HYPERCONFORMAL TRANSFORMATIONS

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HYPERCONFORMAL TRANSFORMATIONS.

1. Introduction.

An outstanding feature of recent developments in Quantum Mechanics and the Theory of Relativity is the important role played by the element of arc. The possibilities of the quadratic form with coefficients dependent upon the properties of the space in question have received exhaustive consideration. On the other hand so far as the author is aware, nothing has been done in the way of investigating the possibilities of other forms. One purpose of the present paper is to consider the characteristics in space of n dimensions of an n-ic form in the differentials of the coordinates which may be resolved into n linear factors.

As a preliminary justification of the concept just indicated it may be well to point out some of the more important consequences entailed. Liouville has shown that the transformations which preserve angles in space of three dimensions are trivial, being comprised under translations, rotations, expansions, reflections and inversions. An intimately related fact is the absence of a concept analogous to the complex variable in space of three dimensions. These remarks apply to hyperspace in general. In an attempt to construct hyperspace analogs of plane conformal transformations one is naturally led to assume a composite differential form of arc element. Indeed all plane transformations which preserve angles in magnitude and sense are completely characterized by admitting as relative invariants the factors of the quadratic element of arc. This requirement entails the preservation of the isotropic lines and the absolute. The introduction of a composite n-ic form as the element of arc in space of n dimensions leads at once to the generalization of these concepts. Transformations exist which admit the factors of the arc element as relative invariants and preserve a degenerate absolute of class n. These transformations will be shown to have all the properties of plane conformal transformations. We shall call them hyperconformal transformations.

In regard to the extension of the complex variable concept, it may be remarked that in the work of Volterra (1) and Rainich (2) certain group properties of the analytic function have been sacrificed. The plan just outlined entails the preservation of all of these properties. The consequent breakdown of the systems of functions considered into a set of analytic functions of ordinary complex variables is not at all surprising. Indeed this very circumstance affords a striking demonstration of the unique character of such analytic functions.

In the above sketch we have confined our attention to considerations of affine geometry. This is in keeping with the purpose of the present paper. The concepts in question admit of projective generalization by the introduction of a non-degenerate absolute of class n in space of n dimensions. It is the intention of the author to make these considerations the subject of a subsequent paper.

11. Formulation of the Problem.

We define the element of arc by the equation;

$$ds^n = \prod_{k=1}^{n} du^k \tag{1}$$

where;

$$u^k = \sum_{j=1}^{n} a_j^k x_j \qquad |a_j^k| \neq 0 \qquad (k = 1 \ldots n)$$

We define an isotropic hyperplane to be one on which the element of arc (1) vanishes. Thus there are n isotropic hyperplanes defined by the equations;

$$u^k = 0 \qquad (k = 1 \ldots n)$$

The transformations in question may be defined by the equations;

$$\bar{u}^k = F^k(u^k) \qquad (k = 1 \ldots n)$$

where the functions $F^k(u^k)$ are analytic and single valued and;

$$\bar{u}^k = \sum_{j=1}^{n} a_j^k \bar{x}_j \qquad (k = 1 \ldots n)$$

The x_j being the coordinates of a generic point x and the $\bar{x}_j$ those of the transform-point which is uniquely determined under the conditions imposed. The transformations in question are completely characterized by the requirement that the forms du^k be relative invariants, that is;

$$d\bar{u}^k = \lambda^k(x_1, x_2, \ldots x_n) du^k \quad (k = 1..n)$$

The necessary conditions are;

$$(2) \quad \frac{1}{a_1^k}\frac{\partial \bar{u}^k}{\partial x_1} = \cdots = \frac{1}{a_j^k}\frac{\partial \bar{u}^k}{\partial x_j} = \cdots \frac{1}{a_n^k}\frac{\partial \bar{u}^k}{\partial x_n} \quad (k = 1...n)$$

Conversely, any transformation of the form;

$$\bar{x}_j = f_j(x_1, x_2 \ldots x_n) \quad (j = 1...n)$$

where the $\bar{x}_j$ satisfy equations (2) is a hyperconformal transformation admitting the isotropic hyperplanes;

$$u^k = 0 \quad (k = 1...n)$$

For each of the above ratios is readily seen to be equal to; $\dfrac{d\bar{u}^k}{du^k}$

and may be set equal to an arbitrary function of u^k. Equations (2) are the necessary and sufficient conditions for the existence of the derivatives;

$$\frac{d\bar{u}^k}{du^k} \qquad (k=1...n)$$

They may be viewed as a generalization of the Cauchy-Riemann conditions.

