TECHNICAL RESEARCH REPORT

Institute for Systems Research

Adaptive Control of Nonlinear Systems via Approximate Linearization

by R. Ghanadan and G.L. Blankenship

The Institute for Systems Research is supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

TR 93-23

Adaptive Control of Nonlinear Systems via Approximate Linearization

R. Ghanadan† and G. L. Blankenship†

February 22, 1993

Abstract

We present a direct adaptive tracking control scheme for nonlinear systems that do not have a well defined (vector) relative degree and hence are not feedback linearizable. This technique uses feedback and coordinate changes to transform a nonlinear system with parameter uncertainty into an *approximate* input-output linearized one. Our result is also applicable to *slightly* non-minimum phase nonlinear systems with unknown parameters. We prove that the presented adaptive design scheme results in an asymptotically stable closed loop system and show that the controller can achieve adaptive tracking of reasonable trajectories with bounds on the tracking error. We also present a state regulation scheme based on state approximate linearization. We demonstrate the adaptive approximate tracking results using a simplified model of an aircraft which is slightly non-minimum phase. The usefulness of our approach is also illustrated on a "benchmark" example that is not feedback linearizable.

I. Introduction

Over the last decade, geometric nonlinear control theory has provided powerful tools for systematic design of nonlinear feedback systems [Isi89, Nvds90]. Most of the available methods for nonlinear tracking control system design are based on linearizing the input-output response of a nonlinear system using state feedback, or exact state linearization using a coordinate change $z = T(x)$ and a state feedback [HSM83, Isi89, Nvds90]. Major limitations to these approaches come from the fact that they require certain regularity conditions such as involutivity, existence of a (vector) relative degree, minimum phase property, and that they rely on exact cancellation of nonlinear terms. Alternatively, there have been several successful approaches that aim to *approximately* linearize a nonlinear system by relaxing one or more of these restrictions. Krener [Kre84, Kre86], Karahan [Kar88], and co-workers

†Institute for Systems Research, University of Maryland, College Park, MD 20742, E-mail: rezag@src.umd.edu, gilmer@eng.umd.edu. This work was supported in part by U.S. Army TAcom under order No. TCN92 − 457 with Battelle and in part by the National Science Foundation under Grant NSFD CDR-8803012.

[KKHF87, KK92] first introduced the approximate linearization approach as an alternative to exact state linearization. This approach has recently been successfully applied to chemical reactor control problem for non-involutive systems [DM90]

Using Taylor expansion, a nonlinear system can always be approximated by a linear system to first degree. However, the idea here is to approximate a nonlinear system up to the highest degree possible. This will increase the validity of the approximation to the highest order possible with small error terms, causing minimal performance degradation that may be ignored in some neighborhood of the equilibrium. In the extreme case, there is no error term and exact state linearization is achieved. Other schemes include *extended linearization* introduced by Baumann and Rugh [BR86] and Rugh [Rug86], *pseudolinearization* by Champetier and Reboulet [CRM84, RC84], and recently by Wang and Rugh [WR89], and *uniform system approximation* by Hauser [Hau91] and co-workers [HSK92]. A survey on general applicability and properties of these approximate linearization approaches vs. exact linearization technique for chemical reactor control was reported in [DM92]. In [HSM89, HSK92], it was shown that one can approximately (input-output) linearize a nonlinear system and design a stable *approximate* tracking controller under much weaker conditions than those needed for tracking design based on *exact* feedback linearization schemes.

A major deficiency, however, in design schemes based on exact or approximate feedback linearization is caused by *parametric uncertainty* in the system dynamics where exact cancellation of nonlinearities may not be possible. Parameter adaptive control theory [NA88, SI89, TKMK89, KKM91b, TKKS91, BS91, KKM91a] has offered a promising approach to compensate for this parameter mismatch problem. Unfortunately, the available geometric nonlinear adaptive control schemes are based on exact feedback linearization theory and suffer from the same limitations due to stringent regularity conditions required for exact feedback linearization. In this paper, we attempt to extend parameter adaptive schemes developed for feedback linearizable systems to *approximate* linearizable ones and hence, avoid several restrictions that limit the general applicability of these schemes.

We first review the approximate linearization technique for nonlinear systems. In section III, we present the design procedures for adaptive input-output approximate linearization and tracking. The adaptive regulation counterpart using approximate state linearization is addressed in section IV, and a systematic design procedure for adaptive *quadratic* linearization is presented. In section V, we apply our adaptive scheme to a simplified model of the Harrier aircraft studied in [HSM89] which is not a minimum phase system. We also show the usefulness of our adaptive scheme on a "benchmark" example of adaptive control design case where the system is not feedback linearizable and can not be transformed into the so-called "parametric-pure-feedback form."

II. Review of Approximate Linearization

Consider the following nonlinear system:

$$\dot{x}(t) \quad = \quad f(x) + \sum_{1}^{m} g_i(x) \cdot u_i \tag{2.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, and $f(0) = 0$. Krener [Kre84] gave necessary and sufficient conditions for the existence of transformations:

$$\begin{aligned} z &= z(x) \\ v &= v(x, u) = \alpha(x) + \beta(x) \cdot u \end{aligned} \tag{2.2}$$

which transforms the nonlinear system (2.1) into an *approximate* linear system:

$$\begin{aligned} \dot{z} &= Az + Bv + O(x, u)^{\rho+1} \\ A &= \frac{\partial f}{\partial x}(0) \ , \ B = g(0) \end{aligned} \tag{2.3}$$

with an error of order $\rho + 1$, where $\rho \geq 1$ is the order of approximation[1]. This nonlinear transformation can then be followed by linear transformations of the states to obtain any canonical form representation of (2.3), such as the Brunovsky form, at the cost of loosing the physical significance of the states.

Theorem 2.1 ([Kre84]) *The nonlinear system (2.1) can be transformed into order ρ linear system (2.3) where (A, B) is a controllable pair with controllability indices $k_1 \geq \ldots \geq k_n$ iff:*

(i). Distribution D^k has an order ρ local basis at 0 consisting of:

$$\left\{ ad_f^l g_j : 0 \leq l < min(k_j, k); j = 1, \ldots, m \right\}$$

(ii). $D^{k_j - 1}$ is order ρ involutive at 0 for $j = 1, \ldots, m$.

Note that it is always possible to find such transformation where $\rho = 1$ using the Taylor expansion.

When an output is specified and control objective is output tracking for the system of the form:

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x) \cdot u \\ y(t) &= h(x(t)) \end{aligned} \tag{2.4}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $h \in \Re^p$ is the output with $f(0) = 0, h(0) = 0$, one seeks input-output linearization scheme [Isi89, Nvds90] in order to linearize the input-output map $v \to y$. To simplify the notation consider the SISO case. After taking subsequent derivatives of the output until the control appears one gets:

$$y^{(r)} = \mathcal{L}_f^r h(x) + u \cdot \mathcal{L}_g \mathcal{L}_f^{r-1} h(x)$$

for some $r > 0$. Then if the relative degree r of this system is well-defined in a neighborhood $U_\epsilon\ (0)$, i.e. $\forall x \in U_\epsilon\ (0)$:

$$\begin{aligned} \mathcal{L}_g \mathcal{L}_f h_{i-1}(x)(x) &= 0 \quad \forall i < r \\ \mathcal{L}_g \mathcal{L}_f h_{r-1}(x)(x) &\neq 0 \end{aligned} \tag{2.5}$$

[1]Recall that a function $\psi(z)$ is $O(z)^n$ if $lim_{|z| \to 0} \frac{|\psi(z)|}{|z|^n}$ exists and is not zero. $O(z)^0$ is referred to as $O(1)$.

the input-output linearization is achieved by applying the following control law:

$$u = \frac{v - \mathcal{L}_f^r h(x)}{\mathcal{L}_g \mathcal{L}_f h_{r-1}(x)} \tag{2.6}$$

However if the nonlinear system (2.4) does not have a well-defined relative degree in a neighborhood of the nominal operating point of interest, control law (2.6) is not feasible. In this case, one can proceed with *approximate* input-output linearization scheme, introduced by Hauser [HSK92], to seek a smooth function $\phi_1(x)$ that approximates output y:

$$y = h(x) = \phi_1(x) + \psi_0(x)$$

where ψ_0 is of second or higher order with respect to the equilibrium manifold. An *approximate* input-output linearized system is obtained by ignoring the second or higher order terms in subsequent Lie derivatives of the approximate output ϕ_1.

Definition 2.1 (Robust Relative Degree) A nonlinear system (2.4) has a *robust relative degree* of γ about $x = 0$ if there exists smooth functions $\phi_i(x)$, $i = 1, \ldots, \gamma$ such that:

$$\begin{aligned} h(x) &= \phi_1(x) + \psi_0(x) \\ \mathcal{L}_{f+gu}\phi_i(x) &= \phi_{i+1}(x) + \psi_i(x,u) \qquad i = 1, \ldots, \gamma - 1 \\ \mathcal{L}_{f+gu}\phi_\gamma(x) &= \tilde{b}(x) + \tilde{a}(x) \cdot u + \psi_\gamma(x,u) \end{aligned} \tag{2.7}$$

where functions $\psi_i(x,u), i = 0, \ldots, \gamma$ are $O(x,u)^2$ and $\tilde{a}(x)$ is $O(1)$. Also $\psi_i(x,u) = O(x)^2 + O(x)^1 \cdot u$.

Definition 2.2 (Uniformly Higher Order) A function $\psi : \Re^n \times \Re \to \Re$ is said to be uniformly higher order on $U_\epsilon \times B_\sigma \subset \Re^n \times \Re, \epsilon > 0$, if for some $\sigma > 0$, there exists a monotone increasing function of $\epsilon, K(\epsilon)$ such that:

$$|\psi(x,u)| \leq \epsilon K(\epsilon)(|x| + |u|) \forall x \in U_\epsilon \ , |u| \leq \sigma \tag{2.8}$$

Remark 2.1 As shown in [HSK92], the robust relative degree of system (3.1) is equal to the relative degree of its Jacobian linearization. Moreover, the functions $\xi_i(x) = \mathcal{L}_{f(x)}^{i-1} h(x), i = 1, \ldots, \gamma$ are independent in a neighborhood of the equilibrium x_e.

The advantage of this scheme is that if the nonlinear system (2.4) does not have a well-defined realive degree but it is linearly controllable we can *approximate* (2.4) with an input-output linearized one. Let's say (2.4) has a singular point x_s *i.e.* $\mathcal{L}_g \mathcal{L}_f^{r-1} h(x_s) = 0$ [HD87], but that it has a robust relative degree of γ in U_ϵ (x_s). Consider the following two local diffeomorphism $\Phi(x)$ of $x \in \Re^n$:

$$\begin{aligned} (\xi^T, \eta^T)^T &= (\xi_i = \mathcal{L}_f^{i-1} h(x), i = 1, 2, \ldots, r, \quad \eta_1, \ldots, \eta_{n-r})^T \\ (\tilde{\xi}^T, \tilde{\eta}^T)^T &= (\xi^T, \tilde{\xi}_i = \mathcal{L}_f^{i-1} h(x), i = r+1, \ldots, \gamma, \quad \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-\gamma})^T \end{aligned} \tag{2.9}$$

We have for the *true system* when $x \in U_\epsilon(x_s)$:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} + \psi_{r-1}(x) \cdot u \\
\dot{\tilde{\xi}}_{r+1} &= \tilde{\xi}_{r+2} + \psi_r(x) \cdot u \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma + \psi_{\gamma-2}(x) \cdot u \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x) \cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned}
\tag{2.10}
$$

where $\psi_i(x)$ are $O(x)^1, \tilde{a}(x) \triangleq \mathcal{L}_g \mathcal{L}_f^{\gamma-1} h(x), \tilde{b}(x) \triangleq \mathcal{L}_f^\gamma h(x)$, and $\tilde{a}(x_s)$ is $O(1)$. The approximate system is:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x) \cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned}
\tag{2.11}
$$

This represents an *approximate* input-output linearized description of the true system (2.4) obtained by neglecting some high order terms in some neighborhood U_ϵ of the singular state x_s (*i.e.* $x \in U_\epsilon(x_s)$). When system (2.4) is operating in U_ϵ , where (2.11) is a valid approximation, one may design a feedback control law to achieve approximate output tracking [HSK92]. The control law will, in fact, be the exact tracking control law using the approximate description (2.11). With the above notation in mind, we say (2.4) is slightly non-minimum phase if the true system, described by (2.10), is non-minimum phase but its approximate linearization, described by (2.11) is minimum phase [HSM89].

Approximate Tracking is achieved by choosing the control law u:

$$u = \frac{1}{\tilde{a}(\xi, \eta)}[-\tilde{b}(\xi, \eta) + v] \tag{2.12}$$

with:

$$v = y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \tilde{\xi}_\gamma) + \ldots + \alpha_0(y_d - \tilde{\xi}_1) \tag{2.13}$$

where α_i are chosen so that $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \ldots + \alpha_0$ is a Hurwitz polynomial. Thus the control law u in (2.12) approximately linearizes the system (2.4) from input v to the output y up to the order ϵ (say $O(x, u)^2$).

Theorem 2.2 ([HSK92]) *Let U_ϵ , be a family of operating envelopes and suppose that the zero dynamics of the approximate system are exponentially stable, $\tilde{q}$ is Lipschitz in $\tilde{\xi}$ and $\tilde{\eta}$ on $\Phi(U_\epsilon)$ for each ϵ, and the functions $\psi_i(x, u)$ are uniformly higher order on $U_\epsilon \times B_\sigma$. Then, for ϵ sufficiently small and for desired trajectories with sufficiently small values and* *derivatives* $(y_d, \dot{y}_d, \ldots, y_d^{(\gamma)})$*, the states of the closed loop system and control (2.12) will remain bounded and the tracking error will be* $O(\epsilon)$.

