

ABSTRACT

Title of dissertation: A Study of Equivalence of SUSY Theories
using Adinkras and Super Virasoro Algebras

Isaac Samuel Chappell II, Doctor of Philosophy, 2012

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Supersymmetry (SUSY) theories describe a wide number of quantum field theories with supersymmetric particles interacting. By using two methods, Adinkras and Super Virasoro algebras (SVAs), more information is gained about SUSY theories: (a.) when two representation may be considered equivalent, that is, describing the same physics, and (b.) the derivation of OPE's that do not rely on Wick rotations. Adinkras [1] are objects that encode important information about the theory in graphs. These graphs can be translated into matrices through what is now called a Garden Algebra. In a specific example, ($d=4$, $N=4$ SUSY theories,) it is found that there are six classes of SUSY theories through studying the Adinkras by one definition. However, using a criterion that is motivated by physical considerations of four dimensional field theories, this number is reduced to only three. Super Virasoro Algebras are close relatives of Super Conformal Algebras that contain a Lie algebra. They can be used to find Operator Product Expansions which are related to two-point correlation functions. By comparison of two different realizations of SVAs

(the Geometrically Realization \mathcal{GR} and the one developed by Hasiewicz, Thielemans, Troost [2],) we show that one is contained inside the other which allows some new OPEs to be calculated.

A Study of Equivalence of SUSY Theories
Using Adinkras and Super Virasoro Algebras

by

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Dedication

To Debra:

Your love and support has been the difference in getting this done.

To Bootz, Houdini, Jack, and Shadoe:

To know them, is to love them.

And to love them, is to know them.

And not to know them, is to love them...from afar.

Table of Contents

List of Tables	vi
List of Figures	vi
1 Introduction	1
1.1 Supersymmetry and Adinkras	1
1.2 Superspace and Geometrically Realized Super Virasoro Algebra (\mathcal{GR} SVA)	3
1.3 Equivalence Classes of Adinkras	5
1.4 Calculation of Short Distance Operator Expansions by Co-adjoint Method	5
1.5 Relationship between Adinkra and SVAs	6
1.6 Outline of Dissertation	6
2 Adinkras	8
2.1 Basic Supersymmetry	8
2.1.1 Basic Concepts of Supersymmetry	8
2.2 Defintion of Adinkras	9
2.2.1 $N = 1$ Adinkras	10
2.2.2 $N = 2$ Adinkras	11
2.2.3 $N = 3$ Adinkras	13
2.2.4 $N = 4$ Adinkras and Three Supersymmetric Multiplets	14
2.2.4.1 Chiral Multiplet	15
2.2.4.2 Tensor Multiplet	15
2.2.4.3 Vector Multiplet	16
2.3 Introduction of the Garden Algebra ($\mathcal{GR}(d, N)$)	17
2.3.1 Shorthand Notation for L matrices	20
3 Discussion on the Equivalence Classes of Adinkras	24
3.1 Finding the Solutions to the (4,4) Garden Algebra	24
3.1.1 Using Mathematica TM to Find Solutions	24
3.1.2 Generation of Adinkrizable Solutions in the Space of $d = 4$, $N = 4$ Adinkras	25

3.2	Analysis of Adinkrizable Solutions in the $d = 4, N = 4$ Adinkras	29
3.3	Adinkra Equivalence Classes Determined by Transformations	32
3.3.1	Edge-Color Swap	34
3.3.2	Dashing Flip	34
3.3.3	Node Swap	34
3.3.4	Node Sign Flip	35
3.3.5	Klein Flip	35
3.4	Discussion of Equivalence of the $d=4, N=4$ Adinkras	36
3.4.1	A New Permutation Group Based Definition of Adinkra Equivalence Classes and Implications	36
3.4.2	Application of the Hodge Duality and the $*$ -map to the Fields	40
3.4.3	Comparison of the Klein Flip and Matrix Transposition Arguments for Equivalence	44
3.4.4	Combination with Previously Defined Equivalence Classes	45
4	The \mathcal{GR} SVA, Coadjoint Method, and OPEs	47
4.1	Introduction	47
4.2	Getting to the Geometrical Realization (\mathcal{GR}) of the 1D, $N = 4$ Extended Super Virasoro Algebra	47
4.2.1	The Super Conformal Algebra (SCA)	47
4.2.2	Redefinition of Operators for \mathcal{GR} SVA	50
4.3	Explanation of the Coadjoint Orbit Method of Deriving the Operator Product Expansion	52
4.3.1	Description of the Use of Coadjoint Orbits	52
4.3.2	Relationship to OPE	54
4.3.3	Pulling it all together	56
4.4	Application of the Coadjoint Orbit Method to the \mathcal{GR} SVA	57
4.4.1	Preliminaries	57
4.4.2	Example: $L * \tilde{L}$	58
4.4.3	Calculation of \mathcal{GR} SVA OPE's	59
4.4.4	Observations	61
4.4.5	Relationship between D_I and G_A^I	62

5	The Hasiewicz, Thielemans, and Troost (HTT) SVA, the Clifford Algebra, and Operator Product Expansions (OPE)	64
5.1	The HTT formulation of SVA	64
5.2	The Clifford Algebra Correspondence	68
5.3	Calculation of HTT SVA OPEs	71
5.4	Comparison of \mathcal{GR} and HTT SVA OPEs	77
5.5	Expansion of \mathcal{GR} SVA OPEs by HTT realization	80
5.6	$SU(2)$, $SO(3)$, and Clifford Algebra Correspondence	83
6	Conclusions	86
A	Mathematical Concepts	88
A.1	Grassman Variables	88
A.2	Clifford Algebras	88
A.3	Cartan Matrix	89
B	List of L-Matrices Solutions of the $(d = 4, N = 4)$ Garden Algebra	90
C	Other Group Theory Concepts Related to the Equivalence Classes of Adinkras	94
C.1	The $O(4)$ Group	94
C.2	The Group D_2	95
C.3	Graph Theory Ideas	95
D	Valise Adinkras of L-Matrices Solutions of the $(d = 4, N = 4)$ Garden Algebra	98
	Bibliography	103

List of Tables

3.1	Binary/Permutation Element Decomposition of L-matrices	29
4.1	SCA Generators and Their Associated Symmetries and Derivations . .	49
5.1	Correspondence between \mathcal{GR} Super Virasoro Algebra Generators . . .	70
5.2	Binary/Permutation Element Decomposition	85

List of Figures

1.1	An adinkra	2
2.1	N=1 Adinkras	10
2.2	Two $N = 2$ adinkras, depicting two supermultiplets	12
2.3	A $N = 3$ adinkra	14
2.4	The $N = 4$ Chiral Supermultiplet adinkra	16
2.5	The $N = 4$ Tensor Supermultiplet adinkra	17
2.6	The $N = 4$ Vector Supermultiplet adinkra	18
3.1	Space of $\mathcal{GR}(4, 4)$ matrices.	38
D.1	The $N = 4$ Chiral Supermultiplet Valise Adinkra	99
D.2	The $N = 4$ Tensor Supermultiplet Valise Adinkra	100
D.3	The $N = 4$ Vector Supermultiplet Valise Adinkra	100
D.4	The $N = 4$ VM1 Valise Adinkra	101
D.5	The $N = 4$ VM2 Valise Adinkra	101
D.6	The $N = 4$ VM3 Valise Adinkra	102

Chapter 1

Introduction

1.1 Supersymmetry and Adinkras

This melody of Supersymmetry has an innate beauty that can be expressed in many ways.

In this thesis two separate issues are discussed and resolutions are given to their respective questions.

A method for understanding off-shell representations of supersymmetry has been developed by various collaborations surrounding my advisor, S.J.Gates. This methodology has included a special class of algebras (given the names $\mathcal{GR}(d,N)$ or ‘garden algebras’ [3]), graphical representations (given the names ‘Adinkras’ [1,4], Figure 1.1) and the use of computer codes [5]. He has sometimes referred to this as the GAAC methodology (Garden Algebra/Adiinkras/Codes).

This is a purely algebraic approach which strives to develop an understanding of the realization of off-shell supersymmetry that is comparable to that which has long been the standard for the use of Lie groups in particle physics where notions like root and weight space diagrams are well understood tools for constructing representations. Some years ago this led to the notion that all of the information required to describe a four dimensional supersymmetrical theory is holographically stored in one dimensional supersymmetrical theories. This idea is called ‘SUSY holography.’

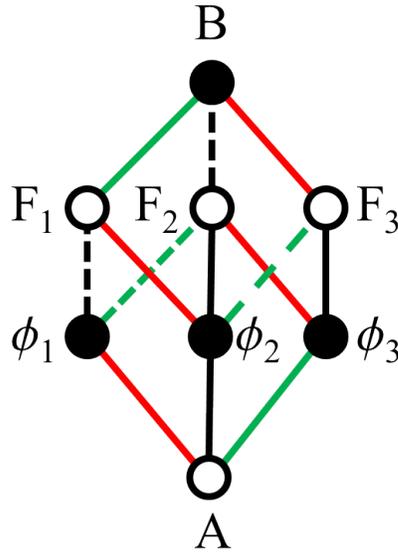


Figure 1.1: An adinkra

Along this line of approach, there has long been a desire to establish an intrinsic definition relying solely on algebraic structures intrinsic to the methodology and logically divorced from input of the four dimensional theory. Following this desire, such structure needed to be unambiguously identified. This thesis made a major breakthrough here. In the following it will be shown that the purely algebraic structures of the GAAC methodology contains a notion of class structure that follows from analyzing an involution operator acting on both the Garden Algebras representations and the Adinkra representations that appears to be the one dimensional remnant of the Hodge operator map acting in four dimensional field theories. The intrinsic definition takes advantage of Coxeter Algebras and a partitioning of the permutation group in a way that to our knowledge has never previously been observed in either the mathematical or physics literature.

The second question resolved within this thesis is one involving the development of the standard Operator Product Expansion (OPE). Throughout many works on superstring theory, there can be prominent use of OPE expansions as this is by now a standard tool. However, the standard derivation of this approach relies on the Wick rotation to the complex plane as an intermediary step. This is just fine as long as the rotation leads to hermitian Lagrangian after its implementation.

However, there are one class of theories for which this fails. Any Lagrangian that begins with chiral bosons described in the Siegel approach will fail to be hermitian after the Wick rotation. There was thus a desire to establish arguments that do not rely on the Wick rotation and yet permit the development of the the OPE type approach to investigate these theories.

For this purpose, Gates et. al. have developed tools called ‘geometrical representations’ [6,7] and ‘short distance expansions’ [8] which achieve the goal described above. In a second portion of this thesis these tools are applied to a particular class of supersymmetrical system and it is demonstrated that the goal of deriving the usual OPE methods in systems that utilize the Siegel description of chiral bosons can be implemented.

1.2 Superspace and Geometrically Realized Super Virasoro Algebra (\mathcal{GR} SVA)

One of the major tools used in SUSY theories is the concept of superspace. Just as supersymmetry uses specific relationships to place fermionic and bosonic

fields on the same footing, the concept of superspace creates a “fermionic” mathematical space that is on the same footing with regular (or “bosonic”) space. The mathematics uses special numbers called “Grassmann variables” with the property of anticommutivity, the same as fermion fields. The Grassmann variables allow fermion and boson fields to be calculated with respect to superspace variables and give a deeper understanding to the physics.

SUSY theories use the inherent symmetries in superspace. The most common space used for SUSY theory is based on the common Minkowski space of 3 spatial and 1 temporal dimensions. The natural symmetries of Minkowski space can also be expressed as algebra of transformations (spatial translations, spatial rotations, and boosts). This set of transformations makes a Poincaré algebra. But there are additional useful symmetries of superspace that can be included. By adding two elements that represent dilations and the special conformal transformation, the algebra becomes the Conformal algebra. A Super Conformal Algebra (SCA) is formed when two specific supersymmetry elements are put with the other members of the algebra. These elements of the SCA can be recast in a form that highlights a Super Virasoro Algebra (SVA) in the set of elements. The Virasoro algebra is more familiar in conformal field theory and string theory. The algebra can be represented in a number of ways. One can use operators, vectors, and matrices. In a number of papers ([6], [7]), a particular representation was used called the Geometrically Realized representation. This is because it is based on mathematical properties of the algebra with no assumptions about the theory described.

1.3 Equivalence Classes of Adinkras

Ideally, one would want any difference between equivalence classes to describe a different supersymmetric theory. There are some natural equivalence classes based on combinatorics and signs of the fields and links. It will be shown here that the combinatorial factors are fixed with respect to the solutions of the Garden Algebra equations. There are 6 combinatorial groups of matrices that form solution groups. There are fixed sets of sign factors that are related to those solutions. The underlying cycle representations are the basis of natural equivalence classes of the solutions under the operation of taking the transpose matrix.

1.4 Calculation of Short Distance Operator Expansions by Co-adjoint Method

Conformal Field Theory (CFT) is a field theory based on the operators from the conformal algebra mentioned above. The key point about a CFT is that it is a field theory based on an algebra of transformations that keep the metric of Minkowski space the same up to a scale factor. A familiar technique from CFT is the Operator Product Expansion (OPE) as it is closely related to the calculation of two-point correlation functions which themselves are related to the propagation and interaction of fields represented in SUSY theories.

The method used for calculating these OPEs is the Coadjoint action method discussed by [7]. It uses ideas about coadjoints from A. A. Kirillov [9] and is built upon the elements of Lie algebras and their realizations. The method involves going

from the closed algebra of operators and symmetries of the space to elements of a vector space. This vector space is then expanded with the dual space and a bilinear metric between the two. These objects are then used to find the coadjoint orbits which can be related to how the fields act under symmetry transformations. It is through these transformations, given by the adjoint action on coadjoint elements, that OPEs can be calculated from purely algebraic principles without the presence of an action from a physical theory.

1.5 Relationship between Adinkra and SVAs

How are adinkras and SVAs related? Both are tools for looking at SUSY theories and determining if they are alike. Adinkras use the language of graph theory and matrices to talk about equivalence of theories. Different representations of the SVA, which can describe different SUSY theories, must have the same elements and commutators underneath. The two concepts compliment each other: adinkras describe static structure of a theory and SVA OPEs describe dynamical formulas between SUSY fields. There exists a lot of opportunity gain from understanding both. This dissertation aims to highlight some findings in both.

1.6 Outline of Dissertation

The outline of the paper is as follows. Chapter 2 will introduce supersymmetry and Adinkras. Chapter 3 will focus on the most recent work on finding equivalence classes for Adinkras. In Chapter 4, the focus will be on the \mathcal{GR} SVA, the Coadjoint

method, and its application to calculate various short distance Operator Product Expansions for the algebra. In Chapter 5, a different representation of a SVA developed by [2] will be described and related to the GR SVA through the use of Clifford algebras and a discussion of some of the implications of the results of the research.

Chapter 2

Adinkras

2.1 Basic Supersymmetry

2.1.1 Basic Concepts of Supersymmetry

A supersymmetric theory is supersymmetric quantum field theory. In quantum field theory (QFT), particles correspond to various representations of the Lorentz group. By extending the Lorentz transformations with a supersymmetric transformation, new equivalence classes are created. Particles related by a supersymmetry transformation are called superpartners. If there are more than two particles, the set is called a supermultiplet.

Another important concept of QFT and SUSY theories, has to do with the equations of motion of the particles. If the physical particles obey the equations of motion given by the theory through the Lagrangian, then the particles are considered on-shell. If they do not obey the equations of motion, then they are off-shell. On-shell particles are mainly associated with real physical particles. An example of an off-shell particle is a virtual particle created in an interaction.

A SUSY theory associated with this research is the \mathcal{N} -extended SUSY theory. The \mathcal{N} relates to the count of the smallest spinors in the theory. In most of the theories discussed here, there will be $\mathcal{N} = 1$ spinor which will be related to the

$N = 4$ supercharges. This other N that represents the number of supersymmetric dimensions, and thus supercharges, in the theory. Basic SUSY theory only has $N = 1$ supercharge. If it has multiple supercharges, it is considered a N -extended SUSY theory.

2.2 Defintion of Adinkras

An adinkra is a graphical representation of a supermultiplet. An adinkra has nodes, colored black for fermions and white for bosons. Nodes of the two distinct color types must never appear at the same heights in a diagram. All heights are integrally spaced vertically. An adinkra has links between nodes and the links are colored with respect to the degree of \mathcal{N} -extended supersymmetry they represent. The links are also solid for a positive value and dashed for a negative value. The adinkras also satisfy a closed loop rule such that any closed loop of 4 links must have an odd number of negative signs (dashed lines).

The value in adinkras come from translating between the graphs into supersymmetric relationships between fields and thus supersymmetric theories. Each node represents a supersymmetric field, fermionic or bosonic in nature. The links represent the supersymmetry relations between the fields. The relation can be described using a super-charge operator Q_I or a covariant superspace derivative D_I acting the fields. The subscript I represents which of the number of super-charges is being specified and takes on values from 1 to N in the theory.

To remove the extra issue of having to deal with many different engineering

dimensions, one can write an adinkra where all the fermions are on one level and all the bosons are on a different level. This is called a valise adinkra. It allows one to focus on the supersymmetric connections between the fields but can make it hard to discriminate between different adinkras.

2.2.1 $N = 1$ Adinkras

The simplest case of supersymmetry is described by the $N = 1$ adinkras. There is only one supersymmetric operator, D , which turns fermions into bosons and bosons into fermions. There are two separate cases. In one, a scalar boson ϕ is transformed into a chiral fermion ψ . The equation describing the relationship is

$$D\phi = \psi \tag{2.1a}$$

$$D\psi = i\frac{d}{dt}\phi. \tag{2.1b}$$

The other case, a fermion χ is turned into a boson B . The supersymmetric relation is given by

$$D\chi = B \tag{2.2a}$$

$$DB = i\frac{d}{dt}\chi. \tag{2.2b}$$

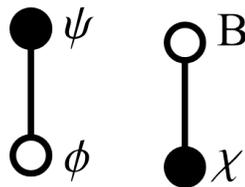


Figure 2.1: $N=1$ Adinkras

The adinkras for these two case are given in Figure 2.1. The two adinkras look similar. The key is to describe that similarity using mathematics in such a way as to be able to apply it to higher dimensional SUSY theories.

2.2.2 $N = 2$ Adinkras

The next case is $N = 2$ with two supersymmetric pairs of partners. A couple of additional rules are apparent in this case:

1. Every node has exactly $N = 2$ links corresponding to the number of different supersymmetric operators that can act on that field.
2. Every closed loop in the Adinkra must have an odd number of minus links in its path. A link can be positive or negative.
3. Every node can only have one unique link to a supersymmetric partner field, you can not have multiple links between the same two fields.

These rules only allow two basic types of Adinkras at $N = 2$, which may be called the bowtie and the diamond, respectively, in Figure 2.2.

The first adinkra has two pairs, a bosonic pair (ϕ_1, ϕ_2) and a fermionic pair (ψ_1, ψ_2) . The boson fields have the same engineering dimension $[\phi_1] = [\phi_2]$. The fermion field have the same engineering dimension $[\psi_1] = [\psi_2]$ but $[\psi_{\hat{k}}] = [\phi_i] + \frac{1}{2}$, for $i = 1, 2$ and $\hat{k} = 1, 2$. The fields are related supersymmetrically by Equations

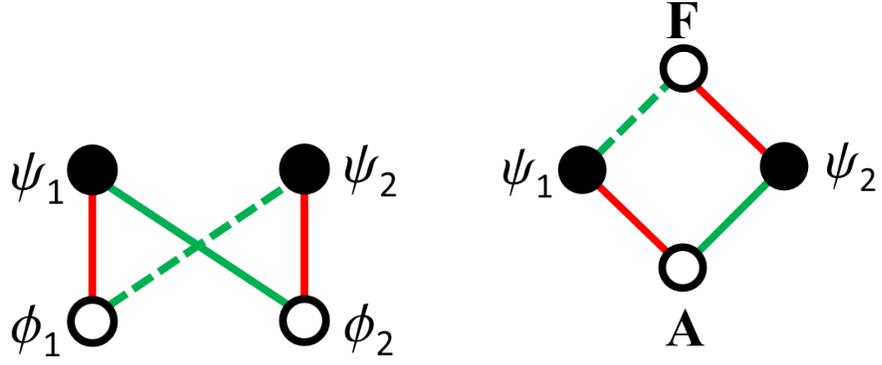


Figure 2.2: Two $N = 2$ adinkras, depicting two supermultiplets

2.3.

$$D_1\phi_1 = \psi_1 \quad (2.3a)$$

$$D_2\phi_1 = -\psi_2 \quad (2.3b)$$

$$D_1\phi_2 = \psi_2. \quad (2.3c)$$

$$D_2\phi_2 = \psi_1. \quad (2.3d)$$

The second consists of a scalar field (A), two spin-1/2 fermions fields (ψ_1, ψ_2), and a second scalar field F that possesses a different engineering dimension from the other boson (i.e. ϕ) in the multiplet. The engineering dimensions now satisfy $[F] - \frac{1}{2} = [\psi_1] = [\psi_2] = [A] + \frac{1}{2}$. The supersymmetry relations are in Eqns. 2.4.

$$D_1A = \psi_1 \quad (2.4a)$$

$$D_2A = \psi_2 \quad (2.4b)$$

$$D_1\psi_1 = F. \quad (2.4c)$$

$$D_2\psi_2 = -F. \quad (2.4d)$$

There are no other *inequivalent* Adinkras that can be formed that satisfy the rules. It may be better said that any other $N = 2$ Adinkra is equivalent to one of these up to certain symmetry operations. The known list of automorphisms acting on these graphs include

- Renaming the supercharges (red \leftrightarrow green)
- ‘Flipping’ solid links for dashed ones and vice-versa, while preserving an odd number of dashed links
- Renaming the node variables at the same fixed height ($\phi_1 \leftrightarrow \phi_2, \psi_1 \leftrightarrow \psi_2$)
- Changing the signs of all fields ($+\phi \leftrightarrow -\phi$)

Later on, these transformations will be discussed in mathematical terms. Adinkras use Graph Theory to describe SUSY theories. Graph Theory has ties to groups and algebras through the mathematical description of the transformations of graphs. Therefore, adinkras also allow SUSY theories to be explored with transformations of graphs. In particular, the adinkras and the SUSY theories are related to an algebra of matrices that will be described in the next section.

