

## ABSTRACT

Title of dissertation: The Multivariate Variance Gamma Process and Its Applications in Multi-asset Option Pricing

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Dependence modeling plays a critical role in pricing and hedging multi-asset derivatives and managing risks with a portfolio of assets. With the emerge of structured products, it has attracted considerable interest in using multivariate Lévy processes to model the joint dynamics of multiple financial assets. The traditional multidimensional extension assumes a common time change for each marginal process, which implies limited dependence structure and similar kurtosis on each margin.

In this thesis, we introduce a new multivariate variance gamma process which allows arbitrary marginal variance gamma (VG) processes with flexible dependence structure. Compared with other multivariate Lévy processes recently proposed in the literature, this model has several advantages when applied to financial modeling. First, the multivariate process built with any marginal VG process is easy to simulate and estimate. Second, it has a closed form joint characteristic function which largely simplifies the computation problem of pricing multi-asset options. Last, it can be applied to other time changed Lévy processes such as normal inverse gaussian (NIG)

process.

To test whether the multivariate variance gamma model fits the joint distribution of financial returns, we compare the model performance of explaining the portfolio returns with other popular models and we also develop Fast Fourier Transform (FFT)-based methods in pricing multi-asset options such as exchange options, basket options and cross-currency foreign exchange options.

THE MULTIVARIATE VARIANCE GAMMA PROCESS AND ITS  
APPLICATIONS  
IN MULTI-ASSET OPTION PRICING

by

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## Dedication

To My parents and Min.

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## List of Abbreviations

$\alpha$	alpha
$\beta$	beta
VG	variance gamma
NIG	normal inverse Gaussian
FFT	fast Fourier transform
MLE	maximum likelihood estimation
RMSE	root mean squared error
FX	Foreign Exchange
ATM	at the money

# Chapter 1

## Introduction

### 1.1 Background

Dependence modeling plays a central role in pricing multi-asset derivatives and managing risks exposed to multiple financial assets. Before the emerging of alternative copula based models, the study of multivariate time series and stochastic processes has been dominated by elliptic models, like multivariate normal or  $t$  distributions. Their popularity only results from their mathematical tractability and is questioned by empirical financial data.

The classical approach to model dependence is through constructing multivariate Brownian motions or diffusion based processes such as log-normal processes. Using a high-dimensional correlated Brownian motion may be the most natural way to build the dependence, but it also has many limitations on the generated distribution. Besides its very limited symmetric dependence structure, the marginal processes were questioned for many years to explain the dynamics of a single asset. The well-documented heavy tail phenomena of the stock returns and the volatility skew effects observed in the option market provided strong evidences to support the use of non-normal distributions. A vast literature on more sophisticated models such as stochastic volatility models (e.g. Heston model [29], SABR model [26]) and Lévy based models (e.g. VG model [42], NIG model [3]) emerged in the last decade

to incorporate these effects.

Lévy processes have attracted considerable attention amongst practitioners and academics for the primary reason that the flexibility of their distributions is well-suited to financial asset returns. Such Lévy models including the variance gamma model by Madan and Seneta [42], the normal inverse gamma(NIG) model by Barndorff-Nielsen [3] and the CGMY model by Carr, Geman, Madan and Yor [9] have been developed over the last decade. In the recent structured products market, it has become quite usual that the payoff function is determined by more than one assets. While these models successfully explain the dynamics of a single price process, modeling a higher dimensional Lévy process is not so straightforward as the case of multivariate Brownian motion. Recently, there has been an increasing interest in the multivariate Lévy process modeling. For example, Tankov [59] introduced the Lévy copula model, which characterizes the joint law of multivariate Lévy processes by applying the idea of copula on the Lévy measure. Cont and Tankov [12], Luciano and Schoutens [40] studied and tested the multivariate time changed Brownian motion by a common subordinator. Semeraro [54], Luciano and Semeraro [39] proposed a similar model with multivariate subordinators.

In this dissertation, we propose a new multivariate VG model based on decomposition of marginal VG processes into independent components. The model has arbitrary VG marginal processes and flexible dependence structure. Its closed-form joint characteristic function simplifies the calculation of multi-asset option pricing by FFT. The idea can also be applied to other time changed Brownian motions such as the NIG process. The outline of this thesis is as follows. In chapter one, we

review the basics of a Lévy process, its use as a financial model and the technique of change of numeraire in option pricing. We also introduce the Carr-Madan FFT method as a standard engine of pricing options under Lévy based models. In chapter two, we present the new multivariate variance gamma model and its properties. We discuss the estimation and simulation scheme of the model. In chapter three, we study the performance of the multivariate variance gamma model in explaining the joint dynamics of stock returns. We report the chi-square test statistics on randomly generated portfolio returns and compare the test results with the popular full-rank Gaussian copula method. In the last chapter, we study the problem of pricing multi-asset options such as exchange option, spread option, basket option and cross-currency option.

## 1.2 Lévy Processes in Finance

### 1.2.1 Lévy Processes and Lévy-Khintchine Representation

Lévy processes, named after the French mathematician Paul Lévy, have been used in mathematical finance for a long period of time. Brownian motion, the best known of all Lévy processes, was introduced as a model for stock prices in early 1900s by Bachelier. Though most of the financial models developed in the following several decades were driven by Brownian motions, non-normal Lévy processes were widely studied and became increasingly popular in the last decade. It was Mandelbrot [44] who studied the first non-normal exponential Lévy process in 1960s and introduced the  $\alpha$ -stable Lévy motion with index  $\alpha < 2$ . Later, models based

on more general pure jump Lévy processes such as variance gamma(VG), normal inverse gaussian(NIG) and CGMY, were developed and studied.

Generally speaking, Lévy processes are stochastic processes with independent and stationary increments. They can be thought of as analogues of random walks in continuous time. Every Lévy process has a càdlàg (means "right continuous with left limits") modification which is itself a Lévy process. Therefore, we always work with this càdlàg version of the process. The formal definition can be written as follows:

**Definition 1.1.** *A càdlàg stochastic process  $(X_t)_{t \geq 0}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $X_0 = 0$  is called a Lévy process if it possesses the following properties:*

- *Independent increments: for any  $0 < t_0 < t_1 < \dots < t_n$ , the random variables  $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$  are independent.*
- *Stationary increments: the law of  $X_{t+h} - X_t$  does not depend on  $t$ .*
- *Stochastic continuity:  $\forall \epsilon > 0, \lim_{h \rightarrow 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$ .*

If we sample a Lévy process at any fixed time intervals with equal increments, we obtain a random walk. Since this can be done for any sampling interval, the distribution of a Lévy process at any time  $t$  has some special properties. This connects closely to the concept of infinitely divisible distribution.

**Definition 1.2.** *If, for every positive integer  $n$ , the characteristic function  $\phi_X(u)$  is also the  $n$ th power of a characteristic function, we say that the distribution is*

*infinitely divisible.*

In other words, an infinitely divisible distribution  $F$  can be written as the distribution of the sum of  $n$  independent and identically distributed random variables for any positive integer  $n$ . The following proposition shows the relationship between Lévy processes and infinitely divisible distributions.

**Proposition 1.3.** *For a Lévy process  $(X_t)_{t \geq 0}$ ,  $X_t$  has an infinitely divisible distribution at any time  $t$ . Conversely, if  $F$  is an infinitely divisible distribution, then there exists a Lévy process  $(X_t)$  such that the distribution of  $X_1$  is given by  $F$ .*

By the infinite divisibility, the characteristic function  $\phi_X(u)$  of Lévy process  $X_t$  can be expressed in a simple form. If we denote  $\phi_X(u) = e^{\psi_X(u)}$ ,  $\psi_X(u)$  is called the characteristic exponent of  $X$ . We then have the following fact:

$$\phi_{X_t}(u) = E(e^{iuX_t}) = e^{t\psi_{X_1}(u)} \quad (1.1)$$

where  $\psi_{X_1}(u)$  is the characteristic exponent of the Lévy process at unit time. It is now possible to characterize all Lévy processes by looking at their characteristic functions, which leads to the famous Lévy-Khintchine formula.

**Theorem 1.4. (Lévy-Khintchine Representation)** *Let  $(X_t)_{t \geq 0}$  be a Lévy process on  $\mathcal{R}$ . The Lévy-Khintchine formula gives the expression for characteristic exponent  $\psi_{X_1}(u)$  as follows:*

$$\psi_{X_1}(u) = bui - \frac{1}{2}\sigma^2 u^2 + \int_{\mathcal{R} \setminus \{0\}} (1 - e^{iux} + iux1_{|x| < 1}) \nu(dx) \quad (1.2)$$

with

$$\int_{\mathcal{R} \setminus \{0\}} (1 \wedge x^2) \nu(dx) < \infty. \quad (1.3)$$

From Lévy-Khintchine representation, we can easily see that a Lévy process can be decomposed into three independent components: a deterministic drift with rate  $b$ , a continuous path diffusion with volatility  $\sigma$  and a jump process with the Lévy measure  $\nu(dx)$ . If the Lévy measure is of the form  $\nu(dx) = k(x)dx$ , we call  $k(x)$  the Lévy density. Hence, a Lévy process can be fully characterized by the combined Lévy triplet  $(b; \sigma; \nu)$ .

The path property of a Lévy process is determined by the Lévy triplet  $(b; \sigma; \nu)$ . For example, if  $b = 0$ ,  $\nu = 0$ , then the Lévy process becomes a standard Brownian motion with continuous random paths. In the case of  $\sigma^2 = 0$ , the Lévy process has no diffusion part and becomes a pure jump process. If the Lévy measure also satisfies  $\nu(dx) = \lambda\delta(1)$ , where  $\delta(1)$  is the Dirac function at 1, then it is a Poisson process with rate parameter  $\lambda$ .

Lévy processes with only jump components can also be divided into two categories by the arrival rate of jumps. A Lévy process is called of finite activity if  $\int_{\mathcal{R}\setminus\{0\}} \nu(dx) < \infty$ . If  $\int_{\mathcal{R}\setminus\{0\}} \nu(dx) = \infty$  instead, then the Lévy process has infinite activity, which means its arrival rate of jumps is infinity.

### 1.2.2 Change of measure for Lévy processes

When pricing a contingent claim traded in the financial market, the probability measure we use is usually different from the statistical measure we observe. In probability theory, the Girsanov theorem tells how stochastic processes change under changes in measure. Therefore, it is the key theorem in the Black-Scholes model in

connecting the physical measure with the risk-neutral one. Since all option pricing should be done under the risk-neutral measure by non-arbitrage pricing theory, it can be shown that the Black-Scholes option pricing formula does not depend on the drift term under the physical measure given the following Girsanov theorem:

**Theorem 1.5. (*Girsanov theorem for Brownian motion*)** *Let  $W_t$  be a Brownian motion on  $(\Omega, \mathcal{F}_T, \mathbb{P})$ , and let  $X_t$  be a measurable process adapted to the filtration of  $W_t$ . Let  $[X]_t$  be the quadratic variation of the process  $X$ . Let  $Z$  be the associated exponential martingale*

$$Z_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$$

*If  $Z_t$  is a martingale under  $\mathbb{P}$ , then a new probability measure  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$  can be defined by the Radon-Nikodym derivative:*

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t \tag{1.4}$$

*Furthermore if  $Y_t$  is a  $\mathbb{P}$  local martingale, then  $Y_t - [W, X]_t$  is a  $\mathbb{Q}$  local martingale.*

To change measures for general Lévy processes, one needs to find equivalent martingale measures. The detailed discussion of equivalent martingale measures for Lévy processes can be found in [31, 52]. Here, we only state the results on the change of measure for pure jump processes. Jacod and Shiryaev [31] show that one can explicitly compute out the change of the measure given its physical and risk-neutral Lévy measure. Assume that we have the pure jump processes with Lévy densities  $k_{\mathbb{P}}(x)$  and  $k_{\mathbb{Q}}(x)$  under the  $\mathbb{P}$  and  $\mathbb{Q}$  measures, respectively. If they are

equivalent measures, the Radon-Nikodym derivative is given by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp\left(-t \int_{-\infty}^{\infty} (Z(x) - 1)k_{\mathbb{P}}(x)dx\right) \prod_{s \leq t} (\Delta X(s)) \quad (1.5)$$

where  $Z(x)$  is given by

$$k_{\mathbb{Q}} = Z(x)k_{\mathbb{P}}$$

Given the explicit form of measure change, we may infer the measure change from both measures.

### 1.2.3 Lévy-based financial models

There are many reasons to introduce Lévy processes to financial modeling. One of the most important reasons is that the historical log returns of stocks/indices are not normally distributed as in the Black-Scholes model. To price and hedge derivative securities, it is crucial to have a good model of the probability distribution of the underlying product. Lévy processes have similar nice features, i.e. with independent and stationary increments, as Brownian motions but with more flexible distribution. The distributions of most Lévy processes can exhibit various of types of skewness and excess kurtosis. Examples of such models include the Variance Gamma (VG), the Normal Inverse Gaussian (NIG), the CGMY, the Generalized Hyperbolic Model. The stock price models driven by Lévy processes assume the market consists of one riskless asset with a price process  $B_t = e^{rt}$ , and one risky asset. The model for the risky asset is

$$S_t = S_0 e^{X_t} \quad (1.6)$$

where the log returns  $\ln(S_t/S_0) = X_t$  can be any Lévy process. Lévy models can fit the distribution very well to the historical returns. However, pricing vanilla options under these models is not so straightforward as the diffusion-based ones since the uniqueness of equivalent martingale measures is not kept in most of the realistic Lévy models. Thus, the Lévy financial models lead to incomplete markets in which there are infinitely many equivalent martingale measures and perfect hedge exists. To price an option under these models, one needs to first choose the risk-neutral measure from many equivalent martingale measures available. There are several methods proposed in the literature, including Esscher transform, mean-correcting martingale measure, minimal entropy measure or indifference pricing. One of the most convenient choices is to use the mean-correcting martingale measure. We assume:

$$S_t = S_0 e^{r t} \frac{e^{X_t}}{E(e^{X_t})} \quad (1.7)$$

It is easy to check that  $S_t$  is a martingale given  $X_t$  is any Lévy process. It is called the mean-correcting martingale measure as it is equivalent to  $X_t$  is mean-corrected by  $X_t + r - \ln \phi(-i)$  assuming interest rate  $r$  and no dividend yield.

We now discuss two important Lévy processes: the VG process and the NIG process in details. For other Lévy processes and Lévy based models, we refer the readers to [53] [12].

### 1.3 The Variance Gamma Process

The class of variance gamma distribution was first introduced by Madan and Seneta in the late 1980s. The symmetric case of VG process was proposed and developed by Madan and Seneta [42] and Madan and Milne [41] as a model for studying stock returns and option pricing. The general case with skewness was late introduced by Madan *et al.* [43]. Since the original symmetric VG process can be considered as a special case of the general one with  $\theta = 0$ , we always refer to the general case when we talk about the VG process from now on. The VG process has become one of the most popular Lévy models in both literature and practice.

#### 1.3.1 The VG process and its properties

A VG process can be considered as a drifted Brownian motion time changed by an independent gamma process. Namely, it can be represented as:

$$X_t = \theta G_t + \sigma W_{G_t} \tag{1.8}$$

where  $W = (W_t; t \geq 0)$  is a standard Brownian motion and the independent subordinator (i.e. an increasing, positive Lévy process)  $G_t$  is a gamma process with unit mean rate and variance rate  $\nu$ .

As a Lévy process, the dynamics of a VG process is determined by the distribution of  $X_t$  at unit time. The random variable of a VG process at unit time follows a 3-parameter  $\text{VG}(\theta, \sigma, \nu)$  probability law with characteristic function in a simple

form:

$$\phi_{VG}(u) = (1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu)^{-1/\nu}. \quad (1.9)$$

This distribution is infinitely divisible and the VG process thus has independent and stationary increments for which the increment  $X_{t+s} - X_s$  follows a  $VG(\sigma\sqrt{t}, \nu/t, t\theta)$  law.

It is worth noting that the idea of time change has strong economic intuitions. We know that the financial market does not evolve identically every day. To be more specific, the trading volume is not uniform during the day and the trading activities vary a lot from time to time. Intuitively, one can regard the original clock as the calendar time and a random clock as the business time. A more active business day implies a faster business clock. Therefore, the concept of business time is used to distinguish from the calendar time and describe the trading activity evolution. A VG process is thus a Brownian motion run under a random gamma business clock.

An alternative parametrization of the VG model was discussed as a special case of the CGMY model with  $Y = 0$ . With the parametrization in terms of  $C$ ,  $G$  and  $M$ , the characteristic function of  $X_{VG}(1)$  reads as follows:

$$\phi_{VG}(u) = \left( \frac{GM}{GM + (M - G)iu + u^2} \right)^C. \quad (1.10)$$

The characterization also allows the Lévy measure of a VG process to be in a more elegant form:

$$k_{VG}(x) = \begin{cases} \frac{C \exp(Gx)}{|x|}, & x < 0 \\ \frac{C \exp(-Mx)}{x}, & x > 0 \end{cases} \quad (1.11a)$$

$$(1.11b)$$

where:

$$\begin{aligned}
C &= 1/\nu, \\
G &= \left( \sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1} \\
M &= \left( \sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1}
\end{aligned}$$

With this parametrization, it is clear that VG process can be decomposed into two processes with only positive and negative jumps controlled by parameters  $G$  and  $M$  respectively. Hence it can be written as the difference of two independent Gamma processes. This fact leads to a straight-forward simulation algorithm of the VG process by simulating two independent gamma processes.

$$X_{VG} = X_{gamma}(C; 1/M) - X_{gamma}(C; 1/G) \quad (1.12)$$

There are some other remarkable properties of the VG process. For instance, the Lévy measure has infinite mass, and hence a VG process has infinitely many jumps in any finite time interval. The VG process also has paths of finite variation with no Brownian component.

The popularity of the VG process lies in its flexibility of handling the skewness and excess kurtosis exhibited from the historical data of stock prices. While the parameter  $\sigma$  still plays a similar role to the volatility parameter in Black-Scholes world, the other parameters add much flexibility to the distribution. Generally speaking, the parameter  $\theta$  controls the skewness of the distribution and  $\nu$  determines the kurtosis of the distribution. For example, for the vanilla option market, a

negative  $\theta$  accounts for the negative slope in the volatility curve. For  $(C, G, M)$  parametrization,  $C = \frac{1}{\nu}$  controls the kurtosis, and both  $G$  and  $M$  determine the skewness. In the special case of  $G = M$ , the distribution is symmetric.

### 1.3.2 The VG Stock Price Model

The VG stock price model is constructed by replacing the Brownian motion in the Black-Scholes model by a VG process. By assuming a VG process on the stock log-returns, the VG model can capture the well-documented volatility smile/skew observation. Assume there is no dividend, we choose the risk-neutral measure by mean correcting the original VG process, and can write the stock price process as:

$$S_t = S_0 \exp(rt + X_t + wt) \quad (1.13)$$

where

$$w = -\log\phi(-i) = \frac{1}{\nu} \log(1 - \theta\nu - \frac{1}{2}\sigma^2\nu). \quad (1.14)$$

The density function of the log return in VG model, like many other pricing models, can only be expressed in terms of integrals or special function.