The approximate feedback linearization results of theorems (2.1) and (2.2) are clear design alternatives to the more restrictive schemes of exact feedback linearization approach. These results have already been applied to the design of chemical engineering systems [DM90, DM92], and automatic flight control systems [HSM89]. In the next two sections, we present a direct adaptive tracking and adaptive regulation scheme for nonlinear systems that are *approximately* feedback linearizable in the sense of [Kre84, Hau91] and hence, are subject to milder involutivity restrictions, and are not necessarily minimum phase with a well-defined (vector) relative degree as assumed in most current adaptive control strategies for nonlinear systems.

In this paper, for notational consistency, we use $O(x,u)^\rho$ to denote a uniformly higher order function of the form $O(x)^\rho + O(x)^{\rho-1} \cdot u$.

III. Adaptive Tracking

Consider a SISO nonlinear system of the form (2.4) under parameter uncertainty:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + g(x,\theta) \cdot u \\ y(t) &= h(x,\theta) \end{aligned} \tag{3.1}$$

where $x \in \Re^n$ is the state, $u \in \Re$ is the input, $y \in \Re$ is the output, $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, f, g, and h are smooth functions on $\Re^n$. We assume (3.1) has relative degree r around the equilibrium x_e, but not necessarily has a well defined relative degree at x_e. We further assume system (3.1) has a *robust relative degree* γ in $U_\epsilon(x_e)$:

Assumption 3.1 (Relative Degree) *System (3.1) has a robust relative degree of* γ *on* $U_\epsilon(x_e)$ *, an open neighborhood of the equilibrium point* x_e*. i.e.* $\forall x \in U_\epsilon(x_e)$, $\forall \hat{\theta} \in U_\sigma(\theta)$:

$$\begin{array}{ll} \mathcal{L}_{g(x)}\mathcal{L}^i_{f(x)}h(x) = 0 & i = 0, \cdots, r-2 \\ \mathcal{L}_{g(x)}\mathcal{L}^j_{f(x)}h(x) \text{ are of order } \epsilon & j = r-1, \cdots, \gamma-2 \\ \mathcal{L}_{g(x)}\mathcal{L}^{\gamma-1}_{f(x)}h(x) \neq 0 & \end{array}$$

where $0 < r < \gamma$ *is the relative degree of (3.1) outside* $U_\epsilon(x_e)$ *but not necessarily well defined at every point inside* $U_\epsilon(x_e)$ *. Moreover, terms of order* ϵ *could be either* $O(x)^2$ *or small bounded terms when* $x \in U_\epsilon(x_e)$ *.*

Assumption 3.2 (Linear Parameter Dependence) *The vector fields* f *and* g *in (3.1) are unknown but may be parametrized linearly in unknown parameters* θ*:*

$$\begin{aligned} f(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot f_i(x) \\ g(x,\theta) &= \sum_{i=1}^{p} \theta_i \cdot g_i(x) \end{aligned} \tag{3.2}$$

where vector fields f_i, and g_i are known functions of x.

By Frobenius theorem, there exists $n-\gamma$ functions $\eta_i(x,\theta)$ such that $\mathcal{L}_{g(x,\theta)}\eta_i(x,\theta)=0$. The resulting local diffeomorphism of $x \in \Re^n$ is:

$$(\xi^T,\eta^T)^T = (\xi_i = \mathcal{L}^{i-1}_{f(x,\theta)}h(x), i=1,2,\ldots,r,\ldots,\gamma, \quad \eta_1,\ldots,\eta_{n-\gamma})^T \tag{3.3}$$

transforms the system (3.1) to an approximate input-output linearized system given in (2.11). However, this transformation can not be used directly in the design scheme since it depends on unknown parameters θ. We replace the transformation ξ with its estimate $\hat{\xi}$ by replacing all unknown parameters θ_i appearing in ξ by their estimates $\hat{\theta}_i$:

$$(\hat{\xi}^T,\eta^T)^T = (\hat{\xi}_i = \mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x), i=1,2,\ldots,r,\ldots,\gamma, \quad \eta_1,\ldots,\eta_{n-\gamma})^T \tag{3.4}$$

The dynamics of (3.1) under this (time-varying) transformation along the solution trajectories of (3.1) is:

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \mathcal{L}_{f(x,\theta)}h(x) \\
\dot{\hat{\xi}}_2 &= \mathcal{L}_{f(x,\theta)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_{r-1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-2}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_{r-1}(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
\dot{\hat{\xi}}_r &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r-1}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_r(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
\dot{\hat{\xi}}_{r+1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{r}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{r}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_r(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_{\gamma-1} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{\gamma-2}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{\gamma-2}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_{\gamma-1}(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
\dot{\hat{\xi}}_{\gamma} &= \mathcal{L}_{f(x,\theta)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x) + \mathcal{L}_{g(x,\theta)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_{\gamma}(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
\dot{\eta} &= q(\hat{\xi},\eta)
\end{aligned} \tag{3.5}$$

From assumption (3.2), we have:

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \sum_{i=1}^{p}(\theta_i-\hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}h(x) \\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \sum_{i=1}^{p}(\theta_i-\hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}\mathcal{L}_{f(x,\hat{\theta})}h(x) + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \\
&\vdots \\
\dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \sum_{i=1}^{p}(\theta_i-\hat{\theta}_i)\cdot\mathcal{L}_{f_i(x)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \\
&\quad +\mathcal{L}_{g(x,\theta)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x)\cdot u + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}}\cdot\dot{\hat{\theta}} \quad i=r,\ldots,\gamma-1
\end{aligned} \tag{3.6}$$

From assumption (3.1), and applying the control law:

$$u_{ad} = \frac{1}{\mathcal{L}_{g(x,\hat{\theta})}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x)}[-\mathcal{L}^{\gamma}_{f(x,\hat{\theta})}h(x) + v_{ad}] \tag{3.7}$$

with:

$$v_{ad} = y_m^{(\gamma)} + \alpha_{\gamma-1}(y_m^{(\gamma-1)} - \hat{\xi}_\gamma) + \ldots + \alpha_0(y_m - \hat{\xi}_1) \tag{3.8}$$

and α_i chosen such that $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \ldots + \alpha_0$ is a Hurwitz polynomial, we can rewrite (3.5) in a more compact form with $\phi = \theta - \hat{\theta}$:

$$\begin{array}{rcl}
\dot{\hat{\xi}}_1 & = & \hat{\xi}_2 + w_1(x,\hat{\theta}) \cdot \Phi \\
\dot{\hat{\xi}}_2 & = & \hat{\xi}_2 + w_2(x,\hat{\theta}) \cdot \Phi + \frac{\partial\hat{\xi}_2(x,\hat{\theta})}{\partial\hat{\theta}} \cdot \dot{\hat{\theta}} \\
 & \vdots & \\
\dot{\hat{\xi}}_r & = & \hat{\xi}_{r+1} + w_r(x,\hat{\theta}) \cdot \Phi + \frac{\partial\hat{\xi}_r(x,\hat{\theta})}{\partial\hat{\theta}} \cdot \dot{\hat{\theta}} + \psi_r(x,\theta,u) \\
 & \vdots & \\
\dot{\hat{\xi}}_{\gamma-1} & = & \hat{\xi}_\gamma + w_{\gamma-1}(x,\hat{\theta}) \cdot \Phi + \frac{\partial\hat{\xi}_{\gamma-1}(x,\hat{\theta})}{\partial\hat{\theta}} \cdot \dot{\hat{\theta}} + \psi_{\gamma-1}(x,\theta,u) \\
\dot{\hat{\xi}}_\gamma & = & v_{ad} + w_\gamma(x,\hat{\theta},u_{ad}) \cdot \Phi + \frac{\partial\hat{\xi}_\gamma(x,\hat{\theta})}{\partial\hat{\theta}} \cdot \dot{\hat{\theta}} \\
\dot{\eta} & = & q(\hat{\xi},\eta)
\end{array} \tag{3.9}$$

where:

$$w_\gamma(x,\hat{\theta},u) \cdot \Phi = \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i) \cdot \left[\mathcal{L}_{f_i(x)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x) + u \cdot \mathcal{L}_{g_i(x)}\mathcal{L}^{\gamma-1}_{f(x,\hat{\theta})}h(x)\right]$$

and $\psi_i(x,u) = \mathcal{L}_{g(x,\theta)}\mathcal{L}^{i-1}_{f(x,\hat{\theta})}h(x) \cdot u$. Finally:

$$\begin{array}{rcl}
\dot{\hat{\xi}} & = & A \cdot \hat{\xi} + B \cdot v + W \cdot \Phi + M \cdot \dot{\hat{\theta}} + \Psi(x,u) \\
\dot{\eta} & = & q(\hat{\xi},\eta)
\end{array} \tag{3.10}$$

where:

$$A = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, W = \begin{bmatrix} w_1 \\ \vdots \\ \\ w_\gamma \end{bmatrix} \tag{3.11}$$

$$\Psi(x,u) = [0,\ldots,\psi_r(x),\ldots,\psi_{\gamma-1},0]^T \quad , \quad M = \frac{\partial\hat{\xi}(x,\hat{\theta})}{\partial\hat{\theta}}$$

The design objective is to force the output y of system (3.1) to asymptotically track a known reference signal y_m. For this, the control law and the parameter update law must be independent of unknown parameters θ and initial conditions $\hat{\xi}(0)$. Moreover, all the closed-loop signals must remain bounded. The error signal e is defined as:

$$e_i = \hat{\xi}_i - y_m^{(i-1)} \qquad i = 1,\ldots,\gamma \tag{3.12}$$

with $e_1 = \hat{\xi}_1 - y_m = y - y_m$. Therefore, for approximate tracking, we require $e_1(t) \rightarrow B_\epsilon(0)$ as $t \rightarrow \infty$.

Assumption 3.3 (Reference Signal) *The reference trajectory $y_m(t)$ and its first γ derivatives are bounded. i.e $|y_m^{(i)}| \leq b_m \quad i = 0, 1, \ldots, \gamma$ for some $b_m > 0$.*

Remark 3.1 Often, as in model reference adaptive control, the control objective is to force the states ξ to track the states ξ_m of an asymptotically stable linear reference model with a relative degree equal to that of system (3.1):

$$\dot{\xi}_m = A_m \cdot \xi_m + b_m \cdot r$$

where A_m and b_m are in controllable canonical form and $r(t)$ is a bounded reference input. In this case the error may be defined as: $e = \hat{\xi} - \xi_m$ with $v = \sum_{i=1}^{\gamma} \alpha_i \hat{\xi}_i + r(t)$ replacing (3.8).

To determine a parameter update law $\dot{\hat{\theta}}$ that assures the stability of the closed-loop system we first construct a regressor like equation from (3.10) by cancelling M using an auxiliary system in the adaptive loop and factoring θ in $\Psi(x, u)$. Let $\bar{\xi}$ be a new signal generated as the solution trajectory of the following state observer system which is a modified version of the system used in [PP89, Akh89] and the semi-indirect adaptive scheme of [TKKS91]:

$$\begin{aligned} \dot{\bar{\xi}} &= A \cdot \hat{\xi} + B \cdot v_{ad} + M \cdot \dot{\hat{\theta}} + \hat{\Psi}(x, u, \hat{\theta}) + \bar{A} \cdot (\bar{\xi} - \hat{\xi}) \\ \bar{\xi}(0) &= \bar{\xi}_0 = \hat{\xi}(0) \end{aligned} \tag{3.13}$$

where $\hat{\xi} \in \Re^{\gamma}$ from (3.4), M as in (3.10), $\hat{\Psi}(x, u, \hat{\theta})$ is an estimate of $\Psi(x, u)$ in (3.11) evaluated at $\theta = \hat{\theta}$, and:

$$\bar{A} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{\gamma-1} \end{bmatrix}$$

with α_i chosen as in control (3.8) and $\dot{\hat{\theta}}$ still to be determined. In (3.13), if $\hat{\xi}(0)$ is not available $\bar{\xi}_0$ is set to an estimate of $\hat{\xi}(0)$. Let's define the augmented error $s(t)$ as:

$$s = \bar{\xi} - \hat{\xi} \tag{3.14}$$

where $\hat{\xi}$ is defined by (3.4) and its dynamics is given by (3.10) subject to control $v = v_{ad}$ as in (3.8). Observe that $s(t)$ satisfies:

$$\begin{aligned} \dot{s} &= \bar{A} \cdot s - W \cdot \Phi + \hat{\Psi}(x, u, \hat{\theta}) - \Psi(x, u, \theta) \\ &= \bar{A} \cdot s + W_2 \cdot \Phi \end{aligned} \tag{3.15}$$

where $W_2(x, u)$ is formed by factoring parameters θ in $\Psi(\cdot)$ and regrouping all parameter dependent nonlinearities.

The following assumption is needed to provide the internal stability of the plant:

Assumption 3.4 (Zero Dynamics) *The Zero Dynamics of the approximate input-output linearized system (2.11), or equivalently (3.10), are locally exponentially stable with q locally Lipschitz in $\hat{\xi}$ and η.*

Remark 3.2 The zero dynamics of the approximate system are a subsystem of the zero dynamics of the true system. Assumption (3.4) does not ask for the *plant* (3.1) to be minimum phase which is required in most adaptive control design schemes (e.g. [TKKS91, BS91, KKM91a, SI89, TKMK89]). In contrast, our scheme can handle *slightly* non-minimum phase systems which by definition have minimum phase *approximate* linearization.

We are now ready to state the main result of this section. The following theorem provides a parameter update law that guarantees adaptive approximate tracking and gives an upper bound on the tracking error.

Theorem 3.1 (Adaptive Approximate Tracking) *Consider the system of (3.1) satisfying robust relative degree assumption (3.1) and the zero dynamics assumption (3.4) with the vector fields f and g parameterized as in assumption (3.2). Suppose that the system (3.10) is formed and assume that $\Psi(x, u_{ad}), W(x, u_{ad}, \hat{\theta})$ and $M(x, u_{ad}, \hat{\theta})$ are locally Lipschitz continuous. Then, given a reference trajectory y_m satisfying assumption (3.3) with sufficiently small b_m, it follows that for ϵ sufficiently small the control law u_{ad} in (3.7) achieves adaptive approximate tracking of order ϵ; i.e.,*

$$|y - y_m| \leq k\epsilon$$

for some $k < \infty$, with the parameter update law:

$$\dot{\hat{\theta}} = -\Omega \cdot W_2^T \cdot P \cdot s \tag{3.16}$$

Furthermore, all the signals in the resulting closed loop adaptive system remain bounded.

