THE EFFECT OF BEHAVIORAL OBJECTIVES ON MEASURES OF LEARNING AND FORGETTING ON HIGH SCHOOL ALGEBRA

by

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Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1972

APPROVAL SHEET

Title of Thesis: The Effect of Behavioral Objectives on Measures of Learning and Forgetting on High School Algebra

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Doctor of Philosophy, 1972

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ABSTRACT

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Elwood Lockert Loh, Doctor of Philosophy, 1972

Thesis directed by: Dr. Henry H. Walbesser
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During the past decade, the number of educators who advocate the use of behavioral objectives in education has increased. The increase in the number of advocates of behavioral objectives has been followed by an increasing awareness of the need for empirical research to give credence to such a viewpoint. At present, there is not a substantial number of research studies in which behavioral objectives have been used as a manipulated variable.

In previously reported learning studies in which behavioral objectives have been used as an experimental variable, measures of learning and measures of forgetting have been derived from achievement scores. The results obtained in the learning studies have not been singular in support of the use of behavioral objectives, however, the

results obtained in forgetting studies have consistently supported their use.

This two part study investigated the effect of presenting behavioral objectives to students during the initial phase of a learning program. There were six criterion variables observed: index of learning, rate of learning, index of forgetting, rate of forgetting, index of retention, and index of efficiency.

Two 2-year algebra one classes with a total of 52 students were randomly partitioned into two treatment groups for the learning phase of the study. The classes were further randomly partitioned into three retention groups for the forgetting phase of the study.

The instructional materials were programmed within the framework of a learning hierarchy. The use of the learning hierarchy facilitated the use of a procedure for separating behaviors not yet possessed by a student from behaviors previously acquired. This was accomplished by presenting students with preassessment tasks prior to instruction for a behavior in the learning hierarchy. If the subject's response to the preassessment task indicated that he possessed the behavior, instruction was not given for that behavior. If the response indicated that the subject had not previously acquired the behavior, instruction was presented. The measures of the time needed to acquire the behavior were subsequently used to compute the six experimental measures.

Three retention periods of 7 calendar days, 14 calendar days, and 15 to 21 calendar days were used for the forgetting phase of the study. The results of the three retention periods were pooled for the two forgetting measures, the index of retention, and the index of efficiency.

The data collected in the study were analyzed by six separate tests using a one-way analysis of variance. A 0.05 level of significance was used for each of the six tests. The following results were obtained:

1. The index of learning for students who were informed of behavioral objectives during the initial phases of the learning program was not greater than the index of learning for students who were not so informed.

2. The rate of learning for students who were informed of behavioral objectives during the initial phases of the learning program was not greater than the rate of learning for students who were not so informed.

3. The index of forgetting for students who were informed of behavioral objectives during the initial phases of the learning program was not less than the index of forgetting for students who were not so informed.

4. The rate of forgetting for students who were informed of behavioral objectives during the initial phases of the learning program was not

less than the rate of forgetting for students who were not so informed.

5. The index of retention for students who were informed of behavioral objectives during the initial phases of the learning program was not greater than the index of retention for students who were not so informed.

6. The index of efficiency for students who were informed of behavioral objectives during the initial phases of the learning program was not greater than the index of efficiency for students who were not so informed.

It was concluded that the results of the study do not support the use of behavioral objectives as a procedure for improving either measures of learning or measures of forgetting which are functions of the time needed to reach criterion in a learning program using programmed instruction for teaching an algebraic topic to below average mathematics students in senior high school. It was recommended that further research is needed to determine a reliable and valid procedure for measuring learning and forgetting. It was also recommended that alternatives to programmed instruction be considered for learning and forgetting studies.

ACKNOWLEDGEMENTS

It is probably impossible to thank everyone who is in some way deserving of thanks for assistance and encouragement during the various stages of a task such as this. However, one would be derelict not to make an attempt to thank some of the people who so generously gave of their time.

First, and foremost, I should like to express my heartfelt thanks to Dr. Henry Walbesser for his assistance and encouragement. Throughout this task, Dr. Walbesser has been an excellent mentor always giving plenty of rope but never allowing a hanging.

Secondly, words alone seem less than sufficient for me to thank Dr. Clayton Stunkard whose wise counsel I value so much. Dr. Stunkard deserves credit for any degree of clarity that may be found in this study as the result of the many hours of time he so willingly gave.

I should also like to thank Dr. Allen Chotiner, the principal of the school in which this study was conducted. Two others who deserve many thanks are Mrs. Laurie Yeomans and Mr. Ron Imbriale, the two teachers who provided the assistance in the classroom that made the experiment possible. Fifty-two thank yous go to the students in the study.

A very special thanks goes to the lady who is responsible for the beautiful typing of this thesis, Mrs. Beverly Gladd. Without the assistance of this very capable person, this thesis would be little more than a pile of papers with barely intelligible scribbles.

Last, but not least, I thank my wife, Norma, and my daughter, Janine, who tolerated me and my pile of scrap paper during this project. May they henceforth be able to use my desk without fear of disturbing my "junk."

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Chapter 1

INTRODUCTION AND REVIEW OF THE LITERATURE

Do students learn faster as a result of some "treatment effect" during the initial phase of a period of instruction? Do students forget more slowly as a result of the same "treatment effect"?

The answers to these questions are not as definitive as they may appear. The questions tend to beg either a "yes" or a "no" response. However, the law of the excluded middle does not apply and a third alternative response is "the questions are moot." It is, probably, this third response that makes the questions important.

The questions are important to students, parents, educators, and all others concerned with learning because they focus on several problems of learning and forgetting. In particular, how does an experimenter "know" that learning has occurred as a result of instruction? If learning has occurred, how does he measure only the learning that is the result of a specific instructional period? What is the relevance of "learning faster"? Is the "treatment effect" one that may readily be used in a classroom learning situation? Analogous questions may be asked about forgetting. Hence, the comments that follow apply to both learning and forgetting. If there are exceptions, they will be so noted.

How does an experimenter "know" that learning and/or forgetting has occurred? With specific reference to learning, Hilgard (1951:517) has stated

> Learning is always an inference, derived from changes in performance, and learning is not the only factor that can cause these changes. Performances change as organisms grow older, they change as organisms become fatigued, they change with the state of the motivation. Only by appropriate controls can it be ascertained that the changes studied are surely to be classified as learning, or, more strictly, to be used to make inferences about learning.

It seems fair to argue that forgetting is also an inference, derived from changes in performance.

When measurements of learning and forgetting are made, they are measures of changes in performance. The performance must be in the form of some observable behavior. If a behavior is not observable, any measurement of change in performance is virtually impossible. Therefore, it may be fair to conclude that one "knows" that learning and/or forgetting has occurred by observing a change in the behavior of a subject.

The question, "How does an experimenter measure only the learning that is a result of a specific instructional period?" is a question of procedure. The answer to this question may vary greatly from situation to situation.

In the present study, the procedure to be used will be facilitated by a learning hierarchy which partitions a task to be learned into a collection of subordinate tasks. By pretesting a subject at each subordinate task, it will be possible to delete from a measurement of learning those behaviors which the subject has acquired prior to instruction. The measure of learning during a period of instruction will be the sum of the measures of learning subordinate behaviors during the period of instruction.

Because not all subordinate tasks are of equal difficulty, a provision will be made to compensate for unequal difficulty in the measurement of learning subordinate tasks. The procedure described above provides a method for separating the measurement of "new" learning from "old" learning and also includes a provision for unequal difficulty. Measures of forgetting can be accomplished similarly.

The question, "What is the relevance of 'learning faster'?" is a question of priorities. The answer to this question will, undoubtedly, vary from individual to individual. For example, if an individual places a high priority on time, then "learning faster" is relevant to time.

The school systems, necessarily, place high priorities on time because of financial burdens and artificial constraints such as the Carnegie unit. If the question is relevant to time, variables that help increase the rate at which an individual learns are clearly of importance. Similarly, if the rate of forgetting can be reduced, then the time spent relearning may be used for new learning.

The "treatment effect" may be any one of a vast number of manipulated variables. In the present study, the "treatment variable" will be knowledge of behavioral objectives prior to instruction. This treatment is easily adapted to a classroom learning situation and may be used for a large number of learning models.

The question to be answered by this study may be stated as follows: Does the apprising of instructional objectives in behavioral terms prior to instruction optimize measurements related to learning and/or forgetting?

A MODEL FOR LEARNING

Experiments in learning have been conducted for at least 90 years. The classical experiments of the German psychologist Ebbinghaus are reported by Bugelski (1964:16) to have been the first recorded learning experiments. Ebbinghaus began with the assumption that retention was a function of effective learning. Further, he assumed that by discovering the factors of retention and forgetting he would be able to find principles which would lead to more effective learning and a lessening of forgetting.

Although Ebbinghaus did not contribute much to solving the problems of improving classroom study operations, which were of interest to him, he did significantly contribute to experimental psychology by providing a "method," or perhaps more appropriately stated, a "model," for studying learning. His nonsense syllable approach, in effect, greatly simplified a much more complex problem. Since the 1885 publication of Memory by Ebbinghaus (and its 1913 English translation) there has been a plethora of learning experiments based on his model.

The theoretical model for many learning experiments is the Ebbinghaus paired-associate learning model. There are several assumptions, either explicit or implicit, in a paired-associate learning experiment. Most of these assumptions are related to measures of learning. Some of these assumptions are as follows:

1. The level of difficulty of any pair of items in a list of paired-associates is equal.

2. No associates in a list of paired-associates to be learned are known by the learner prior to the period of learning.

3. The rate of learning is either the number of correct associations per unit of time or the number of correct associations per number of trials.

A Laboratory Learning Model--A Discussion

The appropriateness of the assumptions above for classroom learning experiments is questionable. Several reasons are readily evident for such questioning. In general, the appropriateness of the theoretical learning model to a classroom setting is questionable. The following remark by Gagné and Rohwer (1969:381) is revealing.

> Remoteness of applicability to instruction, we note with some regret, characterizes many studies of human learning, retention, and transfer, appearing in the most prestigious of psychological journals. The findings of many studies of human learning presently cannot be applied directly to instructional design for two major reasons: (a) the conditions under which the learning is investigated, such as withholding knowledge of learning goals from the subject and the requiring of repetition of responses, are often unrepresentative of conditions under which most human learning occurs; and (b) the tasks set for the learner (e.g., the verbatim reproduction of verbal responses, the guessing of stimulus attributes chosen by the experimenter, among many others) appear to cover a range from the merely peculiar to the downright esoteric. This is not to imply that such studies do not

> further an understanding of the learning process. However, it would seem that extensive theory development centering upon learning tasks and learning conditions will be required before one will be able to apply such knowledge to the design of instruction for representative human tasks.

The implication of the statement by Gagné and Rohwer is that the methodology of human learning experiments is not always adaptable to a classroom. The information to be inculcated upon the learner in human learning experiments is not always representative of classroom information. The attempt to control many variables during a learning experiment is seldom practiced in a classroom.

To be useful to classroom research the methodology of human learning experiments must be modified to be used in a classroom setting. The modifications may, however, be so severe as to literally make the methodology impotent. A closer look at the three assumptions above makes it clear that they are not appropriate for a classroom learning experiment.

<u>Assumption one</u>. The first assumption, obviously, cannot be met. The materials to be learned in a classroom may vary in difficulty from the very easy to the very difficult. Thus, a measure of difficulty for each part of the learning sequence needs to be established.

Assumption two. The second assumption that a subject will enter a learning situation not knowing any of the information to be learned is clearly fallacious in a classroom learning situation. In an instructional program using programmed materials, the criteria often used in constructing the program is determined by the subject of least ability. Some frames of the program will be unnecessary for some subjects because they already possess the behavioral objective of the frame. Any measure of learning which is to be restricted to learning that results from a particular program of instruction will be in error if prior knowledge is not considered.

Assumption three. The third assumption is not valid in a classroom learning situation. The assumption that rate of learning is the number of correct associations per unit of time or, alternatively, the number of correct associations per number of trials, is suggestive of an equal level of difficulty for each part of the instructional program. Such a suggestion must be rejected. Thus, this assumption about the rate of learning is not appropriate for a classroom learning study.

A Classroom Learning Model

Although the theoretical model of human learning based on the paired-associate model due to Ebbinghaus is not directly applicable, a model due to Gagné (1965) is applicable. Gagné (1961, 1962, 1963a, 1963b) argues that it is necessary to learn a collection of primitive concepts before one can learn a higher level concept that is dependent upon the subordinate concepts. Following the argument of Gagné, the learning of a complex behavior may be divided into a finite number of subordinate behaviors. Each of the subordinate behaviors can again be divided into a finite number of lower level subordinate behaviors. This procedure can be repeated until low level subordinate behaviors are reached that are known to have been acquired by every learner. This non-unique partitioning of a complex behavior is a hierarchy of learning dependencies.

In an address given before the National Council of Teachers of Mathematics, Gagné (1963a:622) described his hierarchical learning sequence as follows:

> The method of making an analysis to arrive at the set of subordinate knowledges arranged in a hierarchy . . . should be briefly mentioned. One begins with the terminal class (or classes) of tasks for which learning is being undertaken. For each of these, one asks the question, "What must the learner already know how to do, in order to achieve this (new) performance, assuming that he is to be given only instructions?" The latter part of this question assumes that the

> instructions will have the functions of the frames previously mentioned, with the exception of task repetition. The answer to this question defines one or more elements of subordinate knowledge. The question is then applied to each of these in turn, thus identifying the entire hierarchy. The process ends when one arrives at subordinate knowledge which can be assumed to be possessed by every learner for who the learning program is intended.

Implicit in the Gagné model is the use of behavioral objectives to specify the behavior to be exhibited and the criterion for acceptance or rejection of the exhibited behavior. Clearly, one of the largest and most influential programs that has been developed using the Gagné hierarchy of behavioral objectives is the American Association for the Advancement of Science elementary school project, Science--A Process Approach.

MEASURES OF LEARNING

Definitions in mathematics are generally consistent in usage by all mathematicians. Unfortunately, the same cannot be said for learning theory. The point is well made by Restle and Greeno (1970:254), who write

> For example, in learning theory it is common to refer to the learning rate as a constant and to trials as a variable. From one experiment to another the learning rate certainly varies, and it may well vary from one subject to another and from one group to another within an experiment. Meanwhile, the same trials (1,2,3,...) are used in all experiments. What then is meant by constant and variable?

Because of the lack of consistent definitions, measurement continues to be a persistent problem in learning studies.

Carrol (1963) presented a model for school learning. The reader is cautioned that the model of learning being presented should not be confused with what is ordinarily called "learning theory." The emphasis of the model is the development of a rationale for the degree of learning. Degree of learning is described as a function of the ratio of the time spent learning to the time needed to learn. Studies by Sjogren (1967) and Shores (1961) support Carrol's argument for a model of learning in which degree of learning is a function of time.

Wang (1968) investigated several procedures for measuring and making predictions about the rate of learning in the classroom. The study by Wang was concerned with rate of learning in classrooms in which the mode of instruction was individualized. The study involved 609 elementary school children who worked in six different units of Individually Prescribed Instruction Project (IPI) mathematics. The results of the study suggest there are no clear cut answers concerning the rate of learning in IPI classrooms. Wang suggests that the major contribution of the study may be interpreted as a warning against making the too easy assumption that such simple measures as number of units or number of skills completed per unit of time are a valid measure of rate of learning. An additional point of emphasis is that the results of the study indicate that the rate of learning is specific to a given task and is not a general factor characterizing student performance in all learning situations.

Various procedures have been used in measuring difficulty in learning sequences in which programmed materials have been used for instruction. The most commonly reported procedure has been in terms of "size-of-step" of a frame. Homme and Glaser (1959) define difficulty in terms of size-of-step as being the number of steps required to take the learner from the initial to the terminal stages in a program. Lumsdaine (1959:351) considered several meanings of size-of-step as related to difficulty:

> Size-of-step may refer to the difficulty of giving a correct response to any item. Also, it may refer to the difference or increment in difficulty between the concept, relationships, or terminology between successive steps or set of steps in the program.

Carr (1959:543) describes size-of-step as,

> . . . when the term is used as a dependent variable, it is usually specified by the percentage of incorrect responses. Thus, if learners make few error responses on a given program, the size-of-step is inferred to be small.

Another procedure for defining difficulty is due to Green (1962). Green defined difficulty in terms of the "density of a program." He defined "density of a program" as the ratio of the total number of different responses in a program to the total number of responses.

Moore and Smith (1965:7) report, "Green's (1962) concern with developing an adequate description of a program, and correspondingly, of the difficulty of a frame, reflects the inadequacies of existing empirical methods for this purpose."

Several procedures for measuring difficulty of a program frame are described by Moore and Smith (1965). In one procedure, each subject is asked to check one of five levels of difficulty (very easy, easy, average, hard, very hard) for each frame in the program. In another procedure, instructional frames were presented in pairs. The two frames present the same information but in slightly different ways. The subject is to select the frame that he thinks is the easier of the two. Moore and Smith (1965:72) conclude, " . . . the learner's perceived difficulty of a program is a function of his level of achievement on the program."

The general lack of a universally accepted collection of definitions in classroom learning experiments allows for each experimenter to use those definitions most useful for his needs or to construct new definitions. The latter shall be done in this study following the arguments of Carrol (1963) and heeding the warning of Wang (1968).

FORGETTING

An experimental study of forgetting represents a two-stage experiment. The subjects must first learn a task and then, secondly, subsequent to a specified interval, retention is measured. If all subjects learn to exactly the same degree then it may be reasonably assumed that the effect of the degree of learning will be the same on all retention measurements.

Underwood (1964:112), after many years of researching short term memory and forgetting, made the following point:

> Very often in learning research an independent variable is introduced during the first stage to test some hypothesis about its influence on the second stage. . . . If the variable introduced in the first stage influences the rate of change in performance in this stage, a warning about a potential confounding in later stages should be automatic. For in fact, if rate of learning in the first stage differs, final level of performance attainment may vary.

Retention Measures and Behavioral Objectives

Although the manipulation of behavioral objectives in learning studies has not produced singular results, their effect on retention has been positive. The results of the studies by Engel (1968), Cook (1969), and J. Smith (1970) all indicate that those subjects who have been given behavioral objectives prior to a period of instruction have significantly higher retention scores. A search of the literature did not reveal any studies with results that indicate that behavioral objectives negatively affect retention.

The retention measures in the studies cited above may have been confounded by their measures of learning. In each of the three studies, the mode of instruction was individualized. The measure of learning for each student began at the lowest level of the instructional format and proceeded through the terminal behavior. Thus, no provision was made for those behaviors that a subject had previously acquired if the behavior was a part of the learning sequence. An assumption seems to have been made that subjects would be acquiring new learning throughout the learning sequence.

If a subject has all of the behaviors of a learning sequence before a period of instruction, then the degree of learning for that subject as a result of the instruction is zero. In such a case, a measure of retention is clearly inappropriate.

BEHAVIORAL OBJECTIVES

A review of the educational literature of the past few years reveals a tendency for educators to become more behavioristic. Educational researchers, especially, continue to move toward the point of view of stating behavioral objectives and then proceeding towards specific goals. It may reasonably be expected that the use of behavioral objectives in education will accelerate as "how to do it" books about preparing behavioral objectives increase in number. Among the "how to do it" books presently available are those by Walbesser (1968a), Drumheller (1970), and Yelon and Scott (1971).

The overall effect of behavioral objectives in education will probably not be known for several generations. The use of behavioral objectives clearly delimits a learning task, since once an end is prescribed only the means to that end need be considered. Rosove (1971:37) makes this point when he states,

> Those who assert the importance of stating behavioral objectives in education believe, if I may speak for them, that in the performance of <u>any</u> task it is salutary to have a good idea of what one's goals or objectives are. Why should this be good practice for every activity <u>except</u> education? Why is it desirable to state clearly one's objectives in advance of undertaking any complex task? Because once goals are clearly stated (to put a man on the moon by 1970; to invade

> Normandy on June 6, 1944; to become President of the United States; etc.), one is then in a position to determine how the goal will be achieved.

Also, according to Mager (1962:1), "You cannot concern yourself with the problems of selecting the most efficient route to your destination until you know what your destination is."

Unfortunately, though many have in words accepted the use of behavioral objectives, they have not done so in deeds. A notable exception is the American Association for the Advancement of Science (AAAS). The elementary school science project, Science--A Process Approach, was developed by the AAAS. In Science--A Process Approach instructional units are based on behavioral objectives that enhance learning as well as provide a means of evaluating the program. The existence of such a large scale curriculum project is evidence of the feasibility of developing curricula based on behavioral objectives.

Even though large curriculum projects have been developed and evaluation (Walbesser, 1968b) has demonstrated that students do learn, there is not universal acceptance of behavioral objectives as a means of improving learning. For example, Silberman (1970) is a critic of the viewpoint of those who advocate the use of behavioral objectives. Although Silberman criticizes the use of behavioral objectives, he does not offer any evidence of research which demonstrates that their use has had any negative effect on learning.

Eisenberg (1970:19) reports that " . . . , not a single instance of behaviorally stated objectives hindering performance was encountered in the literature." Also, a later search of the literature published up to June of 1971 did not uncover any research that indicates that behavioral objectives have an adverse effect on learning.

Although there are numerous references to behavioral objectives and arguments for their use in education, nearly all arguments are unsubstantiated by empirical research. The use of behavioral objectives as manipulated variables in educational research is not extensively reported. The results of the published research studies, in which behavioral objectives were an independent variable, are inconsistent. The studies restricted to some measure of learning have also had varying results when behavioral objectives were manipulated.

Studies Not Supporting the Use of Behavioral Objectives

In a study by Baker (1969) no significant difference in achievement was reported between students who were given objectives behaviorally and students who were given objectives non-behaviorally. The lack of significant differences in the study may have been due to a procedural problem. Baker (1969:5) remarks, "Teachers' faulty understanding of objectives, indicated by their inability to provide relevant classroom practice and to identify, when asked, test items measuring given objectives, may have accounted for lack of differences."

Slow learner junior high school students were divided into two groups in a study by S. Smith (1967). Those in one group received behavioral objectives prior to a period of instruction. Those in the second group did not receive behavioral objectives. Instruction for both groups was programmed. The programmed materials were the same for both groups. The findings indicated there was no significant difference in achievement between the two groups.

Rowan (1971) conducted a study with 92 fifth grade students as subjects. The students were from three different counties in Maryland. Each of the three classes were randomly separated into two treatment groups. One treatment group was given programmed materials with behavioral objectives and the other group was given programmed materials without behavioral objectives. The results of the research indicated there was no significant difference in the order of completion of the two treatment groups. Also, in Rowan's study, the treatment effect of behavioral objectives on achievement was not significant.

Similarly, studies by Cook (1969) and J. Smith (1970) do not support the argument that using behavioral objectives will result in significantly improved learning. Self-paced individualized instruction was used for each of these two studies. The results of the study by Cook may be erroneous because of a non-valid learning hierarchy. The study by J. Smith may also have results which are suspect. In J. Smith's study, students were allowed to work independently and unsupervised. No provision was made to keep treatment groups separated. Therefore, confounding of treatments may well have occurred.

No provision was made in any of these studies to separate "prior" learning from "new" learning. Hence, the measures of learning used in the studies were not necessarily restricted to new learning. The omission of such a provision may possibly account for the results of the studies which do not support the use of behavioral objectives as a positive modifier of learning time.

Studies Which Support the Use of Behavioral Objectives

There are now and have been for several years many who strongly endorse the viewpoint of using behavioral objectives as a means of improving instruction. One of the early leaders was Tyler (1934) who recognized the need for specificity of instruction if valid achievement tests were to be constructed. A recent text by Popham and Baker (1970) is evidence of continued interest today.

An interest in the use of behavioral objectives has been shown by the armed forces of the United States for the purpose of streamlining training procedures. In reviewing the objectives of the military, R. Smith (1964:3) wrote:

> The performance required by the soldier in his job is the basic source of training objectives. The key question is, "What must this soldier be able to do in order to do his job well?" It is highly important that the soldier be taught only the things he needs for doing his job; teaching him things that are irrelevant to the job or teaching him the wrong things can be very costly.

The empirical research evidence supporting the use of behavioral objectives to improve some aspect of learning is sparse. The following studies are among the few reported with results supporting the use of behavioral objectives as a positive modifier of learning.

Engel (1968) conducted a study that was very similar to the previously reported study by S. Smith. Interestingly, the results of Engel's study supported the use of behavioral objectives to improve achievement whereas the results of the study by S. Smith did not. The difference of results may have been caused by any one of several differences in the two studies. S. Smith's subjects were slow learner junior high school students, whereas the subjects in Engel's study were college students majoring in elementary education. Hence, the learning priorities of the two sets of subjects may have been markedly different.

Another difference in the two studies by S. Smith and Engel that may account for the inconsistency of their results was the procedure for ascertaining whether or not the subjects were aware of the behavioral objectives in each unit of instruction. S. Smith's study did not include a procedure for determining if students were aware of the specified learning behaviors. Engel's study did include a rather novel procedure for making subjects aware of the behavioral objective of each lesson.

Another study that provides evidence in support of use of behavioral objectives as a positive modifier of learning was conducted by Dalis (1970). In the study by Dalis, tenth-grade health and safety students who were given precisely stated behavioral objectives prior to a period of instruction achieved significantly higher than students who were given either vaguely stated behavioral objectives or no objectives.

The results of the studies by Engel and by Dalis are consistent in support of the use of behavioral objectives to improve achievement. It is of interest to note that their modes of instruction were different. Engel used programmed instruction whereas Dalis used the traditional classroom setting.

The effects of task instruction on set solutions of different classes of anagrams was reported by Maltzman and Morrisett (1953). In their study, Maltzman and Morrisett assigned 50 subjects to five groups. Two groups were given a hint or clue as to the type of solution for a particular class of anagram. The other groups did not receive this extra information. The hint or clue given to the two groups acted, in essence, as a form of a behavioral objective. The results of the study indicated that the use of "behavioral objectives" significantly effected the solutions of anagrams.

Although the results of using behavioral objectives as an experimental variable are contrary, the fact that they are not contradictory is encouraging.

SUMMARY OF CHAPTER ONE

The Gagné learning hierarchy is appropriate for a classroom learning situation. The construction of the hierarchy enables one to partial out previous learning from new learning, thereby providing a relatively accurate measure of learning.

The model by Carrol (1963) and the studies by Wang (1968), Homme and Glaser (1959), Lumsdaine (1959), Carr (1959), Green (1962), and Moore and Smith (1965) provide a background upon which a procedure for measuring learning as a function of time and difficulty may be devised.

The studies by Engel (1968), Cook (1969), and J. Smith (1970) suggest that the use of behavioral objectives does improve retention. The comment by Eisenberg (1970) and the studies by Engel (1968), Dalis (1970), and Maltzman and Morrisett (1953) suggest the use of behavioral objectives as a "non-negative" modifier of learning. The results of these studies provide evidence in support of advocates of the use of behavioral objectives.

As previously stated, the purpose of this study is to provide information about the use of behavioral objectives as a means of improving the rate of learning and/or the rate of forgetting. The rationale for the study is the need to extend the knowledge of the effect of behavioral objectives on learning when measures of learning are restricted to "new" learning. Previously reported studies have not indicated that any procedures have been used to restrict learning to "new" learning. Also--previous studies of learning, in which behavioral objectives have been used as a manipulated variable, have not considered differences in difficulty of acquiring various behaviors prescribed in a program of instruction. Hopefully, the present study will extend the knowledge of the effect of behavioral objectives on learning and forgetting and will do so with the previous studies as models of what to do and as indicators of possible corrections needed in the measurement of learning.

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Chapter 2

PROBLEM AND HYPOTHESIS

This study is concerned with the problem of determining if behavioral objectives affect the rate of learning and/or the rate of forgetting. To accomplish the goals of the study, a validated learning hierarchy will be used to construct instructional materials. The instructional materials will be individualized for each student in the sense of being self paced. The time needed by a student to complete the tasks in an individual cell in the learning hierarchy will be used to construct a measure of difficulty. Measures of difficulty will then be used to determine measures of learning. Measures of learning will be restricted to new learning. Three different retention periods will be used to study the rate of forgetting.

There are six research questions to be answered by this study. The first two research questions are concerned with measures of learning. The second two research questions are concerned with measures of forgetting. The last two research questions are concerned with measures of retention and efficiency of learning.

DEFINITIONS--A DISCUSSION

The definitions that follow in the next section have been formulated in an effort to clarify and provide a basis for analysis of the experimental measurements of this study. The discussion that follows provides rationale for these definitions.

If a measure of learning is to be made in an experimental study then the measure should be restricted to only that learning which occurs within the constraints of the study. Failure to provide some means of partialling out prior knowledge from new knowledge will clearly confound the measure of learning affected by the manipulated variables of the experiment. In studies in which learning occurs within the framework of a hierarchy, determining the cells of the hierarchy for which a subject can, prior to instruction, correctly respond to an assessment of the prescribed behavior would seem to identify prior learning. Hence a definition of base set has been constructed to identify the cells of the hierarchy for which a subject needs instruction in order to acquire the prescribed behavior.

The phrase "rate of learning" suggests some measure of learning over some interval of time. In paired-associate learning studies, the measures of learning has been specified as a function of the ratio of the number of correct associations to the number of trials or some interval of time. Such a specification of the measure of learning appears to assume that the degree of difficulty of any two paired-associate responses is equivalent. This assumption cannot logically be assumed in a classroom learning situation where some behaviors will require more time to be acquired than will other behaviors. Therefore, some measure of difficulty of the acquisition of a behavior of a particular cell in the hierarchy is needed. This measure of difficulty must also become an integral component of the learning measure. The measure difficulty of acquisition will be constructed "hierarchy cell by hierarchy cell" in terms of the ratio of the mean hierarchy cell time to the mean time needed to proceed through the entire learning sequence.

The measure of learning by a subject as a result of the instructional sequence will be called the index of learning. The index of learning for an individual subject will be a sum of the measures of cell difficulty of acquisition for the subject's base set. It should be noted that any two subjects of the same group who have the same base set will have an equal index of learning.

The rate of learning will be defined as the ratio of a subject's index of learning to his total time needed to acquire the set of behaviors specified by his base set.

Measures of forgetting also require some modification in learning studies constructed within the framework of a learning hierarchy. The reasons are similar to those given for measures of learning. In particular, consideration must be given to differences in difficulty of acquiring individual behaviors. Thus index of forgetting and rate of forgetting will be defined as analogous to the definitions of index of learning and rate of learning.

It should be noted that "index" is being used rather than "degree." The choice of index was made because of connotations carried by the word degree. Index is being used in the sense of a measure associated with a percentage of some quantity.

A measure of retention will be defined in the naive sense of the difference of index of learning and index of forgetting. Similarly, a measure of efficiency will be defined in a naive sense. It should seem fair to consider efficiency as a ratio of retention and learning.

DEFINITION OF TERMS

1. Individualized Instruction: Individualized Instruction is a learning situation using programmed materials in which a student works at his own pace.

2. Preassessment task: An assessment task of a behavior prior to instruction for the behavior to determine if the behavior has been previously acquired by the subject.

