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On Phase Information in Multivariable Systems

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ABSTRACT

The "median phase" and "phase spread" of a matrix are defined and properties are derived. The question of robust stability under uncertainty with phase information is addressed and a corresponding necessary and sufficient condition is given. This condition involves a "phase sensitive singular value". A computable upper bound to this quantity is obtained. The case when the uncertainty is block-structured is also considered.

Key Words: matrix phase, robust stability, structured singular value.

1. INTRODUCTION

Extension of the concept of phase of a transfer function to multivariable systems has been considered by various authors in the last ten years (e.g., [1–5]). In connection with this, various definitions have been proposed for the phase of a complex matrix. In this note we consider a somewhat more general issue, that of robustness under uncertainty with phase information.

As a starting point we define the concepts of "median phase" and "phase spread" of a matrix (the latter was previously considered by Owens [5]). We then consider linear time-invariant systems affected by uncertainty, in the now popular "uncertainty in the feedback loop" setup. The "phase sensitive singular value" μ^θ of a matrix is defined (inspired by Doyle's structured singular value), yielding a necessary and sufficient condition for robust stability for an uncertainty set with information on the phase. For block-structured uncertainty sets with phase information, a similar condition is obtained, based on the phase sensitive "structured" singular value $\mu^\theta_{\mathcal{K}}$. Finally, computable upper bounds on μ^θ and $\mu^\theta_{\mathcal{K}}$ are obtained. This extends results previously obtained in the scalar case [6].

2. MEDIAN PHASE AND PHASE SPREAD

Given a complex matrix A, let $\mathcal{N}(A) \subset \mathbb{C}$ be its numerical range, i.e.,

$$\mathcal{N}(A) = \{\langle x, Ax\rangle : \; x \in \partial B\} \subset \mathbb{C}$$

where $\partial B = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$ and $\|\cdot\|_2$ is the Euclidean norm. This set is known to be convex.

Definition 1. Let A be a complex square matrix such that $0 \notin \mathcal{N}(A)$. The *median phase* $\mathrm{MP}(A)$ of A is the phase, taken in $(-\pi, \pi]$, of the ray bisecting the convex sector generated by $\mathcal{N}(A)$; the *phase spread* $\mathrm{PS}(A)$ of A is the measure of the arc intercepted by this sector. $\square$

Thus $\mathrm{MP}(A) \in (-\pi, \pi]$ and $\mathrm{PS}(A) \in [0, \pi]$ (see Fig. 1). Below we will refer to the pair $(\mathrm{MP}(A), \mathrm{PS}(A))$ as *(matrix) phase* of A. If $0 \in \mathcal{N}(A)$, the phase of A is undefined.

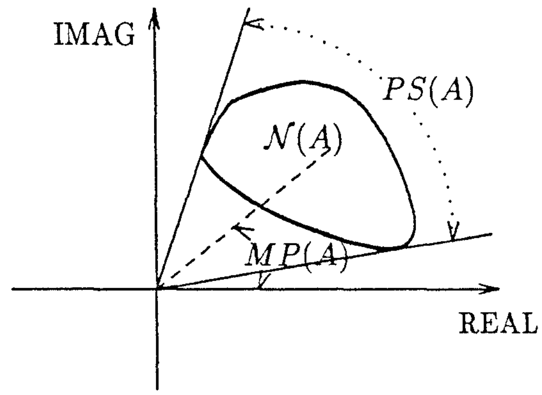

Figure 1: Phase information of a matrix

Note that, in the case of a complex number $a = \rho e^{j\phi}$ with $\rho > 0$ and $\phi \in (-\pi, \pi]$, the (matrix) phase of a is $(\phi, 0)$. Also, clearly, the phase of a matrix is invariant under multiplication of the matrix by a positive scalar and, if A is Hermitian positive definite, both $\mathrm{MP}(A)$ and $\mathrm{PS}(A)$ are zero. Finally, it is readily checked that the phase of a matrix is invariant under unitary similarity transformations.

