ and expanding $e^{-\frac{\Delta\beta(\Omega)}{2}(i\alpha_H+1)} \approx 1 - \frac{\Delta\beta(\Omega)}{2}(i\alpha_H + 1)$, I get

$$\begin{aligned} E_{\text{out}} &\approx \frac{-e^{-\frac{\beta_s}{2}(i\alpha_H+1)}}{2} \left(\Delta g_\Omega (i\alpha_H + 1) A_p - 2i(\tilde{A}_s(\Omega) e^{i\theta_{\text{in}}}) \right) \\ &= \frac{-e^{-\frac{\beta_s}{2}(i\alpha_H+1)}}{2} \left(-\frac{2(1-\Gamma_s) A_p^2 \sin(\theta_{\text{in}})}{1 + \Gamma_s A_p^2 + i\Omega\tau} (i\alpha_H + 1) - 2ie^{i\theta_{\text{in}}} \right) \tilde{A}_s(\Omega). \end{aligned} \tag{E.27}$$

E.3 Parametric Gain

I define

$$\tilde{E}_{\text{in}}(\Omega) \equiv \tilde{A}_s(\Omega) e^{i\theta_{\text{in}}}. \tag{E.28}$$

E_{out1} can then be rewritten

$$E_{\text{out1}} = e^{-\frac{\beta_s}{2}(i\alpha_H+1)} \left(\frac{(1-\Gamma_s) p \sin(\theta_{\text{in}})}{1 + \Gamma_s p + i\Omega\tau P_{\text{sat}}} (i\alpha_H + 1) + ie^{i\phi} \right) \tilde{A}_s(\Omega), \tag{E.29}$$

where P is the power of the input pump on each arm of the NMZI: $p = A_p^2$. I reintroduce P_{sat}. Substituting for $\tilde{E}_{\text{in}}$, I get

$$E_{\text{out1}} = e^{-\frac{\beta_s}{2}(i\alpha_H+1)} \left(\frac{(1-\Gamma_s)p\left(\tilde{E}_{\text{in}}(\Omega) - \tilde{E}^*_{\text{in}}(\Omega)\right)}{P_{\text{sat}} + \Gamma_s p + i\Omega\tau P_{\text{sat}}}(i\alpha_H + 1) + i\tilde{E}_{\text{in}}(\Omega) \right). \quad \text{(E.30)}$$

I define μ and ν such that

$$E_{\text{out1}} = \mu\tilde{E}_{\text{in}}(\Omega) + \nu\tilde{E}^*_{\text{in}}(\Omega). \quad \text{(E.31)}$$

Therefore, I have

$$\mu \equiv ie^{-\frac{\beta_s}{2}(i\alpha_H+1)} \left[1 - \frac{1}{2}(i\alpha_H + 1)\frac{(\Gamma - 1)}{\Gamma_s p + P_{\text{sat}} + i\Omega\tau P_{\text{sat}}}p\right] \quad \text{(E.32)}$$

and

$$\nu \equiv ie^{-\frac{\beta_s}{2}(i\alpha_H+1)} \left[\frac{1}{2}(i\alpha_H + 1)\frac{(\Gamma - 1)}{\Gamma p + P_{\text{sat}} + i\Omega\tau P_{\text{sat}}}p\right]. \quad \text{(E.33)}$$

I can be seen that for $\Omega = 0$, equation (E.32) and (E.33) are equivalent to equation (6.42) and (6.43) I can now find the parametric gain, which is

$$\begin{aligned} \Gamma(\Omega) &= \frac{|E_{\text{out}}|^2}{|\tilde{A}_s(\Omega)|^2} \\ &= |\mu(\Omega)e^{i\theta_{\text{in}}} + \nu(\Omega)|e^{-i\theta_{\text{in}}}|^2. \end{aligned} \quad \text{(E.34)}$$

Appendix F

Quantum Mechanical Model for Interaction of Light with Saturable Absorber

F.1 Solving for $\hat{\sigma}_+$

I multiply equation (7.21) by $e^{-(i\omega_{21}-\gamma)t}$

$$
\begin{aligned}
e^{-(i\omega_{21}-\gamma)t}\frac{\mathrm{d}\hat{\sigma}_+(t)}{\mathrm{d}t} = e^{-(i\omega_{21}-\gamma)t}\left(i\omega_{21}-\gamma\right)\hat{\sigma}_+(t) + \frac{i\kappa^*}{\hbar}\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}_1(t)e^{-(i\omega_{21}-\gamma)t} \\
+ \hat{\mathbf{F}}_+(t)e^{-(i\omega_{21}-\gamma)t},
\end{aligned}
\tag{F.1}
$$

which is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\sigma}_+(t)e^{-(i\omega_{21}-\gamma)t} = \frac{i\kappa^*}{\hbar}\hat{\mathbf{A}}^\dagger\hat{\mathbf{N}}_1(t)e^{-(i\omega_{21}-\gamma)t} + \hat{\mathbf{F}}_+(t)e^{-(i\omega_{21}-\gamma)t}. \tag{F.2}
$$

Therefore,

$$
\begin{aligned}
\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{(i\omega_{21}-\gamma)t} + \frac{i\kappa^*}{\hbar}e^{(i\omega_{21}-\gamma)t}\int_0^t \hat{\mathbf{A}}^\dagger(t')\hat{\mathbf{N}}_1(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t' \\
+ e^{(i\omega_{21}-\gamma)t}\int_0^t \hat{\mathbf{F}}_+(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t'.
\end{aligned}
\tag{F.3}
$$

I define

$$
\hat{\mathbf{F}}^\dagger_{\mathrm{ab}} \equiv \frac{i\kappa}{\hbar}e^{-i\omega_0 t}\int_0^t \hat{\mathbf{F}}_+(t')e^{-(i\omega_{21}-\gamma)(t-t')}\mathrm{d}t'. \tag{F.4}
$$

I use this definition in the expression of $\hat{\sigma}_+(t)$, I get

$$
\begin{aligned}
\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{(i\omega_{21}-\gamma)t} + \frac{i\kappa^*}{\hbar}e^{(i\omega_{21}-\gamma)t}\int_0^t \hat{\mathbf{A}}^\dagger(t')\hat{\mathbf{N}}_1(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t' \\
+ \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}.
\end{aligned}
\tag{F.5}
$$

I assume sufficient time lapse so that the initial conditions are forgotten by the system. I obtain

$$\hat{\sigma}_+(t) = \frac{i\kappa^*}{\hbar} e^{(i\omega_{21}-\gamma)t} \int_0^t \hat{\mathbf{A}}^\dagger(t')\hat{\mathbf{N}}_1(t')e^{-(i\omega_{21}-\gamma)t'}\mathrm{d}t' + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{F.6}$$

let $\hat{\mathbf{A}}(t) \equiv \hat{\mathbf{B}}(t)e^{-i\omega_0 t}$. $\hat{\mathbf{N}}_1(t)$ and the envelope $\hat{\mathbf{B}}(t)$ are pretty much constant relative to the oscillation $e^{(i\omega_{21}-\gamma)t}$. The equation becomes

$$\hat{\sigma}_+(t) = \frac{i\kappa^*}{\hbar} e^{(i\omega_{21}-\gamma)t}\hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{N}}_1(t) \int_0^t e^{-(i(\omega_{21}-\omega_0)-\gamma)t'}\mathrm{d}t' + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{F.7}$$