3. Base set (B_{ij}): The hierarchy cells for which the ith subject of the jth group receives instruction as a result of failing to respond correctly to two preassessment tasks.

4. Subject cell time (SCT_{ijk}): The number of classroom clock minutes used by the ith subject of the jth group to acquire the behavioral objectives of the kth cell of the learning hierarchy.

5. Group cell time (G_{jk}): The mean number of classroom clock minutes used by the jth group to acquire the behavioral objective of the kth cell in the learning hierarchy.

6. Group time (GT_j): The group time for the jth group is the sum of their group cell times for all cells in

the learning hierarchy, i.e., $GT_j = \sum_{k=1}^{n} G_{jk}$ where n is the number of cells in the learning hierarchy.

7. Difficulty of acquisition (DA_{jk}): The difficulty of acquisition of the behavior specified in the kth cell of the learning hierarchy by the jth group is the ratio of G_{jk} to GT_j, i.e., $DA_{jk} = G_{jk}/GT_j$.

8. Index of learning (IL_{ijk}): The index of learning for the ith subject of the jth group is 100 times the sum of the measures of difficulty of acquisition on the subject's base set, i.e., $IL_{ijk} = 100 \sum_{k \in B_{ij}} DA_{jk}$.

9. Rate of learning (RL_{ij}): The rate of learning for the ith subject of the jth group is the ratio of the subject's index of learning to the sum of the subject's cell time on the subject's base set, i.e.,

$$RL_{ij} = \frac{IL_{ij}}{\sum_{k \in B_{ij}} SCT_{ijk}}$$

10. Retention test: An alternate form of an assessment task for an acquired behavior, in particular--for the terminal behavior.

11. Retention time (R_{ij}): The number of calendar days from the day on which the ith subject of the jth group

acquires the terminal behavior until and including the day of a retention test for the terminal behavior.

12. Index of forgetting (IF_{ij}): 100 times the sum of the measures of difficulty of acquisition of the cells beginning with the terminal task and going back to the last cell for which the ith subject of the jth group fails to correctly respond to a retention test.

13. Rate of forgetting (RF_{ij}): The rate of forgetting of the ith subject of the jth group is the ratio of IF_{ij} to R_{ij}, i.e., $RF_{ij} = IF_{ij}/R_{ij}$.

14. Index of retention (IR_{ij}): The difference between the index of forgetting and the index of learning for the ith subject of the jth group, i.e., $IR_{ij} = IL_{ij} - IF_{ij}$.

15. Index of efficiency (IE_{ij}): The index of efficiency is the ratio of the index of retention to the index of learning for the ith subject of the jth group, i.e., $IE_{ij} = IR_{ij}/IL_{ij}$.

STATEMENT OF THE PROBLEM

It was previously stated that the question to be answered by this study is: Does the apprising of instructional objectives in behavioral terms prior to instruction optimize the measurements related to learning and/or forgetting?

The question above suggests the following research questions.

1. Is the index of learning higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

2. Is the rate of learning higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

3. Is the index of forgetting less for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

4. Is the rate of forgetting less for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

5. Is the index of retention higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

6. Is the index of efficiency higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed?

RESEARCH HYPOTHESES

These research questions suggest the following global hypothesis: Students who are apprised of instructional objectives in behavioral terms prior to instruction will have higher learning measures and lower measures of forgetting than students not so informed.

The following research hypotheses will be tested to provide answers for the individual research questions.

> Hypothesis I: The index of learning is higher for students who are informed of behavioral objectives in a learning hierarchy and for students who are not so informed.
>
> Hypothesis II: The rate of learning is higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.
>
> Hypothesis III: The index of forgetting is lower for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.
>
> Hypothesis IV: The rate of forgetting is lower for students who are informed of behavioral

objectives in a learning hierarchy than for students who are not so informed.

Hypothesis V: The index of retention is higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

Hypothesis VI: The index of efficiency is higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

SUMMARY OF CHAPTER TWO

The research questions to be investigated in this study have been presented in this chapter. Also included in this chapter are the operational definitions to be used and a rationale for the definitions. The chapter concludes with a global hypothesis and six research hypotheses.

Chapter 3

EXPERIMENTAL DESIGN OF THE INVESTIGATION

The study was a two-stage experiment. The first stage of the experiment was concerned with the measurement of learning a set of behaviors as a result of instruction. The second stage of the experiment was concerned with the measurement of forgetting during an interval of time following the period of instruction.

The instructional materials used in the study were individualized in the sense of being self paced. Each student was furnished with a packet of instructional materials that allowed him to proceed at his own rate, independently of the other students. An assumption that was made was that all subjects would be able to read and comprehend the written instructional materials without additional assistance.

Information about the experimental subjects, the experimental variables, the instructional hierarchy, the experimental procedures, and the planned experimental analysis follow. Included are the learning hierarchy and the validation data for the hierarchy. Additional information about the actual conduct of the experiment will be presented in the next chapter. Also, any modifications in the planned experimental analysis will be presented.

SUBJECTS

The subjects used in this study are tenth and eleventh grade students in a senior high school. The students are enrolled in the second half of a two-year sequence of algebra one.

The students in the study live in a relatively high socio-economic area in the suburbs of Washington, D.C. The school is located adjacent to the U.S. Department of Agriculture Research Center and the U.S. Naval Ordinance Laboratory. The University of Maryland is less than 6 miles away. Many of the parents of the students in this study are employed by the federal government or the University of Maryland. However, not all of these parents have a college education although they are employed in an "academic" environment.

The two-year algebra one sequence was introduced into the school system for those students who desired to enroll in academic mathematics in grade nine but who were judged to have learning difficulties that make enrollment in a typical algebra one class inadvisable. The assignment of students to the two-year algebra one sequence suggests, but does not prove, that these students have less than "average" ability in mathematics. Nearly all of the students who enroll in the two-year algebra one program have indicated that they intend to go to college and, hence, they "need" as much secondary school academic mathematics as they can "successfully" complete.

These subjects were selected for the study for the following reasons. They have completed, roughly, one-half of a first-year algebra one, so they have acquired some computational skills and have been exposed to several concepts that are assumed in the lowest level of the instructional sequence. They had not studied the topics presented in the instructional sequence. Hence, they were adjudged to possess the competencies necessary for the program but not to possess the behaviors specific to the learning hierarchy.

The subjects used in the study were partitioned into two treatment groups. The subjects were enrolled in two separate classes. The subjects were randomly assigned to the two treatment groups within classes for the first stage of the experiment. The two treatment groups were identified as:

1. Subjects informed of behavioral objectives for each cell in the learning hierarchy.
2. Subjects not informed of behavioral objectives for any cell in the learning hierarchy.

EXPERIMENTAL VARIABLES

Independent Variables

Two variables were manipulated in this study. The first variable manipulated was knowledge of behavioral objectives. The effect of this variable on both phases of the experiment was ascertained. The second variable manipulated was the length of the period of retention. The effect of this variable on the second stage of the experiment was investigated.

Behavioral objectives. Knowledge of behavioral objectives was manipulated to determine whether or not it differentially affects the rate of learning and/or the rate of forgetting. This variable was dichotomized into two states: the subject was presented with the behavioral objective of a specific task or the subject was not presented with the behavioral objective.

Length of the period of retention. The length of the period of retention was manipulated to determine if the manipulation of knowledge of behavioral objectives differentially affects the rate of forgetting for periods of retention of different lengths. This variable has three states: 7 calendar days, 14 calendar days, and 21 calendar days.

Dependent Variables

There were two dependent variables in the experiment. Both were measures of rate. The first was the rate of learning. The second was the rate of forgetting. Rate of learning and rate of forgetting were operationally defined on pages 35 and 36. The definitions of both dependent variables are based on the assumption that rate of learning and rate of forgetting over a period of time will vary directly as a function of the difficulty of the behavior to be acquired.

Controlled Variables

In an experimental study in a classroom, several variables occur that may confound the effect(s) of the treatment(s). One recognized variable is the teacher. Another recognized variable is the time of day if more than one class of students is used in the experiment.

To control for the teacher variable, instruction was not given by the teacher. Rather, instruction for the acquisition of a specified behavior not yet possessed by a subject was written in the form of individualized self paced instruction. Since not all students read at the same rate, the constraint of time was also removed as the materials were self paced.

Variables such as time of day and other factors that may confound treatment effects were controlled by randomly assigning subjects to treatment groups.

INSTRUCTIONAL HIERARCHY

The materials used in this study were prepared by the experimenter. The materials were prepared such that each subject could proceed through the program of instruction at his own pace. The instructional materials used in this study may be found in Appendix A.

The procedure used in sequencing the instructional materials is due to Gagné (1965). After selecting a terminal behavior for the learning sequence, the behaviors subordinate to the terminal behavior were identified and arranged in a hierarchy of learning dependencies.

For example, in this study, the terminal behavior selected required that a student write the equation of a circle in the form $(X - h)^2 + (Y - k)^2 = r^2$ given the coordinates of the center (h,k) and the radius r. Before a student could write the equation in the specified form given the information indicated, he must have acquired the subordinate behaviors of being able to identify the ordered pair of real numbers with the location of a point in the plane and identify the radius r as the distance from the center of the circle to a point on the circle. Thus an initial development of a hierarchy might be figuratively represented as follows on the next page.

The behaviors of identifying the two suggested subordinate behaviors, identify the coordinates of the center of a circle and identify the radius of the circle, in turn have subordinate behaviors. The learning hierarchy used in preparing the instructional materials for this study may be found on page 52.

The preparation of a learning hierarchy may be described as a series of approximations. The first approximation is, usually, an "armchair" construction of a hierarchy of learning dependencies. This first approximation is then tested with a small group of subjects to ascertain whether or not the "sequencing" of the hypothesized learning dependencies is reasonable. If the results of this test seem reasonable, a more thorough test is needed. If the

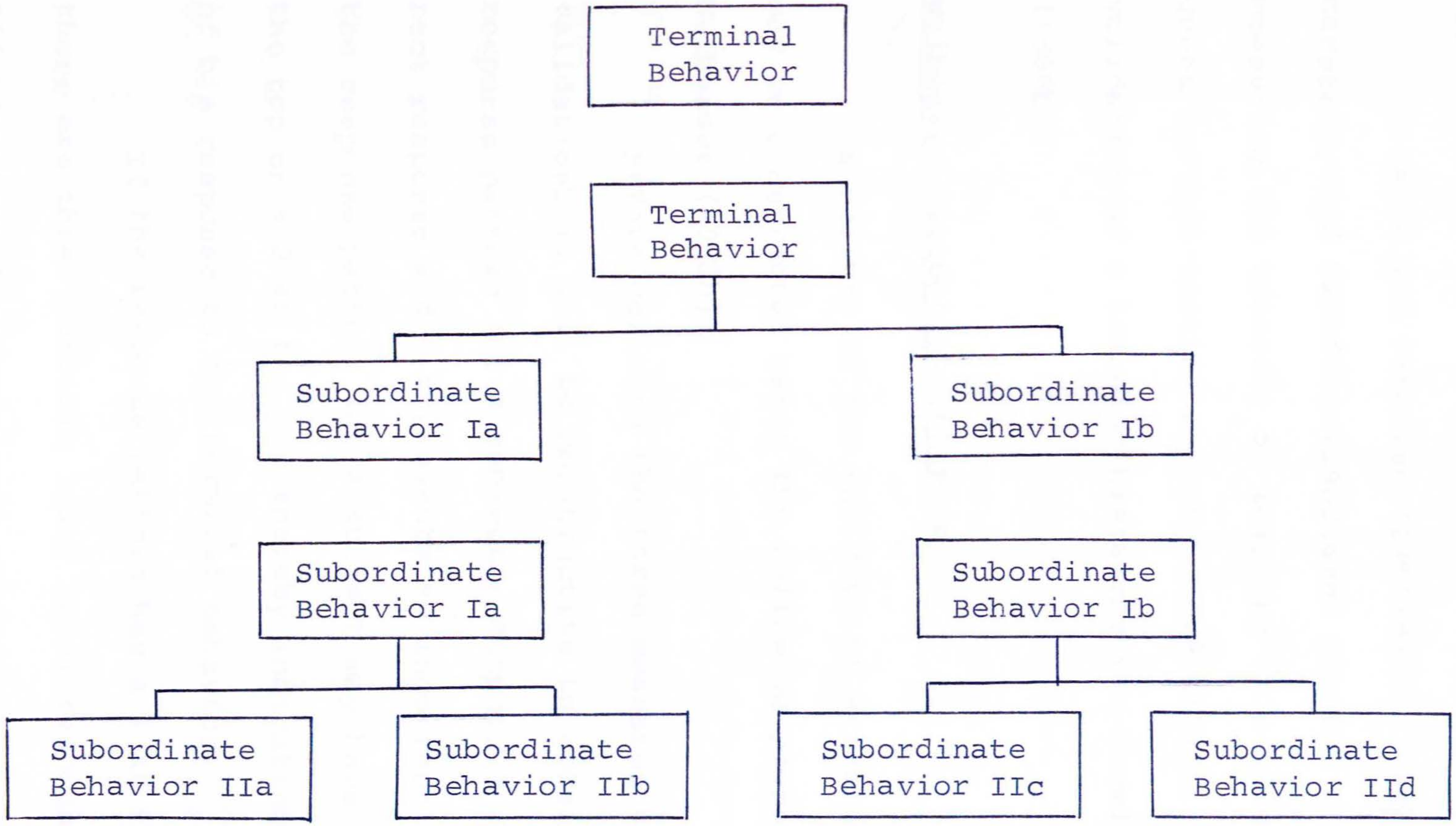


Figure 1

A Representation of the Development of A Hierarchy results of the first test are not reasonable, a new "first" approximation is needed before proceeding.

Gagné and Paradise (1961:9-10), Gagné, Mayor, Garstens, and Paradise (1962:6-8) present a procedure for measuring the transfer of learning in a hierarchy. Subsequent to this initial work by Gagné et al., a procedure for validation of a learning hierarchy was developed by Walbesser (1968b).

Walbesser Validation Criteria

A measure of the validity of a learning hierarchy may be constructed using the following procedure due to Walbesser (1968b).

Before defining the three measures used for the validation, it will be constructive to consider the following response pattern for a subject. Suppose 1 represents a correct response and 0 represents an incorrect response. Thus, the response pattern for a subject may have either a 1 at the top or a 0 at the top thereby indicating the correctness of his response to the terminal behavior.

If the response pattern has a 1 at the top, then there are three possible lower level response patterns. If all responses to lower level subordinate tasks are correct, the lower level response pattern will consist of all ones.

If all responses to subordinate tasks were incorrect, the lower level response pattern will consist of all zeros. If, however, some responses were correct and some incorrect, the lower level response pattern will consist of both zeros and ones.

Similarly, if the response pattern has a zero at the top there are three possible lower level response patterns. Therefore, there are a total of six response patterns. Each of these six patterns is illustrated below for a simple 3 cell hierarchy.

```
  1      1      1      0      0      0
 1 1    0 1    0 0    1 1    0 1    0 0
```

The frequency for each subject's response pattern is needed for the validation procedure. The three measures constructed from the response pattern frequencies are:

1. Consistency ratio: a measure of the consistency of the dependency of the terminal behavior to the hypothesized subordinate behaviors.
2. Adequacy ratio: a measure of how adequate the subordinate behaviors are in inducing the terminal behavior.
3. Completeness ratio: a measure of the completeness of instruction.

If the response patterns are identified as above and the frequency of each response pattern is denoted as $\sum\begin{pmatrix}\text{response}\\\text{pattern}\end{pmatrix}$ then the above ratios are defined as follows:

$$\text{Consistency ratio} = \frac{\sum\binom{1}{1\ 1}}{\sum\binom{1}{1\ 1} + \sum\binom{1}{0\ 1} + \sum\binom{1}{0\ 0}}$$

$$\text{Adequacy ratio} = \frac{\sum\binom{1}{1\ 1}}{\sum\binom{1}{1\ 1} + \sum\binom{0}{1\ 1}}$$

$$\text{Completeness ratio} = \frac{\sum\binom{1}{1\ 1}}{\sum\binom{1}{1\ 1} + \sum\binom{0}{0\ 0} + \sum\binom{0}{0\ 1}}$$

The Walbesser validation criteria was used to validate the hierarchy used in this study. The validity measures were collected by the experimenter in a tenth-grade geometry class. The students in the geometry class were considered to be of "average" or "below average" ability in mathematics (based on previous grades and ITED scores). The validity measures for the hierarchy are presented in the table on the following page. No additional validity measures will be constructed for the learning hierarchy to be used in the present study.

Table 1

Hierarchy Validity Measures

	Response Pattern					
	1 1 1	1 0 1	1 0 0	0 1 1	0 0 1	0 0 0
Frequency	18	2	0	0	0	0

$$\text{Consistency ratio} = \frac{18}{18 + 2 + 0} = .90$$

$$\text{Adequacy ratio} = \frac{18}{18 + 0} = 1.00$$

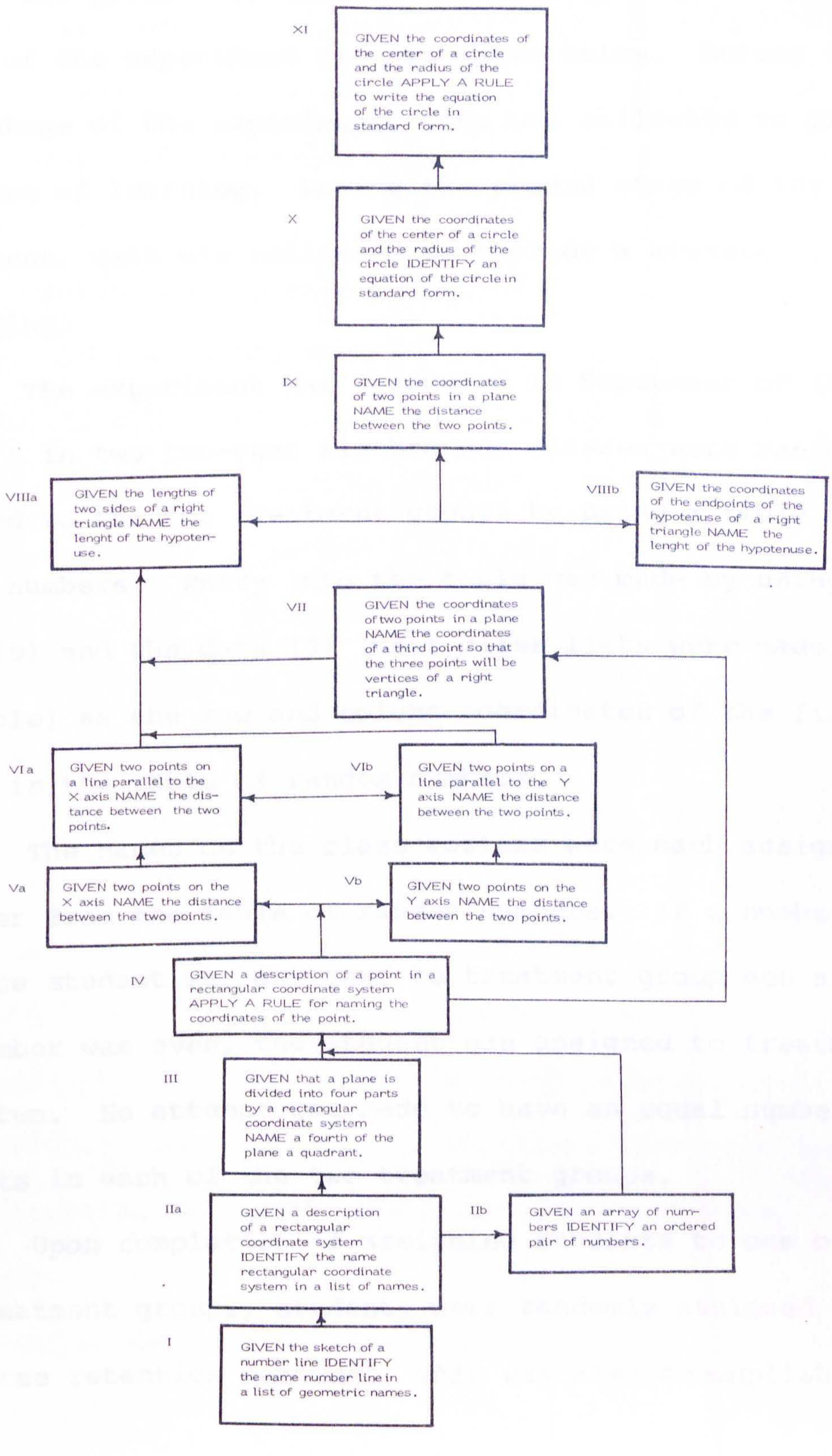
$$\text{Completeness ratio} = \frac{18}{18 + 0 + 0} = 1.00$$

In this study, the hierarchy was considered to be valid if at least 90 per cent of the subjects acquired at least 90 per cent of the behaviors. The ratios above indicate that 90 per cent of the subjects acquired 100 per cent of the behaviors, therefore the learning hierarchy was considered valid.

The learning hierarchy is presented in Figure 2 on the following page.

Figure 2

LEARNING SEQUENCE



EXPERIMENTAL PROCEDURES

The procedures used for collecting data for the two stages of the experiment are presented below. During the first stage of the experiment, data was collected to provide a measure of learning. During the second stage of the experiment, data was collected to provide a measure of forgetting.

The experiment was conducted in September of 1971. Students in two two-year algebra one classes were randomly assigned to the two treatment groups by using a table of random numbers. Entry into the table was made by using the month (9) and the date (3) (that class lists were made available) as the row and column coordinates of the first number in the table of random numbers.

The names on the class rosters were each assigned a number from the table of random numbers. If a number was odd, the student was assigned to treatment group one and if the number was even, the student was assigned to treatment group two. No attempt was made to have an equal number of subjects in each of the two treatment groups.

Upon completion of assigning students to one of the two treatment groups, students were randomly assigned to the three retention periods. This was also accomplished by using a table of random numbers. Entry into the table was made by using the hour and nearest minute (on the date that this assignment was made) as the coordinates of the row and the column in the table. Again, the names on the class rosters were each assigned a number from the table of random numbers.

Assignment was made to one of the three retention periods. If a zero was encountered in the table, it was omitted. If the digit encountered in the table yielded a remainder of one when divided by three, the student was assigned to retention period 1. If the digit yielded a remainder of 2 when divided by 3, the subject was assigned to retention period 2, and if the remainder was 0, the subject was assigned to retention period 3.

Before a subject was presented with instructional materials, an attempt was made to determine the behaviors in the learning hierarchy which were already possessed by the subject. This was accomplished by presenting the subject with three preassessment tasks for each behavior prior to instructional materials for the behavior. For each behavior in the learning hierarchy, a set of similar preassessment tasks were prepared.

If a subject correctly responded to at least two of the three preassessment tasks, it was assumed that the subject had previously acquired the behavior. If a subject was assumed to possess a particular behavior, then no instruction for that behavior was given. Instead, the subject was given another set of three preassessment tasks to determine whether or not he possessed the next behavior in the learning hierarchy.

This procedure was used for each cell in the hierarchy, thereby separating prior knowledge from new knowledge. If, however, the subject did not correctly respond to at least two of the three preassessment tasks for a behavior in the learning hierarchy, he was presented with an instructional program designed to induce the prescribed behavior. The hierarchy cells for which a subject required instruction were his base set.

When a subject was presented with an instructional program, the time in hours, minutes, and seconds (to the nearest five) were recorded and as soon as the subject indicated he was ready for an assessment task, the time was again recorded in hours, minutes, and seconds (to the nearest five). The lapsed time was rounded to the nearest minute. Timing was recorded by using a direct read digital dial clock that allowed the viewer to read the hour, minute, and second (in five-second graduations). The lapsed time for acquisition of a particular behavior was used to construct a measure of difficulty for the behavior. It is recognized that such a procedure for timing allowed for an error of measurement. However, the construction of a ratio of times tends to suppress the error. Also, although an error is present, the criticism of assuming that each behavior is of equal difficulty to acquire is removed.

The instructional materials for each of the two treatment groups was identical except for the first paragraph. The first paragraph for subjects in the first treatment group provided subjects with an explicit statement of the objective of the lesson. The first paragraph for subjects in the second treatment group was a rather uninformative statement informing students only that the lesson to follow is an extension of previous lessons and that the present lesson is needed for later lessons. To ascertain whether or not subjects in the first group read the first paragraph, subjects were asked to tell the objective of the lesson after they returned the instructional materials and the time had been recorded. Whether this procedure tended to reinforce the behavior is not known; therefore no assumption about possible reinforcement was made.

Subsequent to all subjects in a group attaining the terminal behavior, a computer program developed by the experimenter was used to compute measures needed to determine the rate of learning for each subject. The program is included in Appendix B. The computed measures are presented in Table 11, on page 249.

Subjects were randomly assigned to one of three retention periods. The retention periods were designed to prevent the date of a retention test from falling on a weekend. One, or two, or three calendar weeks after attaining the terminal behavior a subject was given a retention test. The retention test was a set of assessment tasks.

The first task was an assessment task similar to the terminal task used in the learning phase of the study. If a subject responded correctly, he had completed his retention testing. However, if a subject did not answer the first task correctly, he was given a second assessment task similar to the assessment task for the behavior that preceded the terminal task. If he responded correctly, he then had completed his retention testing. If he did not respond correctly, he was given another assessment task similar to the assessment task for the behavior that preceded the behavior for which he had just been tested.

This procedure was continued for a subject until he finally responded correctly to an assessment task similar to one of the original assessment tasks--or--until he reached level 4 of the learning hierarchy. If a student was not able to respond correctly to any assessment task for a behavior in the learning hierarchy above or at level 4, it was assumed he had no retention of the behaviors in the learning sequence.

The index of forgetting for a subject was computed by summing the measures of difficulty of acquisition for those behaviors for which he failed to respond correctly. Thus, a student who correctly responded to the first retention test had zero as his index of forgetting. A student who failed to respond to, say, five retention tests had an index of forgetting equal to the sum of the measures of difficulty of acquisition for the last five behaviors in the learning hierarchy.

EXPERIMENTAL ANALYSIS

The raw data collected during the experiment was used to generate six measures. These measures were (1) index of learning, (2) rate of learning, (3) index of forgetting, (4) rate of forgetting, (5) index of retention, and (6) index of efficiency. These six measures were computed for each subject in each treatment group. The measures in each case were on at least an interval scale.

The assignment of treatments to subjects was a random procedure. Therefore, the observations were considered to be independent. The two treatment samples were of approximately equal size. However, no assumption was made about the distribution of the experimental population nor was any assumption made about homogeneity of variance for the two treatment groups on the basis of the demographic information available.

Even though these last assumptions could not necessarily be met, a parametric test was used in the analysis of the data. This decision was made after noting that violations of normality and/or homogeneity of variance apparently do not significantly affect the analysis of variance. In particular, Dayton (1970:35) reported,

> There is a good deal of evidence that the analysis of variance is virtually unaffected by violations of normality and homogeneity of variance _if_ the samples entering into the analysis are of the same, or approximately the same, size. Results reported by Box (1954) and Norton (cited in Lindquist, 1953) directly support this contention for analysis-of-variance designs, and a study by Boneau (1960) gives similar evidence for the two-sample case (_t_ test).

The preplanned experimental analysis was six _t_ tests. The six _t_ tests were carried out using the University of Miami MANOVA two-sample analysis of variance test.

SUMMARY OF CHAPTER THREE

The purpose of this chapter was to present the design of the experiment. The chapter has five major subdivisions. They are (1) subjects, (2) experimental variables, (3) instructional hierarchy, (4) experimental procedures, and (5) experimental analysis.

Demographic information has been included about the subjects to be used in this study.

The two independent variables are knowledge of behavioral objectives and length of the period of retention. The dependent variables are rate of learning and rate of forgetting.

The instructional hierarchy has been constructed within the framework of a learning sequence due to Gagné (1965) and validated by a procedure due to Walbesser (1968b). A brief discussion of the procedures has been included. The validity measures for the hierarchy are also included.

The procedures planned for the conduct of the experiment are included for both phases of the study.

The planned experimental analysis of the data to be collected in the study is presented. Also included is the rationale and the justification for the analysis to be used.

FOOTNOTES--CHAPTER THREE

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Chapter 4

THE EXPERIMENT AND RESULTS

The data for the learning phase of this study were collected between September 20th and September 28th, 1971. The procedure used for collecting the data did not vary from the planned procedure.

Fifty-four students from two 2-year algebra one classes were randomly assigned to the two treatment groups. By this procedure 28 students were assigned to group one and 26 students were assigned to group two. One student from each group withdrew from the class during the experiment, thereby leaving 27 students in group one and 25 students in group two. The 52 students who completed the experiment were randomly assigned to the three retention periods.

The distribution of subjects from classes to treatment groups and retention groups is presented in Table 2.

The experiment was conducted with the assistance of two teachers. One of the two teachers assisted during one of the two class periods. At the beginning of the experiment

Table 2

Distribution of Subjects from Classes to Treatment Groups
and Retention Groups

	Class Period					
	1		2		Totals	
Retention Group	Treatment Group		Treatment Group		Treatment Group	
	1	2	1	2	1	2
1	2	3	5	2	7	5
2	3	3	4	4	7	7
3	7	7	6	6	13	13
Totals	12	13	15	12	27	25

students were informed that they would not need to bring their books to class for the next week and if they were cooperative, they would not be given homework assignments during that period of time. The students were not informed of the purpose for this change in "normal" classroom practice. They were told only that what was to follow would be used to measure their rate of learning new material. No mention was made of the second phase of the study.

Students were then asked if they had any questions. Students did ask if they would be given a "major" test on the materials to be learned. They were informed that it would not be necessary since the materials they were to learn had been "broken up" into small parts and they would be tested on each of the parts as they progressed. Several students asked what would happen if they "flunked." An attempt was made to assure them that no one would "flunk"--in fact, everyone would receive a test grade of "A" if they would only do their best and be cooperative. The promise of an "A" seemed to create a mood of "let's go!"

Each student was given the first preassessment task by the assisting teacher. While this was being done, the purpose of the preassessment tasks was explained. Students were told to take their preassessment tasks to the experimenter to be checked when they had completed them. Students who passed were given the next preassessment task by the assisting teacher. Students who did not pass were given instructional materials for the behavior and were informed to read the materials "very carefully." Further, they were informed that when they had completed the instructional materials and were ready for a test (assessment task) they were to come directly to the experimenter so that their time could be recorded. If the student had no questions, the time was recorded on the instructional materials and the student began. If the student had questions, they were answered as completely as possible without providing any instruction for the specific behavior to be learned from the instructional materials.

When a student brought his instructional materials to the experimenter, the time was recorded immediately. The assisting teacher then asked the student what was the objective of the lesson. Students in group one were required to give explicit answers whereas students in group two were not required to do so. No coaching was provided. If a student from group one could not give an explicit answer, he was told to get his instructional materials, have the time recorded, and study them again. This was only required of 7 students in group one and this occurred only once, with one exception. One student failed to give an acceptable response twice.

Students who gave acceptable responses to the question of the objective of the lesson were given an assessment task by the assisting teacher. These students were told to work carefully and when they had completed the assessment task, they were to bring them to the experimenter for checking. All preassessment tasks and assessment tasks were checked by the experimenter.

