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High-Order Averaging on Lie Groups and Control of an Autonomous Underwater Vehicle[1]

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Abstract

In this paper we extend our earlier results on the use of periodic forcing and averaging to solve the constructive controllability problem for drift-free left-invariant systems on Lie groups with fewer controls than state variables. In particular, we prove a third-order averaging theorem applicable to systems evolving on general matrix Lie groups and show how to use the resulting approximations to construct open loop controls for *complete* controllability of systems that require up to depth-two Lie brackets to satisfy the Lie algebra controllability rank condition. The motion control problem for an autonomous underwater vehicle is modelled as a drift-free left-invariant system on the matrix Lie group $SE(3)$. In the general case, when only one translational and two angular control inputs are available, this system satisfies the controllability rank condition using depth-two Lie brackets. We use the third-order averaging result and its geometric interpretation to construct open loop controls to arbitrarily translate and orient an autonomous underwater vehicle.

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1 Introduction

Drift-free systems with fewer controls than state variables arise in a variety of control problems including motion planning for wheeled robots subject to nonholonomic constraints, spacecraft attitude control and the motion control of autonomous underwater vehicles. The basic state-space model takes the form

$$\dot{x} = \sum_{i=1}^{m} F_i(x)u_i, \quad x \in \Re^n, \;\; u = (u_1, \ldots, u_m) \in \Re^m, \;\; n > m \; . \tag{1}$$

It is well known that if the vector fields F_i satisfy a Lie algebra rank condition, then there exists a control u that drives the system to the origin from any initial state. However, unlike the linear setting where the controllability Grammian yields constructive controls, here the rank condition does not lead immediately to an explicit procedure for constructing controls. As a result, recent research has focused on constructing controls to achieve complete controllability [1, 2, 3, 4, 5, 6]. In particular, constructive procedures based on periodically time-varying controls have proven successful [3, 4, 5, 6].

Our interest in this paper is in constructive controllability using periodic forcing of drift-free left-invariant systems of the form

$$\dot{X} = \epsilon XU, \qquad U(t) = \sum_{i=1}^{n} A_i u_i(t), \tag{2}$$

evolving on matrix Lie groups. Here $X(t)$ is a curve in a matrix Lie group G of dimension n, $U(t)$ is a curve in the Lie algebra $\mathcal{G}$ of G, and $\{A_1, \ldots, A_n\}$ a basis for $\mathcal{G}$. (For an introduction to matrix Lie groups and Lie algebras see [7]). The $u_i(\cdot)$ are assumed to be periodic functions of common period T. $\epsilon > 0$ is a small parameter such that $\epsilon u_i(\cdot)$ are interpreted as the small-amplitude periodic controls, although some of the $u_i(\cdot)$ may be identically zero.

Equation (2) describes the kinematics of several types of important systems. For example, equation (2) describes rigid spacecraft kinematics if we interpret $U(t)$ as the time-dependent skew symmetric matrix of spacecraft angular velocity such that X evolves on

$G = SO(3)$, the special orthogonal group, where

$$SO(n) \triangleq \{A \in \Re^{n \times n} | A^T A = I,\ det(A) = 1\}.$$

Similarly, equation (2) describes the kinematics of an underwater vehicle if we interpret $U(t)$ as the time-dependent matrix of vehicle angular and translational velocities such that X evolves on $G = SE(3)$, the special Euclidean group, where

$$SE(n) \triangleq \{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \in \Re^{(n+1)\times(n+1)} | A \in SO(n), b \in \Re^n \}.$$

We state formally the complete constructive controllability problem for system (2) where $u_i(t) \equiv 0,\ i = m+1, \ldots, n$:

(P) Given an initial condition $X_i \in G$, a final condition $X_f \in G$ and a time $t_f > 0$, find $u(t) = (u_1(t), \ldots, u_m(t)),\ t \in [0, t_f]$, such that $X(0) = X_i$ and $X(t_f) = X_f$.

Our objective is to prescribe means to solve (P) using small-amplitude periodic controls. One strategy is to use periodic controls to provide open loop control of the system and apply intermittent feedback corrections to make finer adjustments in system behavior. This strategy allows us to take advantage of a priori knowledge of the system and prescribe efficient open loop controls to drive the system as desired without sacrificing accuracy and sensitivity reduction associated with feedback control. We use averaging theory for systems of the form (2) as a means to specify open loop periodic control. The goal of averaging in this context is to describe an approximate solution to (2) that evolves on the matrix group G and remains close to the actual solution, but gives rise to straightforward procedures for achieving complete constructive controllability.

First and second-order averaging theorems have been proved for systems of the form (2) [8, 6]. The first-order averaging results reveal the effect of the dc component of the periodic forcing on the behavior of the system, but are only useful for complete constructive controllability if none of the controls $u_i(\cdot)$ is identically zero. However, the second-order average approximation provides a formula for achieving complete constructive controllability using fewer than n periodic controls if the controllability Lie algebra rank condition is satisfied for a system of the form (2) using up to depth-one Lie brackets (i.e., single brackets). In this case the formula solves (P) with $O(\epsilon^2)$ accuracy which could be improved with intermittent feedback if desired. Additionally, the second-order average approximation admits a geometric interpretation as an area rule. This was used to advantage in the design of open loop controls for the spacecraft attitude control problem with only two controls available and for the unicycle motion planning problem (c.f. [6]).

In this paper we prove a third-order averaging theorem for systems of the form (2) and develop the associated geometric interpretation. This facilitates the design of open loop controls to solve (P) with $O(\epsilon^3)$ accuracy for systems (2) which require up to depth-two Lie brackets (i.e., double brackets) to satisfy the controllability Lie algebra rank condition. We apply this result to the problem of specifying controls to drive an autonomous underwater vehicle to a desired position and orientation when only three controls are available (two rotational and one translational).

Autonomous underwater vehicles can potentially be sent into environments too risky for a manned vehicle and too deep for a tethered vehicle. Thus, they are expected to play an increasingly larger role in oceanic exploration and exploitation, for example, in geological surveying, data collection, drill support, construction, maintenance, etc. [9]. Similarly, there is great potential for their use in other types of hazardous environments such as in nuclear reactor vessels, e.g., for inspection and maintenance. Further, with the advent of micro-machining and micro-technologies comes the prospect of using micro-scale autonomous underwater vehicles for micro-scale underwater tasks such as in medical applications, for example, to send through blood vessels or arteries for organ inspection or repair.

To achieve autonomy of the underwater vehicle in each of these settings, the nonlinear behavior of the vehicle must be controlled. As described above, system (2) models the motion control problem if we can interpret the vehicle angular and translational velocities as our control inputs. This interpretation means that we assume that we can independently actuate these velocities (or at least some of them) as desired. For example, by controlling a propeller at the back of the vehicle, stern and bow planes on the sides of the vehicle and rudder planes at the back of the vehicle, the three angular velocities and one translational velocity can be controlled [10].

In the special case of a micro-scale underwater vehicle or a relatively small vehicle in a highly viscous fluid, angular and translational velocities can be effected simply by cyclic body deformations. This special case is the case of low Reynolds number ($\mathrm{Re} \ll 1$) in which frictional forces between the vehicle and the fluid dominate while inertial forces are negligible. Motion in this context has been studied by physicists interested in understanding how microorganisms such as paramecia swim [11, 12, 13, 14]. In imitation of the flagella or cilia used by microorganisms for maneuvering, actuators such as flapping flexible oars or rotating corkscrews could be used to generate angular and translational velocities for the vehicle at low Reynolds number.

In [10] a globally stable nonlinear tracking controller was developed using three angular velocities and one translational velocity as control inputs. For this controller, the kinematic equation was given by Euler angles and the reference trajectory was assumed to have a non-zero velocity. In [15] an exponentially convergent stabilizing control law was presented using again three angular velocities and one translational velocity as control inputs. In this case the kinematics were modelled as in (2). In this paper we need only require authority over two angular velocities and one translational velocity to translate and orient an underwater vehicle as desired. The low number of controls required to achieve complete constructive controllability provides a measure of redundancy to the control system. This redundancy can also be interpreted as the means for the controller to "adapt" to a failure in the system that reduces the control authority, by continuing to provide complete control over the position and orientation of the vehicle.