2.2.3 $N = 3$ Adinkras

For completeness, a $d=4, N=3$ adinkra is presented. From the diagram, one can easily see that the associated multiplet has a scalar field A , a fermion ϕ , a vector field F , and another fermion field K . The engineering dimensions are different at each

height of the adinkra with A being the lowest engineering dimension, $[A] = [\phi_i] - \frac{1}{2} = [F_i] - 1 = [K] - \frac{3}{2}$. It is important to note that the adinkra follows the same rules as in the previous section for $N=2$ Adinkras.

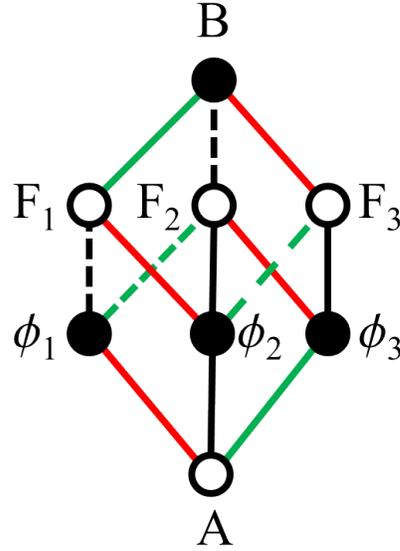


Figure 2.3: A $N = 3$ adinkra

2.2.4 $N = 4$ Adinkras and Three Supersymmetric Multiplets

There are many ways to make $N = 4$ adinkras. In [10], six adinkras were introduced. There were three supermultiplets each having an on-shell and off-shell representation. The off-shell cases will be presented for future reference. There remains a question if there are any other supermultiplets which describe the same theory.

2.2.4.1 Chiral Multiplet

The d=4, N=4 chiral multiplet (Figure 2.4) consists of 5 fields:

- a scalar field, A,
- a pseudoscalar field, B,
- a Majorana fermion field, ϕ_a ,
- a scalar auxiliary field, F, and
- a pseudoscalar auxiliary field, G.

The adinkra indicates the relationship between the fields. The superspace derivative D_a acting on each of the fields yields Eqns. 2.5e.

$$D_a A = \phi_a \tag{2.5a}$$

$$D_a B = i(\gamma^5)_a^b \phi_b \tag{2.5b}$$

$$D_a \phi_b = i(\gamma^\mu)_{ab} \partial_\mu A - i(\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - iC_{ab} F + (\gamma^5)_{ab} G \tag{2.5c}$$

$$D_a F = (\gamma^\mu)_a^b \partial_\mu \phi_b \tag{2.5d}$$

$$D_a G = i(\gamma^5 \gamma^\mu)_a^b \partial_\mu \phi_b \tag{2.5e}$$

2.2.4.2 Tensor Multiplet

The d=4, N=4 Tensor multiplet (Figure 2.5) has only 3 fields:

- a scalar field, φ ,
- a Majorana fermion, χ_a , and

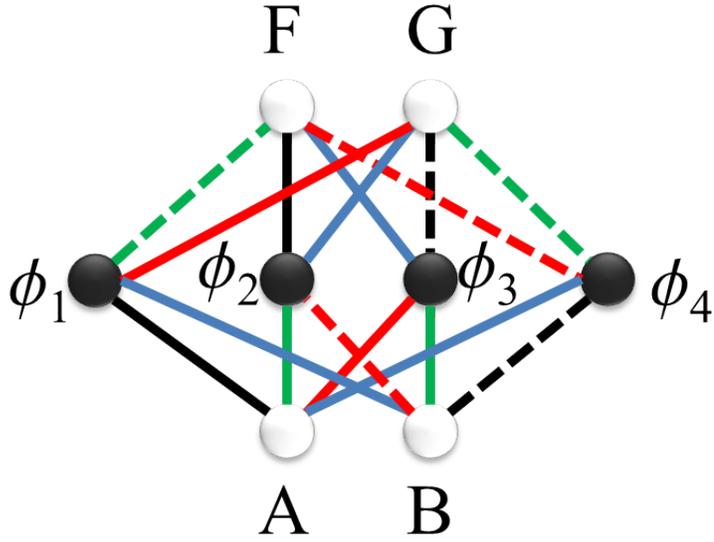


Figure 2.4: The $N = 4$ Chiral Supermultiplet adinkra

- a second-rank skew-symmetric tensor. $B_{\mu\nu}$

They are related by Eqn. 2.6c.

$$D_a \varphi = \chi_a \quad (2.6a)$$

$$D_a \chi_b = i(\gamma^\mu)_{ab} \partial_\mu \varphi - (\gamma^5 \gamma^\mu)_{ab} \epsilon^{\rho\sigma\tau} \partial_\rho B_{\sigma\tau} \quad (2.6b)$$

$$D_a B_{\mu\nu} = -\frac{1}{4} ([\gamma_\mu, \gamma_\nu])_a^b \chi_b \quad (2.6c)$$

2.2.4.3 Vector Multiplet

The $d=4, N=4$ Vector multiplet also has 3 field but of different types:

- an auxiliary pseudoscalar field, d ,
- a Majorana fermion, λ_b , and

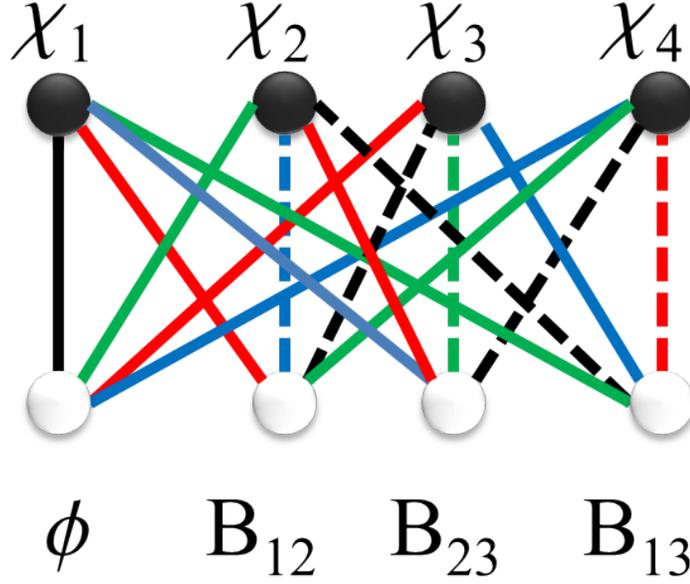


Figure 2.5: The $N = 4$ Tensor Supermultiplet adinkra

- a vector field, A_μ .

The superspace variations are given by Eqn. 2.7c

$$D_a d = i (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \lambda_b \quad (2.7a)$$

$$D_a \lambda_b = -i \frac{1}{4} ([\gamma_\mu, \gamma_\nu])_{ab} \partial_{[\mu} A_{\nu]} + (\gamma^5)_{ab} d \quad (2.7b)$$

$$D_a A_\mu = (\gamma_\mu)_a^b \lambda_b \quad (2.7c)$$

All of the multiplets are “off-shell” and some contain auxiliary fields.

2.3 Introduction of the Garden Algebra ($\mathcal{GR}(d, N)$)

In [10], a review of six supersymmetric multiplets (off-shell versus on-shell were counted as inequivalent for the purposes of the study) was given in terms of the fields

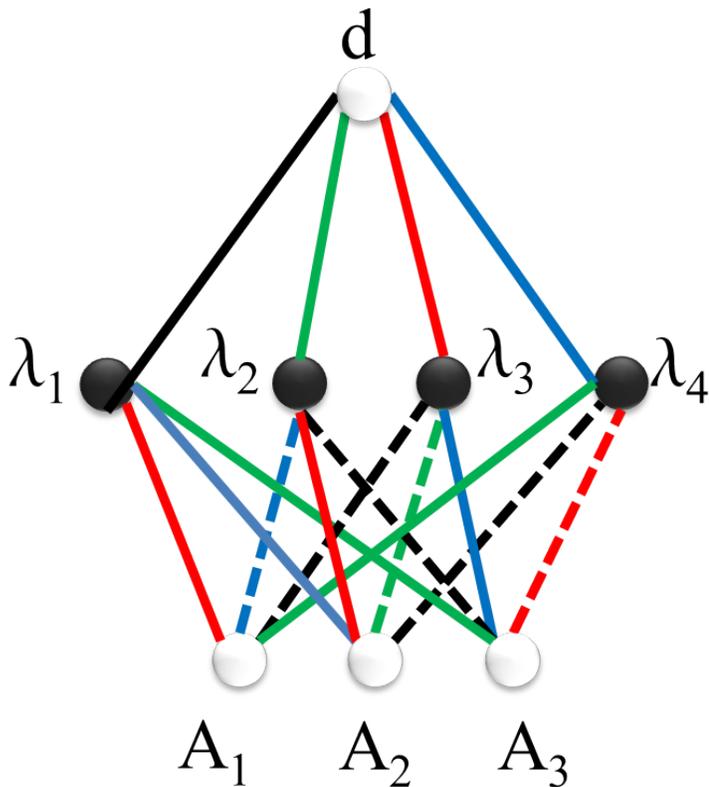


Figure 2.6: The $N = 4$ Vector Supermultiplet adinkra

and the superspace covariant derivative. The resulting equations for the supersymmetric relations between fermionic and bosonic fields were condensed into matrix equations. For example, the 1D, $N = 4$ chiral multiplet consists of the bosonic fields A, B, F, G and the fermionic fields $\psi_i (i = 1..4)$ with the superspace covariant derivatives D_I and time derivative ∂_t . By defining new fields Φ_i and Ψ_i in terms of the ordinal bosonic and fermionic fields, the supersymmetric system of equations can be written as

$$D_I \Phi_i = i (L_I)_{i\hat{k}} \Psi_{\hat{k}} . \quad (2.8)$$

and

$$D_I \Psi_{\hat{k}} = (R_I)_{\hat{k}i} \frac{d}{dt} \Phi_i . \quad (2.9)$$

The (L_I) and (R_I) are matrices which contain the relationship information about the fields. The matrices are related by the equation

$$(R_I) \equiv [(L_I)]^T \quad (2.10)$$

so that each R-matrix is fully specified in terms of the corresponding L-matrix, which in turn satisfy

$$(L_I)^T = (L_I)^{-1} \quad (2.11)$$

For the chiral multiplet, the L-matrices are given by

$$\begin{aligned} (L_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} , & (L_2)_{i\hat{k}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \\ (L_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} , & (L_4)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \end{aligned} \quad (2.12)$$

These matrices, along with their R-matrices counterparts, satisfy the equations

$$\begin{aligned} (L_I)_i^{\hat{j}} (R_J)_{\hat{j}}^k + (L_J)_i^{\hat{j}} (R_I)_{\hat{j}}^k &= 2 \delta_{IJ} \delta_i^k , \\ (R_I)_i^{\hat{j}} (L_J)_{\hat{j}}^{\hat{k}} + (R_J)_i^{\hat{j}} (L_I)_{\hat{j}}^k &= 2 \delta_{IJ} \delta_i^{\hat{k}} , \\ (R_I)_{\hat{j}}^k \delta_{ik} &= (L_I)_i^{\hat{k}} \delta_{\hat{j}\hat{k}} , \end{aligned} \quad (2.13)$$

which define the “ $\mathcal{GR}(d, N)$ ” or “(d, N) Garden Algebra.” The “d” is for the dimension ($d \times d$) of the matrices. This is also related to the number of fermionic (d_f) and bosonic (d_b) fields. The numbers d_f and d_b do not have to be equal but are ($d_f = d_b = d$) in this analysis. The “N” refers to the number of supersymmetric partners in the theory. For the majority of calculations used in this research, $N = 4$. There will be N L-matrices and N R-matrices in the algebra.

2.3.1 Shorthand Notation for L matrices

By inspection of the rules for making an Adinkra, it follows that for every ($d \times d$) L_I matrix we can write for every fixed I

$$(L_I)_{i\hat{k}} = (\mathcal{S}^{(I)})_i^{\hat{\ell}} (\mathcal{P}_I)_{\hat{\ell}\hat{k}} \quad \text{for each fixed } I = 1, 2, \dots, N \text{ and } i, \hat{\ell}, \hat{k} = 1 \dots d \quad (2.14)$$

where $\mathcal{S}^{(I)}$ is a diagonal matrix whose non-zero elements are ± 1 and \mathcal{P}_I is some matrix representation of the permutation group of d objects. The observations above allows us to easily count the number of possible Adinkrizable matrices that satisfy the Garden Algebra with the result

$$\#(L_I) = 2^d d! \quad (2.15)$$

However, looking at the requirements in 2.14 for these matrices, clearly given one set of matrices that solve all the conditions, multiplying this set by minus one will produce another solution. So we must divide by a factor of two to obtain the number of linearly independent solutions thus we arrive at

$$span[(L_I)] = 2^{d-1} d! \quad (2.16)$$

To clarify the meaning of the term “span,” let us consider the permutation group of four elements.

This permutation group has $4! =$ twenty-four elements. A natural representation of any one of these elements is given by a four by four matrix acting on a vector space whose bases are the elements to be permuted. The number of linearly independent four by four matrices is obviously sixteen. But there are twenty-four elements in the permutation group. So we use the word span above, we refer to a collection of matrices whose number is determined by the counting rules for independent permutations. We will use the shorthand notation to describe the 2 L matrices in the case of $d = 2, N = 2$ adinkras. From 2.8, the shorthand notation has a permutation part $\mathcal{P}_{(1)}$ and binary part, $\mathcal{S}^{(1)}$. We start with the permutation part and will use a subscript p to denote the elements in permutation notation. There are only 2 possible matrices for $\mathcal{P}_{(1)}$: $(12)_p$ and $(21)_p$.

There are only two possible $\mathcal{S}^{(1)}$ matrices up to an overall minus sign:

$$(\mathcal{S}^{(1)})_{i\hat{k}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (\mathcal{S}^{(2)})_{i\hat{k}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (2.17)$$

We can represent these matrices in another way:

$$(\mathcal{S}^{(1)})_{i\hat{k}} = \begin{bmatrix} (-1)^0 & 0 \\ 0 & (-1)^0 \end{bmatrix}, (\mathcal{S}^{(2)})_{i\hat{k}} = \begin{bmatrix} (-1)^0 & 0 \\ 0 & (-1)^1 \end{bmatrix} \quad (2.18)$$

However, there is an even more efficient way to store this data. Each of these matrices can be written in the form

$$(\mathcal{S}^{(I)})_{i\hat{k}} = \begin{bmatrix} (-1)^{p_{I1}} & 0 \\ 0 & (-1)^{p_{I2}} \end{bmatrix} . \quad (2.19)$$

for a string composed from the bits according to $(p_{I_1}p_{I_2})$. We will use the subscript b to denote the binary notation. We note that there is a unique natural number \mathcal{R}_I associated with this word via the map

$$(\mathcal{R}_I)_b = p_{I_1} 2^0 + p_{I_2} 2^1 \quad . \quad (2.20)$$

Writing the matrices in this fashion will be referred to as the Binary/Permutation method. This method highlights the different parts of the mathematical structures in the matrices. There is another method that will be used called the Overbar-Bracket notation. We replace the binary word with a overbar($\bar{}$) over the permutation positions that are negative. For example, $(10_b)(4231_p)$ is $\langle 4\bar{2}3\bar{1} \rangle$. The benefit of this notation is its conciseness in listing large numbers of matrices. The notations are equivalent on the permutation part, $(1234)_p \equiv \langle 1234 \rangle$ and will be used interchangeably on such.

We can quickly analyze the $N = 2$ case by hand. Using the notation (2.8)–(2.9) and restricting ourselves to the left-hand side adinkra in Figure 2.2, we read off the 2 ($= N$) L-matrices:

$$\mathbf{D}_1 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (\mathbf{L}_1)_i^{\hat{k}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftrightarrow \langle 12 \rangle ; \quad (2.21)$$

$$\mathbf{D}_2 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad (\mathbf{L}_2)_i^{\hat{k}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longleftrightarrow \langle 2\bar{1} \rangle ; \quad (2.22)$$

The corresponding words are $\mathbf{L}_1 = (0)_b \langle 12 \rangle = \langle 12 \rangle$ and $\mathbf{L}_2 = (2)_b \langle 21 \rangle = \langle 2\bar{1} \rangle$.

The total number of L-matrices is then $2^2 2! = 8$. Removing the overall minus sign

redundancy, we have 4 matrices

$$\langle 12 \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \langle 1\bar{2} \rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \langle 21 \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \langle 2\bar{1} \rangle = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (2.23)$$

All other L-matrices can be generated by an overall minus sign. It will prove useful to note that these matrices can be written in terms of Pauli matrices. Chapter 5 has more detail. Specifically, for the adinkras shown above we find $\mathcal{S}^{(1)} = \{\mathbb{1}_2, \boldsymbol{\sigma}^3\}$ and $\mathcal{P}_{(I)} = \{\mathbb{1}_2, \boldsymbol{\sigma}^1\}$, for $I = 1, 2$.

Chapter 3

Discussion on the Equivalence Classes of Adinkras

3.1 Finding the Solutions to the (4,4) Garden Algebra

3.1.1 Using MathematicaTM to Find Solutions

This investigation used MathematicaTM to investigate the space of L-matrices as solutions that provide realizations of the Garden Algebra equations 2.14. It was used to find all the solutions to the Garden Algebra equation provided by 4×4 matrices. In order to begin we concentrate on a special class of representations of the L-matrices. In many of our previous studies we have concentrated on ‘adinkrizable’ representations. In such a representation a single field via a supersymmetry transformation is mapped into solely one other field. Thus the form of the L_i contain one non-vanishing element in each row and one non-vanishing element in each column. Every non-zero element in the matrix is a ± 1 to represent edge parity.

This allows us to count the number of such matrices as a starting point from which to construct solutions.

3.1.2 Generation of Adinkrizable Solutions in the Space of $d = 4$,

$N = 4$ Adinkras

The matrices are generated algorithmically in two steps. The first step is to create the individual unsigned 4×4 matrices. Because of assumptions made above, the matrices can be generated from a permutation of 4 objects giving $4!$ or 24 matrices that represent the elements of the permutation group S_4 . The next step is to introduce all possible combinations of minus signs to generate all the matrices. This is done by creating a set of 4×4 diagonal matrices that have every possible combination of ± 1 as elements. There are $2^4 = 16$ of these. By taking a product of these two sets of matrices, all the possible $4! \times 2^4$, or 384, matrices are generated. But only 192 are linearly independent because of the overall minus sign.

We used a shorthand notation early on to describe these matrices. From (6), the shorthand notation has a permutation part $(\mathcal{P})_I$ and binary, $\mathcal{S}^{(I)}$. We start with the permutation part. A natural mapping exists between the matrix and the elements of S_4 used to generate it. We use cycle notation for S_4 for a shorthand and a subscript p to denote these elements are in permutation notation. For example the matrices listed above 2.12 for the chiral multiplet are $(1423)_p$, $(2314)_p$, $(3241)_p$, and $(4132)_p$.