The analogy of the transformations in question with plane conformal transformations consists in the preservation of n-1 cross ratios which will be taken as measures of angles. We proceed to demonstrate the existence of these invariants. From equations (2), we have;

$$d\bar{u}^k = \frac{1}{a^k_s}\frac{\partial \bar{u}^k}{\partial x_s} du^k \qquad (k=1...n) \tag{3}$$

where a^k_s is any non-vanishing coefficient of u^k. Now the ratios;

$$R^k = \frac{du^k}{du^n} \qquad (k=1..n-1)$$

define the linear element dx at the point x. The transformation induces a one to one correspondence between the linear elements at x and the linear elements at $\bar{x}$. We denote by $d\bar{x}$ the linear element at $\bar{x}$ corresponding to dx and by $\bar{R}^k$ the ratios which define it. From equations (3) we have;

$$\frac{\bar{R}^k}{\bar{R}^k} = \frac{a^n_r}{a^k_s} \frac{\frac{\partial \bar{u}^k}{\partial x_s}}{\frac{\partial \bar{u}^n}{\partial x_r}} \qquad (k = 1 \ldots n-1)$$

$$a^k_s \neq 0 \qquad a^n_r \neq 0$$

The right members of the last equations are functions of position only. Hence if we consider a second linear element at x defined by the ratios S^k and the corresponding linear element at $\bar{x}$ defined by the ratios $\bar{S}^k$, we have;

$$\bar{R}^k : R^k = \bar{S}^k : S^k \qquad (k = 1 \ldots n-1)$$

The cross ratios; $R^k : S^k$ are therefore absolute invariants under the transformation. To give a geometrical interpretation of this invariance, we consider the pencils of hyperplanes;

$$u^k = \lambda^k u^n \qquad (k = 1 \ldots n-1)$$

The linear element at x defined by the ratios R^k determines a hyperplane corresponding to the value R^k of the parameter λ^k, and the linear element at x defined by the ratios S^k determines a second hyperplane corresponding to the value S^k of the parameter. Denoting these hyperplanes by V^k and W^k respectively, we have;

$$R^k : S^k = (u^k ; u^n ; V^k : W^k) \qquad (k = 1 \ldots n-1)$$

Hence the cross ratios R^{k}; S^{k} may be taken as the measures of the hyperangle between the hyperplanes V^{k} and W^{k}. The hyperangle is thus an n-1 dimensional vector which is invariant under the transformation.

We conclude our preliminary study of these transformations by two remarks of fundamental importance. First, as to the group property. Since the transformations are completely characterized by the invariance of the isotropic hyperplanes, it follows without difficulty that all transformations admitting the same isotropic hyperplanes form a group. This conclusion is readily verified by consideration of the equations;

$$\bar{\bar{u}}^{k} = F^{k}(\bar{u}^{k}) = G^{k}(u^{k}) \qquad (k=1...n)$$

Second as to the complete set of invariants. Of the $\frac{1}{2}n(n-1)$ absolute invariants at our disposal we have made an arbitrary choice of n-1. The remaining $\frac{1}{2}(n-1)(n-2)$ are dependent on these n-1. The relative invariant $\prod du^{k}$ defined by equation (1) furnishes the metric of our geometry.

111. Normalization.

We have seen that any two linear elements at the point x determine n-1 cross ratios which are invariant under the transformation. We shall take these cross ratios as measures of n-1 angles. The n-1 dimensional vector thus defined will be called the hyperangle between the two linear elements in question. In order to define these concepts with precision, we proceed to introduce a system of normal functions adapted to our metric.

We define the hypernorm of the linear element dx by the equation;

$$(4) \qquad \overline{N(dx)}^{n} = \prod \sum_{1}^{n} a_{j}^{k} \, dx_{j}$$

This linear element may be determined by n-1 arguments; $\theta_1 \ldots \theta_{n-1}$. Let us set;

$$(5) \qquad dx_{j} = N(dx) f_{j}(\theta_1 \ldots \theta_{n-1}) \qquad (j = 1 \ldots n)$$

where the functions f_j are arbitrary. We have;

$$\prod \sum a_{j}^{k} \, dx_{j} = \overline{N(dx)}^{n} \prod \sum a_{j}^{k} f_{j}.$$