In Section 2 we summarize the results from our work on first-order and second-order averaging [8, 6, 16]. In Section 3 we discuss the basic ideas behind high-order averaging and prove a third-order averaging theorem for general matrix Lie groups. Our main result is an "area-moment rule" (Theorem 5) for systems on groups. We discuss the geometric interpretation of this rule and the consequences for control illustrating how to use the averaging result to achieve complete constructive controllability for systems which meet the controllability rank condition with up to depth-two Lie brackets. In Section 4 we examine our main result for the Lie group $SE(3)$ in the context of underwater vehicle control. An example is given for achieving complete constructive controllability when only one translational and two angular controls are available.

2 First and Second-Order Averaging

Since there are no explicit global representations of the solution to (2) we make use of local representations: the product of exponentials representation given by Wei and Norman [17] and the single exponential representation given by Magnus [18]. The basic idea which we use for first and second-order averaging as well as for high-order averaging is to derive classical averaging theory approximations for the local representation and then transfer such estimates to the group level for solutions to (2). We begin by defining the Wei-Norman representation and summarizing the associated averaging results.

Lemma 1 (Wei and Norman). Let $X(t)$ be the solution to (2) and $X(0) = I$. Then $\exists t_0 > 0$ such that for $|t| < t_0$, $X(t)$ can be expressed in the form

$$X(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} \cdots e^{g_n(t)A_n} . \tag{3}$$

The Wei-Norman parameters $g = (g_1, \ldots, g_n)^T$ satisfy

$$\dot{g} = \epsilon M(g) u \, , \quad \text{for } |t| < t_0 \, , \tag{4}$$

where $g(0) = 0$ and $M(g)$ is a real analytic matrix-valued function of g. If $\mathcal{G}$ is solvable then there exists a basis of $\mathcal{G}$ and an ordering of this basis for which (4) holds globally, i.e., for all t. $\square$

It is customary to refer to components of g as the canonical coordinates of the second kind for G. Let W be the open neighborhood of $0 \in \Re^n$ such that $\forall g \in W$, $M(g)$ is well-defined. Let $\Phi : \Re^n \rightarrow G$ define the mapping

$$\Phi(g) = e^{g_1 A_1} e^{g_2 A_2} \cdots e^{g_n A_n} \tag{5}$$

and define $V \equiv \Phi(W) \subset G$. Then, the Wei-Norman formulation provides a local representation of the solution to (2) for initial condition $X(0) \in V \subset G$. Now let S be the largest neighborhood of $0 \in \Re^n$ contained in W such that $\Psi \equiv \Phi|_S : S \rightarrow G$ is one-to-one. Let $Q \equiv \Psi(S) \subset V$. Then $\Psi : S \rightarrow Q$ is a diffeomorphism and we can define a metric $\tilde{d} : Q \times Q \rightarrow \Re_+$ by

$$\tilde{d}(Y, Z) = d(\Psi^{-1}(Y), \Psi^{-1}(Z)) \tag{6}$$

where $d : \Re^n \times \Re^n \rightarrow \Re_+$ is given by

$$d(\alpha, \beta) = \|\alpha - \beta\|_1 = \sum_{i=1}^{n} |\alpha_i - \beta_i|. \tag{7}$$

The averaged system associated with (4) is defined as

$$\begin{aligned} \dot{\bar{g}} &= \epsilon M(\bar{g})(\frac{1}{T}\int_0^T u(\tau)d\tau) \\ &\triangleq \epsilon M(\bar{g})u_{av}, \quad \bar{g}(0) = \bar{g}_0 \end{aligned} \tag{8}$$

where we assume that $M(\bar{g}_0)$ is well-defined. The average solution $\bar{X}(t)$ associated with the solution $X(t)$ of (2) is defined as

$$\dot{\bar{X}} = \epsilon \bar{X} U_{av}, \quad U_{av} = \sum_{i=1}^{n} A_i u_{avi} \tag{9}$$

where $u_{av} = (u_{av1}, \ldots, u_{avn})^T$. Thus,

$$\bar{X}(t) = \bar{X}(0) e^{\epsilon U_{av} t}. \tag{10}$$

The first-order averaging theorem can now be stated (c.f. Theorem 2 of [8]).

Theorem 1. Let $\epsilon > 0$ be a small parameter. Let $D = \{g \in \Re^n \mid \|g\| < r\} \subset S$. Assume that $u(t) \in \Re^n$ is periodic in t with period $T > 0$ and has continuous derivatives up to second order for $t \in [0, \infty)$. Let $X(t)$ be the solution to (2) represented by (3) where $g(t,\epsilon)$ is the solution to (4) with $\Psi(g(0,\epsilon)) = X(0)$ and $g(0,\epsilon) \in D$. Let $\bar{X}(t)$ be the solution to (9) and let $\bar{g}(t,\epsilon)$ be the solution to (8) with $\Psi(\bar{g}(0,\epsilon)) = \bar{X}(0)$.

$$\text{If } \bar{g}(t,\epsilon) \in D, \quad \forall t \in [0, b/\epsilon] \quad \text{and} \quad \|g(0,\epsilon) - \bar{g}(0,\epsilon)\| = O(\epsilon)$$

$$\text{then } \tilde{d}(X(t), \bar{X}(t)) = O(\epsilon), \quad \forall t \in [0, b/\epsilon]. \quad \Box$$

Next we show the form of the second-order average approximation $\bar{\bar{X}}(t)$ to the solution $X(t)$ of (2). Let $\bar{\bar{g}} = (\bar{\bar{g}}_1, \ldots, \bar{\bar{g}}_n)^T$ be the second-order average approximation of the solution $g(t)$ to (4). We define the second-order approximation $\bar{\bar{X}}$ on the group level as

$$\bar{\bar{X}}(t) = e^{\bar{\bar{g}}_1(t)A_1} e^{\bar{\bar{g}}_2(t)A_2} \cdots e^{\bar{\bar{g}}_n(t)A_n}, \tag{11}$$

which is well-defined for $\bar{\bar{g}}(t)$ well-defined. To isolate the second-order effect we assume that $u_{av} = 0$. We define $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T$ by

$$\tilde{u}_i(t) = \int_0^t u_i(\tau)d\tau. \tag{12}$$

So $u = \dot{\tilde{u}}$ and $\tilde{u}$ is periodic in t with common period T. Let $\tilde{U} = \sum_{i=1}^n \tilde{u}_i A_i$. Next we define $Area_{ij}(T)$ to be the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period, i.e., from $t = 0$ to $t = T$. By Green's Theorem we can express this area as

$$Area_{ij}(T) = \frac{1}{2} \int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))d\sigma. \tag{13}$$

This area can be interpreted as the projection onto the i-j plane of the area enclosed by the curve $(\tilde{u}_1, \ldots, \tilde{u}_n)$ in one period. Finally, we define the structure constants Γ_{ij}^k associated with the basis $\{A_1, \ldots, A_n\}$ for the Lie algebra $\mathcal{G}$ of G by

$$[A_i, A_j] = \sum_{k=1}^n \Gamma_{ij}^k A_k, \quad i,j = 1, \ldots, n \tag{14}$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathcal{G}$ defined by $[A,B] = AB - BA$.

The second-order averaging theorem can now be stated (c.f. Theorem 2 of [6]).

Theorem 2 (Area Rule). Let $\epsilon > 0$ be a small parameter. Let $D = \{g \in \Re^n \mid \|g\| < r\} \subset S$. Assume that $u(t) \in \Re^n$ is periodic in t with period $T > 0$ and has continuous derivatives up to third order for $t \in [0, \infty)$. Suppose that $u_{av} = 0$. Let $X(t)$ be the solution to (2) represented by (3) where $g(t,\epsilon)$ is the solution to (4) with $g(0,\epsilon) = g_0 \in D$ such that $\Psi(g_0) = X(0)$ and $\|g_0\| = O(\epsilon)$. Define

$$\bar{z}_k(t,\epsilon) = \frac{\epsilon^2 t}{T} \sum_{i,j=1;i<j}^{n} Area_{ij}(T)\Gamma_{ij}^k, \quad k = 1,\ldots,n, \tag{15}$$

$$\bar{\bar{g}} = \bar{z} + \epsilon\tilde{u} + \bar{\bar{g}}_0, \tag{16}$$

$$\bar{\bar{X}}(t) = e^{\bar{\bar{g}}_1(t)A_1} e^{\bar{\bar{g}}_2(t)A_2} \cdots e^{\bar{\bar{g}}_n(t)A_n}, \tag{17}$$

where $\|g_0 - \bar{\bar{g}}_0\| = O(\epsilon^2)$. If $(\bar{z}(t,\epsilon) + \bar{\bar{g}}_0) \in D$, $\forall t \in [0, b/\epsilon]$ then

$$\tilde{d}(X(t), \bar{\bar{X}}(t)) = O(\epsilon^2), \quad \forall t \in [0, b/\epsilon]. \quad \square$$