After evaluating the integral, I get

$$\hat{\sigma}_+(t) = \frac{\kappa^*}{i\hbar}\frac{\hat{\mathbf{B}}^\dagger(t)\hat{\mathbf{N}}_1(t)}{(i(\omega_{21}-\omega_0)-\gamma)}\left(e^{i\omega_0 t} - e^{(i\omega_{21}-\gamma)t}\right) + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{F.8}$$

Assuming that γ far more significant than the oscillation ω_{21}, I get

$$\hat{\sigma}_+(t) = \frac{\kappa^*}{\hbar}\frac{\hat{\mathbf{A}}^\dagger(t)\hat{\mathbf{N}}_1(t)}{((\omega_0-\omega_{21})-i\gamma)} + \frac{\hbar}{i\kappa}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)e^{i\omega_0 t}. \tag{F.9}$$

F.1.1 Properties of Noise Source $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$

$\hat{\mathbf{F}}_{\mathrm{ab}}(t)$ is chosen so that $\left[\hat{\mathbf{A}}(t), \hat{\mathbf{A}}^\dagger(t)\right] = 1$ for all values of t. Without $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$, I have

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\mho\hat{\mathbf{A}}(t)\hat{\mathbf{N}}_1(t) + i\omega_0\hat{\mathbf{A}}(t), \tag{F.10}$$

where $\mho \equiv \mho_r + i\mho_i$. I set $\hat{\mathbf{C}} \equiv \mho\hat{\mathbf{N}}_1(t) + i\omega_0$. This gives

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\hat{\mathbf{C}}\hat{\mathbf{A}}(t). \tag{F.11}$$

I solve this equation and get

$$\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(0)e^{-\hat{\mathbf{C}}t}. \tag{F.12}$$

Therefore,

$$\begin{aligned}\left[\hat{\mathbf{A}}(t),\hat{\mathbf{A}}^{\dagger}(t)\right] =& \left[\hat{\mathbf{A}}(0),\hat{\mathbf{A}}^{\dagger}(0)\right]e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t} \\ &= e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t},\end{aligned} \tag{F.13}$$

which is obviously wrong as it has to be one (1) regardless of t To fix this, I add the following the noise term back

$$\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} = -\hat{\mathbf{C}}\hat{\mathbf{A}}(t)+\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t} \tag{F.14}$$

and I require that that $\left[\hat{\mathbf{F}}_{\mathrm{ab}}(t),\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t')\right] = A\delta(t-t')$. The solution is

$$\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(0)e^{-\hat{\mathbf{C}}t}+\int_0^t e^{-\hat{\mathbf{C}}(t-t')-i\omega_0 t'}\hat{\mathbf{F}}_{\mathrm{ab}}(t')\mathrm{d}t'. \tag{F.15}$$

I compute the commutator

$$\begin{aligned}\left[\hat{\mathbf{A}}(t),\hat{\mathbf{A}}^{\dagger}(t)\right] =& \left[\hat{\mathbf{A}}(0),\hat{\mathbf{A}}^{\dagger}(0)\right]e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t} \\ &+A\int_0^t\int_0^t e^{-\hat{\mathbf{C}}(t-t')-i\omega_0 t'}e^{-\hat{\mathbf{C}}^{\dagger}(t-t'')+i\omega_0 t''}\delta(t'-t'')\mathrm{d}t'\mathrm{d}t''.\end{aligned} \tag{F.16}$$

After some algebra, I get

$$1 = e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t}+A\int_0^t e^{-(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger})(t-t')}\mathrm{d}t'. \tag{F.17}$$

I evaluate the integral

$$1 = e^{-\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)t}+A\left(1-e^{-(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger})t}\right)\left(\hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger}\right)^{-1}. \tag{F.18}$$

Therefore,

$$\begin{aligned}A &= \hat{\mathbf{C}}+\hat{\mathbf{C}}^{\dagger} \\ &= 2\Re(\mho)\hat{\mathbf{N}}_1(t) \\ &= 2\mho_r\hat{\mathbf{N}}_1(t).\end{aligned} \tag{F.19}$$

Therefore,

$$\left[\hat{\mathbf{F}}_{\text{ab}}(t), \hat{\mathbf{F}}^{\dagger}_{\text{ab}}(t')\right] = 2\mho_r \hat{\mathbf{N}}_1(t)\delta(t-t'). \tag{F.20}$$

Remembering that I wrote $\hat{\mathbf{N}}_1(t)$ because $\hat{\mathbf{N}}_2(t) \approx 0$. In reality it was $\hat{\mathbf{N}}_2(t) - \hat{\mathbf{N}}_1(t)$. Therefore,

$$\left[\hat{\mathbf{F}}_{\text{ab}}(t), \hat{\mathbf{F}}^{\dagger}_{\text{ab}}(t')\right] = -2\mho_r \left(\hat{\mathbf{N}}_2(t) - \hat{\mathbf{N}}_1(t)\right)\delta(t-t'). \tag{F.21}$$

Therefore,

$$\begin{aligned} \left\langle \hat{\mathbf{F}}^{\dagger}_{\text{ab}}(t)\hat{\mathbf{F}}_{\text{ab}}(t')\right\rangle &= 2\mho_r \left\langle \hat{\mathbf{N}}_2(t)\right\rangle \delta(t-t') \\ &= 0 \end{aligned} \tag{F.22}$$

and

$$\left\langle \hat{\mathbf{F}}_{\text{ab}}(t)\hat{\mathbf{F}}^{\dagger}_{\text{ab}}(t')\right\rangle = 2\mho_r \left\langle \hat{\mathbf{N}}_1(t)\right\rangle \delta(t-t'). \tag{F.23}$$

$\hat{\mathbf{F}}_{\text{ab}}(t)$ is a lowering operator.