Students waiting to have either a preassessment task or an assessment task checked were generally well behaved and cooperative. Most waited quietly and conducted themselves in a courteous manner. Students who were not cooperative were seated apart from the other students in the classroom. This seemed to improve the situation.

As students completed the learning sequence, they were given dittoed pages of traditional algebraic materials. This was done to keep everyone working until all students had completed the learning sequence.

The three retention periods planned were 7 days, 14 days, and 21 days. Students were given individual retention tests according to the date on which they completed the learning sequence. The first and second retention schedules were followed. The school calendar precluded the possibility of giving retention tests according to the third retention schedule. Therefore students in the third retention period were tested over varying intervals of time ranging from 15 days to 21 days.

Retention tests were given to students without previous notification. They were given to students while the classes were doing "homework" at their desks. No explanation was offered for the additional testing although several students did ask for an explanation. If a student insisted on an answer, he was informed that the purpose was to determine whether or not he remembered the materials previously studied individually. Student cooperation was excellent for this part of the study.

EXPERIMENTAL MEASURES DERIVED FROM RAW DATA

Prior to the beginning of the experiment, a computer program was written and debugged to convert the raw data collected during the experiment into experimental measures. The experimental measures are index of learning, rate of learning, index of forgetting, rate of forgetting, index of retention, and index of efficiency. The program was designed to provide a "print out" of the experimental measures and also to punch the experimental measures on data processing cards for subsequent statistical analysis. The program was "run" at the Computer Science Center at the University of Maryland on an UNIVAC 1108. A copy of the program and its "output" of experimental measures may be found in Appendix B.

The computed experimental measures were subsequently used for testing the research hypotheses of this study. In addition, the experimental measures were used to construct forgetting curves and other geometric interpretations. These graphical representations may be found in Chapter 5.

ANALYSIS OF EXPERIMENTAL MEASURES

It is well known that the t test is logically equivalent to a one-way analysis of variance for the two sample case. (For example, see Winer [1962:208] or Ferguson [1966: 293].) Therefore, the availability of "canned" statistical programs at the University of Maryland's Computer Science Center was sufficient cause for using the one-way ANOVA programs for the analysis of the experimental data.

One of the "canned" programs available is the University of Miami's MANOVA (Multiple Analysis of Variance) program. The MANOVA program is not restricted to multiple analysis. The program may be used for a one-way analysis of variance with two or more groups.

It was noted in the previous chapter that violations of the assumptions of normality and homogeneity of variance have been shown to have little effect on the results of an analysis of variance provided sample sizes are equal or approximately equal. A judgment was made that the sample sizes of 27 and 25 were approximately equal, hence the available MANOVA program was used for the analysis of the data.

The MANOVA program provides the user with a probability measure for decision making without need for additional tables. Prior to the analysis, a 0.05 level of significance was selected.

In the section that follows, the means and standard deviations for the experimental measures are presented. Following the table of means and standard deviations are summary tables of the statistical analysis for each of the experimental measures.

<u>Experimental Measures</u>

The first experimental measure computed was the index of learning. The index of learning is a measure of the percentage of the total instructional sequence learned by a subject as a result of instruction. The second experimental measure computed was the rate of learning. The rate of learning is the ratio of the subject's index of learning to the total time needed by the subject to achieve the terminal behavior. The third experimental measure computed was the index of forgetting. The index of forgetting is a measure of the percentage of the instructional sequence to which a subject responds incorrectly on retention tests before responding correctly. The fourth experimental measure computed was the rate of forgetting. The rate of forgetting is the ratio of a subject's index of forgetting to the number of days in his retention period. The fifth experimental measure computed was the index of retention. The index of retention is the difference between the index of learning and the index of forgetting. The sixth experimental measure computed was the index of efficiency. The index of efficiency is the ratio of the index of retention to the index of learning.

The means and standard deviations of the six measures for each of the treatment groups follows in Table 3.

Table 3

Means and Standard Deviations of the Six Experimental Measures by Treatment Groups

Experimental Measure	Treatment Group: Number 1 Mean	Number 1 Standard Deviation	Number 2 Mean	Number 2 Standard Deviation
Index of Learning	64.0	15.33	64.0	15.55
Rate of Learning	1.32	0.418	1.40	0.662
Index of Forgetting	22.9	26.21	17.1	28.89
Rate of Forgetting	1.94	2.567	1.31	2.437
Index of Retention	41.0	29.39	46.8	33.24
Index of Efficiency	0.633	0.415	0.720	0.461

It was hypothesized that the index of learning would be higher for subjects who were informed of the behavioral objectives of the instructional sequence than the index of learning for subjects not so informed. The analysis of the measure index of learning is presented in Table 4.

Table 4

The Analysis of the Measure Index of Learning

Source	df	MS	F	p <
Within	50	238.279		
Between	1	0.000	0.000	0.999

The F ratio of 0.000 indicates there is no significant difference in the index of learning between the two treatment groups. Therefore the first research hypothesis is not supported.

The second hypothesis tested concerned the rate of learning. It was hypothesized that the rate of learning for subjects informed of the behavioral objectives of the instructional sequence would be higher than the rate of learning for subjects not so informed. The analysis of the measure rate of learning is presented in Table 5.

Table 5

The Analysis of the Measure Rate of Learning

Source	df	MS	F	p <
Within	50	0.301		
Between	1	0.090	0.297	0.588

The F ratio of 0.297 is not in the region of rejection of the null hypothesis. Therefore the second research hypothesis is not supported.

The third hypothesis tested concerned the index of forgetting. It was hypothesized that the index of forgetting would be lower for subjects who were informed of behavioral objectives of the instructional sequence than the index of forgetting for subjects not so informed. The analysis of the measure index of forgetting is presented in Table 6.

Table 6

The Analysis of the Measure Index of Forgetting

Source	df	MS	F	p <
Within	50	757.610		
Between	1	437.566	0.578	0.451

The F ratio of 0.578 is not in the region of rejection of the null hypothesis. Therefore the third research hypothesis is not supported.

The fourth experimental measure is rate of forgetting. It was hypothesized that the rate of forgetting would be lower for subjects who were informed of the behavioral objectives of the instructional sequence than for the subjects not so informed. The analysis of the measure rate of forgetting is presented in Table 7.

Table 7

The Analysis of the Measure Rate of Forgetting

Source	df	MS	F	p <
Within	50	6.275		
Between	1	5.166	0.823	0.369

The F ratio of 0.823 is not in the region of rejection of the null hypothesis. Therefore the fourth research hypothesis is not supported.

The fifth experimental measure is the index of retention. It was hypothesized that the index of retention would be higher for subjects informed of the behavioral objectives of the instructional sequence than for subjects not so informed. The analysis of the measure index of retention is presented in Table 8.

Table 8

The Analysis of the Measure Index of Retention

Source	df	MS	F	p <
Within	50	979.470		
Between	1	425.150	0.434	0.513

The F ratio of 0.434 is not in the region of rejection of the null hypothesis. Therefore the fifth research hypothesis is not supported.

The sixth experimental measure is index of efficiency. It was hypothesized that the index of efficiency would be higher for subjects who were informed of the behavioral objectives of the instructional program than for subjects not so informed. The analysis of the measure index of efficiency is presented in Table 9.

Table 9

The Analysis of the Measure Index of Efficiency

Source	df	MS	F	p <
Within	50	0.192		
Between	1	0.098	0.513	0.477

The F ratio of 0.513 is not in the region of rejection of the null hypothesis. Therefore the sixth research hypothesis is not supported by the data.

SUMMARY OF CHAPTER FOUR

The means and standard deviations of the experimental measures computed from the raw data collected during the experiment are presented in this chapter. Summary tables of the analyses of each of the experimental measures are also presented. The following conclusions were obtained.

1. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a higher index of learning than subjects not so informed.

2. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a higher rate of learning than subjects not so informed.

3. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a lower index of forgetting than subjects not so informed.

4. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a lower rate of forgetting than subjects not so informed.

5. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a higher index of retention than subjects not so informed.

6. Subjects who are informed of behavioral objectives during the initial phase of a learning program do not have a higher index of efficiency than subjects not so informed.

FOOTNOTES--CHAPTER FOUR

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Chapter 5

CONCLUSIONS AND DISCUSSION

The purpose of this investigation was to determine whether or not measures of learning and/or forgetting are improved as a result of students being apprised of behavioral objectives during the initial phase of a learning program. The global hypothesis was:

> Students who are apprised of instructional objectives in behavioral terms prior to instruction will have higher learning measures and lower forgetting measures than students not so informed.

The global hypothesis was tested by the following six research hypotheses:

> Hypothesis I: The index of learning is higher for students who are informed of behavioral objectives in a learning hierarchy and for students who are not so informed.

> Hypothesis II: The rate of learning is higher for students who are informed of behavioral objectives

in a learning hierarchy than for students who are not so informed.

Hypothesis III: The index of forgetting is lower for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

Hypothesis IV: The rate of forgetting is lower for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

Hypothesis V: The index of retention is higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

Hypothesis VI: The index of efficiency is higher for students who are informed of behavioral objectives in a learning hierarchy than for students who are not so informed.

The six respective conclusions which were reached are presented below:

1. The first hypothesis is not supported. The index of learning for students who are informed of

behavioral objectives during the initial phases of the learning program is not higher than the index of learning for students who are not so informed.

2. The second hypothesis is not supported. The rate of learning is not increased by informing students of behavioral objectives during the initial phase of the learning program.

3. The third hypothesis is not supported. The index of forgetting is not decreased by informing students of behavioral objectives during the initial phase of the learning program.

4. The fourth hypothesis is not supported. The rate of forgetting is not decreased by informing students of behavioral objectives during the initial phase of the learning program.

5. The fifth hypothesis is not supported. The index of retention is not higher for students who are informed of behavioral objectives during the initial phase of the learning program.

6. The sixth hypothesis is not supported. The index of efficiency is not increased by informing students of behavioral objectives during the initial phase of the learning program.

DISCUSSION OF THE CONCLUSIONS

Informing students of behavioral objectives of a learning program increases the quantity of information a person must "input." Similarly, the more information a person "inputs" the greater the quantity of information available to be forgotten. In this study, informing students of behavioral objectives during the initial phase of the learning program did not affect either measures of learning or measures of forgetting.

The data do not support the use of behavioral objectives as a procedure for improving either the rate of learning or the rate of forgetting. It should also be noted that the data do not support the non-use of behavioral objectives.

There are several possible explanations for the results of this study. Perhaps, behavioral objectives do not affect measures of learning and/or forgetting when the measures are a function of time. The use of programmed instruction may tend to suppress any differences in the measures of learning and/or forgetting that exist. The subjects in the study may have contaminated the experimental measures of the two treatment groups.

Behavioral Objectives
and Non-effect

The results of the studies by Engel (1968) and Dalis (1970) indicate that knowledge of behavioral objectives prior to instruction improves achievement. The results of the studies by Engel (1968), Cook (1969), and J. Smith (1970) indicate that the use of behavioral objectives improves retention scores. In each of these studies learning and forgetting were measured by using achievement tests.

Unlike the learning and forgetting studies noted above, in this study measures of learning and forgetting were a function of the length of time needed to reach criterion. Achievement test scores were not used because they provide only a measure of "how much" is learned and not a measure of the time "needed" to reach criterion. In the present study, rate of learning was being investigated and therefore a measure of time "needed" to reach criterion was needed.

The results of the studies by Engel, Cook, Dalis, and J. Smith suggest that behavioral objectives do affect measures of learning and forgetting provided the measures are derived from achievement scores. It may be, therefore, that measures of learning and forgetting which are a function of the length of time needed to reach criterion tend to suppress treatment effects more than measures of learning and forgetting derived from achievement scores.

Programmed Instruction

The use of instructional materials that are programmed so that each subject may proceed at his own rate may have suppressed differences in measures of learning and/or forgetting. Rowan (1971:83), who also used programmed instruction, writes

> It might be hypothesized that pupils proceed through an instructional sequence at about the same rate, whether they are informed of the goals or not. A possible explanation for this might be that pupils tend to find a work speed which is comfortable to them and then to use that speed despite the availability of other information. . . . Speed of completion on materials such as were used in this study may be more a function of reading ability than other factors.

The use of programmed instruction virtually assures every subject that he will reach criterion. Also, each behavior subordinate to the terminal behavior will be acquired by the learner who ultimately acquires the terminal behavior. This suggests that the level of learning will be relatively high whether or not subjects are informed of the objectives of the instruction.

An article by Underwood (1964) is directed to the problem mentioned above. However, he is particularly concerned with verbal learning. Underwood presents several studies which indicate that if learning measures, expressed as percentages, are near 100 per cent, then differences in degree of learning cannot be measured. He further argues that even though differences in learning measures cannot be measured when near 100 per cent, they will markedly influence retention.

The comments by Underwood probably do not apply to this study since the average (mean) index of learning (i.e., degree of learning to Underwood) for each group was approximately 64 per cent. However, the comments by Rowan may be applicable to this study. Possibly, as suggested by Rowan, the use of programmed instruction tends to have a leveling effect on the manipulated variables, thereby reducing differences that may result with a different instructional mode.

Subject Effect on Measurement

The students who were subjects in this study may have contaminated the measurements made during the learning phase of the study. Although no evidence is available to indicate that students did communicate answers to one another during the learning phase of the study, observations of these students since the time of experiment suggest that such a possibility exists.

A geometric interpretation of the four "curves" that are presented on the following pages is that there was no interaction between treatment groups over the three periods of retention. Another geometric interpretation of the "curves" is that the subjects in the second retention period had greatest mean index of forgetting regardless of which treatment group they were in during the learning phase of the study. No explanation is readily apparent for this latter interpretation.

The subjects used in this study have been identified by previous teachers, guidance personnel, and by standardized test scores as students with less than average ability in mathematics. They day to day performance is highly erratic. For example, on any given day, one or more of these students can correctly solve a set of twenty algebraic problems, whereas on the following day several of these students cannot solve any problem from the same set of problems.

These students tend to be absent more often than students in the regular academic mathematics program. Self reliance does not seem to be evident among these students. They appear to work well in groups of three or four but not alone. Day to day classwork can be considered acceptable for approximately 75 per cent of these students.

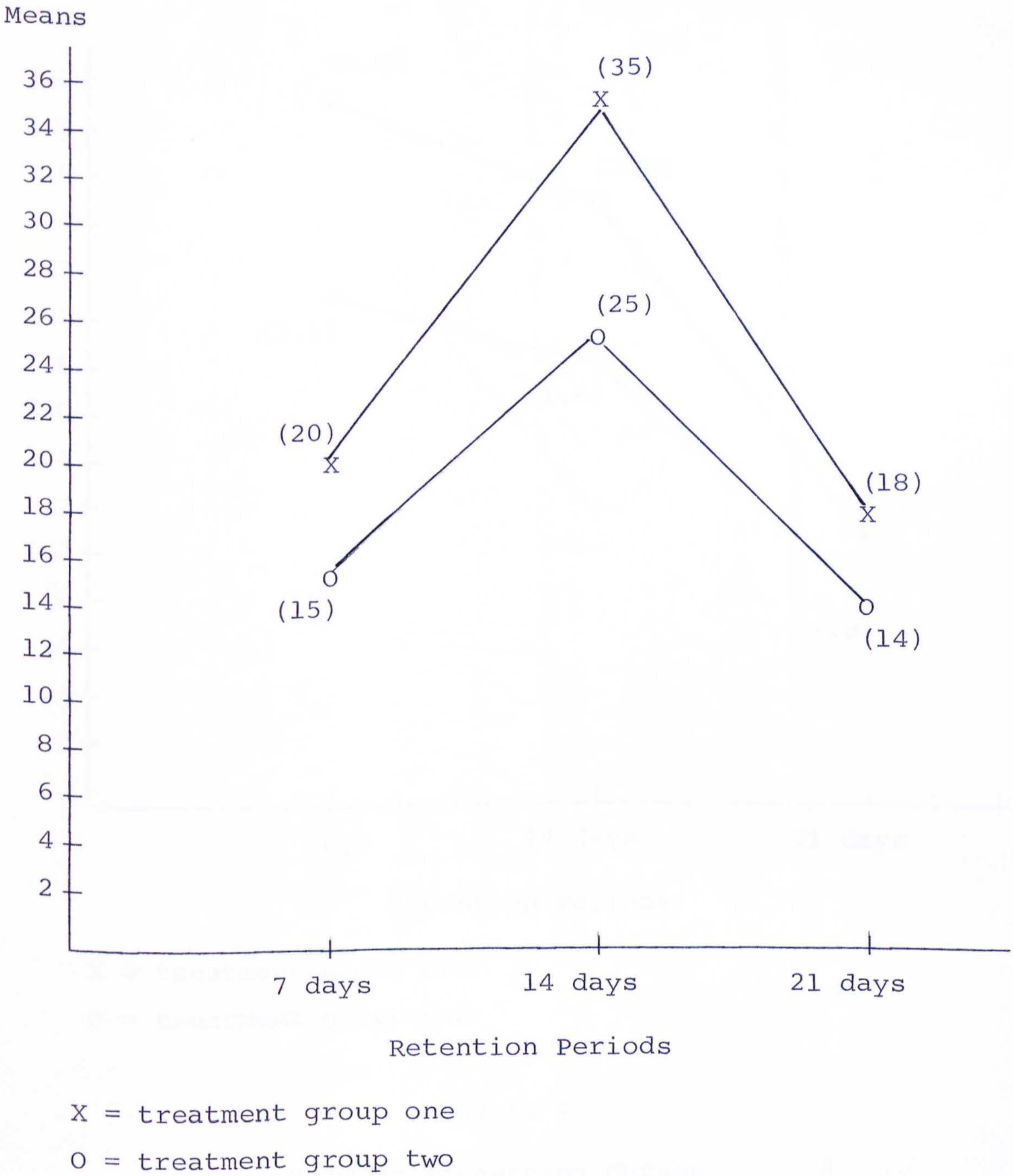


Figure 3

Index of Forgetting Curves

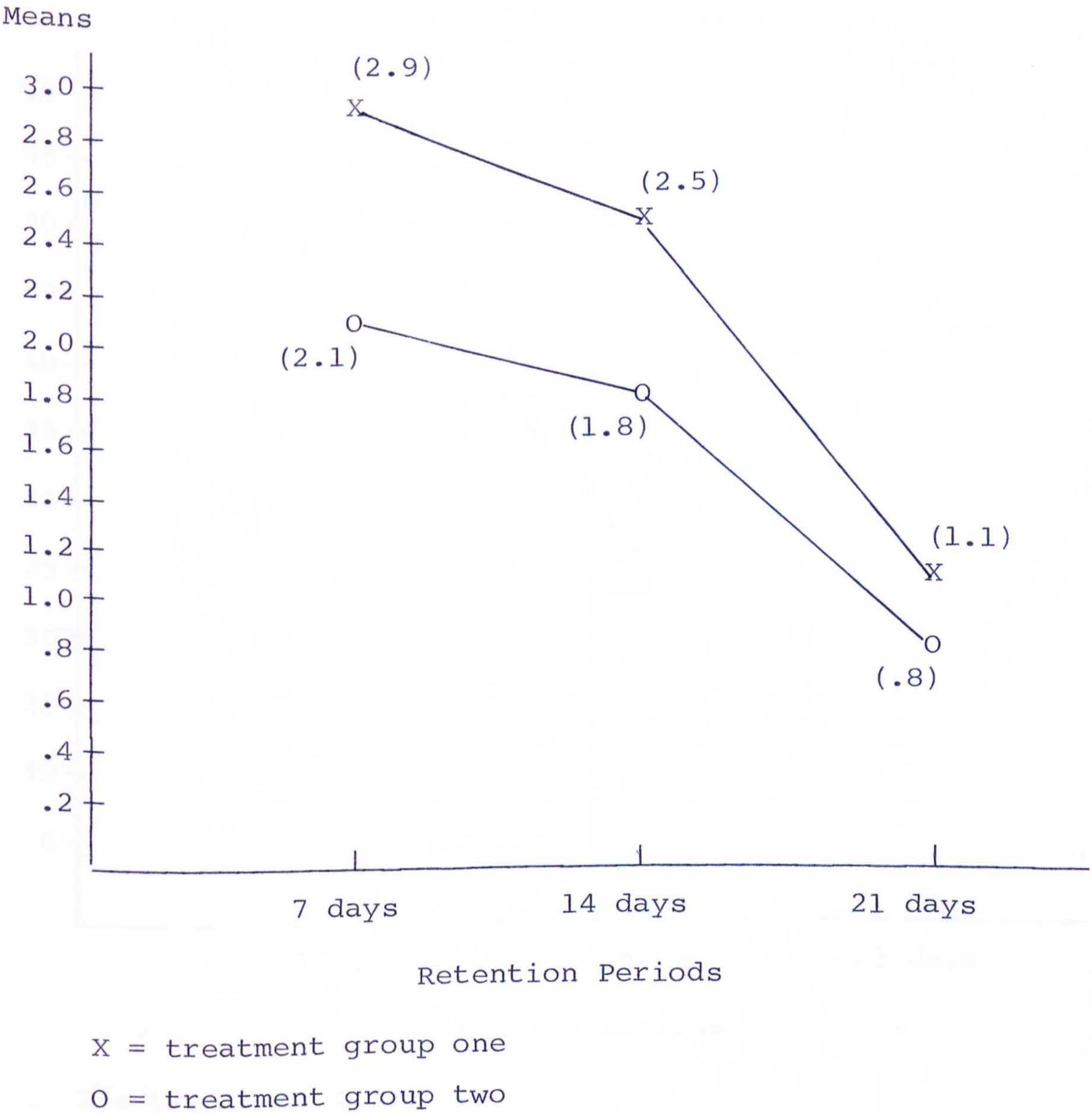
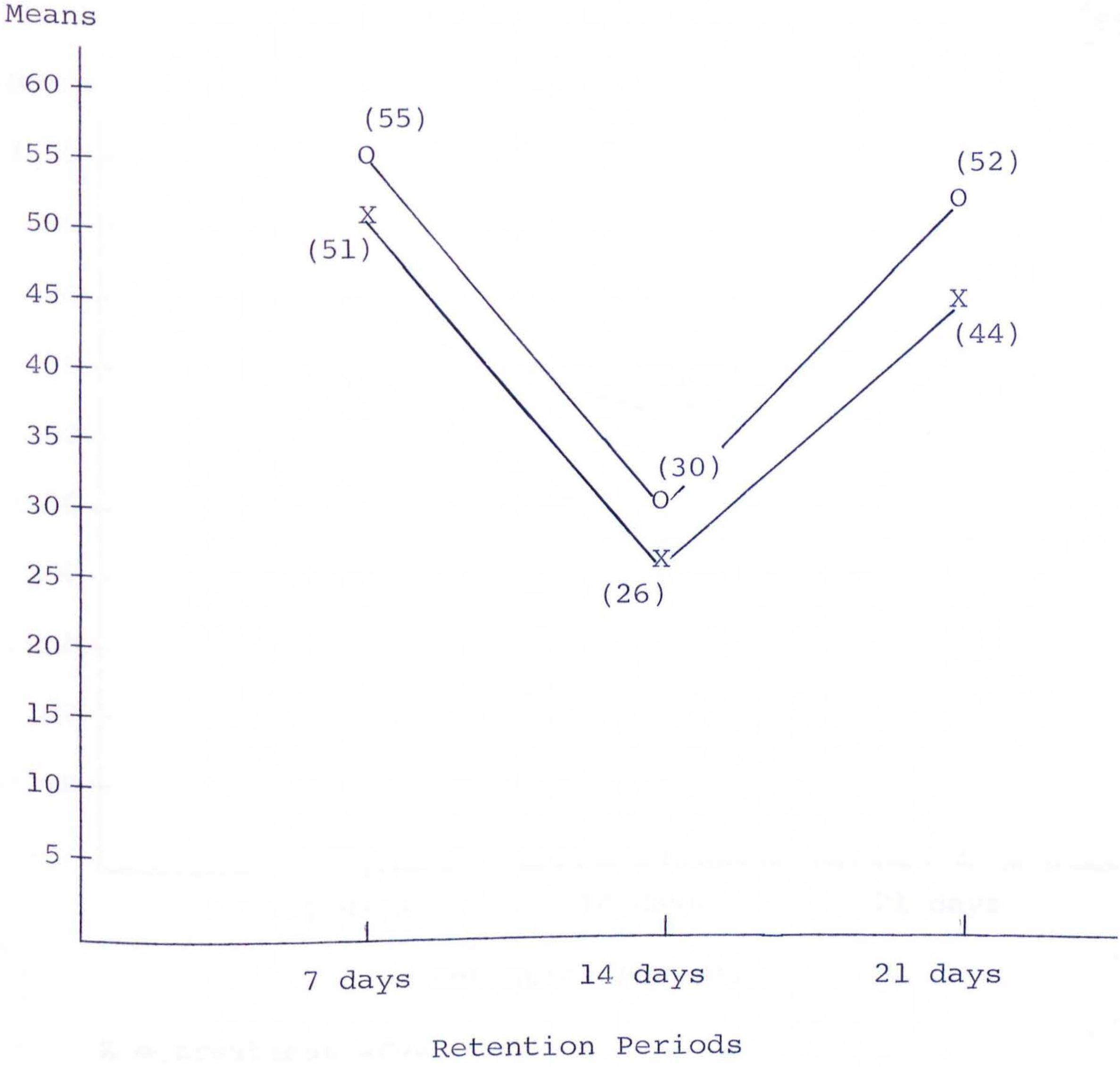


Figure 4

Rate of Forgetting Curves



X = treatment group one

O = treatment group two

Figure 5

Index of Retention Curves

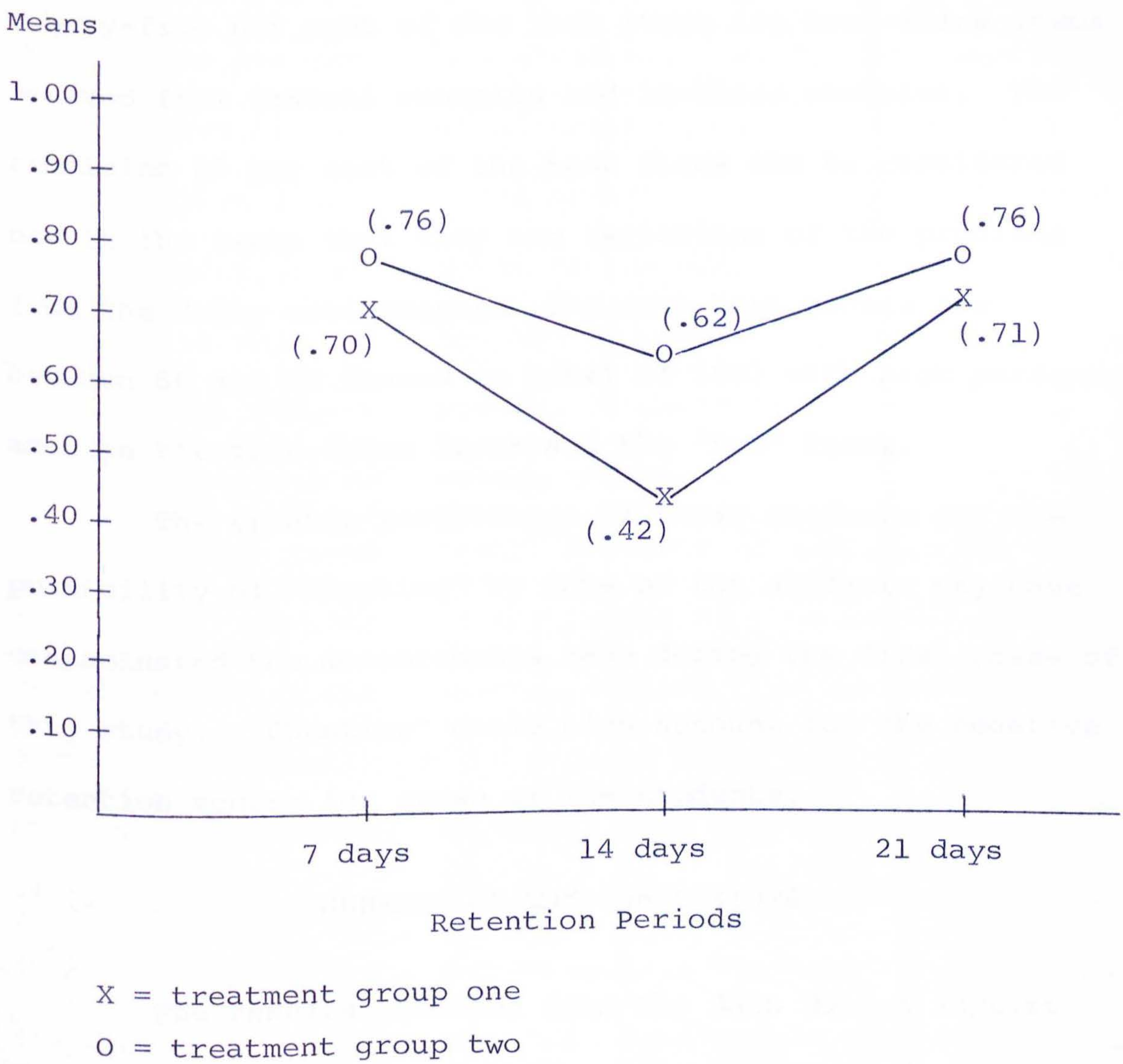


Figure 6

Index of Efficiency Curves

Biweekly tests are constructed so that at least 50 per cent of the test items are from the daily assignments. Twenty-five per cent of the test items are true-false items derived from textual examples and in-class examples. The remaining 25 per cent of the test items can be considered new in the sense that they are variations of the problems from the daily assignments. The mean test scores are between 60 and 70 (based on total of 100) with high performance on the true-false items and the "new" items.

The erratic performance of these students and the possibility of "cheating" by some of the students may have contaminated the measurements made during the first phase of this study. "Cheating" could also account for the negative retention scores for seven of the students.

SUMMARY OF THE CONCLUSIONS

The results obtained from the data do not support the use of behavioral objectives as a procedure for improving either measures of learning or measures of forgetting in a learning program using programmed instruction for teaching an algebraic topic to "below average" mathematics students in senior high school.

RECOMMENDATIONS

Each of the three subsections in the discussion of the conclusions contains implications for further research. It was noted in the first of these three subsections that some previous studies of the effect of knowledge of behavioral objectives on measures of learning and forgetting have had results supporting the use of behavioral objectives. In all of those studies, measures of learning and forgetting were derived from achievement scores.

In the present study, measures of learning and forgetting were functions of the length of time needed to reach criterion rather than measures derived from achievement scores. The results of the present study indicating that the use of behavioral objectives does not affect measures of learning or measures of forgetting suggests there may be inconsistencies in the two procedures for measuring learning and forgetting. Therefore it is recommended that additional research be conducted to ascertain which (if either) procedure for measuring learning and forgetting is more reliable. Also, the question of validity of measurement should be investigated for the two techniques for measuring learning and forgetting.

