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Matrix-Geometric Solution For Two Node Tandem Queueing Systems With Phase-Type Servers Subject To Blocking And Failures

by

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ABSTRACT

A two node tandem queueing system with phase-type servers and *Bernoulli* arrivals is considered in *discrete-time* when servers are subject to *blocking* and *failures*. The invariant probability vector of the the underlying *finite* state Quasi-Birth-and-Death process is shown to admit a *matrix-geometric* representation for all values of the arrival rate λ. The corresponding rate matrix is given *explicitly* in terms of the model parameters and the resulting closed-form expression provides the basis for an *efficient* calculation of the invariant probability vector. The cases $\lambda = 1$ and $\lambda < 1$ are studied separately and the irreducibilty of the underlying Markov chain is discussed for each case. The continuous-time formulation is briefly discussed and only major differences with the discrete-time results are pointed out. Some numerical examples are also provided.

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1. INTRODUCTION

Tandem queueing systems with *finite* buffers and *blocking* are an essential modelling component of many manufacturing facilities and communication networks. To fix the ideas, consider a production line where parts sequentially require work from several machines in a specified order. Each machine is attended by a finite capacity buffer and *blocking* may thus occur. This blocking phenomenon affects the performance of the system in an essential way, especially when the various machines are subject to *failures* or to various other *interruptions*, as is often the case in real systems. Queueing networks with blocking have been studied by researchers from different research communities, and are typically difficult to analyze. To date strikingly few results are available, with the bulk of the work focusing on *continuous-time* models. A brief classification of the blocking systems discussed in the literature is provided in [8], where an annotated bibliography of some of these papers has been compiled.

The aim of this paper is to gain understanding of the blocking phenomenon through the study of *simple* models for which easily computable analytical results can be obtained. The simple model with blocking analyzed here is the two node *tandem* queueing system, depicted in Figure 1.1, with *finite* capacity intermediate buffers. *Both* nodes are attended by *single* servers which operate according to the FCFS (first-come-first-served) queueing discipline. The service time distributions at each node are *phase type (PH)*. Jobs arrive into the system according to a *Bernoulli/Poisson* process and some work is performed on a job at the first node. The job is then either passed on to the second node or is fed back to the first buffer with a certain probability. Upon completion of the service at the second node, the job is either fed back to the intermediate buffer or is ejected from the system.

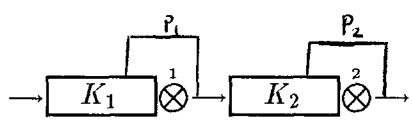

Figure 1.1.

It is assumed that the server in the second node is *never blocked* and the *immediate* blocking stategy is adopted for the first node server in that the blocking of this server occurs as soon as the intermediate buffer becomes full. Note that this event occurs necessarily at a service completion. The server remains blocked until the congestion is reduced at the second node, at which time the blocked server resumes service and *begins* to process its next job (if any). The methodology developed in this paper also applies to two node systems under the *non-immediate* blocking policy whereby the first node server is blocked at a service completion time if the job that has just completed service, cannot proceed to the intermediate buffer due to congestion. When the congestion is reduced downstream, this job proceeds to the next buffer without receiving any further service, and the blocked server resumes service and begins processing its next job (if any).

The discussion is given here for the discrete-time model while the results for the continuous-time formulation are only briefly mentioned for sake of completeness. The modelling of time as a discrete parameter is motivated by the fact that service times in manufacturing systems are usually constant, the only source of randomness being introduced by the possibility of server failures and random arrivals.

Interest in the simple system of Figure 1.1 is two-fold. Firstly, this simple model can be used as a *building* block for generating approximation algorithms to analyze more complex tandem queueing systems. Indeed, many of the approximation schemes developed for general tandem queueing systems, say by Altıok [1], Brandwajn and Jow [4], Gün and Makowski[10], Hillier and Boling [11] and Sheskin [15], to name a few, are based on *decomposition* and *isolation* ideas that reduce the problem to one of simultaneously solving several simpler systems which are special cases of the model studied here. These ideas clearly motivate a careful study of the *general* two node tandem systems with blocking, with a view towards providing exact analytical results in order to enhance existing and future approximation schemes. Secondly, an efficient analysis of such two node systems is of independent interest as they can be used to model some practical situations [16]. In an earlier paper, in the case of *exponentialy* distributed service times, Konheim and Reiser [12] obtained an *algorithmic* solution for the *joint* queue-length distribution of the system depicted in Figure 1.1 at steady state when $K_1 = \infty$ One of the contributions of this paper is to identify a class of two node tandem systems with blocking for which this *joint* queue length distribution can be obtained in *closed-form.*

The paper is organized as follows: The model of interest is described precisely in Section 2. In Section 3, the model is analyzed when the Bernoulli arrival stream has parameter λ by making use of the fact that the underlying Markov chain is a *finite* state Quasi-Birth-and-Death (QBD) process. As pointed out by Neuts [14], QBD processes enjoy interesting structural properties which can be used to advantage in the computations. Indeed, under fairly general assumptions, the stationary probability vector $\pi = (\pi_0, \pi_1, \ldots)$ of a QBD process with *countably infinite* state space exhibits the *matrix-geometric* property [14], i.e., there exists a matrix R such that

$$\pi_{k+1} = \pi_k R, \quad k \geq 0 \tag{1.1}$$

where the matrix R is the minimal nonnegative solution of a matrix quadratic equation.

For *finite* state QBD processes, the situation is somewhat different owing to the presence of *boundary* states, and it is not possible in general to assert that the invariant distribution exhibits a matrix-geometric property. However, as pointed out by the authors in [7] and [9], it is sometimes possible to obtain a closed-form solution with a matrix-geometric form similar to (1.1). The properties that define such QBD processes are present in a number of well-known queueing problems and can be used to provide an *explicit* expression for a matrix R in terms of the model parameters.

The solution techniques presented in [7] and [9] apply here to obtain closed-form matrix-geometric expressions for the invariant probabilities of the system states when $\lambda = 1$ and $\lambda < 1$, respectively. In the case $\lambda = 1$, the system is effectively reduced to a *saturated* two node system in that the input queue in Figure 1.1 contains K_1 jobs at all times, so that the first node server is *never starved.* While the Markov chain may have several ergodic classes if $\lambda = 1$, it is shown to *always* have a unique ergodic class for $\lambda < 1$.