**Theorem 1.6.** *The density for the log return  $z_t = \ln(\frac{S_t}{S_0})$ , where the process follows (1.8), is given by the following:*

$$f(z) = \frac{2\exp(\theta x/\sigma^2)}{\nu^{t/\nu} \sigma \sqrt{2\pi} \Gamma(\frac{t}{\nu})} \left( \frac{x^2}{2\sigma^2/\nu} + \theta^2 \right)^{\frac{t}{2\nu} - \frac{1}{4}} K_{\frac{t}{\nu} - \frac{1}{2}} \left( \frac{\sqrt{x^2(2\sigma^2/\nu + \theta^2)}}{\sigma^2} \right) \quad (1.15)$$

where  $K$  is the modified Bessel function of the second type,

$$x = z - rt - \frac{t}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2). \quad (1.16)$$

Madan *et al* [43] derived a closed-form formula for pricing the European call option with strike  $K$ . The option price formula is of the similar form to the Black-Scholes formula, but it is numerical demanding to compute the Bessel function of the second type involved. A more efficient way to compute the option prices using FFT is now used as the market standard in pricing options under Lévy models. We will discuss this important pricing method in the next section.

**Theorem 1.7.** *Under the risk-neutral price process, the European call option price on a stock is*

$$\begin{aligned}
c(S_0; K, t) = & S_0 \Psi\left(d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \frac{t}{\nu}\right) \\
& - Ke^{-rt} \Psi\left(d\sqrt{\frac{1-c_2}{\nu}}, (\alpha s)\sqrt{\frac{\nu}{1-c_2}}, \frac{t}{\nu}\right)
\end{aligned} \tag{1.17}$$

where

$$\begin{aligned}
d &= \frac{1}{s} \left[ \ln(S(0)/K) + rt + \frac{t}{\nu} \ln\left(\frac{1-c_1}{1-c_2}\right) \right], \\
\alpha &= -\frac{\theta}{\sigma \sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2 \frac{\nu}{2}}}, \\
c_1 &= \frac{\nu(\alpha+s)^2}{2}, \\
c_2 &= \frac{\nu\alpha^2}{2},
\end{aligned}$$

the function  $\Psi$  is defined in terms of the modified Bessel function of the second kind and the degenerate hypergeometric function of two variables.

## 1.4 The Normal Inverse Gaussian Process

The normal inverse Gaussian model was first introduced by Barndorff-Nielsen [3, 4] and applied to option valuation. It is another important class of Lévy processes which shares many similarities with the VG process. The normal inverse Gaussian distribution is defined as a variance-mean mixture of a normal distribution with the inverse Gaussian as the mixing distribution. As an infinitely divisible distribution, it determines a Lévy process, which can be represented as a time changed Brownian motion subordinated by the inverse Gaussian process.

Like the VG distribution, the density function of a NIG distribution has complex Bessel functions involved and is hard to work with. However, the characteristic function of the normal inverse Gaussian distribution  $NIG(\alpha, \beta, \delta)$  with parameters  $\alpha > 0, \beta \in (-\alpha, \alpha], \delta > 0$  is given in an elegant form:

$$\phi_{NIG}(u) = \exp(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2})). \quad (1.18)$$

From a different point of view, one can generate a NIG process with parameters  $\alpha, \beta$  and  $\delta$  by time changing a Brownian motion. We can write:

$$X_t = \beta\delta^2 I_t + \delta W_{I_t} \quad (1.19)$$

where  $W = W_t, t > 0$  is a standard Brownian motion and  $I = I_t, t > 0$  is an Inverse Gaussian process with the mean rate of 1 and shape parameter  $\delta\sqrt{\alpha^2 - \beta^2}$ , with  $\alpha > 0, -\alpha < \beta < \alpha$  and  $\delta > 0$ . An inverse Gaussian process has independent and stationary inverse Gaussian distributed increments. It is called "inverse" in that, while the Gaussian describes the distribution of distance at fixed time in Brownian

motion, the inverse Gaussian describes the distribution of the time a Brownian Motion with positive drift takes to reach a fixed positive level.

Financial models based on the NIG process are pretty much the same as those based on the VG process. Empirical studies on the stock returns show both two distributions have significant improvements in explaining stock returns. For more details of the NIG process, we refer the reader to [3] [4].

## 1.5 The Fast Fourier Transform Method and Option Pricing

One of the most important problems all Lévy models face is to find an efficient way of pricing European options. Closed form solutions under these models either do not exist or involve complicated functions which are difficult to evaluate even numerically. Since the 1990s, a lot of attention has been paid on the use of characteristic functions and Fourier analysis for understanding the proposed processes. Given the characteristic function of a stochastic process, Heston [29] showed how to numerically value standard European options by using Lévy's inversion formula for the distribution function. It takes two Fourier transforms to compute two probabilities in the call option pricing formula, which was later improved significantly by Carr and Madan. By analytically relating the Fourier transform of an option price to its characteristic function, Carr and Madan [10] showed how to use the Fast Fourier Transform method to price European options. This method has become a standard calibration engine due to its fast speed of computation.

### 1.5.1 The Carr-Madan FFT Method

The Carr-Madan FFT method evaluates the value of an option by doing an inverse Fourier transform to the characteristic function of the log price. The method is much faster than using the analytic formula for VG models in which a numerical integration of the modified Bessel function of the second type is needed. Since the only thing required for using this method is the closed-form characteristic function of the log price, the Carr-Madan FFT method is widely used for most of Lévy models and stochastic volatility models. We sketch the method as follows:

Let  $k$  be the log of the strike price  $K$ , and let  $C_T(k)$  be the value of a call option with maturity  $T$ . Let the  $\phi_T(u)$  be the characteristic function of the log price  $S_T$  under the chosen risk neutral measure. To solve the problem of the singularity in the integrand, Carr and Madan included  $\exp(-\alpha k)$  as a dampening factor. They considered the Fourier transform of  $c_T = \exp(-\alpha k)C_T$  with respect to  $k$  defined by:

$$\psi_T(\nu) = \int_{-\infty}^{\infty} e^{i\nu k} c_T(k) dk.$$

Since an analytical expression for  $\psi_T(\nu)$  can be derived, they obtained call prices numerically using the inverse transform

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-i\nu k} \psi_T(\nu) d\nu \quad (1.20)$$

where  $\psi_T(\nu)$  can be computed in terms of  $\phi_T(u)$ :

$$\psi_T(\nu) = \frac{e^{-rT} \phi_T(\nu - (\alpha + 1)i)}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha + 1)}. \quad (1.21)$$

To apply for the FFT method to compute the integral in the equation (1.24), one can approximate it using the trapezoidal rule on a well-defined grid. Let  $\eta$  be

the step size for the grid of the characteristic function  $\phi$ .  $N$  is chosen to be a power of 2 to take the full advantage of FFT. Then  $a = \eta N$  is the upper limit of the integration. The grid is chosen on  $\nu_j = (j-1)\eta, j = 1, 2, \dots, N$ . Also let  $\lambda$  be the step size of the log strike  $k$ , then the log strikes change from  $-b$  to  $b$  and on the grid of  $k_u = -b + \lambda(u-1)$ , for  $u = 1, 2, \dots, N$ . These parameters satisfy  $\lambda\eta = \frac{2\pi}{N}$ .

We have the following approximation of (1.24).

$$C_T(k_u) = \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-\frac{2\pi i}{N}(j-1)(u-1)} e^{ib\nu_j} \psi(\nu_j) \frac{\eta}{3} [3 + (-1)^j - \delta_{j-1}] \quad (1.22)$$

where  $\delta_n$  is the Kronecker delta function that is unity for  $n$  and zero otherwise. The summation in formula (1.26) can be computed using the FFT. By making the appropriate choices for  $\eta$  and  $\alpha$ , one may compute the option prices very efficiently. For one single run, the FFT method calculates the option prices across all the strikes, which makes the calibration of Lévy model to market data incredibly fast.

## 1.5.2 The Greeks

The Greeks represent the sensitivities of financial derivatives to a change in underlying parameters. As vital tools in risk management, the Greeks are extremely important for hedging purpose. Financial portfolios are often rebalanced accordingly to achieve a desired exposure by using the Greeks. We discuss the calculations for the Delta (the rate of change of option value with respect to changes in the underlying asset's price), the Gamma (the rate of change of the delta with respect to changes in the underlying asset's price ) and the Rho(sensitivity to the applicable interest rate). Other Greeks such as the Vega, the sensitivity of price with respect to its

implied volatility in Black-Scholes model, are not available in Lévy based models. It is worth noting that since Lévy models describe incomplete markets, a perfect hedge no longer exists.

The option price using FFT method is given in (1.24) and (1.25). Differentiation with respect to variables such as  $S_0$ , and  $r$  only has an impact on the function  $\psi$ . Hence the following result can be derived.

**Proposition 1.8.** *The Greeks are computed by FFT in the following form:*

$$\frac{\exp(-\alpha k)}{\pi} \int_0^\infty e^{-i\nu k} \psi_T(\nu) d\nu \quad (1.23)$$

where

for  $\Delta = \frac{\partial C(K,T)}{\partial S_0}$ ,  $\psi$  is given by

$$\psi_T^\Delta(\nu) = \frac{e^{-rT} \phi_T(\nu - (\alpha + 1)i)}{S_0(\alpha + i\nu)},$$

for  $\Gamma = \frac{\partial^2 C(K,T)}{\partial S_0^2}$ ,  $\psi$  is given by

$$\psi_T^\Gamma(\nu) = \frac{e^{-rT} \phi_T(\nu - (\alpha + 1)i)}{S_0^2},$$

for  $\rho = \frac{\partial C(K,T)}{\partial r}$ ,  $\psi$  is given by

$$\psi_T^\rho(\nu) = \frac{T e^{-rT} \phi_T(\nu - (\alpha + 1)i)}{\alpha + 1 + i\nu}.$$

By changing the corresponding function  $\psi$  in formula (1.26), the FFT method computes the Greeks from the characteristic function across all strikes in one run.

## 1.6 Change of Numeraire and Option Pricing

Risk-neutral pricing method has become the market standard in pricing financial derivatives since the celebrated Black-Scholes work. Later, Harrison and Kreps

[27] completed the non-arbitrage asset pricing theory by arguing that the absence of arbitrage implies the existence of a risk-adjusted probability  $Q$  such that the current price of any security should equal to its discounted expectation of future values. The riskless money account  $B(t) = e^{rt}$ , also referred to as the numeraire, is the relative benchmark account associated with this measure  $Q$ .

However, Geman *et al* [24] noted that the risk-neutral measure  $Q$  is not necessarily the most natural choice for pricing a contingent claim. Changing the benchmark account (numeraire) to a more convenient one may largely simplify the option pricing problem. In such cases, the change of numeraire has surprisingly helped reduce the complexity in pricing derivatives, especially in multi-asset option pricing or models with multiple underlying such as the fixed income and FX market. Although the idea of numeraire was used in Margrabe's formula as early as 1970s, Geman *et al* [24] formally developed the general framework for the change of numeraire technique and introduced the following definition.

**Definition 1.9.** *A numeraire is any positive non-dividend-paying asset.*

As different numeraires are associated with different equivalent martingale measures, option prices are invariant under any of these numeraires. Hence, by choosing the most convenient numeraire, pricing options can be largely simplified. The main result we will use later in multi-asset option pricing is the following theorem which can be found in [49]:

**Theorem 1.10.** *Assume there exists a numeraire  $N$  and a probability measure  $Q^N$  equivalent to the initial measure  $Q_0$ , such that the price of any traded asset  $X$  relative*

to  $N$  is a martingale under  $Q^N$ , i.e.,

$$\frac{X_t}{N_t} = E^N\left[\frac{X_T}{N_T} \mid \mathcal{F}_t\right], 0 \leq t \leq T. \quad (1.24)$$

Let  $U$  be an arbitrary numeraire. Then there exists a probability measure  $Q^U$ , equivalent to the initial  $Q_0$ , such that the price of any attainable claim  $Y$  normalized by  $U$  is a martingale under  $Q^U$ , i.e.,

$$\frac{Y_t}{U_t} = E^U\left[\frac{Y_T}{U_T} \mid \mathcal{F}_t\right], 0 \leq t \leq T. \quad (1.25)$$

Moreover, the Radon-Nikodym derivative defining the measure  $Q^U$  is given by

$$\frac{dQ^U}{dQ^N} = \frac{U_T N_0}{U_0 N_T}. \quad (1.26)$$

The choice of a convenient numeraire determines the complexity of computation for many problems. The general rule is conducted as follows. A payoff  $f(X_T)$  depending on an underlying variable  $X$  at time  $T$  is priced under the risk-neutral numeraire with the money-market account  $B(t) = \exp(rt)$ . By using the above theorem, the formula under a new numeraire  $U$  is given by:

$$E_0\left(\frac{h(X_T)}{B(T)}\right) = U_0 E^{Q^U}\left(\frac{h(X_T)}{U_T}\right) \quad (1.27)$$

Hence, we look for a numeraire  $U$  with the following properties:

- $X_t U_t$  is a tradable asset.
- The quantity  $\frac{h(X_T)}{U_T}$  is simple.

The standard applications of the above method are Margrabe's formula for exchange options, quanto derivative pricing, caps and swaptions pricing in LIBOR market model, etc. Change of numeraire is especially useful in yield curve modeling and interest rate derivative pricing.

## Chapter 2

### The New Multivariate Variance Gamma Model

#### 2.1 Correlating Lévy Processes: An Overview

Lévy processes have been increasingly popular in financial modeling due to their flexibility of incorporating the jump dynamics. Many Lévy models including variance gamma (VG), normal inverse gaussian (NIG) and CGMY have been developed over the last decade. While these models successfully explain the dynamics of a single price process, modeling a multivariate Lévy process usually does not lead to an elegant form as a multivariate Brownian motion. Madan and Seneta [42] first introduced the multivariate symmetric VG process by subordinating a multivariate Brownian motion without a drift by a common gamma process. Similarly, Barndorff-Nielsen [4] studied the multivariate case of the NIG process using a common subordinator. The extension to an asymmetric case is developed in Cont and Tankov [12], Luciano and Schoutens [40]. They studied multivariate Lévy processes with VG components in the following settings:

$$X_i(t) = \theta_i \Gamma(t) + \sigma_i W_i(\Gamma(t)) \quad (2.1)$$

where  $W_i$  and  $W_j$  are correlated with correlation  $\rho_{ij}$ .

The linear correlation of  $X_i, X_j$  is then:

$$\text{corr}(X_i, X_j) = \frac{\theta_i \theta_j \text{Var}(\Gamma(t)) + \sigma_i \sigma_j \rho_{ij} E(\Gamma(t))}{\sqrt{\text{Var}(X_i(t)) \text{Var}(X_j(t))}} \quad (2.2)$$

These models are easy to construct and work with. It has been noted, however, that this model does not accommodate independence, and linear correlation cannot be fitted once the marginals are fixed. A more serious problem may be, as noted in [39], that sharing the same parameter  $\nu$  on all the marginal processes puts a strict restriction on the joint process. It may cause great difficulty in the joint calibration to option prices on the marginals.

To allow the dependence built on arbitrary marginal VG processes, Semeraro [54], Luciano and Semeraro [40] studied the multivariate subordination to multivariate Brownian motions. The general model proposed in these papers uses the following marginal processes:

$$X_i(t) = \theta_i G_i(t) + \sigma_i W_i(G_i(t)), i = 1..n \quad (2.3)$$

where  $W_1, \dots, W_n$  are independent Brownian motions and  $G(t) = (G_1(t), \dots, G_n(t))$  is a multivariate subordinator with the following components:

$$G_i(t) = Y_i(t) + a_i Z(t) \quad (2.4)$$

where,  $Y_i(t)$  and  $Z(t)$  are independent gamma processes.

The correlation of  $X_i, X_j$  is then:

$$\text{corr}(X_i, X_j) = \frac{a_i a_j \theta_i \theta_j \text{Var}(Z(t))}{\sqrt{\text{Var}(X_i(t)) \text{Var}(X_j(t))}} \quad (2.5)$$

Luciano and Semeraro built on this formulation in order to extend it to other time changed Brownian motions, like the NIG process and the CGMY process. Their model captures the case of full independence, when the subordinators are all independent. The correlation can be fitted by choosing the parameters of the common

component of the subordinator. However, the closed-form joint characteristic function, which plays a critical role in option pricing and parameter estimation, can only be found in the case of independent Brownian motions. With independent Brownian motions, the dependence mainly comes from the drift part and is sometimes too weak for financial modeling purpose.

Eberlein and Madan [17] worked on the model to correlate Lévy processes by time changing multivariate Brownian motions by independent gamma processes. This model can be considered as a special case of Semeraro's multivariate subordination models. By matching the sample correlation with the theoretical one, the implied correlation among Brownian motions can be estimated quickly. They then tested the model on performance of portfolio returns.

Recently, a totally different track of modeling dependence using Lévy processes is proposed by Kallen and Tankov [35], Tankov [59]. Analogous to the idea of copula, Tankov introduced the Lévy copula which provides the connection between the joint Lévy measure and its marginal Lévy measures. It separates the marginal Lévy measure from the dependence structure of marginal jumps. It is a natural way to build multi-dimensional Lévy processes since Lévy copula guarantees that the resulting process is a Lévy process. Despite the elegant theory of the Lévy, applying a Lévy copula to the financial data is so far still a difficult problem. Both estimation and simulation can be numerically heavy, and we refer readers to [12] [35] for more details and progress on the Lévy copula.

## 2.2 The Multivariate Variance Gamma Process

In this thesis, we introduce a new multi-variate VG process with the following nice features:

- It is a multidimensional Lévy process with arbitrary VG marginal processes and flexible dependence structure.
- It is easy to construct and simulate. The joint characteristic function can be derived in a closed form.
- It can be easily applied to other Lévy processes which are time-changed Brownian motions.
- It fits the empirical joint returns better compared with other popular models.

### 2.2.1 Definition and Properties

A VG process  $\text{VG}(\theta, \sigma, \nu)$  can be considered as a Brownian motion  $\theta t + \sigma B_t$  time-changed by a gamma process  $\Gamma(t; 1, \nu)$ . Here, a gamma process  $\Gamma(t; 1, \nu)$  with unit mean rate and variance rate  $\nu$  has independent gamma increments. The VG process is one of most popular models in modeling financial asset returns. The additional parameters in the drift of Brownian motion and volatility of time change provide control over the skewness and kurtosis of the return distribution, which makes it more flexible than the classical Black-Scholes model in modeling asset returns.

We construct our multi-variate VG process given arbitrary VG marginal processes. For simplicity, we first consider the two-dimensional case. We reparameterize the parameters in the marginal VG process as follows:  $VG(\theta, \sigma, \nu) = vg(a, b, c)$  where,  $a = \theta\nu, b^2 = \sigma^2\nu, c = \frac{1}{\nu}$ . Then the characteristic function can be written as

$$\Phi_{VG(t)}(u) = \left( \frac{1}{1 - iau + (b/2)u^2} \right)^{tc} \quad (2.6)$$

To interpret the parameters in this new parametrization, we start with a drifted Brownian motion  $at + bB_t$  with mean rate  $a$  and variance rate  $b^2$ . The subordinating gamma process  $\Gamma(t; c, c)$  has mean rate  $c$  and variance rate  $c$ . Because of the scaling property of the gamma processes, this particular setting of the subordinating gamma process does not put any restrictions on the generating VG process. We can derive the characteristic function of this VG process in terms of  $(a, b, c)$  exactly as (2.4).