We can state analogous first and second-order averaging results based on the single exponential local representation of solutions to (2). By Theorem III of [18], assuming a certain convergence criterion is met, the solution to (2) with $X(0) = I$ can be expressed as

$$X(t) = e^{Z(t)} \tag{18}$$

where $Z(t) \in \mathcal{G}$ is given by the infinite series (we show terms up to $O(\epsilon^3)$):

$$\begin{aligned} Z(t) &= \epsilon \int_0^t U(\tau)d\tau + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau \\ &+ \frac{\epsilon^3}{4}\int_0^t [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau + \frac{\epsilon^3}{12}\int_0^t [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau + \ldots \end{aligned} \tag{19}$$

Satisfying the convergence criterion means limiting the duration of validity of the single exponential representation (see [16] for details). Assuming the convergence requirement is met, then $Z(t)$ is the solution to

$$\dot{Z} = \epsilon U + \frac{\epsilon^2}{2}[\tilde{U}, U] + \frac{\epsilon^3}{4}[\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau, U] + \frac{\epsilon^3}{12}[\tilde{U}, [\tilde{U}, U]] + \ldots, \quad Z(0) = 0. \tag{20}$$

Let $\hat{\Phi} : \mathcal{G} \to G$ define the mapping

$$\hat{\Phi}(Z) = e^{Z}. \tag{21}$$

Let $\hat{S}$ be the largest neighborhood of $0 \in \mathcal{G}$ such that $\hat{\Psi} \equiv \hat{\Phi}|_{\hat{S}} : \hat{S} \to G$ is one-to-one. Let $\hat{Q} \equiv \hat{\Psi}(\hat{S}) \subset G$. Then $\hat{\Psi} : \hat{S} \to \hat{Q}$ is a diffeomorphism and we can define a metric $\hat{d} : \hat{Q} \times \hat{Q} \to \Re_{+}$ by

$$\hat{d}(X,Y) = d(\hat{\Psi}^{-1}(X), \hat{\Psi}^{-1}(Y)) \tag{22}$$

where d is given by (6).

The second-order averaging theorem based on the single exponential representation is given below (c.f. Theorem 4 [16]). The first-order averaging theorem using this representation gives the same result as Theorem 1.

Theorem 3 (Single Exponential Area Rule). Let $\epsilon > 0$ be a small parameter. Let $D = \{Z \in \mathcal{G} | \|Z\| < r\} \subset \hat{S}$. Assume that $u(t) \in \Re^n$ is periodic in t with period $T > 0$ and has continuous derivatives up to third order for $t \in [0, \infty)$. Suppose $u_{av} = 0$. Let $b > 0$ be such that the convergence requirement for (19) is met $\forall t \in [0, b/\epsilon]$. Let $X(t)$ be the solution to (2), with $X(0) = I$, represented by the single exponential (18) where $Z(t,\epsilon) \in \mathcal{G}$ is the solution to (20). Define

$$\bar{\bar{Z}}(t,\epsilon) = \frac{\epsilon^2 t}{T} \sum_{k=1}^{n} \left(\sum_{i,j=1; i<j}^{n} Area_{ij}(T) \Gamma_{ij}^{k} \right) A_k, \tag{23}$$

$$\bar{\bar{X}}_S(t) = e^{\bar{\bar{Z}} + \epsilon \bar{U}}. \tag{24}$$

If $\bar{\bar{Z}}(t,\epsilon) \in D$, $\forall t \in [0, b/\epsilon]$ then

$$\hat{d}(X(t), \bar{\bar{X}}_S(t)) = O(\epsilon^2) \text{ on } [0, b/\epsilon]. \quad \Box$$

It is clear from equation (9) that the first-order approximation $\bar{X}(t)$ describes the effect of the dc component of the control input u on the system (2). Thus, the approximation provides a formula for complete constructive controllability only if none of the controls $u_i(\cdot)$ is identically zero. On the other hand, the second-order average approximation provides a formula for complete controllability even when some of the controls are identically zero. Specifically, the second-order average approximation $\bar{\bar{X}}(t)$ given by equations (15) - (17) (or similarly $\bar{\bar{X}}_S$ given by (23) and (24)), in providing more information about the actual solution to (2), captures the effect of the group level version of depth-one Lie brackets. This effect is stated in the next theorem (c.f. Theorem 3 of [6]).

Theorem 4. Suppose that system (2) satisfies the Lie algebra controllability rank condition with up to depth-one Lie brackets. Then the complete constructive controllability problem (P) can be solved with $O(\epsilon^2)$ accuracy using the formula for $\bar{\bar{X}}(t)$ given by (15) - (17) or the formula for $\bar{\bar{X}}_S(t)$ given by (23) and (24). $\square$

Basically, this theorem tells us that in the formula (17) for $\bar{\bar{X}}$ (and analogously for $\bar{\bar{X}}_S$), each $\bar{\bar{g}}_k$ will be the linear combination of terms like $\tilde{u}_k$ and terms like $Area_{ij}(T)$ and no $\bar{\bar{g}}_k$ will be identically zero (or constant). It is then easy to see how to construct open loop controls since we know the geometric meaning of the terms $Area_{ij}(T)$, i.e., that $Area_{ij}(T)$ is the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period. In particular, if we choose $\tilde{u}_i$ and $\tilde{u}_j$ to be sinusoids that are in phase then $Area_{ij}(T) = 0$. Alternatively, if they are chosen out of phase then $Area_{ij}(T) \neq 0$ can be computed based on the signal magnitudes and their phase difference. An algorithm can then be derived based on this geometric reasoning.

The spacecraft attitude control problem, where it is assumed that two angular velocities are available as control inputs (e.g., in the case where there are two reaction wheels and no external torque is applied), can be modelled as (2) with $n = 3$ and $u_3(\cdot) = 0$. It can be shown that this system satisfies the hypothesis of Theorem 4 and with the appropriate choice of basis elements $\{A_1, A_2, A_3\}$, $\bar{\bar{X}}$ takes the form

$$\begin{aligned} \bar{\bar{X}}(t) &= e^{\bar{\bar{g}}_1 A_1} e^{\bar{\bar{g}}_2 A_2} e^{\bar{\bar{g}}_3 A_3} \\ &= e^{(\epsilon \tilde{u}_1 + \bar{\bar{g}}_{01}) A_1} e^{(\epsilon \tilde{u}_2 + \bar{\bar{g}}_{02}) A_2} e^{(\frac{\epsilon^2 t}{T} Area_{12}(T) + \bar{\bar{g}}_{03}) A_3}. \end{aligned} \tag{25}$$

Similarly, the unicycle motion planning problem can be modelled by (2) with $G = SE(2)$, $n = 3$ and $u_3(\cdot) = 0$. This system also satisfies the hypothesis of Theorem 4 and $\bar{\bar{X}}$

again takes the form (25). Details on an algorithm that was derived for these systems as well as corresponding simulation results can be found in [6].

3 Third-Order Averaging

Higher-order average approximations to the solution to (2) naturally provide successively more information about the actual solution, $X(t)$. The nature of this information can be gleaned from the infinite series expansion of $Z(t)$ in (19), where it is noted that by (18) $Z(t)$ is the logarithm of $X(t)$. The $O(\epsilon^2)$ term in (19) is a depth-one Lie bracket and as verified in Theorem 4 of the previous section the $O(\epsilon^2)$ approximation completely captures the effect of the depth-one Lie brackets in the context of controllability. It is expected that the $O(\epsilon^p)$ approximation for $p \geq 2$ of $X(t)$ will completely capture the effect of depth-$(p-1)$ Lie brackets in the context of controllability. (Note that a depth-$(p-1)$ Lie bracket is defined as $(p-1)$ iterated brackets, e.g., a depth-two bracket is of the form $[A,[B,C]]$, a depth-three bracket is of the form $[A,[B,[C,D]]]$, $A,B,C,D \in \mathcal{G}$, etc.) In this section we prove this result for $p = 3$. Additionally, we show that the third-order approximation has a geometric interpretation based on a higher-order geometric object which can be described as a first moment. The first moment plays a role analogous to the role played by area in the second-order average approximation.