For the $\mathcal{S}^{(I)}$ matrix, a mapping to binary numbers with $+1$ going to 0 and -1 going to 1 will work. We can calculate $\mathcal{S}^{(I)}$ for each of the matrices in 2.12 by using

2.14 with the results

$$\begin{aligned}
 (\mathcal{S}^{(1)})_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, & (\mathcal{S}^{(2)})_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
 (\mathcal{S}^{(3)})_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, & (\mathcal{S}^{(4)})_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.1}
 \end{aligned}$$

A clearly equivalent way to represent these is

$$\begin{aligned}
 (\mathcal{S}^{(1)})_{i\hat{k}} &= \begin{bmatrix} (-1)^0 & 0 & 0 & 0 \\ 0 & (-1)^1 & 0 & 0 \\ 0 & 0 & (-1)^0 & 0 \\ 0 & 0 & 0 & (-1)^1 \end{bmatrix} \\
 (\mathcal{S}^{(2)})_{i\hat{k}} &= \begin{bmatrix} (-1)^0 & 0 & 0 & 0 \\ 0 & (-1)^0 & 0 & 0 \\ 0 & 0 & (-1)^1 & 0 \\ 0 & 0 & 0 & (-1)^1 \end{bmatrix} \\
 (\mathcal{S}^{(3)})_{i\hat{k}} &= \begin{bmatrix} (-1)^0 & 0 & 0 & 0 \\ 0 & (-1)^1 & 0 & 0 \\ 0 & 0 & (-1)^1 & 0 \\ 0 & 0 & 0 & (-1)^0 \end{bmatrix}
 \end{aligned}$$

$$(\mathcal{S}^{(4)})_{i\hat{k}} = \begin{bmatrix} (-1)^0 & 0 & 0 & 0 \\ 0 & (-1)^0 & 0 & 0 \\ 0 & 0 & (-1)^0 & 0 \\ 0 & 0 & 0 & (-1)^0 \end{bmatrix} . \quad (3.2)$$

Then these matrices can be rewritten in the simple form

$$(\mathcal{S}^{(I)})_{i\hat{k}} = \begin{bmatrix} (-1)^{p_{I1}} & 0 & 0 & 0 \\ 0 & (-1)^{p_{I2}} & 0 & 0 \\ 0 & 0 & (-1)^{p_{I3}} & 0 \\ 0 & 0 & 0 & (-1)^{p_{I4}} \end{bmatrix} . \quad (3.3)$$

for a string composed from the bits according to $(p_{I1}p_{I2}p_{I3}p_{I4})$. The unique natural number \mathcal{R}_I associated with this word via the map

$$(\mathcal{R}_I)_b = p_{I1} 2^0 + p_{I2} 2^1 + p_{I3} 2^2 + p_{I4} 2^3 . \quad (3.4)$$

For example, the L-matrices for the chiral multiplet as given in Ref. [10] may be decomposed as:

$$(\mathbf{L}_I)_i^{\hat{k}} = (\mathcal{C}^{(I)})_i^{\hat{\ell}} \times (\mathcal{P}_{(I)})_{\hat{\ell}}^{\hat{k}} = (\mathcal{R}_I)_b (p_1 p_2 p_3 p_4)_p$$

$$(\mathbf{L}_1)_i^{\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (10)_b(1423)_p = \langle 1\bar{4}2\bar{3} \rangle ;$$

(3.5a)

$$(\mathbf{L}_2)_i^{\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (12)_b(2314)_p = \langle 23\bar{1}\bar{4} \rangle ;$$

(3.5b)

$$(\mathbf{L}_3)_i^{\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = (6)_b(3241)_p = \langle 3\bar{2}\bar{4}1 \rangle ;$$

(3.5c)

$$(\mathbf{L}_4)_i^{\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (0)_b(4132)_p = \langle 4132 \rangle .$$

(3.5d)

So the sign-numbers of the L-matrices shown here are $(10)_b$, $(12)_b$, $(6)_b$, and $(0)_b$;

they pertain to the 1D dimensional reduction of the chiral supermultiplet.

Each matrix is then checked with all the other matrices to see if they satisfy 2.14. It was found that for every matrix, there were 12 other matrices that were solutions. The list of solutions was stored and iterately checked to find a complete set of 4 matrices that solved 2.14 for all pairings of matrices. In the end, there were 16 sets of 4 matrices for every single matrix that solved 2.14. When only the unique tetrads of solutions were counted, there were 1536 solution tetrads to Eq. 2.14. This will be the starting point for looking at equivalence classes.

In terms of the Binary/Permutation Element decomposition, the three off-shell multiplets of [2] can be seen in the following table.

	L_1	L_2	L_3	L_4
CM	$(10)_b(1423)_p$	$(12)_b(2314)_p$	$(6)_b(3241)_p$	$(0)_b(4132)_p$
VM	$(10)_b(2413)_p$	$(12)_b(1324)_p$	$(0)_b(4231)_p$	$(6)_b(3142)_p$
TM	$(14)_b(1342)_p$	$(4)_b(2431)_p$	$(8)_b(3124)_p$	$(2)_b(4213)_p$

Table 3.1: Binary/Permutation Element Decomposition of L-matrices

3.2 Analysis of Adinkrizable Solutions in the $d = 4, N = 4$ Adinkras

Looking at the set of groups of solution matrices, we found an interesting pattern. The solution sets broke into smaller partitions of the permutation group of S_4 . More specifically, we had 6 sets composed of 4 matrices each that generate solutions to the Garden Algebra once the proper signs are included. These are given

by:

- $(1423)_p, (2314)_p, (3241)_p, (4132)_p$
- $(2413)_p, (1324)_p, (4231)_p, (3142)_p$
- $(1342)_p, (2431)_p, (3124)_p, (4213)_p$
- $(4123)_p, (1432)_p, (2341)_p, (3214)_p$
- $(3421)_p, (4312)_p, (2134)_p, (1243)_p$
- $(3412)_p, (4321)_p, (1234)_p, (2143)_p$

As we will make use of this partitioning later, it is useful to introduce some notation for the partitioned sets of quartets of elements of the permutation group as

$$\begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \\ \{CM\} \equiv \{ \langle 1423 \rangle, \langle 2314 \rangle, \langle 3241 \rangle, \langle 4132 \rangle \} , & & & (3.6a) \end{array}$$

$$\{VM\} \equiv \{ \langle 2413 \rangle, \langle 1324 \rangle, \langle 4231 \rangle, \langle 3142 \rangle \} , \quad (3.6b)$$

$$\{TM\} \equiv \{ \langle 1342 \rangle, \langle 2431 \rangle, \langle 3124 \rangle, \langle 4213 \rangle \} , \quad (3.6c)$$

$$\{VM_1\} \equiv \{ \langle 4123 \rangle, \langle 1432 \rangle, \langle 2341 \rangle, \langle 3214 \rangle \} , \quad (3.6d)$$

$$\{VM_2\} \equiv \{ \langle 3421 \rangle, \langle 4312 \rangle, \langle 2134 \rangle, \langle 1243 \rangle \} , \quad (3.6e)$$

$$\{VM_3\} \equiv \{ \langle 3412 \rangle, \langle 4321 \rangle, \langle 1234 \rangle, \langle 2143 \rangle \} , \quad (3.6f)$$

and it is interesting to note that if we use a matrix representation for each of element of the permutations indicated above, the following condition is satisfied by

all six sets,

$$\sum_{I=1}^4 \text{Tr}(\mathcal{S}_I) = 4 \quad . \quad (3.7)$$

The solution that describes the chiral multiplet is in the first set. The second contains the vector multiplet solution, and the third has the tensor multiplet solution. We can put all the L-matrices tetrad solutions in to valise adinkra graphs. These can be seen in Appendix C.3. We now have a definition of equivalence class with respect to removing the signs from the L-matrices.

We can now change the question and ask what are the equivalence classes with respect to these permutation elements. We start with the first set which corresponds to the chiral multiplet. Because the elements are fixed inside this group, we can just focus on a single element in this group, $(2314)_p$. For this element, there are 256 unique solution groups that solve 2.14. We can look at two methods of creating equivalence classes from the binary part.

In the first method, we focus on fixing a single permutation element to reduce the number of solutions. We can factor out 16 sets of groups as being the same initial matrix $(2314)_p$ multiplied by all possible ± 1 matrices $(i)_b, \{i \in 0 \dots 15\}$. Keeping with the solution from the chiral multiplet, we are left with 16 groups of 4 matrices that all contain $(12)_b(2314)_p$.

Looking at the sign codes of the other matrices in the solution groups, we find that there are only 6 sign codes. For $(2314)_p(12)_b$, they are $(0)_b, (5)_b, (6)_b, (9)_b, (10)_b$, and $(15)_b$. Upon closer inspection, we find that 3 are the exact opposite sign of the other three: $(0)_b = -(15)_b, (5)_b = -(10)_b$, and $(6)_b = -(9)_b$. So finally,

there are 3 sets of sign representations with a matrix and its opposite sign or flipped matrix. If we look at the solution groups for the opposite sign of $(12)_b$, which is $(3)_b$, we find the exact same solution group. This accounts for all the possible differences between solution groups.

The second method focuses on solutions without overall sign flips since that changes the “sign parity” of the matrices but not the underlying structure of the permutation part. Starting once again with the 256 and taking only solutions tetrads with only even parity binary matrices (e.g. $(0)_b$, $(2)_b$, $(4)_b$, etc.), this reduced the number of tetrads to 16 binary solution sets. With 16 binary solution sets for each of the 6 permutation tetrads, there exist 96 unique, even parity binary/permutation solution tetrads. The complete list can be found in App. A.

3.3 Adinkra Equivalence Classes Determined by Transformations

As mentioned in Section 2.2.2, there are a number of different transformations that can be done on Adinkras to get other Adinkras:

Edge-Color Swap: Renaming the supercharges, *i.e.*, swapping red \leftrightarrow green

Dashing Flip: ‘Flipping’ solid links for dashed ones and vice-versa, while preserving an odd number of dashed links

Node Swap: Renaming the nodes variable at the same fixed height ($\phi_1 \leftrightarrow \phi_2$, $\psi_1 \leftrightarrow \psi_2$)

Node Sign Flip: Changing the signs of some fields/nodes ($+\phi \leftrightarrow -\phi$)

Klein Flip: swapping the color of all nodes white \leftrightarrow black, *i.e.*, swapping bosons \leftrightarrow fermions

The first two of these correspond to outer automorphisms acting on the supercharges. The next two correspond to inner automorphisms acting on the fields of the representation. The final one corresponds to a Klein transformation that exchanges bosons for fermions and *vice versa* throughout the supermultiplet.

Now that we have all the solutions (and a simple way to talk about them), we can clearly denote the transformation and its effect on the solution matrices. The benefit of the binary/permutation representation is the to simplicity of dealing with some of the combinatorics associated with adinkra transformations. For example, switching the labels of the 1st and 2nd nodes in an adinkra correspond to a transposition of the 1st and 2nd elements in the state, *i.e.* $(abcd) \rightarrow (bacd)$. If we let $(abcd)$ represent a vector of the 4 bosons in a theory and $(\kappa\lambda\mu\nu)$ represent a vector of the 4 superpartner fermions and ask how does the supersymmetric variation map the bosons into the fermions. For a theory with a L-matrix of $L_2 = (12)_b(2314)_p$, we can apply (2.8) to see the following:

$$D_2(a, b, c, d)^t = i(L_2)_i^{\hat{k}} (\kappa, \lambda, \mu, \nu)^t \quad (3.8)$$

$$= i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{bmatrix} = i \begin{bmatrix} \lambda \\ \mu \\ -\kappa \\ -\nu \end{bmatrix}. \quad (3.9)$$

What this means is for drawing the adinkra for this case, the boson a is linked to fermion λ , b to μ , c to $-\kappa$, and d to $-\nu$. Eqn.(2.9) calls for the use of the corresponding $(R_2) = (L_2)^t$ matrix, acting on the vector of bosons and giving the supersymmetry transformation of the fermion vector. We now discuss each transformation.

3.3.1 Edge-Color Swap

The first transformation is simply a relabeling of the Adinkra. It is effectively relabeling the colors of the Adinkra. In terms of the L-matrices, it is shifting the indices so $L_1 \rightarrow L_2$, et cetera. This does not change the solution group that the original matrices were in.

3.3.2 Dashing Flip

The second transformation is equivalent to multiplying all the L_i matrices by -1 . Here again, the sign representations of the L_i would change. However, because the original solution group contains both the original and -1 flipped versions of the sign representations, the solution group is effectively the same.

3.3.3 Node Swap

The third transformation is a relabelling of the fields at a certain height. This corresponds to changing the order of the elements in one of the states, $(a_1 a_2 \dots a_i \dots a_j \dots) \rightarrow (a_1 a_2 \dots a_j \dots a_i \dots)$. The transformation is a permutation, \mathcal{P} , that can be applied to the

L_I matrix or the other state vector, $(\mu_1\dots)$. Applying it the L_I and more specifically the cycle part of the representation, definitely changes the matrices and therefore the solution. By applying \mathcal{P} to the other field state, it doesn't change the L_I and the solution group doesn't change.

3.3.4 Node Sign Flip

The fourth transformation involves changing the sign of one or more fields. This would involve a transformation of the sign representation of the L-matrices. This would not change the cycle part of the solution group but would change the sign part. As shown above, all the possible sign combinations are already a part of the solution group. Thus this would not effectively change the solution group.

3.3.5 Klein Flip

The fifth listed transformation switches the bosons for fermions and fermions for bosons. Mathematically, this exchanges the vectors Φ_i and $\Psi_{\hat{k}}$ in equations (2.8) and (2.9). To relate to the original formulation, we would have to switch the L_I 's for the R_I 's in the definitions. This is effectively mapping the matrix L_I to its transpose matrix $[(L_I)]^T$. One would think that this does not change the solution group. However upon inspection of all the permutation solution groups, we find something interesting.

The 1st solution group (which contains $(1432)_p$) is mapped to the 3rd solution group (which contains $(1342)_p$, the transpose of $(1432)_p$ in S_4). This gives a

relationship between the chiral multiplet and the tensor multiplet. All of the other solution groups, including the solution group for the Vector multiplet, map back to themselves under the operation of taking the transpose of the L- matrices.

Out of all these transformations, the third and fifth transformations are the only transformations that may change the cycle representation of the solution group. The fifth transformation only changes two of the solution groups into each other. The third transformation is the only one that changes the solution group completely. All the other transformations at most change the signs inside the solution group.

3.4 Discussion of Equivalence of the d=4, N=4 Adinkras

3.4.1 A New Permutation Group Based Definition of Adinkra Equivalence Classes and Implications

We can take things a step further by analyzing *only* the node swap and the Klein flip, and their effects in changing between permutation solution sets. The node swap can clearly change one of the 6 solution sets into another depending on the reassignment of fields. We cannot define an equivalence class around this because the transformation makes no distinction between the solution sets: we can map any solution set into any other solution set with no loss of generality. We return to these transformations at the end of this section.

The Klein flip however breaks the solution sets into three definite classes:

1. the two solution sets, $\{CM\}$ and $\{TM\}$, which are exchanged by the Klein flip;

2. the three solution sets, $\{VM\}$, $\{VM_1\}$ and $\{VM_2\}$, which the Klein flip maps to themselves, albeit up to some edge-color swapping;
3. the one solution set, $\{VM_3\}$, which the Klein flip leaves fully unchanged.

Let us consider this situation further. The action of transposition can also be considered directly on the permutation factors, $\mathcal{P}_{(1)}$. If one begins with one element of the permutation group \mathcal{A} , then the transposed element $*\mathcal{A}$ is simply the inverse, $*\mathcal{A} = \mathcal{A}^{-1}$, owing to Eq. (2.11). Under the action of this transposition operator, we find the sets satisfy

$$\begin{aligned}
*\{VM\} &= \{VM^{(c)}\} , \\
*\{CM\} &= \{TM^{(c)}\} , & *\{VM_1\} &= \{VM_1^{(c)}\} , & *\{VM_3\} &= \{VM_3\} . \\
*\{TM\} &= \{CM^{(c)}\} ; & *\{VM_2\} &= \{VM_2^{(c)}\} , & &
\end{aligned} \tag{3.10}$$

The “^(c)” superscript denoted that the L-matrices *within* the set have been permuted.

For the purposes of visualization, the space of 384 (192, up to an overall -1 sign) matrices (representing the elements of the Coxeter group BC_4) can be illustrated in terms of a pie chart where the sets $\{CM\}$, $\{TM\}$, $\{VM\}$, $\{VM_1\}$, $\{VM_2\}$, and $\{VM_3\}$ each occupy one-sixth of the area.

The Klein flip operation acting on the adinkras is in 1–1 correspondence with the $*$ -operation acting on the elements of the both the signed and the unsigned permutation groups, BC_4 and S_4 . Therefore, the partitioning (3.10) described also in the above enumeration as well as depicted in the pie-chart in figure 3.1 are all perfectly *intrinsic* to both BC_4 and S_4 , and so also to the complete solution set for the $\mathcal{GR}(4, 4)$ matrix algebra.

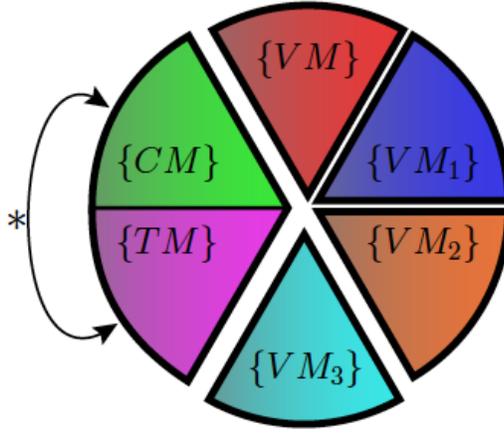


Figure 3.1: Space of $\mathcal{GR}(4,4)$ matrices.

In fact, this partitioning (3.10) induced by the action of the $*$ -map also follows from the elementary properties of the elements of the group of unsigned permutations, S_4 . Considering just the permutation factors of the $\{CM\}$, $\{TM\}$ and $\{VM\}$ sets in table 3.1 and the $\{VM_1\}$, $\{VM_2\}$, and $\{VM_3\}$ sets in Appendix A.3, we find:

1. The $\{CM\}$ and $\{TM\}$ permutation factors are all order-3, *i.e.*, their 3rd power equals $\mathbb{1}_4$. Moreover, each $\{CM\}$ permutation factor is the square of some $\{TM\}$ permutation factor, and also the other way around. This property pairs them, perfectly in line with the $*$ -map pairing (3.10) also depicted in figure 3.1.
2. The $\{VM\}$, $\{VM_1\}$ and $\{VM_2\}$ sets each have two permutation factors of order-2 and two of order-4, *i.e.*, their 2nd and 4th power equals $\mathbb{1}_4$, respectively.
3. Only the $\{VM_3\}$ set has the identity $\mathbb{1}_4$ as one of the permutation factors, and the remaining three are of order-2, *i.e.*, they square to $\mathbb{1}_4$.

Considering next only the sign-matrices, represented by their sign-numbers, we find:

1. The $\{CM\}$, $\{TM\}$ and $\{VM_3\}$ sets only use the odd permutations of the sign-number tetrads $\{(0)_b, (6)_b, (10)_b, (12)_b\}$ and $\{(2)_b, (4)_b, (8)_b, (14)_b\}$, a total of 24 sign-tetrads.
2. Furthermore, each of these 24 sign-tetrads appears in two of the $\{CM\}$, $\{TM\}$ and $\{VM_3\}$ sets, none in all three. Stated differently, eight of the 24 sign-tetrads appear in $\{CM\}$ and $\{TM\}$, eight in $\{CM\}$ and $\{VM_3\}$, and the last eight in $\{TM\}$ and $\{VM_3\}$.

On the other hand,

3. The $\{VM\}$, $\{VM_1\}$ and $\{VM_2\}$ sets only use the even permutations of the sign-number tetrads $\{(0)_b, (6)_b, (10)_b, (12)_b\}$ and $\{(2)_b, (4)_b, (8)_b, (14)_b\}$, a total of 24 sign-tetrads.
4. Furthermore, each of these 24 sign-tetrads appears in two of the $\{VM\}$, $\{VM_1\}$ and $\{VM_2\}$ sets, none in all three. Stated differently, eight of the 24 sign-tetrads appear in $\{VM\}$ and $\{VM_1\}$, eight in $\{VM\}$ and $\{VM_2\}$, and the last eight in $\{VM_1\}$ and $\{VM_2\}$.

This partitioning of the 48 sign-tetrads (all the permutations of $\{(0)_b, (6)_b, (10)_b, (12)_b\}$ and of $\{(2)_b, (4)_b, (8)_b, (14)_b\}$, taken up to overall sign) is consistent with the partitioning (3.10) of the (unsigned) permutations. Therefore, that the partitioning (3.10), as depicted in figure 3.1 extends from the (unsigned) permutation group S_4 to the full signed permutation group, BC_4 , and thus also to the space of matrix representations of $\mathcal{GR}(4, 4)$ and the corresponding adinkras. Finally, since adinkras faithfully depict 1D supermultiplets of N -extended supersymmetry which admit a

basis of component fields wherein each supercharge transforms each component fields into another component field or its derivative, the same partitioning also extends to these supermultiplets.

It is then highly suggestive to expect various different equivalence classes of $\mathcal{GR}(4, 4)$ representations—such as those depicted in figure 3.1—to in fact correspond to different supermultiplets. It has been shown in this paper that combinatorial factors are fixed with respect to the solutions of the Garden Algebra equations. There are 6 combinatorial sets of 4 matrices that form solution sets. There are fixed sets of sign factors that are related to those solutions. The underlying permutation representations are the basis of natural equivalence classes of the solutions under the $*$ -map operation, of taking the transpose matrix.