Now we have the following property: For two independent VG processes  $vg(a, b, c_1)$  and  $vg(a, b, c_2)$ ,

$$vg(a, b, c_1) + vg(a, b, c_2) \stackrel{D}{=} vg(a, b, c_1 + c_2)$$

The property indicates the sum of two independent VG processes with the same parameters  $a, b$  is still a VG process. The result stems from the addition property of the gamma distribution and can be easily verified by comparing their characteristic functions. Now for an arbitrary VG process we can decompose it into two independent VG components. By correlating one of them using a common time change, we derive the following result:

**Proposition 2.1.** *Given two marginal VG processes  $X_1 \sim VG(\theta_1, \sigma_1, \nu_1)$  and  $X_2 \sim$*

$VG(\theta_2, \sigma_2, \nu_2)$ , we can build the dependence with two additional parameters  $\rho$  and  $\nu_0$  as follows:

$$X_1 = A_1 + Y \quad (2.7)$$

$$X_2 = A_2 + Z \quad (2.8)$$

$$A_1 \sim VG\left(\theta_1 \frac{\nu_1}{\nu_0}, \sigma_1 \sqrt{\frac{\nu_1}{\nu_0}}, \nu_0\right), Y \sim VG\left(\theta_1 \left(1 - \frac{\nu_1}{\nu_0}\right), \sigma_1 \sqrt{1 - \frac{\nu_1}{\nu_0}}, \frac{1}{\frac{1}{\nu_1} - \frac{1}{\nu_0}}\right) \quad (2.9)$$

$$A_2 \sim VG\left(\theta_2 \frac{\nu_2}{\nu_0}, \sigma_2 \sqrt{\frac{\nu_2}{\nu_0}}, \nu_0\right), Z \sim VG\left(\theta_2 \left(1 - \frac{\nu_2}{\nu_0}\right), \sigma_2 \sqrt{1 - \frac{\nu_2}{\nu_0}}, \frac{1}{\frac{1}{\nu_2} - \frac{1}{\nu_0}}\right) \quad (2.10)$$

where,  $(A_1, A_2)$ ,  $Y$  and  $Z$  are independent.  $(A_1, A_2)$  is a 2-dimensional  $\rho$ -correlated Brownian motion with associated mean and covariance matrix subordinated by a common gamma process  $\Gamma(t; 1, \nu_0)$ . The parameter  $\nu_0$  satisfies  $\nu_0 \geq \max(\nu_1, \nu_2)$ .

The two-dimensional process is constructed by decomposing marginal processes into two parts. We correlate the parts with common time-change parameter by subordinating a two-dimensional Brownian motion and leave the other parts independent. As a process in modeling dependence of asset returns, this setting has strong economic intuitions. The dependent part  $(A_1, A_2)$  stands for a systematic factor or a global factor which governs the big co-movements of individual assets, while the independent part represents the individual factor of each asset. From now on, we usually refer  $(A_1, A_2)$  as the systematic part and  $(Y, Z)$  as the independent part of the process.

The new two-dimensional process we introduce here has independent and stationary increments. The distribution of  $(X_1, X_2)$  at any time  $t$  is infinitely divisible.

Since the process can be decomposed into two independent parts with known characteristic functions, we can derive the closed-form joint characteristic function. This is a very nice aspect of this process in modeling asset returns, as characteristic functions play a critical role in the option pricing.

**Proposition 2.2.** *The joint characteristic function of the two-dimensional variance gamma process is:*

$$\begin{aligned} \phi_{X_1(t), X_2(t)}(u_1, u_2) &= \left( \frac{1}{1 - iu_1\theta_1\nu_1 - iu_2\theta_2\nu_2 + u^T\Sigma u/2} \right)^{\frac{t}{\nu_0}} \\ &\quad \cdot \left( \frac{1}{1 - i\theta_1 u_1\nu_1 + (\sigma_1^2\nu_1/2)u_1^2} \right)^{\frac{t}{\nu_1} - \frac{t}{\nu_0}} \\ &\quad \cdot \left( \frac{1}{1 - i\theta_2 u_2\nu_2 + (\sigma_2^2\nu_2/2)u_2^2} \right)^{\frac{t}{\nu_2} - \frac{t}{\nu_0}} \end{aligned} \quad (2.11)$$

$$\text{where } u = (u_1, u_2)^T, \Sigma = \begin{pmatrix} \sigma_1^2\nu_1 & \sigma_1\sigma_2\rho\sqrt{\nu_1\nu_2} \\ \sigma_1\sigma_2\rho\sqrt{\nu_1\nu_2} & \sigma_2^2\nu_2 \end{pmatrix}$$

*Proof.* It suffices to derive the joint characteristic function of the systematic part  $(A_1, A_2)$  of the process. Note this process can be considered as a two-dimensional Brownian motion subordinated by a common gamma process  $\Gamma_t$ . We compute the characteristic function through conditioning on the gamma time change. From (2.7) and (2.8), we get:

$$\begin{aligned} \phi_{A_1(t), A_2(t)}(u_1, u_2) &= E(\exp(i(u_1A_1 + u_2A_2))) \\ &= E(E(\exp(i(u_1A_1 + u_2A_2)) | \gamma_t = z)) \\ &= E(\exp(iu_1\theta_1\frac{\nu_1}{\nu_0}z + iu_2\theta_2\frac{\nu_1}{\nu_0}z + u^T\Sigma uz/2\nu_0)) \\ &= \left( \frac{1}{1 - iu_1\theta_1\nu_1 - iu_2\theta_2\nu_2 + u^T\Sigma u/2} \right)^{\frac{t}{\nu_0}} \end{aligned}$$

Where,  $\Sigma = \begin{pmatrix} \sigma_1^2 \nu_1 & \sigma_1 \sigma_2 \rho \sqrt{\nu_1 \nu_2} \\ \sigma_1 \sigma_2 \rho \sqrt{\nu_1 \nu_2} & \sigma_2^2 \nu_2 \end{pmatrix}$ .

The joint characteristic function of  $(X_1, X_2)$  is just the product of the two parts by independence. □

To derive the Lévy measure of the process, it suffices to find the Lévy measure of the systematic part  $(A_1, A_2)$ , since the independent part is the same as the single dimensional VG process and the sum of two independent Lévy processes has the Lévy measure as the sum of two Lévy measures. We use the result for subordination of a Lévy process (see [12] page 108 for a complete proof):

The Lévy measure  $\rho^S$  for  $S_t$ , which can be written as a two-dimensional Brownian  $W_t$  with drift  $\theta$  and volatility rate  $\Sigma$  time-changed a common subordinator  $\Gamma_t(1, \nu)$ , is given by:

$$\rho^S(B) = \int_0^\infty p_s^W(B) \rho(ds), \forall B \in \mathcal{B}(R^2).$$

where  $\rho(ds)$  is the Lévy measure for the subordinator and  $p_s^W$  is the probability distribution of  $W_s$ .

In the VG case, we have the Lévy measure of the gamma process with unit mean rate and variance rate  $\nu$  as:

$$\rho(ds) = \frac{1}{\nu} \frac{e^{-\frac{s}{\nu}}}{s} ds.$$

Then the Lévy density is written in the following integral form:

$$\rho^Y(dx) = \left( \int_0^\infty f_s(x) \frac{1}{\nu} \frac{e^{-\frac{s}{\nu}}}{s} ds \right) dx.$$

where  $f_s(x)$  is the probability density function of the multivariate normal distribution with mean  $\theta s$  and variance matrix  $\Sigma s$ .

We need the following identity:

$$e^{-|x|^a} = \int_0^{+\infty} \frac{a}{\sqrt{2\pi y^3}} e^{-\frac{a^2}{2y} - \frac{x^2}{2}y} dy$$

Then the Lévy density is:

$$\begin{aligned} m_Y(x) &= \int_0^\infty \frac{1}{(2\pi)^{n/2} \sqrt{|\Sigma|} s} \exp\left(-\frac{(x - \theta s)^T \Sigma^{-1} (x - \theta s)}{2s}\right) \frac{1}{\nu} \frac{e^{-\frac{s}{\nu}}}{s} ds \\ &= \frac{\exp(\theta^T \Sigma^{-1} x)}{\nu (2\pi)^{(n-1)/2} \sqrt{|\Sigma|}} \int_0^\infty \frac{1}{\sqrt{2\pi s^3}} \exp\left(-\frac{x^T \Sigma^{-1} x}{2s} - \frac{(\theta^T \Sigma^{-1} \theta + \frac{2}{\nu})s}{2}\right) ds \\ &= \frac{\exp(\theta^T \Sigma^{-1} x)}{\nu (2\pi)^{(n-1)/2} \sqrt{|\Sigma|} \sqrt{x^T \Sigma^{-1} x}} \exp\left(-\sqrt{\left(\theta^T \Sigma^{-1} \theta + \frac{2}{\nu}\right)(x^T \Sigma^{-1} x)}\right) \end{aligned}$$

The Lévy density for the two-dimensional multivariate VG process is thus,

$$m_A(x_1, x_2) + \rho_Y(x_1) + \rho_Z(x_2)$$

where  $m_A$  is given above with VG parameters described in (2.9)-(2.10),  $\rho_Y$  and  $\rho_Z$  are Lévy measures for single VG processes with VG parameters in (2.9)-(2.10).

To see the flexibility of the dependence structure, we can analyze the impact of two dependence parameters  $\nu_0$  and  $\rho$ . As  $\nu_0 \rightarrow \infty$ ,  $X_1$  and  $X_2$  become independent VG processes. When  $\nu_0 = \nu_1 = \nu_2$  and  $\rho = 1$ ,  $X_1$  and  $X_2$  are fully dependent. In the general case  $\nu_1 = \nu_2$ ,  $X_1$  and  $X_2$  achieve the maximal dependence when  $\nu_0 = \max(\nu_1, \nu_2)$  and  $\rho = 1$ .

**Proposition 2.3.** *The linear dependence between  $X_1$  and  $X_2$  at any time  $t$  is:*

$$\text{Corr}(X_1, X_2) = \frac{\theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0} + \sigma_1 \sigma_2 \rho \frac{\sqrt{\nu_1 \nu_2}}{\nu_0}}{\sqrt{\theta_1^2 \nu_1 + \sigma_1^2} \sqrt{\theta_2^2 \nu_2 + \sigma_2^2}} \quad (2.12)$$

*Proof.* To compute the correlation between two random variable, we first derive the covariance  $Cov(X_1, X_2)$  at time  $t$ . Note here  $(X_1, X_2)$  denotes the random variables of the process at time  $t$ , though we did not write  $t$  explicitly. We will see the correlation is independent with the time horizon  $t$ . For easy use of the notations, we denote the systematic part  $(A_1, A_2)$  by  $(\theta_1 \frac{\nu_1}{\nu_0} t + \sigma_1 \sqrt{\frac{\nu_1}{\nu_0}} W_t^1, \theta_2 \frac{\nu_2}{\nu_0} t + \sigma_2 \sqrt{\frac{\nu_2}{\nu_0}} W_t^2)$  time changed by  $\gamma_t$ . The correlation of  $W_1$  and  $W_2$  is  $\rho$ .

$$\begin{aligned}
Cov(X_1, X_2) &= E(X_1 X_2) - E(X_1)E(X_2) \\
&= E((A_1 + Y)(A_2 + Z)) - E(A_1 + Y)E(A_2 + Z) \\
&= E(A_1 A_2) - E(A_1)E(A_2) \\
&= E(E(A_1 A_2 | \gamma_t = z)) - \theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0^2} t^2 \\
&= E(\theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0^2} z^2 + \sigma_1 \sigma_2 \rho \frac{\sqrt{\nu_1 \nu_2}}{\nu_0} z) - \theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0^2} t^2 \\
&= \theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0^2} (t^2 + \nu_0 t) + \sigma_1 \sigma_2 \rho \frac{\sqrt{\nu_1 \nu_2}}{\nu_0} t - \theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0^2} t^2 \\
&= (\theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0} + \sigma_1 \sigma_2 \rho \frac{\sqrt{\nu_1 \nu_2}}{\nu_0}) t
\end{aligned}$$

Therefore, the correlation of  $X_1$  and  $X_2$  is

$$Corr(X_1, X_2) = \frac{\theta_1 \theta_2 \frac{\nu_1 \nu_2}{\nu_0} + \sigma_1 \sigma_2 \rho \frac{\sqrt{\nu_1 \nu_2}}{\nu_0}}{\sqrt{\theta_1^2 \nu_1 + \sigma_1^2} \sqrt{\theta_2^2 \nu_2 + \sigma_2^2}}$$

□

## 2.2.2 Extensions to the High-Dimensional Case

The process can be extended easily to the case of a higher dimension. We can write the  $n$ -dimensional VG process with one systematic part as follows:

$$X_i(t) = A_i(t) + Y_i(t), i = 1..n \quad (2.13)$$

where  $A_i(t) = W_i(\Gamma(t))$  are time-changed  $n$ -dimensional Brownian motion by a common gamma process  $\Gamma(t)$  with parameter  $\nu_0$ .  $Y_i(t), i = 1..n$  are the independent VG parts needed to match the marginal process. The parameters for marginal processes can be arbitrary, but the parameters for dependence need to satisfy the condition  $\nu_0 \geq \max(\nu_1, \nu_2, \dots, \nu_n)$ .

The pairwise correlation between  $X_i$  and  $X_j$  is:

$$\text{Corr}(X_i, X_j) = \frac{\theta_i \theta_j \frac{\nu_i \nu_j}{\nu_0} + \sigma_i \sigma_j \rho \frac{\sqrt{\nu_i \nu_j}}{\nu_0}}{\sqrt{\theta_i^2 \nu_i + \sigma_i^2} \sqrt{\theta_j^2 \nu_j + \sigma_j^2}} \quad (2.14)$$

### 2.3 Estimation

Estimation approaches are based on finding model parameters in order to fit the observed returns. One of the widely used methods for estimating a parametric model is the maximum likelihood estimation (MLE) method. The maximum likelihood estimators  $\hat{\theta}$  are known to be asymptotically unbiased and efficient. Given a functional form  $f(x; \theta)$  for the density of the log-returns  $r_i$ , the idea of MLE is to choose the model parameters so that the likelihood of the observed data in the model is maximized:

$$\max_{\theta} \prod_{i=1}^N f(r_i; \theta)$$

It is thus equivalent to maximizing the sum of the log-likelihood functions:

$$l(\theta) = \sum_{t=1}^N \ln f(r_i; \theta).$$

The maximum likelihood estimators can be written as

$$\hat{\theta} = \operatorname{argmax}_{\theta} \sum_{t=1}^N \ln f(r_t; \theta).$$

Unlike the normal distribution or other distribution families whose maximum likelihood estimators can be solved in closed form, for most of the Lévy processes, the difficulty in implementing this method lies in the fact that the density function is usually not known analytically. Therefore, the computation must be done numerically and the optimization is usually computed by a gradient descent based algorithm. For one dimensional VG process, the density function can be computed by inverting the characteristic function using FFT to avoid computation of the Bessel functions. For large datasets, it is also usually a good idea to bin the data into small intervals to simplify the computation. For example, we bin the log return data into one hundred equally spaced intervals within five standard deviations of the return distribution. The MLE procedure can be done fast and efficiently.

For a general high-dimensional estimation problem, the MLE method works in principle, but may lead to a complicated numerical computation problem especially when the analytic joint density function is not known. High-dimensional FFT or numerical integration are numerically demanding to compute which often leads to inaccurate results as well. Meanwhile, the optimization part of the procedure often has too many parameters involved. Searching for the optimal value in a high dimensional space is very sensitive to the initial point of your optimization algorithm. Other estimation methods such as generalized methods of moments (GMM) [22], methods based on empirical characteristic function [32, 56] or Markov chain Monte

Carlo (MCMC) [38] are popular in many econometrics contexts. Our construction of the multivariate VG model circumvents this problem by using a two-stage method. Since adjusting the dependence parameters will not affect the marginal distribution, we can introduce a two-step procedure which is relatively easy to implement.

First we estimate the marginal VG parameters by MLE as described above on each marginal distribution. This makes sure the more important and reliable information on margins is incorporated in the process. Then with fixed marginal parameters, we estimate the dependence parameter  $\nu_0$  and the correlation matrix  $\Sigma_0$ . It turns out that the parameter  $\nu_0$  is hard to estimate out from the model given that the likelihood function with respect to  $\nu_0$  is almost flat. This is because a full rank- $n$  Brownian motion has  $n$  independent factors and we are actually using  $n + 1$  factors to estimate out  $n$  variables with the independent factor. Hence, we suggest that one chooses  $\nu_0 = \max_i(\nu_i)$  for most problems in applications and estimates the matrix  $\Sigma_0$  by matching the pairwise correlations with sample correlations. Since the pairwise correlations are given in (2.14),  $\Sigma_{ij}$  in the correlation matrix is given by

$$\Sigma_{ij} = \frac{\tilde{\rho}_{ij} \sqrt{\theta_i^2 \nu_i + \sigma_i^2} \sqrt{\theta_j^2 \nu_j + \sigma_j^2} - \theta_i \theta_j \frac{\nu_i \nu_j}{\nu_0}}{\sigma_i \sigma_j \frac{\sqrt{\nu_i \nu_j}}{\nu_0}} \quad (2.15)$$

where  $\tilde{\rho}_{ij}$  is the sample correlation between stock  $i$  and  $j$ .

In chapter 3 and section 4.6.2, we show more results on estimating the process with this method and the performance on stock portfolio returns compared with other popular methods.

## 2.4 Simulation

Monte Carlo simulation is widely used in financial engineering when an analytic solution for one problem is not available. Its applications range from pricing, hedging, risk managing, etc. Compared with other methodologies in option pricing, Monte Carlo simulation is straightforward and easy to implement. Though Monte Carlo simulation is often considered as the last choice in pricing derivatives, for derivatives with very complicated payoff structures, such as mountain range derivatives or hybrid products, it is often the only feasible approach for pricing purpose. Therefore, to derive an easy and effective simulation scheme for our new multi-variate VG process is very important.

For a single dimensional VG process, there are several efficient methods including sequential sampling and bridge sampling techniques for constructing sample paths of a VG process. Sequential sampling based methods rely on different representations of VG process presented in the previous chapter. One is to write VG process as time changed Brownian motion and the other is to decompose one VG process into two independent gamma processes. Bridge sampling based methods samples the end of the path first, and then fills in the rest of the path as needed. This technique is based on the theory of Brownian bridge and gamma bridge for time-changed Brownian motions. For the details of simulating VG process in one-dimensional case, we refer readers to [21, 25].

To simulate the multivariate VG model, one can again use sequential sampling and bridge sampling techniques. Though there is no difference between statistical

properties of random samples generated by these two techniques, the bridge sampling method has potential advantage when it is used with variance reduction techniques and low-discrepancy methods. See [51] for a detailed discussion. By considering systematic part of the process as time-changed Brownian motions, we here first briefly describe the procedure of simulating the two-dimensional VG process using sequential sampling. It extends to any higher dimensional case naturally. We use the same parametrization as (2.7)-(2.10), the simulation scheme is then:

### Sequential sampling of Multivariate VG process:

**Input:** parameters  $\theta_{1,2}, \sigma_{1,2}, \nu_{1,2}, \rho, \nu_0$ ; time spacing  $\Delta t_1, \dots, \Delta t_n$  with  $\sum_{i=1}^N t_i = T$ .