The results are extended in Section 4 to *unreliable* servers with PH-type service and repair distributions. The *effective* service time distribution of such servers is shown to admit a PH-type represention of higher order so that the methods discussed in Section 3 apply. To illustrate this point, the case $\lambda = 1$ is discussed under the assumption that idling servers are allowed to fail; the case where only operational servers fail is then obtained as a special case. In section 5, the results are supplemented with several numerical examples. In Section 6, the continuous-time formulation is briefly discussed and major differences with the discrete-time results are pointed out.

In order to fix the terminology used in the paper, several definitions and results from the theory of nonnegative matrices are collected in the Appendix.

A word on the notation used hereafter: The $r \times r$ identity matrix is denoted by I_r and the $r \times 1$ column vector of ones is denoted by e_r, while the $r \times r$ matrix and the $1 \times r$ dimensional row vector with all zero entries are denoted by $0_{r \times r}$ and 0_r, respectively. The notation $\overline{x}$ is used to denote $1 - x$ for $0 \leq x \leq 1$. The notation $y > 0$ is used if each entry of the $1 \times r$ vector y is nonnegative and $y \neq 0_r$. The symbol $\otimes$ denotes the Kronecker product of matrices.

2. TWO NODE TANDEM SYSTEM WITH PH-TYPE SERVERS AND FEEDBACK

The model consists of two nodes with *finite* capacity buffers of size K_1 and K_2 in front of the first and second node servers, respectively, inclusive of the jobs in service. Each node is attended by a *single* server. The service times at each node are assumed to be *independent* and *identically distributed (i.i.d)* with common PH-distribution given by the *irreducible* representations (α, A) and (β, B) for the first and second node servers, respectively. The service times at different servers are also assumed mutually independent. The row vectors α and β, and the matrices A and B have dimensions $1 \times l$, $1 \times m$, $l \times l$ and $m \times m$, respectively, and the corresponding $l \times 1$ and $m \times 1$ column vectors of absorption (service completion) probabilities for the first and the second node server are denoted by a and b, respectively. The reader is refered to [14, Chap. 2] for the probabilistic interpretation of the notation used here. It is assumed, without loss of generality, that $\alpha_{l+1} = \beta_{m+1} = 0$. Moreover, the matrices $(I_l - A)$ and $(I_m - B)$ are both assumed *nonsingular* or equivalently, that service completion, from any initial state, is certain [14, p. 45].

A *Bernoulli* arrival stream with parameter λ feeds into the first buffer under the assumption that an arrival which sees a full buffer is *lost*. It is also assumed that the second node server is never blocked, and that the *immediate* blocking strategy is enforced on the first node server. A job whose service is completed in the i^{th} node server receives another service from this server with probability p_i, $0 \leq p_i < 1$, $i = 1, 2$, i. e., a job serviced at station 1 joins the intermediate buffer with probability $\overline{p}_1$ and a job serviced at station 2 leaves the system with probability $\overline{p}_2$.

Under this feedback mechanism, the *effective* service time distibutions are again of PH-type with *irreducible* representations $(\alpha, \widetilde{A})$ and $(\beta, \widetilde{B})$ where $\widetilde{A} = A + p_1 a\alpha$ and $\widetilde{B} = B + p_2 b\beta$. The corresponding effective absorption probability vectors are now given by $\widetilde{a} = \overline{p}_1 a$ and $\widetilde{b} = \overline{p}_2 b$. In order to avoid simplify the notation, feedback will be directly incorporated into the PH-type representation as indicated above, i. e., there is no loss of generality in assuming $p_1 = p_2 = 0$, so that $\widetilde{A} = A$, $\widetilde{B} = B$, etc...

The state space of the system is naturally defined to be the set E given by

$$E = \bigcup_{k_1=0}^{K_1} E_{k_1} \tag{2.1a}$$

where the sets E_{k_1}, $0 \leq k_1 \leq K_1$, are defined by

$$\begin{aligned} E_0 :=& \left\{ e \in E : \; e = (0, k_2, j), \;\; 1 \leq k_2 \leq K_2 \right\} \cup \{(0,0)\} , \\ E_{k_1} :=& \left\{ e \in E : \; e = \left\{ \begin{array}{ll} (k_1, 0, i), & k_2 = 0 \\ (k_1, k_2, i, j), & 1 \leq k_2 < K_2 \\ (k_1, K_2, j), & k_2 = K_2 \end{array} \right\} \right\} , 1 \leq k_1 \leq K_1, \end{aligned} \tag{2.1b}$$

for $1 \leq i \leq l$ and $1 \leq j \leq m$. Here, k_1 and k_2 represent the numbers of jobs in buffer one and two, respectively, while i and j represent the service phases at the first and second node servers, respectively. The phase of the first node server is not defined when it has no job to process or when it is blocked, while the phase of the second node server is not defined when the second buffer is empty. Note that the sets E_{k_1}, $0 \leq k_1 \leq K_1$, form a partition of the state space E of the Markov chain. It will be convenient to use the notation

$$r := (K_2 - 1)lm + l + m \quad \text{and} \quad s := K_2 m + 1$$

in the forthcoming discussion.

The one-step probability transition matrix of the underlying Markov chain defined on the set E is denoted by P. The purpose of this paper is to find any invariant probability vector π of P in a computationaly efficient way, i. e., any $1 \times (K_1 r + s)$ row vector π, which satisfies the equations

$$\pi = \pi P \quad \text{and} \quad \pi e_{K_1 r + s} = 1. \tag{2.2}$$

Any such vector π is partitioned into $K_1 + 1$ blocks of components, say $\pi = (\pi_0, \pi_1, \ldots, \pi_{K_1})$, with π_0 being a $1 \times s$ vector and π_{k_1}, $0 < k_1 \leq K_1$, being $1 \times r$ row vectors. Each block entry π_{k_1}, $0 \leq k_1 \leq K_1$, is of the form $\pi_{k_1} = (\pi_{k_1 0}, \pi_{k_1 1}, \ldots, \pi_{k_1 K_2})$ where the vector $\pi_{k_1 k_2}$ is of dimension

$$\begin{cases} lm \;, & 1 \leq k_1 \leq K_1, \;\; 1 \leq k_2 < K_2, \\ l \;, & 1 \leq k_1 \leq K_1, \;\; k_2 = 0, \\ m \;, & k_1 = 0, \;\; 1 \leq k_2 \leq K_2, \text{ or} \\ & 1 \leq k_1 \leq K_1, \;\; k_2 = K_2, \\ 1 \;, & k_1 = 0, \;\; k_2 = 0, \end{cases}$$

and the entries of each vector $\pi_{k_1 k_2}$, $1 \leq k_1 \leq K_1$, $1 \leq k_2 < K_2$, are ordered as $(1,1), (2,1), \ldots,$ $(l,1), (1,2) \ldots, (i,j), \ldots, (l-1,m), (l,m)$.