Going back to [10], we ask what are the implications of this definition of equivalence class based on the transpose matrix operation. The vector multiplet as defined there turns up in the class (3.6b) which is inert under the action of matrix transposition. Similarly, the chiral multiplet and tensor multiplet (as identified in Ref. [10]) turn up in the distinct pair of classes (3.6a) and (3.6c), which are mapped into each other by the $*$ -map, implemented as the matrix transposition operation on the L-matrices.

3.4.2 Application of the Hodge Duality and the $*$ -map to the Fields

The Hodge Duality is related to the mapping of fields to each other based on their dimensionality with respect to the overall dimensionality of the space they are

in. A specific example will be instructive.

For a 4-dimensional space, we can start with a scalar field ϕ , a vector field A_μ , and a second rank antisymmetric tensor $b_{\mu\nu}$ (i.e. $B_{\nu\mu} = -B_{\mu\nu}$). These fields can be used as potentials for other fields:

$$f_\mu = \partial_\mu \phi, \quad (3.11)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \text{ and} \quad (3.12)$$

$$h_\mu = \frac{1}{3!} \epsilon^{\kappa\lambda\mu\nu} \partial_\lambda B_{\mu\nu}. \quad (3.13)$$

We can use mathematical identities to construct simple algebraic equations for all of these new fields:

$$\partial_\mu f_\nu - \partial_\nu f_\mu = 0 \quad (\text{symmetry of the double derivative}) \quad (3.14)$$

$$\epsilon^{\kappa\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 \quad (\text{Bianchi identity}) \quad (3.15)$$

$$\partial_\mu h_\nu = 0 \quad (\text{symmetrization of antisymmetric tensor}) \quad (3.16)$$

From a physics perspective, an action can be constructed for each of the new fields also.

$$S_0 = -\frac{1}{2} \int d^4x f^\mu f_\mu \quad (3.17)$$

$$S_1 = -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu} \quad (3.18)$$

$$S_2 = \frac{1}{2} \int d^4x h^\mu h_\mu \quad (3.19)$$

$$(3.20)$$

We can vary each one of these actions with respect to the respective fields to get equations of motion. We find that the equations of motion are exactly the

algebraic identities found above in Equation 3.16 with the following mappings:

$$\frac{\delta S_0}{\delta \phi} = 0 \longrightarrow \partial^\mu f_\mu = 0 \quad \Longrightarrow \quad f_\mu \leftrightarrow h_\mu \quad (3.21)$$

$$\frac{\delta S_1}{\delta A_\mu} = 0 \longrightarrow \partial^\mu F_{\mu\nu} = 0 \quad \Longrightarrow \quad F_{\mu\nu} \leftrightarrow \epsilon^{\kappa\lambda\mu\nu} F_{\mu\nu} \quad (3.22)$$

$$\frac{\delta S_2}{\delta B_{\mu\nu}} = 0 \longrightarrow \partial_\mu h_\nu - \partial_\nu h_\mu = 0 \Longrightarrow h_\mu \leftrightarrow f_\mu \quad (3.23)$$

It is obvious that the roles of the equations of motion and algebraic identities become “switched” as we go from the scalar field theory to the rank two tensor one. The field f_μ which was associated with a scalar field ϕ is now mapped to h_μ , associated with the tensor $B_{\mu\nu}$ and vice-versa. The field $F_{\mu\nu}$ with vector field A_μ is mapped to $\epsilon^{\kappa\lambda\mu\nu} F_{\mu\nu}$ which implies a \tilde{A} . All this is only possible in $d=4$ with the antisymmetric Levi-Civita tensor with 4 indices. It is this kind of mapping based on the dimensionality of the space that is called the Hodge duality. The fact that in the presence of the equations of motion (i.e. on-shell) these two systems are exactly the same implies that they both ϕ and $B_{\mu\nu}$ describe a spin-0 degree of freedom.

So the bottom line is that a Hodge duality transformation switches the fields according to:

$$\phi \leftrightarrow B_{\mu\nu} \quad , \quad A_\mu \leftrightarrow \tilde{A}_\mu \quad (3.24)$$

and when one works out the consequences for the electromagnetic case, the effect is the exchange of the electric with the magnetic field and vice-versa.

This observation comes together beautifully with the structure seen in (3.10) if we identify the dual map defined on the elements of the permutation group with a Hodge star-like map acting on the space of fields in the four dimensional field theory. Under this duality, a chiral supermultiplet is replaced by a tensor supermultiplet and

vice-versa. We may consider a mapping between the fields in the two multiplets and we find that $A \leftrightarrow \varphi$ and $\psi_a \leftrightarrow \chi_a$ by inspection. This would further imply that all the fields B , F , and G of the chiral multiplet are mapped to the components¹ B_{ij} of the skew-symmetric tensor $B_{\mu\nu}$. Furthermore under this duality, a vector supermultiplet maps into another vector supermultiplet. All of these observations are consistent with the equations seen in (3.10).

So what we have shown is that equivalence classes defined by the Klein flip and degree flip on the L_I 's for the $d = 4, N = 4$ supermultiplets are the same as the action of the Hodge $*$ -map on the sets of fields of the supermultiplets. Thus, just as the Klein flip turns the valise adinkra and matrix solutions of the Chiral supermultiplet into the valise adinkra and matrix solutions of the Tensor supermultiplet, the Hodge $*$ -map maps the supersymmetric fields of the Chiral multiplet directly into the supersymmetric fields of the Tensor multiplet. The Klein flip on 3 of the Vector supermultiplets rearranges the matrices inside the solution tetrad but doesn't fundamentally change the number or types of fields, exactly like how the $*$ -map takes vector fields to vector fields. The remaining Vector supermultiplet is left invariant by the Klein flip and the $*$ -map.

¹ Recall that in the construction of any adinkra for a component gauge field, only the field components in the Coulomb gauge occur in an adinkra

3.4.3 Comparison of the Klein Flip and Matrix Transposition Arguments for Equivalence

To see how the matrix transposition argument and the Klein flip argument are really one and the same, one starts with the differential matrix equations which define the Garden Algebra (2.8) and (2.9)

$$D_I \Phi_i = i (L_I)_i^{\hat{k}} \Psi_{\hat{k}} , \quad (3.25)$$

$$D_I \Psi_{\hat{k}} = (R_I)_{\hat{k}}^i \frac{d}{dt} \Phi_i . \quad (3.26)$$

and now after applying a Klein flip (where Φ always denotes a boson and Ψ always denotes a fermion) operator, this becomes

$$D_I \Psi_i = (L_I)_i^{\hat{k}} \Phi_{\hat{k}} , \quad (3.27)$$

$$D_I \Phi_{\hat{k}} = i (R_I)_{\hat{k}}^i \frac{d}{dt} \Psi_i . \quad (3.28)$$

However, in order to have the bosons at the lowest level of the adinkra, one must make a redefinition using a degree flip

$$\Phi_{\hat{k}} \rightarrow \frac{d}{dt} \Phi_{\hat{k}} , \quad (3.29)$$

and now after applying a Klein flip and the degree flip, the equations in (3.25) and (3.26) become

$$D_I \Phi_{\hat{k}} = i (R_I)_{\hat{k}}^i \Psi_i , \quad (3.30)$$

$$D_I \Psi_i = (L_I)_i^{\hat{k}} \frac{d}{dt} \Phi_{\hat{k}} . \quad (3.31)$$

So that the net is that one accomplishes the interchange

$$(L_I)_i^{\hat{k}} \leftrightarrow (R_I)_{\hat{k}}^i \quad (3.32)$$

of transposing the L -matrices where the final valise has bosons at the bottom level and fermions at the top level as did the starting one. In this final result, the ‘hatted latin indices’ are associated with the bosons and the ‘unhatted latin indices’ are associated with the fermions.

3.4.4 Combination with Previously Defined Equivalence Classes

In [14], a different set of equivalence classes was found. These are the cis- and trans-Adinkras based on calculations of traces of the L_I matrices. Specifically,

$$\text{Tr}[L_I(L_J)^T] = 4(n_c + n_t)\delta_{IJ} \quad (3.33)$$

$$\text{Tr}[L_I(L_J)^T L_K(L_L)^T] = 4(n_c + n_t)(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \quad (3.34)$$

$$+ 4(n_c - n_t)\epsilon_{IJKL} \quad (3.35)$$

The cis-Adinkra has $n_c = 1$ and $n_t = 0$ while the trans-Adinkra has $n_t = 1$ and $n_c = 0$. The “physical” interpretation is that the cis-Adinkra is a mirror reflection of the trans-Adinkra about a particular color axis. The numbers n_c and n_t are called SUSY enantiomer number, a reference to the enantiomer values in chemistry. The Chiral Supermultiplet is in the cis-Adinkras class, and the Vector and Tensor Adinkras are in the trans-Adinkra class. These classes are the key building blocks to building larger supermultiplets.

Now, combined with the equivalence classes just described in the previous section, we can break the 3 supermultiplets into individual classes. The Chiral multiplet has $n_c = 1$ and turns into the Tensor multiplet under transposition of the L_I matrices. The Tensor multiplet turns into the Chiral multiplet under transpo-

sition but has $n_t = 1$. The Vector multiplet has $n_t = 1$ but maps into itself under transposition.

Thus, for a given Adinkra, one can write the L_I matrices that solve the Garden Algebra equations. From these matrices, one can calculate the SUSY enantiomer values and take their transposes and know what kind of basic Adinkra one has. This will also help with classifying larger SUSY multiplets in the future.

Chapter 4

The \mathcal{GR} SVA, Coadjoint Method, and OPEs

4.1 Introduction

In this Chapter, the Geometrically Realized Super Virasoro Algebra and the Coadjoint method, will be introduced and used to generate dynamical statements about SUSY theories through the Operator Product Expansion. The work was pioneered in [6,7,8]. The research done here aimed at understanding their work and expanding it in the next Chapter.

4.2 Getting to the Geometrical Realization (\mathcal{GR}) of the 1D, $N = 4$ Extended Super Virasoro Algebra

4.2.1 The Super Conformal Algebra (SCA)

The Geometrical Realization of the Extended Super Virasoro Algebra is really a different representation of the Super Conformal Algebra (SCA). One can get to the SCA by starting with a Kac-Moody Lie algebra¹ using T_{IJ} as generators. For this work, the Kac-Moody Lie algebra used is $SO(N)$. Adding translations generated by momentum generators P , the dilations, Δ , special conformal transformations,

¹ A Kac-Moody Lie algebra is a Lie Algebra, usually infinite dimensional, with elements related by a Cartan matrix. See Appendix A.

K , the supersymmetry generators Q_I and S_I , finally make up the algebra. It has some peculiar properties found in research. For $N \leq 4$, it is known how to close this algebra without additional generators [6] but for $N > 4$ closure requires the presence of additional operators.

The operators can be represented by derivations of the one dimensional time variable and its derivative, τ and ∂_τ , and the $N = 4$ superspace variables and their derivatives, ζ^I and ∂_I . The time variable and its derivative are real and commute with everything. The superspace coordinates are real Grassmann variables. [See Appendix A for more on Grassmann variables.] The algebra is defined by its commutation relations. There are 36 possible combinations but only thirteen are nonzero:

$$[\Delta, P] = -iP \quad , \quad [\Delta, Q_I] = -i\frac{1}{2}Q_I \quad , \quad [\Delta, K] = iK \quad , \quad (4.1)$$

$$[\Delta, S_I] = i\frac{1}{2}S_I \quad , \quad [P, S_I] = iQ_I \quad , \quad [K, Q_I] = -iS_I \quad , \quad (4.2)$$

$$[Q_I, Q_J] = 4\delta_{IJ}P \quad , \quad [S_I, S_J] = 4\delta_{IJ}K \quad , \quad [P, K] = -i2\Delta \quad (4.3)$$

$$[Q_I, S_J] = 4\delta_{IJ}\Delta + 2T_{IJ} \quad , \quad (4.4)$$

$$[T_{IJ}, Q_K] = -i\delta_{IK}Q_J + i\delta_{JK}Q_I \quad , \quad (4.5)$$

$$[T_{IJ}, S_K] = -i\delta_{IK}S_J + i\delta_{JK}S_I \quad , \quad (4.6)$$

$$[T_{IJ}, T_{KL}] = i\delta_{JK}T_{IL} - i\delta_{JL}T_{IK} + i\delta_{IL}T_{JK} + i\delta_{IK}T_{JL} \quad (4.7)$$

The generators and their corresponding symmetries are listed in Table I.

This algebra can be deformed in $N = 4$ with the addition of a Levi-Civita tensor, ϵ_{IJKL} , and a parameter, ℓ , that measures the deformation. It only affects

Generators	Symmetry	Derivation	No. of generators
P	Translations	$i \partial_\tau$	1
Δ	Dilations	$i(\tau \partial_\tau + \frac{1}{2} \zeta^I \partial_I)$	1
K	Special Conformal	$i(\tau^2 \partial_\tau + \tau \zeta^I \partial_I)$	1
Q_I	Supersymmetry	$i(\partial_I - i 2 \zeta_I \partial_\tau)$	$4 = N$
S_I	S-supersymmetry	$i\tau \partial_I + 2\tau \zeta_I \partial_\tau + \zeta_I \zeta^J \partial_J$	$4 = N$
T_{IJ}	SO(N)	$i(\zeta_I \partial_J - \zeta_J \partial_I)$	$6 = [N(N - 1)/2]$

Table 4.1: SCA Generators and Their Associated Symmetries and Derivations

three of the six operators:

$$S_I(\ell) \equiv i\tau \partial_\tau + 2\tau \zeta_I \partial_\tau + 2\zeta_I \zeta^J \partial_J + \ell \epsilon_{IJKL} (\zeta^J \zeta^K \partial^L - \frac{1}{3!} \zeta^J \zeta^K \zeta^L \partial_\tau) \quad (4.8)$$

$$K(\ell) \equiv i(\tau^2 \partial_\tau + \tau \zeta^I \partial_I - i 2 \ell \epsilon^{IJKL} [\frac{1}{4} \zeta_I \zeta_J \zeta_K \partial_L + \zeta_I \zeta_J \zeta_K \zeta_L \partial_\tau]) \quad (4.9)$$

$$T_{IJ}(\ell) \equiv i\zeta_{[I} \partial_{J]} - i\ell \epsilon_{IJKL} \zeta_K \partial_L \quad (4.10)$$

This changes the last three of the commutation relations

$$[T_{IJ}, Q_K] = -i\delta_{IK} Q_J + i\delta_{JK} Q_I + i\ell \epsilon_{IJKL} Q_L \quad (4.11)$$

$$[T_{IJ}, S_K] = -i\delta_{IK} S_J + i\delta_{JK} S_I + i\ell \epsilon_{IJKL} S_L \quad , \quad (4.12)$$

$$\begin{aligned} [T_{IJ}, T_{KL}] &= \frac{1}{2} (\ell^2 + 3) [i\delta_{JK} T_{IL} - i\delta_{JL} T_{IK} + i\delta_{IL} T_{JK} + i\delta_{IK} T_{JL}] \\ &+ \frac{1}{2} (\ell^2 - 1) [i\delta_{JK} Y_{IL} - i\delta_{JL} Y_{IK} + i\delta_{IL} Y_{JK} + i\delta_{IK} Y_{JL}] \end{aligned} \quad (4.13)$$

with $Y_{IJ} \equiv i\zeta_{[I} \partial_{J]} + i\ell \epsilon_{IJKL} \zeta_K \partial_L$. For $\ell \pm = 1$, there are no Y_{IJ} terms in the last

commutation relation.

4.2.2 Redefinition of Operators for \mathcal{GR} SVA

The next step is to recast the previous generators in terms of a Virasoro algebra with the supersymmetric generators and Lie algebra. This is done by choosing the forms

$$L_m \equiv -[\tau^{m+1}\partial_\tau + \frac{1}{2}(m+1)\tau^m\zeta\partial_\zeta] \quad , \quad H_r \equiv -[\tau^{r+1}\partial_\tau + \frac{1}{2}(r+1)\tau^r\zeta\partial_\zeta] \quad (4.14)$$

$$F_m \equiv i\tau^{m+\frac{1}{2}}[\partial_\zeta - i2\zeta\partial_\tau] \quad , \quad G_r \equiv i\tau^{r+\frac{1}{2}}[\partial_\zeta - i2\zeta\partial_\tau] \quad (4.15)$$

where $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. The L and H are the same except L takes integers and H takes half integers. The F and G forms follow the same pattern. H is fermionic and L is bosonic because L exists in the $N = 0$ case.

These new generator pairs can be combined using a different notation with simple commutation relations:

$$\left(\begin{array}{l} L_{\mathcal{A}} \equiv (L_m, H_r) \\ G_{\mathcal{A}} \equiv (F_m, G_r) \end{array} \right) \rightarrow \left(\begin{array}{l} [L_{\mathcal{A}}, L_{\mathcal{B}}] = (\mathcal{A} - \mathcal{B})L_{\mathcal{A}+\mathcal{B}} \\ [G_{\mathcal{A}}, G_{\mathcal{B}}] = -i4L_{\mathcal{A}+\mathcal{B}} \\ [L_{\mathcal{A}}, G_{\mathcal{B}}] = (\frac{1}{2}\mathcal{A} - \mathcal{B})G_{\mathcal{A}+\mathcal{B}} \end{array} \right) \quad (4.16)$$

with \mathcal{A}, \mathcal{B} taking values in \mathbb{Z} and $\mathbb{Z} + \frac{1}{2}$. For $N = 1$, this pair of generators is closed under graded commutation. In the $N = 4$ exceptional Super Virasoro algebra, an index I for the supersymmetric levels has to be added and the ℓ -deformed terms must be put in properly, including a ℓ -deformed supersymmetric $T_{IJ}(\ell)$ generator. For the 1D , N=4 exceptional Super Virasoro algebra, the set of generators ($L_{\mathcal{A}}(\ell)$,

$G_{\mathcal{A}}^I(\ell), T_{\mathcal{A}}^{IJ}(\ell)$ closes under graded commutation. These generators are

$$\begin{aligned} L_{\mathcal{A}} &\equiv -[\tau^{\mathcal{A}+1}\partial_{\tau} + \frac{1}{2}(\mathcal{A}+1)\tau^{\mathcal{A}}\zeta^I\partial_I] \\ &+ i\ell\mathcal{A}(\mathcal{A}+1)\tau^{\mathcal{A}-1}[\zeta^{(3)I}\partial_I + i4\zeta^{(4)}\partial_{\tau}] \end{aligned} \quad (4.17)$$

$$\begin{aligned} G_{\mathcal{A}}^I &\equiv \tau^{\mathcal{A}+\frac{1}{2}}[\partial^I - i2\zeta^I\partial_{\tau}] + 2(\mathcal{A} + \frac{1}{2})\tau^{\mathcal{A}-\frac{1}{2}}\zeta^I\zeta^K\partial_K \\ &+ \ell(\mathcal{A} + \frac{1}{2})\tau^{\mathcal{A}-\frac{1}{2}}[\epsilon^{IJKL}\zeta_J\zeta_K\partial_L \\ &- i4\zeta^{(3)I}\partial_{\tau}] + i4\ell(\mathcal{A}^2 - \frac{1}{4})\tau^{\mathcal{A}-\frac{3}{2}}\zeta^{(4)}\partial^I \end{aligned} \quad (4.18)$$

$$T_{\mathcal{A}}^{IJ} \equiv \tau^{\mathcal{A}}[\zeta^{[I}\partial^{J]} - \ell\epsilon^{IJKL}\zeta_K\partial_L] - i2\ell\mathcal{A}\tau^{\mathcal{A}-1}[\zeta^{(3)[I}\partial^{J]} - \ell\epsilon^{IJKL}\zeta_K^{(3)}\partial_L] \quad (4.19)$$

Their supercommutation relations are

$$[L_{\mathcal{A}}, L_{\mathcal{B}}] = (\mathcal{A} - \mathcal{B})L_{\mathcal{A}+\mathcal{B}} + \frac{1}{8}c(\mathcal{A}^3 - \mathcal{A})\delta_{\mathcal{A}+\mathcal{B},0} \quad (4.20)$$

$$[L_{\mathcal{A}}, G_{\mathcal{B}}^I] = (\frac{\mathcal{A}}{2} - \mathcal{B})G_{\mathcal{A}+\mathcal{B}}^I \quad (4.21)$$

$$[L_{\mathcal{A}}, T_{\mathcal{B}}^{IJ}] = -\mathcal{B}T_{\mathcal{A}+\mathcal{B}}^{IJ} \quad (4.22)$$

$$\{G_{\mathcal{A}}^I, G_{\mathcal{B}}^J\} = -i4\delta^{IJ}L_{\mathcal{A}+\mathcal{B}} - i2(\mathcal{A} - \mathcal{B})T_{\mathcal{A}+\mathcal{B}}^{IJ} - ic(\mathcal{A}^2 - \frac{1}{4})\delta_{\mathcal{A}+\mathcal{B},0}\delta^{IJ} \quad (4.23)$$

$$[T_{\mathcal{A}}^{IJ}, G_{\mathcal{B}}^K] = 2(\delta^{JK}G_{\mathcal{A}+\mathcal{B}}^I - \delta^{IK}G_{\mathcal{A}+\mathcal{B}}^J) \quad (4.24)$$

$$\begin{aligned} [T_{\mathcal{A}}^{IJ}, T_{\mathcal{B}}^{KL}] &= T_{\mathcal{A}+\mathcal{B}}^{IK}\delta^{JL} - T_{\mathcal{A}+\mathcal{B}}^{IL}\delta^{JK} + T_{\mathcal{A}+\mathcal{B}}^{JL}\delta^{IK} - T_{\mathcal{A}+\mathcal{B}}^{JK}\delta^{IL} \\ &- 2c(\mathcal{A} - \mathcal{B})(\delta^{I[K}\delta^{JL]}) \end{aligned} \quad (4.25)$$

4.3 Explanation of the Coadjoint Orbit Method of Deriving the Operator Product Expansion

4.3.1 Description of the Use of Coadjoint Orbits

The coadjoint orbit method uses ideas developed by A. A. Kirillov [9]. Kirillov's original work relates the irreducible representations of a Lie algebra to the coadjoint orbits. His work also showed these coadjoint orbits allow a symplectic invariant structure. It is the combination of these two points that form the first part of the method.