**Initialization:** Set  $X_1(0) = X_2(0) = 0$ .

**Loop** from  $i=1$  to  $N$ :

1. Generate  $\Delta G_i^0 \sim \Gamma(\Delta t_i/\nu_0, \nu_0), \Delta G_i^1 \sim \Gamma(\frac{\Delta t_i}{\frac{1}{\nu_1}-1}, \frac{1}{\frac{1}{\nu_1}-1}), \Delta G_i^2 \sim \Gamma(\frac{\Delta t_i}{\frac{1}{\nu_2}-1}, \frac{1}{\frac{1}{\nu_2}-1})$ .
2. Generate multivariate normal vector  $\Delta W_i \sim N(0, \Sigma)$ , where  $\Sigma$  is given in (2.1).
3. Generate normal variables  $Z_{1i} \sim N(0, \sigma_1 \sqrt{1 - \frac{\nu_1}{\nu_0}})$  and  $Z_{2i} \sim N(0, \sigma_2 \sqrt{1 - \frac{\nu_2}{\nu_0}})$ .
4. Return:

$$X_1(t_{i+1}) = X_1(t_i) + \frac{\theta_1 \nu_1}{\nu_0} G_i^0 + \sqrt{G_i^0} \Delta W_{i1} + \theta_1 (1 - \nu_1/\nu_0) G_i^1 + \sqrt{G_i^1} Z_{1i}$$

$$X_2(t_{i+1}) = X_2(t_i) + \frac{\theta_2 \nu_2}{\nu_0} G_i^0 + \sqrt{G_i^0} \Delta W_{i2} + \theta_2 (1 - \nu_2/\nu_0) G_i^2 + \sqrt{G_i^2} Z_{2i}$$

For higher dimensional simulation scheme, it is the same as the two-dimensional case following the model (2.7)-(2.10). Given a large correlation matrix, the simplest way to generate dependent normal distributed vector in step 2 is to use the Cholesky decomposition on the correlation matrix  $\Sigma$ . Let  $\Sigma = LL^T$ . If  $v$  is a  $n$ -dimensional

independent normal random vector, then  $Lv$  is a normal random vector with correlation matrix  $\Sigma$ .

Bridge sampling can also be used to simulate the multivariate VG model. The main idea of bridge sampling is to find the conditional distribution of a stochastic process  $X_t$  at time  $t \in (0, T)$ , given  $X_0$  and  $X_T$ . That leads to the theory of Brownian bridge and gamma bridge.

**Proposition 2.4. *Brownian Bridge:*** *For a Brownian motion  $B_t$  with drift  $\mu$  and volatility  $\sigma$ , the distribution of  $B_t, 0 \leq t \leq T$  conditional on  $B_0$  and  $B_T$  is:*

$$B_t \sim \mathcal{N}\left(\frac{T-t}{T}B_0 + \frac{t}{T}B_T, \frac{t(T-t)}{T}\sigma^2\right) \quad (2.16)$$

***Gamma Bridge:*** *For a gamma process  $\gamma_t$  with mean rate  $\mu$  and variance rate  $\nu$ , the distribution of  $\gamma_t, 0 \leq t \leq T$  conditional on  $\gamma_0$  and  $\gamma_T$  is written as:*

$$\gamma_t \sim \gamma_0 + (\gamma_T - \gamma_0)Y \quad (2.17)$$

where  $Y \sim \beta(t/\nu, (T-t)/\nu)$  is beta distribution with mean  $\frac{t}{T}$  and variance  $\frac{t(T-t)\nu}{T^2(T+\nu)}$ .

For multivariate Brownian bridge, it is straightforward from the one-dimensional case. We may construct independent Brownian bridges on each marginal and correlate them through linear transformation. Namely for a multivariate Brownian motion  $X_t$  with mean  $\mu$  and covariance matrix  $\Sigma$ , it can be written as

$$X_t \sim \mu t + LZ_t$$

where  $LL^T = \Sigma$  and  $Z_t$  are independent normal variables.

In our multivariate VG mode, the independent components can be simulated via bridge sampling as one dimensional VG processes separately. Thus, it suffices to

show how to simulate the systematic part of the process using the Brownian bridge. In the following we describe a simulation scheme in an informal way to explain the procedure of simulating systematic part  $X_t$ . It can be written as  $\theta G_t + W_{G_t}$  where  $W_t$  is a multivariate Brownian motion with covariance matrix  $\Sigma$  and  $\theta$  is the drift vector term.

1. Generate a gamma bridge  $\gamma_{t_i}$  on the grids  $t_i, i = 0 \dots 2^N$  ( $t_0 = 0$  and  $t_{2^N} = T$ ).
2. Generate  $n$  independent Brownian bridges  $Z_i^j, j = 1 \dots n$  on the same time grids.
3. Decompose the covariance matrix by Cholesky decomposition  $\Sigma = LL^T$ .
4. For  $i = 1$  to  $2^N$ , return  $X_i = (\gamma_{t_i} - \gamma_{t_{i-1}})\theta + \sqrt{\gamma_{t_i} - \gamma_{t_{i-1}}}LZ_i$ .

In section 2.6, we give the simulated paths the of a pair of stock prices based on the multivariate VG process.

## 2.5 The Multivariate NIG Process

The multivariate VG case can be extended to the normal inverse gaussian (NIG) process and other time-changed Brownian motions. The characteristic function for  $NIG(\alpha, \beta, \delta)$  is:

$$\Phi_{NIG}(u) = \exp(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}))$$

We may decompose the marginal processes into two independent pieces  $NIG(\alpha, \beta, \delta_1)$  and  $NIG(\alpha, \beta, \delta_2)$  satisfying the condition  $\delta = \delta_1 + \delta_2$ . By correlating the ones with

the same time change parameter to be subordinated Brownian motions, one can build the dependent part of the processes. The multivariate NIG model shares similar properties to the VG case and we only present the basic construction of the process here.

**Theorem 2.5.** *Given two marginal NIG processes  $X_1 \sim NIG(\alpha_1, \beta_1, \delta_1)$  and  $X_2 \sim NIG(\alpha_2, \beta_2, \delta_2)$ , we can build the dependence with two additional parameters  $\delta_0$  and  $\rho$  as follows:*

$$X_1 = A_1 + Y \tag{2.18}$$

$$X_2 = A_2 + Z \tag{2.19}$$

$$A_1 \sim NIG(\alpha_1, \beta_1, \delta_0), Y \sim NIG(\alpha_1, \beta_1, \delta_1 - \delta_0) \tag{2.20}$$

$$A_2 \sim NIG(\alpha_2, \beta_2, \delta_0), Z \sim NIG(\alpha_2, \beta_2, \delta_2 - \delta_0) \tag{2.21}$$

where,  $(A_1, A_2)$ ,  $Y$  and  $Z$  are independent.  $(A_1, A_2)$  is a 2-dimensional  $\rho$ -correlated Brownian Motion with associated mean and covariance matrix subordinated by a common inverse gamma process  $I(t; 1, \delta_0 \sqrt{\alpha^2 - \beta^2})$ . The parameter  $\delta_0$  satisfies  $\delta_0 \leq \min(\delta_1, \delta_2)$ .

## 2.6 The Risk Neutral Multivariate Stock Price Model

In this section, we study a risk neutral multivariate stock price model based on the multivariate VG process. Under the risk neutral measure, the discounted stock prices will be martingales by non-arbitrage pricing theory. If we assume  $n$  assets  $S_1 \dots S_n$  are driven by exponential multivariate VG processes, we have to adjust the

drift term so that the discounted marginal processes will be martingales. We can write:

$$S_1 = \exp(rt + X_1(t) - w_1(t)); X_1 = A_1 + Z_1$$

$$S_2 = \exp(rt + X_2(t) - w_2(t)); X_2 = A_2 + Z_2$$

...

$$S_n = \exp(rt + X_n(t) - w_n(t)); X_n = A_n + Z_n$$

Where,  $w_i$  are compensators for exponential VG processes, which can be written as  $w_i = \frac{1}{\nu_i} \log(1 - \theta_i \nu_i - \sigma_i^2 \nu_i / 2); i = 1 \dots n$ ,  $A_i$  and  $Z_i$  are systematic and independent factors, respectively.

The following figures show the two-dimensional simulation paths for multivariate VG stock prices with different parameters. The first one has a relatively stronger correlation as the dependent component of the process plays a major role. The correlation in the second one is weak and the sample paths behave almost independently with each other.

Figure 2.1: Simulation path I

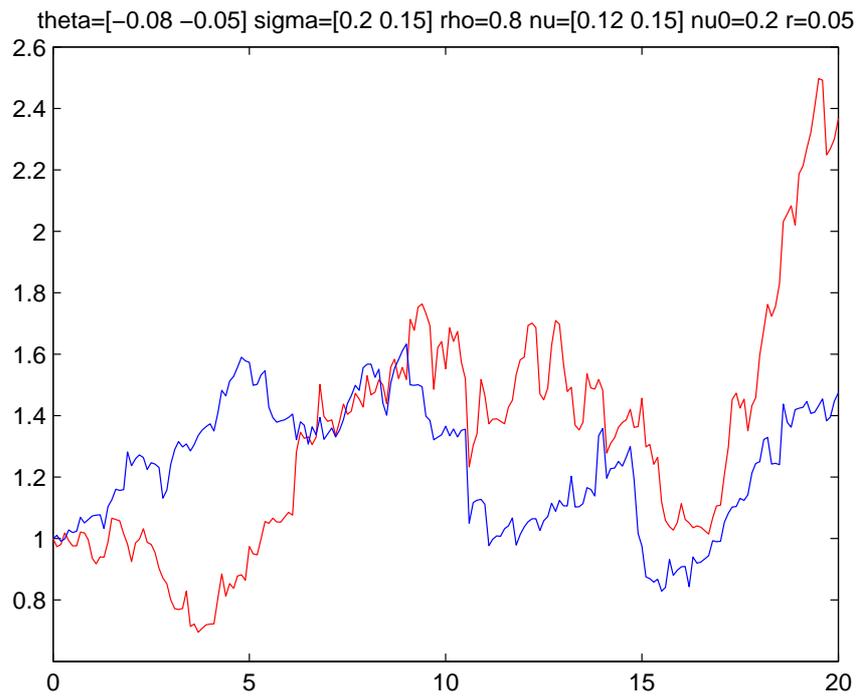
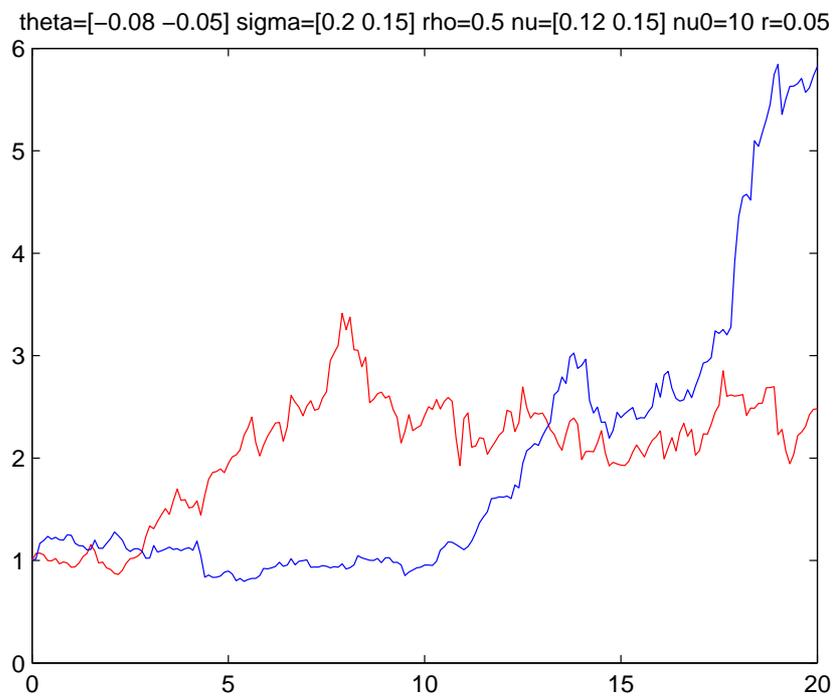


Figure 2.2: Simulation path II



## Chapter 3

### Dependence Modeling with Multivariate Variance Gamma Model

#### 3.1 Overview of Dependence Modeling

Modeling the dependence structure among a number of financial assets is an important topic in mathematical finance. It plays a critical role in pricing and hedging multi-asset derivatives, controlling of risk clustering and managing risks with a portfolio of assets. In the standard framework of Gaussian distribution, the multivariate distribution takes the form of an exponential of a quadratic form  $-X\Sigma^{-1}X$ , where  $X$  is the vector of asset returns and  $\Sigma$  is their covariance matrix. The beauty and simplicity of the Gaussian case is that the symmetric covariance matrix determines the whole joint dynamics of the random vector. Risk is then completely embodied by the variance of the portfolio return, which is the basis of Markovitz portfolio theory [47]. However, the reason for the wide use of the Gaussian distribution and Brownian motion is not because it is a good model for financial data in many cases, but due to its tractability. Ever since Fama [20], it is well known that asset returns do not follow a normal distribution. The dependence structure arising from a multi-dimensional Brownian motion is also too simple to capture the sophisticated comovement dynamics of several assets in the financial market. For instance, multivariate Brownian motions generate very limited symmetric dependence structure and zero tail dependence, while many empirical studies indicate that one

needs models with more flexible dependence structures to explain the joint dynamics of multiple price processes in the financial market. Here, we review one important stream of dependence modeling approach in the literature: copula methods.

### 3.1.1 Copula Methods

The concept of copula was introduced by Sklar [57]. A copula is used as a general way of formulating a multivariate distribution such that various general types of dependence can be represented. It is a function which generates a joint distribution from several marginal distributions. Hence, it makes possible to separate the dependence structure from the marginal distributions. By using a copula, one can easily construct a multivariate distribution with any marginal distribution and any pre-specified dependence structure. This flexibility gains the copula models, which appear as a natural modeling device in a non-Gaussian world, widely popularity in financial applications. A copula was first introduced to model default correlation in credit derivatives side by Li [37]. Later, copula methods were widely used in risk management [19], in option pricing [14], in credit derivative pricing such as CDS and CDO.

**Definition 3.1.** *A copula is a function  $C: [0, 1]^n \rightarrow [0, 1]$  such that:*

- $C(u) = 0$  whenever  $u \in [0, 1]^n$  at least one component equal to 0.
- $C(u) = u_i$  whenever  $u \in [0, 1]^n$  has all the components equal to 1 except the  $i$ -th one, which is equal to  $u_i$ .
- $C(u)$  is  $n$ -increasing. (Any  $n$ -dimensional distribution function is  $n$ -increasing)

The approach of formulating a multivariate distribution using a copula is based on the idea that a simple transformation can be made of each marginal variable in such a way that each transformed marginal variable has a uniform distribution. The following elegant result shows that the study of the dependence of random variables can be performed independently of the behavior of the marginal distributions.

**Theorem 3.2. (Sklar)** *Let  $X$  and  $Y$  be random variables with joint distribution function  $H$  and marginal distribution functions  $F$  and  $G$ , respectively. Then there exists a copula  $C$  such that  $H(x, y) = C(F(x), G(y))$  for all  $x, y$  in  $R$ . Conversely, if  $C$  is a copula and  $F$  and  $G$  are distribution functions, then the function  $H(x, y) = C(F(x), G(y))$  is a joint distribution function with margins  $F$  and  $G$ .*

Copulas of increasing or decreasing transforms of continuous random variables are easily written in terms of the copula of these variables. In particular, copulas are invariant with respect to increasing transforms.

**Theorem 3.3.** *Let  $X_1, X_2$  be continuous random variables with marginal distribution functions  $F_1, F_2$  and copula  $C$ . If  $I_1, I_2$  are two increasing transformations, the random variables  $I_1(X_1), I_2(X_2)$ , which have marginal distribution functions  $H_1 = F_1(I_1^{-1}), H_2 = F_2(I_2^{-1})$  and joint one*

$$H(x_1, x_2) = Pr(I_1(X_1) \leq x_1, I_2(X_2) \leq x_2)$$

*has copula  $C$  too:*

$$H(x_1, x_2) = C(H_1(x_1), H_2(x_2))$$

*This is also true for more than two random variables.*

The above theorem provides a powerful way of studying scale-invariant measures of associations. It is also a natural starting point for the construction of multivariate distributions and provides the theoretical justification of the method of determination of multivariate distributions that we use below.

To describe the multivariate distribution, we need the measures of the dependence between random variables. The most commonly used one in practice is linear correlation (Pearson's correlation). Although correlation works fine for elliptical distributions, it turns out to be a bad measure when it comes to non-elliptical margins. Thus, there are other copula-based measures of dependence including concordance measures and tail dependence. Concordance is used to describe the global trend. Two popular concordance measures are Kendall's  $\tau$  and Spearman's  $\rho$ .

Tail dependence describes asymptotical dependence between extreme events. It thus studies the dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a joint distribution. The rigorous definition of tail dependence reads as follows:

**Definition 3.4.** *Let  $(X, Y)$  be a random vector with marginal distributions  $F$  and  $G$ . The lower tail dependence index is defined as:*

$$\lambda_L = \lim_{\nu \rightarrow 0^+} \mathbb{P}(Y \leq G^{-1}(\nu) | X \leq F^{-1}(\nu)),$$

*given the limit  $\lambda_L$  exists.*

*Similarly, the upper tail dependence index is defined as;*

$$\lambda_U = \lim_{\nu \rightarrow 1^-} \mathbb{P}(Y > G^{-1}(\nu) | X > F^{-1}(\nu)),$$

given the limit  $\lambda_L$  exists.  $X$  and  $Y$  are said to be asymptotically dependent in the lower(upper) tail if  $\lambda_L \in (0, 1](\lambda_U \in (0, 1])$ ;  $X$  and  $Y$  are said to be asymptotically independent in the lower(upper) tail if  $\lambda_L = 0(\lambda_U = 0)$ .

Popular examples of copula families applied in finance include the Gaussian copula, t copula and the Archimedean family of copulas. In the next section, we are going to discuss more details of the Gaussian copula and the method of implementing it. For more discussion of other copulas, we refer the readers to [14].

While the copula method provides a simple way of modeling dependence structure without specifying the marginal distribution, one big restriction in applying it is the difficulty of generating stochastic processes from random variables. Copula is constructed and estimated in random variable level at a certain time and it is not clear so far whether stochastic processes consistent with given random variables can be easily constructed.

### 3.1.2 Full-rank Gaussian Copula Method

One of the most popular copulas applied in finance is the Gaussian copula which is the copula implied by the multivariate Gaussian distribution. Therefore, the parameters in a  $n$ -dimensional Gaussian copula are just  $n(n - 1)/2$  entries in the symmetric correlation matrix. For a bivariate Gaussian copula, it only has one parameter  $\rho$ , which is commonly known as the correlation coefficient. It can be represented as:

$$C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) \quad (3.1)$$

where,  $u, v \in [0, 1]$ ,  $\Phi$  denotes the standard normal cumulative distribution function and  $\Phi_\rho$  stands for the standard bivariate normal cumulative distribution function with correlation  $\rho$ .

As one of the benchmark models in pricing structured credit derivatives, the methodology of applying the Gaussian copula to credit derivatives is said to be one of the reasons behind the global financial crisis of 2008. Despite it is tractable and simple, the fact that the Gaussian copula has no tail dependence leads to a serious misestimate of the tail event of a whole portfolio. Empirical studies in finance indicate that stock pairs from the same industry often has a strong tail dependence of large negative movements. For example, the big price drop for one stock usually coincides with the slump of the stock price in the same sector. However, the Gaussian copula is not able to capture this effect and thus always underestimates the probability of the big losses of an asset pool significantly. Unfortunately, the rare events of the huge loss misestimated by the model can cause disasters in the financial market. A popular article in wired magazine of Feb 2009 named the Gaussian copula as "The Formula That Killed Wall Street"

To test the Gaussian copula method with historical stock data and compare the model performance of explaining portfolio returns with our multivariate VG model, we here implement the full-rank Gaussian copula (FGC) approach which follows the idea of Malevergne and Sornette [45, 46].

Given historical returns on  $n$  different stocks, we first transform marginal returns  $x_i, i = 1 \dots n$  on stock  $i$  into the standard normal variables  $z_i, i = 1 \dots n$  by

using the following formula.

$$z_i = N^{-1}(F_i(x_i)) \quad (3.2)$$

Where  $F_i$  stands for the parametric distribution function estimated from the data (e.g. VG distribution),  $N$  is the cumulative distribution function for standard normal random variables.