By ordering the states in this way, the matrix P can be put in the form

$$P = \begin{pmatrix} B_1 & B_0 & & & & & \\ B_2 & A_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_2 & A_1 & A_0 \\ & & & & & A_2 & C_1 \end{pmatrix}, \tag{2.3}$$

where the block entries B_0, B_1, and B_2 are of dimensions $s \times s$, $s \times r$ and $r \times s$, respectively, and the matrices A_i, $0 \le i \le 2$, and C_1 are of dimensions $r \times r$. These matrices are given by

$$B_0 = \lambda \begin{pmatrix} \alpha & & & & & \\ b \otimes \alpha & B \otimes \alpha & & & & \\ & b\beta \otimes \alpha & B \otimes \alpha & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & b\beta \otimes \alpha & B \otimes \alpha & \\ & & & & b\beta \otimes \alpha & B \end{pmatrix},$$

$$B_1 = \overline{\lambda} \begin{pmatrix} 1 & & & & \\ b & B & & & \\ & b\beta & B & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & b\beta & B \\ & & & & b\beta & B \end{pmatrix},$$

$$B_2 = \overline{\lambda} \begin{pmatrix} 0 & \beta \otimes a & & & & \\ & b\beta \otimes a & B \otimes a & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & b\beta \otimes a & B \otimes a & \\ & & & & b\beta \otimes a & B \otimes a \\ & & & & & 0 \end{pmatrix},$$

$$A_0 = \lambda \begin{pmatrix} A & & & & & \\ b \otimes A & B \otimes A & & & & \\ & b\beta \otimes A & B \otimes A & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & b\beta \otimes A & B \otimes A & \\ & & & & b\beta \otimes \alpha & B \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \overline{\lambda}A & \lambda(\beta\otimes a\alpha) & & & & \\ \overline{\lambda}(b\otimes A) & A_d & \lambda(B\otimes a\alpha) & & & \\ & \overline{\lambda}(b\beta\otimes A) & A_d & \lambda(B\otimes a\alpha) & & \\ & & \cdot & \cdot & \cdot & \\ & & & \overline{\lambda}(b\beta\otimes A) & A_d & \lambda(B\otimes a) \\ & & & & \overline{\lambda}(b\beta\otimes\alpha) & \overline{\lambda}B \end{pmatrix},$$

$$A_2 = \overline{\lambda}\begin{pmatrix} 0 & \beta\otimes a\alpha & & & & \\ & b\beta\otimes a\alpha & B\otimes a\alpha & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & b\beta\otimes a\alpha & B\otimes a\alpha & \\ & & & & b\beta\otimes a\alpha & B\otimes a \\ & & & & 0 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} A & \lambda(\beta\otimes a\alpha) & & & & \\ b\otimes A & C_d & \lambda(B\otimes a\alpha) & & & \\ & b\beta\otimes A & C_d & \lambda(B\otimes a\alpha) & & \\ & & \cdot & \cdot & \cdot & \\ & & & b\beta\otimes A & C_d & \lambda(B\otimes a) \\ & & & & b\beta\otimes\alpha & B \end{pmatrix},$$

with the diagonal entries A_d and C_d being defined by

$$A_d = \overline{\lambda}\,(B\otimes A) + \lambda\,(b\beta\otimes a\alpha)\ ,$$
$$C_d = B\otimes A + \lambda\,(b\beta\otimes a\alpha)\ .$$

In the following sections, this model will be analyzed in detail when $\lambda = 1$ and $\lambda < 1$. The results will then be generalized to capture the situation of *unreliable* servers with PH-type service and repair time distributions.

3. ANALYSIS OF THE MODEL

3.1. $\lambda = 1$

When $\lambda = 1$, the block matrices B_1, B_2 and A_2 are all equal to the identically zero matrix with appropriate dimensions, and it is easy to see that the subset E_{K_1} of E, given by (2.1b), is an absorbing set of the Markov chain with one-step transition matrix P. Therefore the system can be viewed as a two node system with an *infinite* supply of customers in front of the first node server, whence the intermediate queue size and the phases of each server suffice to characterize the system behavior. The vector π_{K_1}, as defined in Section 2, is an invariant probability vector for the Markov chain corresponding to this new system with state space E_{K_1} and one-step transition matrix C_1.

The matrix C_1 has the same QBD structure as the matrix P. The similarity is emphasized by denoting its block entries by the same letters as the corresponding ones for the matrix P but in calligraphic style, so that

$$C_1 = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_0 & & & & & \\ \mathcal{B}_2 & \mathcal{A}_1 & \mathcal{A}_0 & & & & \\ & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_0 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{C}_0 \\ & & & & & \mathcal{C}_2 & \mathcal{C}_1 \end{pmatrix}. \tag{3.1.1}$$

The block entries $\mathcal{B}_0$, $\mathcal{B}_1$, and $\mathcal{B}_2$ now have dimensions $l \times lm$, $l \times l$ and $lm \times l$, respectively. The matrices $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ are all of dimensions $lm \times lm$, while the matrices $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ have dimensions $lm \times m$, $m \times m$ and $m \times lm$, respectively, and are given by

$$\mathcal{A}_0 = B \otimes a\alpha, \qquad \mathcal{A}_1 = B \otimes A + b\beta \otimes a\alpha, \qquad \mathcal{A}_2 = b\beta \otimes A, \tag{3.1.2a}$$

$$\mathcal{B}_0 = \beta \otimes a\alpha, \qquad \mathcal{B}_1 = A, \qquad \mathcal{B}_2 = b \otimes A, \tag{3.1.2b}$$

$$\mathcal{C}_0 = B \otimes a, \qquad \mathcal{C}_1 = B, \qquad \mathcal{C}_2 = b\beta \otimes \alpha. \tag{3.1.2c}$$

When $K_2 = 1$, i. e., there is no intermediate buffer, the matrix C_1 is always *irreducible* and the invariant probability vector π_{K_1} is given explicitly by

$$\pi_{K_1 0} = c\alpha (I_l - A)^{-1} \quad \text{and} \quad \pi_{K_1 1} = c\,\beta\,(I_m - B)^{-1} \,,$$

with $c = (ES_1 + ES_2)^{-1}$. Here ES_i denotes the expected service time of server i, $i = 1, 2$.

However, when $K_2 > 1$, the matrix C_1 can have several ergodic classes as pointed out in [6, Chap. 3] where necessary and sufficient conditions are given for the irreducibility of the matrix C_1. Necessary conditions for any invariant probability vector of C_1 are obtained through the solution technique of [9], provided the properties (P0)-(P2) below hold for the corresponding QBD process.