Median phase and phase spread are related to the concept of principal phases introduced by Postlethwaite *et al.* [1]. Namely, for any square complex matrix A,

$$\mathrm{MP}(A) - \frac{1}{2}\mathrm{PS}(A) \le \psi_{\min}(A) \le \psi_{\max}(A) \le \mathrm{MP}(A) + \frac{1}{2}\mathrm{PS}(A)$$

where $\psi_{\min}(A)$ and $\psi_{\max}(A)$ are the minimum and maximum principal phases of A, respectively. This result, stated differently, was obtained by Owens [5] (who also used the term "phase spread").

3. UNCERTAINTY WITH PHASE INFORMATION

Consider now a linear time-invariant model affected by uncertainty. It is well known (see, e.g., [7]) that in many cases of interest such a system can be represented in "feedback" form as shown in Fig. 2, with $P(\cdot)$ and $\Delta(\cdot)$ in $H_\infty^{n \times n}$ and $\|\Delta\|_\infty \le 1$. If no other information is available concerning $\Delta(\cdot)$, a necessary and sufficient condition for robust stability is that $\|P\|_\infty < 1$.

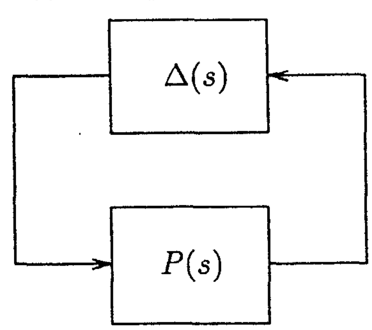

Figure 2: Feedback representation of uncertainty

In the classical structured singular value (SSV) framework the uncertainty $\Delta(\cdot)$ is assumed to be block diagonal. This reflects the assumption that the uncertainty is "structured" in the sense that it is tied to specific subsystems. The necessary and sufficient condition for robust stability becomes

$$\sup_{\omega} \mu_{\mathcal{K}}(P(j\omega)) < 1$$

where $\mu_{\mathcal{K}}$ is the structured singular value corresponding to the given block-structure $\mathcal{K}$ [8].

In this paper we consider the case where phase information is available concerning $\Delta(\cdot)$. Specifically, given two continuous functions $\Phi : \mathbb{R} \to (-\pi, \pi]$ and $\Theta : \mathbb{R} \to [0, \pi]$, we assume that $\Delta(\cdot)$ is known to lie in the set (see Fig. 3)

$$\mathcal{X}^{\Phi\Theta} = \{0\} \bigcup \{\Delta \in H_{\infty}^{nxn} : \mathrm{MP}(\Delta(j\omega)) + \frac{1}{2}\mathrm{PS}(\Delta(j\omega)) \leq \Phi(\omega) + \Theta(\omega),$$
$$\mathrm{MP}(\Delta(j\omega)) - \frac{1}{2}\mathrm{PS}(\Delta(j\omega)) \geq \Phi(\omega) - \Theta(\omega) \quad \forall \omega \in \mathbb{R}\}.$$

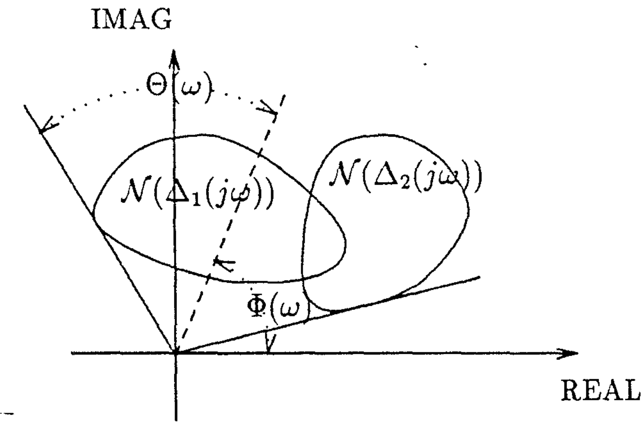

Figure 3: Uncertainty with phase information

To address the robustness question in this context, we introduce the concept of "phase sensitive singular value" as follows (this concept was previously used in [6] in the scalar case). Let $\sigma_{\max}$ denote the largest singular value of a matrix and, for $\theta \in [0, \pi]$, let