First, the adjoint and coadjoint representations have to be defined. A Lie algebra can act on itself through the Lie Bracket. A function can be defined for a fixed element of the Lie algebra that acts on other Lie algebra elements.

$$x, y \in \mathbf{G}, f_x(y) \equiv [x, y] = xy - yx. \quad (4.26)$$

This function is the adjoint function $Ad_x(y)$. When the elements of the Lie Algebra are expressed as matrices, this forms the adjoint representation of the algebra.

A Lie Algebra can also be considered as a space of vector fields on a manifold. As a vector space, there exists a dual vector space that represents linear functionals on these vector fields. For the Lie Algebra \mathbf{G} , and its dual \mathbf{G}^* , an inner product is defined between the two spaces in terms of basis elements of the vector space and its dual:

$$e_i \in \mathbf{G}, \omega_j \in \mathbf{G}^* : \langle \omega_j, e_i \rangle = \delta_{ij} \quad (4.27)$$

This inner product extends to the adjoint and coadjoint representation. Specifically,

if \mathcal{F}_I is an adjoint vector and \mathcal{B}_J is a coadjoint vector, there exists $\langle \mathcal{F}_I | \mathcal{B}_J \rangle$. This inner product only equals a Kronecker delta function when the coadjoint element is the dual of the adjoint element, i.e.

$$\mathcal{L}_i^* \text{ is dual to } \mathcal{L}_j \rightarrow \langle \mathcal{L}_i^* | \mathcal{L}_j \rangle = \delta_{ij} \quad (4.28)$$

We now look at the algebraic action of general adjoint element on coadjoint element. What we get is new coadjoint element in terms of our old adjoint and old coadjoint elements. The coadjoint orbit is the set of coadjoint vectors that can be reached with this adjoint transformation. This is similar to how a latitudinal great circle is the orbit of a coadjoint vector acted on by a latitudinal transformation.

Since the coadjoint action tells us how the fields change under a transformation, we can say that it is equal to the difference between the old field and the new field: $\delta_F \mathcal{B}$.

The question may be asked at this point “why care about the coadjoint orbit?” Since the algebra represents transformations on a space², the dual of those transformations represents linear functions on that space, or fields. The coadjoint action tells us how the fields change from an algebraic standpoint with respect to the adjoint action, or transformations.

As an aside, one of the uses of coadjoint orbits is relate the classification of the orbits to the classification of another related mathematical structure. For example, if G is the set of all linear $n \times n$ real invertible matrices, then the classification of coadjoint orbits is equivalent to the classification of matrices up to similarity. The

² Usually this is all done on a circle, S_1 , which allows a central extension (a 2-cocycle which tells us how close the action is to closing under multiplication). We will ignore it for now but remember that it exists and is a value in the reals.

analysis of the coadjoint orbits allows one to classify two dimensional conformal field theories (2D-CFT's).

We go back to the inner product of adjoint and coadjoint elements to define the coadjoint action. The inner product was shown to be equal to a delta function which is a constant. The variation of a constant is zero. However, the variation of the inner product, specifically the action of the adjoint action, acts on the inner product like the two-input function it is and gives two terms. One term is the adjoint action of the adjoint element and the other term is the coadjoint action on the coadjoint element. We can rearrange this equation to define the coadjoint action in terms of the adjoint action which we can definitely calculate:

$$\langle \mathcal{L}_I^* | \mathcal{L}_J \rangle = \delta_{IJ} \quad (4.29)$$

$$Ad_{\mathcal{L}}(\mathcal{L}_I^*) = \mathcal{L} * \mathcal{L}_I^* \quad (4.30)$$

$$\delta_{\mathcal{L}}(\langle \mathcal{L}_I^* | \mathcal{L}_J \rangle) = \delta_{\mathcal{L}}(\delta_{IJ}) \quad (4.31)$$

$$\langle \mathcal{L} * \mathcal{L}_I^* | \mathcal{L}_J \rangle + \langle \mathcal{L}_I^* | \mathcal{L} * \mathcal{L}_J \rangle = 0 \quad (4.32)$$

$$\langle \mathcal{L} * \mathcal{L}_I^* | \mathcal{L}_J \rangle = -\langle \mathcal{L}_I^* | \mathcal{L} * \mathcal{L}_J \rangle \quad (4.33)$$

The right hand side of the equation exists because Kirillov showed it exists as a symplectic invariant in his work [9]. Now we can read off the coadjoint orbit from the equation above.

4.3.2 Relationship to OPE

The Operation Product Expansion (OPE) is an expression of the product of two operators as a sum of singular functions of other operators. This is useful

when calculating the product of field operators at the same point. Wilson and Zimmerman [13] have a discussion of the use of OPEs in Quantum Field Theory. In this case, the operators are tensor fields. The general form of an OPE is

$$A(y)B(x) \sim \sum_i C_i(x)(y-x)^{-i} + (\text{non singular terms}) \quad (4.34)$$

where C_i is a member of a complete set of operators. The non-singular terms are not important because the singular terms determine the properties of the product of operators. These products are further related to useful field theory quantities such as propagators and mass terms.

The goal is to express the product of fields that represent the underlying algebra in terms of functions of other fields which represent other elements in the algebra. In this case, it is the new coadjoint field that will be expressed in terms of the old adjoint and coadjoint fields.

In the previous subsection, the action of the adjoint transformation on the coadjoint field was found. That action also represented the change or variation of the coadjoint field with respect to the adjoint field. We will use that and the definition of the inner product to calculate the OPE.

Starting with Eqn. 4.33, we can interpret the inner product as an integral over some space following the analogy of Eqn. 4.27. However, we have to use the associated fields with the adjoint and coadjoint representation for this to make sense. Thus, for an adjoint field $A(x)$ associated with \mathcal{F} and a coadjoint field Λ associated with \mathcal{B}

$$\langle \mathcal{B}_\Lambda | \mathcal{F}_A \rangle = \int dx \Lambda(x) A(x). \quad (4.35)$$

Applying this to both sides of 4.33 with A and Λ as the respective adjoint and coadjoint fields, we find

$$\int dy (A(x) * \Lambda_I(x)) A_J(y) = - \int dy \Lambda_I(y) (A(x) * A_J(x)) \quad (4.36)$$

where the integration is over a dummy variable y and adjoint action is taken over the same spatial location x .

This is tied to the OPE in the following way. The element in parentheses on the LHS of Eqn. 4.36 is the new coadjoint element calculated from the coadjoint action. The element in parentheses on the RHS is the corresponding adjoint pair of fields we are interested in for the OPE. We take the dual of the RHS to get the right form for the coadjoint fields we want for the OPE. We now can pull out a delta function of the correct dimensionality to fix the dummy variable in the fields and we can now simply read off the OPE.

4.3.3 Pulling it all together

The actual use of the method flows from the following steps:

1. Choose an coadjoint field and an adjoint action on it. This gives the variation of the physical field with respect to some transformation.
2. Compare to the integral form of the inner product of the new coadjoint field.

The OPE will be the equivalent expression of the previous step once it has been put in the associated integral form. This will involve the use of delta functions on the space (a line in the 1D case) and its derivatives.

Typically, one needs an action to determine the useful field theory quantities such as correlation functions. However, these quantities are dependent on the symmetries found in the theory and not necessarily obvious in the action. The Coadjoint Orbit method allows for these quantities to be calculated without an action and totally based on the underlying symmetries of the theory being studied.

4.4 Application of the Coadjoint Orbit Method to the \mathcal{GR} SVA

4.4.1 Preliminaries

The methods used are found in [6,7,8,12]. Applying this process to the algebra of interest, the adjoint vector of the 1D $N = 4$ \mathcal{GR} SVA is $L = (L_A, G_B^I, T_C^{JK})$. The adjoint acting on this gives

$$\begin{aligned} ad((L_{\mathcal{M}}, G_N^K, T_P^{LM}))(L_A, G_B^I, T_C^{JK}) &= (L_{\mathcal{M}}, G_N^K, T_P^{LM}) * (L_A, G_B^I, T_C^{JK}) \\ &= (L_{\mathcal{Q},\text{new}}, G_{\mathcal{R},\text{new}}^H, T_{\mathcal{S},\text{new}}^{\text{FG}}) \end{aligned} \quad (4.37)$$

The coadjoint element is $\tilde{L} = (\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_P^{JK})$ and correspondingly gives

$$\begin{aligned} ad((L_{\mathcal{M}}, G_N^K, T_P^{LM}))(\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_P^{JK}) &= (L_{\mathcal{M}}, G_N^K, T_P^{LM}) * (\tilde{L}_A, \tilde{G}_B^I, \tilde{T}_P^{JK}) \\ &= (\tilde{L}_{\mathcal{Q},\text{new}}, \tilde{G}_{\mathcal{R},\text{new}}^H, \tilde{T}_{\mathcal{S},\text{new}}^{\text{FG}}) \end{aligned} \quad (4.38)$$

and the inner product is

$$\langle (\tilde{L}_M, \tilde{G}_N^K, \tilde{T}_P^{LM}) | (L_A, G_B^I, T_C^{JK}) \rangle = \delta_{M,A} + \delta_{N,B} \delta_K^I + \delta_{P,C} \delta_{LM}^{JK} \quad (4.39)$$

To calculate the OPEs, one needs the expression of $\delta_L \tilde{L} = L * \tilde{L}$ where L is an adjoint vector and \tilde{L} is a coadjoint vector. Using the fact that $\langle \tilde{L} | L \rangle$ is an invariant and $L * \tilde{L}$ can be calculated from $\langle L' * \tilde{L} | L \rangle$, one can use the Leibnitz rule on the

invariant form and get

$$\langle L * \tilde{L} | L \rangle = -\langle \tilde{L} | L * L \rangle \quad (4.40)$$

Since L and \tilde{L} are made up of components (L, G, T), it is easier to calculate pairs of adjoint elements acting on coadjoint elements. This reduces the number of calculations greatly. The list of adjoint/coadjoint pairs are

$$\begin{aligned} \delta \tilde{L} &= L * \tilde{L} + G * \tilde{G} + T * \tilde{T} \\ \delta \tilde{G} &= L * \tilde{G} + G * \tilde{L} + G * \tilde{T} + T * \tilde{G} \\ \delta \tilde{T} &= L * \tilde{T} + G * \tilde{G} + T * \tilde{T} \end{aligned}$$

This checks against the calculations from [7]. Note that there is no $\tilde{T} * L$ term in the list of changes to the coadjoint vector.

Using a realization of the algebra as tensor fields, the adjoint representation elements are $F = (\eta, \chi^I, t^{\text{RS}})$, which are general elements of the Virasoro, Kac-Moody, and so(4) algebras respectively. The coadjoint fields are $B = (D, \psi^I, A^{\text{RS}})$, a rank two pseudo tensor, a set of 4 spin-3/2 fields, and the 6 so(4) gauge fields.

The coadjoint action can be seen as generating the changes in the fields. It acts as

$$F * \tilde{B} = \delta_F \tilde{B} = (\eta, \chi^J, t^{\text{KL}}) * (D, \psi^I, A^{\text{JK}}) = (\delta D, \delta \psi^I, \delta A^{\text{JK}}). \quad (4.41)$$

4.4.2 Example: $L * \tilde{L}$

Choosing $L * \tilde{L}$ as an example, the physical field representation is used :

$$L * \tilde{L} \leftrightarrow \delta_\eta D \quad (4.42)$$

$$L_\eta * \tilde{L}_D \rightarrow \tilde{L}_{\tilde{D}} \implies \tilde{D} = -D'\eta - 2D\eta' \quad (4.43)$$

$$\delta_\eta D = \tilde{D} = \eta * D \quad (4.44)$$

$$\int dy [\eta(x)D(y)]\eta(y) = - \int dy D(y)(\eta(x)\eta(y)) \quad (4.45)$$

$$\int dy [\tilde{D}(x, y)]\eta(y) = \int dy (-D'(x)\eta(y) - 2D(x)\eta'(y))\eta(y). \quad (4.46)$$

$$\int dy D(y)(\eta(x)\eta(y)) \leftrightarrow \int dy \eta(y)(D(y)D(x)) \quad (4.47)$$

Using the 1D formula for the delta function,

$$\delta(y - x) = \frac{1}{2\pi i(y - x)} \quad (4.48)$$

and integration by parts to separate out $\eta(x)$ terms,

$$\int dy [D(y)D(x)]\eta(x) = \int dy \left(\partial_x D(x) \frac{-1}{2\pi i(y - x)} + D(x) \frac{-1}{\pi i(y - x)^2} \right) \eta(x) \quad (4.49)$$

4.4.3 Calculation of \mathcal{GR} SVA OPE's

By taking pairs of individual adjoint elements acting on individual coadjoint elements, the OPE's can found.

1. $D(y)O(x)$

$$L_\eta * \tilde{L}_D = \tilde{L}_{\tilde{D}} \rightarrow \tilde{D} = -D'\eta - 2D\eta' \quad (4.50)$$

$$L_\eta * \tilde{G}_{\psi^{\bar{Q}}} = \tilde{G}_{\tilde{\psi}^{\bar{Q}}} \rightarrow \tilde{\psi}^{\bar{Q}} = -\left(\frac{3}{2}\eta'\psi^{\bar{Q}} - \eta(\psi^{\bar{Q}'})\right) \quad (4.51)$$

$$L_\eta * \tilde{T}_{A\bar{I}\bar{J}}^{\bar{R}\bar{S}} = \tilde{T}_{\tilde{A}\tilde{I}\tilde{J}}^{\tilde{R}\tilde{S}} \rightarrow \tilde{A}^{\tilde{I}\tilde{J}} = -(A^{RS})'\eta - \eta' A^{RS} \quad (4.52)$$

These expressions yield the following OPEs:

$$D(y)D(x) = \frac{-1}{\pi i(y-x)^2}D(x) - \frac{1}{2\pi(y-x)}\partial_x D(x) \quad (4.53)$$

$$D(y)\psi^Q(x) = \frac{-3}{4\pi i(y-x)^2}\psi^Q(x) - \frac{1}{2\pi i(y-x)}\partial_x \psi^Q(x) \quad (4.54)$$

$$D(y)A^{RS}(x) = \frac{-1}{2\pi i(y-x)^2}A^{RS}(x) - \frac{1}{2\pi i(y-x)}\partial_x A^{RS}(x) \quad (4.55)$$

2. $\psi(y)O(x)$

$$G_{\chi^I}^I * \tilde{L}_D = 4i\tilde{G}_{\tilde{\chi}^I}^I \rightarrow \tilde{\chi}^I = -\chi^I D \quad (4.56)$$

$$G_{\chi^I}^I * \tilde{G}_{\psi^{\bar{Q}}}^{\bar{Q}} = \frac{\delta^{I\bar{Q}}}{2}\tilde{L}_{\tilde{D}} + \tilde{T}_{\tilde{A}^I\tilde{Q}}^{\bar{Q}} \rightarrow \tilde{D} = [(\psi^{\bar{Q}})'\psi^I - 3(\psi^I)'\psi^{\bar{Q}}] \quad (4.57)$$

$$\tilde{A}^{I\bar{Q}} = 2(\chi^I\psi^{\bar{Q}} - \chi^{\bar{Q}}\psi^I)$$

$$G_{\chi^I}^I * \tilde{T}_{\tau\bar{R}\bar{S}}^{\bar{R}\bar{S}} = \delta^{[RS][I\bar{Q}]} \tilde{G}_{\psi^{\bar{Q}}}^{\bar{Q}} \rightarrow \psi^{\bar{Q}} = 2(\chi^I)'t^{RS} + \chi^I(t^{RS})' \quad (4.58)$$

The OPEs are

$$\psi^I(y)D(x) = \frac{-3}{4\pi i(y-x)^2}\psi^I(x) - \frac{i}{4\pi(y-x)}\partial_x \psi^I(x) \quad (4.59)$$

$$\psi^I(y)\psi^Q(x) = \frac{-4i}{(y-x)}\delta^{IQ}D(x) \quad (4.60)$$

$$\psi^A(y)A^{RS}(x) = \frac{\pi}{i(y-x)}(\delta^{AR}\delta^{LS} - \delta^{AS}\delta^{LR})\psi^L(x) \quad (4.61)$$

3. $A(y)O(x)$

$$T_{t^{IJ}}^{IJ} * \tilde{G}_{\psi^{\bar{Q}}}^{\bar{Q}} = 2\delta^{QI}G_{\phi^J}^J - 2\delta^{QJ}G_{\phi^I}^I \rightarrow \psi^{\bar{Q}} = t^{IJ}\psi^Q \quad (4.62)$$

$$T_{t^{IJ}}^{IJ} * \tilde{T}_{\tau\bar{R}\bar{S}}^{\bar{R}\bar{S}} = -\delta^{[\bar{R}\bar{S}]}(\delta_{RS}^{JK} + \delta_{RS}^{KJ})\tilde{T}_{(t^{JK})'\tau\bar{R}\bar{S}}^{\bar{R}\bar{S}} - \tilde{L}_{\tilde{D}}\delta^{[\bar{R}\bar{S}][JK]} \quad (4.63)$$

$$\rightarrow D = (t^{JK})'\tau^{\bar{R}\bar{S}}$$

Note that there is no $T*\tilde{L}$ term. However the $A^{JK}(y)D(x)$ and $A^{JK}(y)A^{RS}(x)$

terms are generated from the $T * \tilde{T}$ action. The OPE that follow are

$$A^{JK}(y)D(x) = \frac{1}{4\pi i(y-x)^2}(\delta^{RS}\delta^{JK} - \delta^{RK}\delta^{LS})A_{RS}(x) \quad (4.64)$$

$$A^{AB}(y)\psi^C(X) = \frac{-1}{\pi i(y-x)}(\delta^{AC}\psi^B(x) - \delta^{AB}\psi^C(x)) \quad (4.65)$$

$$A^{JK}(y)A^{RS}(x) = \frac{1}{4\pi i(y-x)}\delta_{AB}^{JKRS}A^{AB}(x) \quad (4.66)$$

4.4.4 Observations

A number of interesting points can be found here. In previous papers [6] [7], the non-deformed ($\ell = 0$) 1D, $N = 4$ \mathcal{GR} Super Virasoro algebra is used to generate OPEs. This algebra is the “large” $N = 4$ algebra which has a 16-dimensional representation. It does not close unless two more *sets* of generators (U’s and R’s, which are related to the T’s,) are added. The $\ell = \pm 1$ cases of the ℓ -extended algebra map the generators to a 8-dimensional representation which does not need the other generators to close. This can be easily seen when instead of using derivations to represent the generators, an appropriately sized Clifford algebra is used [2]. The use of a Clifford algebra will allow more insight into the difference. This and the difference between using the “small” and the “large” $N = 4$ algebras will be discussed in Section 5.1.

Another point is whether the central extension should be dropped in the equations. From [8], the closure of the algebra is found to be related to the existence of a central extension, specifically if the central extension is eliminated for $N > 2$. Because $N = 4$, it is a valid question to ask if a central extension may exist too. The Jacobi Identity on $(G_{\mathcal{A}}^I, U_B^{IJ}, G_P^K)$ was used before in [6] to answer this question.

Because the supercommutators have the same form as the $N > 2$, it would seem that the answer would be true. But there are no longer $U_{\mathcal{A}}^{IJ}$ generators in the algebra. The Jacobi identity for the other generators must be analyzed to check if a central extension is allowed. Although this could be addressed now, this question will be revisited later when the Clifford representation of the generators is presented. For now, c will be set to zero.

In the non-extended version of the algebra [6,7,8,12], there are extra generators that must be added to close the algebra. When the Coadjoint Orbit method is applied, these extra generators correspond to fields and have their own OPEs. The fields ω and ρ , which correspond to the U and R operators respectively, have 44 and 11 independent components. The spin of the fields are varied, either being 0 or $\frac{1}{2}$ depending on the structure of the individual operator. This also true for the general extended $\ell \neq \pm 1$ case. However, the $\ell = \pm 1$ case does not have these fields or their OPEs. Thus there is no difference between the regular ($\ell=0$) and extended ($\ell \neq 0$) cases except when $\ell = \pm 1$. These cases reduce the number of operators and fields necessary to fully describe the theory.