(P0): *The matrices $I_l - \mathcal{B}_1$ and $I_m - \mathcal{C}_1$ are nonsingular.*

(P1): *There exists $lm \times lm$ matrices X and Y such that the equalities*

$$\mathcal{A}_0 X = \mathcal{A}_2 Y = 0_{lm \times lm} \;, \quad \mathcal{B}_0 X = 0_{l \times lm} \quad \text{and} \quad \mathcal{C}_2 \mathrm{Y} = 0_{\mathrm{m} \times \mathrm{lm}}$$

$$(I_{lm} - \mathcal{A}_1)(I_{lm} - X)(I_{lm} - Y) = \mathcal{A}_0(I_{lm} - Y) + \mathcal{A}_2(I_{lm} - X)$$

$$\mathcal{B}_2\,(I_l - \mathcal{B}_1)^{-1}\,\mathcal{B}_0(I_{lm} - Y) = \mathcal{A}_2(I_{lm} - X)$$

and

$$XY = YX$$

hold, and either one of the $lm \times lm$ matrices M and N defined by

$$N := (I_{lm} - \mathcal{A}_1) X + \mathcal{A}_0 \ , \quad M := (I_{lm} - \mathcal{A}_1) Y + \mathcal{A}_2 \tag{3.1.3}$$

is invertible.

(P2) : *There exists an $l \times lm$ matrix V such that $\mathcal{B}_2 V = \mathcal{A}_2$.*

Property (P0) is trivially satisfied under the irreducibility assumptions made earlier on the PH-distributions since $\mathcal{B}_1 = A$ and $\mathcal{C}_1 = B$. Except for the invertibility of the matrices M and N, it is an easy exercise to check that the properties (P1) and (P2) are also satisfied by choosing

$$X = I_m \otimes (I_l - e_l \alpha), \qquad Y = (I_m - e_m \beta) \otimes I_l \quad \text{and} \quad V = \beta \otimes I_l. \tag{3.1.4}$$

The matrices M and N defined in (3.1.3) can be written in terms of the system parameters as

$$M = (I_m - e_m \beta) \otimes I_l + (e_m \beta - B) \otimes A \tag{3.1.5a}$$

$$N = I_m \otimes (I_l - e_l \alpha) + B \otimes (e_l \alpha - A) \ . \tag{3.1.5b}$$

The following necessary and sufficient conditions for the invertibility of these matrices follows from Lemma 2.3.17 and Corollary 2.3.18 of [6] combined to the irreducibility of the PH-representations (α, A) and (β, B). To that end, let $\mathcal{D}$ be the open disc centered at $(\frac{1}{2}, 0)$ in the complex plane with radius $\frac{1}{2}$ and let $Sp(Z)$ denote the spectrum of the matrix Z.

Lemma 3.1.1. *If $Sp(A) \subseteq \mathcal{D}$ (resp. $Sp(B) \subseteq \mathcal{D}$) then the matrix M (resp. N) defined by (3.1.5) is nonsingular. However, the matrix M (resp. N) is singular if the matrix A (resp. B) is singular.*

The following theorem, stated for the case when the matrix M is invertible and $K_2 > 1$, follows immediately from Theorem A.13 of the Appendix and from the *structural* results obtained in [9].

Theorem 3.1.2. *Let the $1 \times l$ row vector x be the unique solution to the equation*

$$x(A + a\alpha) = x, \qquad x e_l = 1.$$

If the matrix M is invertible, then any invariant probability vector π of P is of the form $\pi = (0_s, 0_r, \ldots, 0_r, \pi_{K_1})$. The $1 \times r$ probability vector $\pi_{K_1} = (\pi_{K_1 0}, \pi_{K_1 1}, \ldots, \pi_{K_1 K_2})$ has the matrix-geometric property

$$\pi_{K_1 k_2} = \begin{cases} \pi_{K_1 0}\, S\, R^{k_2 - 1} \ , & 1 \leq k_2 < K_2 \ , \\ \pi_{K_1 0}\, S\, R^{K-2} (B \otimes a)(I_m - B)^{-1} \ , & k_2 = K_2 \ , \end{cases} \tag{3.1.6}$$

where the $lm \times lm$ matrix R and the $l \times lm$ matrix S are defined by

$$R := N M^{-1} \qquad \text{and} \qquad S := (\beta \otimes (I_l - A))\, M^{-1} \ ,$$

and the $1 \times l$ *vector* $\pi_{K_1 0}$ *satisfies the linear equation*

$$x = \pi_{K_1 0} \left[I_l + S \sum_{k=1}^{K-1} R^{k-1}(I_l \otimes e_m) + S R^{K-2} C_0 (I_m - C_1)^{-1} e_m x \right] . \tag{3.1.7}$$

When $\lambda = 1$, it is shown in [6] that in statistical equilibrium the probability of finding k_2 jobs in the intermediate buffer coincides with probability of finding $K_2 - k_2$ jobs in the case when the order of the servers is *reversed.* Melamed [13] provides a probabilistic interpretation for this kind of *reversibility* [13] by viewing the vacant buffer locations (holes) of the original system as "occupied" by fictitious *dual jobs.* As *regular* jobs march through the buffer in one direction, the holes march in the opposite direction and they both receive *identical* service times.

3.2. $0 < \lambda < 1$

A necessary and sufficient condition for the *irreducibility* of the Markov chain associated with the matrix P of one-step transition probabilities, given by (2.3), is first obtained. Here, unlike for the case $\lambda = 1$, the *uniqueness* of the corresponding invariant probability vector is seen to hold. Since $0 < \lambda < 1$, the following three observations are easily verified:

(i) The directed graphs of the matrices A_1 and C_1 have the same topological structure by virtue of Lemma A.3.

(ii) The set $E_{k_1 - 1}$ is reachable from *every* state in the set E_{k_1} and vice versa, for $1 \leq k_1 \leq K_1$.

(iii) The state $(0,0)$ is reachable from every state in the set E_0.

The next theorem readily follows from observations (i)-(iii) and the irreducibility of the PH-representation (β, B).

Theorem 3.2.1. *If the matrix* C_1 *given by (3.1.1) is irreducible, so is the matrix* P.