As in the case of second-order averaging in Theorem 3, we use the single exponential representation (18) of solutions to (2) with the associated differential equation for $Z(t)$ given by (20) as a means to do third-order averaging. As in the case of second-order averaging we assume that $u_{av} = 0$. Let $\tilde{u}$ be as defined by (12). Define

$$a_{ij}(t) = \frac{1}{2}\int_0^t (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))d\sigma. \tag{26}$$

Then $a_{ij}(t)$ is of the form

$$a_{ij}(t) = \frac{Area_{ij}(T)t}{T} + f(t),$$

where $f(t+T) = f(t)$, $f(0) = 0$, T is the period of the control u and $Area_{ij}(T)$ is given by (13). So, in particular,

$$a_{ij}(qT) = qArea_{ij}(T) \tag{27}$$

where q is a positive integer. Let

$$m_{ijk}(T) = \frac{1}{3}\int_0^T (\tilde{u}_i(\sigma)\dot{\tilde{u}}_j(\sigma) - \tilde{u}_j(\sigma)\dot{\tilde{u}}_i(\sigma))\tilde{u}_k(\sigma)d\sigma. \tag{28}$$

Now consider the closed curve C defined by $\tilde{u}_i(t)$, $\tilde{u}_j(t)$ and $\tilde{u}_k(t)$ over one period, i.e., from $t = 0$ to $t = T$. From (28) we get that

$$m_{ijk}(T) = \frac{1}{3}\oint_C \tilde{u}_i\tilde{u}_k d\tilde{u}_j - \tilde{u}_j\tilde{u}_k d\tilde{u}_i. \tag{29}$$

Let A be any oriented surface with boundary $\partial A = C$. Then by Stokes' Theorem,

$$m_{ijk}(T) = \frac{1}{3}\int_A -\tilde{u}_i d\tilde{u}_j d\tilde{u}_k - \tilde{u}_j d\tilde{u}_k d\tilde{u}_i + 2\tilde{u}_k d\tilde{u}_i d\tilde{u}_j. \tag{30}$$

So $m_{ijk}(T)$ as described by (30) can be interpreted as the first moment, i.e., a linear combination of $\tilde{u}_i$ integrated over the area of the projection of A onto the j-k plane, $\tilde{u}_j$ integrated over the area of the projection of A onto the k-i plane and $\tilde{u}_k$ integrated over the area of the projection of A onto the i-j plane.

Next we define depth-two structure constants θ^p_{ijk} associated with basis $\{A_1, \ldots, A_n\}$ for the Lie algebra $\mathcal{G}$ in terms of the structure constants Γ^k_{ij} defined by (14) as

$$\theta^p_{ijk} \triangleq \sum_{l=1}^{n} \Gamma^l_{ij}\Gamma^p_{lk}. \tag{31}$$

This definition comes from the computation of structure constants for depth-two Lie brackets as follows:

$$[[A_i, A_j], A_k] = [\sum_{l=1}^{n} \Gamma^l_{ij}A_l, A_k] = \sum_{l=1}^{n} \Gamma^l_{ij}[A_l, A_k] = \sum_{p=1}^{n}\sum_{l=1}^{n} \Gamma^l_{ij}\Gamma^p_{lk}A_p = \sum_{p=1}^{n} \theta^p_{ijk}A_p. \tag{32}$$

Theorem 5 (Area-Moment Rule). Let $\epsilon > 0$ be a small parameter. Let $D = \{Z \in \mathcal{G} | \|Z\| < r\} \subset \hat{S}$. Assume that $u(t) \in \Re^n$ is periodic in t with period $T > 0$ and has continuous derivatives up to fourth order for $t \in [0, \infty)$. Let $b > 0$ be such that the convergence requirement for (19) is met $\forall t \in [0, b/\epsilon]$. Let $X(t)$ be the solution to (2), with $X(0) = I$, represented by the single exponential (18) where $Z(t,\epsilon) \in \mathcal{G}$ is the solution to (20). Define

$$Z^{(3)}(t,\epsilon) = \sum_{p=1}^{n}(\epsilon\tilde{u}_p(t) + \epsilon^2 \sum_{i,j=1;i<j}^{n} a_{ij}(t)\Gamma_{ij}^{p} - \frac{\epsilon^3 t}{T}\sum_{k=1}^{n}\sum_{i,j=1;i<j}^{n} m_{ijk}(T)\theta_{ijk}^{p})A_p, \qquad (33)$$

$$X^{(3)}(t) = e^{Z^{(3)}(t)}. \qquad (34)$$

If $Z^{(3)}(t,\epsilon) \in D$, $\forall t \in [0, b/\epsilon]$,

$$\hat{d}(X(t), X^{(3)}(t)) = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon].$$

Proof. Let $s = \epsilon t$. Then from (20),

$$\begin{aligned} \frac{dZ}{ds} &= U + \frac{\epsilon}{2}[\tilde{U}, U] + \frac{\epsilon^2}{4}[\int_0^s [\tilde{U}(\tau), U(\tau)]d\tau, U] + \frac{\epsilon^2}{12}[\tilde{U}, [\tilde{U}, U]] + \dots \\ &\triangleq f(s,\epsilon). \end{aligned} \qquad (35)$$

Let $Z_0(s,\epsilon)$, $Z_1(s,\epsilon)$ and $Z_2(s,\epsilon)$ be the solutions, respectively, to

$$\frac{dZ_0}{ds} = f(s,0) = U(s), \quad Z_0(0,\epsilon) = 0, \qquad (36)$$

$$\frac{dZ_1}{ds} = \frac{\partial f}{d\epsilon}(s,0) = \frac{1}{2}[\tilde{U}, U](s), \quad Z_1(0,\epsilon) = 0, \qquad (37)$$

$$\frac{dZ_2}{ds} = \frac{\partial^2 f}{d\epsilon^2}(s,0) = \frac{1}{4}[\int_0^s [\tilde{U}(\tau), U(\tau)]d\tau, U(s)] + \frac{1}{12}[\tilde{U}, [\tilde{U}, U]](s), \quad Z_2(0,\epsilon) = 0. \qquad (38)$$

Then by standard perturbation theory (c.f. Theorem 7.1 [19]), if $Z_0(s,\epsilon) \in D$, $\forall s \in [0,b]$, then $\exists \epsilon^* > 0$ such that $\forall |\epsilon| < \epsilon^*$ (35) has the unique solution $Z(s,\epsilon)$ defined on $[0,b]$ such that

$$\|Z(s,\epsilon) - (Z_0(s,\epsilon) + \epsilon Z_1(s,\epsilon) + \epsilon^2 Z_2(s,\epsilon))\| = O(\epsilon^3), \quad \forall s \in [0,b].$$

This implies that

$$\|Z(t,\epsilon) - (Z_0(t,\epsilon) + \epsilon Z_1(t,\epsilon) + \epsilon^2 Z_2(t,\epsilon))\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon], \qquad (39)$$

where by (36) - (38) and since $ds = \epsilon dt$,

$$\dot{Z}_0 = \epsilon U(t), \quad Z_0(0,\epsilon) = 0, \qquad (40)$$

$$\dot{Z}_1 = \frac{\epsilon}{2}[\tilde{U}, U](t), \quad Z_1(0,\epsilon) = 0, \tag{41}$$

$$\dot{Z}_2 = \frac{\epsilon}{4}[\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau, U(t)] + \frac{\epsilon}{12}[\tilde{U}, [\tilde{U}, U]](t), \quad Z_2(0,\epsilon) = 0. \tag{42}$$

Now let $Y \triangleq \epsilon^2 Z_2$. Then

$$\|Z - (Z_0 + \epsilon Z_1 + Y)\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon] \quad \text{and} \tag{43}$$

$$\dot{Y} = \frac{\epsilon^3}{4}[\int_0^t [\tilde{U}(\tau), U(\tau)]d\tau, U(t)] + \frac{\epsilon^3}{12}[\tilde{U}, [\tilde{U}, U]](t), \quad Y(0,\epsilon) = 0. \tag{44}$$

Let $\bar{Y}(t,\epsilon)$ be the solution to

$$\dot{\bar{Y}} = \frac{\epsilon^3}{4T}\int_0^T [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau + \frac{\epsilon^3}{12T}\int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau, \quad \bar{Y}(0,\epsilon) = 0. \tag{45}$$

By classical averaging (c.f. Theorem 7.4 [19]), if $\bar{Y} \in D$, $\forall t \in [0, b/\epsilon]$ and ϵ is small enough then $\|Y(t,\epsilon) - \bar{Y}(t,\epsilon)\| = O(\epsilon^3)$ on $[0, b/\epsilon]$. So by (43) and the triangle inequality

$$\|Z(t,\epsilon) - Z^{(3)}(t,\epsilon)\| = O(\epsilon^3), \quad \forall t \in [0, b/\epsilon], \tag{46}$$

where

$$\begin{aligned} Z^{(3)}(t,\epsilon) &\triangleq Z_0(t,\epsilon) + \epsilon Z_1(t,\epsilon) + \bar{Y}(t,\epsilon) \\ &= \epsilon\tilde{U}(t) + \frac{\epsilon^2}{2}\int_0^t [\tilde{U}, U](\tau)d\tau + \frac{\epsilon^3 t}{4T}\int_0^T [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau)]d\tau \\ &+ \frac{\epsilon^3 t}{12T}\int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]]d\tau, \end{aligned} \tag{47}$$

by (40), (41) and (45). Thus, by (46) and the definition of $\hat{d}$, $\hat{d}(X(t), X^{(3)}(t)) = O(\epsilon^3)$, $\forall t \in [0, b/\epsilon]$, where $X^{(3)}$ is defined by (34).