Proof. Consider the following Lyapunov candidate function for the system (3.15):

$$V(s, \phi) = s^T P s + \Phi^T \Omega^{-1} \Phi \tag{3.17}$$

where Ω is a constant diagonal gain matrix and $P = P^T$ is the positive definite solution to the Lyapunov equation $\bar{A}^T P + P\bar{A} = -\lambda \cdot I$ with $\lambda > 0$, and $\bar{A}$ asymptotically stable as in (3.13). The derivative of V along the solution trajectories of (3.15) is:

$$\dot{V} = -\lambda s^T \cdot s + 2s^T P W_2^T \Phi + 2\Phi^T \Omega^{-1} \dot{\Phi}$$

From (3.16), we have:

$$\dot{V} = -\lambda |s|^2 \leq 0$$

Hence:

$$s(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{3.18}$$

To establish a bound on $|s(t)|$ and $|\Phi(t)|$, let $\lambda_{min}(P) > 0$ be the minimum eigenvalue of P in (3.17). Then $\forall t \geq 0$:

$$\begin{aligned} \lambda_{min}(P) \cdot |s(t)|^2 &\leq s^T(t)Ps(t) \leq V(s(t), \Phi(t)) \leq V(s(0), \Phi(0)) \\ &\leq \frac{1}{g_{min}} \cdot |\Phi(0)|^2 + \lambda_{max}(P) \cdot |S_0|^2 \end{aligned}$$

where $g_{min} > 0$ is the minimum gain entry in Ω. Hence:

$$\begin{aligned} |s(t)| &\leq \lambda_s \\ |\Phi(t)| &\leq \lambda_\phi \end{aligned} \tag{3.19}$$

where λ_ϕ and λ_s are some positive constants with magnitudes depend on the error in initial estimates $\hat{\theta}(0)$ and $\bar{\xi}_0$ of θ and $\hat{\xi}(0)$. Note that if $\hat{\xi}(0)$ is available we have $s(0) = 0, \lambda_s = (g_{min} \cdot \lambda_{min}(P))^{-1/2} \cdot |\Phi(0)|$, and $\lambda_\phi = (g_{max}/g_{min})^{1/2} \cdot |\Phi(0)|$. Hence, with the update law (3.16), $\hat{\theta}(t)$ remains bounded. To show convergence $\Phi(t) \to c$ for some constant c, not necessarily zero, we need to show that $s(t) \to 0$ as $t \to \infty$. A sufficient condition for this is that $\dot{s}(t) \in \mathcal{L}_\infty$. We also need to show boundedness of states x and $u_{ad}(x)$ so that $W_2(x, u, \hat{\theta})$ in (3.15) and (3.16) remains bounded. This will be shown next together with approximate tracking requirement in (3.12).

The tracking error signal e defined in (3.12) satisfies the following differential equation:

$$\begin{aligned} \dot{e} &= \bar{A} \cdot e + W^T \cdot \Phi + M \cdot \dot{\hat{\theta}} + \Psi(x, u) \\ &= \bar{A} \cdot e + W^T \cdot \Phi - M \cdot \Omega \cdot W_2^T \cdot P \cdot s + \Psi(x, u) \end{aligned} \tag{3.20}$$

To show $e_1 \to B_\epsilon(0)$, let's consider the total error defined as:

$$r \stackrel{\triangle}{=} e + s \tag{3.21}$$

or equivalently $r_i = \bar{\xi}_i - y_m^{(i-1)}$, and note that from (3.20) and (3.15), $r(t)$ satisfies the following differential equation:

$$\begin{aligned} \dot{r} &= \bar{A} \cdot r + M \cdot \dot{\hat{\theta}} + \hat{\Psi}(x, u, \hat{\theta}) \\ &= \bar{A} \cdot r - M \cdot \Omega \cdot W_2^T \cdot P \cdot s + \hat{\Psi}(x, u, \hat{\theta}) \end{aligned} \tag{3.22}$$

Remark 3.3 Equation (3.22) may be interpreted as a linear time-varying filter under small perturbation $\Psi(\cdot)$ with bounded input $s(t)$ (from (3.18)) and subject to the internal dynamics: $\dot{\eta} = q(x, \theta)$ driven by r. Let's define the output of this filter, from (3.21), as:

$$\begin{aligned} \dot{r} &= \bar{A} \cdot r - \left[M \cdot \Omega \cdot W_2^T \cdot P\right] \cdot s(t) + \hat{\Psi}(x, u, \hat{\theta}) \\ e(t) &= r - s(t) \\ \dot{\eta} &= q(x, \theta) = q(\hat{\xi}, \eta, \theta) \end{aligned} \tag{3.23}$$

Next we will analyze the stability properties of this filter in order to show $e_1 \to B_\epsilon(0)$. More specifically we establish $e(t)$ as the output of an asymptotically stable linear filter (3.23) with stable internal dynamics $q(x, \theta)$. This requires that x is bounded, or equivalently,

$\hat{\xi}$ and η are bounded. To show $\hat{\xi}$ and η are bounded, we will first show that r and η are bounded using a suitable Lyapunov candidate function for (3.23):

$$V(r, \eta) = r^T \bar{P} r + \mu v_2(\eta) \tag{3.24}$$

where $\mu > 0$ is a constant to be determined later, $\bar{P} > 0$ is such that $\bar{A}^T \bar{P} + \bar{P} \bar{A} = -I$, and $v_2(\eta)$ is a Lyapunov function for the system $\dot{\eta} = q(0, \eta)$. From assumption (3.4), a converse Lyapunov argument assures the existence of v_2 with following properties:

$$\begin{array}{rcl} k_1|\eta|^2 \leq & v_2(\eta) & \leq k_2|\eta|^2 \\ \frac{\partial v_2}{\partial \eta} q(0, \eta) & \leq & -k_3|\eta|^2 \\ \left|\frac{\partial v_2}{\partial \eta}\right| & \leq & k_4|\eta| \end{array} \tag{3.25}$$

for some positive constants k_1, k_2, k_3, and k_4. The time derivative of $V(r, \eta)$ along the solution trajectories of (3.23) is:

$$\dot{V} = -|r|^2 - 2r^T \bar{P}[M \cdot \Omega \cdot W_2^T \cdot P] \cdot s + 2r^T \bar{P} \hat{\Psi} + \mu \frac{\partial v_2}{\partial \eta} q(\hat{\xi}, \eta) \tag{3.26}$$

From (3.12) and assumption (3.3) we have:

$$|\hat{\xi}| \leq |e| + b_m \tag{3.27}$$

and from the definition of r in (3.21):

$$|e| \leq |r| + |s| \tag{3.28}$$

Since x is a local diffeomorphism of $(\hat{\xi}, \eta)$, we have:

$$\begin{array}{rcl} |x| & \leq & l_x(|\hat{\xi}| + |\eta|) \\ & \leq & l_x(|e| + b_m + |\eta|) \end{array} \tag{3.29}$$

Since W_2 is assumed locally Lipschitz continuous we have:

$$|2\bar{P}[M \cdot \Omega \cdot W_2^T \cdot P]| \leq l_W|x| + c_1 \tag{3.30}$$

where $c_1 > 0$ is a constant.

Because $\Psi(x, u)$ is $O^2(x, u)$, we have for some constants $l_\epsilon > 0$ and $\delta > 0$:

$$\begin{array}{rcl} |2\bar{P}\Psi(x, u)| & \leq & l_\epsilon|x|^2 \quad \forall x : |x| \leq \epsilon, |u| \leq \delta \\ & \leq & l_\epsilon \epsilon |x| \end{array} \tag{3.31}$$

Note that since u is a function of $x(t)$ and $y_m(t)$, size of δ depends on ϵ and b_m. This immediately suggests that $\alpha_i s$ in (3.8) should be chosen such that the assigned poles are not too far left. Otherwise, the resulting higher control magnitude $|u|$ will push the state $|x|$ outside its approximating region.

From assumption (3.4):

$$|q(\hat{\xi}, \eta) - q(0, \eta)| \leq l_q|\hat{\xi}|$$

Hence:

$$\begin{aligned}\frac{\partial v_2}{\partial \eta} q(\hat{\xi},\eta) &= \frac{\partial v_2}{\partial \eta} q(0,\eta) + \frac{\partial v_2}{\partial \eta}(q(\hat{\xi},\eta) - q(0,\eta)) \\ &\le -k_3|\eta|^2 + k_4 l_q |\eta|(|e| + b_m)\end{aligned} \tag{3.32}$$

Substituting above inequalities in (3.26) yields:

$$\begin{aligned}\dot{V} &\le -|r|^2 + |r|(l_W|x| + c_1)|s| + \epsilon l_\epsilon |r||x| \\ &\quad +\mu(-k_3|\eta|^2 + k_4 l_q|\eta|(|e| + b_m)) \\ &\le -|r|^2 + l_x l_W |r|(|r| + |s| + b_m + |\eta|)|s| \\ &\quad +c_1|r||s| + \epsilon l_\epsilon |r||x| + \mu(-k_3|\eta|^2 + k_4 l_q|\eta|(|r| + |s| + b_m))\end{aligned}$$

Let $c_2 \triangleq b_m + \frac{c_1}{l_x l_W}$, then:

$$\begin{aligned}\dot{V} &\le -\left(\tfrac{|r|}{2} - l_W l_x(|s| + c_2)|s|\right)^2 + (l_W l_x(|s| + c_2))^2|s|^2 \\ &\quad + \left(\tfrac{|r|}{2} - (l_W l_x|s| + \epsilon l_\epsilon l_x + \mu k_4 l_q)|\eta|\right)^2 + (l_W l_x|s| + \epsilon l_\epsilon l_x + \mu k_4 l_q)^2|\eta|^2 \\ &\quad - \left(\tfrac{|r|}{2} - \epsilon l_\epsilon l_x(|s| + b_m)\right)^2 + (\epsilon l_\epsilon l_x(|s| + b_m))^2 \\ &\quad -\mu k_3\left(\tfrac{|\eta|}{2} - \tfrac{k_4}{k_3} l_q(|s| + b_m)\right)^2 + \tfrac{\mu}{k_3}(k_4 l_q(|s| + b_m))^2 \\ &\quad -(1/4 - l_W l_x|s| - \epsilon l_\epsilon l_x)|r|^2 - \tfrac{3}{4}\mu k_3|\eta|^2 \\ &\le -(1/4 - l_W l_x\lambda_s - \epsilon l_\epsilon l_x)|r|^2 - [\tfrac{3}{4}\mu k_3 - (l_W l_x\lambda_s + \epsilon l_\epsilon l_x + \mu k_4 l_q)^2]|\eta|^2 \\ &\quad (l_W l_x(\lambda_s + c_2))^2{\lambda_s}^2 + (\epsilon l_\epsilon l_x(\lambda_s + b_m))^2 + \tfrac{\mu}{k_3}(k_4 l_q(\lambda_s + b_m))^2\end{aligned}$$

Define:

$$\mu \triangleq \frac{k_3}{4(l_W l_x + k_4 l_q + l_\epsilon l_x)^2} \tag{3.33}$$

Then for $\mu \le \mu_0, \epsilon \le min\left\{\frac{1}{8[l_W l_x + l_x l_\epsilon]}, \mu\right\}$, and $|\Phi(0)|$ and $|s(0)|$ small enough such that $\lambda_s \le \epsilon$ we have:

$$\begin{aligned}\dot{V} &\le -\tfrac{|r|^2}{8} - \tfrac{\mu k_3}{2}|\eta|^2 + \tfrac{\mu}{k_3}(k_4 l_q(\lambda_s + b_m))^2 \\ &\quad +(l_W l_x(\lambda_s + c_2))^2{\lambda_s}^2 + (\epsilon l_\epsilon l_x(\lambda_s + b_m))^2\end{aligned}$$

Hence when $|r|$ and $|\eta|$ are large $\dot{V} \le 0$. Therefore, $|r|$ and $|\eta|$ are bounded which by (3.28) implies that $|e|$ is bounded. From (3.27), this shows that $|\hat{\xi}|$ is bounded and from (3.29), $|x|$ is bounded. This guarantees that M and W_2 are bounded which by (3.15) implies that $\dot{s}$ is bounded. This together with (3.18) implies that $s \to 0$ as $t \to \infty$. We have shown that (3.23) is an exponentially stable linear filter with stable internal dynamics $q(x, \theta)$ under bounded ϵ-order perturbation Ψ and input $s \to 0$. Hence , its output e converges to a ball of order ϵ. i.e. $|y - y_m| \le k\epsilon$ for some constant k. This completes the proof. □

The adaptive design scheme developed in this section can also be applied to the multi-input multi-output (MIMO) nonlinear systems of the form:

$$\begin{aligned}\dot{x}(t) &= f(x,\theta) + \sum_{i=1}^{m} g_i(x,\theta)\cdot u_i \\ y_i(t) &= h_i(x) \qquad i = 1,\ldots,m\end{aligned} \tag{3.34}$$

where the vector relative degree $(r_1, \ldots, r_m)$ at some point of interest is ill-defined, or that the decoupling matrix A_r:

$$A_r = \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^{r_1-1}h_1(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{r_1-1}h_1(x) \\ \vdots & \vdots & \vdots \\ \mathcal{L}_{g_1}\mathcal{L}_f^{r_m-1}h_m(x) & \ldots & \mathcal{L}_{g_m}\mathcal{L}_f^{r_m-1}h_m(x) \end{bmatrix} \tag{3.35}$$

is almost singular due to the presence of small terms. In this case, with assumptions (3.1)-(3.4), we first apply the dynamic extension algorithm [Isi89, Nvds90] to the approximate model in order to get a nonsingular decoupling matrix A_γ in (3.35) with robust vector relative degree $(\gamma_1, \ldots, \gamma_m)$. The same procedure as in the SISO case can then be applied to the resulting extended system. We demonstrate these procedures in the section V where we apply our adaptive design scheme to a simplified model of an aircraft which is slightly non-minimum phase with an almost singular decoupling matrix.