F.1.2 Photon Number Equation

By definition $\hat{\mathbf{n}}(t) = \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{A}}(t)$. Taking the derivative of this equation, I get

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = \hat{\mathbf{A}}^{\dagger}(t)\frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\hat{\mathbf{A}}(t)}{\mathrm{d}t}\hat{\mathbf{A}}(t). \tag{F.24}$$

After substituting equation (F.14), I get

$$\frac{\mathrm{d}\hat{\mathbf{n}}(t)}{\mathrm{d}t} = -2\mho_r \hat{\mathbf{N}}_1(t)\hat{\mathbf{n}}(t) + \hat{\mathbf{A}}^{\dagger}(t)\hat{\mathbf{F}}_{\text{ab}}(t)e^{-i\omega_0 t} + \hat{\mathbf{F}}^{\dagger}_{\text{ab}}\hat{\mathbf{A}}(t)e^{i\omega_0 t}. \tag{F.25}$$

The term $-2\mho_r\hat{\mathbf{N}}_1(t)\hat{\mathbf{n}}(t)$ represent absorption. Using equation (F.15), I can compute $\left\langle \hat{\mathbf{A}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t}\right\rangle$ and $\left\langle \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)\hat{\mathbf{A}}(t)e^{i\omega_0 t}\right\rangle$, which contains absorbed noise. Taking the Hermitian conjugate of (F.15), I get

$$\hat{\mathbf{A}}^\dagger(t) = \hat{\mathbf{A}}^\dagger(0)e^{-\hat{\mathbf{C}}^\dagger t} + \int_0^t e^{-\hat{\mathbf{C}}^\dagger(t-t')+i\omega_0 t'}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\mathrm{d}t'. \tag{F.26}$$

I Multiply the equation by $\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t}$

$$\hat{\mathbf{A}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t} = \hat{\mathbf{A}}^\dagger(0)\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-(\hat{\mathbf{C}}^\dagger+i\omega_0)t} + \int_0^t e^{-(\hat{\mathbf{C}}^\dagger+i\omega_0)(t-t')}\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\hat{\mathbf{F}}_{\mathrm{ab}}(t)\mathrm{d}t'. \tag{F.27}$$

Since

$$\left\langle \hat{\mathbf{A}}^\dagger(0)\hat{\mathbf{F}}_{\mathrm{ab}}(t)\right\rangle = 0 \tag{F.28}$$

and

$$\left\langle \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\hat{\mathbf{F}}_{\mathrm{ab}}(t)\right\rangle = 0, \tag{F.29}$$

I have

$$\left\langle \hat{\mathbf{A}}^\dagger(t)\hat{\mathbf{F}}_{\mathrm{ab}}(t)e^{-i\omega_0 t}\right\rangle = 0. \tag{F.30}$$

Similarly,

$$\left\langle \hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t)\hat{\mathbf{A}}(t)e^{-i\omega_0 t}\right\rangle = 0. \tag{F.31}$$

F.2 Noise From a Single Atom

I added $\hat{\mathbf{F}}_N(t)$ to equation (7.17) empirically. I remove it from (7.65). I consider separately the population decay. This is done by turning off the electric field, setting

$\hat{\mathbf{A}}(t) = 0$ in equation (7.65), and letting the system relax. For one single atom, I have:

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = \frac{1 - \hat{\mathbf{N}}_{1j}(t)}{\tau}. \tag{F.32}$$

I also know that

$$\hat{\mathbf{N}}_{3j}(t) = 1 - \hat{\mathbf{N}}_{1j}(t) - \hat{\mathbf{N}}_{2j}(t), \tag{F.33}$$

because the number of carrier is conserved. Since $\hat{\mathbf{N}}_{2j}(t) \approx 0$, I have

$$\hat{\mathbf{N}}_{3j}(t) = 1 - \hat{\mathbf{N}}_{1j}(t). \tag{F.34}$$

Therefore our original equation, can be rewritten

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = \frac{\hat{\mathbf{N}}_{3j}(t)}{\tau}. \tag{F.35}$$

Also,

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = -\frac{\mathrm{d}\hat{\mathbf{N}}_{3j}(t)}{\mathrm{d}t}. \tag{F.36}$$

Therefore, the equation can be rewritten

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{3j}(t)}{\mathrm{d}t} = -\frac{\hat{\mathbf{N}}_{3j}(t)}{\tau}. \tag{F.37}$$

For a small Δt, I have

$$\begin{aligned} \hat{\mathbf{N}}_{3j}(t + \Delta t) &= \hat{\mathbf{N}}_{3j}(t) - \frac{\hat{\mathbf{N}}_{3j}(t)}{\tau}\Delta t \\ &= \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{\Delta t}{\tau}\right). \end{aligned} \tag{F.38}$$

Since $\hat{\mathbf{N}}_{3j}(t)\hat{\mathbf{N}}_{3j}(t) = \hat{\mathbf{N}}_{3j}(t)$ for any t, I have

$$\begin{aligned} \left(\hat{\mathbf{N}}_{3j}(t + \Delta t)\right)^2 &= \left(\hat{\mathbf{N}}_{3j}(t)\right)^2 \left(1 - \frac{\Delta t}{\tau}\right)^2 \\ &\approx \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{2\Delta t}{\tau}\right). \end{aligned} \tag{F.39}$$

Therefore,

$$\hat{\mathbf{N}}_{3j}(t+\Delta t) = \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{2\Delta t}{\tau}\right). \tag{F.40}$$

Since I started with

$$\hat{\mathbf{N}}_{3j}(t+\Delta t) = \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{\Delta t}{\tau}\right), \tag{F.41}$$

something is wrong. Let's add a noise term $\hat{\mathbf{F}}_{Nj}(t)$ (phonon)

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = \frac{1-\hat{\mathbf{N}}_{1j}(t)}{\tau} + \hat{\mathbf{F}}_{Nj}(t), \tag{F.42}$$

which can be rewritten

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{3j}(t)}{\mathrm{d}t} = -\frac{\hat{\mathbf{N}}_{3j}(t)}{\tau} + \hat{\mathbf{F}}_{Nj}(t). \tag{F.43}$$

I require that

$$\left\langle \hat{\mathbf{F}}_{Nj}(t)\hat{\mathbf{F}}_{Nj}(t')\right\rangle = A\delta(t-t'). \tag{F.44}$$

Therefore,

$$\hat{\mathbf{N}}_{3j}(t+\Delta t) = \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{\Delta t}{\tau}\right) + \int_t^{t+\Delta t}\hat{\mathbf{F}}_{Nj}(t')\mathrm{d}t'. \tag{F.45}$$

Multiplying this equation by itself, I get

$$\begin{aligned}\hat{\mathbf{N}}_{3j}(t+\Delta t) = \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{2\Delta t}{\tau}\right) + \hat{\mathbf{N}}_{3j}(t)\left(1 - \frac{2\Delta t}{\tau}\right)\int_t^{t+\Delta t}\hat{\mathbf{F}}_{Nj}(t')\mathrm{d}t' \\ + \int_t^{t+\Delta t}\int_t^{t+\Delta t}\hat{\mathbf{F}}_{Nj}(t')\hat{\mathbf{F}}_{Nj}(t'')\mathrm{d}t'\mathrm{d}t''.\end{aligned} \tag{F.46}$$

When I take the average of the equation

$$\begin{aligned}\left\langle \hat{\mathbf{N}}_{3j}(t+\Delta t)\right\rangle = \left\langle \hat{\mathbf{N}}_{3j}(t)\right\rangle\left(1 - \frac{2\Delta t}{\tau}\right) \\ + \int_t^{t+\Delta t}\int_t^{t+\Delta t}\left\langle \hat{\mathbf{F}}_{Nj}(t')\hat{\mathbf{F}}_{Nj}(t'')\right\rangle \mathrm{d}t'\mathrm{d}t'',\end{aligned} \tag{F.47}$$