The proof is complete if we show that $Z^{(3)}$ defined by (47) agrees with (33). By definition, we have that $\epsilon\tilde{U}(t) = \sum_{p=1}^n \epsilon\tilde{u}_p(t)A_p$. Next we show that the second term on the right side of (47) is equivalent to the second term on the right side of (33).

$$\begin{aligned} \frac{\epsilon^2}{2}\int_0^t [\tilde{U}, U](\sigma)d\sigma &= \frac{\epsilon^2}{2}\int_0^t [\sum_{i=1}^n \tilde{u}_i(\sigma)A_i, \sum_{j=1}^n \dot{\tilde{u}}_j(\sigma)A_j]d\sigma \\ &= \frac{\epsilon^2}{2}\int_0^t \sum_{i,j=1;i<j}^n (\tilde{u}_i\dot{\tilde{u}}_j - \dot{\tilde{u}}_i\tilde{u}_j)(\sigma)[A_i, A_j]d\sigma \\ &= \sum_{p=1}^n (\epsilon^2 \sum_{i,j=1;i<j}^n a_{ij}(t)\Gamma_{ij}^p)A_p, \end{aligned} \tag{48}$$

where we have used the fact that $[A_i, A_j] = -[A_j, A_i]$ and the definitions of $a_{ij}(t)$ and Γ^p_{ij}.

Next we examine the third term on the right side of (47).

$$
\begin{aligned}
& \frac{\epsilon^3 t}{4T} \int_0^T [\int_0^\tau [\tilde{U}(\sigma), U(\sigma)] d\sigma, U(\tau)] d\tau \\
= \;& \frac{\epsilon^3 t}{4T} \int_0^T [\int_0^\tau [\sum_{i=1}^n \tilde{u}_i(\sigma) A_i, \sum_{j=1}^n \dot{\tilde{u}}_j(\sigma) A_j] d\sigma, \sum_{k=1}^n \dot{\tilde{u}}_k(\tau) A_k] d\tau \\
= \;& \frac{\epsilon^3 t}{4T} \sum_{k=1}^n \sum_{i,j=1;i<j}^n (\int_0^T (\int_0^\tau (\tilde{u}_i \dot{\tilde{u}}_j - \dot{\tilde{u}}_i \tilde{u}_j)(\sigma) d\sigma) \dot{\tilde{u}}_k(\tau) d\tau) [[A_i, A_j], A_k] \\
= \;& -\sum_{p=1}^n \frac{3}{4} \frac{\epsilon^3 t}{T} \sum_{k=1}^n \sum_{i,j=1;i<j}^n \frac{1}{3} \int_0^T (\tilde{u}_i(\tau) \dot{\tilde{u}}_j(\tau) - \tilde{u}_j(\tau) \dot{\tilde{u}}_i(\tau)) \tilde{u}_k(\tau) d\tau \theta^p_{ijk} A_p \\
= \;& -\sum_{p=1}^n (\frac{3}{4} \frac{\epsilon^3 t}{T} \sum_{k=1}^n \sum_{i,j=1;i<j}^n m_{ijk}(T) \theta^p_{ijk}) A_p, \qquad (49)
\end{aligned}
$$

where we have used (32), integration by parts and the definition of $m_{ijk}(T)$.

The last term on the right side of (47) can be similarly expanded.

$$
\begin{aligned}
& \frac{\epsilon^3 t}{12} \int_0^T [\tilde{U}(\tau), [\tilde{U}(\tau), U(\tau)]] d\tau \\
= \;& \frac{\epsilon^3 t}{12} \int_0^T [\sum_{k=1}^n \tilde{u}_k(\tau) A_k, [\sum_{i=1}^n \tilde{u}_i(\tau) A_i, \sum_{j=1}^n \dot{\tilde{u}}_j(\tau) A_j]] d\tau \\
= \;& \frac{\epsilon^3 t}{4} \sum_{k=1}^n \sum_{i,j=1;i<j}^n \frac{1}{3} \int_0^T (\tilde{u}_i(\tau) \dot{\tilde{u}}_j(\tau) - \tilde{u}_j(\tau) \dot{\tilde{u}}_i(\tau)) \tilde{u}_k(\tau) d\tau [A_k, [A_i, A_j]] \\
= \;& \frac{\epsilon^3 t}{4} \sum_{k=1}^n \sum_{i,j=1;i<j}^n m_{ijk}(T) [A_k, [A_i, A_j]] \\
= \;& -\frac{\epsilon^3 t}{4} \sum_{k=1}^n \sum_{i,j=1;i<j}^n m_{ijk}(T) [[A_i, A_j], A_k] \\
= \;& -\sum_{p=1}^n (\frac{1}{4} \frac{\epsilon^3 t}{T} \sum_{k=1}^n \sum_{i,j=1;i<j}^n m_{ijk}(T) \theta^p_{ijk}) A_p, \qquad (50)
\end{aligned}
$$

Therefore, substituting (48), (49) and (50) into (47) yields the expression for $Z^{(3)}$ given by (33). □.

The terms of $Z^{(3)}(t)$ (33) can be characterized as follows. The first term on the right side of (33) is an $O(\epsilon)$ periodic term. The second term is a secular term (linear in

t) with an $O(\epsilon^2)$ periodic term superimposed. The secular term is proportional to the areas $Area_{ij}(T)$ and the structure constants Γ^p_{ij} associated with the choice of basis for $\mathcal{G}$. The third term of (33) is purely a secular term proportional to the first moments $m_{ijk}(T)$ and the depth-two structure constants θ^p_{ijk} associated with choice of basis for $\mathcal{G}$. This interpretation makes Theorem 5 an area-moment rule.

It should be noted that the formula of Theorem 5 is clearly basis independent. Additionally, because system (2) is left-invariant, Theorem 5 actually gives the formula for the third-order approximation $X^{(3)}(t)$ to the solution $X(t)$ of (2) for any initial condition $X(0) \in G$. Let $X_I(t)$ and $X_I^{(3)}(t)$ correspond to the actual and approximate solutions, respectively, of (2) with $X_I(0) = I \in G$. By left-invariance of (2), $X(t) = X(0)X_I(t)$ and $X^{(3)}(t) = X(0)X_I^{(3)}(t)$ is an $O(\epsilon^3)$ approximation of $X(t)$ on an $O(1/\epsilon)$ time interval.

As discussed earlier, $X^{(3)}(t)$ captures the effect of depth-two Lie brackets. This effect in the context of controllability is summarized in the next theorem.

Theorem 6. Suppose that system (2) satisfies the Lie algebra controllability rank condition with up to depth-two Lie brackets. Then the complete constructive controllability problem (P) can be solved with $O(\epsilon^3)$ accuracy using the formula for $X^{(3)}(t)$ given by (33) and (34).