IV. Adaptive Regulation

When the objective is state regulation, the design procedure becomes simpler. In this case, we attempt to approximately (state) linearize the nonlinear system and then design a controller such that the closed loop system is asymptotically stable. The key here is to linearize the system to the highest order possible. Consider a nonlinear system with no output specified:

$$\dot{x}(t) \quad = \quad f(x,\theta) + \sum_1^m \; g_i(x,\theta) \cdot u_i \tag{4.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ is the vector of unknown constant parameters, f and g_i are smooth functions on $\Re^n$ with $f(0,\theta) = 0$. The parameter update law for our adaptive regulation scheme is assumed to be of the following form:

$$\dot{\hat{\theta}} = g(x, \hat{\theta}, u) \tag{4.2}$$

where $g(\cdot)$ is $O(x,u)^{\rho_g}$ and will be determined later.

Lemma 4.1 *Consider system (4.1) satisfying assumption (3.2). Let θ_0 be the unknown nominal value of θ and assume that $\forall \theta \in B_\sigma(\theta_0)$:*

i. Distribution D^k has an order ρ_d local basis at 0 consisting of:

$$\left\{ ad_f^l g_j : 0 \le l < min(k, k_j); j = 1, \ldots, m \right\}$$

for a set of $k_1, \ldots, k_m$, controllability indices.

ii. D^{k_j-1} is order ρ_d involutive at 0 for $j = 1, \ldots, m$.

Then there exists a local transformation $z = T(x, \hat{\theta}), T(0, \hat{\theta}) = 0$ and a nonlinear feedback $u_i(x, \hat{\theta})$ such that with the choice of the parameter update law in (4.2) the nonlinear system

(4.1) is transformed into the following regressor form approximate linear system, with $\rho \triangleq min\{\rho_d, \rho_g\}$:

$$\dot{z}_j = \begin{cases} z_{j+1} & + \quad w_j^T(x,\hat{\theta})(\theta_0 - \hat{\theta}) + O(z,u)^{\rho+1} \quad if\ j \neq k_1 + \ldots + k_i \\ v_i & + \quad w_j^T(x,\hat{\theta},u)(\theta_0 - \hat{\theta}) + O(z,u)^{\rho+1} \quad if\ j = k_1 + \ldots + k_i \end{cases} \tag{4.3}$$

Proof It is clear from theorem (2.1) that for a fixed parameter θ and any $\rho_d > 0$, conditions (i) and (ii) are necessary and sufficient for the existence of a transformation $z = T(x,\theta)$ which transforms (4.1) into the following form:

$$\dot{z}_j = \begin{cases} z_{j+1} & + \quad O(x,u)^{\rho_d+1} \quad if\ j \neq k_1 + \ldots + k_i \\ v_i & \quad if\ j = k_1 + \ldots + k_i \end{cases} \tag{4.4}$$

When parameter vector θ is not known and is replaced by an estimate $\hat{\theta}$ with an update law of the form (4.2), which is of order ρ_g, we get:

$$\dot{z}_j = \begin{cases} z_{j+1} & + \quad w_j^T(x,\hat{\theta})(\theta_0 - \hat{\theta}) + O(x,u)^{\rho_d+1} + \frac{\partial T(x,\hat{\theta})}{\partial \hat{\theta}} \cdot g(x,u,\hat{\theta}) \\ & \qquad if\ j \neq k_1 + \ldots + k_i \\ v_i & + \quad w_j^T(x,\hat{\theta},u)(\theta_0 - \hat{\theta}) + \frac{\partial T(x,\hat{\theta})}{\partial \hat{\theta}} \cdot g(x,u,\hat{\theta}) \\ & \qquad if\ j = k_1 + \ldots + k_i \end{cases} \tag{4.5}$$

where we used the fact that in (4.4): $z_{j+1} = \mathcal{L}_{f(x,\theta)} z_j \ \ j \neq k_1 + \ldots + k_i$, and:

$$w_j^T = \left\{ \begin{array}{l} \left(\mathcal{L}_{f_1(x)} z_j, \ldots, \mathcal{L}_{f_p(x)} z_j\right) \qquad if\ j \neq k_1 + \ldots + k_i \\ \left(\mathcal{L}_{f_1(x)} z_j + \sum_{i=1}^{m} u_i \cdot \mathcal{L}_{g_{1,i}(x)} z_j \ , \ldots, \ \mathcal{L}_{f_p(x)} z_j + \sum_{i=1}^{m} u_i \cdot \mathcal{L}_{g_{p,i}(x)} z_j\right) \\ \quad if\ j = k_1 + \ldots + k_i \end{array} \right\} \tag{4.6}$$

Finally, since $z = z(x)$ is a diffeomorphism with $z(0) = 0$, we have a term $O(x,u)^l$ if and only if it is $O(z,u)^l$. This together with the fact that $g(\cdot)$ is $O(x,u)^{\rho_g}$ results in (4.3). $\square$.

The transformed system can be written as:

$$\dot{z} = A \cdot z + B \cdot v + W^T \cdot \Phi + O(z,u)^{\rho+1} \tag{4.7}$$

where W is a matrix with columns w_j defined in (4.6), and (A, B) are in Brunovsky form. The following control law can then be used to assign stable poles:

$$v_j = \sum_{i=k_{j-1}+1}^{k_j} \alpha_{i,j} z_i \tag{4.8}$$

where $\alpha_{i,j}$ chosen such that $s^{k_j} + \alpha_{k_j-1,j} s^{k_j-1} + \ldots + \alpha_{1,j}$ is a Hurwitz polynomial. The resulting feedback system is of the form:

$$\dot{z} = \bar{A} \cdot z + W^T \cdot \Phi + O(z)^{\rho+1} \tag{4.9}$$

where $\bar{A}$ is an asymptotically stable matrix. Let P be the unique solution to the Lyapunov equation $\bar{A}^T P + P\bar{A} = -I$ which is guaranteed to exist. The following theorem provides the regulation counterpart of the tracking result in theorem (3.1).

Theorem 4.2 (Adaptive Regulation) *Consider system (4.1) satisfying assumption (3.2) and conditions of lemma (4.1) on $U_\epsilon(0)$, an open neighborhood of the origin, with:*

$$g(x, \hat{\theta}, u) = -\Omega \cdot W^T \cdot P \cdot z \tag{4.10}$$

Then there exist an open neighborhood $B(\theta_0)$ such that $\forall \hat{\theta}(0) \in B_\sigma(\theta_0)$ and ϵ sufficiently small, $\hat{\theta}$ remains bounded, $x \to 0$ as $t \to \infty$ and the equilibrium $x = 0, \hat{\theta} = \theta_0$ of the resulting closed-loop adaptive system is uniformly stable.

proof Consider the following Lyapunov candidate function:

$$V(z, \phi) = z^T P z + \Phi^T \Omega^{-1} \Phi$$

where Ω is a constant diagonal gain matrix. Derivative of V along the solution trajectories of (4.9) is:

$$\begin{array}{rcl} \dot{V} & = & z^T(\bar{A}^T P + P\bar{A})z + 2z^T PW^T \Phi + 2z^T P \cdot O(z)^{\rho+1} + \Phi^T \Omega^{-1} \Phi \\ & \leq & -z^T z + O(z)^{\rho+2} \end{array} \tag{4.11}$$

Since $\rho \geq 1$, by definition there exist a constant $l > 0$ such that: $O(z)^{\rho+2} \leq l|z|^{\rho+2} \leq l\epsilon^\rho |z|^2 \quad \forall z \in B_\epsilon(0)$. Hence:

$$\dot{V} \leq (-1 + l \cdot \epsilon^\rho) \cdot |z|^2 \tag{4.12}$$

which is negative semidefinite for:

$$\forall \epsilon \;\; s.t. \quad \epsilon^\rho < \epsilon_0 \stackrel{\triangle}{=} \frac{1}{l} \tag{4.13}$$

This shows that for ϵ and $|\Phi(0)|$ sufficiently small, $z(t), \phi(t)$ and $x(t)$ remain bounded and $z \in \mathcal{L}_2$. Also, from (4.7), $\dot{z}$ is bounded and by Barbalat's Lemma $z(t) \to 0$ as $t \to \infty$. Hence, since $z = T(x)$ is a diffeomorphism on $B_\epsilon(0) \in U_\epsilon$ with $T(0) = 0$, $x(t) \to 0$ as $t \to \infty$. This proves that the regulation of state $x(t)$ is achieved for all initial conditions: $\Phi(0) \in B_\sigma(\theta_0), x(0) \in T^{-1}(B_\epsilon(0))$. It is also clear that the equilibrium:

$$x = 0 \quad , \quad \hat{\theta} = \theta_0$$

is uniformly stable. This completes the proof $\square$.

Remark 4.1 Due to the *approximate* nature of our nonlinear analysis, the feasibility domain of our adaptive scheme is generally *local.* We, however, note that in the proof of the above theorem, the feasibility domain obtained for regulating $x(t)$ depends monotonically on the order ρ of our approximation. This is true since for ϵ *small*, ϵ^ρ is a decreasing function of $\rho \geq 1$ and consequently (4.13) holds for a larger range of ϵ.

We now consider the case where $\rho = 2$ for a single input nonlinear system:

$$\dot{x}(t) \;\; = \;\; f(x, \theta) + g(x, \theta) \cdot u \tag{4.14}$$

Kang and Krener [KK92] proved that *any* linearly controllable nonlinear system is feedback linearizable to second degree by a dynamic state feedback. This is in contrast to the results of [CLM88] showing that if a single input nonlinear system is not *exactly* feedback linearizable, then it is not linearizable by a dynamic state feedback. We next remove the linear parameter dependence assumption of theorem (4.2) and give a systematic design scheme for ***adaptive quadratic regulation*** which can be applied to any linearly controllable nonlinear system. The following definition and theorem is due to [KK92]:

Definition 4.1 (Quadratic Linearization) If we can find a dynamic state feedback for system (4.14) such that the resulting extended system is linearly controllable and it can be transformed into:

$$\dot{z} = Fz + Gv + O(z,v)^3 \tag{4.15}$$

by a change of coordinates:

$$\begin{bmatrix} x \\ \omega \end{bmatrix} = z + \psi^{[2]}(z) \tag{4.16}$$

where $\psi^{[2]}(z)$ is a polynomial vector field of order two, then system (4.14) is called *quadratically linearizable* by a dynamic state feedback.

Theorem 4.3 ([KK92]) *Any linearly controllable system (4.14) is quadratically linearizable by an (n-1)-dimensional dynamic state feedback of the form:*

$$\dot{\omega} = A\omega + Bv, \qquad u = \omega_1 + \gamma^{[1]}(x,\omega) + \gamma^{[2]}(x,\omega) \tag{4.17}$$

where (A,B) is in Brunovsky form.

We now give a systematic design scheme to achieve quadratic regulation under parameter uncertainty:

Step One: Construct the second jet of system (4.14) around the reference point 0. This will give an approximate system up to the second order:

$$\dot{x} = A(\bar{\theta})x + B(\bar{\theta})u + f^{[2]}(x,\bar{\theta}) + g^{[1]}(x,\bar{\theta})\cdot u + O(x,u)^3 \tag{4.18}$$

where $f^{[2]}(\cdot)$ and $g^{[1]}(\cdot)$ are n-dimensional polynomial vector fields of order two and one in the components of x. Note that in this series expansion, all the unknown parameters θ of system (4.14) now appear, possibly after reparameterization to $\bar{\theta}$, *linearly* in the approximate model (4.18), i.e. in linear and quadratic terms: $Ax, Bu, f^{[2]}(x)$ and $g^{[1]}(x)\cdot u$ of system (4.14).

Step Two: Since (4.14) was assumed linearly controllable, the pair (A,B) is a controllable pair. Perform the following *linear* change of coordinates and *linear* state feedback:

$$\begin{aligned} z &= Est.(T)\cdot x \ , \ T \triangleq \left[B_c, A_cB_c, \ldots, A_c^{n-1}B_c\right]\cdot\left[B, AB, \ldots, A^{n-1}B\right]^{-1} \\ u &= \gamma^{[1]}(z,\hat{\theta}) + \mu \end{aligned} \tag{4.19}$$

to transform (4.18) into:

$$\dot{z} = Fz + B_c\mu + f^{[2]}(z) + g^{[1]}(z)\cdot(\gamma^{[1]} + \mu) + W_1(x)\cdot\Phi + O(z,\mu)^3 \tag{4.20}$$

where $\phi = \bar{\theta} - \hat{\theta}$, (F, B_c) is in Brunovsky form, and $f^{[2]}(z), g^{[1]}(z)$ are of the form:

$$f_i^{[2]} = \sum_{j=1,k=1}^{n} a_{ijk} \cdot x_j x_k \quad , \quad g_i^{[1]} = \sum_{j=1}^{n} b_{ij} x_j \tag{4.21}$$

Since, in general, transformation above T depends on some unknown parameters $\bar{\theta}$, we used its estimate with parameter update laws still to be determined. Note that these parameter update laws will be some order 3 smooth functions in the state and have been included in $O(z, \mu)^3$ terms.