I get

$$\left\langle \hat{\mathbf{N}}_{3j}(t+\Delta t) \right\rangle = \left\langle \hat{\mathbf{N}}_{3j}(t) \right\rangle \left(1 - \frac{2\Delta t}{\tau}\right) + A\Delta t. \tag{F.48}$$

I choose

$$A = \frac{\left\langle \hat{\mathbf{N}}_{3j}(t) \right\rangle}{\tau}. \tag{F.49}$$

This implies that

$$\left\langle \hat{\mathbf{N}}_{3j}(t+\Delta t) \right\rangle = \left\langle \hat{\mathbf{N}}_{3j}(t) \right\rangle \left(1 - \frac{\Delta t}{\tau}\right), \tag{F.50}$$

which works. Therefore

$$\begin{aligned} \left\langle \hat{\mathbf{F}}_{Nj}(t) \hat{\mathbf{F}}_{Nj}(t') \right\rangle &= \frac{\left\langle \hat{\mathbf{N}}_{3j}(t) \right\rangle}{\tau} \delta(t-t') \\ &= \frac{1 - \left\langle \hat{\mathbf{N}}_{1j}(t) \right\rangle}{\tau} \delta(t-t'). \end{aligned} \tag{F.51}$$

F.3 Noise From a Collection of Atom

I have for a single atom

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = \frac{1 - \hat{\mathbf{N}}_{1j}(t)}{\tau} + \hat{\mathbf{F}}_{Nj}(t). \tag{F.52}$$

Therefore, for a collection of atoms, I have

$$\sum_{j=1}^{N} \frac{\mathrm{d}\hat{\mathbf{N}}_{1j}(t)}{\mathrm{d}t} = \sum_{j=1}^{N} \frac{1 - \hat{\mathbf{N}}_{1j}(t)}{\tau} + \hat{\mathbf{F}}_{Nj}(t), \tag{F.53}$$

which gives

$$\frac{\mathrm{d}\hat{\mathbf{N}}_{1}(t)}{\mathrm{d}t} = \frac{N - \hat{\mathbf{N}}_{1}(t)}{\tau} + \sum_{j=1}^{N} \hat{\mathbf{F}}_{Nj}(t). \tag{F.54}$$

I define

$$\hat{\mathbf{F}}_N(t) \equiv \sum_{j=1}^{N} \hat{\mathbf{F}}_{Nj}(t). \tag{F.55}$$

Then,

$$\left\langle \hat{\mathbf{F}}_N(t)\hat{\mathbf{F}}_N(t') \right\rangle = \sum_{j=1}^{N}\sum_{k=1}^{N} \left\langle \hat{\mathbf{F}}_{Nj}(t)\hat{\mathbf{F}}_{Nk}(t') \right\rangle. \tag{F.56}$$

Since $\hat{\mathbf{F}}_{Nj}(t)$ and $\hat{\mathbf{F}}_{Nk}(t)$ are uncorrelated for $j \neq k$, all the cross terms of the double sum are zero. Therefore,

$$\begin{aligned} \left\langle \hat{\mathbf{F}}_N(t)\hat{\mathbf{F}}_N(t') \right\rangle &= \sum_{j=1}^{N} \left\langle \hat{\mathbf{F}}_{Nj}(t)\hat{\mathbf{F}}_{Nj}(t') \right\rangle \\ &= \sum_{j=1}^{N} \frac{1 - \left\langle \hat{\mathbf{N}}_{1j}(t) \right\rangle}{\tau} \delta(t-t') \\ &= \frac{N - \left\langle \hat{\mathbf{N}}_{1}(t) \right\rangle}{\tau} \delta(t-t'). \end{aligned} \tag{F.57}$$

Appendix G

Solving Differential Equation of SA

G.1 Solution for the Fluctuating Terms

Substituting equation (8.26) in equation (7.77), I get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[B(t)\left(1+\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\right)\Big] = &-\frac{B(t)}{2T}\left(\hat{\beta}(t)(i\alpha_H+1)\right)\left(1+\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\right)\\ &+\hat{\mathbf{F}}_{\mathrm{ab}}(t).\end{aligned} \tag{G.1}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}B(t)}{\mathrm{d}t}\big(1+\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\big)+B(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\big) = &-\frac{B(t)}{2T}\left(\hat{\beta}(t)(i\alpha_H+1)\right)\\ &\times\left(1+\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\right)\\ &+\hat{\mathbf{F}}_{\mathrm{ab}}(t).\end{aligned} \tag{G.2}$$

It is easy to show that by substituting $\hat{\mathbf{B}}(t) = B(t)+\delta\hat{\mathbf{B}}(t)(t)$ and $\hat{\beta}(t) = \beta(t)+\delta\hat{\beta}(t)$ in equation (7.77) and separating the average terms from the fluctuating terms, I get for the average terms

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = -\frac{1}{2T}\beta(t)\left(1+i\alpha_H\right)B(t). \tag{G.3}$$

Substituting equation (G.3) in equation (G.2), I get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\big) = &-\frac{\delta\hat{\beta}(t)}{2T}\big((i\alpha_H+1)\big)\big(1+\hat{\mathbf{x}}(t)+i\hat{\mathbf{y}}(t)\big)+\frac{\hat{\mathbf{F}}_{\mathrm{ab}}(t)}{B(t)}\\ \approx &-\frac{\delta\hat{\beta}(t)}{2T}\big((i\alpha_H+1)\big)+\frac{\hat{\mathbf{F}}_{\mathrm{ab}}(t)}{B(t)}.\end{aligned} \tag{G.4}$$

where the last step is a first order approximation. Taking the Hermitian conjugate of this equation, I get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\mathbf{x}}(t) - i\hat{\mathbf{y}}(t)\big) = \frac{\delta\hat{\beta}(t)}{2T}\big((i\alpha_H - 1)\big) + \frac{\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}(t)}{B^{*}(t)}. \tag{G.5}$$

Summing the equation (G.4) and (G.5) and dividing the result by 2, I get

$$\frac{\partial\hat{\mathbf{x}}(t)}{\mathrm{d}t} = -\frac{1}{2T}\delta\hat{\beta}(t) + \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} + B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2} \tag{G.6}$$

Subtracting the equation (G.4) and (G.5) and dividing the result by $2i$, I get

$$\frac{\mathrm{d}\hat{\mathbf{y}}(t)}{\mathrm{d}t} = \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} - B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2i} - \frac{\alpha_H}{2T}\delta\hat{\beta}(t), \tag{G.7}$$

Check:

I assume that $n_{\mathrm{sat}} \to \infty$, which is the case of unsaturable loss. In that case equation (8.19) gives

$$\tau\frac{\mathrm{d}\delta\hat{\beta}(t)}{\mathrm{d}t} = -\delta\hat{\beta}(t) + \tau\hat{\mathbf{F}}_L(t). \tag{G.8}$$