$$X^{\theta} = \{\Delta \in \mathbb{C}^{nxn} : \ \mathrm{MP}(\Delta) + \frac{1}{2}\mathrm{PS}(\Delta) \leq \theta, \ \mathrm{MP}(\Delta) - \frac{1}{2}\mathrm{PS}(\Delta) \geq -\theta\}.$$

Definition 2. Given $\theta \in [0, \pi]$, the *phase sensitive singular value* $\mu^{\theta}(M)$ of $M \in \mathbb{C}^{nxn}$ with respect to phase θ is $\mu^{\theta}(M) = 0$ if there is no $\Delta \in X^{\theta}$ such that $\det(I - \Delta M) = 0$, and

$$\mu^{\theta}(M) = \left(\min_{\Delta \in X^{\theta}} \{\sigma_{\max}(\Delta) : \det(I - \Delta M) = 0\}\right)^{-1}$$

otherwise. □

Theorem 1. The feedback system of Fig. 2 is well-formed and internally stable for all $\Delta \in \mathcal{X}^{\Phi\Theta}$, $\|\Delta\|_{\infty} \leq 1$ if, and only if,

$$\sup_{\omega} \mu^{\Theta(\omega)}(e^{j\Phi(\omega)}P(j\omega)) < 1.$$

□

The above can readily be extended to the case when $\Delta(\cdot)$ is also known to be block diagonal with, possibly, some real scalar blocks (corresponding to parametric uncertainty). In Theorem 1, $e^{j\Phi(\omega)}$ must then be replaced by a diagonal matrix with phase shifts corresponding to the specific structure, and μ^θ must be replaced by the phase sensitive *structured* singular value $\mu^\theta_{\mathcal{K}}$, where $\mathcal{K}$ is the block-structure.

4. COMPUTABLE UPPER BOUND

As is the case for the structured singular value, it appears that computation of the phase sensitive singular value is a nontrivial issue. As a first step toward addressing this issue, the phase sensitive SSV can be expressed as the optimal value of a smooth constrained optimization problem as follows. Here x^H and A^H indicate the Hermitian transpose of vector x and matrix A, respectively.

Theorem 2.

$$\mu^\theta(M) = \begin{cases} 0 & \text{if } \mathcal{S}^\theta(M) = \emptyset; \\ \sup\limits_{x \in \mathcal{S}^\theta(M)} \|Mx\|_2 & \text{otherwise.} \end{cases}$$

where $\mathcal{S}^\theta(M)$ is defined as follows:

(*i*) $\mathcal{S}^0(M) = \{x \in \partial B : x^H(M - M^H)x = 0, \quad x^H(M + M^H)x > 0\}$

(*ii*) for $\theta \in (0, \frac{\pi}{2})$,

$$\mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M + M^H)x > 0, \\ x^H[(1 + j\beta)M + (1 - j\beta)M^H]x \geq 0 \;\; \forall \beta \in \{\pm \cot\theta\}\}$$

(*iii*) for $\theta = \frac{\pi}{2}$,

$$\mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M + M^H)x \geq 0\}$$

(*iv*) for $\theta \in (\frac{\pi}{2}, \pi)$

$$\mathcal{S}^\theta(M) = \{x \in \partial B : jx^H(M - M^H)x \geq 0, \\ x^H[(1 + j\beta)M + (1 - j\beta)M^H]x \geq 0 \;\; \beta = -\cot\theta\} \\ \bigcup \{x \in \partial B : jx^H(M - M^H)x \leq 0, \\ x^H[(1 + j\beta)M + (1 - j\beta)M^H]x \geq 0 \;\; \beta = \cot\theta\}.$$