Step Three: Apply the following p-dimensional dynamic state feedback to (4.20):

$$\dot{\omega} = A\omega + Bv, \quad \mu = \omega_1 + \gamma^{[2]}(x, \omega, \hat{\theta}) \tag{4.22}$$

where (A, B) is in Brunovsky form with dimension p:

$$p \triangleq max\{j - i \; ; \; a_{ijk} \neq 0 \; , \; n + 1 - i \; ; \; b_{ij} \neq 0\} \tag{4.23}$$

where a_{ijk} and b_{ij} are as in (4.21). The resulting extended system is in the form:

$$\begin{bmatrix} \dot{z} \\ \dot{\omega} \end{bmatrix} = A_{n+p} \cdot \begin{bmatrix} z \\ \omega \end{bmatrix} + B_{n+p} \cdot v + \begin{bmatrix} B_c \gamma^{[2]}(z, \omega) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{f}^{[2]}(z, \omega) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{W}_1(x, \omega) \\ 0 \end{bmatrix} \cdot \phi + O(z, \omega, v)^3 \tag{4.24}$$

where $\bar{f}^{[2]}(z, \omega) = f^{[2]}(z) + g^{[1]}(z) \cdot (\gamma^{[1]}(z) + \omega_1)$.

Step Four: Consider the following change of coordinates:

$$\begin{aligned} \xi_1 &= z_1 \\ \xi_k &\triangleq \textit{Linear and quadratic parts of } \dot{z}_{k-1} \quad \textit{except terms containing } \phi \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq k \leq n \end{aligned} \tag{4.25}$$

where we ignore all the terms containing $\dot{\phi}$ and $\dot{\hat{\theta}}$ since they will be of order 3. Also:

$$\xi_k = \omega_{k-n} \qquad n + 1 \leq k \leq n + p \tag{4.26}$$

The resulting system is transformed to:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + w_1(x, \omega)\phi \\ \dot{\xi}_2 &= \xi_3 + w_2(x, \omega)\phi + O(\xi, \mu)^3 \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + w_{n-1}(x, \omega)\phi + O(\xi, \mu)^3 \\ \dot{\xi}_n &= \omega_1 + \gamma^{[2]}(\xi) + \psi^{[2]}(x, \omega_1, \ldots, \omega_n) + w_n(x, \omega)\phi + O(\xi, \mu)^3 \end{aligned} \tag{4.27}$$

where $\psi^{[2]}(x, \omega_1, \ldots, \omega_n)$ is a homogeneous polynomial of second degree. Let:

$$\gamma^{[2]}(x) \triangleq [-\psi^{[2]}(x, \omega_1, \ldots, \omega_n)] \tag{4.28}$$

then we have:

$$\begin{array}{rcl}
\dot{\xi}_1 & = & \xi_2 + w_1(x,\omega)\phi \\
\dot{\xi}_2 & = & \xi_3 + w_2(x,\omega)\phi + O(\xi,v)^3 \\
 & \vdots & \\
\dot{\xi}_{n-1} & = & \xi_n + w_{n-1}(x,\omega)\phi + O(\xi,v)^3 \\
\dot{\xi}_n & = & \xi_{n+1} + w_n(x,\omega)\phi + O(\xi,v)^3 \\
\dot{\xi}_{n+1} & = & \xi_{n+2} \\
 & \vdots & \\
\dot{\xi}_{n+p} & = & v
\end{array} \tag{4.29}$$

which can be rewritten in the following compact form:

$$\dot{\xi} = F\xi + Gv + W(x,\omega,u)\cdot\phi + O(\xi,v)^3 \tag{4.30}$$

where (F,G) is in the $n+p$ dimensional Brunovsky form.

Step Five: Choose the control law v and the parameter update law as:

$$\begin{array}{rcl}
v & = & -K\cdot\xi \\
\dot{\hat{\theta}} & = & -\Omega\cdot W^T(x,\omega,u)\cdot P\cdot\xi
\end{array} \tag{4.31}$$

where K is such that $F_c \triangleq (F - G\cdot K) < 0$, P is the solution to $F_c^T P + PF_c = -I$, and Ω is a constant $n\times n$ diagonal gain matrix.

Remark 4.2 The procedures indicated above in obtaining the transformation $\xi(x)$ are similar to those of [KK92], where all parameters are assumed known and the system is assumed to be in the *extended quadratic controller normal form.* Here, with an additional integrator, we avoid solving the set of linear equations [KK92]:

$$[Fz + G(\omega_1 + \gamma^{[1]}(z,\omega)), \phi^{[2]}(z,\omega)] + \frac{\partial\phi^{[2]}}{\partial\omega}A\omega = G\gamma^{[2]}(z,\omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1 + \gamma^{[1]}(z,\omega))$$

where under parameter uncertainty a solution is not feasible.

The following theorem summarizes our results in this section on adaptive quadratic regulation. The adaptive quadratic model following can also be shown using the observer system (3.13).

Theorem 4.4 (adaptive quadratic regulation) *For any linearly controllable nonlinear system (4.14) with unknown parameters θ, adaptive quadratic regulation can be achieved following the steps described above.*

Proof The resulting closed-loop system can be written in the following compact form:

$$\dot{\xi} = F_c\xi + W(x,\omega,u)\cdot\phi + O(\xi,v)^3 \tag{4.32}$$

and an argument analogous to the one used in the proof of theorem (4.2) holds. Hence, for $|x(0)|$ and $|\Phi(0)|$ sufficiently small, $x(t)$ and $\hat{\theta}(t)$ remain bounded and $x(t)\to 0$ as $t\to\infty$. Moreover, the equilibrium:

$$x = 0 \quad , \quad \hat{\theta} = \theta_0$$

is uniformly stable. □.

In section V.2, we will demonstrate the usefulness of the above scheme with the help of a "benchmark" example of state regulation problem where the system is not feedback linearizable. We also illustrate the case where unknown parameters do not appear linearly in the system.

V. Simulations

In this section, we illustrate the features of our adaptive design scheme with the help of three examples. First, we consider a simplified model of the Harrier aircraft studied in [HSM89] under some parameter uncertainty in mass and moment of inertia. We compare the performance of our adaptive tracking controller to the performance of the non-adaptive tracking controller. In the second and third examples, we discuss adaptive nonlinear regulation of a system that violates the conditions of some other adaptive schemes.

V.I Applications to Flight Control Systems

The equations of motion for the prototype PVTOL (planar vertical takeoff and landing) aircraft considered in [HSM89] are given by:

$$\begin{aligned} m\ddot{x} &= -\sin\theta \cdot u_1 + \epsilon \cos\theta \cdot u_2 \\ m\ddot{y} &= \cos\theta \cdot u_1 + \epsilon \sin\theta \cdot u_2 + mg \\ J\ddot{\theta} &= u_2 \\ y_1 &= x \ , \ y_2 = y \end{aligned} \tag{5.1}$$

where outputs x and y give the position of the aircraft center of mass, θ is the angle of the aircraft relative to the x-axis, u_1 and u_2 are the thrust and the rolling moment, g is the gravitational acceleration normalized to -1, and ϵ is a small coupling coefficient between the rolling moment u_2 and the lateral acceleration of the aircraft $\ddot{x}$ and $\ddot{y}$. The objective is to find a feedback law that decouples outputs x and y under some parameter uncertainty in m and J. We require x to track a smooth trajectory while y remains at zero. The decoupling matrix for system (5.1) is:

$$\frac{1}{m}\begin{bmatrix} -\sin\theta & \epsilon\cos\theta \\ \cos\theta & \epsilon\sin\theta \end{bmatrix} \tag{5.2}$$

which is nonsingular with a small determinant $-\epsilon/m$. This will result in a relatively *large* decoupling control law:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = m \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta/\epsilon & \sin\theta/\epsilon \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 + 1 \end{bmatrix} \tag{5.3}$$

which input-output linearizes system (5.1):

$$\begin{aligned} \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{\theta} &= \tfrac{1}{\epsilon m g}(\sin\theta + \cos\theta \cdot v_1 + \sin\theta \cdot v_2) \end{aligned} \tag{5.4}$$

Since ϵ is small (typically around 0.01), the exact decoupling control (5.3) will require a large control effort and hence is not feasible in practice. Moreover, this control law renders an unstable internal dynamics since the resulting input-output linearized system (5.4) is non-minimum phase and there is no global tracking controller that can be designed for such a system. Hauser [HSM89] proposed a tracking controller design scheme for this system by ignoring the ϵ-dependent terms in the model (5.1). The resulting approximate system overlooks the small coupling between rolling moments and lateral acceleration (set $\epsilon = 0$):

$$\begin{aligned} m\ddot{x} &= -\sin\theta \cdot u_1 \\ m\ddot{y} &= \cos\theta \cdot u_1 + mg \\ J\ddot{\theta} &= u_2 \\ y_1 &= x \ , \ y_2 = y \end{aligned} \tag{5.5}$$

We now, following the procedures in section III, design an adaptive approximate tracking controller for system (5.1) under parameter uncertainty in m and J.

Remark 5.1 The neglected nonlinearities in (5.5) are not $O(x,u)^2$ which is required for approximate linearization analysis. However, since ϵ is a small parameter, one may still apply the approximate linearization technique to achieve approximate tracking. In this case, the loss of performance is less for smaller values of ϵ, and as shown in [HSM89], this approximation gives desirable results for ϵ up to around 0.6. The same argument holds for our adaptive tracking scheme.

In order for output y to be independent of the neglected nonlinear terms used in our approximation, we first remap the controls u_1 and u_2. This is mainly done to provide *exact* tracking of the altitude (y) where PVTOL aircraft are designed to be maneuvered close to the ground. Note that under parameter uncertainty in mass m, which is common in aircrafts due to the fuel consumption and load variation, good tracking in y-output is not possible unless the controller is robust against this uncertainty and can adapt to the parameter variations in the system dynamics. Let $\tilde{u}_1$ and $\tilde{u}_2$ be new controls such that:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{pmatrix} 1 & -\epsilon \ \tan\theta \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \tag{5.6}$$

then (5.1) can be rewritten as:

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \theta_1 \cdot \begin{bmatrix} -\sin\theta & \epsilon/\cos\theta \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \\ \ddot{\theta} &= \theta_2 \cdot \tilde{u}_2 \end{aligned} \tag{5.7}$$

where for simplicity we have redefined the unknown parameters in (5.1) with the help of the following notation:

$$\theta_1 \triangleq 1/m \ , \ \theta_2 \triangleq 1/J \tag{5.8}$$

The approximate system is then given by:

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \theta_1 \cdot \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} \\ \ddot{\theta} &= \theta_2 \cdot \tilde{u}_2 \end{aligned} \tag{5.9}$$

which has a singular decoupling matrix. Therefore, we first need to apply the dynamic extension algorithm where we consider $\tilde{u}_1$ as a new state. The resulting extended approximate system is:

$$\begin{aligned} \dot{\eta} &= \begin{bmatrix} \eta_2 \\ -\theta_1\eta_7\sin\eta_5 \\ \eta_4 \\ -1+\theta_1\eta_7\cos\eta_5 \\ \eta_6 \\ 0 \\ \eta_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \theta_2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} \\ y &= \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix} \end{aligned} \tag{5.10}$$

where $\eta \triangleq (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}, \tilde{u}_1, \dot{\tilde{u}}_1)^T$.This system has a robust vector relative degree $(4,4)$ and is therefore, a minimum phase system. Consider the following local diffeomorphism of η:

$$\begin{aligned} \xi_1 &= \eta_1 \\ \xi_2 &= \eta_2 \\ \xi_3 &= -\theta_1\sin\eta_5\eta_7 \\ \xi_4 &= -\theta_1\eta_6\eta_7\cos\eta_5 - \theta_1\eta_8\sin\eta_5 \\ \xi_5 &= \eta_3 \\ \xi_6 &= \eta_4 \\ \xi_7 &= -1+\theta_1\eta_7\cos\eta_5 \\ \xi_8 &= -\theta_1\eta_6\eta_7\sin\eta_5 + \theta_1\eta_8\cos\eta_5 \end{aligned} \tag{5.11}$$

which transforms the approximate system (5.10) into the following input-output linearized system:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= \theta_1\sin\eta_5\eta_7\eta_6{}^2 - 2\theta_1\cos\eta_5\eta_6\eta_8 - \theta_1\sin\eta_5\cdot w - \theta_1\theta_2\cos\eta_5\eta_7\cdot\tilde{u}_2 \\ \dot{\xi}_5 &= \xi_6 \\ \dot{\xi}_6 &= \xi_7 \\ \dot{\xi}_7 &= \xi_8 \\ \dot{\xi}_8 &= \theta_1\cos\eta_5\eta_7\eta_6{}^2 - 2\theta_1\sin\eta_5\eta_6\eta_8 - \theta_1\cos\eta_5\cdot w - \theta_1\theta_2\sin\eta_5\eta_7\cdot\tilde{u}_2 \end{aligned} \tag{5.12}$$

Since transformation (5.11) depends on unknown parameters θ_i, we consider the following local diffeomorphism which is an estimate of the transformation in (5.11):

$$\begin{aligned} \hat{\xi}_1 &= \eta_1 \\ \hat{\xi}_2 &= \eta_2 \\ \hat{\xi}_3 &= -\hat{\theta}_1\sin\eta_5\eta_7 \\ \hat{\xi}_4 &= -\hat{\theta}_1\eta_6\eta_7\cos\eta_5 - \hat{\theta}_1\eta_8\sin\eta_5 \\ \hat{\xi}_5 &= \eta_3 \\ \hat{\xi}_6 &= \eta_4 \\ \hat{\xi}_7 &= -1+\hat{\theta}_1\eta_7\cos\eta_5 \\ \hat{\xi}_8 &= -\hat{\theta}_1\eta_6\eta_7\sin\eta_5 + \hat{\theta}_1\eta_8\cos\eta_5 \end{aligned} \tag{5.13}$$

where the update laws for $\hat{\theta}_1$ and $\hat{\theta}_2$ will be determined later. The equations describing the dynamics of $\hat{\xi}$ are given by:

$$
\dot{\hat{\xi}} = \begin{bmatrix} \eta_2 \\ -\theta_1 \sin \eta_5 \eta_7 \\ -\hat{\theta}_1 \eta_6 \eta_7 \cos \eta_5 - \hat{\theta}_1 \eta_8 \sin \eta_5 \\ \hat{\theta}_1 \sin \eta_5 \eta_7 {\eta_6}^2 - 2\hat{\theta}_1 \cos \eta_5 \eta_6 \eta_8 \\ \eta_4 \\ -1 + \theta_1 \eta_7 \cos \eta_5 \\ -\hat{\theta}_1 \eta_6 \eta_7 \sin \eta_5 + \hat{\theta}_1 \eta_8 \cos \eta_5 \\ -\hat{\theta}_1 \cos \eta_5 \eta_7 {\eta_6}^2 - 2\hat{\theta}_1 \sin \eta_5 \eta_6 \eta_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\hat{\theta}_1 \sin \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \cos \eta_5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hat{\theta}_1 \cos \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \sin \eta_5 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ 0 \\ -\eta_7 \sin \eta_5 \\ -\eta_6 \eta_7 \cos \eta_5 - \eta_8 \sin \eta_5 \\ 0 \\ 0 \\ \eta_7 \cos \eta_5 \\ -\eta_6 \eta_7 \sin \eta_5 + \eta_8 \cos \eta_5 \end{bmatrix} \cdot \dot{\hat{\theta}}_1 \tag{5.14}
$$