Proof. Consider a system of the form (2) with $u_{m+1}(\cdot) = \ldots = u_n(\cdot) = 0$, $m \leq n$. Without loss of generality we can assume that $X_i = I \in G$ and $X_f \in \hat{Q} \subset G$ is such that $Z_f = \hat{\Psi}^{-1}(X_f) = O(\epsilon^2)$. This is possible due to the left-invariance of the system and the fact that Theorem 5 can be applied repeatedly. Let

$$\mathcal{C} = \{C \mid C = A_p \text{ or } C = [A_i, A_j], \text{ or } C = [[A_i, A_j], A_k],\ p, i, j, k = 1, \ldots, m\}.$$

By hypothesis,

$$\mathcal{G} = \text{span}\mathcal{C} = \{\sum_{p=1}^{m} c_p A_p + \sum_{i,j=1}^{m} c_{ij}[A_i, A_j] + \sum_{i,j,k=1}^{m} c_{ijk}[[A_i, A_j], A_k],\ c_p, c_{ij}, c_{ijk} \in \Re\}$$

$$= \{\sum_{p=1}^{m} c_p A_p + \sum_{p=1}^{n} (\sum_{i,j=1;i<j}^{m} c_{ij}\Gamma_{ij}^{p} + \sum_{k=1}^{m} \sum_{i,j=1;i<j}^{m} c_{ijk}\theta_{ijk}^{p}) A_p, \; c_p, c_{ij}, c_{ijk} \in \Re\}.$$

Therefore, since $Z_f \in \mathcal{G}$, $\exists c_p, c_{ij}, c_{ijk} \in \Re$, $p, i, j, k = 1, \ldots, m$ such that

$$Z_f = \sum_{p=1}^{m} c_p A_p + \sum_{p=1}^{n} (\sum_{i,j=1;i<j}^{m} c_{ij}\Gamma_{ij}^{p} + \sum_{k=1}^{m} \sum_{i,j=1;i<j}^{m} c_{ijk}\theta_{ijk}^{p}) A_p. \tag{51}$$

Also, from (33) and the assumption that $u_i(\cdot) = 0$ for $i = m+1, \ldots, n$ we have that

$$Z^{(3)}(t,\epsilon) = \sum_{p=1}^{m} \epsilon \tilde{u}_p(t) A_p + \sum_{p=1}^{n} (\sum_{i,j=1;i<j}^{m} \epsilon^2 a_{ij}(t)\Gamma_{ij}^{p} - \sum_{k=1}^{m} \sum_{i,j=1;i<j}^{m} \frac{\epsilon^3 t}{T} m_{ijk}(T)\theta_{ijk}^{p}) A_p. \tag{52}$$

So if we choose $u_p(t)$, $t \in [0, t_f]$, $p = 1, \ldots, m$ such that

$$\epsilon \tilde{u}_p(t_f) = c_p, \;\; p = 1, \ldots, m, \tag{53}$$

$$\epsilon^2 a_{ij}(t_f) = c_{ij}, \;\; i, j = 1, \ldots, m, \quad \text{and} \tag{54}$$

$$-\frac{\epsilon^3 t_f}{T} m_{ijk}(T) = c_{ijk}, \;\; i, j, k = 1, \ldots, m, \tag{55}$$

then from (51) and (52) $Z^{(3)}(t_f) = Z_f$. This implies that $X^{(3)}(t_f) = \hat{\Psi}(Z^{(3)}(t_f)) = \hat{\Psi}(Z_f) = X_f$. So, by Theorem 5, $\|X(t_f) - X_f\| = O(\epsilon^3)$.

It remains to show that (53) - (55) can be met. This becomes clear by recognizing the geometric meaning of the terms $a_{ij}(t)$ and $m_{ijk}(T)$. For q an integer, $a_{ij}(qT) = qArea_{ij}(T)$ and $Area_{ij}(T)$ is the area bounded by the closed curve described by $\tilde{u}_i$ and $\tilde{u}_j$ over one period. In particular, if we choose $\tilde{u}_i$ and $\tilde{u}_j$ to be sinusoids that are in phase then $Area_{ij}(T) = 0$. Alternatively, if they are chosen out of phase then $Area_{ij}(T) \neq 0$ is a function of the product of the signal magnitudes and phase difference. The terms $m_{ijk}(T)$ are first moments as described above. In particular, if we choose $\tilde{u}_i$ and $\tilde{u}_j$ to be sinusoids that are in phase then $m_{ijk}(T) = 0$. Only when $\tilde{u}_i$ and $\tilde{u}_j$ are nonzero and out of phase and $\tilde{u}_k$ is also nonzero will $m_{ijk} \neq 0$. In this case $m_{ijk}(T)$ will be proportional to the product of the signal magnitudes. Based on this reasoning the final values of each of these terms can be matched independently, i.e., (53) - (55) can be met. The timing can be controlled by choosing the frequency and amplitudes of the sinusoids appropriately. □

4 Underwater Vehicle Control

Consider an autonomous underwater vehicle and let $(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$ be coordinates fixed on the vehicle. Let $(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ be inertial coordinates. Then we define $X(t) \in SE(3)$ where

$$X(t) = \left[\begin{array}{c|c} X_R(t) & x_T(t) \\ \hline 0\,0\,0 & 1 \end{array}\right], \quad X_R(t) \in SO(3),\ x_T(t) \in \Re^3,$$

such that

$$X(t)\begin{bmatrix} \mathbf{r_i} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} X_R(t) & x_T(t) \\ \hline 0\,0\,0 & 1 \end{array}\right]\begin{bmatrix} \mathbf{r_i} \\ 1 \end{bmatrix} = \begin{bmatrix} X_R(t)\mathbf{r_i} + x_T(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b_i} \\ 1 \end{bmatrix}.$$

That is, $X(t)$ describes the orientation and position of the vehicle at time t. Let $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$. Define $\hat{}: \Re^3 \to so(3)$ where $so(3)$ is the space of skew symmetric matrices and $x = (x_1, x_2, x_3)^T$, by

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Let

$$A_i = \begin{cases} \left[\begin{array}{c|c} \hat{e}_i & 0 \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 1,2,3 \\ \left[\begin{array}{c|c} 0 & e_{i-3} \\ \hline 0\,0\,0 & 0 \end{array}\right] & i = 4,5,6. \end{cases}$$

Then $\{A_1, \ldots, A_6\}$ defines a basis for $\mathcal{G} = se(3)$, the Lie algebra associated with $SE(3)$. Now let $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ define the angular velocity of the vehicle and $v = (v_1, v_2, v_3)^T$ the vehicle translational velocity all with respect to $(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3})$. Then $X(t)$ satisfies

$$\dot{X} = X\left(\sum_{i=1}^{3} \Omega_i(t) A_i + \sum_{i=4}^{6} v_i(t) A_i\right). \tag{56}$$

Based on the discussion in the introduction, we assume that we can interpret $\Omega(t)$ and $v(t)$ as controls such that (56) is of the form (2). Specifically, suppose that we have four controls available, i.e., $\epsilon u_i(t) = \Omega_i(t)$, $i = 1,2,3$ and $\epsilon u_4(t) = v_1(t)$. Let $F_i(X) = XA_i$, $i = 1, \ldots, 4$, then

$$\dot{X} = \epsilon X(\sum_{i=1}^{4} A_i u_i) = \epsilon \sum_{i=1}^{4} F_i(X) u_i. \tag{57}$$

Now, the Lie bracket of left-invariant vector fields on a matrix Lie group can be expressed in terms of the Lie bracket on the associated Lie algebra as $[F_i(X), F_j(X)] = [XA_i, XA_j] = X[A_i, A_j]$. Thus, since $[A_3, A_4] = A_5$ and $[A_4, A_2] = A_6$, the system is completely controllable with the Lie algebra rank condition requiring only depth-one Lie brackets. However, if only three controls were available, e.g., if one of the controls failed, then the controllability situation changes. Certainly, if the translational control u_4 is lost then position control is lost. However, if one of the rotational controls is lost the system is still controllable. Specifically, if u_1 is lost then the system still satisfies the Lie algebra rank condition with depth-one brackets. On the other hand, if u_2 or u_3 is lost, depth-two Lie brackets are needed to satisfy the Lie algebra rank condition. For example, suppose u_1, u_2 and u_4 are the controls available. Then $[A_1, A_2] = A_3$, $[A_4, A_2] = A_6$ and $[[A_1, A_2], A_4] = A_5$ show the system is controllable using one depth-two Lie bracket. Thus, in the general case where one translational and two angular controls are available, the third-order average formula (and *not* the second-order average formula) provides a means to derive controls for complete control of the vehicle.

In effect, one can think of the third-order average formula as providing an "adaptive" control law for translating and orienting an autonomous underwater vehicle. Specifically, under normal conditions with four controls available, the control algorithm could be based on the second-order average formula. In the event of a failure that reduces control authority to three controls, the control algorithm could be switched to one based on the third-order average formula. In this scenario the controller would adapt to the failure and continue to effect complete control over the vehicle's position and orientation.