The use of programmed instruction may have a suppressing effect on differences in treatments in a learning experiment. Every student who completes the instructional program reaches criterion and therefore has a high level of learning. As stated by Underwood, if the learning measures are near 100 per cent, differences in the degree of learning cannot be measured. Therefore the use of programmed instruction may preclude the possibility of measuring differences in learning.

If differences in learning are not measurable then subsequent measures of forgetting may be incorrect, especially if differences in learning actually exist. Therefore it is recommended that research be conducted to ascertain the effect of programmed instruction on measures of learning and forgetting. In particular, it is recommended that careful consideration be given to alternatives to programmed instruction.

The possibility of students communicating with one another during a classroom learning study suggests that an alternative procedure should be established. Computers could be used for both the instruction and testing and such a procedure would almost certainly remove any type of student interaction. It is recommended that the present study be replicated with the following modifications.

The preassessment tasks, the instructional program, and the assessment tasks should be prepared for computer administration. The preassessment tasks and the assessment tasks would probably have to be constructed in a multiple-choice format due to the physical characteristics of an input/output device such as a teletype. If possible, the computer program that contains the information just mentioned should include a subroutine to record the time needed for the acquisition of each behavior in the instructional program.

SUMMARY OF CHAPTER FIVE

The global hypothesis and the six research hypotheses of this study with their respective conclusions are presented in this chapter. A brief discussion is presented which includes possible explanations for the results. Several recommendations for further research are presented.

FOOTNOTES--CHAPTER FIVE

Cook, James Marvin. 1969. "Learning and Retention by Informing Students of Behavioral Objectives and Their Place in the Hierarchical Learning Sequence." Unpublished Doctor's dissertation, University of Maryland, College Park, Maryland.

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Smith, John M. 1970. "Relations Among Behavioral Objectives, Time of Acquisition, and Retention." Unpublished Doctor's dissertation, University of Maryland, College Park, Maryland.

Underwood, Benton J. 1964. "Degree of Learning and the Measurement of Forgetting," Journal of Verbal Learning and Verbal Behavior, 3:112-129.

APPENDIX A

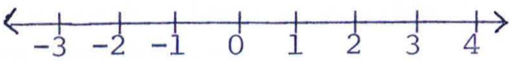
PREASSESSMENT TASKS, INSTRUCTIONAL MATERIALS AND ASSESSMENT TASKS FOR TREATMENT GROUPS ONE AND TWO

TREATMENT GROUP ONE

P-AT I, 1

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

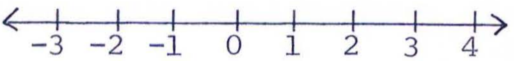
a) double-ray
b) number line
c) directed line
d) directed segment
e) line



P-AT I, 2

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

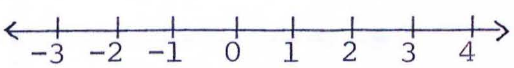
a) directed line
b) directed segment
c) double-ray
d) line
e) number line



P-AT I, 3

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

a) number line
b) double-ray
c) directed line
d) directed segment
e) line



TASK I

The objective of this task is to learn to identify a number line given a sketch.

If you lay a ruler (straight edge) on a piece of paper and then using a pencil draw a mark along one edge, what you would see is like what you see below.

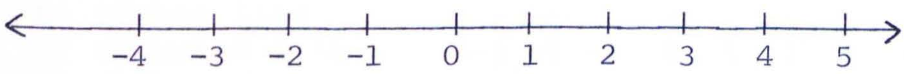
What you should see is a pencil mark that may be described as being a "straight line." A "line" does not end as our sketch above, so we indicate this by placing "arrow-heads" like > and < on the right and left ends of our drawing. Our new sketch looks like

Now, if you select any point on the line and call it "the point related to the number zero" we will have started a process for relating points on lines with real numbers.

We need to relate one more point on our line with a number (different from zero since we have already used zero). The choice of a second point on our line may be either to the left or right of zero. <u>Usually</u> we select a point on the right of the point we related to zero. Suppose we do this and also relate the number one to this new point. (We could have selected any number other than zero, but by choosing one we will avoid many difficulties.) Our new sketch now looks like

If we now use the distance from zero to one as a "unit" distance, we can mark off any number of points to the right and left of these first two points. Our sketch should now look like

We can now relate more numbers to the points on our line. 2 is related to the point one unit to the right of the point related to 1. 3 is related to the point one unit to the right of the point related to 2. 4 is related to the point one unit to the right of the point related to 3. And so on. Similarly, -1 is related to the point one unit to the left of the point related to 0. -2 is related to the point one unit to the left of the point related to -1. And so on. Following this pattern, our sketch should look like

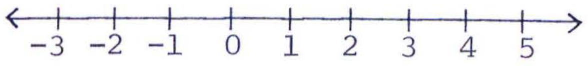


The sketch we have constructed is called a number line.

ASSESSMENT TASK 1

IDENTIFY the name of the drawing by circling the letter to the left of its name in the list below.

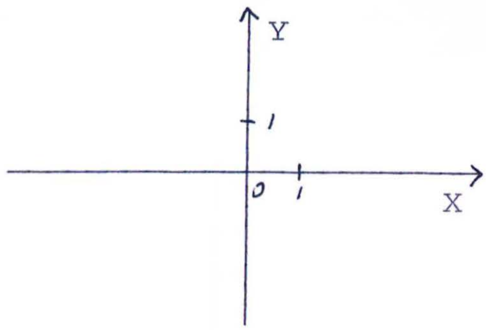
a) double-ray
b) number line
c) directed line
d) directed segment
e) line



P-AT IIa, 1

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

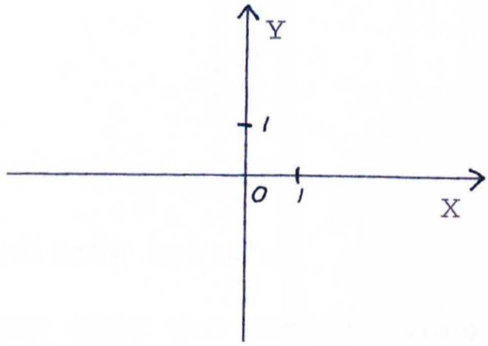
a) cartesian product
b) xy graph
c) graph
d) rectangular coordinate system
e) grid



P-AT IIa, 2

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

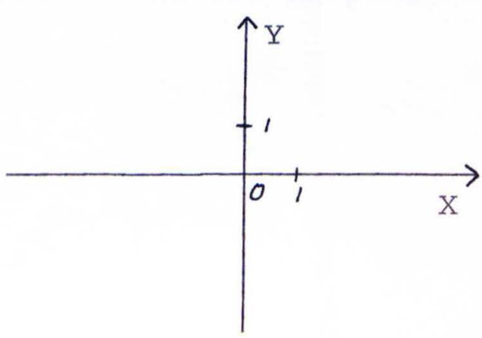
a) grid
b) graph
c) rectangular coordinate system
d) cartesian product
e) xy graph



P-AT IIa, 3

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

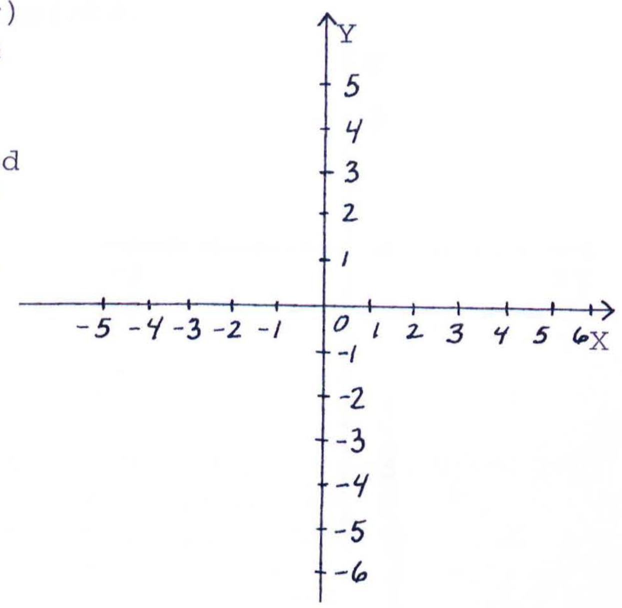
a) grid
b) graph
c) xy graph
d) rectangular coordinate system
e) cartesian product



TASK IIa

The objective of this task is to learn to name a rectangular coordinate system given a description.

In the sketch at the right, two number lines intersect (i.e., they cross one another) at the point numbered zero on each line.



The horizontal line is labeled X and is called the "X axis," the vertical line is labeled Y and is called the "Y axis."

The lines intersect at right angles. (That is, line X is perpendicular to line Y and line Y is perpendicular to line X.)

We call the numbers on the number lines <u>coordinates</u>.

NOTE: In the name rectangular coordinate system;

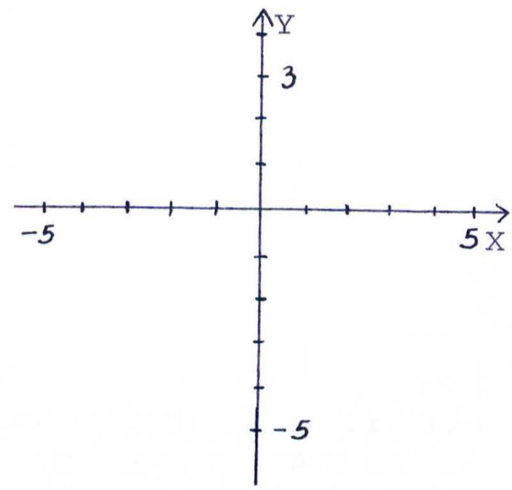
<u>rectangular</u> is used to tell us that the number lines intersect at a right angle.

<u>coordinate</u> is used to tell us that the points on the line have been assigned numbers, and

<u>system</u> is used to tell us that we have some systematic (or precise) way of naming the location of any point in a plane by making reference to the two perpendicular number lines.

ASSESSMENT TASK 2a

NAME the drawing below by writing the name in the space provided. The two number lines intersect at right angles at their zero points.



P-AT IIb, 1

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to left of the correct response.

a) $\{-7,3\}$

b) $[-7,3]$

c) (-73)

d) $(-7,3)$

e) $\langle -7,3 \rangle$

P-AT IIb, 2

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to the left of the correct response.

a) $(5-4)$

b) $(5,-4)$

c) $[5,-4]$

d) $\{5,-4\}$

e) $\langle 5,-4 \rangle$

P-AT IIb, 3

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to the left of the correct response.

a) (128)

b) $\{12,8\}$

c) $(12,8)$

d) $\langle 12,8 \rangle$

e) $[12,8]$

TASK IIb

The objective of this task is to learn to identify an ordered pair of real numbers in a set of numbers.

We will be naming locations of points in a plane by 2 numbers. The numbers will be (by agreement) arranged in a certain order so that if we say the numbers 1 and 2 in one way, such as 2,1 we will not mean the same thing if we say the numbers in the other order 1,2.

We will find it easier to use some method of writing two numbers that are to be ordered so that we can tell at a glance that the order of the numbers is of importance for solving some problem. People who do mathematics indicate that two numbers are ordered by writing them in the following way: (2,1) .

Note that we enclose the pair of numbers in parentheses and that we separate the numbers with a comma.

The parentheses tells us that the numbers enclosed are to be considered as ordered.

The comma tells us where the first number stops and the second number begins.

The importance of the order of numbers will be shown in a later task.

ASSESSMENT TASK 2b

IDENTIFY the correct example of an ordered pair of real numbers by circling the letter to the left of the correct response.

a) $\langle -7,3 \rangle$

b) (-73)

c) (-7,3)

d) (-7,3]

e) -7,3

P-AT III, 1

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Quarter
b) Quadrant
c) Corner
d) Rectangle
e) Coordinate

P-AT III, 2

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Corner
b) Quadrant
c) Coordinate
d) Quarter
e) Rectangle

P-AT III, 3

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Quarter
b) Rectangle
c) Corner
d) Coordinate
e) Quadrant

TASK III

The objective of this task is to learn to NAME a fourth of a plane given that the plane is divided into four parts by a rectangular coordinate system.

Before describing how we may name the location of any point in a plane using our rectangular coordinate system, we will look at the way a rectangular coordinate system divides up a plane.

Recall a line extends indefinitely. This means that if a line is drawn in a plane, the line will separate the plane into how many parts? If your answer is 2, you are correct. If you did not answer 2, perhaps you can convince yourself that 2 is a reasonable answer by thinking of a piece of paper as part of a plane and then draw a line (straight) from one edge of the paper to any other edge of the paper. If you should now cut along this line with a pair of scissors, then wouldn't the paper separate into two pieces?

Now think of the paper as getting longer and wider and the line you cut with the scissors will also get longer, however the paper will still be separated into two pieces.

If a second line is drawn in the plane so that this second line is perpendicular to the first line then the plane will be separated into how many pieces? If your answer is 4, you are correct. If you did not answer 4, again try to convince yourself that 4 is a reasonable answer by using paper with two perpendicular lines drawn on the paper. Then cut along each of the two lines with a scissors and count the number of separated pieces.

Each of the four pieces of the plane is called a quadrant.

When we begin to name the location of points in a plane using our rectangular coordinate system, we will find that points in the upper-right quadrant will be associated with numbers that are both positive, points in the upper-left quadrant will be associated with a negative first number and a positive second number, points in the lower-left quadrant will be associated with numbers that are both negative, and points in the lower-right quadrant will be associated with a positive first number and a negative second number.

ASSESSMENT TASK 3

NAME a fourth of a plane that is formed by dividing a plane into four parts by a rectangular coordinate system. Write the name in the space provided.

____

P-AT IV, 1

A point that is 5 units to the right of the Y axis and 2 units below the X axis has what ordered pair of real numbers for its coordinates?

NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

P-AT IV, 2

A point that is 3 units to the left of the Y axis and 4 units below the X axis has what ordered pair of real numbers for its coordinates?

NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

P-AT IV, 3

A point that is 9 units to the left of the Y axis and 7 units above the X axis has what ordered pair of real numbers for its coordinates?

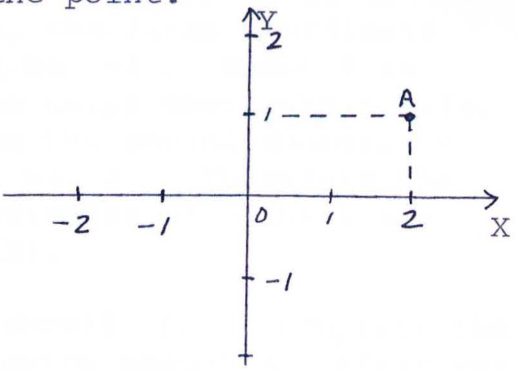
NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

TASK IV

The objective of this task is to learn to apply a rule for naming the coordinates of a point in a plane given a description of the location of the point.

A rectangular coordinate system is shown in the sketch at the right. A point A is located in the upper-right quadrant. Note that A is located 2 units to the right of the Y axis (vertical number line) and also that A is located 1 unit above the X axis (horizontal number line).

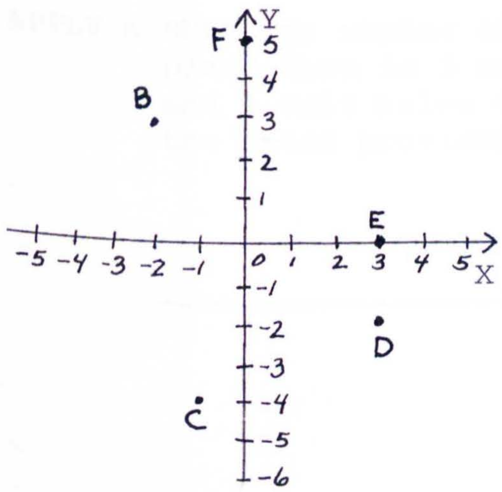


By agreement, mathematicians usually name the location of a point in a plane by an ordered pair of real numbers. We will follow this agreement and name the location of point A by an ordered pair of real numbers. We will often refer to each of these numbers as a coordinate.

The first number in an ordered pair is a number indicating the number of units that the point is to the left or right of the Y axis. The number will be positive if the point is to the right and the number will be negative if the point is to the left. The first number can be read on the X axis since the numbers on the X axis tell us how far to the left or right of the Y axis we are. In our sketch above, point A is two units to the right of the Y axis and notice point A is directly above the point on the X axis numbered 2.

The second number in the ordered pair is a number indicating the number of units that the point is above or below the X axis. The second number can be read on the Y axis since the numbers on the Y axis tell us how far above or below the X axis we are. The second number will be positive if we are above the X axis and it will be negative if we are below the X axis. In our sketch above, point A is one unit above the X axis and notice that point A is directly to the right of the point on the Y axis numbered 1. The coordinates of A are (2,1).

Let us look at some examples:



In the sketch at the left, point B is to the left of the Y axis, hence the first coordinate will be negative since it is two units to the left, the first coordinate will be -2 . Point B is three units above the X axis, hence the second coordinate will be 3 . Therefore the coordinates of point B are (-2,3).

You should try to complete the following examples. After you have completed them, check your answers below.

1) The coordinates of C are ____________.

2) The coordinates of D are ____________.

3) The coordinates of E are ____________.

4) The coordinates of F are ____________.

ANSWERS: 1) (-1,-4) 2) (3,-2) 3) (3,0) 4) (0,5)

ASSESSMENT TASK 4

APPLY A RULE for naming the coordinates of a point in a plane that is 3 units to the right of the Y axis and 1 unit below the X axis. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 4

APPLY A RULE for naming the coordinates of a point in a plane that is 5 units to the left of the Y axis and 4 units above the X axis. Write the answer in the space provided.

____

P-AT Va, 1

If P and Q are points on the X axis and the coordinates of P are (9,0) and the coordinates of Q are (4,0) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Va, 2

If P and Q are points on the X axis and the coordinates of P are (2,0) and the coordinates of Q are (-1,0) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Va, 3

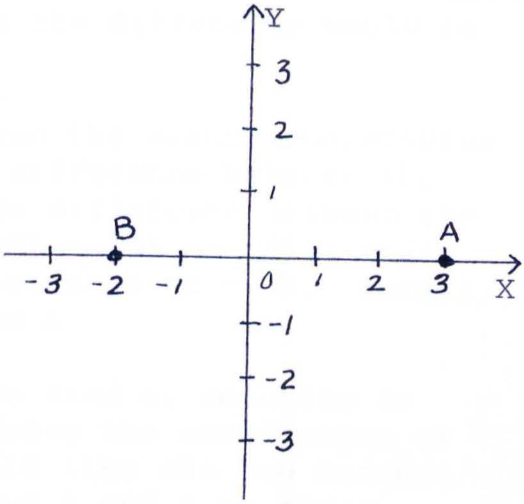
If P and Q are points on the X axis and the coordinates of P are (-3,0) and the coordinates of Q are (7,0) then NAME the distance between P and Q by writing the answer in the space below.

____

TASK Va

The objective of this task is to learn to name the distance between two points on the X axis.

Now that we have a way of naming locations of points in a plane by an ordered pair of numbers, let us consider the problem of finding the distance between two points. We will say that we have found the distance between two points when we can name the number that represents the distance.



Let us begin by looking at the sketch at the right. There are two points indicated, point A and point B.

What is the distance from B to A?

You can find the answer to the question by counting the number of segments between B and A. When I count the segments, I get 5. What do you get?

Is it possible to determine the answer by looking at the coordinates of B and A? In attempting to answer this question, let us begin by writing down the coordinates of B and A.

The coordinates of B are __________.

The coordinates of A are __________.

If you wrote the coordinates of B are (-2,0) and the coordinates of A are (3,0), go on. If you answered differently, go back to the previous tasks and study the examples we completed.

Since we are trying to find the <u>distance between</u> two points, suppose we begin by looking at the <u>difference</u> of the coordinates. At this time, a logical question might be, "How do we find the differences between a pair of coordinates?"

The usual procedure in mathematics is to look at the differences of coordinates by

First--find the difference of the first coordinates. Therefore in our example the difference would be (-2) - (3) <u>or</u> (3) - (-2).

Second--find the difference of the second coordinates. Thus in our example the difference would be (0) - (0).

Since the difference between the second coordinates is zero, we need only look at the difference between the first coordinates. Notice that the difference between the first coordinates can be either (-2) - (3) or (3) - (-2). These two differences can be rewritten as -5 or 5. Recall, we counted 5 segments between B and A.

We would like the answer we find by counting to agree with the answer we find by using the coordinates of A and B. At the same time, we would like the two answers we found by using the coordinates of A and B to agree.

Notice, we can make our two answers agree if we work with their absolute values. That is $|(-2) - (3)| = |-5| = 5$ <u>and</u> $|(3) - (-2)| = |5| = 5$. Note also that we now have an answer that agrees with our answer found by counting.

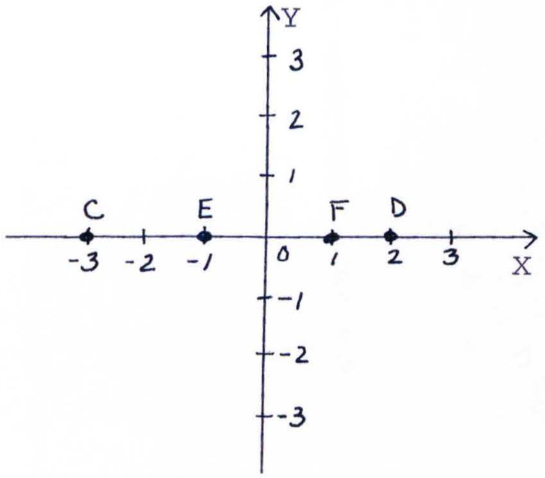
Suppose we look at some examples. In the sketch below, points C, D, E, and F are shown.

The coordinates of C are _____.

The coordinates of D are _____.

The coordinates of E are _____.

The coordinates of F are _____.



The distance from C to D is the absolute value of the difference of the first coordinates of C and D. We should note that we use only the first coordinates because our points are on the X axis. Sometimes we indicate that we are finding the distance from one point to another by using the letters. In this example, CD would mean we are finding the distance from C to D.

CD is equal to $|(-3) - (2)| = |-5| = 5$

CE is equal to $|(-3) - (-1)| = |-2| = 2$

CF is equal to $|(-3) - (1)| = |-4| = 4$

Now you find the following distances. The answers are at the bottom of the page.

EF is equal to ______________________________

ED is equal to ______________________________

FD is equal to ______________________________

ANSWERS: EF = 2, ED = 3, FD = 1

ASSESSMENT TASK 5a

NAME the distance between the following two points on the X axis. Write the answer in the space provided. Point A has coordinates (8,0) and point B has coordinates (1,0).

____

P-AT Vb, 1

If P and Q are points on the Y axis and the coordinates of P are (0,4) and the coordinates of Q are (Q,-1) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Vb, 2

If P and Q are points on the Y axis and the coordinates of P are (0,-2) and the coordinates of Q are (0,5) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Vb, 3

If P and Q are points on the Y axis and the coordinates of P are (0,6) and the coordinates of Q are (0,1) then NAME the distance between P and Q by writing the answer in the space below.

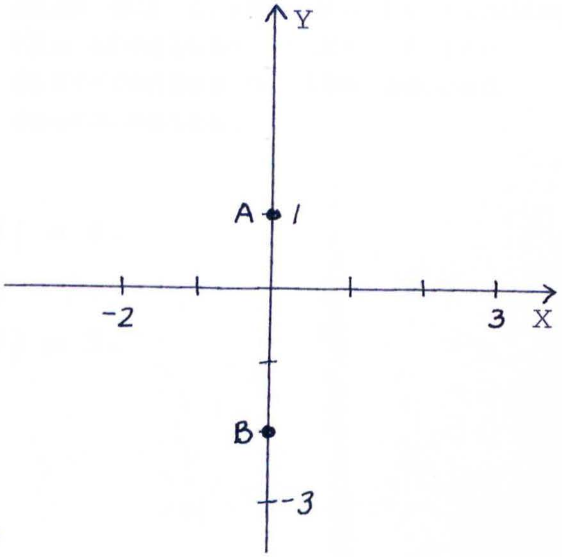
____

TASK Vb

The objective of this task is to learn to name the distance between two points on the Y axis.

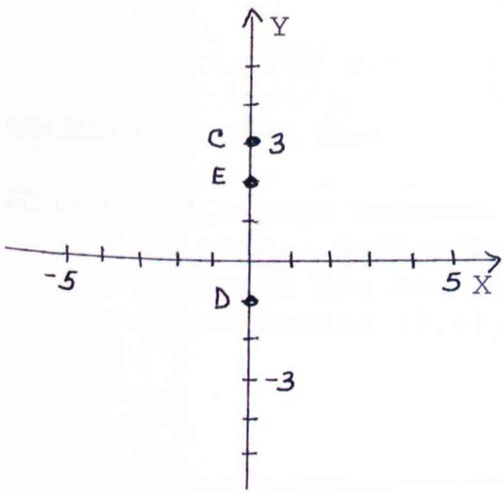
If we are looking for the distance between two points on the Y axis, we should first notice that all points on the Y axis have the same first coordinate, namely zero. This should suggest that the distance between two points on the Y axis can be found by using only the second coordinates.

Suppose we look at the sketch at the right. The points A and B are on the Y axis. If we count the number of segments between A and B we get 3. If we look at the coordinates of A and the coordinates of B we see the second coordinates are 1 and -2. The difference between these two coordinates is (1) - (-2) <u>or</u> (-2) - (1). These two differences then can be simplified to (3) <u>or</u> (-3). Since we want our difference to agree with the answer we find <u>or</u> by counting the segments between A and B, we shall use absolute values.



By using absolute values we can see that our two differences will agree. That is, $|(1) - (-2)| = |3| = 3$ and $|(-2) - (1)| = |(-3)| = 3$.

Suppose we now look at some examples. In the sketch below, the points C, D, and E are on the Y axis.



The coordinates of C are ________.

The coordinates of D are ________.

The coordinates of E are ________.

If we let CD represent the distance from C to D, CE represent the distance from C to E, and DE represent the distance from D to E we can find our distances by finding the absolute value of the differences of the second coordinates.

CD is equal to $|(3) - (-1)| = |4| = 4$.

CE is equal to $|(3) - (2)| = |1| = 1$.

ED is equal to $|(2) - (-1)| = |3| = 3$.

ASSESSMENT TASK 5b

NAME the distance between the following two points on the Y axis. Write the answer in the space provided. Point A has coordinates (0,-3) and point B has coordinates (0,4).

____

P-AT VIa, 1

NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (3,-1) and the coordinates of Q are (10,-1). Write your answer in the space below.

____

P-AT VIa, 2

NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (2,7) and the coordinates of Q are (-4,7). Write your answer in the space below.

____

P-AT VIa, 3

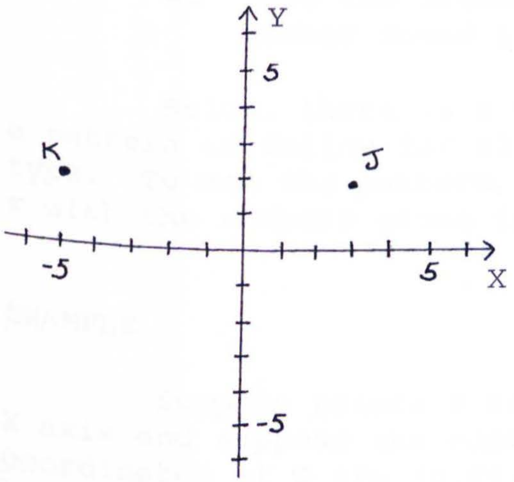
NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (-5,4) and the coordinates of Q are (-1,4). Write your answer in the space below.

____

TASK VIa

The objective of this task is to learn to name the distance between two points on a line parallel to the X axis.

In the sketch below, what is the distance from the point J to the point K? We might also say this as, what is the distance between point J and point K?



By counting we find 8 intervals between J and K.

By using the coordinates of J and K we can use the procedure outlined below.

The coordinates of J are (3,2).

The coordinates of K are (-5,2).

The <u>absolute</u> <u>value</u> of the difference of the first coordinates is $|(3) - (-5)| = |8| = 8$.

The <u>absolute</u> <u>value</u> of the difference of the second coordinates is $|2 - 2| = |0| = 0$.

Therefore, we find the distance is 8 by counting the intervals and also we get 8 by using the coordinates.

You might be wondering what would happen if the difference of our second coordinates had not been zero. If you have, GOOD!! We will soon get to that problem, however for now we shall restrict our attention to points that are on a line parallel to the X axis. By doing so, we will find that every point on a line parallel to the X axis is the same distance above (or below) the X axis and hence will have the same second coordinate. Since the second coordinates are exactly the same, the difference between any two second coordinates will be zero.

Therefore we find that the distance between any two points on a line parallel to the X axis can be determined by finding the "absolute value of the difference of the first coordinates."

It is interesting to note that we "do things" in the opposite order from which we state them. The procedure is as follows.

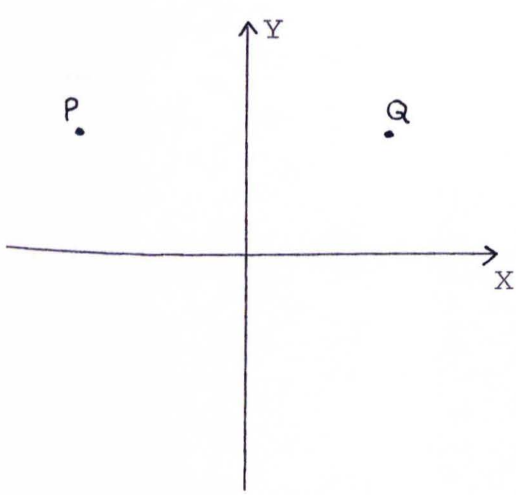
If two points are on a line parallel to the X axis then

1. find the coordinates of the two points.
2. find the difference of the first coordinates of the two points.
3. find the absolute value of the difference found in step 2.
4. name the distance between the two points as the number found in step 3.

Below, there is a general example that will give you a pattern to follow for all problems of this very special type. To use the pattern, you replace the letters p, q, and r with the numbers given in a particular problem.

EXAMPLE

Suppose points P and Q are on a line parallel to the X axis and suppose the coordinates of P are (p,r) and the coordinates of Q are (q,r). (See the sketch below.) Note that the points are on a line parallel to the X axis so the difference of the second coordinates is zero.



Step 1. Coordinates of P are (p,r).
Coordinates of Q are (q,r).

Step 2. The difference of the first coordinates of the two points is (p-q).

Step 3. The absolute value of the difference is $|p - q|$.

Step 4. The distance between P and Q is $|p - q|$.

PRACTICE EXERCISE

Point M has coordinates (4,7) and point N has coordinates (9,7). Name the distance between M and N. (It is helpful to make a sketch. A space has been provided below.)

IF THE POINTS ARE ON A LINE PARALLEL TO THE X AXIS, GO ON.

Step 1. Coordinates of M are

_______.

answer (4,7)

Coordinates of N are

_______.

answer (9,7)

Step 2. The difference of the first coordinates of the two points is

____________________.

answer (4 - 9) or (9 - 4)

Step 3. The absolute value of the difference is

____________________.

answer 5

Step 4. The name of the distance between M and N is _________.

answer 5

ASSESSMENT TASK 6a

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (5,-1) and point B has coordinates (-1,-1).