4.4.5 Relationship between D_I and G_A^I

At first glance, there seems not to be any relationship between the operator D_I found in describing Adinkras and operator G_A^I found in the description of the SVAs. However, by looking at each of these in a particular representation something becomes clear. Choosing 1D, $N = 4$ representation of a general SUSY theory, it is

seen that

$$D_I = \partial_\tau + i\zeta\partial_\zeta \quad (4.67)$$

and

$$G_{-\frac{1}{2}}^I \equiv [\partial^I - i2\zeta^I\partial_\tau] \quad (4.68)$$

It is clear that they have very similar forms. D_a deals with the superspace variation of the different fields. $G_{\mathcal{A}}^I$ contains Q^I which is the supersymmetry operator that changes particles into their superpartners. This is very reminiscent of the parallel between the two formulations of quantum mechanics. There is the Schrödinger picture, where the states have a time component, which corresponds to acting with D_I . There is also the Heisenberg picture where the operators have time components like G . In a way, this correspondence is to be expected since adinkras are projections of quantum field theory down to one dimension, time. One dimensional quantum field theory is just quantum mechanics.

Chapter 5

The Hasiewicz, Thielemans, and Troost (HTT) SVA, the Clifford Algebra, and Operator Product Expansions (OPE)

5.1 The HTT formulation of SVA

Hasiewicz, Thielemans, and Troost (HTT) [2] developed another way to realize a Super Virasoro Algebra. They start with a Kac-Moody Algebra and Lie group with a central extension. A description of their method of generating the SVA will not be given here but their representation of the generators of the SVA will be used. What was interesting is that the elements of the algebra can be related to a Clifford algebra. Applying the Coadjoint method from last chapter relies heavily on being able to write down the generators and their commutation relations. The HTT representation of the SVA will be used to generate OPEs for comparison with the results from last chapter.

Starting with the set of generators $(L_{\mathcal{A}}, G_{\mathcal{A}}^I, T_{\mathcal{A}}^{IJ})$ from the previous chapter,

the following are the supercommutation relations:

$$[L_{\mathcal{A}}, L_{\mathcal{B}}] = (\mathcal{A} - \mathcal{B})L_{\mathcal{A}+\mathcal{B}} + \frac{1}{8}c(\mathcal{A}^3 - \mathcal{A})\delta_{\mathcal{A}+\mathcal{B},0} \quad (5.1)$$

$$[L_{\mathcal{A}}, G_{\mathcal{B}}^{\mathbf{I}}] = (\frac{\mathcal{A}}{2} - \mathcal{B})G_{\mathcal{A}+\mathcal{B}}^{\mathbf{I}} \quad (5.2)$$

$$[L_{\mathcal{A}}, T_{\mathcal{B}}^{\mathbf{IJ}}] = -\mathcal{B}T_{\mathcal{A}+\mathcal{B}}^{\mathbf{IJ}} \quad (5.3)$$

$$\{G_{\mathcal{A}}^{\mathbf{I}}, G_{\mathcal{B}}^{\mathbf{J}}\} = -i4\delta^{\mathbf{IJ}}L_{\mathcal{A}+\mathcal{B}} - i2(\mathcal{A} - \mathcal{B})T_{\mathcal{A}+\mathcal{B}}^{\mathbf{IJ}} - ic(\mathcal{A}^2 - \frac{1}{4})\delta_{\mathcal{A}+\mathcal{B},0}\delta^{\mathbf{IJ}} \quad (5.4)$$

$$[T_{\mathcal{A}}^{\mathbf{IJ}}, G_{\mathcal{B}}^{\mathbf{K}}] = 2(\delta^{\mathbf{JK}}G_{\mathcal{A}+\mathcal{B}}^{\mathbf{I}} - \delta^{\mathbf{IK}}G_{\mathcal{A}+\mathcal{B}}^{\mathbf{J}}) \quad (5.5)$$

$$\begin{aligned} [T_{\mathcal{A}}^{\mathbf{IJ}}, T_{\mathcal{B}}^{\mathbf{KL}}] &= T_{\mathcal{A}+\mathcal{B}}^{\mathbf{IK}}\delta^{\mathbf{JL}} - T_{\mathcal{A}+\mathcal{B}}^{\mathbf{IL}}\delta^{\mathbf{JK}} + T_{\mathcal{A}+\mathcal{B}}^{\mathbf{JL}}\delta^{\mathbf{IK}} - T_{\mathcal{A}+\mathcal{B}}^{\mathbf{JK}}\delta^{\mathbf{IL}} \\ &\quad - 2c(\mathcal{A} - \mathcal{B})(\delta^{\mathbf{I[K}]\delta^{\mathbf{JL]}}) \end{aligned} \quad (5.6)$$

The form of these supercommutators come from their Geometrical Realizations as polynomials and derivatives of complex and Grassmann numbers.

Now we will compare this to the HTT realization of the same algebra. Their method starts with a break down of the Lie superalgebra into smaller, relevant subspaces: a Kac-Moody Lie algebra $KM(L)$ with a Lie algebra L , a Virasoro algebra Vir , and subspaces Q and G with underlying vector spaces respectively, V and W . The underlying vectors spaces of these subspaces (L for $KM(L)$, \mathbb{R} for Vir , V for Q , W for G) are important along with a number of mappings that define the properties of each space. For $w, w' \in W; v, v' \in V; \Sigma, \Sigma' \in L$; there are the

following mappings:

$$[L_m, L_n] = (m - n)L_{m+n} + (m^3 - m)\delta(m + n)c/4 \quad (5.7)$$

$$[L_m, Q_n(v)] = -(m/2 + n)Q_{m+n}(v) \quad (5.8)$$

$$[L_m, G_n(w)] = +(m/2 - n)G_{m+n}(w) \quad (5.9)$$

$$[L_m, T_n(\Sigma)] = -n T_{m+n}(\Sigma) \quad (5.10)$$

$$\{Q_m(v), Q_n(v')\} = -b(v, v')\delta(m + n)c \quad (5.11)$$

$$\begin{aligned} \{G_m(w), G_n(w')\} &= 2B(w, w')L_{m+n} + B(w, w')(m^2 - 1/4)\delta(m + n)c \\ &\quad - (m - n)T_{m+n}(\varphi(w, w')) \end{aligned} \quad (5.12)$$

$$\{G_m(w), Q_n(v)\} = T_{m+n}(\varphi(w, v)) \quad (5.13)$$

$$[T_m(\Sigma), Q_n(v)] = Q_{m+n}(R(\Sigma)v) \quad (5.14)$$

$$[T_m(\Sigma), G_n(w)] = G_{m+n}(\Lambda(\Sigma)w) + m Q_{m+n}(d(\Sigma, w)) \quad (5.15)$$

$$[T_m(\Sigma), T_n(\Sigma')] = T_{m+n}([\Sigma, \Sigma']) - cmK(\Sigma, \Sigma')\delta(m + n) \quad (5.16)$$

with $B, b, R, \varphi, \Lambda$, and d all being mappings and bilinear forms necessary to describe the superconformal Lie superalgebras.

There are special mappings in [2] used to associate with the underlying vector spaces:

$$\varphi_w(w') : w' \in W \rightarrow \varphi(w, w') \in L \quad (5.17)$$

$$d_w(\Sigma) : \Sigma \in L \rightarrow d(\Sigma, w) \in V \quad (5.18)$$

$$i_w(a) : a \in \mathbb{R} \rightarrow aw \in W. \quad (5.19)$$

This set gives the exact series

$$\mathbb{R} \xrightarrow{i_w} W \xrightarrow{\varphi_w} L \xrightarrow{d_w} V \rightarrow 0. \quad (5.20)$$

While this set of mappings and forms

$$\psi_w : v \in V \rightarrow \psi(w, v) \in L \quad (5.21)$$

$$\Lambda_w : \Sigma \in L \rightarrow \Lambda(\Sigma)w \in W \quad (5.22)$$

$$B_w : w' \in W \rightarrow B(w, w') \in \mathbb{R} \quad (5.23)$$

gives the exact series

$$0 \rightarrow V \xrightarrow{\psi_w} L \xrightarrow{\Lambda_w} W \xrightarrow{B_w} \mathbb{R}. \quad (5.24)$$

Note that all the mappings resemble adjoint actions, being based on a fixed element w .

The biggest different between the two representations is the addition of a second supersymmetric space Q with underlying vector space V and conformal dimension $\frac{1}{2}$ to the original supersymmetric space G with vector space W and conformal dimension $\frac{3}{2}$. The two interesting new supercommutators involving Q are the $\{G, Q\}$ equation, which generates a T , and the $[T, G]$ equation, which now generates an additional Q .

One of their most important results is the relationship between the dimensions of the vector spaces:

$$|W| + |V| = |L| + 1 \quad (5.25)$$

With this relationship, one can categorize the type of algebra possible since there are $|W|$ symmetries that exist ($\dim(W) = N$), and $|L|$ is the dimension of the

underlying Lie algebra. We will use it as a check on dimensions of the different spaces.

5.2 The Clifford Algebra Correspondence

The spaces mentioned above define a larger space $S = W \oplus V \oplus L \oplus \mathbb{R}$ of all the elements and an endomorphism Γ that represents a Clifford algebra (see Appendix A) with the mapping B above as a definition:

$$\begin{aligned} \Gamma_w(w' + v + \Sigma + a) &= (aw + \Lambda(\Sigma)w) + d(\Sigma, w) \\ &+ (\varphi(w, w') + \psi(w, v)) + B(w, w') \end{aligned} \quad (5.26)$$

$$\Gamma_w \Gamma'_w + \Gamma'_w \Gamma_w = 2B(w, w'). \quad (5.27)$$

S is also given a metric by θ :

$$\theta(w + v + \Sigma + a, w' + v' + \Sigma' + a') = B(w, w') + b(v, v') - K(\Sigma, \Sigma') - aa' \quad (5.28)$$

What we see here are the functions of all the different mappings (B , b , and K) all define inner products for their respective spaces of W , V , and T . This allows an inner product to be defined for the individual spaces. The mappings $\varphi(w, w')$, $\psi(w, v)$, $d(\Sigma, w)$, and $\Lambda(\Sigma)w$ describe mappings from one space to another. Note that all the mappings resemble adjoint actions, being based on a fixed element w . Combined with the inner product functions, these mappings allow an overall inner product to be defined. It is this new inner product that is used in the Coadjoint method to get OPE's. At this point, a number of similarities to the Coadjoint method shown earlier should be apparent. The elements of S have this particular

form because the elements of all the different spaces are now on an equal footing with each other under the Clifford algebra. The metric has the same form (up to some signs) as the action of the dual element on the vectors describe in Subsection 4.4.1.

The superconformal Lie algebra is built from a vector representing the unit element in the space \mathbb{R} . This element is multiplied by the basis elements of the Clifford algebra to get the other spaces W , V , and L . The previous mappings between spaces allows them to be separated to get the mathematical structure needed.

The $N = 4$ case is presented in their paper [2] for a Clifford algebra signature of $(0, 4)$ explicitly and all other signatures by inference. The choice of $\ell = 1$ corresponds to a 16-dimensional representation of S and the Clifford space. The dimensions of the spaces W , V , and L are 4, 4, and 7 respectively as given by eq. 5.25. The basis vectors for W are

$$w_i = \Gamma_i \tag{5.29}$$

and for V ,

$$v_i = \Gamma_i(\Gamma^5 - \ell) \tag{5.30}$$

where $\Gamma^5 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$, $(\Gamma^2) = 1$, and ℓ real, much like defined in the derivation method. The elements of Lie algebra are given by the ϕ mapping with the addition of one more element:

$$\varphi_{ij} = \varphi(w_i, w_j) = \Gamma_i\Gamma_j (i \neq j) \tag{5.31}$$

$$\sigma = (\Gamma^5 - \ell). \tag{5.32}$$

Original Notation	Condensed Form	from HTT
P, Δ, K	L	$\theta(B, b, K) \in Vir$
Q, S	G	$w_i \in W, v_i \in V$
T	T	$\varphi, \sigma \in L$

Table 5.1: Correspondence between \mathcal{GR} Super Virasoro Algebra Generators

The mappings and bilinear forms from above now take the form

$$\Lambda(\varphi_{ij})w_k = \delta_{jk}w_i - \delta_{ik}w_j + \ell\epsilon_{ijkl}w_l \quad (5.33)$$

$$d(\varphi_{ij}, w_k) = \epsilon_{ijkl}v_l \quad (5.34)$$

$$\psi(w_i, v_j) = \Gamma_i\Gamma_j(\Gamma^5 - \ell) = -\ell\psi_{ij} - \frac{1}{2}\epsilon_{ijkl}\varphi_{kl} - \delta_{ij}\sigma \quad (5.35)$$

$$d(\sigma, w_k) = v_l \quad (5.36)$$

$$R(\varphi_{ij})v_k = \delta_{jk}v_l - \delta_{ik}v_j + \ell\epsilon_{ijkl}v_l \quad (5.37)$$

Now the correspondence between derivation representation and Clifford algebra representation should be clear:

The effects of the extended algebra, which is a function of ℓ , can be seen in the mapping b , the metric term for the vector space V , and the mapping Λ on the six linear combinations $\varphi_{ij} \pm \frac{1}{2}\epsilon_{ijkl}\varphi_{kl}$:

$$b(v_i, v_j) = -\delta_{ij}(1 - \ell^2) \quad (5.38)$$

$$\Lambda(\varphi_{ij} \pm \epsilon_{ijkl}\varphi_{kl})w_m = (1 \mp \ell)(\delta_{jm}w_i - \delta_{im}w_j \mp \ell\epsilon_{ijmn}w_n) \quad (5.39)$$

If $b^2 = 1$, then b is identically equal to 0. The vector space V disappears.

The parameter ℓ can be used to categorize all types of 1D $N = 4$ super Virasoro algebras. When $\ell \neq \pm 1$, the algebra is the “large” $N = 4$ algebra with $so(4) = so(3) \otimes so(3)$. At $\ell = \pm 1$, it collapses to the “small” 8-dimensional $N = 4$ algebra. It is called small because at $\ell = \pm 1$, part of the space is mapped into zero into W . The dimension $\frac{1}{2}$ fields generated by V disappear and the corresponding representation now only has 4 dimension-1/2 fields from W and four dimension-1 fields from the combination of L and Vir .

It is clear that the addition of the ℓ -terms, which also involved the Levi-Civita tensor, has its basis in the Q vector space describing the supersymmetric operators and requires the necessary adjustments to the other operators to close the algebra. The original $[T, Q]$, $[T, S]$, and $[T, T]$ supercommutators reflect this relationship and the close ties between the supersymmetric operators and the Lie algebra underneath.

With the algebra elements written as elements of a Clifford algebra, all of the previous work can be double checked and reanalyzed in a different context. The benefit of going to a Clifford algebra representation is that the Clifford algebras are well-known and well-understood. In [2], there is some discussion about what this would entail and will be investigated for future research.

5.3 Calculation of HTT SVA OPEs

Following the method outlined in Chapter 4, the first thing to note is that adjoint and coadjoint vectors now include the Q operator: $L = (L_m, Q_n(v), G_p(w), T_r(\Sigma))$

and $\tilde{L} = (\tilde{L}_m, \tilde{Q}_n(v'), \tilde{G}_p(w'), \tilde{T}_r(\Sigma'))$. The inner product is now defined by

$$\langle (\tilde{L}_M, \tilde{G}_N^S, \tilde{Q}_P^T, \tilde{T}_R^{UV}) | (L_A, G_B^I, Q_C^J, T_D^{KL}) \rangle = \delta_{M,A} + \delta_{N,B} \delta_S^I + \delta_{P,C} \delta_T^J + \delta_{R,D} \delta_{LM}^{JK}. \quad (5.40)$$

The Leibnitz rule part is still the same:

$$\langle L * \tilde{L} | L \rangle = -\langle \tilde{L} | L * L \rangle. \quad (5.41)$$

The list of changes to the coadjoint operators is of course longer:

$$\begin{aligned} \delta \tilde{L} &= L * \tilde{L} + Q * \tilde{Q} + G * \tilde{G} + T * \tilde{T} \\ \delta \tilde{G} &= L * \tilde{G} + G * \tilde{L} + Q * \tilde{T} + G * \tilde{T} + T * \tilde{G} + T * \tilde{Q} \\ \delta \tilde{Q} &= L * \tilde{Q} + G * \tilde{T} + T * \tilde{Q} \\ \delta \tilde{T} &= L * \tilde{T} + G * \tilde{G} + G * \tilde{Q} + Q * \tilde{Q} + T * \tilde{T} \end{aligned}$$

The associated fields are now:

$$L \rightarrow \xi \quad ; \quad \tilde{L} \rightarrow D \quad (5.42)$$

$$G \rightarrow g \quad ; \quad \tilde{G} \rightarrow \chi \quad (5.43)$$

$$Q \rightarrow f \quad ; \quad \tilde{Q} \rightarrow \phi \quad (5.44)$$

$$T \rightarrow t \quad ; \quad \tilde{T} \rightarrow A \quad (5.45)$$

By taking pairs of individual adjoint elements acting on individual coadjoint elements, the OPE's can found.

1. $D(y)O(x)$

$$\begin{aligned} L_\xi * \tilde{L}_D &= \tilde{L}_{\tilde{D}} \\ \rightarrow \tilde{D} &= -2D\xi' - D'\xi + \frac{C}{4}(\xi''' - \xi') \end{aligned} \quad (5.46)$$

$$\begin{aligned} L_\xi * \tilde{G}(w)_\chi &= \tilde{G}_{\tilde{\chi}}(w') \\ \rightarrow \tilde{\chi} &= -B(w, w')\frac{1}{2}(3\xi'\chi - \xi\chi') \end{aligned} \quad (5.47)$$

$$\begin{aligned} L_\xi * \tilde{Q}_\phi(v) &= \tilde{Q}_{\tilde{\phi}}(v') \\ \rightarrow \tilde{\phi} &= -b(v, v')\frac{1}{2}(2\xi'\phi - \xi\phi') \end{aligned} \quad (5.48)$$

$$\begin{aligned} L_\xi * \tilde{T}_A(\Sigma) &= \tilde{T}_{\tilde{A}}(\Sigma') \\ \rightarrow \tilde{A} &= -K(\Sigma, \Sigma')(\xi'A + A'\xi) \end{aligned} \quad (5.49)$$

These expressions yield the following OPEs:

$$\begin{aligned} D(y)D(x) &= \frac{-1}{\pi i(y-x)^2}D(x) - \frac{1}{2\pi i(y-x)}\partial_x D(x) \\ &- C\left(\frac{-1}{2\pi i(y-x)^2} + \frac{-6}{2\pi i(y-x)^4}\right) \end{aligned} \quad (5.50)$$

$$D(y)\chi(x) = B(w, w')\left(\frac{-3}{4\pi i(y-x)^2}\chi(x) - \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right) \quad (5.51)$$

$$D(y)\phi(x) = \frac{1}{2}b(v, v')\left(\frac{-1}{2\pi i(y-x)^2}\phi(x) - \frac{2}{2\pi i(y-x)}\partial_x\phi(x)\right) \quad (5.52)$$

$$D(y)A(x) = K(\Sigma, \Sigma')\left(\frac{-1}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_x A(x)\right) \quad (5.53)$$

2. $\chi(y)O(x)$

$$\begin{aligned} G_g(w) * \tilde{L}_D &= \tilde{G}_{\tilde{g}}(w') \\ \rightarrow \tilde{g} &= 2B(w, w')[-gD - C(g'' - \frac{1}{4}g)] \end{aligned} \quad (5.54)$$

$$\begin{aligned} G_g(w) * \tilde{G}_\chi(w') &= \tilde{L}_{\tilde{D}} + \tilde{T}_{\tilde{A}}(\Sigma) \\ \rightarrow \tilde{D} &= B(w, w')(g'\chi - 3\chi'g) \\ \rightarrow \tilde{A} &= B(w', \Lambda(\Sigma)w)(g\chi) \end{aligned} \quad (5.55)$$

$$\begin{aligned} G_g(w) * \tilde{Q}_\phi(v) &= \tilde{Q}_{\tilde{\phi}}(v') \\ \rightarrow \tilde{\phi} &= b(v', d(\Sigma, w))(\phi'g - g'\phi) \end{aligned} \quad (5.56)$$

$$\begin{aligned} G_g(w) * \tilde{T}_A(\Sigma) &= \tilde{Q}_\phi(v) + \tilde{G}_\chi(w') \\ \rightarrow \phi &= -K(\Sigma, \phi(w, v))(gA) \\ \rightarrow \chi &= K(\Sigma, \varphi(w, w'))[(2g'A - A'g) + C(g'' - \frac{1}{4}g)] \end{aligned} \quad (5.57)$$