The approximate linearizing control is given by:

$$
\begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} -\hat{\theta}_1 \sin \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \cos \eta_5 \\ \hat{\theta}_1 \cos \eta_5 & -\hat{\theta}_1 \theta_2 \eta_7 \sin \eta_5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} v - \hat{\theta}_1 \eta_7 {\eta_6}^2 \sin \eta_5 + 2\hat{\theta}_1 \eta_6 \eta_8 \cos \eta_5 \\ \hat{\theta}_1 \eta_7 {\eta_6}^2 \cos \eta_5 + 2\hat{\theta}_1 \eta_6 \eta_8 \sin \eta_5 \end{bmatrix} \tag{5.15}
$$

with:

$$
v = x_m^{(4)} + \alpha_3 (x_m^{(3)} - \hat{\xi}_3) + \ldots + \alpha_0 (x_m - \hat{\xi}_1)
$$

and $\alpha_3 = 6, \alpha_2 = 13, \alpha_1 = 12, \alpha_0 = 4$ so that the resulting poles are at -1 and -2. The parameter update law is:

$$
\begin{bmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \end{bmatrix} = - \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \cdot W_2 \cdot P \cdot (\bar{\xi} - \hat{\xi}) \tag{5.16}
$$

where g_1 and g_2 are adaptation gains, P is the solution to $\bar{A}^T P + P\bar{A} = -I$, $\bar{A}$ as in (3.13) with α_i given above, and W_2 and $\bar{\xi}$ are given by:

$$
W_2 = \begin{bmatrix} 0 & -\eta_7 \sin \eta_5 + \epsilon \eta_7 \cos \eta_5 & 0 & 0 & 0 & \eta_7 \cos \eta_5 + \epsilon \tilde{u}_2 \sin \eta_5 & 0 & 0 \\ 0 & 0 & 0 & -\hat{\theta}_1 \eta_7 \tilde{u}_2 \cos \eta_5 & 0 & 0 & 0 & \theta_1 \eta_7 \tilde{u}_2 \sin \eta_5 \end{bmatrix} \tag{5.17}
$$

and:

$$
\dot{\bar{\xi}} = \begin{bmatrix} \eta_2 \\ -\hat{\theta}_1 \eta_7 \sin \eta_5 \\ -\hat{\theta}_1 \eta_6 \eta_7 \cos \eta_5 - \hat{\theta}_1 \eta_8 \sin \eta_5 \\ \hat{\theta}_1 \eta_7 {\eta_6}^2 \sin \eta_5 - 2\hat{\theta}_1 \eta_6 \eta_8 \cos \eta_5 \\ \eta_4 \\ -1 + \hat{\theta}_1 \eta_7 \cos \eta_5 \\ -\hat{\theta}_1 \eta_6 \eta_7 \sin \eta_5 + \hat{\theta}_1 \eta_8 \cos \eta_5 \\ -\hat{\theta}_1 \eta_7 {\eta_6}^2 \cos \eta_5 - 2\hat{\theta}_1 \eta_6 \eta_8 \sin \eta_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\hat{\theta}_1 \sin \eta_5 & -\hat{\theta}_1 \hat{\theta}_2 \eta_7 \cos \eta_5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hat{\theta}_1 \cos \eta_5 & -\hat{\theta}_1 \hat{\theta}_2 \eta_7 \sin \eta_5 \end{bmatrix} \cdot \begin{bmatrix} w \\ \tilde{u}_2 \end{bmatrix}
$$

$$+\begin{bmatrix} 0 \\ 0 \\ -\eta_7 \sin\eta_5 \\ -\eta_6\eta_7\cos\eta_5 - \eta_8 \sin\eta_5 \\ 0 \\ 0 \\ \eta_7 \cos\eta_5 \\ -\eta_6\eta_7 \sin\eta_5 + \eta_8\cos\eta_5 \end{bmatrix} \cdot \dot{\hat{\theta}}_1 + \begin{bmatrix} 0 \\ \epsilon\hat{\theta}_1 \tilde{u}_2/\cos\eta_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \bar{A}(\bar{\xi} - \hat{\xi}) \tag{5.18}$$

with $\bar{\xi}(0) = 0$.

Figures (1) and (2) show the performance of the (non-adaptive) nonlinear controller of [HSM89] for $\epsilon = 0.1$ and $\epsilon = 0.4$ with %25 uncertainty in M and J. While the tracking objective in the x-direction is achieved, the altitude (y) deviation is unacceptable for vertical takeoff and landing purpose which the PVTOL aircraft system design was originally meant for. On the other hand, the decoupling is much improved with the adaptive controller as shown in figures (3) and (4). The altitude deviation is about %90 better and the convergence is faster. The orientation of the PVTOL aircraft, for small ϵ, is the same in both adaptive and non-adaptive case. The simulations demonstrate the advantage of the adaptive controller proposed in section III.

V.II Example 2

We now consider a "benchmark" example of adaptive nonlinear regulation:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 \end{aligned} \tag{5.19}$$

which violates the conditions of [NA88, SI89, TKKS91, KKM91b]. In fact, this system is not feedback linearizable. It is, however, linearly controllable and as shown in [KK92], it is feedback linearizable to second degree by a dynamic state feedback. The linear part of (5.19) is already in Brunovsky form. Applying steps 3 and 4 in section IV gives for the extended system (with a 2-dimensional dynamic state feedback):

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 = x_4 + \gamma^{[2]}(x) + x_1 x_3 \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= v \end{aligned} \tag{5.20}$$

Then:

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= \mathit{Lin.\ \&\ Quad.\ part\ of}\ \dot{\hat{\xi}}_1 = x_2 + \hat{\theta} x_3^2 \\ \xi_3 &= \mathit{Lin.\ \&\ Quad.\ part\ of}\ \dot{\hat{\xi}}_2 = x_3 + 2\hat{\theta} x_3 x_4 \\ \xi_4 &= x_4 \\ \xi_5 &= x_5 \end{aligned} \tag{5.21}$$

$$\dot{\xi}_3 = x_4 + x_3 x_1 + \gamma^{[2]}(x) + 2\hat{\theta} x_4^2 + 2\hat{\theta} x_3 x_5 \tag{5.22}$$

Hence, $\gamma^{[2]}(x)$ may be chosen as:

$$\gamma^{[2]}(x) = -(x_3x_1 + +2\hat{\theta}x_4^2 + 2\hat{\theta}x_3x_5) \tag{5.23}$$

Therefore, the resulting system is transformed into:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + x_3^2 \cdot \phi \\ \dot{\xi}_2 &= \xi_3 + O(\xi, v)^3 \\ \dot{\xi}_3 &= \xi_4 + O(\xi, v)^3 \\ \dot{\xi}_4 &= \xi_5 \\ \dot{\xi}_5 &= v \end{aligned} \tag{5.24}$$

which is in the form given by (4.30). Finally, let:

$$\begin{aligned} v &= -K \cdot \xi \\ \dot{\hat{\theta}} &= -g \cdot W^T \cdot P \cdot \xi \end{aligned} \tag{5.25}$$

where $W^T = [x_3^2, 0, 0, 0, 0]$, g is an adaptation gain, and K is the gain vector that specifies the closed-loop poles. We note that the poles should be chosen not too far left in the left half plane since this will increase the initial magnitude of the control v due to nonzero initial states. This could in turn introduce large perturbations in the system and the approximate model will no longer stays a valid approximation of the true system.

Figure (5) shows the response of the adaptive quadratic regulator (5.25) for the initial state $x(0) = [3, 0, 0]^T$, $\theta = 2$, and %25 uncertainty in the initial estimate of the parameter θ. The Jacobian approximation, in comparison, resulted in an unstable system even when there was no parameter uncertainty present in the system. The non-adaptive quadratic approximation resulted in an unstable system in the presense of this uncertainty. The simulations illustrate the usefulness of the adaptive quadratic control scheme proposed in section IV in providing a parameter robust control for regulation of nonlinear systems that violate the regularity conditions needed to apply adaptive schemes based on exact feedback linearization.

V.III Example 3

Consider the following nonlinear system with unknown parameters θ_1 and θ_2:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1 x_3^2 + e^{\theta_2 x_3} - 1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + x_1 x_3 \end{aligned} \tag{5.26}$$

which is not feedback linearizable but it is linearly controllable and hence, can be quadratically feedback linearized with dynamic state feedback. We now apply the design scheme of section IV. The approximate quadratic system of equation (4.18) is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & \theta_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u + \begin{bmatrix} (\theta_1 + \frac{1}{2}\theta_2^2)x_3^2 \\ 0 \\ x_1x_3 \end{bmatrix} + O(x)^2 \tag{5.27}$$

which has all eigenvalues at zero. The transformation T of (4.19) is ($z = est.\ (T) \cdot x$):

$$T = \begin{bmatrix} 1 & -\theta_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.28}$$

which results in the following Brunovsky form quadratic system with $\bar{\theta}_1 \triangleq \theta_1 + 0.5\theta_2^2, \bar{\theta}_2 \triangleq \theta_2$:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u + \begin{bmatrix} \bar{\theta}_1 x_3^2 \\ 0 \\ x_1 x_3 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix} \cdot \phi_2 + O(x)^2 \tag{5.29}$$

Applying steps two and three, similar to the last example, suggests the following linearizing transformation and control law:

$$\begin{aligned} \xi_1 &= z_1 = x_1 - \hat{\theta}_2 x_2 \\ \xi_2 &= z_2 + \hat{\theta}_1 x_3^2 \\ \xi_3 &= z_3 + 2\hat{\theta}_1 x_3 \omega_1 \\ \xi_4 &= \omega_1 \\ \xi_5 &= \omega_2 \\ u &= \omega_1 + \gamma^{[2]}(x, \omega) \\ \gamma^{[2]}(x, \omega) &= -2\hat{\theta}_1(\omega_1^2 + x_3\omega_2) - x_3 x_1 \end{aligned} \tag{5.30}$$

which transforms (5.26) into:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + (x_3^2, x_3) \cdot \phi + O(\xi, v)^3 \\ \dot{\xi}_2 &= \xi_3 + O(\xi, v)^3 \\ \dot{\xi}_3 &= \xi_4 + O(\xi, v)^3 \\ \dot{\xi}_4 &= \xi_5 \\ \dot{\xi}_5 &= v \end{aligned} \tag{5.31}$$

with $\phi = (\bar{\theta}_1 - \hat{\theta}_1, \bar{\theta}_2 - \hat{\theta}_2)^T$. This system is in the form given in (4.30). Control law v and update law given in (5.25) can then be used here with slight modification in $W(x)$. Figure (6) shows the response to the non-adaptive controller with $\theta_1 = 2, \theta_2 = 1$ for the initial state $x(0) = [1, 0.3, 0.3]^T$. In comparison, figure (7) shows the performance of the adaptive controller which provided a better result for a wider range of uncertainties in θ_1 and θ_2. The Jacobian linearization, however, resulted in an unstable response for this initial state even when there is no parameter uncertainty present.

VI. Conclusion

In this paper we have presented an approach for the adaptive *approximate* feedback linearization of nonlinear systems under parameter uncertainty. The significance of this approach was demonstrated with its potential application to flight control systems where exact linearization approach fails and the non-adaptive controller produces undesirable performance. Compared to adaptive schemes that are based on exact state or input-output linearization, this approach avoids several restrictions such as involutivity, existence of a relative degree, and minimum phase property which are not often met in most complex engineering systems. In section IV, a systematic procedure for adaptive quadratic regulation of any linearly controlable nonlinear system was presented. The quadratic approximate model and the resulting parameter update laws are computed directly in terms of the Taylor series expansion and a simple change of coordinates, and can easily be generated by symbolic programming tools. It is also important to note that the uncertain parameters in the system are not required to appear *linearly* in the system dynamics since they always appear linearly in the *approximate model* possibly after a reparametrization. The feasibility domain of this scheme is local, in general, around a nominal operating point. This limitation is intrinsic of the local nature of approximate feedback linearization technique. The broad applicability of this scheme coupled with its systematic approach motivate its use by control engineers in parameter robust control design for nonlinear systems.

Acknowledgment

The authors are grateful to Drs. A. J. Krener and W. P. Dayawansa for numerous insightful discussions and helpful comments. They would also like to thank Dr. P. V. Kokotovic for valuable discussions and suggesting to consider the aircraft application discussed in section five.