____

P-AT VIb, 1

NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (6,0) and the coordinates of Q are (6,8). Write your answer in the space below.

P-AT VIb, 2

NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (-2,-5), and the coordinates of Q are (-2,3). Write your answer in the space below.

P-AT VIb, 3

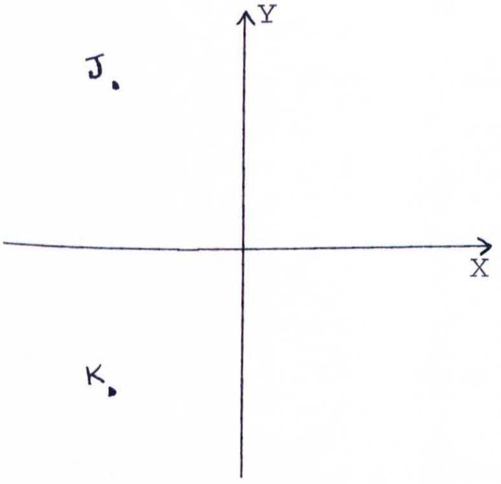
NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (7,-2) and the coordinates of Q are (7,4). Write your answer in the space below.

TASK VIb

The objective of this task is to learn to name the distance between two points on a line parallel to the Y axis.

If we are looking for the distance between two points on a line parallel to the Y axis, we notice that the points on a line parallel to the Y axis all have the same first coordinate since all the points are the same distance to the right (or left) of the Y axis. This suggests that we can find the distance between two points on a line parallel to Y axis by finding the absolute value of the difference between the second coordinates of the two points.

Thus to find the distance between any two points on a line parallel to the Y axis, we can proceed in the following way:



Suppose J and K are two points on a line parallel to the Y axis (see sketch) and suppose the coordinates of J are (t,j) and the coordinates of K are (t,k).

Step 1. The coordinates of J are (t,j).
The coordinates of K are (t,k).

Step 2. The difference between the second coordinates is (j - k) or (k - j).

Step 3. The absolute value of the difference found in step 2 is $|j - k| = |k - j|$.

Step 4. The name of the distance between points J and K is $|j - k|$ or, if you prefer, $|k - j|$.

PRACTICE EXERCISE: Point R has coordinates (6,5). Point S has coordinates (6,2). What is the distance between points R and S?

Step 1. The coordinates of R are _____.
answer (6,5)
The coordinates of S are _____.
answer (6,2)

Step 2. The difference between the second coordinates is ________.
answer (3) or (-3)

Step 3. The absolute value of the difference is ________________.

Step 4. The name of the distance between points R and S is ____________.
answer 3

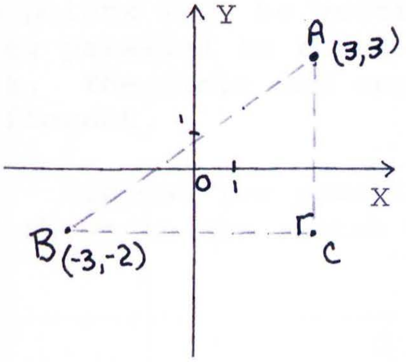
ASSESSMENT TASK 6b

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (2,4) and point B has coordinates (2,-7).

____

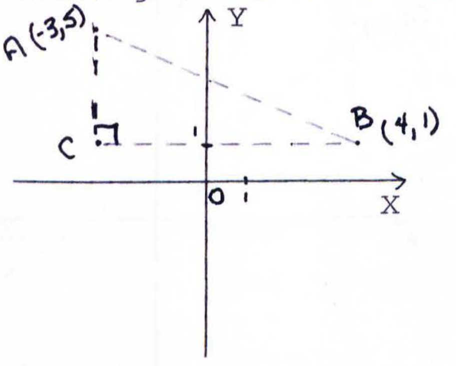
P-AT VII, 1

In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



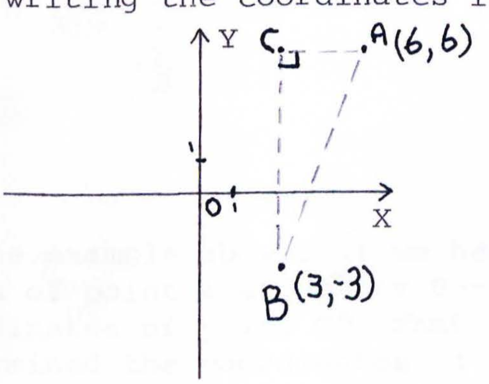
P-AT VII, 2

In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



P-AT VII, 3

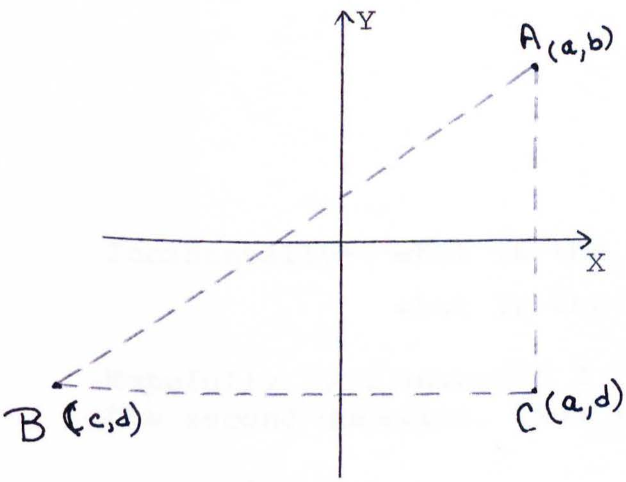
In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



TASK VII

The objective of this task is to learn to name the coordinates of a third point given two points so that the three points will be vertices of a right triangle that has one leg parallel to the X axis and one leg parallel to the Y axis. The first and second coordinate of both points will be different.

Suppose for practice we consider the points A, B, and C shown in the sketch below.



Let us suppose the coordinates of point A are (a,b). Recall the first coordinate tells us how far to the right or left of the Y axis we are and the second coordinate tells us how far above or below the X axis we are.

Further, let us suppose the coordinates of point B are (c,d) and the coordinates of point C are (a,d).

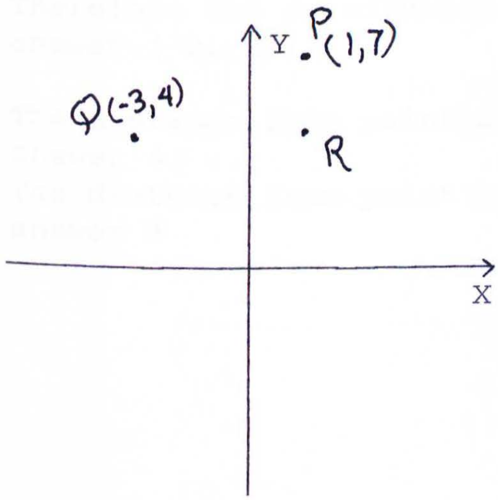
We note that the points B and C are on a line parallel to the X axis and also that the points A and C are on a line parallel to the Y axis.

Since the X axis and the Y axis are perpendicular, we should notice that $\angle BCA$ is a right angle. Hence if we connect the points A and C with segments we have a triangle and we can say it is a right triangle.

In the example above, if we had been given only the coordinates of point A and point B---could we have determined the coordinates of point C? That is, could we have in some way determined the coordinates of point C by using one coordinate from A and one coordinate from B?

NOTICE that the first coordinate of C is the first coordinate of A and that the second coordinate of C is the second coordinate of B.

Let us look at some examples.



In the sketch at the left, the coordinates of point P are (1,7) and the coordinates of point Q are (-3,4).

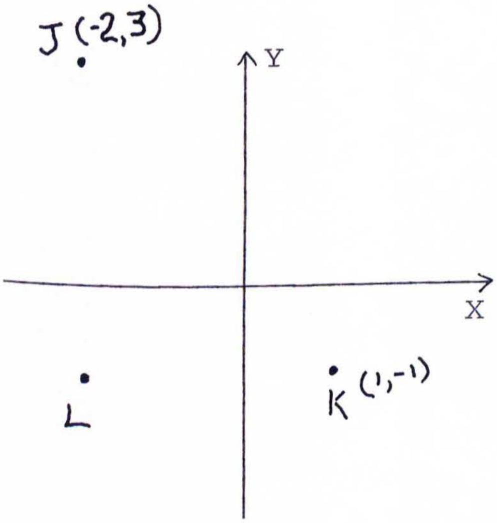
What are the coordinates of point R if we require that $\overline{QR}$ be parallel to the X axis and that $\overline{PR}$ be parallel to the Y axis?

If $\overline{PR}$ is to be parallel to the Y axis, then the first coordinate of R must be the same as the first coordinate of P. If $\overline{QR}$ is to be parallel to the X axis, then the second coordinate of R must be the same as the second coordinate of Q. Therefore, the coordinates of R are (1,4).

Incidentally---what is the distance from P to R? __________

what is the distance from Q to R? __________

Hopefully, you answered 3 for the first question and 4 for the second question. Now let us look at a second example.



In the sketch at the left, the coordinates of point J are (-2,3) and the coordinates of point K are (1,-1).

What are the coordinates of point L if we require that $\overline{JL}$ be parallel to the Y axis and $\overline{LK}$ be parallel to the X axis?

(Fill in the blanks in the paragraph below.)

If $\overline{JL}$ is to be parallel to the Y axis, the first coordinate of L must be the same as the ____________ coordinate of J. If $\overline{LK}$ is to be parallel to the X axis the second coordinate of L must be the same as the ____________ coordinate of K.

Therefore the coordinates of L are (,).
answer (-2, -1)

The distance from point J to point L is ________.
answer 4
The distance from point L to point K is ________.
answer 3

ASSESSMENT TASK 7

NAME the coordinates of a point C given the coordinates of a point A are (4,3) and the coordinates of a point B are (-1,-1) such that AC will be parallel to the X axis and such that BC will be parallel to the Y axis. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 7

Given the coordinates of point A are (,) and the coordinates of point B are (,), NAME the coordinates of a point C such that $\overline{AC}$ will be parallel to the X axis and such that $\overline{BC}$ will be parallel to the Y axis. Write the answer in the space provided.

____

NOTE: The coordinates of point A and of point B are not given. Coordinates were inserted as the assessment tasks were given.

P-AT VIIIa, 1

If $\triangle ABC$ is a right triangle with legs AC = 2 and BC = 3, what is the length of the hypotenuse AB?

P-AT VIIIa, 2

If $\triangle ABC$ is a right triangle with legs AC = 3 and BC = 4, what is the length of the hypotenuse AB?

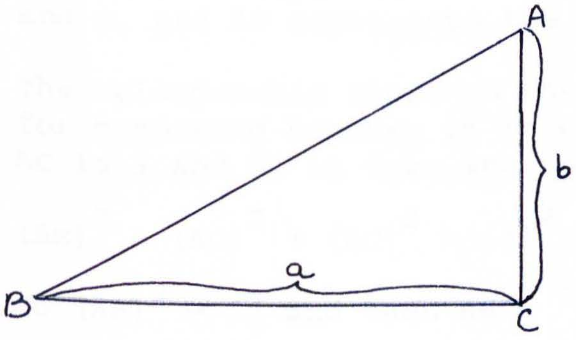
P-AT VIIIa, 3

If $\triangle ABC$ is a right triangle with legs AC = 6 and BC = 9, what is the length of the hypotenuse AB?

TASK VIIIa

The objective of this task is to learn to name the length of the hypotenuse of a right triangle given the length of each of the two legs.

In geometry it is proved that the length of the hypotenuse of a right triangle can be computed if the lengths of the two legs are known.



In the sketch at the left, the angle BCA is a right angle so we can call the triangle a right triangle. $\overline{AB}$ is the side opposite the right angle at vertex C and we call $\overline{AB}$ the hypotenuse of the right $\triangle ABC$. AC and BC are called the legs of right $\triangle ABC$.

The statement that is proved in geometry which relates the lengths of the two legs of a right triangle with the length of the hypotenuse is called the Pythagorean theorem. It is stated as follows:

> In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the two legs.

Suppose we apply this theorem to the right triangle above.

The length of leg $\overline{AC}$ is b. The length of leg $\overline{BC}$ is a.

The square of the length of leg AC is b^2. The square of the length of leg BC is a^2.

The sum of the squares of the lengths of the two legs is $b^2 + a^2$.

The square of the length of the hypotenuse is $b^2 + a^2$.

To complete the task of computing the length of the hypotenuse, we need to find the square root of the square of the length of the hypotenuse.

That is, $\sqrt{b^2 + a^2}$. Therefore the length of AB is $\sqrt{b^2 + a^2}$.

We could also have indicated the relationship between the lengths of the legs of the right triangle and the hypotenuse in the following way:

$$(AC)^2 + (BC)^2 = (AB)^2 \quad \text{or} \quad (AB)^2 = (AC)^2 + (BC)^2$$

Recall that AC represents the length of the segment between A and C, BC represents the length of the segment between B and C, and AB represents the length of the hypotenuse.

The relationship shown on the previous page is often used for computing because it is easy to use. For example, if AC is 3 and BC is 4 we see

$(AB)^2 = (AC)^2 + (BC)^2 = (3)^2 + (4)^2 = 9 + 16 = 25.$

So $(AB)^2 = 25$ and then $AB = \sqrt{(AB)^2} = \sqrt{25} = 5.$

Let us look at this in a step by step fashion.

Step 1: Find the lengths of the legs of the right triangle.

$AC = 3 \quad \text{and} \quad BC = 4$

Step 2: Square the lengths of the legs of the right triangle.

$(AC)^2 = 3^2 = 9 \quad \text{and} \quad (BC)^2 = 4^2 = 16$

Step 3: Add the squares of the lengths of the legs of the right triangle.

$(AC)^2 + (BC)^2 = 9 + 16 = 25$

Step 4: Find the square root of the number found in step 3.

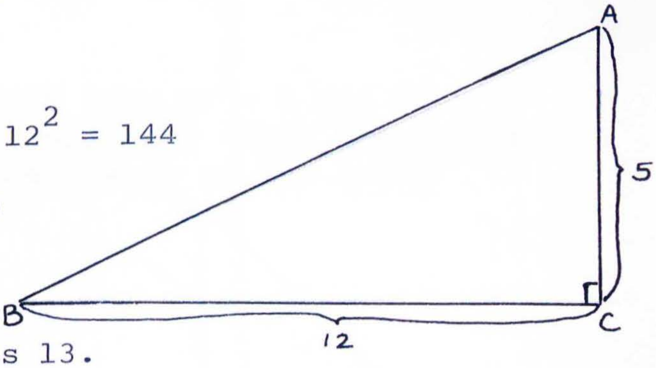
$\sqrt{(AC)^2 + (BC)^2} = \sqrt{25} = 5$

Step 5: State that the length of the hypotenuse is the number found in step 4.

The length of the hypotenuse is 5.

Now let us look at two examples.

Example 1. In the sketch at the right, the lengths of the legs are indicated. What is the length of the hypotenuse?



AC = 5 and BC = 12

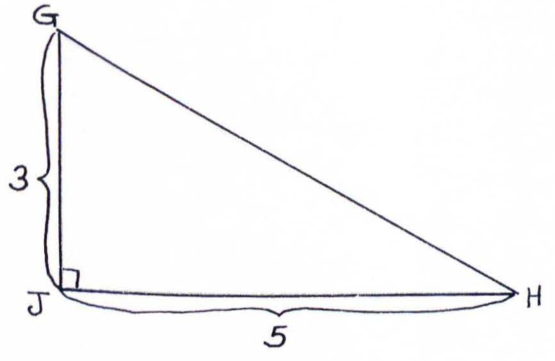
$(AC)^2 = 5^2 = 25$ and $(BC)^2 = 12^2 = 144$

$(AC)^2 + (BC)^2 = 25 + 144 = 169$

$\sqrt{(AC)^2 + (BC)^2} = \sqrt{169} = 13$

The length of the hypotenuse is 13.

Example 2. In the sketch below, the lengths of the legs are indicated. What is the length of the hypotenuse?



GJ = 3 and JH = 5

$(GJ)^2 = 3^2 = 9$ and

$(JH)^2 = 5^2 = 25$

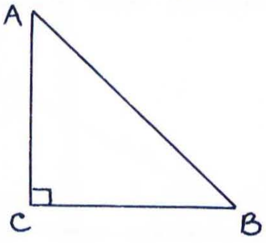
$(GJ)^2 + (JH)^2 = 9 + 25 = 34$

$\sqrt{(GJ)^2 + (JH)^2} = \sqrt{34}$

The length of the hypotenuse is $\sqrt{34}$.

ASSESSMENT TASK 8a

IF $\triangle ABC$ is a right triangle with legs AC = 6 and BC = 5, what is the length of the hypotenuse $\overline{AB}$? NAME the length of the hypotenuse by writing the answer in the space provided.

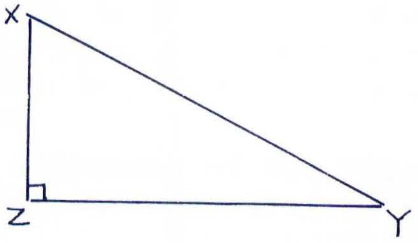


____

ALTERNATE ASSESSMENT TASK 8a

If $\triangle XYZ$ is a right triangle with legs XZ =   and YZ =   , what is the length of the hypotenuse $\overline{XY}$? NAME the length of the hypotenuse by writing the answer in the space provided.

____



NOTE: The lengths of XZ and YZ were given at the time the task was given.

P-AT VIIIb, 1

If the coordinates of the endpoints of the hypotenuse of a right triangle are (2,5) and (-1,1) then what is the length of the hypotenuse?

P-AT VIIIb, 2

If the coordinates of the endpoints of the hypotenuse of a right triangle are (3,2) and (1,-1) then what is the length of the hypotenuse?

P-AT VIIIb, 3

If the coordinates of the endpoints of the hypotenuse of a right triangle are (8,5) and (2,-4) then what is the length of the hypotenuse?

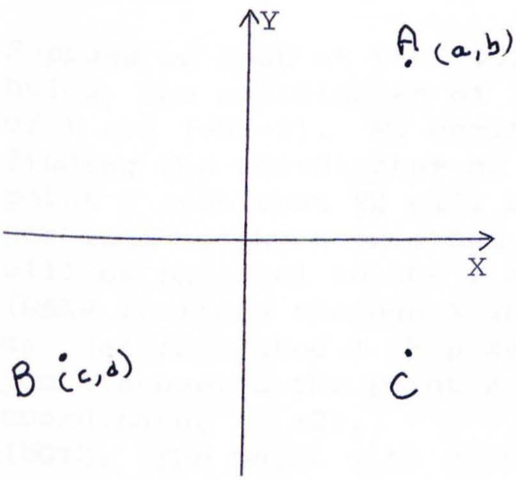
TASK VIIIb

The objective of this task is to learn to apply a rule (the pythagorean theorem) to find the length of the hypotenuse of a right triangle given the coordinates of the endpoints.

In a previous task we found that if we were given the coordinates of two points in a plane that were not on a line parallel to either the X axis or the Y axis, we could determine the coordinates of a third point that would be the right angle vertex of a right triangle. The two original points would then be the coordinates of the endpoints of the hypotenuse of the right triangle.

In the sketch below, we see that the points A and B have their coordinates given. The coordinates of A are (a,b) and the coordinates of B are (c,d). From this information we can determine the coordinates of a point C such that $\overline{AC}$ will be parallel to the Y axis and $\overline{BC}$ will be parallel to the X axis. When we have found C, complete the sketch by drawing in the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. You will then have formed a right triangle. What are the coordinates of point C? ______________

answer (a,d)



Now that the sketch is completed as a right triangle, we note that A and B are the endpoints of the hypotenuse. We also note that the distance from A to B (or, from B to A) is the length of the hypotenuse. Notice also that the distance from A to C is the length of one leg and the distance from B to C is the length of the other leg of the right triangle.

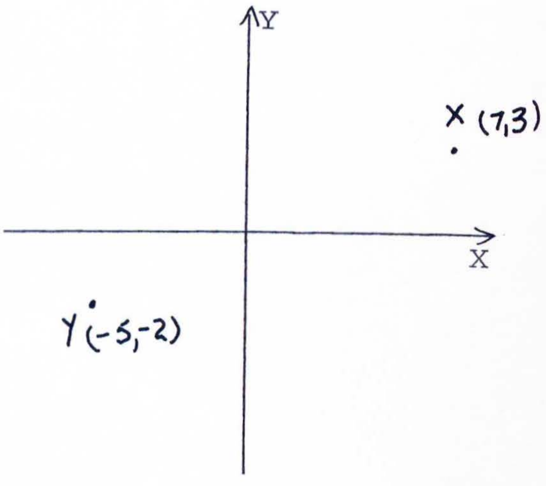
From work we have already completed, we know that we can determine the length of $\overline{AC}$ and we can also determine the length of $\overline{BC}$. The length of $\overline{AC}$ is $|b - d|$ and the length of $\overline{BC}$ is $|c - a|$. (If you do not recall how to find these two lengths, go back to Task VIa and VIb. Also, a review of Task VII may be helpful.)

Since we can find the lengths of the two legs of the right triangle by only knowing the coordinates of endpoints of the hypotenuse, we can use a theorem proved in geometry to find the length of the hypotenuse. The theorem is often stated as

> In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

Therefore, if we know the lengths of 2 sides of a right triangle we can find the length of the third side. In our work we will either know the lengths of the legs of a right triangle or we will be able to determine these lengths without too much work. With this information, we will be able to find the length of the hypotenuse of the right triangle.

Suppose we look at the following example: In the sketch below, the coordinates of X are (7,3) and the coordinates of Y are (-5,-2). We begin by finding the coordinates of a point Z such that $\overline{YZ}$ will be parallel to the X axis and $\overline{XZ}$ will be parallel to the Y axis. (DRAW in lines through X and Y as just indicated.) Hopefully, you discovered the point Z has coordinates (7,-2). (NOTE: The point with coordinates (-5,3) will also satisfy the desired condition.) Next we find the distance between Y and Z and the distance between X and Z. These two distances will be the lengths of the legs of a right triangle with the right angle vertex at Z.



$\overline{YZ} = |(-5) - 7| = |-12| = 12$ and $\overline{XZ} = |3 - (-2)| = |5| = 5$.

Then by the theorem we stated,

$$(XY)^2 = (YZ)^2 + (XZ)^2 = (12)^2 + (5)^2 = 144 + 25 = 169.$$

and finally, the length of the hypotenuse is

$$XY = \sqrt{(XY)^2} = \sqrt{169} = 13.$$

Notice that we could have computed the length of XY by following the steps below.

$$XY = \sqrt{((-5) - 7)^2 + (3 - (-2))^2} = \sqrt{(-12)^2 + (5)^2}$$

$$= \sqrt{144 + 25} = \sqrt{169} = 13.$$

2nd coordinate of Y

2nd coordinate of X

1st coordinate of X

1st coordinate of Y

ASSESSMENT TASK 8b

If the coordinates of the endpoints of the hypotenuse are (3,7) and (-2,0) then NAME the length of the hypotenuse. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 8b

If the coordinates of the endpoints of the hypotenuse are (,) and (,) then NAME the length of the hypotenuse. Write the answer in the space provided.

____

NOTE: Coordinates were inserted in the blank ordered pairs above at the time the task was given.

P-AT IX, 1

Name the distance between the points A and B given that the coordinates of A are (0,3) and the coordinates of B are (4,0).

P-AT IX, 2

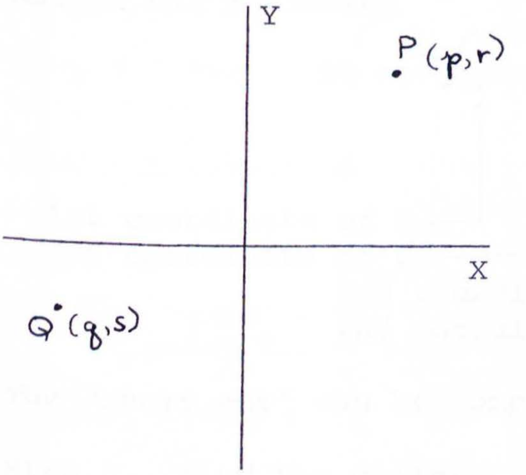
Name the distance between the points A and B given that the coordinates of A are (7,1) and the coordinates of B are (2,13).

P-AT IX, 3

Name the distance between the points A and B given that the coordinates of A are (6,5) and the coordinates of B are (1,1).

TASK IX

The objective of this task is to learn to name the distance between any two points in a plane given the coordinates of the points.



Suppose we are given the sketch at the left with the point P having coordinates (p,r) and the point Q having coordinates (q,s). What are the coordinates of a point R such that $\overline{PR}$ is perpendicular to $\overline{QR}$? That is, find the coordinates of R such that $\overline{QR}$ is parallel to the X axis and $\overline{PR}$ is parallel to the Y axis.

If you answered (p,s) GOOD. If not, go back and review task 7.

What is the distance between points P and R? ____________
answer $|r - s|$ or $|s - r|$

What is the distance between points Q and R? ____________
answer $|p - q|$ or $|q - p|$

If either of these two questions seemed difficult, go back and review task 6a and 6b.

The <u>length</u> of a segment joining two points in a plane is the <u>distance</u> between the two points. Hence, when we find the length of a segment we have found the distance between the endpoints of the segment. Thus (as we have noted before) if the length of segment PR is $|r - s|$ then we also say the distance from P to R is $|r - s|$. Also, we can note that the distance from R to Q is $|p - q|$ and that the length of RQ is $|p - q|$.

To find the length of PQ, we apply the pythagorean theorem.

$$(PQ)^2 = (QR)^2 + (RP)^2 = (p - q)^2 + (r - s)^2$$

REMEMBER when we square a number, we may "drop" the absolute value signs. Now to find PQ we "take the square root" of each side of our equation and get

$$PQ = \sqrt{(p - q)^2 + (r - s)^2}$$

NOTICE in the above work, we found the distance between points P and Q by using only the coordinates of the points P and Q. This suggests we can find a "short cut" for determining the distance between any two points in a plane if we are given the coordinates of the points.

NOTICE the following

$$PQ = \sqrt{(p - q)^2 + (r - s)^2}$$

1st coordinate of P

1st coordinate of Q

2nd coordinate of P

2nd coordinate of Q

The "short cut" can be completed by the following steps:

Step 1. find the difference of the first coordinates of the two points.

Step 2. find the difference of the second coordinates of the two points.

Step 3. square the difference found in step 1.

Step 4. square the difference found in step 2.

Step 5. add the squares found in steps 3 and 4.

Step 6. find the square root of the sum found in step 5.

Step 7. write the answer.

Now that we have a "method" for determining the distance between two points in a plane, suppose we search for a general equation that will name the distance between any two points in a plane. Let us begin by using the steps outlined above.

Let M and N be any two points in a plane and let the coordinates of M be (x_1, y_1) and the coordinates of N be (x_2, y_2). These coordinates are read "x sub 1, y sub 1 and x sub 2, y sub 2."

Step 1. $x_1 - x_2$

Step 2. $y_1 - y_2$

Step 3. $(x_1 - x_2)^2$

Step 4. $(y_1 - y_2)^2$

Step 5. $(x_1 - x_2)^2 + (y_1 - y_2)^2$

Step 6. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Step 7. The distance from M to N is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

PRACTICE EXERCISES: Using the outline above find the distances between the given pairs of points.

1. The coordinates of point A are (0,2) and the coordinates of point B are (4,5)

step 1. $0 - 4$

step 2. $2 - 5$

step 3. $(0 - 4)^2 = 16$

step 4. $(2 - 5)^2 = (-3)^2 = 9$

step 5. $16 + 9 = 25$

step 6. $\sqrt{25} = 5$

step 7. The distance from A to B is 5.

2. The coordinates of point C are (-1,-6) and the coordinates of point D are (2,5).

step 1. ______________

step 2. ______________

step 3. ______________________

step 4. ______________________

step 5. ______________________

step 6. ________________________________

step 7. The distance from point C to point D is $\sqrt{130}$

3. The coordinates of point P are (X,Y) and the coordinates of point C are (2,3).

step 1. ______________

step 2. ______________

step 3. ________________________

step 4. ________________________

step 5. __________________________________

step 6. ____

step 7. The distance from point P to point C is
$\sqrt{(X - 2)^2 + (Y - 3)^2}$

4. The coordinates of point P are (X,Y) and the coordinates of point C are (-3,5).

step 1. ______________

step 2. ______________

step 3. ________________________

step 4. ________________________

step 5. __________________________________

step 6. ____

step 7. The distance from point P to point C is
$\sqrt{(X + 3)^2 + (Y - 5)^2}$

ASSESSMENT TASK 9

NAME the distance between points J and K given that the coordinates of J are (4,7) and the coordinates of K are (-9,5). Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 9

NAME the distance between points M and N given that the coordinates of M are (,) and the coordinates of N are (,). Write the answer in the space provided.

____

NOTE: Coordinates for points M and N were inserted at the time the task was given.

P-AT X, 1

If the center of a circle is at (1,2) and the radius is 3, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X - 1)^2 + (Y - 2)^2 = 3$ b) $(X + 1)^2 + (Y + 2)^2 = 3^2$

c) $(X - 1)^2 + (Y - 2)^2 = 3^2$ d) $(X - 1) + (Y - 2) = 3^2$

P-AT X, 2

If the center of a circle is at (-3,1) and the radius is 5, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X - 3)^2 + (Y + 1)^2 = 5^2$ b) $(X + 3)^2 + (Y - 1)^2 = 5^2$

c) $(X + 3) + (Y - 1) = 5^2$ d) $(X + 3)^2 + (Y - 1)^2 = 5$

P-AT X, 3

If the center of a circle is at (4,-2) and the radius is 2, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X + 4)^2 + (Y - 2)^2 = 2^2$ b) $(X - 4)^2 + (Y + 2)^2 = 2^2$

c) $(X - 4) + (Y + 2) = 2^2$ d) $(X - 4)^2 + (Y + 2)^2 = 2$

TASK X

The objective of this task is to learn to identify the equation of a circle given the coordinates of its center and its radius.

A circle is often defined as a set of points in a plane equally distant from a given point. The given point is called the <u>center</u> of the circle. The distance from the center to any point on the circle is called the <u>radius</u> of the circle.

It is possible to completely describe a circle by writing an equation. The equation that completely describes a circle usually includes the coordinates of the center of the circle and the length of the radius. For example, if the center of a circle is at (7,3) and the radius of the circle is 5, then an equation of the circle is

$$(X - 7)^2 + (Y - 3)^2 = (5)^2$$

NOTE:
1st coordinate of center
2nd coordinate of center
radius

Suppose we look at another example. If (h,k) are the coordinates of the center of a circle and the radius of the circle is r then an equation of the circle is

$$(X - h)^2 + (Y - k)^2 = (r)^2$$

NOTE:
1st coordinate of center
2nd coordinate of center
radius

NOTICE --- in each of the two examples above, we wrote
X <u>minus</u> the 1st coordinate of the center and
Y <u>minus</u> the 2nd coordinate of the center. Then we wrote $(X - h)^2 + (Y - k)^2$ and finally we completed our equation by writing an "equal" sign and then the square of the radius.