The OPEs are

$$\begin{aligned} \chi(y)D(x) &= 2B(w, w')\left[\frac{1}{2\pi i(y-x)}D(x) - C\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\right] \\ \chi(y)\chi(x) &= B(w, w')\frac{1}{2}\left[\frac{3}{2\pi i(y-x)^2}\chi(x) + \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right] \\ &\quad + B(w', \Lambda(\Sigma)w)\frac{3}{2\pi i(y-x)^2}\chi(x) \end{aligned} \quad (5.58)$$

$$\chi(y)\phi(x) = b(v', d(\Sigma, w))\left[\frac{1}{2\pi i(y-x)^2}\phi(x) + \frac{1}{2\pi i(y-x)}\partial_x\phi(x)\right] \quad (5.60)$$

$$\begin{aligned} \chi(y)A(x) &= \left[\frac{2}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_xA(x)\right. \\ &\quad \left.- CK(\Sigma, \varphi(w, w'))\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\right] \\ &\quad + K(\Sigma, \phi(w, v))\frac{1}{2\pi i(y-x)}A(x) \end{aligned} \quad (5.61)$$

3. $\phi(y)O(x)$

$$Q_f(v) * \tilde{L}_D = 0 \quad (5.62)$$

$$Q_f(v) * \tilde{G}_\chi(w) = 0 \quad (5.63)$$

$$\begin{aligned} Q_f(v) * \tilde{Q}_\phi(v') &= \tilde{L}_D + \tilde{T}_A(\Sigma) \\ \rightarrow D &= b(v, v') \times \frac{1}{2}(\phi' f - \phi f') \\ \rightarrow A &= -b(v', R(\Sigma)v)(f\phi) \end{aligned} \quad (5.64)$$

$$\begin{aligned} Q_f(v) * \tilde{T}_A(\Sigma) &= \tilde{G}_\chi(w) \\ \rightarrow \chi(w) &= K(\Sigma, \varphi(w, v))(fA) \end{aligned} \quad (5.65)$$

The OPEs are

$$\begin{aligned} \phi(y)\phi(x) &= b(v, v') \times -\frac{1}{2} \left[\frac{1}{2\pi i(y-x)^2} \phi(x) + \frac{1}{2\pi i(y-x)} \partial_x \phi(x) \right] \\ &\quad + b(v', R(\Sigma)v) \frac{1}{2\pi i(y-x)} \phi(x) \end{aligned} \quad (5.66)$$

$$\phi(y)A(x) = K(\Sigma, \varphi(w, v)) \frac{1}{2\pi i(y-x)} A(x) \quad (5.67)$$

4. $A(y)O(x)$

$$T_A(\Sigma) * \tilde{L}_D = 0 \quad (5.68)$$

$$\begin{aligned} T_A(\Sigma) * \tilde{G}_\chi(w') &= \tilde{G}_{\tilde{\chi}}(w) \\ \rightarrow \tilde{\chi} &= B(w', \Lambda(\Sigma)w)(A\chi) \end{aligned} \quad (5.69)$$

$$\begin{aligned} T_A(\Sigma) * \tilde{Q}_\phi(v) &= \tilde{G}_{\tilde{\chi}}(w) + \tilde{Q}_{\tilde{\phi}}(v') \\ \rightarrow \tilde{\chi} &= b(d(\Sigma, w), v)A'\phi \\ \rightarrow \tilde{\phi} &= b(v', R(\Sigma)v)A\phi \end{aligned} \quad (5.70)$$

$$\begin{aligned} T_A(\Sigma) * \tilde{T}_{\bar{A}}(\Sigma') &= \tilde{T}_{\bar{A}}([\Sigma, \Sigma']) \\ \rightarrow \bar{A}(\Sigma, \Sigma') &= K(\Sigma, \Sigma')(A'A - CA) \end{aligned} \quad (5.71)$$

The OPE that follow are

$$A(y)\chi(x) = B(w', \Lambda(\Sigma)w) \left[\frac{1}{2\pi i(y-x)} \chi(x) \right] \quad (5.72)$$

$$\begin{aligned} A(y)\phi(x) &= b(d(\Sigma, w), v) \left[\frac{-1}{2\pi i(y-x)^2} \phi(x) \right] \\ &+ b(v', R(\Sigma)v) \left[\frac{1}{2\pi i(y-x)} \phi(x) \right] \end{aligned} \quad (5.73)$$

$$A(y)A(x) = K(\Sigma, \Sigma') \left[\frac{C}{2\pi i(y-x)^2} + \frac{1}{2\pi i(y-x)} A(x) \right] \quad (5.74)$$

Two things should be noted here. The first is the inclusion of the central extension terms in the OPE. The central charge terms are the product of the central charge of the Virasoro algebra, now represented by C , multiplied by field related to the adjoint action. These terms are singular in powers of derivatives of the field. When translated into OPEs, these powers of derivatives of the field turn into 1D delta functions and their derivatives. Typically, one leaves these terms out to

simplify the OPE by reducing the number of terms and fields. The only way these terms can contribute to field theory calculations is through integration over the all space.

The second thing to note is that the metric functions b, B , and K from the commutators are carried over from the operators into the OPEs and the fields. The metric functions are all dependent on the underlying vector spaces (L, W, V) . The metrics use elements of the adjoint and coadjoint fields and appear to enforce the inner product of the underlying vector space structure of the different subalgebras of the SVA. The subspaces $(W, V, \text{ and } T)$ their metric functions $(b, B, \text{ and } K)$ and the dimension equation 5.25 define all the other structure in the theory beyond SVA.

5.4 Comparison of \mathcal{GR} and HTT SVA OPEs

To show the equivalence of the two realizations, first we will show that the HTT realization using a Clifford algebra simplifies to the \mathcal{GR} realization by a proper definition of parameters and underlying vector spaces. Next, we will show the OPEs are related using the HTT definitions.

From Chapter 4, $N = 4$ so $\dim(W)=4$. There are no Q operators so $\dim(V)=0$. By equation 5.25, this makes $\dim(L)=3$. This implies that the representation of the supersymmetry part only has three elements besides the unit element. The Lie Algebra is 3 dimensional which limits it to be $so(3)$. The key is to find a representation of both algebras using the Clifford algebra. Such a basis exists and is a subset of the Clifford algebra $Cl(3)$ of dimension $2^3 = 8$ and elements

$\gamma_1, \gamma_2, \gamma_3$. The reason for this is in the next section. The elements of the vector spaces are

$$w_i = (\gamma_0 \equiv 1, \gamma_1, \gamma_2, \gamma_3) \quad (5.75)$$

$$\Sigma^I = (\Sigma^1 \equiv \frac{i}{2}\gamma_2\gamma_3, \Sigma^2 \equiv \frac{i}{2}\gamma_1\gamma_3, \Sigma^3 \equiv \frac{i}{2}\gamma_1\gamma_2) \quad (5.76)$$

The factor of $\frac{i}{2}$ in the Σ 's is necessary for consistency. The metrics are given by the Clifford algebra commutation relations, as given in equations 5.27 and 5.28:

$$B(w_i, w_j) = [w_i, w_j] = [\gamma_i, \gamma_j] = -2\delta_{ij} \quad (5.77)$$

$$K(\Sigma^I, \Sigma^J) = [\Sigma^I, \Sigma^J] = i\epsilon_{IJK}\Sigma^K \quad (5.78)$$

With these definitions, it is straightforward to show that equations (5.7 - 5.16) collapse into equations (5.1 - 5.6).

Now these definitions can be applied to the OPEs, with the metric functions represented by only the constants, the delta and structure functions already accounted for in the inner products.

$$\begin{aligned}
D(y)D(x) &= \frac{-1}{\pi i(y-x)^2}D(x) - \frac{1}{2\pi i(y-x)}\partial_x D(x) \\
&- C\left(\frac{-1}{2\pi i(y-x)^2} + \frac{-6}{2\pi i(y-x)^4}\right) \tag{5.79}
\end{aligned}$$

$$D(y)\chi(x) = -\left(\frac{-3}{4\pi i(y-x)^2}\chi(x) - \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right) \tag{5.80}$$

$$D(y)A(x) = \left(\frac{-1}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_x A(x)\right) \tag{5.81}$$

$$\chi(y)D(x) = -4\left[\frac{1}{2\pi i(y-x)}D(x) - C\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\right] \tag{5.82}$$

$$\begin{aligned}
\chi(y)\chi(x) &= -\frac{1}{2}\left[\frac{3}{2\pi i(y-x)^2}\chi(x) + \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right] \\
&+ \frac{3}{2\pi i(y-x)^2}\chi(x) \tag{5.83}
\end{aligned}$$

$$\begin{aligned}
\chi(y)A(x) &= \left[\frac{2}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_x A(x)\right. \\
&\left.- CK(\Sigma, \varphi(w, w'))\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\right] \tag{5.84}
\end{aligned}$$

$$\begin{aligned}
\phi(y)\phi(x) &= b(v, v') \times -\frac{1}{2}\left[\frac{1}{2\pi i(y-x)^2}\phi(x) + \frac{1}{2\pi i(y-x)}\partial_x\phi(x)\right] \\
&+ b(v', R(\Sigma)v)\frac{1}{2\pi i(y-x)}\phi(x) \tag{5.85}
\end{aligned}$$

$$\phi(y)A(x) = K(\Sigma, \varphi(w, v))\frac{1}{2\pi i(y-x)}A(x) \tag{5.86}$$

$$A(y)\chi(x) = \left[\frac{1}{2\pi i(y-x)}\chi(x)\right] \tag{5.87}$$

$$A(y)A(x) = \left[\frac{C}{2\pi i(y-x)^2} + \frac{1}{2\pi i(y-x)}A(x)\right] \tag{5.88}$$

The OPEs are the same in previous chapter, with the terms associated with the central charge left in and the supersymmetric indices left out. The metric functions B and K are strictly from the inner products and enforce the level of supersymmetry inherent in the underlying vector spaces.

5.5 Expansion of \mathcal{GR} SVA OPEs by HTT realization

In the original HTT paper [2], the $N = 4$ case is presented for a Clifford algebra signature of $(0, 4)$ using a 16-dimensional representation of S and the Clifford space. The dimensions of the spaces W , V , and L are 4, 4, and 7 respectively as given by eq. 5.25. ℓ is set equal to 0 for right now. The basis vectors for W are

$$w_i = \Gamma_i \quad (5.89)$$

and for V ,

$$v_i = \Gamma_i(\Gamma^5) \quad (5.90)$$

where $\Gamma^5 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$, $(\Gamma^2) = 1$. The elements of Lie algebra are given by the ϕ mapping with the addition of one more element:

$$\varphi_{ij} = \varphi(w_i, w_j) = \Gamma_i\Gamma_j (i \neq j) \quad (5.91)$$

$$\sigma = (\Gamma^5). \quad (5.92)$$

The mappings and bilinear forms from above now take the form

$$\Lambda(\varphi_{ij})w_k = \delta_{jk}w_i - \delta_{ik}w_j \quad (5.93)$$

$$d(\varphi_{ij}, w_k) = \epsilon_{ijkl}v_l \quad (5.94)$$

$$\psi(w_i, v_j) = \Gamma_i\Gamma_j(\Gamma^5) = -\frac{1}{2}\epsilon_{ijkl}\varphi_{kl} - \delta_{ij}\sigma \quad (5.95)$$

$$d(\sigma, w_k) = v_k \quad (5.96)$$

$$R(\varphi_{ij})v_k = \delta_{jk}v_l - \delta_{ik}v_j \quad (5.97)$$

It is clear that these mappings represent the relationships between the supersymmetric elements through the antisymmetric properties of the Clifford algebra.

$$\begin{aligned}
D(y)D(x) &= \frac{-1}{\pi i(y-x)^2}D(x) - \frac{1}{2\pi i(y-x)}\partial_x D(x) \\
&- C\left(\frac{-1}{2\pi i(y-x)^2} + \frac{-6}{2\pi i(y-x)^4}\right)
\end{aligned} \tag{5.98}$$

$$D(y)\chi(x) = B(w, w')\left(\frac{-3}{4\pi i(y-x)^2}\chi(x) - \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right) \tag{5.99}$$

$$D(y)\phi(x) = \frac{1}{2}b(v, v')\left(\frac{-1}{2\pi i(y-x)^2}\phi(x) - \frac{2}{2\pi i(y-x)}\partial_x\phi(x)\right) \tag{5.100}$$

$$D(y)A(x) = K(\Sigma, \Sigma')\left(\frac{-1}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_x A(x)\right) \tag{5.101}$$

$$\chi(y)D(x) = 2B(w, w')\left[\frac{1}{2\pi i(y-x)}D(x) - C\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\right] \tag{5.102}$$

$$\begin{aligned}
\chi(y)\chi(x) &= B(w, w')\frac{1}{2}\left[\frac{3}{2\pi i(y-x)^2}\chi(x) + \frac{1}{2\pi i(y-x)}\partial_x\chi(x)\right] \\
&+ B(w', \Lambda(\Sigma)w)\frac{3}{2\pi i(y-x)^2}\chi(x)
\end{aligned} \tag{5.103}$$

$$\chi(y)\phi(x) = b(v', d(\Sigma, w))\left[\frac{1}{2\pi i(y-x)^2}\phi(x) + \frac{1}{2\pi i(y-x)}\partial_x\phi(x)\right] \tag{5.104}$$

$$\begin{aligned}
\chi(y)A(x) &= \left[\frac{2}{2\pi i(y-x)^2}A(x) - \frac{1}{2\pi i(y-x)}\partial_x A(x)\right. \\
&- CK(\Sigma, \varphi(w, w'))\left(\frac{-2}{2\pi i(y-x)^3} - \frac{1}{4}\frac{1}{2\pi i(y-x)}\right)\left. \right] \\
&+ K(\Sigma, \varphi(w, v))\frac{1}{2\pi i(y-x)}A(x)
\end{aligned} \tag{5.105}$$

$$\begin{aligned}
\phi(y)\phi(x) &= b(v, v') \times -\frac{1}{2}\left[\frac{1}{2\pi i(y-x)^2}\phi(x) + \frac{1}{2\pi i(y-x)}\partial_x\phi(x)\right] \\
&+ b(v', R(\Sigma)v)\frac{1}{2\pi i(y-x)}\phi(x)
\end{aligned} \tag{5.106}$$

$$\phi(y)A(x) = K(\Sigma, \varphi(w, v))\frac{1}{2\pi i(y-x)}A(x) \tag{5.107}$$

$$A(y)\chi(x) = B(w', \Lambda(\Sigma)w)\left[\frac{1}{2\pi i(y-x)}\chi(x)\right] \tag{5.108}$$

$$\begin{aligned}
A(y)\phi(x) &= b(d(\Sigma, w), v)\left[\frac{-1}{2\pi i(y-x)^2}\phi(x)\right] \\
&+ b(v', R(\Sigma)v)\left[\frac{1}{2\pi i(y-x)}\phi(x)\right]
\end{aligned} \tag{5.109}$$

$$A(y)A(x) = K(\Sigma, \Sigma')\left[\frac{C}{2\pi i(y-x)^2} + \frac{1}{2\pi i(y-x)}A(x)\right] \tag{5.110}$$

The effects of the extended algebra, which is a function of ℓ , can be seen in the changes to the metric functions and mappings between vector spaces, especially b , the metric term for the vector space V , and the mapping Λ using the six linear combinations $\varphi_{ij} \pm \frac{1}{2}\epsilon_{ijkl}\varphi_{kl}$:

$$b(v_i, v_j) = -\delta_{ij}(1 - \ell^2) \quad (5.111)$$

$$\Lambda(\varphi_{ij})w_k = \delta_{jk}w_i - \delta_{ik}w_j + \ell\epsilon_{ijkl}w_l \quad (5.112)$$

$$\Lambda(\varphi_{ij} \pm \epsilon_{ijkl}\varphi_{kl})w_m = (1 \mp \ell)(\delta_{jm}w_i - \delta_{im}w_j \mp \epsilon_{ijmn}w_n) \quad (5.113)$$

$$d(\varphi_{ij}, w_k) = \epsilon_{ijkl}v_l \quad (5.114)$$

$$d(\sigma, w_k) = v_k \quad (5.115)$$

$$\psi(w_i, v_j) = \Gamma_i\Gamma_j(\Gamma^5 - \ell) = -\ell\varphi_{ij} - \frac{1}{2}\epsilon_{ijkl}\varphi_{kl} - \delta_{ij}\sigma \quad (5.116)$$

$$R(\varphi_{ij})v_k = \delta_{jk}v_l - \delta_{ik}v_j + \ell\epsilon_{ijkl}v_l \quad (5.117)$$

As mentioned earlier, the parameter ℓ can be used to categorize whether the 1D $N = 4$ Super Virasoro Algebras contains the “small” $so(3)$ or “large” $N = 4$ algebra with $so(4) = so(3) \otimes so(3)$. This is reinforced by looking at the OPEs. The only OPEs from 5.98 - 5.110 that are affected are those involving b, ψ, R , and Λ . At both $\ell = \pm 1$, $b = 0$ and a number of OPEs disappear: $D(y)\phi(x)$, $\chi(y)\phi(x)$, $A(y)\phi(x)$, and $\phi(y)\phi(x)$. Two of the four remaining affected OPEs, $\chi(y)\chi(x)$ and $A(y)\chi(x)$, contain Λ which disappears at either value $\ell = \pm 1$. The last two OPEs, $\chi(y)A(x)$ and $\phi(y)A(x)$, are dependent on the elements of the Lie algebra in the Clifford algebra through the compounded metric function $K(\Sigma, \varphi(w, v))$. It is interesting to note that the ϕ function doesn't necessarily disappear by this argument although it

is not used as coadjoint field.

So by using the SVA defined by HTT in [2], we have shown that

- the GR SVA used in [6], [7], [8], and [3] can be written as subset of the HTT SVA,
- the vector space underneath the HTT SVA, which can also be realized as a Clifford Algebra, can be used to explain the structure of the GR SVA.

5.6 $SU(2)$, $SO(3)$, and Clifford Algebra Correspondence

In Chapter 2, it was clearly shown that in the $d=4$, $N=4$ case there existed a $su(2) \otimes su(2)$ algebra in the Garden Algebra that described the 3 supermultiplets. In Chapter 5, it was shown that the HTT SVA contain the “large” $so(4)$ and “small” $so(3)$ groups depending on the value of ℓ in its construction. Basic groups highlights a very interesting connection. $so(4)$ is equivalent to $so(3) \otimes so(3)$. $so(3)$ is equivalent to $su(2)$. Therefore, the construction of the $N = 4$ HTT SVA can be related to the $N=4$ supermultiplets.

It also apparent when looking at the L_I matrices for the different dimension adinkras ($d = 2, 3, 4$). For each of these dimensions, we were able to find a representation of the L_I matrices in terms of Pauli matrices. For $d = 2$, $\mathcal{C}^{(1)} = \{\mathbf{I}_2, \boldsymbol{\sigma}^3\}$, $\mathcal{S}_1 = \{\mathbf{I}_2, -i\boldsymbol{\sigma}^2\}$.

From [1], we have for $d = 3$:

$$L_1 = i\sigma_1 \otimes \sigma_2 \quad (5.118)$$

$$L_2 = i\sigma_2 \otimes \mathbf{I}_2 \quad (5.119)$$

$$L_3 = -i\sigma_3 \otimes \sigma_2 \quad (5.120)$$

In [10], 4×4 matrices denoted by α_I and β_I were used. These also possess decompositions a bit string times an element of the permutation group according to:

$$\begin{aligned} \alpha^1 &= \sigma^2 \otimes \sigma^1 = -i(12)_b(4321)_p , \\ \alpha^2 &= \mathbf{I} \otimes \sigma^2 = -i(10)_b(2143)_p , \\ \alpha^3 &= \sigma^2 \otimes \sigma^3 = -i(6)_b(3412)_p , \\ \beta^1 &= \sigma^1 \otimes \sigma^2 = -i(10)_b(4321)_p , \\ \beta^2 &= \sigma^2 \otimes \mathbf{I} = -i(12)_b(3412)_p , \\ \beta^3 &= \sigma^3 \otimes \sigma^2 = -i(6)_b(2143)_p , \end{aligned} \quad (5.121)$$

and of course the 4×4 identity matrix correspond to

$$\mathbf{I}_4 = (0)_b(1234)_p . \quad (5.122)$$

The significance of these observations is that both the sets define by

$$\{\mathcal{A}\} = \{\mathbf{I}_4, i\alpha_I\} , \quad \{\mathcal{B}\} = \{\mathbf{I}_4, i\beta_I\} , \quad (5.123)$$

also satisfy the conditions of (5). The set of matrices $\{\mathcal{C}\}$, where

$$\{\mathcal{C}\} = \{\mathbf{I}_4, i\alpha_I, i\beta_I, \alpha_I\beta_J\} , \quad (5.124)$$

forms a basis for the expansion of all 4×4 real matrices. It is thus of interest to analyze these completely as representations of S_4 . Our results are summarized in the table below.