I saw in equation (8.24) that under this condition the driving term $\hat{\mathbf{F}}_L(t)$ contains no power and that $\hat{\mathbf{F}}_L(t) = 0$. Therefore, $\delta\hat{\beta}(t) = 0$. Therefore,

$$\frac{\partial\hat{\mathbf{x}}(t)}{\mathrm{d}t} = \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} + B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2}. \tag{G.9}$$

Therefore,

$$\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}(0) + \int_0^T \frac{B^{-1}(t)\hat{\mathbf{F}}_{\mathrm{ab}} + B^{*-1}(t)\hat{\mathbf{F}}^{\dagger}_{\mathrm{ab}}}{2}\mathrm{d}t. \tag{G.10}$$

$$\langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle = \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle + \frac{1}{4T}\int_0^T \frac{\beta(t)}{n(t)}\mathrm{d}t \tag{G.11}$$

Here, $n(t) = n(0)e^{-\frac{1}{T}\beta_0 t}$ and $\beta(t) = \beta_0$

$$\begin{aligned}
\langle\hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle &= \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle + \frac{\beta_0}{4Tn(0)}\int_0^T e^{\frac{1}{T}\beta_0 t}\mathrm{d}t \\
&= \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle + \frac{\left[e^{\frac{1}{T}\beta_0 t}\right]_0^T}{4n(0)} \\
&= \langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle + \frac{e^{\beta_0}-1}{4n(0)}
\end{aligned} \tag{G.12}$$

Multiplying the whole equation by $n(T) = n(0)e^{-\beta_0}$, I get

$$\left\langle \delta\hat{\mathbf{X}}^2(T)\right\rangle = e^{-\beta_0}\left\langle \delta\hat{\mathbf{X}}^2(0)\right\rangle + \frac{1-e^{-\beta_0}}{4}, \tag{G.13}$$

which is what is expected.

G.1.1 Finding an Expression for $\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t')\right\rangle$

At the scale at which I look at this problem, $\hat{\mathbf{F}}_\beta(t)$ is roughly a delta correlated noise. In this section, I will compute the integral of $\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t')\right\rangle$ and rewrite it as a delta correlated function . I can rewrite equation (8.20) as such

$$\left\langle \hat{\mathbf{F}}_L(t)\hat{\mathbf{F}}_L(t')\right\rangle = K\delta(t-t'), \tag{G.14}$$

where

$$K \equiv \frac{T}{n_{\mathrm{sat}}}\left[\frac{\beta_0 + \left(\frac{n(t)}{n_{\mathrm{sat}}}-1\right)\left\langle\hat{\beta}(t)\right\rangle}{\tau^2}\right]. \tag{G.15}$$

Also, equation (8.40) can be rewritten

$$\hat{\mathbf{F}}_\beta(t) = \int_0^t G(\tau_1)\hat{\mathbf{F}}_L(t-\tau_1)\mathrm{d}\tau_1, \tag{G.16}$$

where

$$G(\tau_1) = \frac{1}{2T}e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}. \tag{G.17}$$

Therefore,

$$\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t-\tau_1)\right\rangle = \int_0^t \int_0^{t-\tau_1} G(\tau_2)G(\tau_3)\left\langle \hat{\mathbf{F}}_L(t-\tau_2)\hat{\mathbf{F}}_L(t-\tau_1-\tau_3)\right\rangle \mathrm{d}\tau_2\mathrm{d}\tau_3$$
$$= K\int_0^t \int_0^{t-\tau_1} G(\tau_2)G(\tau_3)\delta(t-\tau_2-t+\tau_1+\tau_3)\mathrm{d}\tau_2\mathrm{d}\tau_3$$
$$= K\int_0^t G(\tau_3+\tau_1)G(\tau_3)\mathrm{d}\tau_3, \tag{G.18}$$

since $t'-t=-\tau_1$. Substituting for $G(\cdot)$, I get

$$\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t-\tau_1)\right\rangle = \frac{K}{4T^2}\int_0^t e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{2\tau_3+\tau_1}{\tau}}\mathrm{d}\tau_3$$
$$= -\frac{\tau K}{8T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}\left[e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{2t+\tau_1}{\tau}} - e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}\right]$$
$$= \frac{\tau K}{8T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1} e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}. \tag{G.19}$$

I look for the total energy of $\hat{\mathbf{F}}_\beta(t)$ as follows

$$\int_{-\infty}^{\infty}\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t-\tau_1)\right\rangle \mathrm{d}\tau_1 = 2\int_0^{\infty}\left\{\frac{\tau K}{8T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1} e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}\right\}\mathrm{d}\tau_1$$
$$\approx \frac{\tau^2 K}{4T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-2}. \tag{G.20}$$

Therefore,

$$\left\langle \hat{\mathbf{F}}_\beta(t)\hat{\mathbf{F}}_\beta(t')\right\rangle = \frac{\tau^2 K}{4T^2}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-2}\delta(t-t') \tag{G.21}$$

G.1.2 Finding an Expression for $\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_\beta(t')\right\rangle$

Equation (8.23) can be rewritten

$$\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_L(t')\right\rangle = K_1\delta(t-t'), \tag{G.22}$$

where

$$K_1 \equiv -\frac{B(t)}{n_{\mathrm{sat}}\tau}\beta(t). \tag{G.23}$$

I take equation (G.16) and multiply it by $\hat{\mathbf{F}}_{\mathrm{ab}}(t)$ and take the average, I get

$$\begin{aligned}
\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t+\tau_1)\right\rangle &= \int_0^{t+\tau_1} G(\tau_2)\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{L}(t+\tau_1-\tau_2)\right\rangle \mathrm{d}\tau_2 \\
&= K_1\int_0^{t-\tau_1} G(\tau_2)\delta(t-t-\tau_1+\tau_2)\mathrm{d}\tau_2 \\
&= K_1 G(\tau_1) \\
&= \frac{K_1}{2T}e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}.
\end{aligned} \tag{G.24}$$

I take the integral of the expression, I get

$$\begin{aligned}
\int_{-\infty}^{\infty}\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t+\tau_1)\right\rangle \mathrm{d}\tau_1 &= 2\int_0^{\infty}\frac{K_1}{2T}e^{-\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)\frac{\tau_1}{\tau}}\mathrm{d}\tau_1 \\
&\approx \frac{\tau K_1}{T}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}.
\end{aligned} \tag{G.25}$$

Therefore,

$$\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t)\hat{\mathbf{F}}_{\beta}(t')\right\rangle \approx \frac{\tau K_1}{T}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}\delta(t-t'). \tag{G.26}$$

Similarly,

$$\left\langle \hat{\mathbf{F}}_{\beta}(t')\hat{\mathbf{F}}_{\mathrm{ab}}(t)\right\rangle \approx \frac{\tau K_1}{T}\left(1+\frac{n(t)}{n_{\mathrm{sat}}}\right)^{-1}\delta(t-t'). \tag{G.27}$$