If the functions f_j are so chosen that their hypernorm is unity, equations (5) will express the components of the linear element dx in terms of its hypernorm and the functions f_j . We choose the system of functions defined by the equations;

(6) $$\sum_{j=1}^{n} a_j^k f_j = e^{P_k(\theta_1 \dots \theta_{n-1})} \qquad (k=1\dots n)$$

where;

$$P_k = \sum_{j=1}^{n-1} r^{jk}\theta_j \qquad (r = e^{\frac{2\pi i}{n}}) \quad (k=1\dots n)$$

We now have for any linear element dx;

(7) $$\sum_{j=1}^{n} a_j^k dx_j = N(dx)\sum_{j=1}^{n} a_j^k f_j = N(dx)e^{P_k(\theta_1 \dots \theta_{n-1})} \qquad (k=1\dots n)$$

Consider now the ratios;

$$R^k = \frac{du^k}{du^n} \qquad (k=1\dots n-1)$$

introduced on page 6. In consequence of equations (7) we have;

$$\log R^k = P_k - P_n \qquad (k=1\dots n-1)$$

We readily find;

(8) $$n\theta_k = \sum_{j=1}^{n-1} r^{(n-k)j}\log R_j \qquad (k=1\dots n-1)$$

Now let ϕ_j be the arguments of the linear element defined by the ratios S .We define the hyperangle between the two linear elements in question to be the n-1 dimensional vector; $\theta - \phi$. The components of this hyperangle are given by the equations;

$$n(\theta_k - \phi_k) = \sum_{j=1}^{n-1} r^{(n-k)j} \log \frac{\rho_j}{\varsigma_j} \qquad (k = 1..n-1) \tag{9}$$

These equations may be viewed as an extension of the Laguerre definition of angle.

We are now in a position to state concisely the characteristic properties of a hyperconformal transformation. We denote by T the vector whose components are; $\frac{d\bar{v}^k}{dv^k}$, by $\alpha_1 \dots \alpha_{n-1}$ the arguments of T, by $\theta_1 \dots \theta_{n-1}$ the arguments of dx and by $\bar{\theta}_1 \dots \bar{\theta}_{n-1}$ the arguments of $d\bar{x}$. In consequence of equations (2),(3) and (7), we have;

$$N(d\bar{x})e^{P_k(\bar{\theta}_1 \dots \bar{\theta}_{n-1})} = N(dx)N(T)e^{P_k(\theta_1 \dots \theta_{n-1}) + P_k(\alpha_1 \dots \alpha_{n-1})} \qquad (k = 1 \dots n)$$

These equations have the solutions;

$$N(d\bar{x}) = N(dx)N(T)$$

$$\bar{\theta}_j = \theta_j + \alpha_j \qquad (j = 1..n-1)$$

The interpretation is that the transform of the element dx is the product of dx and T in the complex variable sense; that is each argument of dx is increased by the corresponding argument of T and the hypernorm of dx is multiplied by the hypernorm of T. The latter hypernorm may be termed the local magnification.

Finally we shall determine the periods of the functions f_j. The conditions necessary and sufficient that the two sets of polar coordinates;

$$(N, \theta_1, \theta_2 \ldots \theta_{n-1}) \qquad (N, \theta_1 + \beta_1, \theta_2 + \beta_2 \ldots \theta_{n-1} + \beta_{n-1})$$

determine the same point, are;

$$f_j(\theta_1, \theta_2 \ldots \theta_{n-1}) = f_j(\theta_1 + \beta_1, \theta_2 + \beta_2 \ldots \theta_{n-1} + \beta_{n-1}) \qquad (j = 1 \ldots n)$$

This necessitates that;

$$P_k(\beta_1 \ldots \beta_{n-1}) = 2N_k \pi i \qquad (k = 1 \ldots n-1)$$

and;

$$P_n(\beta_1 \ldots \beta_{n-1}) = -2\pi i \sum_{1}^{n-1} N_k$$

The solution of these equations gives;

$$\beta_k = \frac{2\pi i}{n} \sum_{j=1}^{n-1} (r^{(n-k)j} - 1) N_j \qquad (k = 1 \ldots n-1)$$

These quantities are the periods of the normal functions.