We now illustrate how to use the third-order average formula (i.e., the area-moment rule) to construct controls to solve problem (P), i.e., to translate and orient an underwater vehicle as desired, in the general case when one translational and two angular controls are available. Specifically, we focus on system (57) where $u_3(\cdot) = 0$, i.e., $u_1,\ u_2$ and u_4 are the controls available, as in the example above. From Theorem 5 we can write down the formula for the third-order approximation $X^{(3)}(t)$ to the solution $X(t) \in SE(3)$ as

$$X^{(3)}(t) = e^{Z^{(3)}(t)}, \quad Z^{(3)}(t) = \sum_{p=1}^{6} \bar{d}_p(t) A_p,$$

$$\begin{aligned} &\bar{d}_1(t) = \epsilon\tilde{u}_1(t) - \tfrac{\epsilon^3 t}{T} m_{212}(T), && \bar{d}_4(t) = \epsilon\tilde{u}_4(t) - \tfrac{\epsilon^3 t}{T} m_{242}(T), \\ &\bar{d}_2(t) = \epsilon\tilde{u}_2(t) - \tfrac{\epsilon^3 t}{T} m_{121}(T), && \bar{d}_5(t) = -\tfrac{\epsilon^3 t}{T}(m_{124}(T) + m_{421}(T)), \\ &\bar{d}_3(t) = \epsilon^2 a_{12}(t), && \bar{d}_6(t) = \epsilon^2 a_{42}(t). \end{aligned} \tag{58}$$

Now suppose without loss of generality that $X(0) = I$ and it is desired that $X(t_f) = X_d$ such that $Z_d = \Psi^{-1}(X_d) = O(\epsilon^2)$ (see the proof of Theorem 6). Let X_{Rd} and x_{Td} be the corresponding desired rotational and translational parts of X_d. To solve the problem (P) with $O(\epsilon^3)$ accuracy, we derive an algorithm such that $X^{(3)}(t_f) = X_d$ and apply Theorem 5.

To simplify our task, we solve the translational part of the problem first and then the rotational part. Recall that $X(t) = \exp(Z(t))$ where we can express $Z(t) = \sum_{p=1}^{6} d_p(t) A_p$. Similarly, $X_d = \exp(Z_d)$ where we can express $Z_d = \sum_{p=1}^{6} d_{dp} A_p$. Thus, the translational part of the problem is to specify controls such that $\bar{d}_p(t_f) = d_{dp},\ p = 4, 5, 6$, and the rotational part of the problem is to specify controls such that $\bar{d}_p(t_f) = d_{dp},\ p = 1, 2, 3$. It can be seen from (58) that we need the third-order ($O(\epsilon^3)$) averaging formula of Theorem 5 to solve the translational part of the problem (otherwise, $\bar{d}_5(t)$ would be identically zero). However, we only need the second-order ($O(\bar{\epsilon}^2)$) averaging formula to solve the rotational part of the problem where we take $\bar{\epsilon} = \epsilon^{3/2}$. In fact, for the rotational problem we can use an algorithm similar to the one used for the attitude control problem of [6] which uses Theorem 2 and the Wei-Norman parametrization. This is because of our choice of basis $\{A_1, \ldots, A_n\}$. The Wei-Norman equations for $G = SE(3)$ with our chosen basis are

$$\begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \\ \dot{g}_4 \\ \dot{g}_5 \\ \dot{g}_6 \end{bmatrix} = \begin{bmatrix} secg_2cosg_3 & -secg_2sing_3 & 0 & 0 & 0 & 0 \\ sing_3 & cosg_3 & 0 & 0 & 0 & 0 \\ -tang_2cosg_3 & tang_2sing_3 & 1 & 0 & 0 & 0 \\ 0 & -g_6 & g_5 & 1 & 0 & 0 \\ g_6 & 0 & -g_4 & 0 & 1 & 0 \\ -g_5 & g_4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \tag{59}$$

and it can be seen that the parameters g_1, g_2, g_3, which parametrize the orientation of the vehicle, correspond to the three parameters in the attitude control problem. The parameters g_4, g_5, g_6 parametrize the position of the vehicle, and it is easy to show that $d_i = g_i$, $i = 4,5,6$. So the desired orientation X_{Rd} is expressed in Wei-Norman parameters (g_{d1}, g_{d2}, g_{d3}) and the desired position x_{Td} is expressed identically in Wei-Norman and single-exponential parameters $(g_{d4}, g_{d5}, g_{d6}) = (d_{d4}, d_{d5}, d_{d6})$. It is additionally assumed that it is desired that u_i, $i = 1,2,4$ be continuous and $u_i(0) = u_i(t_f) = 0$, $i = 1,2,4$.

The following algorithm uses sinusoidal controls and has been derived according to the geometric reasoning outlined in the proof of Theorem 6 (for the translational part of the problem) and the proof of Theorem 4 [6] (for the rotational part of the problem). The time interval $[0, t_f]$ is divided into subintervals $[t_i, t_j]$ such that $t_1 = \frac{\pi}{2\omega} = \frac{T}{4}$, $t_2 = t_1 + qT$, $t_3 = t_2 + rT$, $t_4 = t_3 + \Delta_4 T$, $t_5 = t_4 + \frac{T}{4}$, $t_6 = t_5 + \frac{T}{2}$, $t_7 = t_6 + \frac{T}{4}$, $t_8 = t_7 + sT$, $t_9 = t_8 + \frac{T}{4}$, $t_{10} = t_f = t_9 + \frac{T}{2}$ where q, r, s and Δ_4 are described below. The algorithm defines the controls as follows:

$$\epsilon u_1(t) = \begin{cases} 0 & 0 \le t \le t_1 \\ -a_1\omega sin(\omega(t - t_1)) & t_1 < t \le t_2 \\ 0 & t_2 < t \le t_7 \end{cases}$$

$$\bar{\epsilon} u_1(t) = \begin{cases} -a_4\omega sin(\omega(t - t_7)) & t_7 < t \le t_8 \\ 0 & t_8 < t \le t_9 \\ \frac{1}{2} g_{d1}\omega sin(\omega(t - t_9)) & t_9 < t \le t_{10} \end{cases}$$

$$\epsilon u_2(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ a_2 \omega sin(\omega(t-t_1)) & t_1 < t \leq t_2 \\ a_3 \omega sin(\omega(t-t_2)) & t_2 < t \leq t_3 \\ 0 & t_3 < t \leq t_6 \end{cases}$$

$$\bar{\epsilon} u_2(t) = \begin{cases} g_{d_2} \omega sin(\omega(t-t_6)) & t_6 < t \leq t_9 \\ -\frac{1}{2} g_{d_2} \omega sin(\omega(t-t_9)) & t_9 < t \leq t_{10} \end{cases}$$

$$\epsilon u_4(t) = \begin{cases} d_{d4} \omega sin(\omega t) & 0 \leq t \leq t_5 \\ (d_{d4} \omega cos(\omega \Delta_4 T)) cos(\omega(t-t_4)) & t_4 < t \leq t_5 \\ -\frac{1}{2}(d_{d4} \omega cos(\omega \Delta_4 T)) sin(\omega(t-t_5)) & t_5 < t \leq t_6 \\ 0 & t_6 < t \leq t_{10} \end{cases}$$

Positive integers q, r and s and $a_1, a_2, a_3, a_4 \in \Re$ are selected according to the following rules:

$$a_1 a_2 q = \frac{d_{d5}}{d_{d4}\pi}. \tag{60}$$

$$a_3 r = \frac{d_{d6}}{d_{d4}\pi} - a_2 q. \tag{61}$$

$$a_2^2 q + a_3^2 r = \delta \leq \frac{1}{\pi}. \tag{62}$$

$$a_4 s = \frac{g_{d3}}{\pi g_{d_2}}. \tag{63}$$

Then Δ_4, the period of oscillation T and the frequency ω can be computed as

$$\Delta_4 = \frac{sin^{-1}\{-\pi(a_2^2 q + a_3^2 r)\}}{2\pi},$$

$$T = \frac{t_f}{(q + r + s + \Delta_4 + 2)},$$

$$\omega = \frac{2\pi}{T}.$$

As an example, we use this algorithm to translate and orient an autonomous underwater vehicle to a desired position corresponding to $d_{d4} = 0.2$, $d_{d5} = 0.05$ and $d_{d6} = 0.2$ and a desired orientation corresponding to $g_{d_1} = 0.05$ radians, $g_{d_2} = 0.05$ radians and