G.2 Calculating $\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle$

I take equation (8.43) and use equation (8.39) to substitute the expression of $\hat{\mathbf{N}}_x(t)$, I get

$$\begin{aligned}\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle = \langle \hat{\mathbf{x}}(0)\hat{\mathbf{x}}(0)\rangle\, H_2(0)^2 \\ + \int_0^T\int_0^T H_2(t')H_2(t'')\left\langle \hat{\mathbf{F}}_\beta(t')\hat{\mathbf{F}}_\beta(t'')\right\rangle \mathrm{d}t'\mathrm{d}t'' \\ + \int_0^T\int_0^T H_2(t')H_2(t'')\left(\frac{B^{-1}(t)}{2}\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t')\hat{\mathbf{F}}_\beta(t'')\right\rangle\right. \\ \left. + \frac{B^{*-1}(t)}{2}\left\langle \hat{\mathbf{F}}_\beta(t'')\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t')\right\rangle\right)\mathrm{d}t'\mathrm{d}t'' \\ + \int_0^T\int_0^T H_2(t')H_2(t'')\frac{|B(t')|^{-2}}{4}\left\langle \hat{\mathbf{F}}_{\mathrm{ab}}(t')\hat{\mathbf{F}}^\dagger_{\mathrm{ab}}(t'')\right\rangle \mathrm{d}t'\mathrm{d}t''.\end{aligned} \tag{G.28}$$

Using equations (G.26),(G.27), (G.21) and (7.72) in the above expression, I obtain

$$\begin{aligned}\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle =& \langle \hat{\mathbf{x}}^2(0)\rangle H^2(0) \\ &+ \frac{1}{4T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n(t')} + \frac{\beta_0 + \left(\frac{n(t')}{n_{\mathrm{sat}}} - 1\right)\beta(t)}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}}+1\right)^2}\right)\mathrm{d}t' \\ &+ \frac{1}{T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}}+1\right)}\right)\mathrm{d}t'.\end{aligned} \tag{G.29}$$

check:

I set $n_{\mathrm{sat}} \to \infty$. Then $H(t) \to 1$ and

$$\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle = \langle \hat{\mathbf{x}}^2(0)\rangle + \frac{1}{4T}\int_0^T \left[\frac{\beta(t')}{n(t')}\right]\mathrm{d}t'. \tag{G.30}$$

This gives us

$$\left\langle \delta\hat{\mathbf{R}}(T)\delta\hat{\mathbf{R}}(T)\right\rangle = e^{-\beta_0}\left\langle \delta\hat{\mathbf{R}}(0)\delta\hat{\mathbf{R}}(0)\right\rangle + \frac{1-e^{-\beta_0}}{4} \tag{G.31}$$

as before.

G.2.1 Amplified Initial Noise

The incoming signal's noise is amplified. At the output, it becomes the first expression of equation (G.29), which is

$$\text{Init}_x = H_2^2(0)\langle \hat{\mathbf{x}}^2(0)\rangle. \tag{G.32}$$

Using equation (8.42), I obtain

$$\text{Init}_x = \frac{1}{4}\left(\frac{n(0)+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2 \frac{1}{n(0)}. \tag{G.33}$$

G.2.2 Noise due to Absorption

I compute the first integral of equation (G.29), which is noise due to absorption. I get

$$\begin{aligned}
\text{Noise}_{abs_x} &= \frac{1}{4T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n(t')}\right)\mathrm{d}t' \\
&= \frac{1}{4T}\int_0^T \left(\frac{n(t')+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2\left(\frac{\beta(t')}{n(t')}\right)\mathrm{d}t' \\
&= -\frac{1}{4}\int_{n(0)}^{n(T)}\left(\frac{n+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2\left(\frac{1}{n^2}\right)\mathrm{d}n \\
&= -\frac{1}{4}\frac{\left[2n_{\text{sat}}\ln(n)+n-\frac{n_{\text{sat}}^2}{n}\right]_{n(0)}^{n(T)}}{(n(T)+n_{\text{sat}})^2} \\
&= \frac{1}{4}\frac{2n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right)+\frac{n_{\text{sat}}^2}{n(T)}-\frac{n_{\text{sat}}^2}{n(0)}+n(0)-n(T)}{(n(T)+n_{\text{sat}})^2},
\end{aligned} \tag{G.34}$$

As a check, I set, as $n_{\text{sat}} \to \infty$ (nonsaturable loss). The above expression converges to

$$\begin{aligned}
\text{Noise}_{abs_x} &= \frac{1}{4}\left[\frac{1}{n(T)}-\frac{1}{n(0)}\right] \\
&= \frac{1}{4}\frac{e^{\beta_0}-1}{n(0)},
\end{aligned} \tag{G.35}$$

which is expected.

G.2.3 Noise due to Relaxation

I compute the second integral of equation (G.29), which is noise due to relaxation. I do it as follows

$$\mathrm{Noise}_{rel_x} = \frac{1}{4T}\int_0^T H^2(t')\left(\frac{\beta_0 + \left(\frac{n(t')}{n_{\mathrm{sat}}} - 1\right)\beta(t')}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}} + 1\right)^2}\right)\mathrm{d}t'. \tag{G.36}$$

Since

$$\beta(t) = \frac{\beta_0}{\frac{n(t)}{n_{\mathrm{sat}}} + 1}, \tag{G.37}$$

I can make a change of variable and evaluate the integral as follows

$$\begin{aligned}
\mathrm{Noise}_{rel_x} &= \frac{1}{4T}\int_0^T \beta(t')H^2(t')\left(\frac{1}{n(t') + n_{\mathrm{sat}}} + \frac{\left(\frac{n(t')}{n_{\mathrm{sat}}} - 1\right)}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}} + 1\right)^2}\right)\mathrm{d}t' \\
&= -\frac{1}{4}\int_{n(0)}^{n(T)}\frac{1}{n}\left(\frac{n + n_{\mathrm{sat}}}{n(T) + n_{\mathrm{sat}}}\right)^2\left(\frac{1}{n + n_{\mathrm{sat}}} + \frac{\left(\frac{n}{n_{\mathrm{sat}}} - 1\right)}{n_{\mathrm{sat}}\left(\frac{n}{n_{\mathrm{sat}}} + 1\right)^2}\right)\mathrm{d}n \\
&= \frac{1}{2}\frac{n(0) - n(T)}{\left(n(T) + n_{\mathrm{sat}}\right)^2}.
\end{aligned} \tag{G.38}$$

G.2.4 Beat Noise

I compute the third integral of equation (G.29), which is the beat noise between the relaxation noise and the absorption noise. I evaluate it as follows

$$\begin{aligned}
\mathrm{Noise}_{beat_x} &= \frac{1}{T}\int_0^T H^2(t')\left(\frac{\beta(t')}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}} + 1\right)}\right)\mathrm{d}t' \\
&= -\int_{n(0)}^{n(T)}\frac{1}{n}\left(\frac{n + n_{\mathrm{sat}}}{n(T) + n_{\mathrm{sat}}}\right)^2\left(\frac{1}{n_{\mathrm{sat}}\left(\frac{n}{n_{\mathrm{sat}}} + 1\right)}\right)\mathrm{d}n \\
&= \frac{n(0) - n(T) + n_{\mathrm{sat}}\ln\left(\frac{n(0)}{n(T)}\right)}{\left(n(T) + n_{\mathrm{sat}}\right)^2}.
\end{aligned} \tag{G.39}$$