Next, we estimate the correlation matrix  $\Sigma$  of  $z_i$  by using the sample correlation matrix. When the sample size is small or matrix dimensional is large, a sample covariance or correlation matrix may be not positive definite due to mere sampling fluctuation. In that case, we use Quasi-Newton methods of Qi & Sun [50] to find the closest correlation matrix.

Last, to derive the joint distribution of the original data series, one just needs to convert the estimated correlated normal variables  $z_i$  back to  $x_i$ .

$$x_i = F_i^{-1}(N(z_i)) \quad (3.3)$$

To derive the cumulative distribution function of the VG distribution in step one, one may invert the characteristic function by using FFT. Simple integration leads to the following proposition:

**Proposition 3.5.** *Let  $X$  be a random variable with characteristic function  $\phi(x)$  and let  $e^{-\alpha x}$  be the dampening factor.  $X$  has distribution function  $F(x)$ . Then, the cumulative distribution function  $F(x)$  is given by:*

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \frac{\phi(u + i\alpha)}{-iu + \alpha} du \quad (3.4)$$

It turns out the generated distribution above is equivalent to arbitrary marginal distributions glued by a full-rank Gaussian copula by the invariance principle of copula. The simulation is straightforward given randomly generated correlated Gaussian variables and formula given above. In the next section, we test this method on stock data to using goodness of fit under the full-rank Gaussian copula assumption.

### 3.2 Performance on Fitting Portfolio Returns

In this section, we implement the multivariate VG model to fit the joint returns of the stock prices. We compare the goodness of fit results with the full-rank Gaussian model and show the multivariate variance gamma model has a significant improvement over the Gaussian copula method.

Testing the joint statistical returns of several financial assets is often not easy to implement. One problem we face is that as the dimension of the distribution gets higher, the direct goodness of fit test on the joint density function becomes infeasible. Thus we instead move our attention to the test on the returns of the portfolios. After all, in financial applications, the joint dynamics is mostly important when we consider the interaction among a portfolio of assets. The models we are going to implement and compare are the full-rank Gaussian copula (FGC) approach described before and the multivariate variance gamma (MVG) model we proposed in Chapter 2.

The dataset we select include select eight stocks from technology sector and

seven stocks from the industrial sector. We use time series data on stock prices from 1/1/2002 to 10/1/2008.

The seven industrial stocks we considered as a group are XOM (Exxon Mobil), SUN (Sunoco), XRX (Xerox), WMT (Wal-mart Stores), VZ (Verizon Communications), MMM (3M Company), KO(The Coca-Cola Company) and the eight stocks in the technology sector are AAPL (Apple), AMZN (Amazon.com), CSCO (Cisco Systems), DELL (Dell), IBM (International Business Machines), INTC (Intel), ORCL (Oracle), QCOM (Qualcomm).

We first use MLE to estimate the marginal return distribution on each stock individually with variance gamma distribution. The density function is computed by inverting the characteristic function using FFT. The estimated parameters are reported as follows:

We choose  $\nu_0$  as the largest  $\nu_i$  among all margins and simply estimate the correlation matrix in the Brownian motion by matching correlations of any two stocks in the portfolio with the sample correlations.

We test the model performance on one thousand randomly generated portfolios on stock returns. We form for each sector a thousand randomly generated long-only and long-short portfolios. For the long-short portfolio of  $n$  assets we generated  $n$  independent normal variables and scaled them to be on the unit sphere. For the long only portfolio, we generated  $n$  independent normals and scaled the absolute values by their sum. From the time series we determine the portfolio returns by taking the linear transformations on the data. For the density of portfolio return with weight  $w$  in the model, we first derive the characteristic functions  $\phi(v)$  from

Table 3.1: Estimated VG parameters on marginal laws

TICKER	$\theta$ in basis points	$\sigma$	$\nu$
XOM	-22.34	0.0167	0.6744
SUN	-21.84	0.0237	0.6821
XRX	2.37	0.0309	1.2741
WMT	6.85	0.0179	0.8462
VZ	2.76	0.0188	0.8773
MMM	2.98	0.0159	0.9947
KO	-3.01	0.0150	1.0625
AAPL	11.06	0.0256	0.5165
AMZN	29.22	0.0288	0.9337
CSCO	-8.22	0.0222	0.7667
DELL	-1.70	0.0193	0.7990
IBM	-1.45	0.0146	0.8132
INTC	-6.28	0.0227	0.6967
ORCL	0.19	0.0233	0.8924
QCOM	32.44	0.0241	0.6346

the joint characteristic function  $\Phi(u)$  by choosing appropriate  $u = wv$ , then invert the characteristic function by FFT.

We construct the chi-square statistics on twenty equally spaced intervals within the five standard deviations of the returns and compute p-values. We present empir-

ical complementary distribution function of the p-values across the 1000 portfolios for each sector and each type of portfolio. The graphs of the complementary distribution functions in the following.

From the results of the graph, one can see that MVG model has a much better overall performance than FGC model in both sectors. The full rank Gaussian copula model is especially poor when fitting with long-only portfolios of returns.

### 3.3 Local Correlation

Correlation, or known as Pearson's correlation, is an effective way to represent comovements between variables if they are linked by linear relationships. However, it sometimes may be misleading when the marginal distributions are non-normal. The comovements of variables tend to perform differently for different sample values. To investigate the dependence structure intuitively, we here introduce the idea of local correlation to examine the local dependence structure of the multivariate VG model..

For a two dimensional distribution  $(X_1, X_2)$  on  $R^2$ , we define the local correlation  $\rho_l(x_1, x_2)$  as the correlation locally spanned at  $(x_1 + \epsilon, x_2 + \epsilon)$  for small  $\epsilon$ . To numerically evaluate this number across the  $R^2$  plane, we first derive a closed-form formula for  $\rho_l(x_1, x_2)$  by approximating the joint density  $f(x_1, x_2)$  using the expansion of a joint Gaussian density.

$$g(x_1, x_2) := -2\log f(x_1, x_2) = c(g_{x_1x_1}x_1^2 + 2g_{x_1x_2}x_1x_2 + g_{x_2x_2}x_2^2 + \dots)$$

Figure 3.1: Industrial sector result

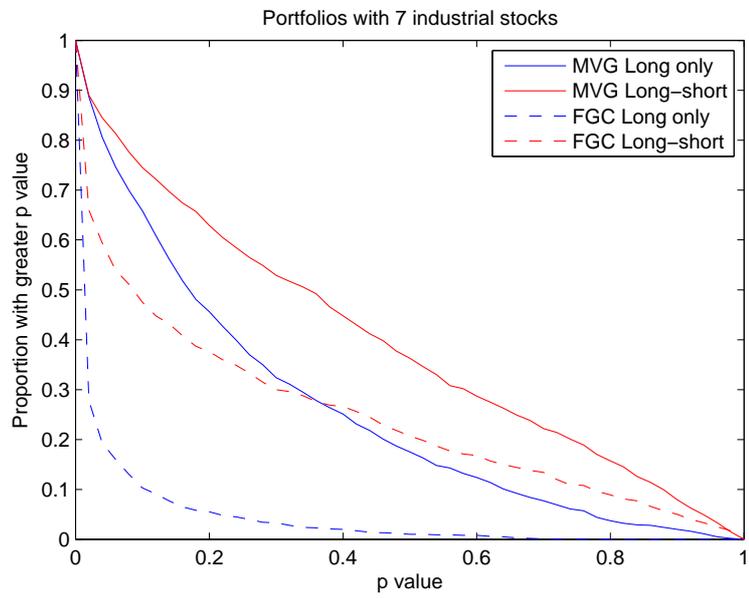
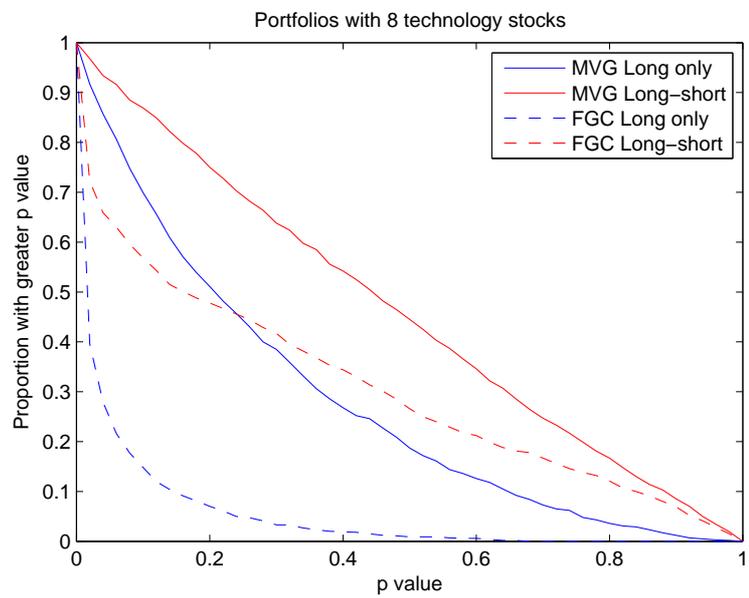


Figure 3.2: Technology sector result



$$\rho_l = \frac{\frac{\partial^2}{\partial x_1 \partial x_2}(-\log(f))}{\sqrt{\frac{\partial^2}{\partial^2 x_1}(-\log(f))} \sqrt{\frac{\partial^2}{\partial^2 x_2}(-\log(f))}}$$

which can be also written as

$$\rho_l = \frac{f_{x_1 x_2} f - f_{x_1} f_{x_2}}{\sqrt{f_{x_1 x_1} f - f_{x_1}^2} \sqrt{f_{x_2 x_2} f - f_{x_2}^2}}$$

where  $f_{x_1}, f_{x_2}$  stand for the first derivatives of  $f$  with respect to  $x_1$  and  $x_2$  and  $f_{x_1 x_1}, f_{x_1 x_2}, f_{x_2 x_2}$  stand for the second derivatives of  $f$  with respect to corresponding variables.

For the multi-variate VG model and many other Lévy models, we do not have the closed form for the joint density even for the two dimensional case. Thus, to evaluate the joint density and its derivatives, one needs to numerically invert the characteristic function by Fourier transform. By FFT, we can compute  $f_{x_1 x_2}, f_{x_1 x_1}, f_{x_2 x_2}, f_{x_1}, f_{x_2}$  in a very efficient way. The following result shows the formulas which compute the local correlation surface in the whole plane.

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} \Phi(u_1, u_2) du_1 du_2 \\ f_{x_1}(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} (-iu_1) \Phi(u_1, u_2) du_1 du_2 \\ f_{x_2}(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} (-iu_2) \Phi(u_1, u_2) du_1 du_2 \\ f_{x_1 x_1}(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} (-u_1^2) \Phi(u_1, u_2) du_1 du_2 \\ f_{x_1 x_2}(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} (-u_1 u_2) \Phi(u_1, u_2) du_1 du_2 \\ f_{x_2 x_2}(x_1, x_2) &= \frac{1}{(2\pi)^2} \iint e^{-iu_1 x_1 - iu_2 x_2} (-u_2^2) \Phi(u_1, u_2) du_1 du_2 \end{aligned}$$

Local correlation surface can be used to determine the relative comovement level of two variables compared with the normal distribution. In the financial market, it is well-documented that asset prices tend to move together when market has big movements. The joint dynamics of multivariate Brownian motion or the Gaussian copula was strongly rejected as a model for comovement of two stock prices. Thus, we expect the local correlation surface of any reasonable bivariate financial models to be non-flat and peaked at the corner of first and third quadrants. The multivariate VG model shows certain desirable features from the correlation surface we constructed. We chose the following two sets of parameters which use different dependence parameters and draw the local correlation surface within the region  $[-10\%, +10\%]^2$ .

Table 3.2: Parameters

	$\theta_1$	$\theta_2$	$\sigma_1$	$\sigma_2$	$\nu_1$	$\nu_2$	$\rho$	$\nu_0$
Set 1	-0.05	-0.05	0.3	0.3	0.5	0.5	1	1
Set 2	-0.05	-0.05	0.3	0.3	0.5	0.5	0.6	0.5

While the linear correlation for these marginal variables are 0.5 and 0.6055 respectively, the local correlation surfaces vary significantly over the plane. The local correlation goes down to a very low level or even negative numbers in the second and fourth quadrants and peaks in the corner of first and third quadrants. This also explains why the multivariate VG model shows great performance in fitting the portfolio returns in the previous section.

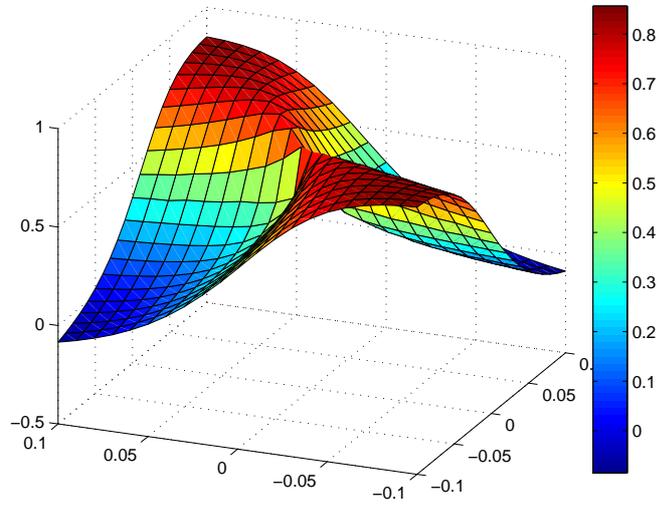


Figure 3.3: Local Correlation Surface I

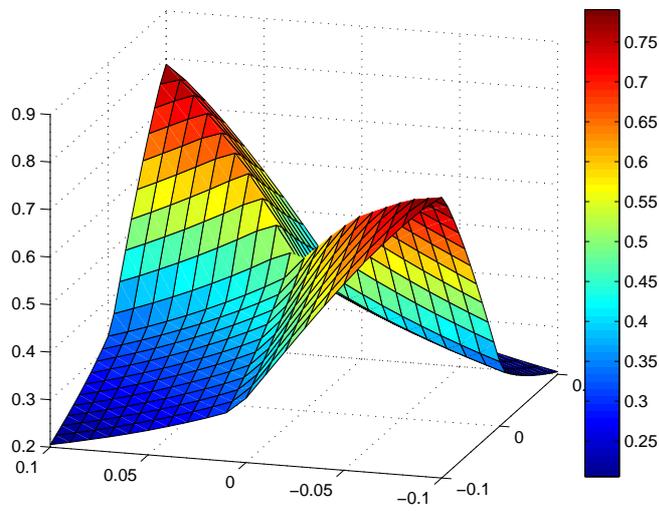


Figure 3.4: Local Correlation Surface II

## Chapter 4

### Applications in Multi-asset Option Pricing

#### 4.1 Overview

Many financial derivatives expose risks to more than one assets, such examples include spread option, basket option, and most structured products. Using a reasonable model to model the joint dynamics of these assets is a must to price and hedge these financial derivatives consistently and correctly. In most of the literature, pricing multi-asset derivatives relies heavily on using multivariate Brownian motions. However, the real market deviates largely from Gaussian based model in two aspects: heavy-tail distributed marginal returns and strong tail dependence. The copula method provides flexible dependence structure, but the difficulty in constructing a process makes pricing and hedging options become difficult. The multivariate variance gamma model we introduce in chapter two provides an alternative to the classic diffusion based model or copula based model.

In this chapter, we discuss the pricing problem of several popular multi-asset options including exchange option, spread option, basket option, rainbow option and cross-currency FX derivatives. Although all these problems can be solved by Monte Carlo simulation, we try to investigate more efficient numerical methods based on the Fourier transform. We derive the analytic closed form for the Fourier transform of the prices and use FFT to numerically compute the result. We show in details

how to price the options and estimate the sensitivities under our multivariate VG model. The FFT method is proved to be much faster and more accurate than Monte Carlo simulation. Meanwhile it is worth noting that these numerical methods are general in the sense that they can be applied to any multivariate model with closed form characteristic function. Due to the computational burden of high-dimensional FFT algorithm, we restrict the case of basket option and rainbow option to the case of only two assets. Though the formula is also given in the context for higher dimensional cases with more than three assets, it is usually not feasible in practice by FFT. Monte Carlo simulation with some variance reduction methods is standard for approaching these products in that case.

## 4.2 Exchange Option

We start our discussion from one of the simplest multi-asset options: exchange option. An exchange option allows the holder of the option to exchange one asset for another and is used commonly in foreign exchange markets, bond markets and stock markets amongst others. The payoff of an exchange option with assets  $S_1$  and  $S_2$  is thus  $(S_1 - S_2)^+$ . It is obvious that exchange option is a special case of spread option with strike equal to zero.

Margrabe [47], in 1978, first introduced the pricing formula for valuing European exchange options assuming two assets follow two correlated log-normal processes. We present this derivation in full for the sake of its importance.

Assume that the risk-neutral dynamics of the two underlying assets are given

by the following stochastic differential equations.

$$dS_1(t) = rS_1(t)dt + \sigma_1dW_1(t) \quad (4.1)$$

$$dS_2(t) = rS_2(t)dt + \sigma_2dW_2(t) \quad (4.2)$$

$$E\{dW_1(t)dW_2(t)\} = \rho dt \quad (4.3)$$

Using the technique of change of numeraire, Margrabe derived the pricing formula for European exchange options.

**Proposition 4.1. (*Margrabe's formula*)** *The price  $p$  of an exchange option with maturity  $T$  is given by*

$$p = S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2) \quad (4.4)$$

where,

$$d_1 = \frac{\ln(S_1(0)/S_2(0)) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln(S_1(0)/S_2(0)) - \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}},$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

The formula is in a similar form to the Black-Scholes formula. For a complete proof, we refer readers to [47]. Margrabe's trick in deriving the formula lies in the use of change of numeraire. By changing the numeraire from the money account to one of these two assets, the problem is greatly simplified to a single dimensional problem. His formula is widely used and can be applied to much more generalized models besides the Black-Scholes model. With this technique of change of numeraire, we

may simplify the problem of pricing an exchange option in multi-variate VG models significantly.

We consider the exchange option pricing problem under the multivariate VG model, where  $S_1$  and  $S_2$  follow an exponential two-dimensional VG process. The joint characteristic function is given by (2.11).

The exchange option price at time 0 is  $e^{-rT} E^Q(S_1 - S_2)^+$ . To simplify the calculation, we change the numeraire from the risk free money account to the asset  $S_2$ . Let  $E^{U_2}$  be the expectation under the measure  $U_2$  with numeraire  $S_2$ . By theorem 1.10 in chapter 1, the Radon-Nykodym derivative of the numeraire change is:

$$\frac{dQ}{dU_2} = \frac{e^{rT} S_2(0)}{S_2(T)} \quad (4.5)$$

By theorem 1.10, the price  $p$  becomes:

$$p = S_2(0) E^{U_2} \left( \frac{S_1(T)}{S_2(T)} - 1 \right)^+ \quad (4.6)$$

We now simplify the problem into an option pricing problem with one underlying. The price of the exchange option becomes an European call option on  $\frac{S_1(T)}{S_2(T)}$  under measure  $U_2$  with strike 1. To apply the FFT method of Carr and Madan, we only need to derive the characteristic function of  $\ln\left(\frac{S_1(T)}{S_2(T)}\right)$  under measure  $U_2$ , which can be derived again by change of numeraire.

$$\begin{aligned}
\phi(u) &= E^{U_2}(e^{iuln(\frac{S_1(T)}{S_2(T)})}) \\
&= E^Q(e^{iuln(\frac{S_1(T)}{S_2(T)})} \frac{dU_2}{dQ}) \\
&= E^Q(\exp(-rT + iuln(S_1(T)) + (1 - iu)ln(S_2(T)) - ln(S_2(0)))) \\
&= \exp(iuln(S_1(0)/S_2(0)) - iuln(\phi_{(X_1, X_2)}(-i, 0)) - (1 - iu)ln(\phi_{(X_1, X_2)}(0, -i))) \\
&\quad \cdot \phi_{(X_1, X_2)}(u, -i - u)
\end{aligned} \tag{4.7}$$

$$\tag{4.8}$$

The exchange option price  $p$  in (4.6) can now be easily computed by the Carr-Madan FFT method on European Call options with  $r = 0$  and characteristic function (4.7).