Next, even when the matrix P is not irreducible, it is now shown to have a *single* ergodic class, whence the invariant probability distribution vector π is always *unique.* Let the $K_1 r \times K_1 r$ matrix $\widetilde{P}$ be obtained by deleting the first s rows and columns of P, i. e., $\widetilde{P}$ is the submatrix of P that governs the transition mechanism within the states in the set $E \setminus E_0$. Assume the directed graph $G(\widetilde{P})$ to induce several ergodic classes and a set of transient states, as would be the case if C_1 had several ergodic classes. In view of (ii), the set E_0 is reachable from all of the ergodic classes of $E \setminus E_0$, whence by (iii), the state $(0,0)$ is reachable from every ergodic class of the set $E \setminus E_0$. On the other hand, if $(0,0) \to e$ for a state e in E, then e will be in the communication *path* between the state $(0,0)$ and the other states of the chain to which $(0,0)$ has access, while if $(0,0) \not\to e$, then e will be a transient state of the chain since $e \to (0,0)$ from the argument given above. Therefore, in the directed graph $G(P)$, the set $E_{(0,0)} := \{e \in E : (0,0) \to e\} \cup \{(0,0)\}$ forms an irreducible class while the set $E \setminus E_{(0,0)}$ forms a transient class.

The next theorem follows from Theorem A.13 and these remarks.

Theorem 3.2.2. *If* $\lambda < 1$, *the Markov chain with one-step transition matrix* P *always has a single ergodic class, and the invariant probability vector* π *as defined above is thus unique.*

In order to obtain a closed-form expression for this unique vector π, nonsingularity of the matrices $(I_s - B_1)$ and A_0 is needed. Since

$$I_s - B_1 = \begin{pmatrix} \lambda & & & & & \\ -\overline{\lambda}\, b & I_m - \overline{\lambda} B & & & & \\ & -\overline{\lambda}\, b\beta & I_m - \overline{\lambda} B & & & \\ & & \ddots & \ddots & & \\ & & & & -\overline{\lambda}\, b\beta & I_m - \overline{\lambda} B \end{pmatrix}$$

and $\lambda > 0$, $I_s - B_1$ is invertible if and only if the matrix $I_m - \overline{\lambda} B$ is invertible, a fact easily established by a direct application of the Gershgorin Circle Theorem [2, p. 500]. This can also be seen by noting that $\rho(B) < 1$ and therefore 0 is not an eigenvalue of the matrix $I_m - \overline{\lambda} B$. On the other hand,

$$\begin{aligned} det(A_0) &= \lambda^r \; det(A)\, det(B)\, [det(B \otimes A)]^{K_2 - 1} \\ &= \lambda^r \; det(A)^{(K_2-1)m+1} \; det(B)^{(K_2-1)l+1} \; , \end{aligned}$$

where the second equality follows from properties of the Kronecker product [5], and the matrix A_0 is invertible if and only if *both* A and B are invertible.

As a result of this discussion, nonsingularity of both A and B will be assumed for the discrete-time model. Although this may seem rather restrictive, many well-known discrete PH-type distributions enjoy this property, including the hypergeometric and negative binomial distributions, to name a few.

Under the invertibility assumption of the matrices A and B, the matrix P is the probability transition matrix of a QBD process of the type discussed in [7] and when $K_1 > 1$, the following matrix-geometric form of the solution is an immediate consequence of the *grouping* technique discussed in this reference.

Theorem 3.2.3. *If the matrices A and B are invertible, then the unique invariant probability vector $\pi = (\pi_0, \pi_1, \ldots, \pi_{K_1})$ has the following structural form*

$$\begin{aligned} \pi_0 &= \pi_1\, B_2\, (I_s - B_1)^{-1} \; , \\ \pi_{k_1} &= \pi_{K_1} \;\; ((I_r - C_1) A_0^{-1}, I_r) \;\; R^{K_1 - k_1 - 1} \;\; \begin{pmatrix} I_r \\ 0_{r \times r} \end{pmatrix} \; , \quad 1 \le k_1 < K_1 \\ \pi_{K_1} &= (xw)^{-1} \; x \; , \end{aligned}$$

where the $1 \times r$ vector x satisfies

$$x\, Z = 0_r \;\; , \quad x > 0_r^T \;\; (or \; x < 0_r^T)$$

with the $2r \times 2r$ and $r \times r$ matrices R and Z and the $r \times 1$ vector w being given by

$$R = \begin{pmatrix} (I_r - A_1)A_0^{-1} & I_r \\ -A_2 A_0^{-1} & 0_{r\times r} \end{pmatrix} ,$$

$$Z = \left((I_r - C_1)\, A_0^{-1}, I_r\right)\, R^{K_1-2} \begin{pmatrix} B_2(I_s - B_1)^{-1}B_0 + A_1 - I_r \\ A_2 \end{pmatrix} ,$$

$$w = \left((I_r - C_1)A_0^{-1}, I_r\right) \left[R^{K_1-2} \begin{pmatrix} I_r \\ 0_{r\times r} \end{pmatrix} B_2(I_s - B_1)^{-1} e_s + \sum_{k_1=1}^{K_1-1} R^{K_1-k_1-1} \begin{pmatrix} e_r \\ 0_r^T \end{pmatrix} \right] + e_r .$$

Note that the scalar (xw) is a normalization constant and the vector w need not be computed as the normalization can be performed after computing all the vectors π_{k_1}, $0 \le k_1 \le K_1$. Much simpler expressions can easily be obtained for the case $K_1 = 1$.

4. EXTENSIONS TO SERVERS SUBJECT TO FAILURES

In this section, a class of unreliable servers with *PH-type* service and repair time distributions is introduced. The *effective* service time distribution of such servers is shown to admit a PH-type represention of higher order so that the methods of the previous section apply. Under the irreducibility assumption on the service and repair PH-distributions, necessary and sufficient conditions for the irreducibility of the effective service representation are established. The situation where even idling servers may be subject to failure is considered. The case when only operational servers can fail is obtained as a special case of this discussion.

4.1. Representation of The Effective Service Time Distribution

Consider the following model for a PH-type server subject to occasional failures: The service and repair distributions have irreducible PH-representations (α^u, A^u) and (α^d, A^d) of order m and n, respectively, with corresponding $m \times 1$ and $n \times 1$ column vectors of absorption probabilities a^u and a^d, respectively. Again, in order to avoid situations of limited interest, take $\alpha^u_{m+1} = \alpha^d_{n+1} = 0$. Let the sets S, R and T be defined by

$$S := \{s_i,\ 1 \le i \le m\}, \quad R := \{r_j,\ 1 \le j \le n\}, \quad T := S \cup R ,$$

where s_i and r_j are the i^{th} service phase and the j^{th} repair phase, respectively. Let C and D be $m \times n$ and $n \times m$ non-negative matrices with entries C_{ij} and D_{ij}, $1 \le i \le m$, $1 \le j \le n$, respectively, with the property that $Ce_n = e_m$ and $De_m = e_n$. Similarly, let the $1 \times m$ column vector f have non-negative entries f_i, $1 \le i \le m$.