G.2.5 Total Amplitude Noise

I get an expression for the total amplitude noise from the saturable absorber, which is

$$
\begin{aligned}
\langle \hat{\mathbf{x}}(T)\hat{\mathbf{x}}(T)\rangle &= \text{Init}_x + \text{Noise}_{abs_x} + \text{Noise}_{rel_x} + \text{Noise}_{beat_x} \\
&= \frac{1}{4}\left(\frac{n(0)+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2 \frac{1}{n(0)} + \frac{1}{4}\frac{2n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\text{sat}}^2}{n(T)} - \frac{n_{\text{sat}}^2}{n(0)} + n(0) - n(T)}{\left(n(T)+n_{\text{sat}}\right)^2} \\
&\quad + \frac{1}{4}\frac{n(0)-n(T)}{\left(n(T)+n_{\text{sat}}\right)^2} + \frac{n(0)-n(T)+n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right)}{\left(n(T)+n_{\text{sat}}\right)^2} \\
&= \frac{1}{4}\left(\frac{n(0)+n_{\text{sat}}}{n(T)+n_{\text{sat}}}\right)^2 \frac{1}{n(0)} + \frac{1}{4}\frac{6n_{\text{sat}}\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\text{sat}}^2}{n(T)} - \frac{n_{\text{sat}}^2}{n(0)}}{\left(n(T)+n_{\text{sat}}\right)^2} \\
&\quad + \frac{3}{2}\frac{(n(0)-n(T))}{\left(n(T)+n_{\text{sat}}\right)^2}.
\end{aligned}
\tag{G.40}
$$

G.3 Calculating $\langle \hat{\mathbf{y}}(T)\hat{\mathbf{y}}(T)\rangle$

To get the phase noise, I use equation (G.7)

$$\frac{\partial \hat{\mathbf{y}}(t)}{\partial t} = \frac{B^{-1}(t)\hat{\mathbf{F}}_{\text{ab}} - B^{*-1}(t)\hat{\mathbf{F}}_{\text{ab}}^{\dagger}}{2i} - \frac{\alpha_H}{2T}\delta\hat{\beta}(t) \tag{G.41}$$

and equation (G.6)

$$\frac{\partial \hat{\mathbf{x}}(t)}{\partial t} = -\frac{1}{2T}\delta\hat{\beta}(t) + \frac{B^{-1}(t)\hat{\mathbf{F}}_{\text{ab}} + B^{*-1}(t)\hat{\mathbf{F}}_{\text{ab}}^{\dagger}}{2}. \tag{G.42}$$

I define

$$\hat{\mathbf{N}}_{pi}(t) \equiv \frac{B^{-1}(t)\hat{\mathbf{F}}_{\text{ab}} - B^{*-1}(t)\hat{\mathbf{F}}_{\text{ab}}^{\dagger}}{2i} \tag{G.43}$$

and

$$\hat{\mathbf{N}}_{pr}(t) \equiv \frac{B^{-1}(t)\hat{\mathbf{F}}_{\text{ab}} + B^{*-1}(t)\hat{\mathbf{F}}_{\text{ab}}^{\dagger}}{2}. \tag{G.44}$$

Therefore, equation (G.6) can be written

$$-\frac{1}{2T}\delta\hat{\beta}(t) = \frac{\partial\hat{\mathbf{x}}(t)}{\partial t} - \hat{\mathbf{N}}_{pr}(t) \tag{G.45}$$

and equation (G.7) can be written

$$\frac{\partial\hat{\mathbf{y}}(t)}{\partial t} = \hat{\mathbf{N}}_{pi}(t) + \alpha_H\left(\frac{\partial\hat{\mathbf{x}}(t)}{\partial t} - \hat{\mathbf{N}}_{pr}(t)\right). \tag{G.46}$$

Algebraic manipulation of equation (G.7) and equation (G.6) gives us

$$\frac{\partial\hat{\mathbf{y}}(t)}{\partial t} - \alpha_H\frac{\partial\hat{\mathbf{x}}(t)}{\partial t} = \hat{\mathbf{N}}_{pi}(t) - \alpha_H\hat{\mathbf{N}}_{pr}(t). \tag{G.47}$$

From this, I can write an expression of the phase noise, which is

$$\hat{\mathbf{y}}(t) - \alpha_H\hat{\mathbf{x}}(t) = \hat{\mathbf{y}}(0) - \alpha_H\hat{\mathbf{x}}(0) + \int_0^t \left(\hat{\mathbf{N}}_{pi}(t) - \alpha_H\hat{\mathbf{N}}_{pr}(t)\right)\mathrm{d}t'. \tag{G.48}$$

The power of the phase noise is

$$\begin{aligned}
\left\langle\hat{\mathbf{y}}(t)^2\right\rangle = &\left\langle\hat{\mathbf{y}}^2(0)\right\rangle + \alpha_H^2\left\langle\hat{\mathbf{x}}^2(0)\right\rangle + \alpha_H^2\left\langle\hat{\mathbf{x}}(t)^2\right\rangle - 2\alpha_H^2\left\langle\hat{\mathbf{x}}(0)\hat{\mathbf{x}}(t)\right\rangle \\
&+ \int_0^t\int_0^t\Bigg[\left\langle\hat{\mathbf{N}}_{pi}(t'')\hat{\mathbf{N}}_{pi}(t')\right\rangle + \alpha_H^2\left\langle\hat{\mathbf{N}}_{pr}(t'')\hat{\mathbf{N}}_{pr}(t')\right\rangle \\
&- \alpha_H\left(\left\langle\hat{\mathbf{N}}_{pr}(t'')\hat{\mathbf{N}}_{pi}(t')\right\rangle + \left\langle\hat{\mathbf{N}}_{pi}(t'')\hat{\mathbf{N}}_{pr}(t')\right\rangle\right)\Bigg]\mathrm{d}t'\mathrm{d}t'' \\
&- \alpha_H^2\int_0^t\Bigg[\left\langle\hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t)\right\rangle + \left\langle\hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pr}(t')\right\rangle \\
&+ \alpha_H\left(\left\langle\hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pi}(t')\right\rangle + \left\langle\hat{\mathbf{N}}_{pi}(t')\hat{\mathbf{x}}(t)\right\rangle\right)\Bigg]\mathrm{d}t'.
\end{aligned} \tag{G.49}$$

I use in the above expression the fact that $\langle\hat{\mathbf{y}}^2(0)\rangle = \langle\hat{\mathbf{x}}^2(0)\rangle = \frac{1}{4n(0)}$,

$$\left\langle\hat{\mathbf{N}}_{pi}(t'')\hat{\mathbf{N}}_{pi}(t')\right\rangle = \left\langle\hat{\mathbf{N}}_{pr}(t'')\hat{\mathbf{N}}_{pr}(t')\right\rangle \tag{G.50}$$

and

$$\left\langle\hat{\mathbf{N}}_{pr}(t'')\hat{\mathbf{N}}_{pi}(t')\right\rangle = -\left\langle\hat{\mathbf{N}}_{pi}(t'')\hat{\mathbf{N}}_{pr}(t')\right\rangle. \tag{G.51}$$