$\alpha_1\beta_1 = (6)_b(1234)_p$	$\alpha_1\beta_2 = (0)_b(2143)_p$	$\alpha_1\beta_3 = (5)_b(3412)_p$
$\alpha_2\beta_1 = (0)_b(3412)_p$	$\alpha_2\beta_2 = (9)_b(4321)_p$	$\alpha_2\beta_3 = (12)_b(1234)_p$
$\alpha_3\beta_1 = (3)_b(2143)_p$	$\alpha_3\beta_2 = (10)_b(1234)_p$	$\alpha_3\beta_3 = (0)_b(4321)_p$

Table 5.2: Binary/Permutation Element Decomposition

When looking closer at the use of Clifford Algebra for each object, we see the following. For adinkras, the Clifford algebra can be found as a combination of the L and R matrices. Specifically,

$$\Gamma_I = \begin{bmatrix} 0 & L_I \\ R_I & 0 \end{bmatrix} \quad (5.125)$$

In this form, the Clifford algebra is connected to how the fields transform through the covariant superspace derivative, D . In the HTT SVA, the Clifford algebra also are used to represent how the fields are used but through the supersymmetry operator, G . As mentioned above, these two are related but it is not clear at this time how this can be made explicit. There exists a path between the two that could be exploited to give the OPEs for the supermultiplets without having a Lagrangian or Action to describe the SUSY theory.

Chapter 6

Conclusions

In the dissertation, the plan was to explore the relationship between representations of SUSY theories. Two methods were used: Adinkras representations of the Garden Algebras and Clifford Algebra Representations of the Super Virasoro Algebra. Using a specific representation of the Garden Algebra, $\mathcal{GR}(4,4)$, Mathematica found all the 4×4 matrices that could be solutions and all the tetrad of solutions that solved $\mathcal{GR}(4,4)$. By analysis of the Adinkras, the tetrad solutions were associated with 3 SUSY multiplets that formed different SUSY theories. Their multiplets were related in a way that allowed unique equivalence classes to be found.

Starting with a geometric representation of the Super Virasoro Algebra, the method of using Coadjoint orbits was used to find Operator Product Expansions of the fields from the SVA. By comparison with another representation of the SVA, that of Hasiewicz, Thielemans, and Troost [2] which uses Clifford Algebras, the Operator Product Expansions were calculated and shown to be equivalent to the ones calculated in the earlier chapter using the Geometrical Realization. This equivalence allowed a quicker calculation of the OPEs for the extended GR SVA by analysis. It also gave insight to the nature of the relationship between underlying vector spaces and supersymmetric properties of the theory.

The key lessons learned were

- $d=4, N=4$ Adinkras break into 6 sets found in S_4
- \mathcal{GR} SVA \subset HTT SVA
- Adinkras and Garden Algebras are related to the SVAs by the Clifford Algebra

One of the goals of the research was not only to discover equivalence relations about Adinkras (and thus relations between SUSY theories) but also to develop an easier way to talk about the dynamics of Adinkras by using \mathcal{GR} SVA and OPEs. This idea was explored but there is still more work to do make the explicit connection to the dynamics of SUSY theories. When this is accomplished, calculations can be done computing the masses and interaction constants of SUSY particles using these methods. These contributions will push the boundaries of theoretical and experimental physics ahead to the next great advance.

Appendix A

Mathematical Concepts

A.1 Grassman Variables

Grassmann numbers were discovered by the German mathematician Hermann Grassmann (1809 - 1877). A Grassman variable is a variable that anticommutes with other Grassmann variables. That is, for Grassmann variables ζ_i, ζ_j

$$\{\zeta_i, \zeta_j\} = \zeta_i\zeta_j + \zeta_j\zeta_i = 0 \quad (\text{A.1})$$

By application of the above formula for ζ_i in both positions, we see that

$$\{\zeta_i, \zeta_i\} = 2\zeta_i\zeta_i = 0 \rightarrow (\zeta_i)^2 = 0. \quad (\text{A.2})$$

It is these properties that allow fermionic fields to be represented correctly mathematically.

A.2 Clifford Algebras

Clifford Algebras were developed by the English mathematician William Kingdon Clifford (1845 - 1879). A Clifford Algebra is an algebra generated by elements denoted γ_i that obey the following rule:

$$\{\gamma_i, \gamma_j\} = \gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij} \quad (\text{A.3})$$

where g_{ij} is a metric. Typically, g_{ij} is equal to δ_{ij} however g_{ij} can have any signature. In [2], 3 particular signatures of a Clifford Algebra are mentioned: (0,4),(1,3),and (2,2).

The number of generators in the Clifford Algebra depends on the dimensionality of the algebra. $\text{Cliff}(2\nu)$ has 2ν generators. $\text{Cliff}(2\nu+1)$ has the same generators as $\text{Cliff}(2\nu)$ but includes a new one, $\Gamma^{2\nu+1}$ that equals the product of the other generators.

$$\Gamma^{2\nu+1} = \prod_{i=1}^{2\nu} \Gamma^i \tag{A.4}$$

A.3 Cartan Matrix

A Cartan matrix is a matrix with the following properties:

- It is square.
- All elements are integers.
- All diagonal elements are equal to 2.
- All off-diagonal elements are non-positive
- An off-diagonal element is zero if and only if the transpose element is also zero.

Appendix B

List of L-Matrices Solutions of the ($d = 4, N = 4$) Garden Algebra

There are 384 matrices that form the solution space of the $\mathcal{GR}(4,4)$ Algebra. Out of that, 1536 unique tetrads of solutions exist. It has been shown that they break into six sets that describe different supermultiplets. Of the 256 elements, one can focus on the even-bitstring solutions since the odd-bit strings can be obtained with a sign flip. This leaves six sets of 16 tetrad solutions, which are listed here.

However, instead of the bit-string/permutation notation used earlier, a more concise notation is used. This notation, called the Bracket-Overbar notation simply replaces the bit-string word with an overbar($\bar{}$) over the permutation position that is negative. Thus $(10_b)(4231_p)$ is now $\langle 4\bar{2}3\bar{1} \rangle$.

$\{CM\}$

$$\begin{array}{cccccccc}
 \langle 1423 \rangle & \langle 23\bar{1}4 \rangle & \langle 3\bar{2}4\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle & \langle 14\bar{2}3 \rangle & \langle 231\bar{4} \rangle & \langle 3\bar{2}\bar{4}\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle \\
 \langle 1423 \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 324\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle & \langle 14\bar{2}3 \rangle & \langle 2\bar{3}14 \rangle & \langle 324\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle \\
 \langle 1\bar{4}23 \rangle & \langle 23\bar{1}4 \rangle & \langle 3\bar{2}4\bar{1} \rangle & \langle 413\bar{2} \rangle & \langle 1\bar{4}23 \rangle & \langle 2314 \rangle & \langle 3\bar{2}4\bar{1} \rangle & \langle 413\bar{2} \rangle \\
 \langle 1\bar{4}23 \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 324\bar{1} \rangle & \langle 413\bar{2} \rangle & \langle 1\bar{4}23 \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 324\bar{1} \rangle & \langle 413\bar{2} \rangle \\
 \langle 142\bar{3} \rangle & \langle 23\bar{1}4 \rangle & \langle 3\bar{2}41 \rangle & \langle 4\bar{1}3\bar{2} \rangle & \langle 142\bar{3} \rangle & \langle 2314 \rangle & \langle 3\bar{2}4\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle \\
 \langle 142\bar{3} \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 324\bar{1} \rangle & \langle 4\bar{1}3\bar{2} \rangle & \langle 142\bar{3} \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 3241 \rangle & \langle 4\bar{1}3\bar{2} \rangle \\
 \langle 1\bar{4}2\bar{3} \rangle & \langle 23\bar{1}4 \rangle & \langle 3\bar{2}4\bar{1} \rangle & \langle 4132 \rangle & \langle 1\bar{4}2\bar{3} \rangle & \langle 231\bar{4} \rangle & \langle 3\bar{2}41 \rangle & \langle 413\bar{2} \rangle \\
 \langle 1\bar{4}2\bar{3} \rangle & \langle 2\bar{3}\bar{1}4 \rangle & \langle 3241 \rangle & \langle 41\bar{3}\bar{2} \rangle & \langle 1\bar{4}2\bar{3} \rangle & \langle 2\bar{3}14 \rangle & \langle 324\bar{1} \rangle & \langle 413\bar{2} \rangle
 \end{array}$$

(B.1)

$\{VM_3\}$

$$\begin{array}{cccc}
 \langle 1234 \rangle & \langle 2\bar{1}43 \rangle & \langle 34\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}34 \rangle & \langle 2143 \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 432\bar{1} \rangle \\
 \langle 1234 \rangle & \langle 2\bar{1}4\bar{3} \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}34 \rangle & \langle 214\bar{3} \rangle & \langle 34\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle \\
 \langle 123\bar{4} \rangle & \langle 2\bar{1}43 \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}3\bar{4} \rangle & \langle 2143 \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle \\
 \langle 123\bar{4} \rangle & \langle 2\bar{1}4\bar{3} \rangle & \langle 34\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}3\bar{4} \rangle & \langle 214\bar{3} \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 432\bar{1} \rangle \\
 \langle 1\bar{2}34 \rangle & \langle 2\bar{1}43 \rangle & \langle 34\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}34 \rangle & \langle 2143 \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle \\
 \langle 1\bar{2}34 \rangle & \langle 2\bar{1}4\bar{3} \rangle & \langle 34\bar{1}2 \rangle & \langle 432\bar{1} \rangle & \langle 1\bar{2}34 \rangle & \langle 214\bar{3} \rangle & \langle 34\bar{1}2 \rangle & \langle 4\bar{3}2\bar{1} \rangle \\
 \langle 1\bar{2}3\bar{4} \rangle & \langle 2\bar{1}43 \rangle & \langle 3\bar{4}\bar{1}\bar{2} \rangle & \langle 432\bar{1} \rangle & \langle 1\bar{2}3\bar{4} \rangle & \langle 2143 \rangle & \langle 34\bar{1}\bar{2} \rangle & \langle 4\bar{3}2\bar{1} \rangle \\
 \langle 1\bar{2}3\bar{4} \rangle & \langle 2\bar{1}4\bar{3} \rangle & \langle 34\bar{1}2 \rangle & \langle 4\bar{3}2\bar{1} \rangle & \langle 1\bar{2}3\bar{4} \rangle & \langle 214\bar{3} \rangle & \langle 3\bar{4}\bar{1}2 \rangle & \langle 4\bar{3}2\bar{1} \rangle
 \end{array}$$

(B.6)

Appendix C

Other Group Theory Concepts Related to the Equivalence Classes of Adinkras

C.1 The $O(4)$ Group

In discussion with a colleague about the equivalence classes, the orthogonal group was discussed as a key to understanding the classes. An element O of the orthogonal group satisfies the equation $O O^T = \mathbf{1}$ where $\mathbf{1}$ is the identity matrix of the proper dimension. An orthogonal matrix can transform a L_i matrix into another matrix by $OL_iO^T = \tilde{L}_i$.

The L_i that are solutions to the Garden algebra are elements of the orthogonal group $O(4)$. The $O(4)$ group has 4 generators which can be mapped directly to the 4 solution matrices, L_i . Since we are free to apply any $O(4)$ transformation, we choose a transformation that takes L_1 and maps it into the identity matrix \mathbf{I}_4 . The solution matrices that make up L_2 , L_3 , and L_4 under the same transformation are now matrices that are orthogonal to the identity matrix and each other. This is the same as a reduction from the $O(4)$ group down to the $O(3)$ subgroup.

C.2 The Group D_2

Another interesting group is the crystal group D_2 . D_2 describes the symmetries of 2 dimensional rectangle with labelled vertices. The symmetric operations in D_2 are rotations around the center of the rectangle and flips along different axes of the rectangle. It is clear that one could apply this group action on the $\mathcal{N} = 2$ adinkra and define equivalence classes. It is not clear if this approach can be taken to higher dimensional adinkras. A similar method as used here could be developed for higher dimensions, focusing on cycle-sign representation of the L-matrices and field state vectors $(a_1 \dots a_{\mathcal{N}})$ and then writing the elements of the group D_2 in these terms.

By studying both of these groups, it may be possible to easily find analytical solutions to higher dimensional Garden Algebra and define the necessary equivalence classes and thus groups of related supersymmetric theories.

C.3 Graph Theory Ideas

Another method discussed [15] relates to graph theory. Graph theory is the study of planar objects consisting of node and links. Obviously an adinkra is a type of graph that follows certain rules. There may be some ideas from graph theory that we can take advantage of to help define equivalence classes.

For every graph, there exists an adjacency matrix which represents the connections between nodes. The row and columns represent the nodes in the graph. The value of each element, γ_{ij} in the matrix is 0 if there is no direct link between

the i^{th} and j^{th} nodes, and non-zero if there is a direct link. The non-zero value can be anything, with the value 1 commonly used for undirected graphs, and ± 1 used for directed graphs.

Adjacency matrices were discussed in [15]. Using the definition from [15] and [10], the following matrix

$$\gamma_I = \begin{bmatrix} 0 & L_I \\ R_I & 0 \end{bmatrix} \quad (C.1)$$

can be taken as an adjacency matrix.

An adjacency matrix can be made for every adinkra. If a fixed method is used to label nodes and links, the adjacency matrix can be used to group similar graphs together. However, there are graphs that may have different looking adjacency matrices but still represent the same underlying supersymmetric theory. Vice versa, there are adinkras which look similar, have similar adjacency matrices but do represent different theories. we would like to separate the later case.

Another method of distinguishing matrices is by using the characteristic polynomial. The characteristic polynomial is a polynomial function generated from a square matrix that contains important information about the matrix. For every graph, there is a characteristic polynomial associated with the adjacency matrix for the the graph.

The characteristical polynomial would seem to be a good choice for determining equivalence classes. We can apply the idea to the adjacency matrix representation of adinkras. For every L_I , we multiply it by a constant t and build an adjacency matrix from all the L_I solutions. The adjacency matrix is now populated with $\pm t$'s.

The characteristic polynomial made from this matrix is a polynomial of rank $2\mathcal{N}$ and only contains even powers. However, this method using the adjacency matrix does not separate the adinkras. All the solutions found for the 4×4 case have the same characteristic polynomial: $256 - 256t^2 + 96t^4 - 16t^6 + t^8$.

Appendix D

Valise Adinkras of L-Matrices Solutions of the $(d = 4, N = 4)$

Garden Algebra

In Chapter 3, we found the solutions of the $\mathcal{GR}(4, 4)$ divided into 6 sets that split the group S_4 . Three of these sets corresponded to the three supermultiplets previously discussed (Chiral, Tensor, Vector). The other sets were referred to as VM1, VM2, and VM3.

The L-matrices can be used to draw adinkras representing the supermultiplets. This can be done using valise adinkras. A valise adinkra is an adinkra with 2 levels, one for fermionic fields and one for bosonic fields on the lowest level.

To move a node from one level to another requires a change of engineering dimension of the requisite field. Since fermionic and bosonic levels differ by half-integer engineering dimension, the move of a field to the next lowest level of the same field is an integer engineering dimension change. This engineering change is accomplished by the use of integrals and derivatives of the fields. Valise adinkras must still obey the Garden Algebra equations so the derivative and integral changes are mapped through the definitions of Φ_i and Ψ_i .

Below are the valise adinkras that correspond to the 6 sets of tetrad solutions shown in Table 3.6f.

For the Chiral Supermultiplet Valise adinkra to have the correct engineering

dimensions and satisfy the Garden Algebra equations, we have the mapping

$$\phi_1 \rightarrow i\Psi_1, \quad \phi_2 \rightarrow i\Psi_2, \quad \phi_3 \rightarrow i\Psi_3, \quad \phi_4 \rightarrow i\Psi_4, \quad (\text{D.1})$$

and

$$\Phi_1 = A, \quad \Phi_2 = B, \quad \partial_0\Phi_3 = F, \quad \partial_0\Phi_4 = G. \quad (\text{D.2})$$

The Tensor Supermultiplet adinkra is a natural Valise adinkra.

For the Vector Supermultiplet valise adinkra to have the correct engineering dimensions and satisfy the Garden Algebra equations, we have the mapping

$$\lambda_1 \rightarrow i\Psi_1, \quad \lambda_2 \rightarrow i\Psi_2, \quad \lambda_3 \rightarrow i\Psi_3, \quad \lambda_4 \rightarrow i\Psi_4, \quad (\text{D.3})$$

and

$$\Phi_1 = A_1, \quad \Phi_2 = A_2, \quad \Phi_3 = A_3, \quad \partial_0\Phi_4 = d. \quad (\text{D.4})$$

The mapping above is applied to the VM1, VM2, and VM3 Valise adinkras also.

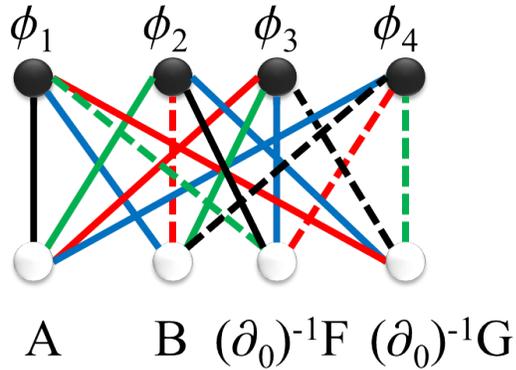


Figure D.1: The $N = 4$ Chiral Supermultiplet Valise Adinkra

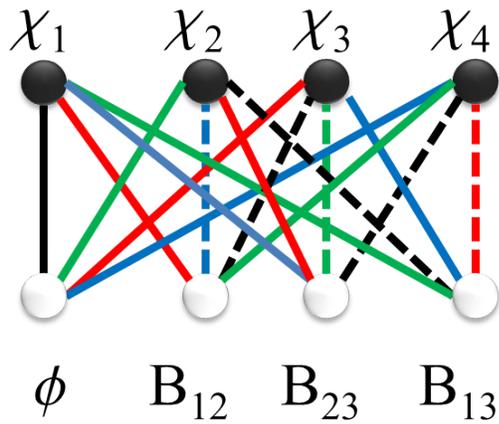


Figure D.2: The $N = 4$ Tensor Supermultiplet Valise Adinkra

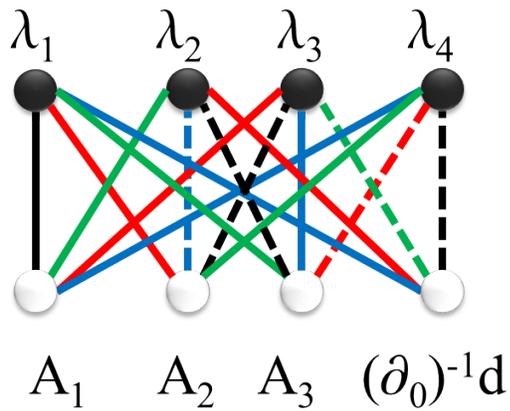


Figure D.3: The $N = 4$ Vector Supermultiplet Valise Adinkra

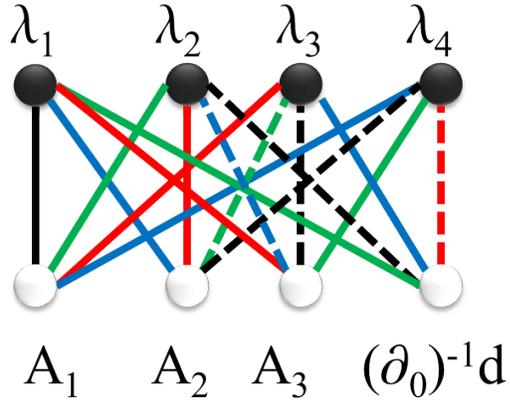


Figure D.4: The $N = 4$ VM1 Valise Adinkra

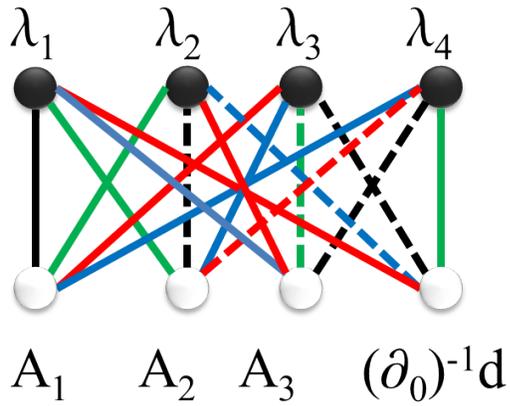


Figure D.5: The $N = 4$ VM2 Valise Adinkra

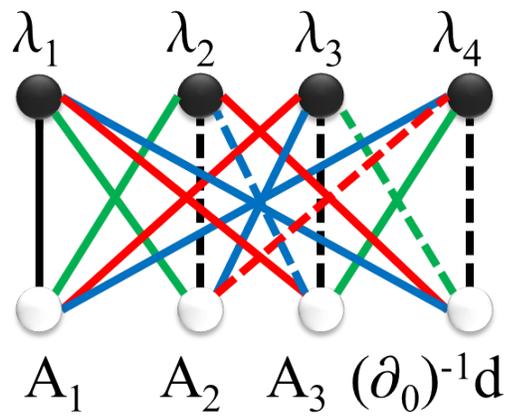


Figure D.6: The $N = 4$ VM3 Valise Adinkra

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