#### 4.2.1 Numerical Results

To see the advantage of using the FFT method, we compare the computational results between FFT method and Monte Carlo simulation for exchange options under the two-dimensional VG model. We tested two groups of VG parameters with a set of  $S_2(0)$ . The algorithm was implemented in Matlab on the same machine. To get an accurate estimation from the simulation, we run 10,000,000 simulation paths each time. The FFT method was much faster compared with Monte Carlo simulation run on the same computer. Besides the computational speed, we also observe that Monte Carlo simulation converges very slowly without using variance reduction techniques. The estimated standard errors are reported in the last column of the table. The FFT method is also able to compute the option prices across different strikes in one

run. In all, the FFT method is shown to have a huge advantage over the simulation method. The results show the accuracy of the FFT method is very reliable.

Table 4.1: Computational Results I for Exchange Options

FFT method		Simulation		
$S_2(0)$	N=2048	N=4096	n=10,000,000	std
80.0	22.4261	22.4260	22.4198	0.0079
90.0	15.0702	15.0688	15.0728	0.0067
100.0	9.5056	9.5056	9.5076	0.0069
110.0	5.9312	5.9300	5.9323	0.0059
120.0	3.7707	3.7701	3.7698	0.0037

$T = 1; \theta_1 = \theta_2 = -0.05; \sigma_1 = \sigma_2 = 0.3; \nu_1 = \nu_2 = 0.5; \nu_0 = 1; \rho = 0.8.$

Table 4.2: Computational Results II for Exchange Options

FFT method		Simulation		
$S_2(0)$	N=2048	N=4096	n=10,000,000	std
80.0	23.7522	23.7519	23.7508	0.0084
90.0	17.3682	17.3668	17.3673	0.0093
100.0	12.6590	12.6590	12.6535	0.0087
110.0	9.3226	9.3219	9.3234	0.0065
120.0	6.9788	6.9684	6.9679	0.0071

$T = 1; \theta_1 = 0.05; \theta_2 = -0.05; \sigma_1 = 0.4; \sigma_2 = 0.3; \nu_1 = 0.8; \nu_2 = 0.5; \nu_0 = 1; \rho = 1.0$

### 4.3 Spread Option

Spread options are popular multi-asset derivatives traded across different sectors of the financial market. By definition, a spread option is an option written on the difference of two asset prices. The payoff function of an European call spread option on two assets  $S_1$  and  $S_2$  with strike  $K$  is  $(S_1(T) - S_2(T) - K)^+$  at maturity  $T$ . Hence, it can be considered as an European call option on the spread of two assets or indices. Spread options are designed to mitigate adverse movements of several indexes. Because of their generic nature, they are used in markets as varied as fixed income markets, currency and foreign exchange markets, commodity futures markets, and energy markets. The definition of spread options sometimes can also be loosened to include all kinds of options written on a linear combination of a finite set of assets.

Pricing spread options is more involved compared with pricing exchange options. By risk-neutral valuation, the price of an European call spread option can be written as  $e^{-\tau T} E^Q(S_1 - S_2 - K)^+$ . The numeraire change technique can not be applied due to the existence of the non-zero strike, and the log return distribution of the difference is usually unknown. It turns out to be a hard problem to derive a pricing formula even under the Black-Scholes framework for a spread option. The main obstacle to a clean pricing methodology lies in the lack of knowledge about the distribution of the difference between two non-trivially correlated stochastic processes. For example, the distribution of the difference of two correlated log-normal processes is no longer log-normal distributed. While various of numerical methods

such as Monte Carlo simulation, PDE solvers are feasible for pricing purpose (See [8] for an overview of pricing and hedge spread options), closed form approximation formulas are much more attractive for practical use. One of the widely-used formulas is derived by Kirk [36] under the Black-Scholes' assumption. Assume two assets or indices follow two correlated log-normal processes as (4.1)-(4.3), Kirk proposed the following closed form approximation for pricing a spread option.

**Proposition 4.2. (*Kirk's Formula*)**

$$p_K = x_1 \Phi\left(\frac{\ln\left(\frac{x_1}{x_2 + Ke^{-rT}}\right) + \frac{\sigma^K}{2}}{\sigma^K}\right) - (x_2 + Ke^{-rT}) \Phi\left(\frac{\ln\left(\frac{x_1}{x_2 + Ke^{-rT}}\right) - \frac{\sigma^K}{2}}{\sigma^K}\right) \quad (4.9)$$

where,

$$x_1 = S_1(0), x_2 = S_2(0), \sigma^K = \sqrt{\sigma_2^2 - 2\rho\sigma_1\sigma_2 \frac{x_2}{x_2 + Ke^{-rT}} + \sigma_1^2 \left(\frac{x_2}{x_2 + Ke^{-rT}}\right)^2}.$$

Instead of deriving such numerical approximation under some certain models, Fourier transform method is considered to be a good candidate to solve more general cases. For most of the option pricing models, the joint characteristic function for asset returns is available analytically and we can write out the price of the spread option as a double integral.

$$p = \iint_{\Omega} e^{-rT} (e^{s_1} - e^{s_2} - K) f_T(s_1, s_2) ds_2 ds_1 \quad (4.10)$$

where the exercise region  $\Omega$  is defined as:

$$\Omega = \{(s_1, s_2 \in \mathbb{R}^2) | e^{s_1} - e^{s_2} - K \geq 0\}.$$

$$s_1 = \ln(S_1(T)), s_2 = \ln(S_2(T)).$$

The exercise region is by nature non-linear, which makes the transformation method of the FFT method difficult to use. Dempster and Hong [16] described a FFT method to find the upper and lower bounds of a spread option. They formed tight upper and lower bounds of the integral by subtracting and collecting pieces on  $N \times N$  different regions and then applied FFT on the regularized region. However, they did not solve the exact form of the transformed price.

The recent progress of solving the problem was made by Hurd and Zhou [30]. They developed a new numerical integration method for computing spread options in two or higher dimensions through FFT with closed-form joint characteristic function. To apply the two-dimensional FFT, they consider the price as a function of initial log stock prices by scaling the strike into 1. The Fourier transform converts the log return space into the initial log strike space with closed form integral function. This formula provided the exact solution of pricing general spread options in terms of a double integral which can be approximated accurately and efficiently by FFT. It thus is a perfect method of computing spread option prices under the two-dimensional VG model. The result stems from the following proposition.

**Proposition 4.3.** *For any real number  $\epsilon = (\epsilon_1, \epsilon_2)$  with  $\epsilon_2 > 0$  and  $\epsilon_1 + \epsilon_2 < -1$*

$$P(x) = (2\pi)^{-2} \iint_{R^2+i\epsilon} e^{iux'} \hat{P}(u) d^2u \quad (4.11)$$

$$\text{where, } \hat{P}(u) = \frac{\Gamma(i(u_1 + u_2) - 1)\Gamma(-iu_2)}{\Gamma(iu_1 + 1)}. \quad (4.12)$$

Here  $\Gamma(z)$  is the complex gamma function defined by the integral  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  for  $Re(z) > 0$ .  $u = (u_1, u_2)$  and  $x = (x_1, x_2)$ .

For any spread option with non-zero strikes, one can assume the strike of the

spread option equals to 1 without loss of generality. By scaling, the payoff function can be written as  $(S_{1T} - S_{2T} - 1)^+$ . Using the result above, the price of this spread option is converted into the following form:

**Proposition 4.4.**

$$\begin{aligned}
P(S_0; T) &= e^{-rT} E[(S_{1T} - S_{2T} - 1)^+] \\
&= (2\pi)^{-2} e^{-rT} \iint_{\mathbb{R}^2 + i\epsilon} E_{X_0}[e^{iuX'_T} \hat{P}(u) d^2u] \\
&= (2\pi)^{-2} e^{-rT} \iint_{\mathbb{R}^2 + i\epsilon} e^{iuX'_0} \Phi(u; T) \hat{P}(u) d^2u \tag{4.13}
\end{aligned}$$

Here,  $X_i = \log S_i$  with  $i = 1, 2$ .  $\Phi(u; T) := E_0[e^{iu(X'_T - X'_0)}]$  is the joint characteristic function of the log returns.

The most efficient way to evaluate the above integral is to use numerical integration by FFT. The double integral can be estimated by a double sum over the lattice:

$$\{\mu(k) = (\mu(k_1), \mu(k_2)) = (-N\eta/2 + k_1\eta, -N\eta/2 + k_2\eta) | k_1, k_2 = 0, 1 \dots N - 1\}$$

In FFT,  $N$  is usually chosen as a power of 2 and lattice spacing  $\eta$  is small in order to keep the error be acceptable. Finally, choose initial values  $X_0 = \log S_0$  to lie on the reciprocal lattice with spacing  $\eta^* = 2\pi/N\eta$ . Compared with Carr-Madan FFT method, the space of log returns is transformed into the space of initial log prices other than the space of the log strike. The new grids after the transform will be on

$$\{x(l) = (x(l_1), x(l_2)) = (-N\eta^*/2 + l_1\eta^*, -N\eta^*/2 + l_2\eta^*) | l_1, l_2 = 0, 1 \dots N - 1\}$$

The approximation of the price is written as

$$\begin{aligned}
P(S_0, T) &\approx \frac{\eta^2 e^{-rT}}{(2\pi)^2} \sum_{k_1, k_2=0}^{N-1} e^{i(\mu(k)+i\epsilon)x'} \Phi(\mu(k) + i\epsilon; T) \hat{P}(\mu(k) + i\epsilon) \\
&= (-1)^{l_1+l_2} e^{-rT} \left(\frac{\eta N}{2\pi}\right)^2 e^{-\epsilon x(l)'} [\text{ifft2}(H)](l)
\end{aligned} \tag{4.14}$$

where,

$$H(k) = (-1)^{l_1+l_2} \Phi(\mu(k) + i\epsilon; T) \hat{P}(\mu(k) + i\epsilon)$$

In the formula, `ifft2` stands for the 2-dimensional inverse discrete Fourier transform, computed with a Fast Fourier transform algorithm. For the  $N \times N$  matrix, the FFT method is shown to have the  $O(N^2 \log N)$  complexity, which is a significant improvement over the normal Fourier transform with complexity of  $O(N^4)$ . With the computational power of the modern computers, the prices can be computed very accurately within fractional seconds.

Similar to computing the Greeks by FFT in Chapter 1, the FFT method can also be applied to approximate the sensitivities for spread options.

### 4.3.1 Numerical Results

In this section, we compare the computational results between the FFT method and Monte Carlo simulation for spread options under the two-dimensional VG model. We tested two groups of VG parameters with a set of strike  $K$ . Note that in the method of Hurd and Zhou, different strikes are equivalent to choosing different initial  $S_1(0)$  and  $S_2(0)$ . The experiment was implemented in Matlab on the same machine. To get an accurate estimation from the simulation, we run 10,000,000

simulation paths each time to control the simulation error into an acceptable range. The FFT method was shown to be significantly faster than simulation. In addition to its fast speed, the FFT method could give the prices with different strike  $K$ ,  $S_1(0)$  and  $S_2(0)$  in one run once the joint process is determined. In the following examples, we use  $r = 0$ ,  $S_1(0) = 100$  and  $S_2(0) = 90$ .

Table 4.3: Computational Results I for Spread Options

		FFT method		Simulation	
$K$	N=512	N=1024	n=10,000,000	std	
5.0	11.8189	11.8200	11.8223	0.0062	
10.0	9.1048	9.1049	9.1078	0.0054	
15.0	6.9512	6.9514	6.9556	0.0057	
20.0	5.2897	5.2911	5.2910	0.0040	
30.0	3.0764	3.0776	3.0785	0.0027	

$T = 1; \theta_1 = \theta_2 = -0.05; \sigma_1 = \sigma_2 = 0.3; \nu_1 = \nu_2 = 0.5; \nu_0 = 1; \rho = 0.8;$

#### 4.4 Basket Option

A basket option is an option whose payoff is linked to a portfolio of stocks, bond, currencies or other assets. It has become very popular over the last few years as part of index-linked products traded in the market. A basket call on a portfolio of  $n$  assets  $S_1 \dots S_n$  with weights  $w_1 \dots w_n$  pays off  $(w_1 S_1(T) + \dots + w_n S_n(T) - K)^+$  at maturity  $T$ . Basket options are often used to protect the risk on the whole portfolio.

Table 4.4: Computational Results II for Spread Options

$K$	FFT method		Simulation	
	N=512	N=1024	n=10,000,000	std
5.0	14.7606	14.7605	14.7648	0.0124
10.0	12.5799	12.5803	12.5812	0.0117
15.0	10.7734	10.7742	10.7709	0.0101
20.0	9.2816	9.2825	9.2805	0.0136
30.0	7.0326	7.0330	7.0315	0.0107

$T = 1; \theta_1 = 0.05; \theta_2 = -0.05; \sigma_1 = 0.4; \sigma_2 = 0.3; \nu_1 = 0.8; \nu_2 = 0.5; \nu_0 = 1; \rho = 1.0;$

Similar to pricing spread options, to evaluate a basket option is not straightforward even under the Black-Scholes lognormal assumption. The reason for this is that the sum of lognormal random variables is not lognormal. Therefore, many approximation techniques and ad hoc rules of thumb have arisen to tackle this problem. Most of the approximation methods rely on either the approximate distribution of the whole basket returns, which includes the lognormal assumption by Hyunh [28], the reciprocal gamma assumption by Milevsky and Posner [48], or the numerical expansion based method as Taylor expansion by Ju [34].

For more complicated option pricing models other than the lognormal assumption, the pricing problem becomes even more complex. Monte Carlo simulation has been almost the standard method of solving this kind of problems. On the other hand, for many option pricing models, the characteristic functions instead of the

density functions are known analytically. Hence, Fourier transform based method can be extremely efficient when the dimension of the problem is small. Similar to Hurd and Zhou's method in pricing spread options, we derive a pricing formula for basket options given the closed form characteristic functions of the joint dynamics of assets. For the multi-variate VG model, we then can use the following results to price the general basket option. The numerical experiments show that the FFT method yields efficient and accurate results for two or three assets, though it may be not suited for the problems with assets number greater than four.

We first derive the formula for basket options with two assets. The payoff function with non zero strike can always be scaled to the following form:  $(S_{1T} + S_{2T} - 1)^+$ . By the fact that the integration region is not regularized, we use the put-call parity to transform the payoff into a different form. We first get the following results:

**Theorem 4.5.** *Consider  $P(x_1, x_2) = (1 - e^{x_1} - e^{x_2})^+$ . For any real numbers  $\epsilon = (\epsilon_1, \epsilon_2)$  with  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$*

$$P(x) = (2\pi)^{-2} \iint_{R^2+i\epsilon} e^{iux'} \hat{P}(u) d^2u \quad (4.15)$$

$$\text{where, } \hat{P}(u) = \frac{\Gamma(-iu_1)\Gamma(-iu_2)}{\Gamma(2 - i(u_1 + u_2))}. \quad (4.16)$$

Here  $\Gamma(z)$  is the complex gamma function defined by the integral  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  for  $Re(z) > 0$ .  $u = (u_1, u_2)$  and  $x = (x_1, x_2)$ .

Proof: See Appendix.

By the put-call parity, we may use the relation between the payoff functions

of call and put options  $(S_{1T} + S_{2T} - 1)^+ - (1 - S_{1T} - S_{2T})^+ = S_{1T} + S_{2T} - 1$ , which leads to the following integral form.

**Proposition 4.6.** *For a basket option with two assets  $S_1$  and  $S_2$ , the basket option formula can be written*

$$\begin{aligned}
Bkt(S_0; T) &= e^{-rT} E[(S_{1T} + S_{2T} - 1)^+] \\
&= e^{-rT} E[S_{1T} + S_{2T} - 1] + e^{-rT} E[(1 - S_{1T} - S_{2T})^+] \\
&= S_{10} + S_{20} - e^{-rT} + (2\pi)^{-2} e^{-rT} \iint_{R^2+i\epsilon} e^{iuX'_0} \Phi(u; T) \hat{P}(u) d^2u \quad (4.17)
\end{aligned}$$

The double integral can again be numerically evaluated using FFT analogous to a spread option. We can implement the same algorithm as that described in section 4.2 by replacing  $\hat{P}$  in (4.13). Meanwhile, the  $\epsilon$  term should also be chosen to satisfy  $\epsilon_i > 0, i = 1, 2$ .

For basket options with more than two assets, the pricing formula can be derived in theory. However, as the complexity of algorithm grows exponentially with the number of the assets  $n$ , it is impractical to compute the price with very large  $n$ .

**Theorem 4.7.** *For the payoff function  $P(x_1, x_2, \dots, x_n) = (1 - e^{x_1} - e^{x_2} - \dots - e^{x_n})^+$ ,  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$  with  $\epsilon_i > 0$  for  $i = 1 \dots n$ ,*

$$P(x) = (2\pi)^{-n} \int \dots \int_{R^n+i\epsilon} e^{iuX'} \hat{P}(u) d^n u \quad (4.18)$$

$$\text{where, } \hat{P}(u) = \frac{\prod_{i=1}^n \Gamma(-iu_i)}{\Gamma(2 - i \sum_{i=1}^n u_i)}. \quad (4.19)$$

**Proposition 4.8.** *For a basket option with  $n$  assets  $S_1 \dots S_n$ , the basket option for-*

mula can be written

$$\begin{aligned}
Bkt(S_0; T) &= e^{-rT} E[(S_{1T} + \dots + S_{nT} - 1)^+] \\
&= e^{-rT} E[S_{1T} + \dots + S_{nT} - 1] + e^{-rT} E[(1 - S_{1T} - \dots - S_{nT})^+] \\
&= S_{10} + \dots + S_{n0} - e^{-rT} + (2\pi)^{-n} e^{-rT} \int \dots \int_{R^n + i\epsilon} e^{iu \cdot X_0^t} \Phi(u; T) \hat{P}(u) d^n u
\end{aligned} \tag{4.20}$$

The implementation of the formula is exactly the same as that of pricing spread options.

For the following numerical experiment, we use  $S_1(0) = 100$ ,  $S_2(0) = 100$ ,  $r = 0$  and test the basket option pricing using FFT for different strikes.

Table 4.5: Computational Results I for Basket Options

$K$	FFT method		Simulation	
	N=512	N=1024	n=10,000,000	std
160.0	45.3253	45.3253	45.3210	0.0163
180.0	31.0607	31.0607	31.0567	0.0178
200.0	20.2533	20.2537	20.2627	0.0134
210.0	12.9385	12.9389	12.9243	0.0101
220.0	8.3032	8.3026	8.3069	0.0127

$T = 1; \theta_1 = 0.05; \theta_2 = -0.05; \sigma_1 = 0.4; \sigma_2 = 0.3; \nu_1 = 0.8; \nu_2 = 0.5; \nu_0 = 1; \rho = 1.0;$

## 4.5 Rainbow Option

A rainbow option usually refers to a call or put option on the best or worst of  $n$  underlying assets, or options which pay the best or worst of  $n$  assets. Sometimes it is also used as a general name of all options whose payoff depends on more than one underlying risky assets. Stulz [58] first derived the pricing formula of options on the maximum or minimum of two assets. It was later generalized to the case of several assets by Johnson [33]. Their models were both based on correlated lognormal processes. In this section, we discuss how to price the option on the maximum or minimum of several assets under general models using FFT method. Again to make use of the Fourier transform, the only requirement on the joint process is the closed form joint characteristic function. We first consider the case of two assets.