IV. Functions of a Hypercomplex Variable.

We shall call the quantities;

$$u^{k} = \sum_{j=1}^{n} r^{(2k-1)(j-1)} x_j \qquad (r = e^{\frac{\pi i}{n}}) \qquad (k=1\ldots n)$$

a complete set of hypercomplex variables in n dimensions. We consider the hyperconformal transformation defined by the equations;

$$\bar{u}^{k} = F^{k}(u^{k}) \qquad \bar{u}^{*k} = F^{k}(u^{*k}) \qquad (k=1\ldots n)$$

where the functions F^{k} are analytic, u^{*k} is the conjugate of u^{k} in the ordinary complex variable sense, and $n = 2m$. In the case of an odd number of dimensions, $n = 2m+1$, we adjoin to the above the equation;

$$\bar{u}^{m+1} = F^{m+1}(u^{m+1})$$

The relative invariant $\prod du^{k}$ takes the form;

$$D(dx_k) = \begin{vmatrix} dx_1 & -dx_n & -dx_{n-1} \ldots & -dx_2 \\ dx_2 & dx_1 & -dx_n \ldots & -dx_3 \\ dx_3 & dx_2 & dx_1 \ldots & -dx_4 \\ \ldots & \ldots & \ldots & \ldots \\ dx_n & dx_{n-1} & dx_{n-2} \ldots & dx_1 \end{vmatrix}$$

Equations (2) are equivalent to the requirement that the Jacobian determinant of the transformation have the form;

$$(10) \qquad D\left(\frac{\partial \bar{x}_k}{\partial x_1}\right)$$

This requirement is the necessary condition for the existence of the derivatives of the functions; $F^k(u^k)$.

Conversely, when this requirement is satisfied, we have;

$$\sum_{j=1}^{n} r^{(2k-1)(j-1)}\, d\bar{x}_j = \sum_{j=1}^{n} r^{(2k-1)(j-1)} \frac{\partial \bar{x}_j}{\partial x_1} \sum_{j=1}^{n} r^{(2k-1)(j-1)}\, dx_j$$

Whence;

$$\frac{dF^k(u^k)}{du^k} = \sum_{j=1}^{n} r^{(2k-1)(j-1)} \frac{\partial \bar{x}_j}{\partial x_1} \qquad (k = 1 \ldots n)$$

Hence the requirement is the necessary and sufficient condition for the existence of the derivatives of the functions $F^k(u^k)$, and may be regarded as the extension of the Cauchy-Riemann conditions in the theory of an ordinary complex variable.

The subgroup for which all the functions F^k are identical may be generated in a manner precisely similar to that employed in the case of plane conformal transformations. Let F(u) be an analytic function of any u, and let the expansion of F(u) be arranged according to powers of r. By equating coefficients of like powers of r in the relation;

$$\bar{u} = F(u)$$

a transformation is obtained which has a Jacobian determinant of the form (10) as is readily verified on differentiation.

Any transformation of the type just considered may be replaced by a set of plane conformal transformations in space of an even number of dimensions with the addition already indicated in case the number of dimensions is odd. The rotation defined by the equations;

$$\sqrt{n}\,(y_{2k-1} + iy_{2k}) = \sqrt{2}\,u^{k} \qquad (k = 1\ldots m)$$
$$(n = 2m)$$

with the addition of the equation;

$$\sqrt{n}\,y_{m+1} = u^{m+1}$$

when $n = 2m + 1$, permits to replace the complete set of hypercomplex variables by a set of ordinary complex variables. Any hyperconformal transformation of the type just considered is therefore the product of this rotation and a suitable set of plane conformal transformations. This conclusion while of a decidedly negative character is not without value as evidencing the unique character of the analytic function.

From the above considerations it is evident that the zeros of an analytic function of a hypercomplex variable are arranged in axes. The same remark applies to the poles. The entire theory of such functions is an immediate extension of the theory of functions of an ordinary complex variable. For integrals taken around a closed variety of one dimension we have;

$$\int_C F(u)du = 0$$

where the contour C does not encircle any axis of poles. The extension of the Cauchy integral formula is;

$$2\pi iF(a) = \int_C \frac{F(u)du}{u-a}$$

Where the closed contour C encircles the axis a and F(a) is the value of F(u) at any point of the axis.