It is assumed that when the server is up and in phase of *service* i, it can fail with probability f_i, $1 \le i \le m$, and the *repair* process is initialized at phase j with probability C_{ij}, $1 \le i \le m$, $1 \le j \le n$. Similarly, when the repair is completed, i.e., whenever there is a transition from a transient phase j to the absorbing phase $n+1$ of the repair distribution, the service restarts at a phase i with probability D_{ji}, $1 \le j \le n$, $1 \le i \le m$. Moreover, a failure at a repair completion epoch is not allowed. In addition to these fairly general transition mechanisms, the server is allowed to fail at a service completion epoch with probability ϕ and reinitialization of the repair phase is then done according to the probability distribution vector α^d. The natural conditions $\phi < 1$ and $f_i < 1,\ 1 \leq i \leq m$, are imposed throughout.

Under these assumptions, it is easy to see that the *effective* service time distribution of such a server admits a PH-type distribution. To that end, consider a discrete-time Markov chain on the state space $T \cup \{s_{m+1}\}$, where s_{m+1} is interpreted as the instantenous state for the PH-representation of the (effective) service. The corresponding one-step transition matrix Q and the initialization probability vector are of the form

$$Q = \begin{pmatrix} A & a \\ 0_{m+n} & 1 \end{pmatrix} \quad \text{and } (\alpha, \alpha_{m+n+1}) \ ,$$

respectively, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m+n})$, where A, a and α are of dimension $(m+n) \times (m+n), (m+n) \times 1$ and $1 \times (m+n)$, respectively. The equalities

$$A = \begin{pmatrix} \Lambda_{\overline{f}}\, A^u & \Lambda_f\, C \\ \Lambda_{a^d}\, D & A^d \end{pmatrix} \ , \tag{4.1.1a}$$

$$a = \begin{pmatrix} \Lambda_{\overline{f}}\, a^u \\ 0_n^T \end{pmatrix} \ , \tag{4.1.1b}$$

$$\alpha = (\overline{\phi}\, \alpha^u \ , \phi\, \alpha^d) \ , \tag{4.1.1c}$$

readily follows, where the notation $\Lambda_x := diag(x_1, \ldots, x_l)$ is used for all x in $I\!R^l$. The relations $\Lambda_f\, e_m = f$ and $\Lambda_{a^d}\, e_n = a^d$ easily follows.

The effective service distribution of such a server thus has a PH-representation (α, A) of order $n+m$. The assumption $\alpha^u_{m+1} = \alpha^d_{n+1} = 0$ implies $\alpha_{m+n+1} = 0$. The irreducibility of the service and the repair distributions does not guarantee the irreducibility of the effective representation (α, A). However, the following neccessary and sufficient conditions are given.

Lemma 4.1.1. *The PH-representation (α, A) is irreducible if and only if every state r_j in the set $R \setminus R^*$ is reachable from the set R^* under the transition mechanism induced by $G(A^d)$, where*

$$R^* := \{ r_j \in R \, : r_j \textit{ is reachable from the set } S \textit{ under } G(A + a\alpha) \ \} \ .$$

Moreover, if a state r_j in $R \setminus R^*$ is **not** *reachable from R^*, then r_j is a* **transient phase** *for the PH-representation (α, A).*

Proof. By definition, the irreducibility of the PH-representation (α, A) is equivalent to the irreducibilty of the $(m+n) \times (m+n)$ matrix $A + a\alpha$, which is readily given by

$$A + a\alpha = \begin{pmatrix} \Lambda_{\overline{f}}\, A^u + \overline{\phi}\, \Lambda_{\overline{f}}\, a^u\, \alpha^u & \Lambda_f\, C + \phi\, \Lambda_{\overline{f}}\, a^u\, \alpha^d \\ \Lambda_{a^d}\, D & A^d \end{pmatrix} .$$

Now, the property $s_i \to s_{i'}$, $1 \le i, i' \le m$, is a direct consequence of the irreduciblity of the matrix $\Lambda_{\overline{f}} A^u + \overline{\phi}\Lambda_{\overline{f}} a^u \alpha^u$, a fact which follows from Lemma A.3, since by assumption $\phi < 1$, $f_i < 1$, $1 \le i \le m$, and the representation (α^u, A^u) is irreducible. Therefore, every state s in S communicates with any other state in S without leaving S. On the other hand, since $(I_n - A^d)$ is assumed invertible, A^d is substochastic and $a^d \ne 0_n^T$. This fact, together with the fact that the states $\{r_1, \ldots, r_n\}$ are all transient for the representation (α^d, A^d) yields the access relation $r_j \to s_i$ for *every* r_j in R and s_i in S.

(Sufficiency) The sufficiency part of the first assertion of the Lemma now follows from the hypothesis, since every r_j in $R \setminus R^*$ is reachable from R^*, thus communicating with the set $S \cup R^*$.

(Necessity) Follows trivally by the definition of irreducibility and the form of the matrix $A + a\alpha$ in that the transitions within the set R are governed by the matrix A^d *only*, due to the assumption that failures are not allowed at repair completions.

The second part of Lemma 4.1.1 also follows easily from the above discussion. Since $a^d \ne 0_n^T$, the underlying Markov chain will eventually leave the set R and enter the set S. If r_j in $R \setminus R^*$ is not reachable from the set R^*, then r_j will never be revisited and will be a transient state for the Markov chain. $\square$

In some applications the server may fail with a positive probability at a service completion epoch, i. e., $\phi > 0$. In that event, since $a^u \ne 0_m^T$, there exists some i, $1 \le i \le m$, such that $a_i^u > 0$, whence $s_i \to r_j$ for every r_j in the set $R_* := \{1 \le j \le n : \alpha_j^d > 0\}$ as the repair phase is initialized according to α^d. Since the representation (α^d, A^d) is irreducible, every state r in $R \setminus R_*$ must be reachable from the set R_* since otherwise r will be a transient state for the matrix $A^d + a^d\alpha^d$ thus contradicting the irreducibility of (α^d, A^d). Therefore, arguments similar to the ones that lead to the sufficiency part of Lemma 4.1.1 show that the representation (α, A) is irreducible.

Since (α, A) is constructed from basic building blocks and not given from the onset, invertibility of the matrix $I_{m+n} - A$ is not automatically guaranteed. A sufficient condition is obtained in the following Lemma.