STUDY the following examples. In each example, you are given the coordinates of the center and the radius. You are to identify the equation in the list that is completely determined by the given information.

Example 1. center (9,2) radius = 4

a) $(X - 9) + (Y - 2) = (4)^2$ b) $(X + 9) + (Y + 2) = (4)^2$

c) $(X^2 - 9) + (Y^2 - 2) = (4)^2$ d) $(X - 9)^2 + (Y - 2)^2 = (4)^2$

e) $(X - 9)^2 + (Y - 2)^2 = (4)$

Before studying the next example, note that $X - (-2) = X + 2$.

Example 2. center (-2,1) radius = 3

a) $(X + 2) + (Y - 1) = (3)^2$ b) $(X + 2)^2 + (Y - 1)^2 = (3)^2$

c) $(X - 2) + (Y + 1) = (3)^2$ d) $(X^2 + 2) + (Y^2 - 1) = (3)^2$

e) $(X + 2)^2 + (Y - 1)^2 = (3)$

Answer ___________.

Example 3. center (-3,0) radius = 5

a) $(X + 3) + (Y - 0) = (5)^2$ b) $(X - 3)^2 + (Y + 0)^2 = (5)^2$

c) $(X^2 + 3) + (Y^2 - 0) = (5)^2$ d) $(X + 3)^2 + (Y - 0)^2 = (5)$

e) $(X + 3)^2 + (Y - 0)^2 = (5)^2$

Answer ___________.

Example 4. center (0,11) radius = 6

a) $(X - 0)^2 + (Y - 11)^2 = (6)^2$ b) $(X^2 - 0) + (Y^2 - 11) = (6)^2$

c) $(X + 0)^2 + (Y + 11)^2 = (6)^2$ d) $(X - 0)^2 + (Y - 11)^2 = (6)^2$

e) $(X + 0) + (Y + 11) = (6)^2$

Answer ___________.

Answers to examples:

1) d 2) b 3) e 4) a

ASSESSMENT TASK 10

IDENTIFY the equation of a circle in the list below given that the center of the circle is at (3,5) and the radius is 4. Place a check mark at the left of the correct response.

a) $(X - 3) + (Y - 5) = (4)^2$

b) $(X - 3)^2 + (Y - 5)^2 = (4)^2$

c) $(X + 3)^2 + (Y + 5)^2 = (4)^2$

d) $(X - 3)^2 + (Y - 5)^2 = (4)$

ALTERNATE ASSESSMENT TASK 10

IDENTIFY the equation of a circle in the list below given that the center of the circle is at (-1,13) and that the radius is 9. Place a check mark at the left of the correct response.

a) $(X - 1)^2 + (Y + 13)^2 = (9)^2$

b) $(X + 1)^2 + (Y - 13)^2 = (9)$

c) $(X + 1) + (Y - 13) = (9)^2$

d) $(X + 1)^2 + (Y - 13)^2 = (9)^2$

P-AT XI, 1

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

P-AT XI, 2

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

P-AT XI, 3

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

NOTE: Coordinates for the center and the measure of the radius were inserted at the time the preassessment task was given.

TASK XI

The objective of this task is to learn to apply a rule to write the equation of a circle in standard form given the coordinates of the center of the circle and the radius of the circle.

In this task we wish to determine the equation of a circle. Hence we shall begin by first reminding ourselves of the definition of a circle.

Definition: A circle is a set of points in a plane equally distant from a given point. The given point is called the center of the circle. The distance from the center to any point on the circle is called the radius of the circle.

Let us begin by deriving or finding the equation of a circle. Suppose the center of the circle is a point O with coordinates (a,b) and suppose the radius of the circle is a positive number r. Let P be any point on the circle and let P have coordinates (X,Y).

We will find that we can proceed very much like we did in a previous task. We can do this because our problem is very much like a previous task in which we found the distance between two points. In this example, the points are O and P. The distinction (or--difference) between our present task and a previous task is that now we are given the distance between O and P. That given distance is r, the radius of our circle.

The following step by step outline was used in a previous task.

step 1. $X - a$

step 2. $Y - b$

step 3. $(X - a)^2$

step 4. $(Y - b)^2$

step 5. $(X - a)^2 + (Y - b)^2$

step 6. $\sqrt{(X - a)^2 + (Y - b)^2}$

step 7. $OP = r = \sqrt{(X - a)^2 + (Y - b)^2}$

To complete the derivation of an equation of a circle, we need to remember what an equation of a circle looks like. Recall--an equation of a circle (in what is referred to as "standard form") looks like the following equation.

$$r^2 = (X - a)^2 + (Y - b)^2$$

The radius of the circle is r, the first coordinate of the center is a and the second coordinate of the center is b. Thus to make our equation in step 7 look like the equation of a circle, we see that <u>squaring each side of our equation</u> in step 7 given us the desired results.

Squaring the left side we get r^2 and squaring the right side we "remove" the square root symbol and get the following equation:

$$r^2 = (X - a)^2 + (Y - b)^2$$

We now see that the equation of a circle is really easier to find than the task of finding the distance between two points, since we do not have to take the square roots. For example, if we wish to determine the equation of a circle with radius 2 and center at (1,3) we can write the equation immediately by- - -

replacing r with 2,
replacing a with 1,
replacing b with 3.

Thus, the equation of our circle is $2^2 = (X - 1)^2 + (Y - 3)^2$.

PRACTICE EXERCISES

1. Write the equation of a circle in standard form given the coordinates of the center are (-2,4) and that the radius is 7.

   In our general equation above, we replace

   a with ________________

   answer -2

   b with ________________

   answer 4

   r with ________________

   answer 7

The equation of the circle is ____________________________

answer $7^2 = (X - (-2))^2 + (Y - 4)^2$

or $7^2 = (X + 2)^2 + (Y - 4)^2$

2. Write the equation of a circle in standard form given the coordinates of the center are (0,-12) and that the radius is 13.

The equation of the circle is ______________________________

TASK XI

If the center of a circle has coordinates (,) and the radius of the circle is then what is an equation of the circle in standard form? Write your answer in the space provided.

____

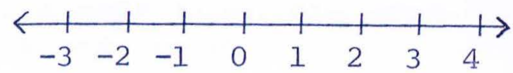
NOTE: The coordinates for the center and the measure of the radius were given at the time the task was given.

TREATMENT GROUP TWO

P-AT I, 1

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

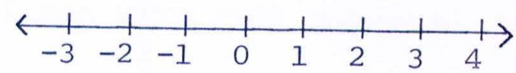
a) double-ray
b) number line
c) directed line
d) directed segment
e) line



P-AT I, 2

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

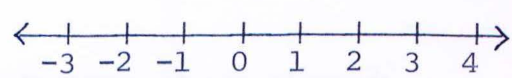
a) directed line
b) directed segment
c) double-ray
d) line
e) number line



P-AT I, 3

IDENTIFY the name of the drawing below by circling the letter to the left of its name in the list below.

a) number line
b) double-ray
c) directed line
d) directed segment
e) line



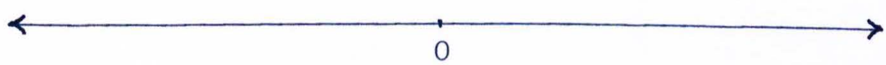
TASK I

The information in this task is needed before you can progress to other parts of the program.

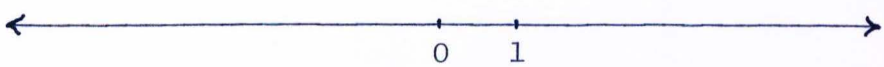
If you lay a ruler (straight edge) on a piece of paper and then using a pencil draw a mark along one edge, what you would see is like what you see below.

What you should see is a pencil mark that may be described as being a "straight line." A "line" does not end as our sketch above, so we indicate this by placing "arrow-heads" like > and < on the right and left ends of our drawing. Our new sketch looks like

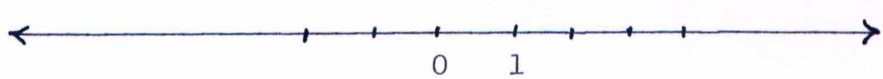
Now, if you select any point on the line and call it "the point related to the number zero" we will have started a process for relating points on lines with real numbers. Your new sketch looks like



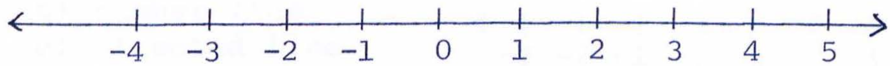
We need to relate one more point on our line with a number (different from zero since we have already used zero). The choice of a second point on our line may be either to the left or right of zero. Usually we select a point on the right of the point we related to zero. Suppose we do this and also relate the number one to this new point. (We could have selected any number other than zero, but by choosing one we will avoid many difficulties.) Our new sketch now looks like



If we now use the distance from zero to one as a "unit" distance, we can mark off any number of points to the right and left of these first two points. Our sketch should now look like



We can now relate more numbers to the points on our line. 2 is related to the point one unit to the right of the point related to 1. 3 is related to the point one unit to the right of the point related to 2. 4 is related to the point one unit to the right of the point related to 3. And so on. Similarly, -1 is related to the point one unit to the left of the point related to 0. -2 is related to the point one unit to the left of the point related to -1. And so on. Following this pattern, our sketch should look like

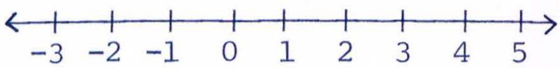


The sketch we have constructed is called a number line.

ASSESSMENT TASK 1

IDENTIFY the name of the drawing by circling the letter to the left of its name in the list below.

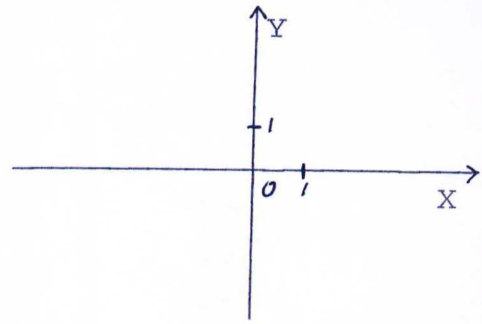
a) double-ray
b) number line
c) directed line
d) directed segment
e) line



P-AT IIa, 1

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

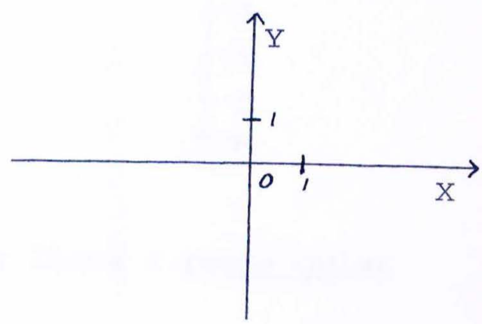
a) cartesian product
b) xy graph
c) graph
d) rectangular coordinate system
e) grid



P-AT IIa, 2

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

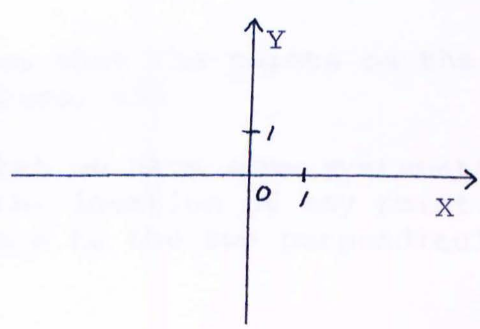
a) grid
b) graph
c) rectangular coordinate system
d) cartesian product
e) xy graph



P-AT IIa, 3

IDENTIFY the name of the sketch below by circling the letter to the left of its name in the list below.

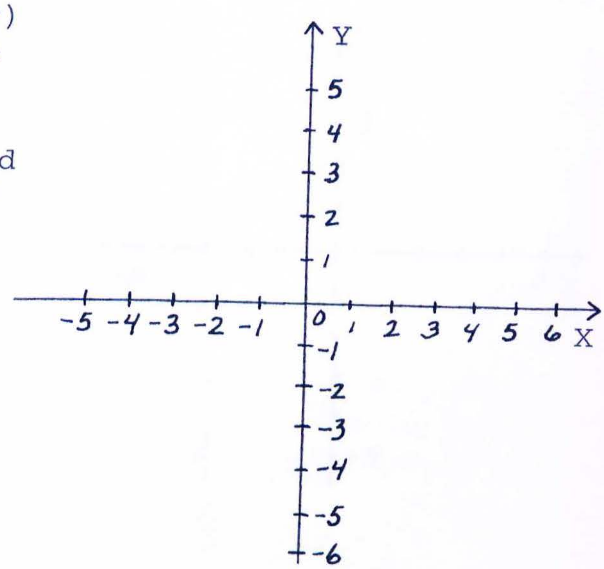
a) grid
b) graph
c) xy graph
d) rectangular coordinate system
e) cartesian product



TASK IIa

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

In the sketch at the right, two number lines intersect (i.e., they cross one another) at the point numbered zero on each line.



The horizontal line is labeled X and is called the "X axis," the vertical line is labeled Y and is called the "Y axis."

The lines intersect at right angles. (That is, line X is perpendicular to line Y and line Y is perpendicular to line X.)

We call the numbers on the number lines <u>coordinates</u>.

We call the two intersecting number lines a <u>rectangular coordinate system</u>.

NOTE: In the name rectangular coordinate system;

<u>rectangular</u> is used to tell us that the number lines intersect at a right angle.

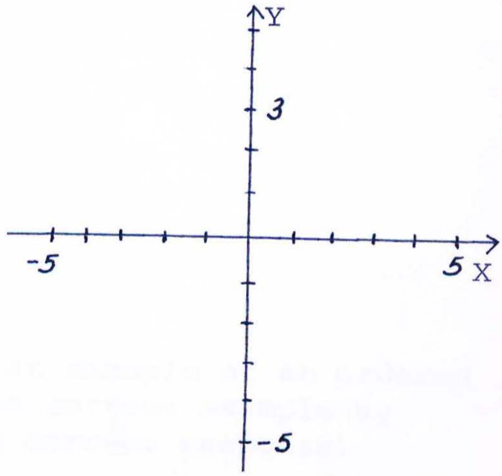
<u>coordinate</u> is used to tell us that the points on the line have been assigned numbers, and

<u>system</u> is used to tell us that we have some systematic (or precise) way of naming the location of any point in a plane by making reference to the two perpendicular number lines.

ASSESSMENT TASK 2a

NAME the drawing below by writing the name in the space provided. The two number lines intersect at right angles at their zero points.

____



P-AT IIb, 1

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to left of the correct response.

a) {-7,3}

b) [-7,3]

c) (-73)

d) (-7,3)

e) ⟨-7,3⟩

P-AT IIb, 2

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to the left of the correct response.

a) (5-4)

b) (5,-4)

c) [5,-4]

d) {5,-4}

e) ⟨5,-4⟩

P-AT IIb, 3

Which of the following is an example of an ordered pair of real numbers? IDENTIFY the correct example by placing a check to the left of the correct response.

a) (128)

b) {12,8}

c) (12,8)

d) ⟨12,8⟩

e) [12,8]

TASK IIb

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

We will be naming locations of points in a plane by 2 numbers. The numbers will be (by agreement) arranged in a certain <u>order</u> so that if we say the numbers 1 and 2 in one way, such as 2,1 we will <u>not</u> mean the same thing if we say the numbers in the other order 1,2 .

We will find it easier to use some method of writing two numbers that are to be ordered so that we can tell at a glance that the order of the numbers is of importance for solving some problem. People who do mathematics indicate that two numbers are ordered by writing them in the following way: (2,1) .

Note that we enclose the pair of numbers in <u>parentheses</u> and that we separate the numbers with a <u>comma</u>.

The parentheses tells us that the numbers enclosed are to be considered as ordered.

The comma tells us where the first number stops and the second number begins.

The importance of the order of numbers will be shown in a later task.

ASSESSMENT TASK 2b

IDENTIFY the correct example of an ordered pair of real numbers by circling the letter to the left of the correct response.

a) ⟨-7,3⟩

b) (-73)

c) (-7,3)

d) (-7,3]

e) -7,3

P-AT III, 1

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Quarter
b) Quadrant
c) Corner
d) Rectangle
e) Coordinate

P-AT III, 2

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Corner
b) Quadrant
c) Coordinate
d) Quarter
e) Rectangle

P-AT III, 3

Which of the following is the name of one fourth of a plane that is divided into four parts by a rectangular coordinate system? IDENTIFY the correct answer by placing a check mark to the left of the correct response.

a) Quarter
b) Rectangle
c) Corner
d) Coordinate
e) Quadrant

TASK III

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

Before describing how we may name the location of any point in a plane using our rectangular coordinate system, we will look at the way a rectangular coordinate system divides up a plane.

Recall a line extends indefinitely. This means that if a line is drawn in a plane, the line will separate the plane into how many parts? If your answer is 2, you are correct. If you did not answer 2, perhaps you can convince yourself that 2 is a reasonable answer by thinking of a piece of paper as part of a plane and then draw a line (straight) from one edge of the paper to any other edge of the paper. If you should now cut along this line with a pair of scissors, then wouldn't the paper separate into two pieces?

Now think of the paper as getting longer and wider and the line you cut with the scissors will also get longer, however the paper will still be separated into two pieces.

If a second line is drawn in the plane so that this second line is perpendicular to the first line then the plane will be separated into how many pieces? If your answer is 4, you are correct. If you did not answer 4, again try to convince yourself that 4 is a reasonable answer by using paper with two perpendicular lines drawn on the paper. Then cut along each of the two lines with a scissors and count the number of separated pieces.

Each of the four pieces of the plane is called a <u>quadrant</u>.

When we begin to name the location of points in a plane using our rectangular coordinate system, we will find that points in the upper-right quadrant will be associated with numbers that are both positive, points in the upper-left quadrant will be associated with a negative first number and a positive second number, points in the lower-left quadrant will be associated with numbers that are both negative, and points in the lower-right quadrant will be associated with a positive first number and a negative second number.

ASSESSMENT TASK 3

NAME a fourth of a plane that is formed by dividing a plane into four parts by a rectangular coordinate system. Write the name in the space provided.

____

P-AT IV, 1

A point that is 5 units to the right of the Y axis and 2 units below the X axis has what ordered pair of real numbers for its coordinates?

NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

P-AT IV, 2

A point that is 3 units to the left of the Y axis and 4 units below the X axis has what ordered pair of real numbers for its coordinates?

NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

P-AT IV, 3

A point that is 9 units to the left of the Y axis and 7 units above the Y axis has what ordered pair of real numbers for its coordinates?

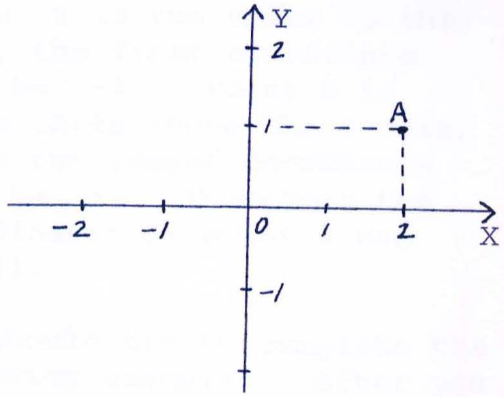
NAME the coordinates of the point by writing the ordered pair of real numbers in the space below.

____

TASK IV

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

A rectangular coordinate system is shown in the sketch at the right. A point A is located in the upper-right quadrant. Note that A is located 2 units to the right of the Y axis (vertical number line) and also that A is located 1 unit above the X axis (horizontal number line).

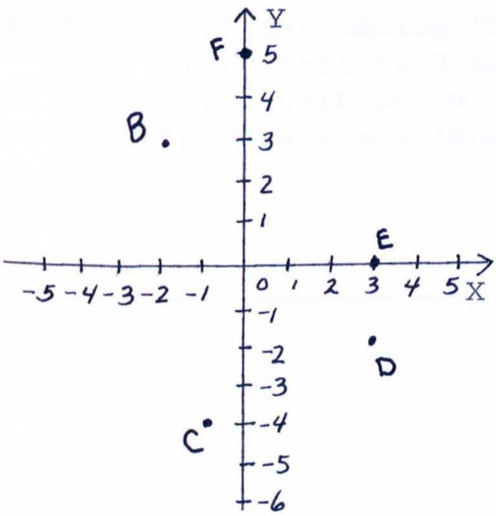


By agreement, mathematicians usually name the location of a point in a plane by an ordered pair of real numbers. We will follow this agreement and name the location of point A by an ordered pair of real numbers. We will often refer to each of these numbers as a coordinate.

The first number in an ordered pair is a number indicating the number of units that the point is to the left or right of the Y axis. The number will be positive if the point is to the right and the number will be negative if the point is to the left. The first number can be read on the X axis since the numbers on the X axis tell us how far to the left or right of the Y axis we are. In our sketch above, point A is two units to the right of the Y axis and notice point A is directly above the point on the X axis numbered 2.

The second number in the ordered pair is a number indicating the number of units that the point is above or below the X axis. The second number can be read on the Y axis since the numbers on the Y axis tell us how far above or below the X axis we are. The second number will be positive if we are above the X axis and it will be negative if we are below the X axis. In our sketch above, point A is one unit above the X axis and notice that point A is directly to the right of the point on the Y axis numbered 1. The coordinates of A are (2,1).

Let us look at some examples:



In the sketch at the left, point B is to the left of the Y axis, hence the first coordinate will be negative since it is two units to the left, the first coordinate will be -2 . Point B is three units above the X axis, hence the second coordinate will be 3 . Therefore the coordinates of point B are (-2,3).

You should try to complete the following examples. After you have completed them, check your answers below.

1) The coordinates of C are ____________.

2) The coordinates of D are ____________.

3) The coordinates of E are ____________.

4) The coordinates of F are ____________.

ANSWERS: 1) (-1,-4) 2) (3,-2) 3) (3,0) 4) (0,5)

ASSESSMENT TASK 4

APPLY A RULE for naming the coordinates of a point in a plane that is 3 units to the right of the Y axis and 1 unit below the X axis. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 4

APPLY A RULE for naming the coordinates of a point in a plane that is 5 units to the left of the Y axis and 4 units above the X axis. Write the answer in the space provided.

____

P-AT Va, 1

If P and Q are points on the X axis and the coordinates of P are (9,0) and the coordinates of Q are (4,0) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Va, 2

If P and Q are points on the X axis and the coordinates of P are (2,0) and the coordinates of Q are (-1,0) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Va, 3

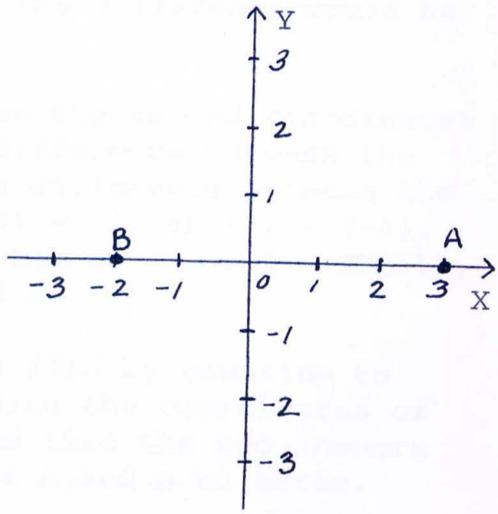
If P and Q are points on the X axis and the coordinates of P are (-3,0) and the coordinates of Q are (7,0) then NAME the distance between P and Q by writing the answer in the space below.

____

TASK Va

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

Now that we have a way of naming locations of points in a plane by an ordered pair of numbers, let us consider the problem of finding the distance between two points. We will say that we have found the distance between two points when we can name the number that represents the distance.



Let us begin by looking at the sketch at the right. There are two points indicated, point A and point B.

What is the distance from B to A?

You can find the answer to the question by counting the number of segments between B and A. When I count the segments, I get 5. What do you get?

Is it possible to determine the answer by looking at the coordinates of B and A? In attempting to answer this question, let us begin by writing down the coordinates of B and A.

The coordinates of B are __________.

The coordinates of A are __________.

If you wrote the coordinates of B are (-2,0) and the coordinates of A are (3,0), go on. If you answered differently, go back to the previous tasks and study the examples we completed.

Since we are trying to find the <u>distance between</u> two points, suppose we begin by looking at the <u>difference</u> of the coordinates. At this time, a logical question might be, "How do we find the differences between a pair of coordinates?"

The usual procedure in mathematics is to look at the differences of coordinates by

First--find the difference of the first coordinates. Therefore in our example the difference would be (-2) - (3) <u>or</u> (3) - (-2).

Second--find the difference of the second coordinates. Thus in our example the difference would be (0) - (0).

Since the difference between the second coordinates is zero, we need only look at the difference between the first coordinates. Notice that the difference between the first coordinates can be either (-2) - (3) or (3) - (-2). These two differences can be rewritten as -5 or 5. Recall, we counted 5 segments between B and A.

We would like the answer we find by counting to agree with the answer we find by using the coordinates of A and B. At the same time, we would like the two answers we found by using the coordinates of A and B to agree.

Notice, we can make our two answers agree if we work with their absolute values. That is $|(-2) - (3)| = |-5| = 5$ <u>and</u> $|(3) - (-2)| = |5| = 5$. Note also that we now have an answer that agrees with our answer found by counting.

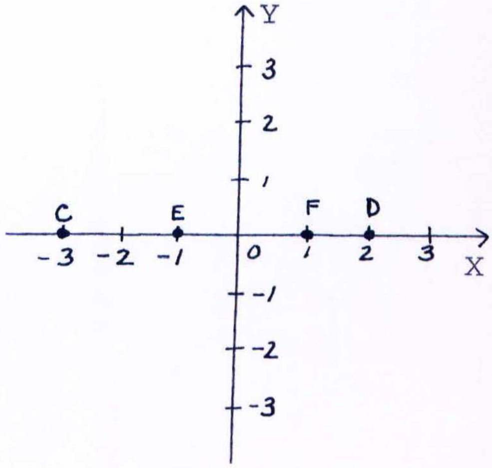
Suppose we look at some examples. In the sketch below, points C, D, E, and F are shown.

The coordinates of C are _____.

The coordinates of D are _____.

The coordinates of E are _____.

The coordinates of F are _____.



The distance from C to D is the absolute value of the difference of the first coordinates of C and D. We should note that we use only the first coordinates because our points are on the X axis. Sometimes we indicate that we are finding the distance from one point to another by using the letters. In this example, CD would mean we are finding the distance from C to D.

CD is equal to $|(-3) - (2)| = |-5| = 5$

CE is equal to $|(-3) - (-1)| = |-2| = 2$

CF is equal to $|(-3) - (1)| = |-4| = 4$

Now you find the following distances. The answers are at the bottom of the page.

EF is equal to ______________________________

ED is equal to ______________________________

FD is equal to ______________________________

ANSWERS: EF = 2, ED = 3, FD = 1

ASSESSMENT TASK 5a

NAME the distance between the following two points on the X axis. Write the answer in the space provided. Point A has coordinates (8,0) and point B has coordinates (1,0).

____

P-AT Vb, 1

If P and Q are points on the Y axis and the coordinates of P are (0,4) and the coordinates of Q are (Q,-1) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Vb, 2

If P and Q are points on the Y axis and the coordinates of P are (0,-2) and the coordinates of Q are (0,5) then NAME the distance between P and Q by writing the answer in the space below.

____

P-AT Vb, 3

If P and Q are points on the Y axis and the coordinates of P are (0,6) and the coordinates of Q are (0,1) then NAME the distance between P and Q by writing the answer in the space below.

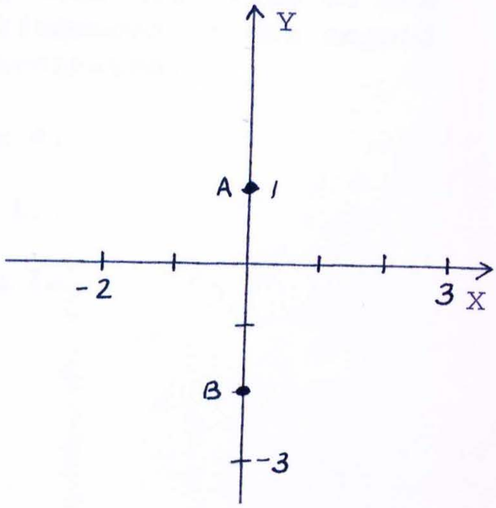
____

TASK Vb

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

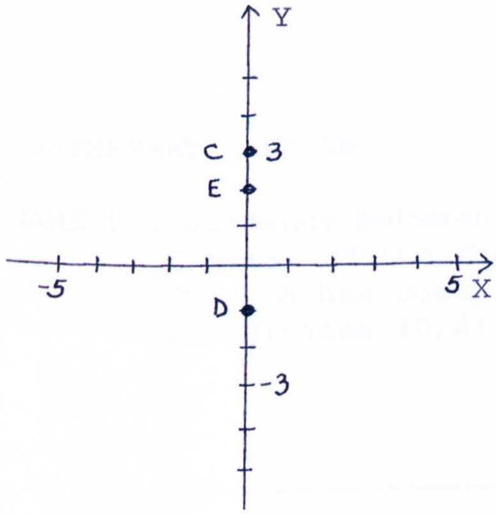
If we are looking for the distance between two points on the Y axis, we should first notice that all points on the Y axis have the same first coordinate, namely zero. This should suggest that the distance between two points on the Y axis can be found by using only the second coordinates.

Suppose we look at the sketch at the right. The points A and B are on the Y axis. If we count the number of segments between A and B we get 3. If we look at the coordinates of A and the coordinates of B we see the second coordinates are 1 and -2. The difference between these two coordinates is (1) - (-2) <u>or</u> (-2) - (1). These two differences then can be simplified to (3) <u>or</u> (-3). Since we want our difference to agree with the answer we find <u>or</u> by counting the segments between A and B, we shall use absolute values.



By using absolute values we can see that our two differences will agree. That is, $|(1) - (-2)| = |3| = 3$ and $|(-2) - (1)| = |(-3)| = 3$.

Suppose we now look at some examples. In the sketch below, the points C, D, and E are on the Y axis.



The coordinates of C are

________.

The coordinates of D are

________.

The coordinates of E are

________.

If we let CD represent the distance from C to D, CE represent the distance from C to E, and DE represent the distance from D to E we can find our distances by finding the absolute value of the differences of the second coordinates.

CD is equal to $|(3) - (-1)| = |4| = 4$.

CE is equal to $|(3) - (2)| = |1| = 1$.

ED is equal to $|(2) - (-1)| = |3| = 3$.

ASSESSMENT TASK 5b

NAME the distance between the following two points on the Y axis. Write the answer in the space provided. Point A has coordinates (0,-3) and point B has coordinates (0,4).

____

P-AT VIa, 1

NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (3,-1) and the coordinates of Q are (10,-1). Write your answer in the space below.

____

P-AT VIa, 2

NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (2,7) and the coordinates of Q are (-4,7). Write your answer in the space below.