With two assets, there are four different call/put options on the maximum or minimum of the prices. While call options on the maximum and put options on minimum are not easy to evaluate directly, the rest two can be solved by the Fourier transform. We then apply the put-call parity to derive the formulas for all four types. The following theorem deals with the payoff functions  $(\min(S_1(T), S_2(T)) - 1)^+$ , which is a call option on the minimum of two stock prices at time  $T$ , and  $(1 - \max(S_1(T), S_2(T)))^+$ , which is a put option on the maximum of two stock prices at time  $T$ .

**Theorem 4.9.** *Consider  $P(x_1, x_2) = (\min(e^{x_1}, e^{x_2}) - 1)^+$ . For any real numbers*

$\epsilon = (\epsilon_1, \epsilon_2)$  with  $\epsilon_1 < 0$  and  $\epsilon_2 < 0$

$$P(x) = (2\pi)^{-2} \iint_{R^2+i\epsilon} e^{iux'} \hat{P}(u) d^2u \quad (4.21)$$

$$\text{where, } \hat{P}(u) = -\frac{1}{u_1 u_2 (i(u_1 + u_2) - 1)}. \quad (4.22)$$

Similarly, consider  $P(x_1, x_2) = (1 - \max(e^{x_1}, e^{x_2}))^+$ . For any real numbers  $\epsilon = (\epsilon_1, \epsilon_2)$  with  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$

$$P(x) = (2\pi)^{-2} \iint_{R^2+i\epsilon} e^{iux'} \hat{P}(u) d^2u \quad (4.23)$$

$$\text{where, } \hat{P}(u) = -\frac{1}{u_1 u_2 (i(u_1 + u_2) - 1)}. \quad (4.24)$$

Proof: See Appendix.

The value of a call option on the minimum of two stocks can be written as the double integral ( $\epsilon_i < 0, i = 1, 2$ ):

$$C = (2\pi)^{-2} e^{-rT} \iint_{R^2+i\epsilon} e^{iuX'_0} \Phi(u; T) \hat{P}(u) d^2u \quad (4.25)$$

The value of a put option on the maximum of two stocks can be written as the double integral ( $\epsilon_i > 0, i = 1, 2$ ):

$$P = (2\pi)^{-2} e^{-rT} \iint_{R^2+i\epsilon} e^{iuX'_0} \Phi(u; T) \hat{P}(u) d^2u \quad (4.26)$$

For a call option on the max or a put option on the min, we derive the formula through the put-call parity.

$$(\max(S_1(T), S_2(T)) - 1)^+ - (1 - \max(S_1(T), S_2(T)))^+ = \max(S_1(T), S_2(T)) - 1$$

$$(\min(S_1(T), S_2(T)) - 1)^+ - (1 - \min(S_1(T), S_2(T)))^+ = \min(S_1(T), S_2(T)) - 1$$

We need an explicit formula for the right side. In other words, we need to compute the expectation of the minimum or the maximum of two stocks at time  $T$ . We use the following identities,

$$\max(e^{X_1}, e^{X_2}) = e^{X_1} + (e^{X_2} - e^{X_1})1_{X_2 > X_1} \quad (4.27)$$

$$\min(e^{X_1}, e^{X_2}) = e^{X_1} - (e^{X_1} - e^{X_2})1_{X_1 > X_2} \quad (4.28)$$

It is shown that the payoff of the minimum or the maximum can be decomposed into a portfolio of a stock and an exchange option. The pricing formula of an exchange option is discussed in full in section 4.1.

For rainbow options with  $n$  ( $n > 2$ ) assets, the following results hold:

**Theorem 4.10.** *Consider  $P(x_1, \dots, x_n) = (\min(e^{x_1}, \dots, e^{x_n}) - 1)^+$ . For any real numbers  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_i < 0$   $i = 1, \dots, n$ .*

$$P(x) = (2\pi)^{-n} \int \dots \int_{R^{n+i\epsilon}} e^{iux'} \hat{P}(u) d^n u \quad (4.29)$$

$$\text{where, } \hat{P}(u) = \frac{1}{(-i)^n u_1 \dots u_n (i(u_1 + \dots + u_n) - 1)}. \quad (4.30)$$

*Similarly, consider  $P(x_1, \dots, x_n) = (1 - \max(e^{x_1}, \dots, e^{x_n}))^+$ . For any real numbers  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  with  $\epsilon_i > 0$   $i = 1, \dots, n$ .*

$$P(x) = (2\pi)^{-n} \int \dots \int_{R^{n+i\epsilon}} e^{iux'} \hat{P}(u) d^n u \quad (4.31)$$

$$\text{where, } \hat{P}(u) = \frac{1}{(-i)^n u_1 \dots u_n (i(u_1 + \dots + u_n) - 1)}. \quad (4.32)$$

Proof: See Appendix.

## 4.6 Foreign Exchange Option

### 4.6.1 Overview

The foreign exchange options (commonly shortened to FX options or currency options) market is the deepest, largest and most liquid market for options in the world. According to Triennial Central Bank Survey by Bank for International Settlements in December 2007, average daily turnover in OTC (over-the-counter) FX derivatives was reported to be over \$4.2 trillion. Different from most equity options traded on stock exchange, FX options are mostly traded OTC and lightly regulated. Market participants are mainly governments, banks, international corporations, mutual and hedge funds and individual investors. With increasing globalization of the world financial system, the role of foreign exchange assets has become more and more important. A rigorous and tractable mathematical model for modeling the FX dynamics is required to price and hedge risks exposed to all kinds of FX rates and derivatives.

The classic model in pricing currency options is the Garman-Kohlhagen model. In 1983, Garman and Kohlhagen [23] extended the Black-Scholes model to cope with the presence of two interest rates (one for each currency). The pricing formula for an European call option in their model is exactly the same formula as the Black-Scholes formula with underlying paying continuous yield dividends, except that now the foreign risk-free interest rate plays the role of dividend rate. Assume the  $r_d$  is the risk-free interest rate to expiry of the domestic currency and  $r_f$  is the foreign currency risk-free interest rate. The FX rates, both strike  $K$  and current spot  $S$ ,

are quoted in terms of "units of foreign currency per unit of domestic currency". Garman and Kohlhagen assume under the risk neutral measure the FX rate follows the following log-normal process:

$$dS_t = S_t(r_d - r_f)dt + \sigma S_t dW_t \quad (4.33)$$

The value of an European call option in the domestic currency is:

$$C(S, K) = S e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2) \quad (4.34)$$

where  $d_1 = \frac{\ln(S/K) + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$ ,  $d_2 = d_1 - \sigma\sqrt{T}$ .

The Garman-Kohlhagen model relies on the same assumption of log-normal distribution as the Black-Scholes model and thus has the same shortcoming. In many cases, the Black-Scholes formula is too idealized and does not capture certain features of the market. One of the most critical features is the non-lognormal distribution of the underlying FX rates. To overcome this drawback and incorporate the well-observed jump dynamics of FX rates, many modified models with stochastic volatility, jump-diffusion and pure jump processes are developed in recent years. Among those pure jump models, the variance gamma option-pricing model was applied to FX options by Daal and Madan [15] and is shown to outperform the Black-Scholes and jump-diffusion model in fitting the implied volatility curves of FX options. One challenging problem that these models face in the FX market is how to model the dependence of currency rates since all main FX rates are naturally correlated. In particular, illiquid currency pairs are often largely affected by other major currencies in the world. For example, AUD-CAD (Australian

Dollar-Canadian Dollar) is mainly governed by the change of two rates USD-AUD and USD-CAD. Other currency derivatives involving two or more currencies are all sensitive to the dependence structure of these FX rates. In the following section, we will use the multivariate VG model to solve some of these pricing problems.

#### 4.6.2 Performance on Joint Dynamics of Multiple FX Rates

Before we proceed to get the pricing formula for FX options, we first use the multivariate VG model to analyze the joint return of the foreign exchange rates under the statistical measure. We will also see why the log-normal processes are not ideal building bricks for FX rate models. Due to the triangular relationship among three currencies, the dynamics of one relatively illiquid pair cross-currency rate is mainly determined by the joint dynamics of the other liquid two. Here, we use the historical data on the exchange rates and compute the daily log-return of these rates over years. The performance of the model explaining the joint dynamics is thus under the physical measure.

The exchange rates data employed was 4684 daily observations of USD-JPY and USD-GBP from 1990 to 2008 taken from WRDS. The marginal distribution is estimated by maximum likelihood estimation of a VG process on the demeaned log return of the exchange rate. We bin the data into 100 equally spaced intervals in the range of  $\pm 5$  standard deviation of the distribution. The return of both FX rates shows highly non-normal pattern with an excess kurtosis. The following graphs show the VG fitting results of the log-return of USD-JPY and USD-GBP.

The red curve and green curve are density functions of the variance gamma and normal distribution estimated from the data. Both are scaled to compare with the histogram of the sample return. For the test of goodness-of-fit, we calculate the chi-square statistic and the p-values using the bins which has probability greater than 1%. The VG parameters are given as follows:

Table 4.6: Estimation Result

	USD-JPY	USD-GBP
$\theta$	5.364-e4	-3.857-e4
$\sigma$	0.00663	0.00555
$\nu$	0.674	0.654
$\chi^2$	36.65	13.04
p-value	0.12	0.99

Notice now the log-return of JPY-GBP is just the difference of that of USD-JPY and USD-GBP. By using the estimation of the joint distribution method described before, we can get the estimator of  $\nu_0$  as 1.96, which implies the correlation in the Brownian motion should be 0.97. The chi-square statistic is 23.81 and the p-value is 0.69. The fitting result is shown below:

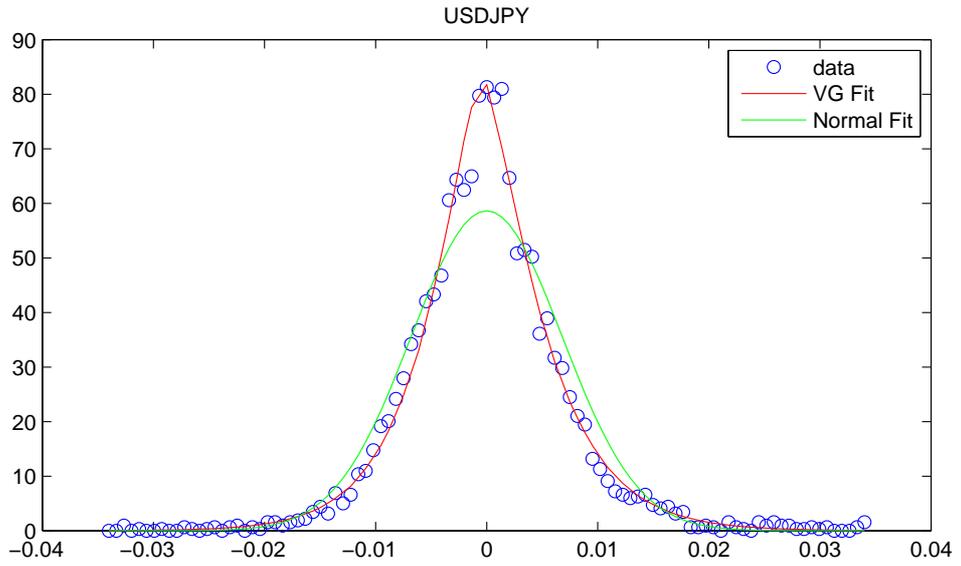


Figure 4.1: Estimation result for USD-JPY daily returns

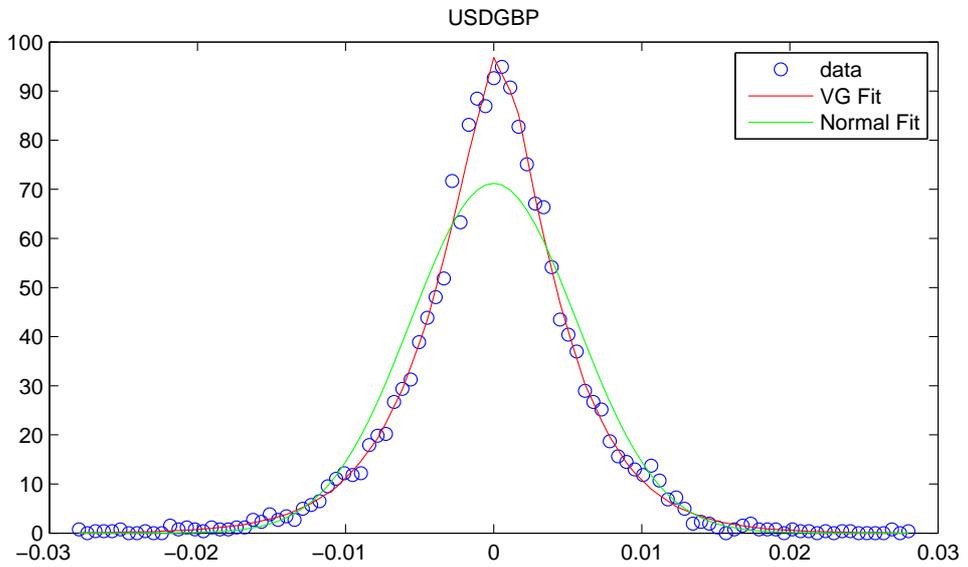


Figure 4.2: Estimation result for USD-GBP daily returns

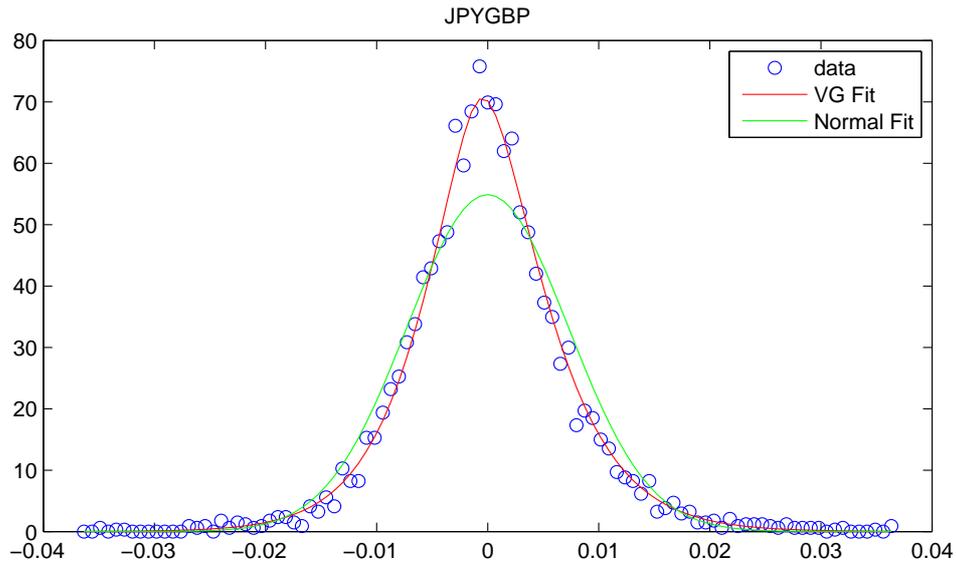


Figure 4.3: Estimation result of JPY-GBP daily returns

### 4.6.3 Cross-currency Option Pricing

One interesting problem in the FX market is using options on two liquid currency pairs to price options on the illiquid cross-currency pair. Because of the special triangular relationship in the FX market, the dynamics of one illiquid FX rate is mainly governed by the rates of each currency with major economies. Using a joint Heston model, Carr and Verma [11] derives the characteristic function of the cross currency rate and applies the FFT option pricing method of Carr and Madan. Options on two liquid currency pairs are assumed to be governed by two Heston processes. To make the process on cross currency rates solvable, they also assume the two rates share the same variance process. The two-dimensional VG model is, to my knowledge, the first model with pure jump processes to deal with this problem. We now use the two-dimensional VG process to price options on an illiquid cross

FX rate given the market prices on two liquid ones.

We consider the options on the illiquid FX pair  $k - j$  (currency  $j$  is quoted in units of currency  $k$ .) which may be computed from the prices of options on liquid pairs  $i - k$  and  $i - j$ . For example, if we are interested in vanilla options on CAD-JPY. We can use the market information on USD-JPY and USD-CAD. For simplicity, we assume deterministic interest rates  $r_i$ ,  $r_j$  and  $r_k$  in three economies. We use  $Q^i$  to denote the risk-neutral measure with numeraire  $i$  which is a money market account invested in currency  $i$ .

Now we define the risk-neutral processes for the two spot FX rates  $S_{ij}(t)$ ,  $S_{ik}(t)$  under measure  $Q^i$  in terms of two VG processes  $X_1(t)$ ,  $X_2(t)$  by:

$$S_{ij}(t) = S_{ij}(0)e^{(r_i - r_j)t} \frac{e^{X_1(t)}}{E^{Q^i}[e^{X_1(t)}]}$$

$$S_{ik}(t) = S_{ik}(0)e^{(r_i - r_k)t} \frac{e^{X_2(t)}}{E^{Q^i}[e^{X_2(t)}]}$$

Here,  $(X_1, X_2)$  is a 2-dimensional VG process with marginal  $VG(\theta_1, \sigma_1, \nu_1)$ ,  $VG(\theta_2, \sigma_2, \nu_2)$  and dependant parameters  $\nu_0$  and  $\rho$ . The characteristic functions for the log returns  $\ln(S_{ij}(t)/S_{ij}(0))$  and  $\ln(S_{ik}(t)/S_{ik}(0))$  are easily written in terms of the joint characteristic function of  $X_1(t)$  and  $X_2(t)$ ,

$$\phi_{(X_1, X_2)}(u_1, u_2) = E(e^{i(X_1(t)u_1 + X_2(t)u_2)})$$

Specifically we have,

$$E(e^{iu \ln(S_{ij}(t)/S_{ij}(0))}) = \exp[iu((r_i - r_j)t - \ln(\phi_{(X_1, X_2)}(-i, 0))]\phi_{(X_1, X_2)}(u, 0)$$

$$E(e^{iu \ln(S_{ik}(t)/S_{ik}(0))}) = \exp[iu((r_i - r_k)t - \ln(\phi_{(X_1, X_2)}(0, -i))]\phi_{(X_1, X_2)}(0, u)$$

The joint characteristic function of two log returns is then given by:

$$E(e^{i(u_1 \ln(S_{ij}(t)/S_{ij}(0)) + u_2 \ln(S_{ik}(t)/S_{ik}(0)))}) = \exp[iu_1((r_i - r_j)t - \ln(\phi_{(X_1, X_2)}(-i, 0))]$$

$$\cdot \exp[iu_2((r_i - r_k)t - \ln(\phi_{(X_1, X_2)}(0, -i))]\phi_{(X_1, X_2)}(u_1, u_2)$$

Since the marginal process can be calibrated to the market prices of options on these two FX rates, we may use the physical correlation to build the dependence and price the cross FX rate options.