□

Since $x^H[(1 + j\beta)M + (1 - j\beta)M^H]x$ is affine in β, the region for β can be replaced by the interval $[-\cot\theta, \cot\theta]$ for the case $\theta \in (0, \frac{\pi}{2})$. For the case $\theta \in (\frac{\pi}{2}, \pi)$, the two inequalities in each of the components of $\mathcal{S}^\theta(M)$ are equivalent to

$$x^H[(1 + j\beta)M + (1 - j\beta)M^H]x \geq 0 \qquad \forall \beta \geq -\cot\theta,$$

and

$$x^H[(1 + j\beta)M + (1 - j\beta)M^H]x \geq 0 \qquad \forall \beta \leq \cot\theta,$$

respectively.

The expressions in Theorem 2 involve optimization problems that typically have many local solutions. Thus they typically provide *lower bounds* on μ^θ. Upper bounds are obtained next. Here $\lambda_{\max}$ denotes the largest eigenvalue (of a Hermitian matrix), and $\overline{\omega}$ denotes the complex conjugate of ω.

Theorem 3.

$$\mu^\theta(M) \le \nu^\theta(M) \le \sigma_{\max}(M)$$

where for $\theta \in [0, \frac{\pi}{2}]$

$$\nu^\theta(M) = \sqrt{\max\{0, \inf_{w\in\mathcal{W}^\theta} \lambda_{\max}(M^H M + wM + \overline{w}M^H)\}}$$

with

$$\mathcal{W}^\theta = \{\alpha(1+j\beta) : \ \alpha \ge 0, \quad |\beta| \le \cot\theta\},$$

and for $\theta \in (\frac{\pi}{2}, \pi)$

$$\nu^\theta(M) =$$

$$\sqrt{\max\{0, \inf_{w\in\mathcal{W}_+^\theta} \lambda_{\max}(M^H M + wM + \overline{w}M^H), \inf_{w\in\mathcal{W}_-^\theta} \lambda_{\max}(M^H M + wM + \overline{w}M^H)\}}$$

with

$$\mathcal{W}_+^\theta = \{\alpha(1+j\beta) : \ \alpha \ge 0, \quad \beta \ge -\cot\theta\}$$

$$\mathcal{W}_-^\theta = \{\alpha(1+j\beta) : \ \alpha \ge 0, \quad \beta \le \cot\theta\}.$$

□

5. DISCUSSION

The extension of Theorems 2 and 3 to the case when the uncertainty is block-structured is straightforward (see also [9]). It is readily checked that for any structure $\mathcal{K}$, as in the case with no phase information, $\mu_{\mathcal{K}}^\theta(M)$ satisfies $\mu_{\mathcal{K}}^\theta(DMD^{-1}) = \mu_{\mathcal{K}}^\theta(M)$ for any nonsingular matrix D which commutes with all $\Delta \in X_{\mathcal{K}}^\theta$, and that furthermore $\inf \nu_{\mathcal{K}}^\theta(DMD^{-1})$ is unaffected when D is restricted further to be Hermitian positive definite. Thus

$$\mu_{\mathcal{K}}^\theta(M) \le \hat{\mu}_{\mathcal{K}}^\theta(M) := \inf\{\nu_{\mathcal{K}}^\theta(DMD^{-1}) : D = D^H > 0, \quad \Delta D = D\Delta\}.$$

The algorithm proposed in [10] can be modified to compute $\hat{\mu}_{\mathcal{K}}^\theta(M)$.

We have considered only uncertainty matrices whose numerical range does not contain the origin. Yet, if this is not the case, the results presented above can still be applied, by making use of the property

$$\mathcal{N}(A+zI) = \mathcal{N}(A) + z \qquad \forall\, z \in \mathbb{C}$$

and of a simple loop transformation on the block diagram of Fig. 2. An upper bound to the gain margin can be computed for each value of the shift z. By taking the infimum over all possible shifts the upper bound can be reduced. This is under investigation.

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