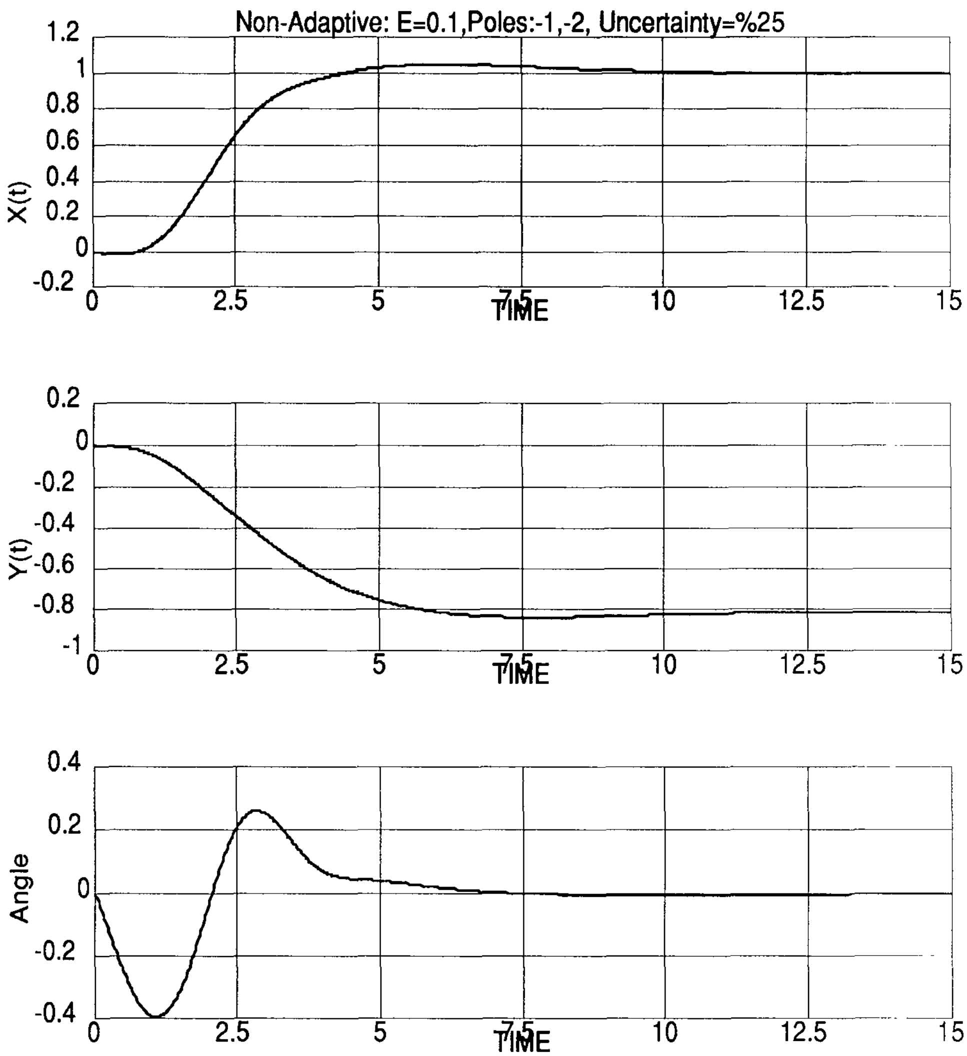

Figure 1: X-Y trajectories and orientation of the PVTOL aircraft in response to the non-adaptive controller with $\epsilon = 0.1$ under %25 parameter uncertainty.

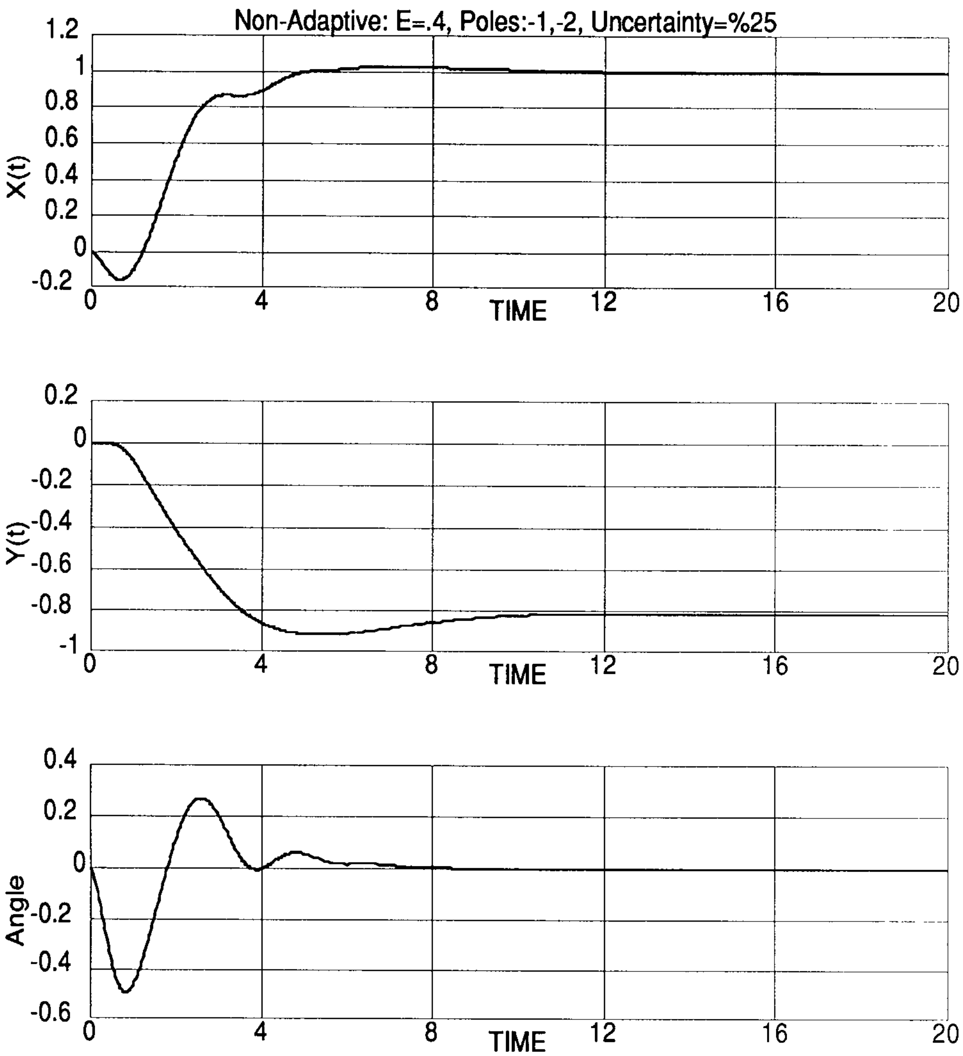

Figure 2: X-Y trajectories and orientation of the PVTOL aircraft in response to the non-adaptive controller with $\epsilon = 0.4$ under %25 parameter uncertainty.

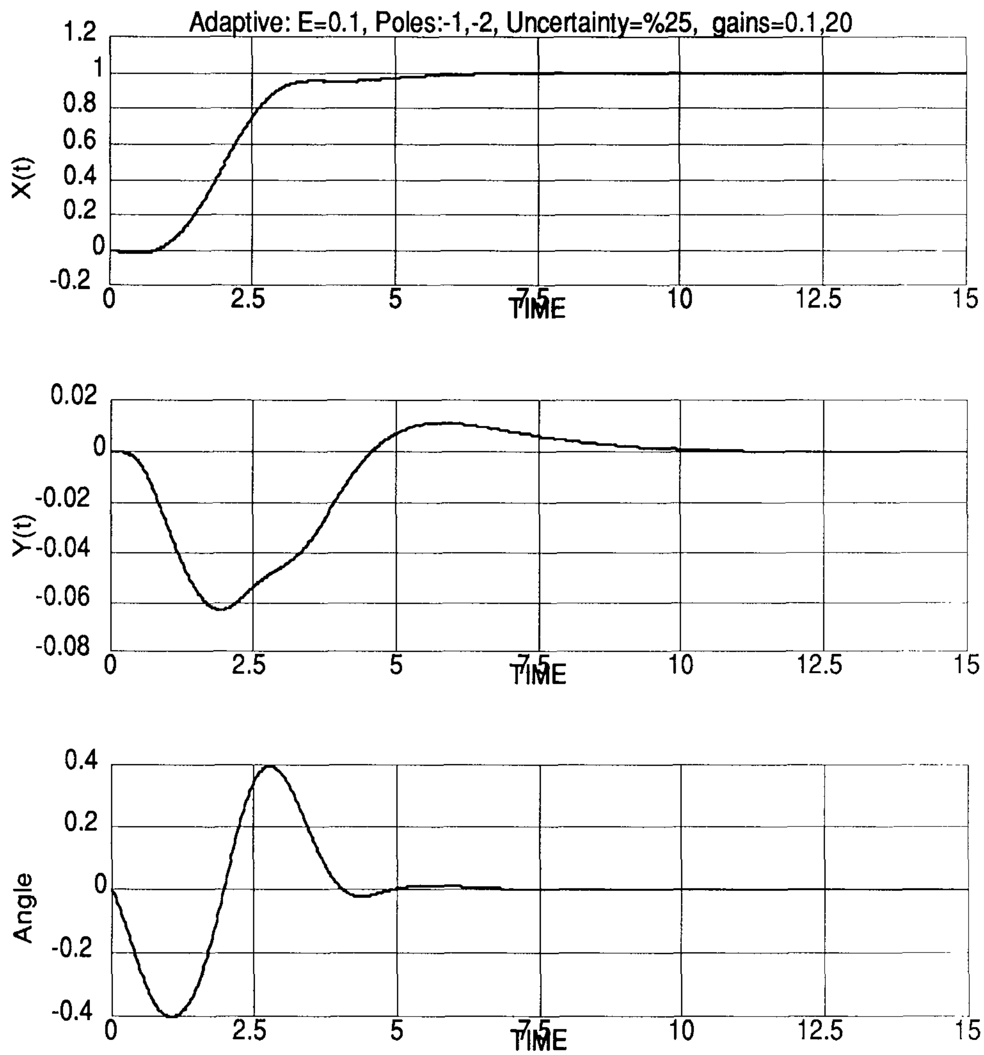

Figure 3: X-Y trajectories and orientation of the PVTOL aircraft in response to the adaptive controller with $\epsilon = 0.1$ under %25 parameter uncertainty.

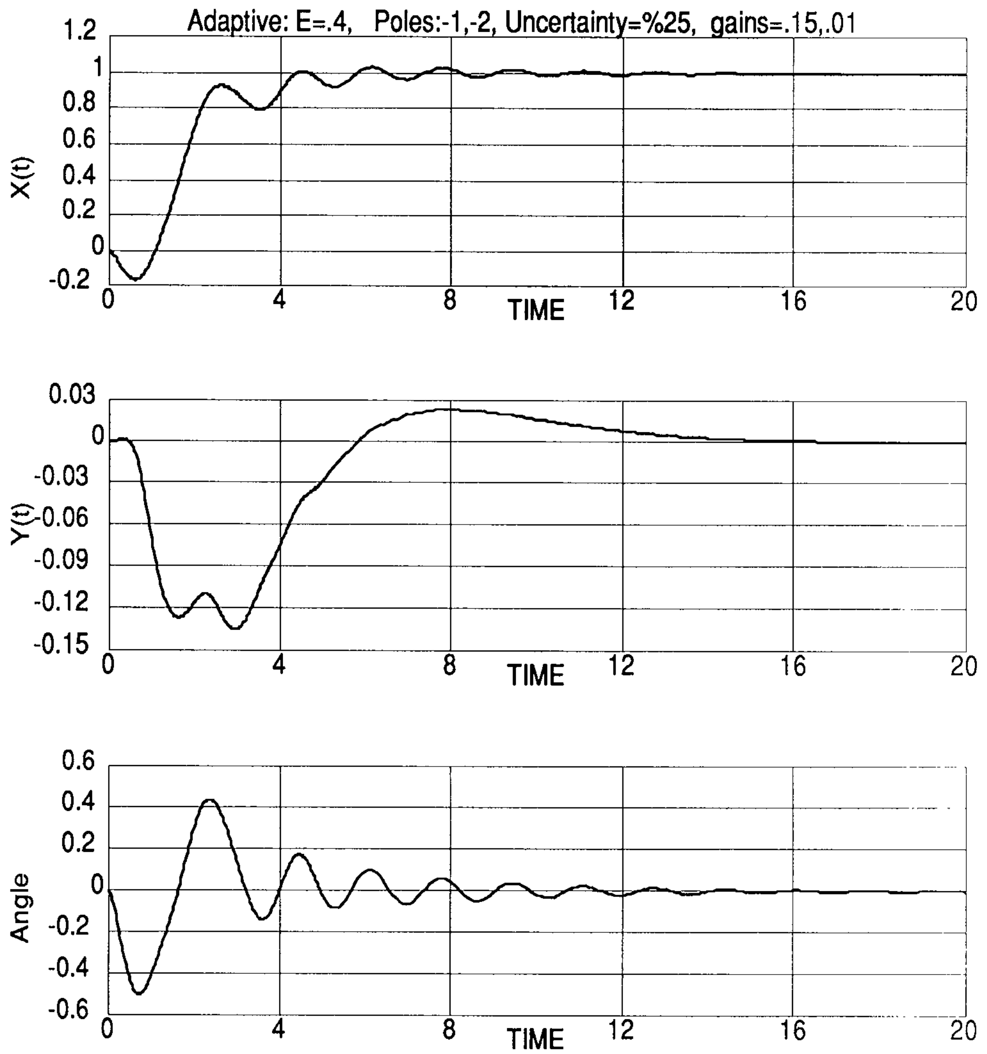

Figure 4: X-Y trajectories and orientation of the PVTOL aircraft in response to the adaptive controller with $\epsilon = 0.4$ under %25 parameter uncertainty.