Lemma 4.1.2. *The matrix $I_{m+n} - A$ is invertible whenever the PH-representation (α, A) is irreducible.*

Proof. The directed graph $G(A)$ may in general induce several ergodic classes. However, the irreducibility of the representation (α, A) implies that all of these classes communicate with each other through s_{m+1} in the access relation induced by $G(A + a\alpha)$, and therefore, the states in T are all transient for the Markov chain with one-step probability transition matrix Q. On the other hand, if $G(A)$ induces a single ergodic class, the states in T are again all transient since s_{m+1} is indeed an absorbing state for Q, i. e., $a \ne 0_{m+n}^T$ from (4.1.1b), owing to the assumption that $a^u \ne 0_m^T$ and $f_i < 1$, $1 \le i \le m$. The invertibility of $(I_{m+n} - A)$ is now immediate since it is equivalent to the statement that the states in T are transient for the Markov chain with one-step probability transition matrix Q [14, Lemma 2.1.1. p. 45]. $\square$

The representation of the effective service time given above for such a faliure type server subsumes the case when the server is *reliable*, i. e., $\phi = 0$ and $f = 0_m^T$, in which case, the matrix $A + a\alpha$ takes the form $\begin{pmatrix} A^u + a^u\alpha^u & 0_{m\times n} \\ \Lambda_{a^d} D & A^d \end{pmatrix}$. This special case also provides a trivial example for the representation (α, A) *not* to be irreducible, although the representation (α^u, A^u) for the service duration is. If the representation (α, A) is not irreducible, Lemma 4.1.1 provides general guidelines for identifying the corresponding irreducible representation. In the following section, the representation (α, A) is assumed *irreducible* so that the matrix $(I_{m+n} - A)$ is invertible.

To conclude, it should be kept in mind that the matrices C and D, corresponding to transition probabilities between the service and repair phases, take special forms depending on the assumptions made. For instance, if upon completion of a repair, the phase of service is reinitialized according to α^u, and similarly if upon a failure the phase of repair is initialized according to α^d, then C and D take the special forms $C = e_m\,\alpha^d$ and $D = e_n\,\alpha^u$. When $C = e_m\,\alpha^d$, it is an easy exercise to see that the representation (α, A) given by (4.1.1) is irreducible regardless of the form of the matrix D.

4.2. Two Node System With PH-Type Failure Servers

The model of Section 2 is considered for the case when $\lambda = 1$. The case $0 < \lambda < 1$ is straightforward and omitted to avoid an already lengthy manuscript. Feedback is again ignored in order not to further complicate the notation as it can easily be incorporated as mentioned in Section 2. The servers are assumed subject to failures even when they are idling. The first and second node servers have irreducible PH-representations of the form (4.1.1), denoted by (α, A) and (β, B), respectively, by using the notation of Section 2. The service and repair PH-representations of the first and the second node server are of order m_i and n_i, $i = 1, 2$, respectively. They are again denoted by the same letters as the effective representations but with superscripts u and d. It is assumed that an idling server fails with probability g_i, $i = 1, 2$, and upon failure the phase of repair is initialized according to the initialization vector α^d and β^d for the first and the second node server, respectively. For sake of compactness, set

$$\begin{aligned} r_i &= m_i + n_i,\ i = 1, 2, & h &= r_1\, r_2, \\ h_1 &= r_1\,(1 + n_2)\,, & h_2 &= r_2\,(1 + n_1)\,. \end{aligned}$$

Dropping the index for the first queue, the state set E of the system contains $|E| = (K_2 - 1)h + h_1 + h_2$ states with

$$E = \begin{cases} (0, i)\,, & k_2 = 0,\ 1 \le i \le r_1, \\ (0, i, j)\,, & k_2 = 0,\ 1 \le i \le r_1 \text{ and } m_2 + 1 \le j \le r_2, \\ (k_2, i, j)\,, & 0 < k_2 < K_2,\ 1 \le i \le r_1 \text{ and } 1 \le j \le r_2, \\ (K_2, i, j)\,, & k_2 = K_2,\ m_1 + 1 \le i \le r_1 \text{ and } 1 \le j \le r_2, \\ (K_2, j)\,, & k_2 = K_2,\ 1 \le j \le r_2. \end{cases}$$

Here, k_2 again indicates the intermediate buffer size, while i and j represent the service or repair phase in the first and the second node server, respectively. The phase of service of the second node server is not defined when it has no jobs to process and the phase of service of the first node server is not defined when the intermediate buffer is full as it is blocked. The pairs $(0,i)$, $1 \le i \le r_1$, and (K_2,j), $1 \le j \le r_2$, correspond to the states where the second and the first node server, respectively, are functional but idle.

By ordering the states as in Section 2, the one-step state transition matrix C_1 of the underlying Markov chain can be obtained in the same block tridiagonal form as in equation (3.1.1). Since the effective service time distribution is still PH-type, the *intermediate* block entries of the matrix C_1 are still given by

$$\mathcal{A}_0 = B \otimes a\alpha, \qquad \mathcal{A}_1 = B \otimes A + b\beta \otimes a\alpha \quad \text{and} \quad \mathcal{A}_2 = b\beta \otimes A,$$

where now each block is an $h \times h$ matrix. For the *boundary* states the entries are given by

$$\mathcal{B}_0 = \begin{pmatrix} \overline{g}_2\,\beta^u & g_2\,\beta^d \\ \Lambda_{b^d}\,D_2 & B^d \end{pmatrix} \otimes a\alpha \qquad h_1 \times h \text{ matrix},$$

$$\mathcal{B}_1 = \begin{pmatrix} \overline{g}_2 & g_2\,\beta^d \\ b^d & B^d \end{pmatrix} \otimes A \qquad h_1 \times h_1 \text{ matrix},$$

$$\mathcal{B}_2 = b\,(\overline{\phi}_2\,, \phi_2\,\beta^d) \otimes A \qquad h \times h_1 \text{ matrix},$$

$$\mathcal{C}_0 = (\overline{\phi}_1\; B \otimes a\;,\; \phi_1\; B \otimes a\,\alpha^d) \qquad h \times h_2 \text{ matrix},$$

$$\mathcal{C}_1 = \begin{pmatrix} \overline{g}_1\,B & g_1\; B \otimes \alpha^d \\ B \otimes a^d & B \otimes A^d \end{pmatrix} \qquad h_2 \times h_2 \text{ matrix}.$$

$$\mathcal{C}_2 = \begin{pmatrix} b\,\beta \otimes (\overline{g}_1 \alpha^u, g_1 \alpha^d) \\ b\,\beta \otimes (\Lambda_{a^d}\,D_1, A^d) \end{pmatrix} \qquad h_2 \times h \text{ matrix},$$

Since the effective service representations (α, A) and (β, B) are both assumed irreducible, the results of [6] can be used to characterize the irreducibility of the Markov chain studied here.