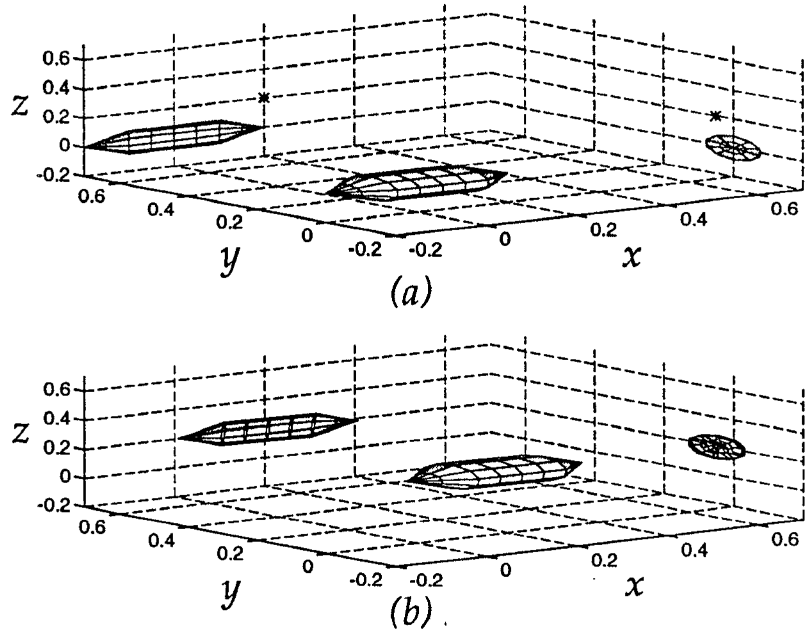

Figure 1: Vehicle in (a) Initial and (b) Desired Position and Orientation.

$g_{d3} = 0.04$ radians in $t_f = 29.2$ units of time. Figure 1(a) shows the vehicle in its initial position and orientation (at the identity of $SE(3)$), and Figure 1(b) shows the vehicle in its desired position and orientation. In each of these figures, the vehicle is shown together with its projected image on the x-z plane (at $y = 0.7$) and on the y-z plane (at $x = 0.7$). The * symbols indicate the desired position of the center of the vehicle projected onto the x-z and y-z planes.

This example is intended to represent one step in a multi-step maneuver. As discussed earlier feedback could be used between steps to improve accuracy. We also note that in using the algorithm above, there is a great deal of flexibility in choosing the constants according to the rules (60) - (63). In this example, the constants were chosen to keep the frequency of the control signals, ω, relatively low. In particular, we used $q = 4$, $r = 4$, $s = 4$, $a_1 \approx 0.11$, $a_2 \approx 0.18$, $a_3 \approx -0.10$, and $a_4 \approx 0.06$ such that $\Delta_4 = 7/12$, $T = 2$ and $\omega = \pi$. Additionally, $\epsilon = 0.2$ and $\bar{\epsilon} = \epsilon^{3/2} \approx 0.09$.

Figure 2 shows the three controls u_1, u_2 amd u_4, as a function of time. Figure 3 shows

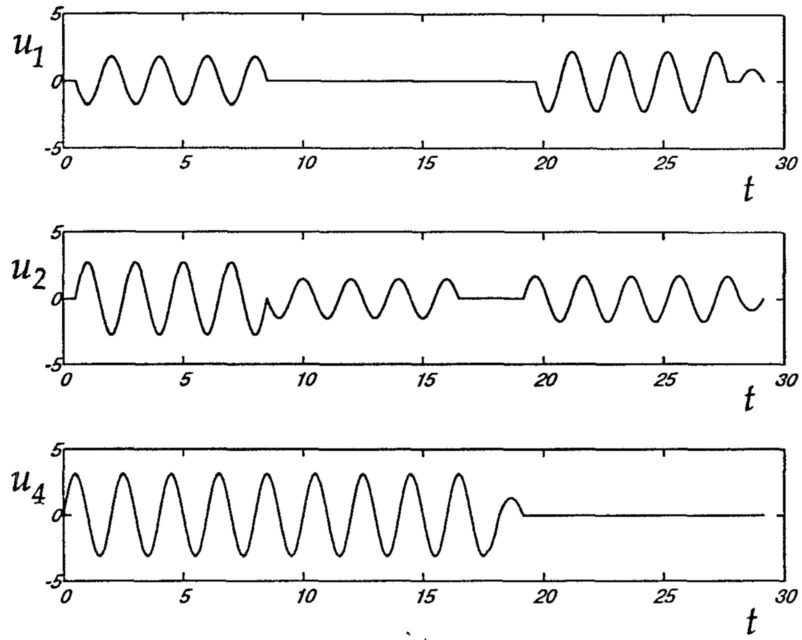

Figure 2: Control Input Signals for Example.

a simulation of the response of the system (solid lines). The simulation was produced by numerically solving the Wei-Norman equations (59) using MATLAB^R. The orientation of the vehicle is given in Figures 3(a), 3(b) and 3(c) which show plots of g_1, g_2, g_3, respectively. The position of the vehicle is given in Figures 3(d), 3(e) and 3(f) which show plots of d_4, d_5, d_6, respectively. The dashed lines represent the corresponding average values of the parameters as a function of time computed directly from the average formula. It is clear, by comparing the solid lines to the dashed lines in Figure 3, that at the end of the simulation $\|X(t_f) - X_d\| = O(\epsilon^3)$, i.e., the vehicle has been moved and oriented as desired with $O(\epsilon^3)$ accuracy.

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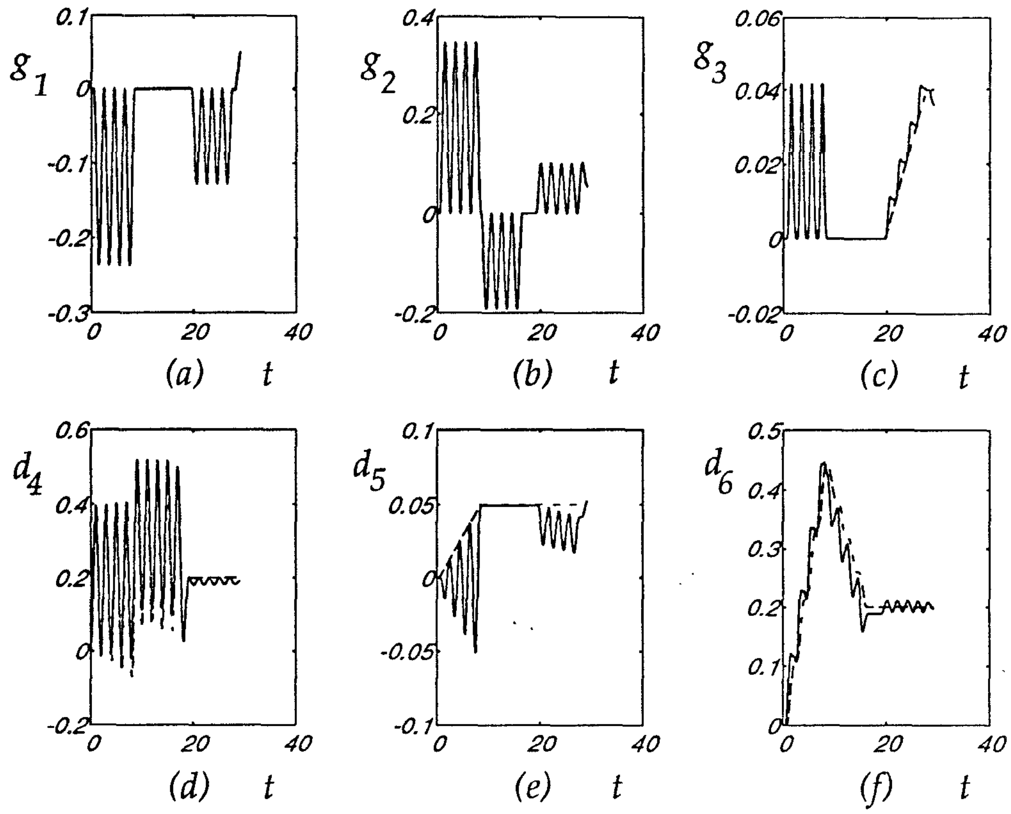

Figure 3: Actual (solid lines) and Average (dashed lines) Parameters for Example.

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