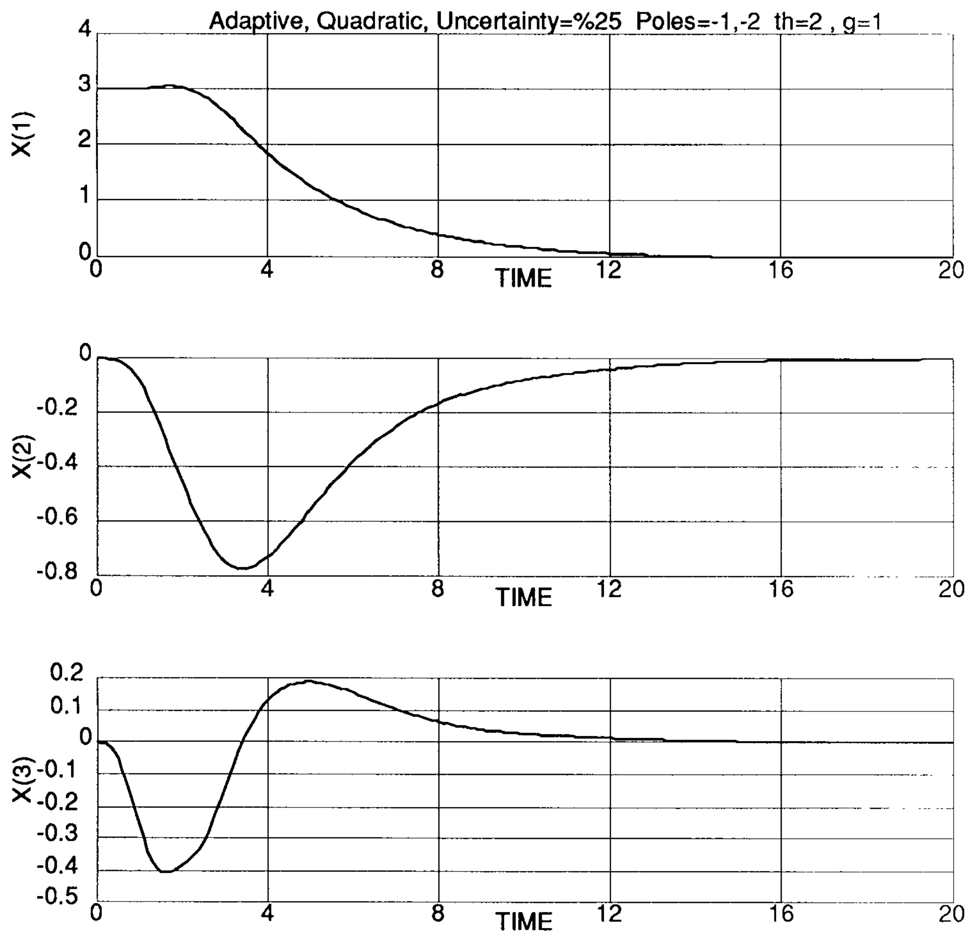

Figure 5: Example 2; state trajectories in response to the adaptive quadratic controller under %25 parameter uncertainty.

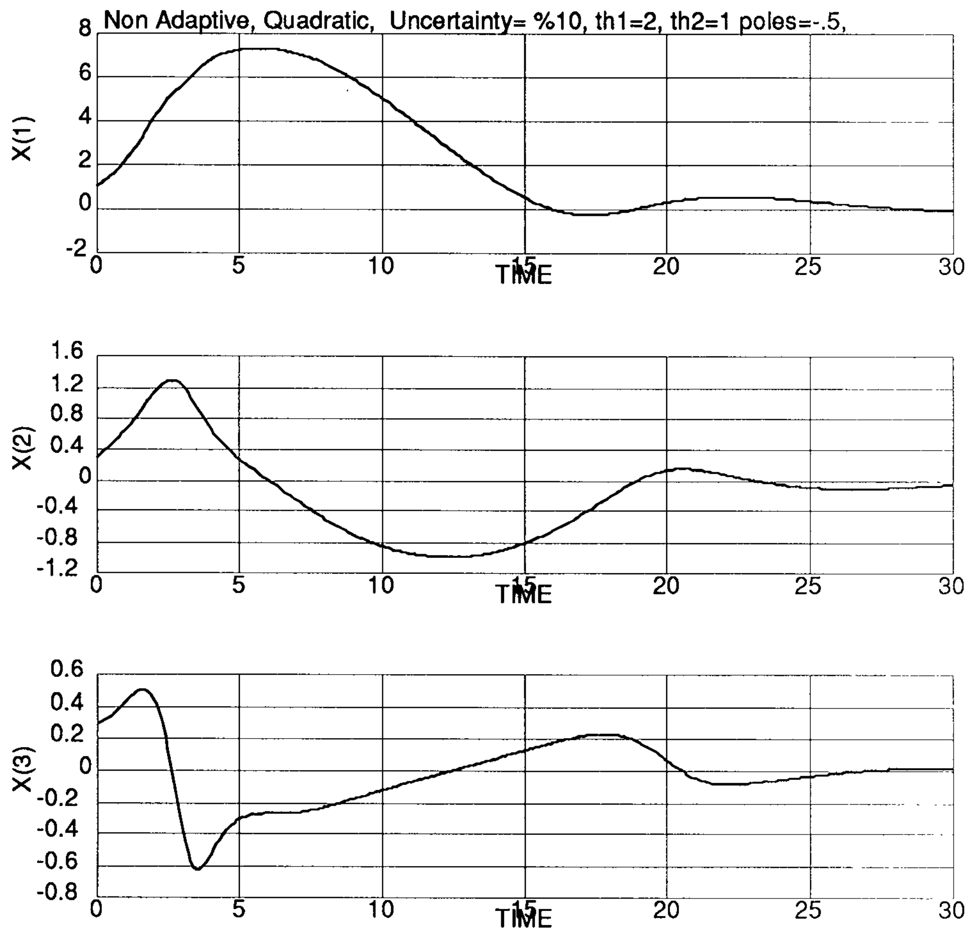

Figure 6: Example 3; state trajectories in response to the non-adaptive quadratic controller under %10 parameter uncertainty.

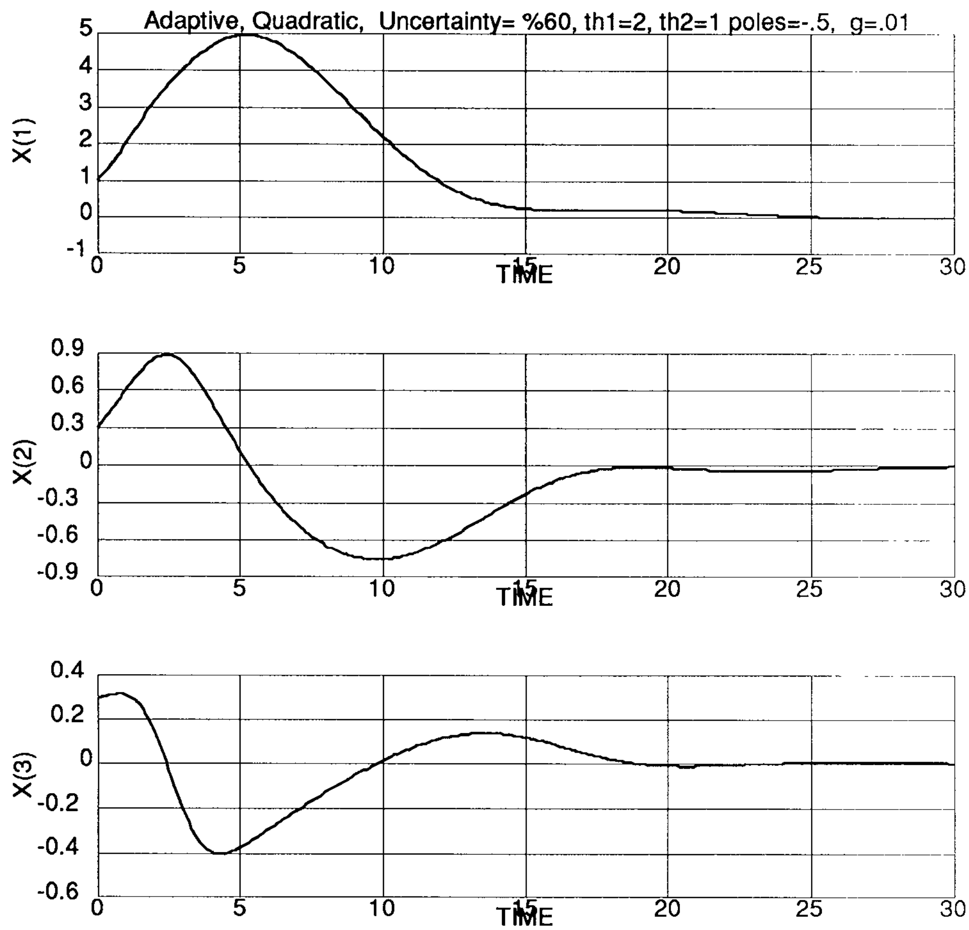

Figure 7: Example 3; state trajectories in response to the adaptive quadratic controller under %60 parameter uncertainty.

References

[Akh89] O. Akhrif. Nonlinear adaptive control with applications to flexible structures. *Ph. D. Dissertation, University of Maryland at College Park*, 1989.

[BR86] W. T. Baumann and W. J. Rugh. Feedback control of nonlinear systems by extended linearization. *IEEE Trans. Automatic Control*, 31,1:40–46, 1986.

[BS91] M. D. Di Benedetto and S. S. Sastry. Adaptive tracking for mimo nonlinear systems. *European Control Conference*, pages 1316–1321, 1991.

[CLM88] B. Charlet, J. Levine, and R. Marino. Dynamic feedback linearization and applications to aircraft control. *In Proceedings of IEEE Conference on Decision and Control, Austin, TX*, pages 701–705, 1988.

[CRM84] C. Champetier, C. Reboulet, and P. Mouyon. pseudolinearization of multi-input nonlinear systems. *In Proceedings of the 23th Conference on Decision and Control*, pages 96–97, 1984.

[DM90] F. J. Doyle and M. Morari. A conic sector-based methodology for nonlinear control design. *In Proceedings of the 1990 American Control Conference, San Diego, CA*, 1990.

[DM92] F. J. Doyle and M. Morari. A survey of approximate linearization approaches for chemical reactor control. *3rd IFAC Symp. on dynamics and control of chemical reactors, distillation columns and batch processes, College Park, Maryland*, April 1992.

[Hau91] J. Hauser. Nonlinear control via uniform system approximation. *System Control lett.*, pages 145–154, 1991.

[HD87] R. M. Hirschorn and J. Davis. Output tracking for nonlinear systems with singular points. *SIAM J. Control Opt.*, vol 25, No. 3:547–557, 1987.

[HSK92] J. Hauser, A. Sastry, and P. Kokotovic. Nonlinear control via approximate input-output linearization: The ball and beam example. *IEEE Trans. Automatic Control*, AC-37, No. 3:392–398, 1992.

[HSM83] L. R. Hunt, R. Su, and G. Mayer. Global transformations of nonlinear systems. *IEEE Trans. Automatic Control*, AC-28:24–31, 1983.

[HSM89] J. Hauser, A. Sastry, and G. Meyer. Nonlinear controller design for flight control systems. *in Proc. IFAC Symp. Nonlinear Control System Design, Capri, Italy*, pages 385–390, June 1989.

[Isi89] A. Isidori. *Nonlinear Control Systems.* Springer Verlag, Berlin, 1989.

[Kar88] S. Karahan. Higher degree approximations to nonlinear systems. *Ph. D. Dissertation, University of California, Davis, CA*, 1988.

[KK92] W. Kang and A. J. Krener. Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems. *SIAM J. Control Opt.*, 30, No. 6:1319–1337, November 1992.

[KKHF87] A. J. Krener, S. Karahan, M. Hubbard, and R. Frezza. Higher order linear approximations to nonlinear control systems. *In Proceedings of the 27th Conference on Decision and Control*, pages 519–523, 1987.

[KKM91a] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse. An extended direct scheme for robust adaptive nonlinear control. *Automatica*, 27:247–255, March 1991.

[KKM91b] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Trans. Automatic Control*, AC-36, No. 11:1241–1253, 1991.

[Kre84] A. J. Krener. Approximate linearization by state feedback and coordinate changes. *System Control lett.*, 5:181–185, 1984.

[Kre86] A. J. Krener. Normal forms for linear and nonlinear systems. *in Differential Geometry, The Interface between Pure and Applied Mathematics, M. Luksik, C. Martin, and W. Shadwick, eds., Contemporary Mathematics, Vol. 68, American Mathematical Society, Providence, Ri,*, pages 157–189, 1986.

[NA88] K. Nam and A. Arapostathis. A model-reference adaptive control scheme for pure-feedback nonlinear systems. *IEEE Trans. Automatic Control*, 33:803–811, September 1988.

[Nvds90] H. Nijmeijer and A. J. van der shaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, Berlin, 1990.

[PP89] J. B. Pomet and L. Praly. Adaptive nonlinear control: an estimation-based algorithm. In *New Trends in Nonlinear Control Theory*. J. Descusse, M. Fliess, A. Isidori, and D. Leborgne Eds., Springer-Verlag, Berlin, 1989.

[RC84] C. Reboulet and C. Champetier. A new method for linearizing nonlinear systems: pseudolinearization. *Int. J. Control*, 40,4:631–638, 1984.

[Rug86] W. J. Rugh. An extended linearization approach for nonlinear systems. *IEEE Trans. Automatic Control*, 31,8:725–733, 1986.

[SI89] S. S. Sastry and A. Isidori. Adaptive control of linearizable systems. *IEEE Trans. Automatic Control*, AC-34:1123–1131, 1989.

[TKKS91] A. Teel, R. Kadiyala, P. V. Kokotovic, and S. S. Sastry. Indirect techniques for adaptive input output linearization of nonlinear systems. *Int. J. Control*, 53:193–222, January 1991.

[TKMK89] D. G. Taylor, P. V. Kokotovic, R. Marino, and I. Kanellakopoulos. Adaptive regulation of nonlinear systems with unmodeled dynamics. *IEEE Trans. Automatic Control*, AC-34:405–412, April 1989.

[WR89] J. Wang and W. J. Rugh. On the pseudolinearization problem for nonlinear systems. *System Control lett.*, 12:161–167, 1989.