V. Special cases.

We shall now give two examples of hyperconformal transformations the first being in space of three dimensions. We select the transformations for which the arc element is defined by the equation;

$$ds^3 = C(dx,dy,dz) = \begin{vmatrix} dx & dz & dy \\ dy & dx & dz \\ dz & dy & dx \end{vmatrix}$$

the restrictions imposed by equations (2) are in this case equivalent to the requirement that the Jacobian determinant of the transformation be of the circulant form;

$$C\left(\frac{\partial\bar{x}}{\partial x}, \frac{\partial\bar{y}}{\partial x}, \frac{\partial\bar{z}}{\partial x}\right)$$

and we have;

$$d\bar{s}^3 = C\left(\frac{\partial\bar{x}}{\partial x}, \frac{\partial\bar{y}}{\partial x}, \frac{\partial\bar{z}}{\partial x}\right)ds^3$$

The isotropic planes are defined by the equations;

$$x + w^k y + w^{2k} z = 0 \qquad (w = e^{\frac{2\pi i}{3}}) \qquad (k = 1,2,3)$$

Such transformations may be generated from the equation;

$$\bar{x} + w\bar{y} + w^2\bar{z} = F(x + wy + w^2 z)$$

by equating coefficients of like powers of w on the two sides. For example the equation;

$$\bar{x} + w\bar{y} + w^2\bar{z} = (x + wy + w^2 z)^2$$

gives rise to the transformation;

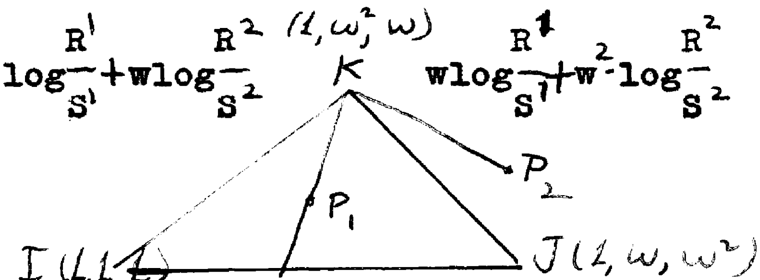
$$\bar{x} = x^2 + 2yz \qquad \bar{y} = z^2 + 2xy \qquad \bar{z} = y^2 + 2xz$$

The normal functions for three dimensions are defined by the equations;

$$f_1(\theta,\phi) + w^k f_2(\theta,\phi) + w^{2k} f_3(\theta,\phi) = e^{w^k\theta + w^{2k}\phi} \qquad (k = 1,2,3)$$

The hyperangle between the linear elements defined by; R^1 R^2 and S^1 S^2 has for components;

$$w^2 \log\frac{R^1}{S^1} + w\log\frac{R^2}{S^2} \qquad w\log\frac{R^1}{S^1} + w^2\log\frac{R^2}{S^2}$$



The isotropic ~~hyper~~planes determine on the plane at infinity a degenerate absolute, the triangle IJK. Denoting by P_1 P_2 the traces of the linear elements above indicated, we have;

$$\frac{R^1}{S^1} = (IJ:JK:JP_1:JP_2) \qquad \frac{R^2}{S^2} = (KI:KJ:KP_1:KP_2)$$

In space of four dimensions we consider the transformations for which;

$$ds^4 = D(dx_1 \,..\, dx_4) = \begin{vmatrix} dx_1 & -dx_4 & -dx_3 & -dx_2 \\ dx_2 & dx_1 & -dx_4 & -dx_3 \\ dx_3 & dx_2 & dx_1 & -dx_4 \\ dx_4 & dx_3 & dx_2 & dx_1 \end{vmatrix}$$

The conformality conditions are equivalent to the requirement that the Jacobian determinant have the form;

$$D\left(\frac{\partial \bar{x}_1}{\partial x_1} \; .... \; \frac{\partial \bar{x}_4}{\partial x_1} \right)$$

The isotropic hyperplanes are defined by the equations;

$$\sum_{j=1}^{4} r^{\sigma} x_j = 0 \qquad (r = e^{\frac{\pi i}{4}}) \qquad (k = 1..4)$$

$$\sigma = (2k-1)(j-1)$$

Such transformations may be generated from the equation;

$$\sum r^{\sigma} \bar{x}_j = F\left(\sum r^{\sigma} x_j \right)$$

by equating coefficients of like powers of r.

VI. Conclusion.

A. If in space of n dimensions we confine ourselves to a metric defined by a quadratic form in n-variables, $n > 2$, we can obtain only conformal transformations of the trivial kind.

B. If we consider an n-ic form in the n differentials a metric is defined which admits transformations for which;

$$d\bar{s}^n = \lambda(x_1, x_2, x_3 \ldots x_n) ds^n$$

These transformations are not trivial.