I also use the following relation

$$\int_0^T \int_0^T \left\langle \hat{\mathbf{N}}_{pi}(t'')\hat{\mathbf{N}}_{pi}(t') \right\rangle \mathrm{d}t'\mathrm{d}t'' = \frac{1}{4T}\int_0^T \left(\frac{\beta(t')}{n(t')}\right) \mathrm{d}t' \\ = -\frac{1}{4}\int_{n(0)}^{n(T)} \left(\frac{1}{n^2}\right) \mathrm{d}n \\ = \frac{1}{4}\left[\frac{1}{n(T)} - \frac{1}{n(0)}\right]. \tag{G.52}$$

Therefore, the expression of the power of the phase noise becomes

$$\left\langle \hat{\mathbf{y}}^2(T) \right\rangle = \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(0)} + \frac{1}{n(T)} - \frac{1}{n(0)}\right] + \alpha_H^2 \left\langle \hat{\mathbf{x}}^2(T) \right\rangle - 2\alpha_H^2 \left\langle \hat{\mathbf{x}}(0)\hat{\mathbf{x}}(t) \right\rangle \\ - \alpha_H^2 \int_0^t \Bigg[\left\langle \hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t) \right\rangle + \left\langle \hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pr}(t') \right\rangle \\ + \alpha_H \left(\left\langle \hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pi}(t') \right\rangle + \left\langle \hat{\mathbf{N}}_{pi}(t')\hat{\mathbf{x}}(t) \right\rangle \right) \Bigg] \mathrm{d}t'. \tag{G.53}$$

I can see that

$$\left\langle \hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pi}(t') \right\rangle = -\left\langle \hat{\mathbf{N}}_{pi}(t')\hat{\mathbf{x}}(t) \right\rangle \tag{G.54}$$

and

$$\left\langle \hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t) \right\rangle = \left\langle \hat{\mathbf{x}}(t)\hat{\mathbf{N}}_{pr}(t') \right\rangle. \tag{G.55}$$

Therefore,

$$\left\langle \hat{\mathbf{y}}^2(T) \right\rangle = \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(T)}\right] + \alpha_H^2 \left\langle \hat{\mathbf{x}}^2(T) \right\rangle - 2\alpha_H^2 \left\langle \hat{\mathbf{x}}(0)\hat{\mathbf{x}}(t) \right\rangle \\ - 2\alpha_H^2 \int_0^T \left\langle \hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t) \right\rangle \mathrm{d}t'. \tag{G.56}$$

I rewrite the integral in the expression as follows

$$2\int_0^T \left\langle \hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t) \right\rangle \mathrm{d}t' = \frac{1}{2T}\int_0^T H(t')\left(\frac{\beta(t')}{n(t')}\right) \mathrm{d}t' \\ + \frac{1}{T}\int_0^T H(t')\left(\frac{\beta(t')}{n_{\mathrm{sat}}\left(\frac{n(t')}{n_{\mathrm{sat}}} + 1\right)}\right) \mathrm{d}t'. \tag{G.57}$$

Using equation (8.42) and our usual change of variable, I have

$$
\begin{aligned}
2\int_0^T \left\langle \hat{\mathbf{N}}_{pr}(t')\hat{\mathbf{x}}(t)\right\rangle \mathrm{d}t' &= \frac{1}{2}\left[\frac{3\ln(n) - \frac{n_{\mathrm{sat}}}{n}}{n(T)+n_{\mathrm{sat}}}\right]_{n(T)}^{n(0)} \\
&= \frac{1}{2}\left[\frac{3\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\mathrm{sat}}}{n(T)} - \frac{n_{\mathrm{sat}}}{n(0)}}{n(T)+n_{\mathrm{sat}}}\right]
\end{aligned}
\tag{G.58}
$$

Substituting this answer in the expression of the phase noise, I get

$$
\begin{aligned}
\left\langle \hat{\mathbf{y}}^2(T)\right\rangle = {} & \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(T)}\right] + \alpha_H^2\left\langle \hat{\mathbf{x}}^2(T)\right\rangle - 2H_2(0)\alpha_H^2\left\langle \hat{\mathbf{x}}^2(0)\right\rangle \\
& - \frac{\alpha_H^2}{2}\left[\frac{3\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\mathrm{sat}}}{n(T)} - \frac{n_{\mathrm{sat}}}{n(0)}}{n(T)+n_{\mathrm{sat}}}\right].
\end{aligned}
\tag{G.59}
$$

Finally, using equation (8.42), I get

$$
\begin{aligned}
\left\langle \hat{\mathbf{y}}^2(T)\right\rangle = {} & \frac{(1+\alpha_H^2)}{4}\left[\frac{1}{n(T)}\right] + \alpha_H^2\left\langle \hat{\mathbf{x}}^2(T)\right\rangle - \frac{\alpha_H^2}{2}\frac{1}{n(0)}\left(\frac{n(0)+n_{\mathrm{sat}}}{n(T)+n_{\mathrm{sat}}}\right) \\
& - \frac{\alpha_H^2}{2}\left[\frac{3\ln\left(\frac{n(0)}{n(T)}\right) + \frac{n_{\mathrm{sat}}}{n(T)} - \frac{n_{\mathrm{sat}}}{n(0)}}{n(T)+n_{\mathrm{sat}}}\right].
\end{aligned}
\tag{G.60}
$$

As a check, I set $n_{\mathrm{sat}} \to \infty$ (unsaturable loss case). Then from equation (G.12), the expression of the amplitude noise is

$$
\begin{aligned}
\left\langle \hat{\mathbf{x}}^2(T)\right\rangle &= \left\langle \hat{\mathbf{x}}^2(0)\right\rangle + \frac{1}{4}\left[\frac{1}{n(T)} - \frac{1}{n(0)}\right] \\
&= \frac{1}{4}\left[\frac{1}{n(T)}\right].
\end{aligned}
\tag{G.61}
$$

Therefore, equation (G.60) becomes

$$
\begin{aligned}
\left\langle \hat{\mathbf{y}}^2(T)\right\rangle &= \frac{(1+2\alpha_H^2)}{4}\left[\frac{1}{n(T)}\right] - \frac{\alpha_H^2}{2}\frac{1}{n(0)} - \frac{\alpha_H^2}{2}\left[\frac{1}{n(T)} - \frac{1}{n(0)}\right] \\
&= \frac{1}{4}\left[\frac{1}{n(T)}\right] + \frac{(\alpha_H^2 - \alpha_H^2)}{2}\left[\frac{1}{n(T)} - \frac{1}{n(0)}\right] \\
&= \frac{1}{4}\left[\frac{1}{n(T)}\right],
\end{aligned}
\tag{G.62}
$$

which is what was expected.

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