____

P-AT VIa, 3

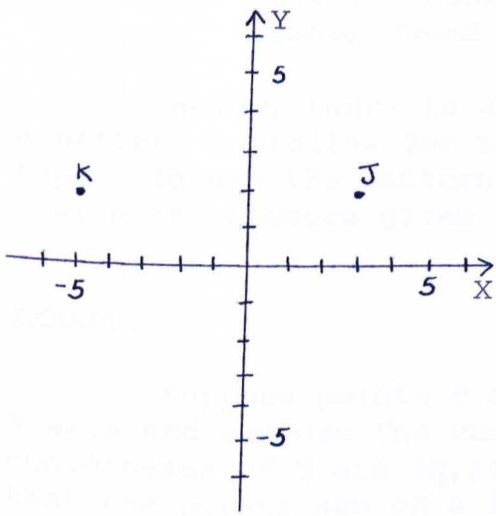
NAME the distance from point P to point Q if P and Q are on a line parallel to the X axis and the coordinates of P are (-5,4) and the coordinates of Q are (-1,4). Write your answer in the space below.

____

TASK VIa

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

In the sketch below, what is the distance from the point J to the point K? We might also say this as, what is the distance between point J and point K?



By counting we find 8 intervals between J and K.

By using the coordinates of J and K we can use the procedure outlined below.

The coordinates of J are (3,2).

The coordinates of K are (-5,2).

The <u>absolute</u> <u>value</u> of the difference of the first coordinates is $|(3) - (-5)| = |8| = 8$.

The <u>absolute</u> <u>value</u> of the difference of the second coordinates is $|2 - 2| = |0| = 0$.

Therefore, we find the distance is 8 by counting the intervals and also we get 8 by using the coordinates.

You might be wondering what would happen if the difference of our second coordinates had not been zero. If you have, GOOD!! We will soon get to that problem, however for now we shall restrict our attention to points that are on a line parallel to the X axis. By doing so, we will find that every point on a line parallel to the X axis is the same distance above (or below) the X axis and hence will have the same second coordinate. Since the second coordinates are exactly the same, the difference between any two second coordinates will be zero.

Therefore we find that the distance between any two points on a line parallel to the X axis can be determined by finding the "absolute value of the difference of the first coordinates."

It is interesting to note that we "do things" in the opposite order from which we state them. The procedure is as follows.

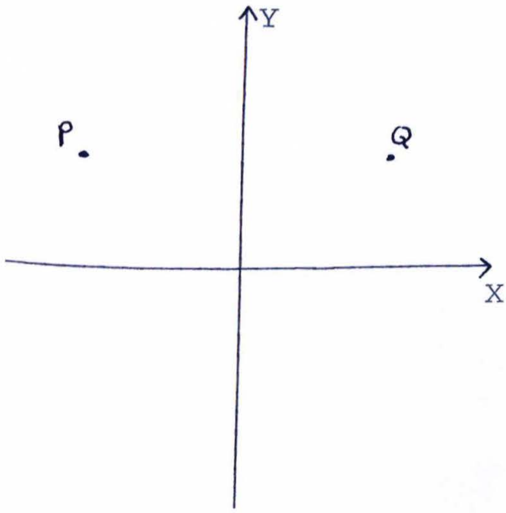
If two points are on a line parallel to the X axis then

1. find the coordinates of the two points.
2. find the difference of the first coordinates of the two points.
3. find the absolute value of the difference found in step 2.
4. name the distance between the two points as the number found in step 3.

Below, there is a general example that will give you a pattern to follow for all problems of this very special type. To use the pattern, you replace the letters p, q, and r with the numbers given in a particular problem.

EXAMPLE

Suppose points P and Q are on a line parallel to the X axis and suppose the coordinates of P are (p,r) and the coordinates of Q are (q,r). (See the sketch below.) Note that the points are on a line parallel to the X axis so the difference of the second coordinates is zero.



Step 1. Coordinates of P are (p,r).
Coordinates of Q are (q,r).

Step 2. The difference of the first coordinates of the two points is (p-q).

Step 3. The absolute value of the difference is $|p - q|$.

Step 4. The distance between P and Q is $|p - q|$.

PRACTICE EXERCISE

Point M has coordinates (4,7) and point N has coordinates (9,7). Name the distance between M and N. (It is helpful to make a sketch. A space has been provided below.)

IF THE POINTS ARE ON A LINE PARALLEL TO THE X AXIS, GO ON.

Step 1. Coordinates of M are ______.
answer (4,7)
Coordinates of N are ______.
answer (9,7)

Step 2. The difference of the first coordinates of the two points is ______.
answer (4 - 9) or (9 - 4)

Step 3. The absolute value of the difference is ______.
answer 5

Step 4. The name of the distance between M and N is ______.
answer 5

ASSESSMENT TASK 6a

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (5,-1) and point B has coordinates (-1,1).

____

P-AT VIb, 1

NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (6,0) and the coordinates of Q are (6,8). Write your answer in the space below.

P-AT VIb, 2

NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (-2,-5), and the coordinates of Q are (-2,3). Write your answer in the space below.

P-AT VIb, 3

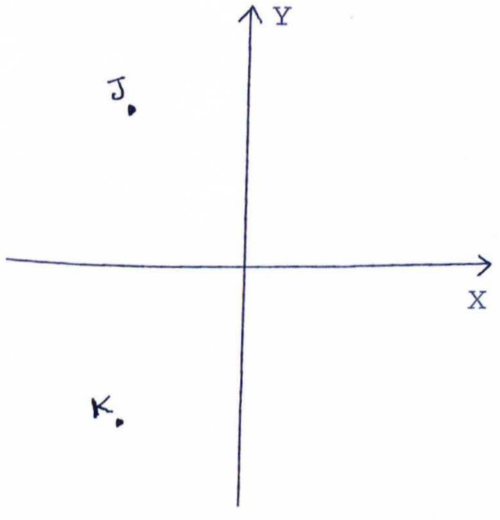
NAME the distance from point P to point Q if P and Q are on a line parallel to the Y axis and the coordinates of P are (7,-2) and the coordinates of Q are (7,4). Write your answer in the space below.

TASK VIb

The information in this task is an extension of previous tasks and is needed before you can progress to other parts of the program.

If we are looking for the distance between two points on a line parallel to the Y axis, we notice that the points on a line parallel to the Y axis all have the same first coordinate since all the points are the same distance to the right (or left) of the Y axis. This suggests that we can find the distance between two points on a line parallel to Y axis by finding the absolute value of the difference between the <u>second</u> coordinates of the two points.

Thus to find the distance between any two points on a line parallel to the Y axis, we can proceed in the following way:



Suppose J and K are two points on a line parallel to the Y axis (see sketch) and suppose the coordinates of J are (t,j) and the coordinates of K are (t,k).

Step 1. The coordinates of J are (t,j). The coordinates of K are (t,k).

Step 2. The difference between the <u>second</u> coordinates is (j - k) or (k - j).

Step 3. The absolute value of the difference found in step 2 is $|j - k| = |k - j|$.

Step 4. The name of the distance between points J and K is $|j - k|$ or, if you prefer, $|k - j|$.

PRACTICE EXERCISE: Point R has coordinates (6,5). Point S has coordinates (6,2). What is the distance between points R and S?

Step 1. The coordinates of R are _____.
answer (6,5)
The coordinates of S are _____.
answer (6,2)

Step 2. The difference between the second coordinates is ________.
answer (3) or (-3)

Step 3. The absolute value of the difference is _______________.

Step 4. The name of the distance between points R and S is ___________.
answer 3

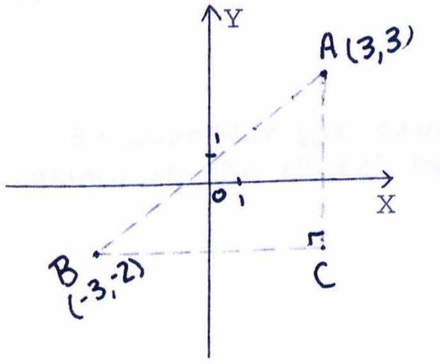
ASSESSMENT TASK 6b

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (2,4) and point B has coordinates (2,-7).

____

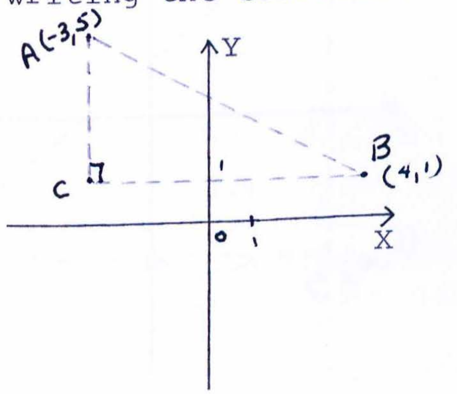
P-AT VII, 1

In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



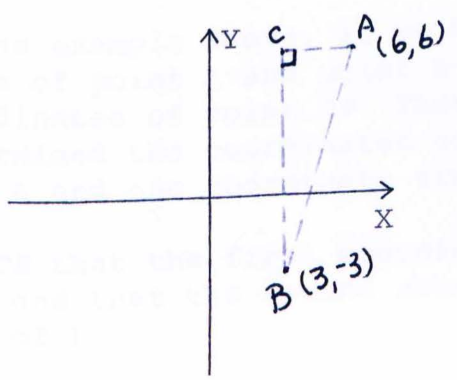
P-AT VII, 2

In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



P-AT VII, 3

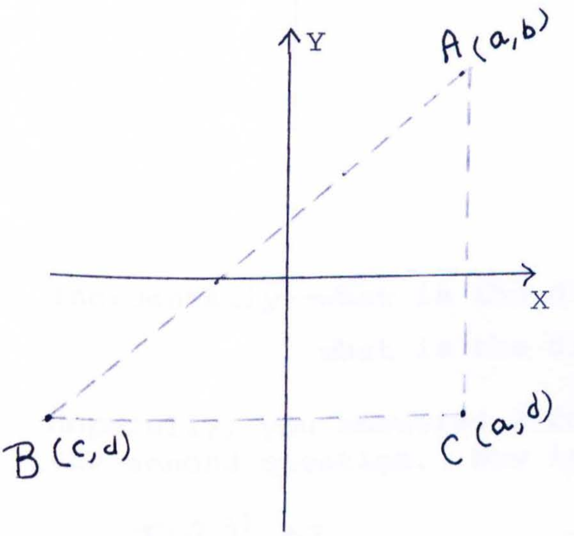
In the sketch below, NAME the coordinates of vertex C by writing the coordinates in the space below.



TASK VII

The information in this task is an extension of previous tasks and is needed to complete the tasks that follow.

Suppose for practice we consider the points A, B, and C shown in the sketch below.



Let us suppose the coordinates of point A are (a,b). Recall the first coordinate tells us how far to the right or left of the Y axis we are and the second coordinate tells us how far above or below the X axis we are.

Further, let us suppose the coordinates of point B are (c,d) and the coordinates of point C are (a,d).

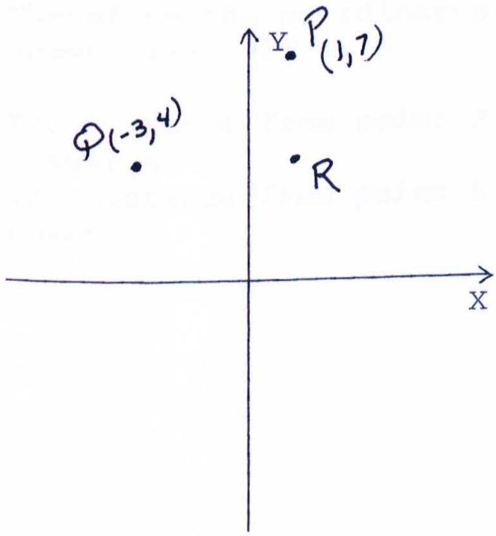
We note that the points B and C are on a line parallel to the X axis and also that the points A and C are on a line parallel to the Y axis.

Since the X axis and the Y axis are perpendicular, we should notice that $\angle BCA$ is a right angle. Hence if we connect the points A and C with segments we have a triangle and we can say it is a right triangle.

In the example above, if we had been given only the coordinates of point A and point B--could we have determined the coordinates of point C? That is, could we have in some way determined the coordinates of point C by using one coordinate from A and one coordinate from B?

NOTICE that the first coordinate of C is the first coordinate of A and that the second coordinate of C is the second coordinate of B.

Let us look at some examples.



In the sketch at the left, the coordinates of point P are (1,7) and the coordinates of point Q are (-3,4).

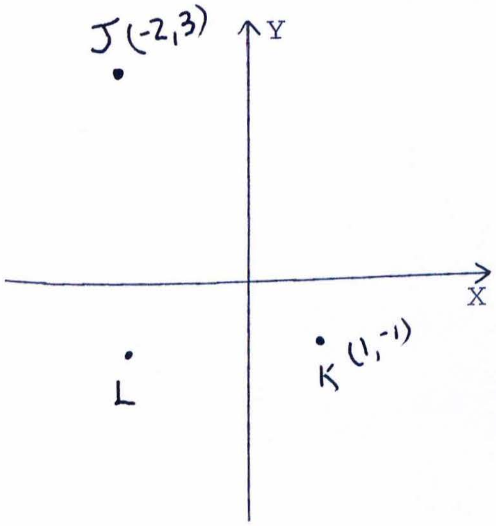
What are the coordinates of point R if we require that $\overline{QR}$ be parallel to the X axis <u>and</u> that $\overline{PR}$ be parallel to the Y axis?

If $\overline{PR}$ is to be parallel to the Y axis, then the first coordinate of R must be the same as the <u>first</u> coordinate of P. If $\overline{QR}$ is to be parallel to the X axis, then the <u>second</u> coordinate of R must be the same as the second coordinate of Q. Therefore, the coordinates of R are (1,4).

Incidentally--what is the distance from P to R? ____________

what is the distance from Q to R? ____________

Hopefully, you answered 3 for the first question and 4 for the second question. Now let us look at a second example.



In the sketch at the left, the coordinates of point J are (-2,3) and the coordinates of point K are (1,-1).

What are the coordinates of point L if we require that $\overline{JL}$ be parallel to the Y axis and $\overline{LK}$ be parallel to the X axis?

(Fill in the blanks in the paragraph below.)

If $\overline{JL}$ is to be parallel to the Y axis, the first coordinate of L must be the same as the ____________ coordinate of J.
If $\overline{LK}$ is to be parallel to the X axis the second coordinate of L must be the same as the ____________ coordinate of K.

Therefore the coordinates of L are (,).
answer (-2, -1)

The distance from point J to point L is __________.
answer 4
The distance from point L to point K is __________.
answer 3

ASSESSMENT TASK 7

NAME the coordinates of a point C given the coordinates of a point A are (4,3) and the coordinates of a point B are (-1,-1) such that AC will be parallel to the X axis and such that BC will be parallel to the Y axis. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 7

Given the coordinates of point A are (,) and the coordinates of point B are (,), NAME the coordinates of a point C such that $\overline{AC}$ will be parallel to the X axis and such that $\overline{BC}$ will be parallel to the Y axis. Write the answer in the space provided.

____

NOTE: The coordinates of A and B were inserted at the time the task was given.

P-AT VIIIa, 1

If $\triangle ABC$ is a right triangle with legs AC = 2 and BC = 3, what is the length of the hypotenuse AB?

P-AT VIIIa, 2

If $\triangle ABC$ is a right triangle with legs AC = 3 and BC = 4, what is the length of the hypotenuse AB?

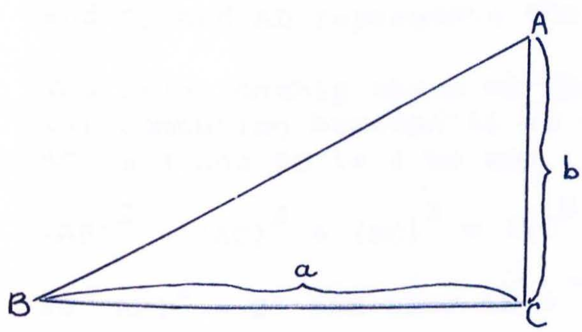
P-AT VIIIa, 3

If $\triangle ABC$ is a right triangle with legs AC = 6 and BC = 9, what is the length of the hypotenuse AB?

TASK VIIIa

The information in this task is an extension of previous tasks and is needed to continue with additional tasks in the program.

In geometry it is proved that the length of the hypotenuse of a right triangle can be computed if the lengths of the two legs are known.



In the sketch at the left, the angle BCA is a right angle so we can call the triangle a right triangle. $\overline{AB}$ is the side opposite the right angle at vertex C and we call $\overline{AB}$ the hypotenuse of the right $\triangle ABC$. AC and BC are called the legs of right $\triangle ABC$.

The statement that is proved in geometry which relates the lengths of the two legs of a right triangle with the length of the hypotenuse is called the Pythagorean theorem. It is stated as follows:

> In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the two legs.

Suppose we apply this theorem to the right triangle above.

The length of leg $\overline{AC}$ is b. The length of leg $\overline{BC}$ is a.

The square of the length of leg AC is b^2. The square of the length of leg BC is a^2.

The sum of the squares of the lengths of the two legs is $b^2 + a^2$.

The square of the length of the hypotenuse is $b^2 + a^2$.

To complete the task of computing the length of the hypotenuse, we need to find the square root of the square of the length of the hypotenuse.

That is, $\sqrt{b^2 + a^2}$. Therefore the length of AB is $\sqrt{b^2 + a^2}$.

We could also have indicated the relationship between the lengths of the legs of the right triangle and the hypotenuse in the following way:

$$(AC)^2 + (BC)^2 = (AB)^2 \quad \text{or} \quad (AB)^2 = (AC)^2 + (BC)^2$$

Recall that AC represents the length of the segment between A and C, BC represents the length of the segment between B and C, and AB represents the length of the hypotenuse.

The relationship shown on the previous page is often used for computing because it is easy to use. For example, if AC is 3 and BC is 4 we see

$(AB)^2 = (AC)^2 + (BC)^2 = (3)^2 + (4)^2 = 9 + 16 = 25.$

So $(AB)^2 = 25$ and then $AB = \sqrt{(AB)^2} = \sqrt{25} = 5.$

Let us look at this in a step by step fashion.

Step 1: Find the lengths of the legs of the right triangle.

$AC = 3$ and $BC = 4$

Step 2: Square the lengths of the legs of the right triangle.

$(AC)^2 = 3^2 = 9$ and $(BC)^2 = 4^2 = 16$

Step 3: Add the squares of the lengths of the legs of the right triangle.

$(AC)^2 + (BC)^2 = 9 + 16 = 25$

Step 4: Find the square root of the number found in step 3.

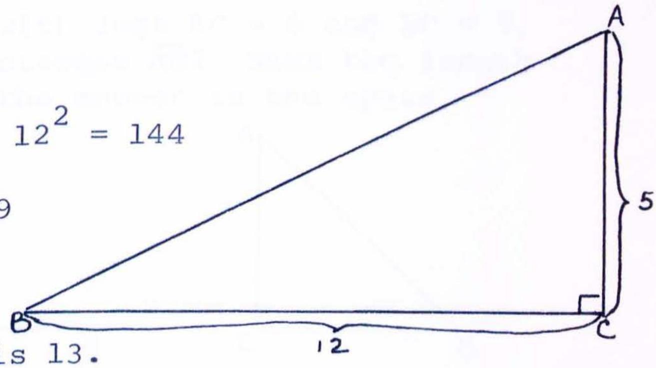
$\sqrt{(AC)^2 + (BC)^2} = \sqrt{25} = 5$

Step 5: State that the length of the hypotenuse is the number found in step 4.

The length of the hypotenuse is 5.

Now let us look at two examples.

Example 1. In the sketch at the right, the lengths of the legs are indicated. What is the length of the hypotenuse?



$AC = 5$ and $BC = 12$

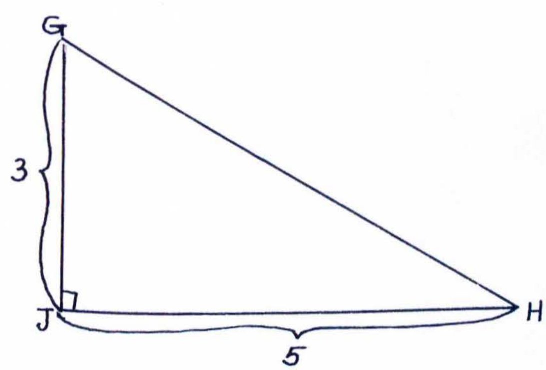
$(AC)^2 = 5^2 = 25$ and $(BC)^2 = 12^2 = 144$

$(AC)^2 + (BC)^2 = 25 + 144 = 169$

$\sqrt{(AC)^2 + (BC)^2} = \sqrt{169} = 13$

The length of the hypotenuse is 13.

Example 2. In the sketch below, the lengths of the legs are indicated. What is the length of the hypotenuse?



$GJ = 3$ and $JH = 5$

$(GJ)^2 = 3^2 = 9$ and

$(JH)^2 = 5^2 = 25$

$(GJ)^2 + (JH)^2 = 9 + 25 = 34$

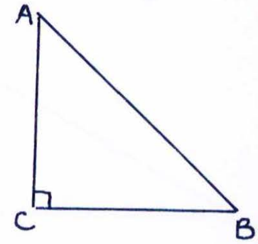
$\sqrt{(GJ)^2 + (JH)^2} = \sqrt{34}$

The length of the hypotenuse is $\sqrt{34}$.

ASSESSMENT TASK 8a

IF $\triangle ABC$ is a right triangle with legs AC = 6 and BC = 5, what is the length of the hypotenuse $\overline{AB}$? NAME the length of the hypotenuse by writing the answer in the space provided.

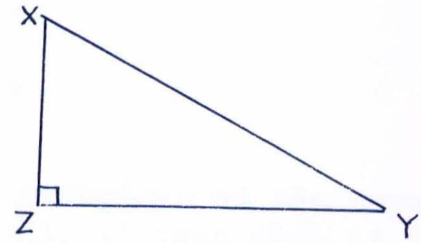
____



ALTERNATE ASSESSMENT TASK 8a

If $\triangle XYZ$ is a right triangle with legs XZ = and YZ = , what is the length of the hypotenuse $\overline{XY}$? NAME the length of the hypotenuse by writing the answer in the space provided.

____



NOTE: The lengths of XZ and YZ were inserted at the time the task was given.

P-AT VIIIb, 1

If the coordinates of the endpoints of the hypotenuse of a right triangle are (2,5) and (-1,1) then what is the length of the hypotenuse?

P-AT VIIIb, 2

If the coordinates of the endpoints of the hypotenuse of a right triangle are (3,2) and (1,-1) then what is the length of the hypotenuse?

P-AT VIIIb, 3

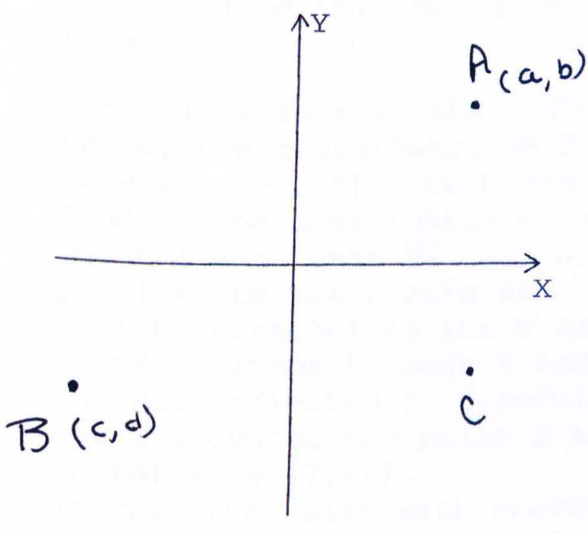
If the coordinates of the endpoints of the hypotenuse of a right triangle are (8,5) and (2,-4) then what is the length of the hypotenuse?

TASK VIIIb

The information in this task is an extension of previous tasks and is needed to continue with additional tasks in the program.

In a previous task we found that if we were given the coordinates of two points in a plane that were not on a line parallel to either the X axis or the Y axis, we could determine the coordinates of a third point that would be the right angle vertex of a right triangle. The two original points would then be the coordinates of the endpoints of the hypotenuse of the right triangle.

In the sketch below, we see that the points A and B have their coordinates given. The coordinates of A are (a,b) and the coordinates of B are (c,d). From this information we can determine the coordinates of a point C such that $\overline{AC}$ will be parallel to the Y axis and $\overline{BC}$ will be parallel to the X axis. When we have found C, complete the sketch by drawing in the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. You will then have formed a right triangle. What are the coordinates of point C? ________________
answer (a,d)



Now that the sketch is completed as a right triangle, we note that A and B are the endpoints of the hypotenuse. We also note that the distance from A to B (or, from B to A) is the length of the hypotenuse. Notice also that the distance from A to C is the length of one leg and the distance from B to C is the length of the other leg of the right triangle.

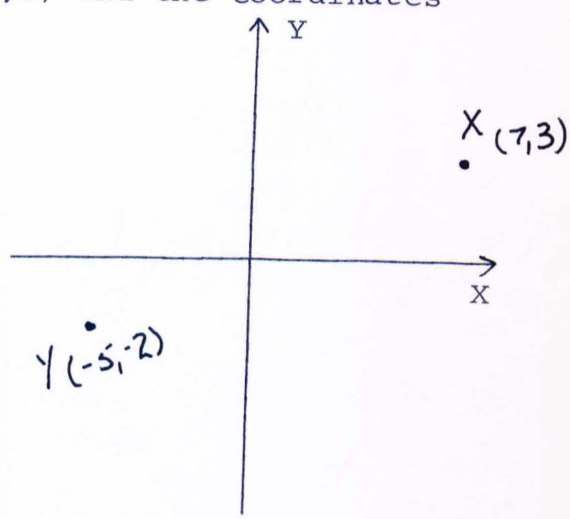
From work we have already completed, we know that we can determine the length of $\overline{AC}$ and we can also determine the length of $\overline{BC}$. The length of $\overline{AC}$ is $|b - d|$ and the length of $\overline{BC}$ is $|c - a|$. (If you do not recall how to find these two lengths, go back to Task VIa and VIb. Also, a review of Task VII may be helpful.)

Since we can find the lengths of the two legs of the right triangle by only knowing the coordinates of endpoints of the hypotenuse, we can use a theorem proved in geometry to find the length of the hypotenuse. The theorem is often stated as

> In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

Therefore, if we know the lengths of 2 sides of a right triangle we can find the length of the third side. In our work we will either know the lengths of the legs of a right triangle or we will be able to determine these lengths without too much work. With this information, we will be able to find the length of the hypotenuse of the right triangle.

Suppose we look at the following example: In the sketch below, the coordinates of X are (7,3) and the coordinates of Y are (-5,-2). We begin by finding the coordinates of a point Z such that $\overline{YZ}$ will be parallel to the X axis and $\overline{XZ}$ will be parallel to the Y axis. (DRAW in lines through X and Y as just indicated.) Hopefully, you discovered the point Z has coordinates (7,-2). (NOTE: The point with coordinates (-5,3) will also satisfy the desired condition.) Next we find the distance between Y and Z and the distance between X and Z. These two distances will be the lengths of the legs of a right triangle with the right angle vertex at Z.



$\overline{YZ} = |(-5) - 7| = |-12| = 12$ and $\overline{XZ} = |3 - (-2)| = |5| = 5$.

Then by the theorem we stated,

$(XY)^2 = (YZ)^2 + (XZ)^2 = (12)^2 + (5)^2 = 144 + 25 = 169.$

and finally, the length of the hypotenuse is

$XY = \sqrt{(XY)^2} = \sqrt{169} = 13.$

Notice that we could have computed the length of $\overline{XY}$ by following the steps below.

$$XY = \sqrt{((-5) - 7)^2 + (3 - (-2))^2} = \sqrt{(-12)^2 + (5)^2}$$

$$= \sqrt{144 + 25} = \sqrt{169} = 13.$$

2nd coordinate of Y

2nd coordinate of X

1st coordinate of X

1st coordinate of Y

ASSESSMENT TASK 8b

If the coordinates of the endpoints of the hypotenuse are (3,7) and (-2,0) then NAME the length of the hypotenuse. Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 8b

If the coordinates of the endpoints of the hypotenuse are (,) and (,) then NAME the length of the hypotenuse. Write the answer in the space provided.

____

NOTE: Coordinates were inserted in the blank ordered pairs above at the time the task was given.

P-AT IX, 1

Name the distance between the points A and B given that the coordinates of A are (0,3) and the coordinates of B are (4,0).

P-AT IX, 2

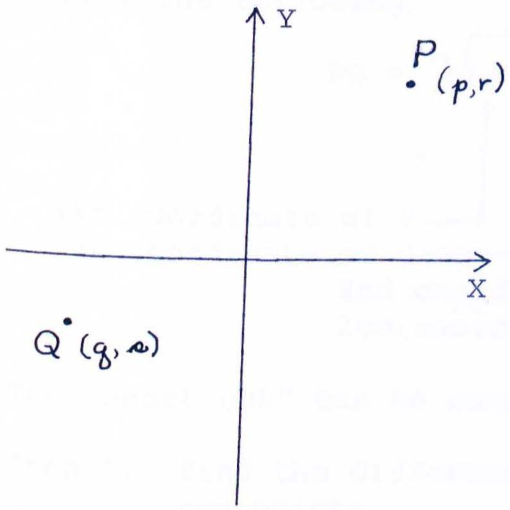
Name the distance between the points A and B given that the coordinates of A are (7,1) and the coordinates of B are (2,13).

P-AT IX, 3

Name the distance between the points A and B given that the coordinates of A are (6,5) and the coordinates of B are (1,1).

TASK IX

The information in this task is an extension of previous tasks and is needed to continue with additional tasks in the program.



Suppose we are given the sketch at the left with the point P having coordinates (p,r) and the point Q having coordinates (q,s). What are the coordinates of a point R such that $\overline{PR}$ is perpendicular to $\overline{QR}$? That is, find the coordinates of R such that $\overline{QR}$ is parallel to the X axis and $\overline{PR}$ is parallel to the Y axis.

If you answered (p,s) GOOD. If not, go back and review task 7.

What is the distance between points P and R? ______________
answer $|r - s|$ or $|s - r|$

What is the distance between points Q and R? ______________
answer $|p - q|$ or $|q - p|$

If either of these two questions seemed difficult, go back and review task 6a and 6b.

The <u>length</u> of a segment joining two points in a plane is the <u>distance</u> between the two points. Hence, when we find the length of a segment we have found the distance between the endpoints of the segment. Thus (as we have noted before) if the length of segment PR is $|r - s|$ then we also say the distance from P to R is $|r - s|$. Also, we can note that the distance from R to Q is $|p - q|$ and that the length of RQ is $|p - q|$.

To find the length of PQ, we apply the pythagorean theorem.

$$(PQ)^2 = (QR)^2 + (RP)^2 = (p - q)^2 + (r - s)^2$$

REMEMBER when we square a number, we may "drop" the absolute value signs. Now to find PQ we "take the square root" of each side of our equation and get

$$PQ = \sqrt{(p - q)^2 + (r - s)^2}$$

NOTICE in the above work, we found the distance between points P and Q by using <u>only</u> the coordinates of the points P and Q. This suggests we can find a "short cut" for determining the distance between <u>any</u> two points in a plane if we are given the coordinates of the points.