C. If in particular, we take the case where;

$$ds^n = \prod_{i=1}^{n} (a^{\bar{i}} dx)$$

we may by a suitable transformation of coordinates view the "a"s as the nth. roots of unity.

D. Transformations of this class may be derived from the equation;

$$\breve{z} = F(z)$$

where F is an arbitrary analytic function and;

$$z = \sum_{j=1}^{n} r^{j-1} x_j \qquad \bar{z} = \sum_{j=1}^{n} r^{j-1} \bar{x}_j \qquad r = e^{\frac{2\pi i}{n}}$$

The equations of the transformation are obtained by identifying the coefficients of like powers of r.

E. It is possible to define an (n-1)-dimensional vector which may be viewed as a hyper-angle, the vector reducing to a scalar for $n = 2$, in which case we obtain the Laguerre definition of angle. Any two linear elements determine such a hyperangle. The transformation; $\bar{z} = F(z)$ defines at the transformed point the two transformed elements which admit the same hyperangle.

F. The Cauchy-Riemann-Weierstrass theory of analytic functions, their singularities, residues and contour integrals may be readily extended to the case of functions of a hypercomplex variable.

G. For any continuous point transformation in S_n we have;

$$d\bar{x}_k = \sum_{j=1}^{n} p_j^k dx_j \quad \left(p_j^k = \frac{\partial \bar{x}_k}{\partial x_j}\right) \qquad (k = 1\ldots n) \tag{11}$$

Denoting by y the trace of the linear element dx on the hyperplane at infinity, these equations become;

$$\bar{y}_k = \sum_{j=1}^{n} p_j^k y_j \qquad (k = 1\ldots n) \tag{12}$$

In the case where the transformation (11) is hyperconformal, equations (12) define transformations in S_{n-1} having the same property. Hence a hyperconformal transformation in space of order n induces n-1 hyperconformal transformations in every subspace of order n-1.

For a sufficiently small region about y, the p_j^k may be regarded as constant, and equations (12) define a collineation. Now the conditions;

$$\sum_{j=1}^{n} a_j^k \, p_s^j = a_s^k \frac{d\bar{u}^k}{du^k} \qquad (k = 1\ldots n) \quad (s = 1\ldots n) \tag{13}$$

show that in the case of hyperconformal transformations the local collineations defined by equations (12) admit the common set of invariant points;

$$a^{k}\,(a^{k}_{1}, a^{k}_{2}, a^{k}_{3} \ldots\ a^{k}_{n}) \qquad (k = 1 \ldots n)$$

Conversely, if a set of points a^{k} be taken such that $|a^{k}_{j}| \neq 0$, the functions u^{k} and $\bar{u}^{k}$ are completely determined, and the equations;

$$\bar{u}^{k} = F^{k}(u^{k}) \qquad (k = 1 .. n)$$

define a group of hyperconformal transformations which preserve the points a^{k} . This property is the extension of the preservation of the circular points at infinity under a plane conformal transformation.

From equations (13) it is evident that the roots of the characteristic equation of the local collineation are the total derivatives; $\dfrac{d\bar{u}^{k}}{du^{k}}$

From a projective point of view the same considerations hold in the case where the hyperplane A of the points a^{k} is arbitrarily chosen. If the hyperplane A and the isotropic hyperplanes be taken as coordinate hyperplanes the equations of the transformation are;

$$\bar{x}_k = F^k(x_k) \qquad (k=1...n)$$

Since the product of the roots of the characteristic equation of the collineations is the Jacobian determinant;

$$J\left(\frac{\partial \bar{x}_k}{\partial x_j}\right)$$

we have for all hyperconformal transformations;

$$\pi\, d\bar{u}^k = J\left(\frac{\partial \bar{x}_k}{\partial x_j}\right) \pi\, du^k$$

It is also evident that there is a one to one correspondence between the groups of hyperconformal transformations in S_n and the collineations in S_{n-1}. The classification of the former is isomorphic with that of the latter.

H. Finally let us remark that the further generalization of conformal transformations is to be sought in the case where the absolute is a non-degenerate variety of class n. The group of transformations would then be defined by the invariance of such a variety. This is equivalent to assuming that the transformations admit as relative invariants a non-decomposable n-ic form in the differentials. We expect to make these considerations the subject of another paper.

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