Now consider the cross FX rate  $S_{kj}(t)$ . By the triangular equation,

$$S_{kj}(t) = \frac{S_{ij}(t)}{S_{ik}(t)}$$

Let  $Q^k$  be the measure equivalent to  $Q^i$  for which a money market account invested in currency  $k$  is the numeraire. The Radon-Nykodym derivative is given by the process  $G$ :

$$\frac{dQ^k}{dQ^i} = G_t = \frac{S_{ik}(t)}{S_{ik}(0)e^{(r_i - r_k)t}}$$

We can derive the characteristic function of the log return  $\ln(\frac{S_{kj}(t)}{S_{kj}(0)})$  under measure  $Q^k$  given by:

$$\begin{aligned}
\Phi(u) &= E^{Q^k} \left( e^{iuln\left(\frac{S_{kj}(t)}{S_{kj}(0)}\right)} \right) \\
&= E^{Q^i} \left( e^{iuln\left(\frac{S_{kj}(t)}{S_{kj}(0)}\right)} \frac{dQ^k}{dQ^i} \right) \\
&= E^{Q^i} \left( e^{iuln\left(\frac{S_{kj}(t)}{S_{kj}(0)}\right)} e^{(r_i-r_k)t+ln\left(\frac{S_{ik}(t)}{S_{ik}(0)}\right)} \right) \\
&= E^{Q^i} \left( e^{iuln\left(\frac{S_{ij}(t)}{S_{ij}(0)}\right)-(iu-1)ln\left(\frac{S_{ik}(t)}{S_{ik}(0)}\right)} e^{-(r_i-r_k)t} \right) \\
&= E^{Q^i} \left( e^{i\left(uln\left(\frac{S_{ij}(t)}{S_{ij}(0)}\right)-(u+i)ln\left(\frac{S_{ik}(t)}{S_{ik}(0)}\right)\right)} e^{-(r_i-r_k)t} \right) \\
&= \phi_{X_1, X_2}(u, -(u+i)) e^{iu(r_k-r_j)t-iuln(\phi_{(X_1, X_2)}(-i, 0))+i(i+u)ln(\phi_{(X_1, X_2)}(0, -i))}
\end{aligned}$$

With the closed form characteristic function of the log return, we can simply apply the FFT method of Carr and Madan to price an European option written on cross FX rate  $S_{kj}(t)$ .

#### 4.6.4 Market Conventions

The foreign exchange options market has its own way to quote and trade options. FX options are quoted by implied volatilities rather than option prices probably due to the dominance of the Black-Scholes formula in the market. For each currency pair, option quotes are provided at a fixed Black-Scholes delta rather than a fixed strike on a certain expiry date. The liquid options are mainly at five levels of deltas: 10  $\delta$  Put, 25  $\delta$  Put, 0  $\delta$  Straddle, 25  $\delta$  Call and 10  $\delta$  Call. Since different deltas correspond to different strikes, option implied volatilities are usually available at five strikes. Similar to options in equity and interest rate derivatives

market, the volatilities are not constant across the strike. This well-documented observation is called "Volatility Skew" or "Volatility Smile".

Because of the volatility skew observed in options on almost every currency pair, the market conventions to quote volatilities in FX market is through Risk Reversals (RR) and Butterflies (BF) or Strangles. In particular, volatilities of FX options can be decomposed into a symmetric part of the smile reflecting the convexity and a skew-symmetric part of the smile reflecting the skew. A 25  $\delta$  Risk Reversal which reflects the skew of the curve is the difference between 25  $\delta$  call and put options. A 25  $\delta$  Butterfly which reflects the convexity is the difference between the average vol of 25  $\delta$  call and put options and the ATM straddle vol. From the market quotes we may compute the implied volatilities of 25  $\delta$  calls and puts by:

$$\begin{aligned}\sigma_c &= ATM + BF + \frac{1}{2}RR \\ \sigma_p &= ATM + BF - \frac{1}{2}RR\end{aligned}\tag{4.35}$$

For Lévy processes we used here to price options, we need both the strike and the price of the market option data. To back out the strikes for 10 or 25  $\delta$  options, one needs to solve it by definition of the delta. For example, we have the following equation to solve the strike  $K_{25\delta p}$  for a 25  $\delta$  put.

$$-e^{-r_f T} \Phi\left(-\frac{\ln\frac{S_0}{K_{25\delta p}} + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = -0.25$$

The strike of at-the-money straddle is really the strike which makes the straddle delta neutral rather than the spot price. Therefore, no delta hedge is needed when trading this straddle. The strike is chosen so that a put and a call have the same delta but with different signs.

#### 4.6.5 Numerical Results

We test this model on two different sets of FX rates and report the results in this section. The option data we use are taken from Bloomberg on the date of Nov. 18, 2008. All options on each pair have a single maturity of three months. For the two sets of FX rates, they are chosen from 1) 3 pairs of USD (US dollar), EUR (Euro) and GBP (Pound sterling); 2) 3 pairs of USD, GBP and NZD (New Zealand dollar). As US dollar is usually considered as the dominating currency, the cross rates in the two sets are EUR-GBP and GBP-NZD respectively. While the options on the cross rate EUR-GBP are still fairly liquid, the bid-ask spread of options on GBP-NZD, however, is quite big. The implied volatility curves in two sets also show quite different patterns. We thus want to show the power of this model to explain these phenomena.

From Bloomberg, we get the Black-Scholes implied volatilities for options with strikes corresponding to the ATM straddle,  $25 \delta$  Call,  $25 \delta$  Put,  $10 \delta$  Call and  $10 \delta$  Put. Therefore, for each FX rate we have five European option prices, which can be used to calibrate the model parameters. Apply the multivariate VG model, we calibrate the model in the following two steps:

1. Calibrate the risk-neutral VG marginal processes to the options on two liquid pairs (e.g. EUR-USD and GBP-USD in set one). For each pair, we apply the FFT method to minimize the absolute errors between model and market prices with three free parameters.
2. Use the marginal processes from step 1 and calibrate two parameters of depen-

dence to the options on the cross rate (e.g. EUR-GBP on set one). The optimization is again done by the FFT method with constraint  $\nu_0 \geq \max(\nu_1, \nu_2)$  and  $0 < \rho < 1$ .

The objective functions in both steps are thus

$$\sum_{alloptions} (\sigma_{market} - \sigma_{model})^2$$

It is shown that, in these two examples, the multivariate VG model successfully captures the smile dynamics of both the marginal processes and its cross-rate process. We report the risk-neutral parameters and fitting RMSE for implied volatilities in the following tables (The two sets share the same GBP-USD pair.):

Table 4.7: Results on options of liquid pairs

	EUR-USD	GBP-USD	NZD-USD
$\theta$	-0.0902	-0.2097	-0.3638
$\sigma$	0.2153	0.2424	0.2683
$\nu$	0.1855	0.1656	0.2007
<i>RMSE</i>	0.0015	0.0011	0.0034

The following table reports the calibration result on the cross-rate options in these two sets.

The model fit of the implied volatilities on the cross-rates:

Figure 4.4: Marginal calibration result of set I

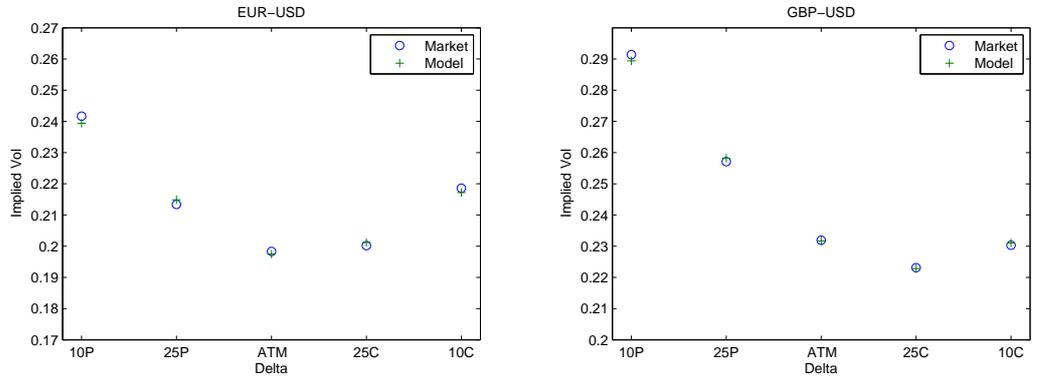


Figure 4.5: Marginal calibration result of set II

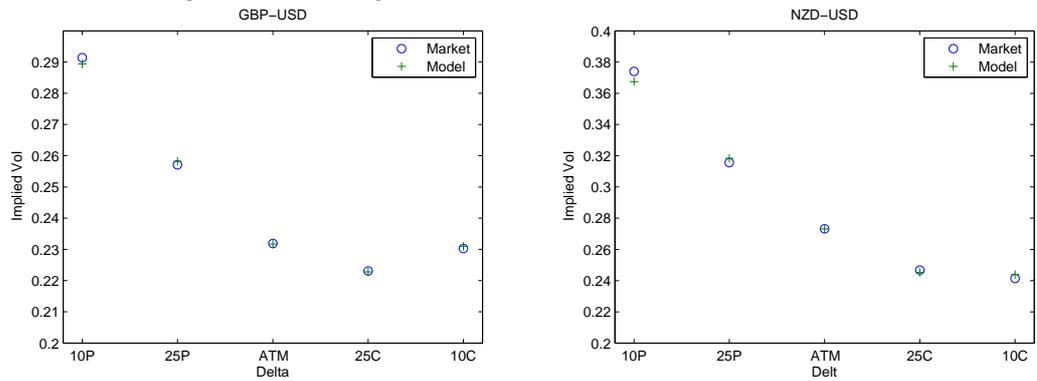


Figure 4.6: Calibration results on the cross-rate options

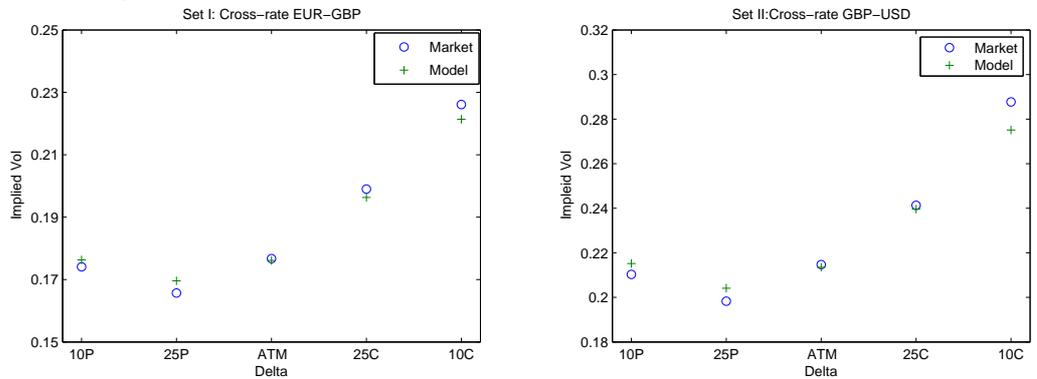


Table 4.8: Results on options of the cross-rate

	$\nu_0$	$\rho$	RMSE
Set I: EUR-GBP	0.1855	0.3929	0.0032
Set II: GBP-NZD	0.2007	0.6695	0.0077

## Chapter 5

### Conclusion and Future Study

In conclusion, we explore how to extend VG processes to multivariate level and study the properties of this new process in this thesis. Compared with the existing models, our model has very flexible dependence structure and is easy to work with. We derive the closed-form joint characteristic function and describe the estimation procedure and simulation scheme. Because of the systematic factor included in this process, we test and find that this process has a significant improvement in explaining the portfolio returns in stock market over the Gaussian copula method. Finally, we derive pricing methods for multi-asset options using simulation and Fourier transform method. We also apply the model to the options on FX currency pairs and show that the multivariate VG model we propose can fit the cross-rate option prices very closely.

The future study will focus on how to extend this model to the stochastic volatility version so that it may fit option price surface with different maturities. Also it will be interesting to investigate the effective methods of pricing multi-asset options in a high dimensional ( $n > 4$ ) case.

## Appendix A

### Proof of Theorems in Chapter 4

#### A.1 Proof of Theorem 4.5

**Theorem A.1.** Consider  $P(x_1, x_2) = (1 - e^{x_1} - e^{x_2})^+$ . For any real numbers  $\epsilon = (\epsilon_1, \epsilon_2)$  with  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$

$$P(x) = (2\pi)^{-2} \iint_{R^2+i\epsilon} e^{iux'} \hat{P}(u) d^2u$$
$$\text{where, } \hat{P}(u) = \frac{\Gamma(-iu_1)\Gamma(-iu_2)}{\Gamma(2 - i(u_1 + u_2))}.$$

Here  $\Gamma(z)$  is the complex gamma function defined by the integral  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  for  $\text{Re}(z) > 0$ .  $u = (u_1, u_2)$  and  $x = (x_1, x_2)$ .

Proof: First we compute,  $\hat{P}$ , the inverse Fourier transform of  $P$ .

$$\begin{aligned} \hat{P}(u) &= \iint_{R^2} e^{-iux} P(x) d^2x \\ &= \int_{-\infty}^0 e^{-iu_1x_1} \left[ \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2x_2} [1 - e^{x_1} - e^{x_2}] dx_2 \right] dx_1 \\ &= \int_{-\infty}^0 e^{-iu_1x_1} (1 - e^{x_1})^{1-iu_2} \left[ \frac{1}{-iu_2} - \frac{1}{1 - iu_2} \right] dx_1 \end{aligned}$$

Let  $z = e^{x_1}$  and  $dz = e^{x_1} dx_1$ . We have:

$$\begin{aligned}\hat{P}(u) &= \frac{1}{(1-iu_2)(-iu_2)} \int_0^1 z^{-iu_1-1} (1-z)^{1-iu_2} dz \\ &= \frac{1}{(1-iu_2)(-iu_2)} \text{Beta}(-iu_1, 2-iu_2) \\ &= \frac{\Gamma(-iu_1)\Gamma(-iu_2)}{\Gamma(2-(u_1+u_2)i)}\end{aligned}$$

$(u_1, u_2)$  will be on the plane  $R^2 + i\epsilon$  satisfying  $\epsilon_{1,2} > 0$  to make the above computation valid.

## A.2 Proof of Theorem 4.7

Proof: We consider the following multivariate integral, where  $u = (u_1, \dots, u_n)$ ,  $x = (x_1, \dots, x_n)$  are n-dimensional vectors.

$$\begin{aligned}\hat{P}(u) &= \int \dots \int_{R^n} e^{-iux} P(x) d^n x \\ &= \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_{-\infty}^{\log(1-e^{x_1}-\dots-e^{x_{n-1}})} e^{-iu_n x_n} [1 - e^{x_1} - e^{x_2} - \dots \\ &\quad - e^{x_n}] dx_1 dx_2 \dots dx_n \\ &= \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_{-\infty}^{\log(1-e^{x_1}-\dots-e^{x_{n-1}})} (1 - e^{x_1} - e^{x_2} - \dots \\ &\quad - e^{x_{n-1}})^{1-iu_n} \left[ \frac{1}{-iu_n} - \frac{1}{1-iu_n} \right] dx_1 dx_2 \dots dx_{n-1} \\ &= \frac{1}{(1-iu_n)(-iu_n)} \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_{-\infty}^{\log(1-e^{x_1}-\dots-e^{x_{n-1}})} (1 - e^{x_1} - e^{x_2} - \dots \\ &\quad - e^{x_{n-1}})^{1-iu_n} dx_1 dx_2 \dots dx_{n-1}\end{aligned}$$

Let  $z = \frac{e^{x_{n-1}}}{1-e^{x_1}-\dots-e^{x_{n-2}}}$ , then  $dz = \frac{e^{x_{n-1}}}{1-e^{x_1}-\dots-e^{x_{n-2}}} dx_{n-1}$ . By substitution, the integral part of  $x_n$  can be written as a Beta function.

$$\begin{aligned}
\hat{P}(u) &= \frac{1}{(1-iu_n)(-iu_n)} \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_0^1 (1-e^{x_1}-e^{x_2}-\dots \\
&\quad -e^{x_{n-2}})^{1-iu_n-iu_{n-1}} z^{-1-iu_{n-1}x_{n-1}} (1-z)^{1-iu_n x_n} dx_1 dx_2 \dots dz \\
&= \frac{\beta(-iu_{n-1}, 2-iu_n)}{(-iu_n)(1-iu_n)} \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_0^1 (1-e^{x_1}-e^{x_2}-\dots \\
&\quad -e^{x_{n-2}})^{1-iu_n-iu_{n-1}} dx_1 dx_2 \dots dx_{n-2} \\
&= \frac{\Gamma(-iu_n)\Gamma(-iu_{n-1})}{\Gamma(2-iu_{n-1}-iu_n)} \int_{-\infty}^0 e^{-iu_1 x_1} \int_{-\infty}^{\log(1-e^{x_1})} e^{-iu_2 x_2} \dots \int_0^1 (1-e^{x_1}-e^{x_2}-\dots \\
&\quad -e^{x_{n-2}})^{1-iu_n-iu_{n-1}} dx_1 dx_2 \dots dx_{n-2}
\end{aligned}$$

By using substitution recursively, we may write the right side integral again as a series of beta functions.

$$\begin{aligned}
\hat{P}(u) &= \frac{\Gamma(-iu_n)\Gamma(-iu_{n-1})}{\Gamma(2-iu_{n-1}-iu_n)} \times \beta(-iu_{n-2}, 2-iu_{n-1}-iu_n) \dots \beta(-iu_1, 2-iu_n-iu_{n-1}-\dots-iu_2) \\
&= \frac{\Gamma(-iu_n)\Gamma(-iu_{n-1}) \dots \Gamma(-iu_1)}{\Gamma(2-iu_n-iu_{n-1}-\dots-iu_1)} \\
&= \frac{\prod_{i=1}^n \Gamma(-iu_i)}{\Gamma(2-i \sum_{i=1}^n u_i)}
\end{aligned}$$

$(u_1, u_2, \dots, u_n)$  will be on the plane  $R^n + i\epsilon$  satisfying  $\epsilon_{1, \dots, n} > 0$  to make the above computation valid.

### A.3 Proof of Theorem 4.9

We compute,  $\hat{P}$ , the inverse Fourier transform of  $P(x_1, x_2) = (\min(e^{x_1}, e^{x_2}) - 1)^+$ .

$$\begin{aligned}\hat{P}(u) &= \int_0^\infty e^{-u_1 x_1} (e^{x_1} - 1) \int_{x_1}^\infty e^{-iu_2 x_2} dx_2 dx_1 + \int_0^\infty e^{-u_2 x_2} (e^{x_2} - 1) \int_{x_2}^\infty e^{-iu_1 x_1} dx_1 dx_2 \\ &= I_1 + I_2\end{aligned}$$

We only need to compute  $I_1$  as two terms are symmetric with  $(x_1, x_2)$ . Let  $z = e^{-x_1}$ ,  $dz = -e^{-x_1} dx_1$ .

$$\begin{aligned}I_1 &= \frac{1}{iu_2} \int_0^\infty e^{-iu_1 x_1 - iu_2 x_1} (e^{x_1} - 1) dx_1 \\ &= \frac{1}{iu_2} \int_0^1 z^{i(u_1 + u_2) - 2} (1 - z) dz \\ &= -\frac{1}{u_2(u_1 + u_2)(i(u_1 + u_2) - 1)}\end{aligned}$$

By symmetry,

$$I_2 = -\frac{1}{u_1(u_1 + u_2)(i(u_1 + u_2) - 1)}$$

We conclude

$$\hat{P}(u) = -\frac{1}{u_1 u_2 (i(u_1 + u_2) - 1)}$$

$(u_1, u_2)$  will be on the plane  $R^2 + i\epsilon$  satisfying  $\epsilon_{1,2} < 0$  to make the above computation valid.

Similarly, for  $P(x_1, x_2) = (1 - \max(e^{x_1}, e^{x_2}))^+$ , the computation of inverse Fourier transform is more or less the same. The result is also the same except  $(u_1, u_2)$  will be on the plane  $R^2 + i\epsilon$  satisfying  $\epsilon_{1,2} > 0$ .

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