NOTICE the following

$$PQ = \sqrt{(p - q)^2 + (r - s)^2}$$

1st coordinate of P
1st coordinate of Q
2nd coordinate of P
2nd coordinate of Q

The "short cut" can be completed by the following steps:

Step 1. find the difference of the first coordinates of the two points.

Step 2. find the difference of the second coordinates of the two points.

Step 3. square the difference found in step 1.

Step 4. square the difference found in step 2.

Step 5. add the squares found in steps 3 and 4.

Step 6. find the square root of the sum found in step 5.

Step 7. write the answer.

Now that we have a "method" for determining the distance between two points in a plane, suppose we search for a general equation that will name the distance between any two points in a plane. Let us begin by using the steps outlined above.

Let M and N be any two points in a plane and let the coordinates of M be (x_1, y_1) and the coordinates of N be (x_2, y_2). These coordinates are read "x sub 1, y sub 1 and x sub 2, y sub 2."

Step 1. $x_1 - x_2$

Step 2. $y_1 - y_2$

Step 3. $(x_1 - x_2)^2$

Step 4. $(y_1 - y_2)^2$

Step 5. $(x_1 - x_2)^2 + (y_1 - y_2)^2$

Step 6. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Step 7. The distance from M to N is $\sqrt{(x_1 = x_2)^2 + (y_1 - y_2)^2}$

PRACTICE EXERCISES: Using the outline above find the distances between the given pairs of points.

1. The coordinates of point A are (0,2) and the coordinates of point B are (4,5)

step 1. $0 - 4$

step 2. $2 - 5$

step 3. $(0 - 4)^2 - 16$

step 4. $(2 - 5)^2 - (-3)^2 = 9$

step 5. $16 + 9 = 25$

step 6. $\sqrt{25} = 5$

step 7. The distance from A to B is 5.

2. The coordinates of point C are (-1,-6) and the coordinates of point D are (2,5).

step 1. ______________

step 2. ______________

step 3. ______________________

step 4. ______________________

step 5. ______________________

step 6. __________________________________

step 7. The distance from point C to point D is $\sqrt{130}$

3. The coordinates of point P are (X,Y) and the coordinates of point C are (2,3).

step 1. ____________

step 2. ____________

step 3. ______________________

step 4. ______________________

step 5. ______________________________

step 6. ____

step 7. The distance from point P to point C is $\sqrt{(X - 2)^2 + (Y - 3)^2}$

4. The coordinates of point P are (X,Y) and the coordinates of point C are (-3,5).

step 1. ____________

step 2. ____________

step 3. ______________________

step 4. ______________________

step 5. ______________________________

step 6. ____

step 7. The distance from point P to point C is $\sqrt{(X + 3)^2 + (Y - 5)^2}$

ASSESSMENT TASK 9

NAME the distance between points J and K given that the coordinates of J are (4,7) and the coordinates of K are (-9,5). Write the answer in the space provided.

____

ALTERNATE ASSESSMENT TASK 9

NAME the distance between points M and N given that the coordinates of M are (,) and the coordinates of N are (,). Write the answer in the space provided.

____

NOTE: Coordinates for points M and N were inserted at the time the task was given.

P-AT X, 1

If the center of a circle is at (1,2) and the radius is 3, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X - 1)^2 + (Y - 2)^2 = 3$ b) $(X + 1)^2 + (Y + 2)^2 = 3^2$

c) $(X - 1)^2 + (Y - 2)^2 = 3^2$ d) $(X - 1) + (Y - 2) = 3^2$

P-AT X, 2

If the center of a circle is at (-3,1) and the radius is 5, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X - 3)^2 + (Y + 1)^2 = 5^2$ b) $(X + 3)^2 + (Y - 1)^2 = 5^2$

c) $(X + 3) + (Y - 1) = 5^2$ d) $(X + 3)^2 + (Y - 1)^2 = 5$

P-AT X, 3

If the center of a circle is at (4,-2) and the radius is 2, identify the equation of the circle in the list below by placing a check mark to the left of the correct response.

a) $(X + 4)^2 + (Y - 2)^2 = 2^2$ b) $(X - 4)^2 + (Y + 2)^2 = 2^2$

c) $(X - 4) + (Y + 2) = 2^2$ d) $(X - 4)^2 + (Y + 2)^2 = 2$

TASK X

The information presented in this task is an extension of previous tasks and is needed for future tasks in the program.

A circle is often defined as a set of points in a plane equally distant from a given point. The given point is called the center of the circle. The distance from the center to any point on the circle is called the radius of the circle.

It is possible to completely describe a circle by writing an equation. The equation that completely describes a circle usually includes the coordinates of the center of the circle and the length of the radius. For example, if the center of a circle is at (7,3) and the radius of the circle is 5, then an equation of the circle is

$$(X - 7)^2 + (Y - 3)^2 = (5)^2$$

NOTE:
1st coordinate of center
2nd coordinate of center
radius

Suppose we look at another example. If (h,k) are the coordinates of the center of a circle and the radius of the circle is r then an equation of the circle is

$$(X - h)^2 + (Y - k)^2 = (r)^2$$

NOTE:
1st coordinate of center
2nd coordinate of center
radius

NOTICE --- in each of the two examples above, we wrote X minus the 1st coordinate of the center and Y minus the 2nd coordinate of the center. Then we wrote $(X - h)^2 + (Y - k)^2$ and finally we completed our equation by writing an "equal" sign and then the square of the radius.

STUDY the following examples. In each example, you are given the coordinates of the center and the radius. You are to identify the equation in the list that is completely determined by the given information.

Example 1. center (9,2) radius = 4

a) $(X - 9) + (Y - 2) = (4)^2$ b) $(X + 9) + (Y + 2) = (4)^2$

c) $(X^2 - 9) + (Y^2 - 2) = (4)^2$ d) $(X - 9)^2 + (Y - 2)^2 = (4)^2$

e) $(X - 9)^2 + (Y - 2)^2 = (4)$

Before studying the next example, note that $X - (-2) = X + 2$.

Example 2. center (-2,1) radius = 3

a) $(X + 2) + (Y - 1) = (3)^2$ b) $(X + 2)^2 + (Y - 1)^2 = (3)^2$

c) $(X - 2) + (Y + 1) = (3)^2$ d) $(X^2 + 2) + (Y^2 - 1) = (3)^2$

e) $(X + 2)^2 + (Y - 1)^2 = (3)$

Answer __________.

Example 3. center (-3,0) radius = 5

a) $(X + 3) + (Y - 0) = (5)^2$ b) $(X - 3)^2 + (Y + 0)^2 = (5)^2$

c) $(X^2 + 3) + (Y^2 - 0) = (5)^2$ d) $(X + 3)^2 + (Y - 0)^2 = (5)$

e) $(X + 3)^2 + (Y - 0)^2 = (5)^2$

Answer __________.

Example 4. center (0,11) radius = 6

a) $(X - 0)^2 + (Y - 11)^2 = (6)^2$ b) $(X^2 - 0) + (Y^2 - 11) = (6)^2$

c) $(X + 0)^2 + (Y + 11)^2 = (6)^2$ d) $(X = 0)^2 + (Y - 11)^2 = (6)^2$

e) $(X + 0) + (Y + 11) = (6)^2$

Answer __________.

Answers to examples:

1) d 2) b 3) e 4) a

ASSESSMENT TASK 10

IDENTIFY the equation of a circle in the list below given that the center of the circle is at (3,5) and the radius is 4. Place a check mark at the left of the correct response.

a) $(X - 3) + (Y - 5) = (4)^2$

b) $(X - 3)^2 + (Y - 5)^2 = (4)^2$

c) $(X + 3)^2 + (Y + 5)^2 = (4)^2$

d) $(X - 3)^2 + (Y - 5)^2 = (4)$

ALTERNATE ASSESSMENT TASK 10

IDENTIFY the equation of a circle in the list below given that the center of the circle is at (-1,13) and that the radius is 9. Place a check mark at the left of the correct response.

a) $(X - 1)^2 + (Y + 13)^2 = (9)^2$

b) $(X + 1)^2 + (Y - 13)^2 = (9)$

c) $(X + 1) + (Y - 13) = (9)^2$

d) $(X + 1)^2 + (Y - 13)^2 = (9)^2$

P-AT XI, 1

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

P-AT XI, 2

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

P-AT XI, 3

Write the equation of a circle in standard form given that the coordinates of the center are (,) and that the radius of the circle is .

NOTE: The coordinates of the center and the measure of the radius were given at the time the task was given.

TASK XI

The information in this task is an extension of previous tasks and is needed to continue and to complete this final task in the program. When you have finished this task, you have completed the entire program of instruction for this sequence of lessons.

In this task we wish to determine the equation of a circle. Hence we shall begin by first reminding ourselves of the definition of a circle.

Definition: A circle is a set of points in a plane equally distant from a given point. The given point is called the center of the circle. The distance from the center to any point on the circle is called the radius of the circle.

Let us begin by deriving or finding the equation of a circle. Suppose the center of the circle is a point 0 with coordinates (a,b) and suppose the radius of the circle is a positive number r. Let P be any point on the circle and let P have coordinates (X,Y).

We will find that we can proceed very much like we did in a previous task. We can do this because our problem is very much like a previous task in which we found the distance between two points. In this example, the points are 0 and P. The distinction (or--difference) between our present task and a previous task is that now we are given the distance between 0 and P. That given distance is r, the radius of our circle.

The following step by step outline was used in a previous task.

step 1. $X - a$

step 2. $Y - b$

step 3. $(X - a)^2$

step 4. $(Y - b)^2$

step 5. $(X - a)^2 + (Y - b)^2$

step 6. $\sqrt{(X - a)^2 + (Y - b)^2}$

step 7. $OP = r = \sqrt{(X - a)^2 + (Y - b)^2}$

To complete the derivation of an equation of a circle, we need to remember what an equation of a circle looks like. Recall--an equation of a circle (in what is referred to as "standard form") looks like the following equation.

$$r^2 = (X - a)^2 + (Y - b)^2$$

The radius of the circle is r, the first coordinate of the center is a and the second coordinate of the center is b. Thus to make our equation in step 7 look like the equation of a circle, we see that <u>squaring each side of our equation</u> in step 7 given us the desired results.

Squaring the left side we get r^2 and squaring the right side we "remove" the square root symbol and get the following equation:

$$r^2 = (X - a)^2 + (Y - b)^2$$

We now see that the equation of a circle is really easier to find than the task of finding the distance between two points, since we do not have to take the square roots. For example, if we wish to determine the equation of a circle with radius 2 and center at (1,3) we can write the equation immediately by- - -

replacing r with 2,
replacing a with 1,
replacing b with 3.

Thus, the equation of our circle is $2^2 = (X - 1)^2 + (Y - 3)^2$.

PRACTICE EXERCISES

1. Write the equation of a circle in standard form given the coordinates of the center are (-2,4) and that the radius is 7.

   In our general equation above, we replace

   a with ____________________
   answer -2
   b with ____________________
   answer 4
   r with ____________________
   answer 7

The equation of the circle is ________________________________

answer $7^2 = (X - (-2))^2 + (Y - 4)^2$

or $7^2 = (X + 2)^2 + (Y - 4)^2$

2. Write the equation of a circle in standard form given the coordinates of the center are (0,-12) and that the radius is 13.

The equation of the circle is ______________________________

TASK XI

If the center of a circle has coordinates (,) and the radius of the circle is then what is an equation of the circle in standard form? Write your answer in the space provided.

____

NOTE: The coordinates of the center and the measure of the radius were given at the time the task was presented.

APPENDIX B

FORTRAN V PROGRAM USED TO COMPUTE

EXPERIMENTAL MEASURES AND

THE EXPERIMENTAL MEASURES

FORTRAN V PROGRAM USED TO COMPUTE EXPERIMENTAL MEASURES

```
      DIMENSION SCT(15,60,2),ISUM(100,2),CCELL(15,2),SUM(2),DA(30,2),
     ADL(60,2),RL(60,2),NFG(60,2),NDAY(60,2),DFG(60,2),RFG(60,2) ,IGP(2)
      INTEGER SCT,CELL
100   FORMAT(312)
101   FORMAT(3X,1512,5X,212)
102   FORMAT(13,2X,15(3X,13),7X,15,3X,2(2X,13))
103   FORMAT('  THE MEAN TIME USED BY GROUP',13,' FOR CELL NUMBER',13,'
     2 IS ',F10.6/)
104   FORMAT(' THE DIFFICULTY OF ACQUISITION OF CELL NUMBER',13,' FOR GR
     3OUP NUMBER ',13,' IS ',5X,F6.5/)
105   FORMAT(5X,' STUDENT ',5X,' GROUP* ',6X,' DEGREE OF ',6X,' RATE OF*
     1 ',6X,' INDEX OF** ',6X,' RATE OF** ',5X,'INDEX OF*',5X,'INDEX OF*
     5**')
106   FORMAT(5X,' NUMBER* ',5X,' NUMBER ',6X,' LEARNING* ',6X,' LEARNING
     2 ',6X,' FORGETTING ',6X,' FORGETTING ' ,4X,'RETENTION',5X,'EFFICIE
     7NCY'/)
107   FORMAT(8X,12,10X,12,12X,F6.0,11X,F6.2,11X,F5.0,13X,F5.2,10X,5F.0,
     69X,F5.2/)
109   FORMAT(' DL ',12,12,F5.0)
110   FORMAT(' RL ',12,12,F5.2)
111   FORMAT(' DFG',12,12,F5.0)
112   FORMAT(' RFG',12,12,F5.2)
113   FORMAT(' RET',12,12,F5.0)
114   FORMAT(' EFF',12,12,F5.2)
57    FORMAT(1H1)
      READ(5,100) (IGP(K),K=1,2),CELL
      WRITE(6,57)
      DO 300 K = 1,2
```

```
      IMAX = IGP(K)
      DO 295 J = 1, IMAX
      ISUM(J,K) = 0
      READ(5,101)(SCT(I,J,K),I=1,CELL),NFG(J,K),NDAY(J,K)
      DO 290  I = 1, CELL
      ISUM(J,K) = ISUM(J,K) + SCT(I,J,K)
290   CONTINUE
      WRITE(6,102)J,(SCT(I,J,K),I=1,CELL),ISUM(J,K),NFG(J,K),NDAY(J,K)
295   CONTINUE
300   CONTINUE
      WRITE(6,57)
      DO 600 K = 1, 2
      IMAX = IGP(K)
      DO 595 I = 1, CELL
      CCELL(I,K) = 0
      DO 590 J = 1, IMAX
      CCELL(I,K) = CCELL(I,K) + SCT(I,J,K)
590   CONTINUE
      CCELL(I,K) = CCELL(I,K)/IGP(K)
      WRITE(6,103) K,I,CCELL(I,K)
595   CONTINUE
      WRITE(6,57)
600   CONTINUE
      WRITE(6,57)
      DO 700 K = 1, 2
      SUM(K) = 0
      DO 690 I = 1, CELL
      SUM(K) = SUM(K) + CCELL(I,K)
```

```
690   CONTINUE
700   CONTINUE
      WRITE(6,57)
      DO 830 K = 1, 2
      DO 820 I = 1, CELL
      DA(I,K) = CCELL(I,K)/SUM(K)
      WRITE(6,104) I,K,DA(I,K)
820   CONTINUE
      WRITE(6,57)
830   CONTINUE
      DO 900 K = 1, 2
      WRITE(6,57)
      WRITE(6,105)
      WRITE(6,106)
      IMAX = IGP(K)
      DO 895 J = 1, IMAX
      DL(J,K) = 0
      DFG(J,K) = 0
      DO 890 I = 1, CELL
      IF (SCT(I,J,K).EQ.0) GO TO 890
      DL(J,K) = DL(J,K) + DA(I,K)
890   CONTINUE
      N = CELL - NFG(J,K) + 1
      DO 892 I=CELL,N,-1
      IF (NFG(J,K).EQ.0) GO TO 892
      DFG(J,K) = DFG(J,K) + DA(I,K)
```

```
892    CONTINUE
       DFG(J,K) = 100*DFG(J,K)
       RFG(J,K) = DFG(J,K)/NDAY(J,K)
       DL(J,K) = 100*DL(J,K)
       RL(J,K) = DL(J,K)/ISUM(J,K)
       RET = DL(J,K) - DFG(J,K)
       EFF = RET/DL(J,K)
       WRITE(6,107) J,K,DL(J,K),RL(J,K),DFG(J,K),RFG(J,K),RET,EFF
       WRITE(1,109) J,K,DL(J,K)
       WRITE(1,110) J,K,RL(J,K)
       WRITE(1,111) J,K,DFG(J,K)
       WRITE(1,112) J,K,RFG(J,K)
       WRITE(1,113) J,K,RET
       WRITE(1,114) J,K,EFF
 895   CONTINUE
 900   CONTINUE
       END
```

Table 10

The Experimental Measure
Index of Learning

Group Number 1		Group Number 2	
Subject	Index of Learning	Subject	Index of Learning
1	75	1	42
2	77	2	58
3	70	3	65
4	37	4	81
5	65	5	85
6	85	6	37
7	63	7	33
8	64	8	67
9	66	9	62
10	61	10	74
11	64	11	80
12	49	12	59
13	49	13	44
14	71	14	76
15	65	15	63
16	81	16	50
17	55	17	86
18	60	18	49
19	75	19	57
20	57	20	62
21	90	21	79
22	54	22	80
23	37	23	85
24	51	24	58
25	38	25	67
26	70		
27	98		

Table 11

The Experimental Measure
Rate of Learning

Group Number 1		Group Number 2	
Subject	Rate of Learning	Subject	Rate of Learning
1	2.02	1	0.73
2	1.51	2	0.74
3	1.18	3	0.95
4	0.69	4	1.73
5	1.45	5	1.42
6	1.18	6	0.68
7	1.54	7	0.92
8	1.06	8	1.11
9	1.60	9	2.21
10	1.34	10	1.02
11	1.57	11	2.42
12	1.59	12	1.00
13	1.35	13	2.45
14	2.44	14	2.25
15	1.23	15	1.46
16	1.77	16	1.46
17	0.90	17	1.31
18	0.92	18	0.95
19	1.57	19	1.51
20	1.04	20	3.11
21	1.01	21	2.03
22	1.81	22	1.06
23	1.24	23	0.80
24	0.94	24	0.73
25	0.71	25	0.98
26	1.27		
27	0.66		

Table 12

The Experimental Measure
Index of Forgetting

Group Number 1		Group Number 2	
Subject	Index of Forgetting	Subject	Index of Forgetting
1	5	1	3
2	61	2	3
3	82	3	3
4	5	4	3
5	5	5	3
6	5	6	3
7	5	7	3
8	61	8	80
9	0	9	69
10	61	10	57
11	23	11	0
12	9	12	3
13	61	13	0
14	5	14	3
15	5	15	69
16	5	16	29
17	5	17	3
18	5	18	3
19	5	19	85
20	5	20	0
21	61	21	0
22	51	22	3
23	23	23	0
24	5	24	0
25	5	25	3
26	51		
27	5		

Table 13

The Experimental Measure
Rate of Forgetting

Group Number 1		Group Number 2	
Subject	Rate of Forgetting	Subject	Rate of Forgetting
1	0.69	1	0.18
2	8.77	2	0.34
3	5.87	3	0.18
4	0.25	4	0.18
5	0.69	5	0.48
6	0.25	6	0.24
7	0.24	7	0.24
8	4.39	8	5.75
9	0.00	9	3.85
10	8.77	10	4.04
11	1.63	11	0.00
12	0.67	12	0.17
13	4.39	13	0.00
14	0.24	14	0.17
15	0.25	15	9.91
16	0.25	16	2.07
17	0.25	17	0.17
18	0.69	18	0.17
19	0.35	19	4.26
20	0.35	20	0.00
21	4.09	21	0.00
22	3.18	22	0.23
23	1.52	23	0.00
24	0.32	24	0.00
25	0.32	25	0.21
26	3.39		
27	0.69		

Table 14

The Experimental Measure
Index of Retention

Group Number 1		Group Number 2	
Subject	Index of Retention	Subject	Index of Retention
1	70	1	39
2	16	2	54
3	-12	3	61
4	32	4	78
5	61	5	82
6	80	6	34
7	58	7	30
8	2	8	-13
9	66	9	- 7
10	- 1	10	18
11	42	11	80
12	40	12	56
13	-13	13	44
14	66	14	73
15	60	15	- 6
16	77	16	21
17	50	17	83
18	55	18	46
19	70	19	-28
20	52	20	62
21	28	21	79
22	4	22	76
23	14	23	85
24	46	24	58
25	33	25	64
26	19		
27	93		

Table 15

The Experimental Measure
Index of Efficiency

Group Number 1		Group Number 2	
Subject	Index of Efficiency	Subject	Index of Efficiency
1	0.94	1	0.92
2	0.20	2	0.94
3	-0.18	3	0.95
4	0.87	4	0.96
5	0.93	5	0.96
6	0.94	6	0.91
7	0.92	7	0.90
8	0.04	8	-0.19
9	1.00	9	-0.12
10	-0.02	10	0.24
11	0.65	11	1.00
12	0.81	12	0.94
13	-0.27	13	1.00
14	0.93	14	0.96
15	0.93	15	-0.10
16	0.94	16	0.41
17	0.91	17	0.96
18	0.92	18	0.93
19	0.94	19	-0.49
20	0.92	20	1.00
21	0.31	21	1.00
22	0.07	22	0.96
23	0.39	23	1.00
24	0.90	24	1.00
25	0.87	25	0.95
26	0.27		
27	0.95		

APPENDIX C

RETENTION TESTS

NAME

DATE

The task below is being given to determine how well you recall the information you learned while proceeding through the individualized learning materials. There is no penalty for not being able to complete the task below. However, please attempt to correctly complete the task. After you have completed the task to the best of your ability, ask your teacher to check it. If you are correct, you are finished. If you are not correct, your teacher will give you another task.

If you complete the second task correctly, you are then finished. If you do not, your teacher will give you a third task and so on. There is no penalty for receiving more than one task.

Task 1.

If the center of a circle has coordinates (,) and the radius of the circle is then what is an equation of the circle in standard form? Write your answer in the space provided.

NOTE: The coordinates of the center and the measure of the radius were given at the time the test was presented.

NAME DATE

Task 2

IDENTIFY the equation of a circle in the list below given that the center of the circle is at (4,-3) and that the radius is 5. Place a check to the left of the correct response.

a) $(X + 4)^2 + (Y - 3)^2 = (5)^2$

b) $(X + 4)^2 + (Y - 3)^2 = (5)$

c) $(X + 4) + (Y - 3) = (5)^2$

d) $(X - 4)^2 + (Y + 3)^2 = (5)^2$

NAME DATE

Task 3

NAME the distance between points J and K given that the coordinates of J are (,) and the coordinates of K are (,). Write the answer in the space provided.

NOTE: The coordinates of J and K were inserted at the time the test was given.

NAME

DATE

Task 4

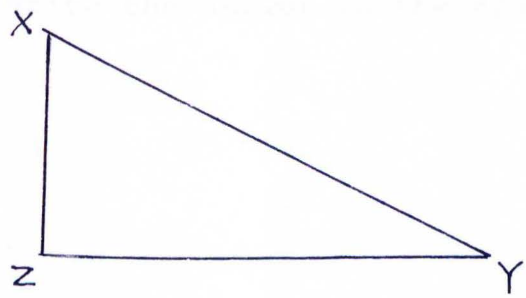
If the coordinates of the endpoints of the hypotenuse of a right triangle are (,) and (,) then NAME the length of the hypotenuse. Write the answer in the space provided.

NOTE: The coordinates of the two points were inserted at the time the test was given.

NAME DATE

Task 5

If $\triangle XYZ$ is a right triangle with legs XZ = and YZ = then what is the length of the hypotenuse XY? NAME the length of the hypotenuse by writing the answer in the space provided.



NOTE: The lengths of XZ and YZ were inserted at the time the test was given.

NAME DATE

Task 6

NAME the coordinates of a point C given the coordinates of a point A are (,) and the coordinates of a point B are (,) such that AC will be parallel to the X axis and such that BC will be parallel to the Y axis. Write the answer in the space provided.

NOTE: The coordinates of A and B were inserted at the time the test was given.

NAME DATE

Task 7

NAME the distance between the following two points. Write the answer in the space provided.

Point A has coordinates (5 ,) and point B has coordinates (5 ,).

NOTE: The second coordinate for A and for B was inserted at the time the test was given.

NAME DATE

Task 8

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (,-4) and point B has coordinates (,-4).

NOTE: The first coordinate for the point A and for the point B was inserted at the time the test was given.

NAME DATE

Task 9

NAME the distance between the following two points. Write the answer in the space provided.

Point A has coordinates (0,) and point B has coordinates (0,).

NOTE: The second coordinate for A and for B was inserted at the time the test was given.

NAME DATE

Task 10

NAME the distance between the following two points. Write the answer in the space provided.
Point A has coordinates (,0) and point B has coordinates (,0).

NOTE: The first coordinate for A and for B was inserted at the time the test was given.

NAME

DATE

Task 11

APPLY A RULE for naming the coordinates of a point in a plane that is units to the right of the Y axis and units below the X axis. Write the answer in the space provided.

NOTE: Numbers were inserted in the spaces provided above at the time the test was given.

APPENDIX D

RAW DATA--LEARNING

AND FORGETTING

Table 16

Learning Data--Treatment Group One

Subject Number	Number of Minutes for Cell Number 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total Time
1	0	0	0	3	0	4	0	0	0	5	6	10	9	0	0	37
2	0	1	1	0	5	4	0	0	0	8	10	19	0	0	3	51
3	0	1	0	0	0	3	0	2	0	23	7	17	0	0	6	59
4	7	4	0	0	0	2	0	0	0	0	4	0	31	0	5	53
5	0	0	4	0	0	7	0	7	0	0	5	8	11	0	3	45
6	0	0	0	0	0	3	5	0	0	13	12	13	19	6	1	72
7	0	0	2	0	0	0	0	0	0	18	8	4	0	0	9	41
8	0	0	8	0	0	0	5	0	0	10	16	11	0	10	0	60
9	0	0	0	0	0	4	0	0	0	14	5	10	0	0	8	41
10	0	0	0	0	0	0	0	2	0	15	0	11	17	0	0	45
11	0	0	3	0	0	0	0	5	0	11	10	8	0	0	4	41
12	0	3	0	0	0	0	0	5	0	10	0	13	0	0	0	31
13	0	2	0	0	0	0	0	0	0	6	10	0	13	5	0	36
14	0	2	0	0	3	3	0	0	0	0	7	3	8	3	0	29
15	0	0	2	0	5	0	0	0	0	7	9	30	0	0	0	53
16	0	3	0	0	0	3	0	0	0	8	8	6	11	7	0	46
17	0	3	0	0	4	0	0	0	0	10	0	44	0	0	0	61
18	0	3	0	0	6	3	0	0	0	10	0	43	0	0	0	65
19	0	0	0	0	10	0	0	0	0	5	0	22	4	5	2	48
20	0	0	0	0	6	0	0	0	0	15	0	25	0	0	9	55
21	0	0	3	0	7	5	0	0	10	14	11	18	14	7	0	89

Table 16 (continued)

Subject Number	Number of Minutes for Cell Number 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total Time
22	0	0	2	0	3	0	0	0	0	11	0	14	0	0	0	30
23	0	0	0	0	0	11	0	0	0	0	0	13	0	6	0	30
24	0	2	0	0	18	0	0	0	0	0	0	28	6	0	0	54
25	0	2	0	0	10	0	0	0	0	0	7	0	31	3	0	53
26	0	0	3	0	6	0	0	0	0	7	10	21	0	0	8	55
27	0	9	2	0	14	17	4	0	15	30	6	9	19	12	11	148

Table 17

Learning Data--Treatment Group Two

Subject Number	Number of Minutes for Cell Number															Total Time
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	0	4	0	3	6	0	0	0	0	0	15	0	30	0	0	58
2	0	0	2	0	15	0	0	0	0	7	13	0	26	15	0	78
3	0	3	0	2	6	0	0	0	0	5	14	28	0	10	0	68
4	0	3	0	0	4	0	6	0	0	7	4	19	4	0	0	47
5	0	0	0	6	6	0	8	0	0	7	4	6	18	5	0	60
6	0	1	0	0	0	0	0	0	0	0	0	40	0	7	7	55
7	0	0	0	0	0	0	5	0	0	7	0	0	24	0	0	36
8	0	3	0	0	11	0	0	0	0	6	14	22	0	4	1	61
9	0	0	0	0	0	8	0	0	0	8	0	10	2	0	0	28
10	0	0	2	0	0	0	0	0	0	6	14	33	18	0	0	73
11	0	2	0	0	3	0	0	0	0	2	1	16	9	0	0	33
12	0	0	0	0	0	6	0	21	3	7	9	13	0	0	0	59
13	0	0	2	0	0	0	0	0	0	5	0	3	0	0	8	18
14	0	1	0	0	0	7	0	0	0	11	7	2	6	0	0	34
15	0	0	0	0	0	0	0	21	0	6	0	3	13	0	0	43
16	0	0	7	0	0	0	0	6	0	0	5	9	0	0	7	34
17	0	0	2	0	2	5	0	0	0	13	10	13	10	0	11	66
18	0	0	2	0	0	4	0	0	0	6	15	0	25	0	0	52
19	0	0	2	0	0	5	0	0	11	2	2	16	0	0	0	38
20	0	0	1	0	1	1	0	0	0	10	3	4	0	0	0	20
21	0	0	0	0	0	0	0	2	0	5	6	17	7	0	2	39

Table 17 (continued)

Subject Number	Number of Minutes for Cell Number 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total Time
22	0	2	0	0	9	0	0	0	0	2	5	37	20	0	0	75
23	0	0	4	0	6	0	0	0	0	7	11	46	18	14	0	106
24	0	0	4	0	6	0	0	0	0	9	11	0	45	4	0	79
25	0	3	0	0	5	0	0	0	0	9	8	28	0	7	9	69

Table 18

Raw Data--Forgetting

Treatment Group One			Treatment Group Two		
Subject Number	Number of Cells Before Recall	Retention Days	Subject Number	Number of Cells Before Recall	Retention Days
1	1	7	1	1	19
2	5	7	2	1	10
3	8	14	3	1	19
4	1	19	4	1	19
5	1	7	5	1	7
6	1	19	6	1	14
7	1	20	7	1	14
8	5	14	8	6	14
9	0	7	9	5	18
10	5	7	10	4	14
11	3	14	11	0	20
12	2	14	12	1	20
13	5	14	13	0	14
14	1	20	14	1	20
15	1	19	15	5	7
16	1	19	16	3	14
17	1	19	17	1	20
18	1	7	18	1	20
19	1	14	19	8	20
20	1	14	20	0	7
21	5	15	21	0	14

Table 18 (continued)

Treatment Group One			Treatment Group Two		
Subject Number	Number of Cells Before Recall	Retention Days	Subject Number	Number of Cells Before Recall	Retention Days
22	4	16	22	1	15
23	3	15	23	0	7
24	1	15	24	0	16
25	1	15	25	1	16
26	4	15			
27	1	7			

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