Except for the invertibility of the matrices M and N, it is a straightforward exercise to show that the properties (P1) and (P2) are again satisfied, by choosing

$$X = I_{r_2} \otimes (I_{r_1} - e_{r_1}\alpha), \qquad Y = (I_{r_2} - e_{r_2}\beta) \otimes I_{r_1} \quad \text{and} \quad V = \beta \otimes I_{r_1}.$$

The matrices M and N are again given by (3.1.5), with the subscripts l and m of I and e now replaced with r_1 and r_2, respectively.

The invertibility of the matrices $I_{h_1} - \mathcal{B}_1$ and $I_{h_2} - \mathcal{C}_1$ is needed in order for the property (P0) to hold. To see this for $I_{h_2} - \mathcal{C}_1$, note that the stochastic matrix $\begin{pmatrix} \overline{g}_1 & g_1\alpha^d \\ a^d & A^d \end{pmatrix}$ has eigenvalues in the closed unit disc, while the matrix B has eigenvalues in the open unit disc of the complex plane. It is an easy exercise to show that if (λ, u) and (μ, v) are right eigenpairs for the matrices B and $\begin{pmatrix} \overline{g}_1 & g_1\alpha^d \\ a^d & A^d \end{pmatrix}$, respectively, then $(\lambda\,\mu,\; u \otimes v)$ is a right eigenpair for the matrix $\mathcal{C}_1$. The eigenvalues of $\mathcal{C}_1$ are therefore all in the open unit disc, or equivalently the eigenvalues of $I_{h_2} - \mathcal{C}_1$ have strictly positive real parts, and the matrix $I_{h_2} - \mathcal{C}_1$ is invertible. The nonsingularity of the matrix $I_{h_1} - \mathcal{B}_1$ can be shown in a similar way.

Therefore, if one of the matrices M or N is invertible, the properties (P0)-(P2) are again satisfied and the same *structural* result of Theorem 3.1.2 also holds true for the case when servers are subject to breakdowns. However, this time the $1 \times h_1$ vector $\pi_{K_1 0}$ satisfies the equation

$$\pi_{K_1 0}\, Z = 0_h \qquad \text{and} \qquad \pi_{K_1 0}\, w = 1, \tag{4.2.1}$$

where the $h_1 \times h$ matrix Z and the $h_1 \times 1$ vector w are given by

$$Z = S \sum_{k_2=1}^{K_2-1} R^{k_2-1}(I_h - (B + b\beta) \otimes (A + a\alpha)) - \mathcal{A}_0 + S\mathcal{A}_2 + SR^{K_2-2}(\mathcal{A}_0 - \mathcal{C}_0(I_{h_2} - \mathcal{C}_1)^{-1}\mathcal{C}_2)$$

$$w = e_{h_1} + S \sum_{k_2=1}^{K_2-1} R^{k_2-1} e_h + SR^{K_2-2}\mathcal{C}_0(I_{h_2} - \mathcal{C}_1)^{-1} e_{h_2} \, .$$

Equation (4.2.1) is obtained by summing the balance equations corresponding to the block entries $\pi_{K_1 k_2}$, $1 \leq k_2 < K_2$, and using the normalization condition $\pi_{K_1} e_{|E|} = 1$.

Special Case – When only operational failures are allowed

The solution for the case when idling servers do not fail can be obtained as a special case of the discussion given above. In this case, even if the servers do not fail when idling, the same state description has to be used as a server can still fail at the time epoch of a service completion. Therefore, the results of this case can be obtained by setting $g_1 = 0$ in $\mathcal{C}_1$ and $\mathcal{C}_2$ and by setting $g_2 = 0$ in $\mathcal{B}_0$ and $\mathcal{B}_1$.

5. NUMERICAL EXAMPLES

In this section, the effects of failures on the invariant probabilities of the model of Section 2 are illustrated through several numerical examples. For a buffer size of $K = 5$, the following three situations are considered when $\lambda = 1$:

(i) The case when both servers are *reliable* and have *Negative Binomial* service time distributions with PH-representations (α^u, A^u) and (β^u, B^u).

(ii) The case when both servers are subject to failures and have service distributions as in (i), and the first server has a *geometric* down time distribution with parameter 0.8, while the second server has *hypergeometric* down time distribution with PH-representation (β^d, B^d). In this case the idling servers are allowed to fail with probabilities $g_1 = 0.2$ and $g_2 = 0.3$.

(iii) In this case both servers are subject to failures and have service and repair distributions as in (ii), but idling servers are *not* allowed to fail.

The following numerical values are considered.

$$A^u = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.3 \end{pmatrix} , \qquad \alpha^u = (1, 0) \, ,$$

$$B^u = \begin{pmatrix} 0.6 & 0.4 \\ 0 & 0.5 \end{pmatrix} , \qquad \beta^u = (1,0) ,$$

$$B^d = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.6 \end{pmatrix} , \qquad \beta^d = (0.4, 0.6) ,$$

The matrices Λ_{f_i}, C_i and D_i, $i =, 1, 2$, are given by

$$\Lambda_{f_1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad \Lambda_{f_2} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} ,$$

and

$$C_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C_2 = e_2\,\beta^d , \qquad D_1 = \alpha^u , \quad D_2 = e_2\,\beta^u ,$$

while the probabilities of a failure at a service completion are $\phi_1 = 0.2$ and $\phi_2 = 0.3$. Therefore the effective service time representations (α, A) and (β, B) are given by

$$A = \begin{pmatrix} 0.72 & 0.18 & 0.1 \\ 0 & 0.27 & 0.1 \\ 0.8 & 0 & 0.2 \end{pmatrix} , \qquad B = \begin{pmatrix} 0.48 & 0.32 & 0.08 & 0.12 \\ 0 & 0.4 & 0.08 & 0.12 \\ 0.5 & 0 & 0.5 & 0 \\ 0.4 & 0 & 0 & 0.6 \end{pmatrix} ,$$

and

$$\alpha = (0.8, 0, 0.2) , \qquad \beta = (0.7, 0, 0.12, 0.18) .$$

If the random variables S_i, R_i and S_i^{eff}, $i = 1, 2$, denote the service time, repair time and the effective service time of the i^{th} server for $i = 1, 2$, respectively, then the above numerical values lead to the following expected values: $E[S_1] = 7$, $E[S_2] = 4$, $E[R_1] = 1.25$, $E[R_2] = 2.3$, $E[S_1^{eff}] = 9.28$, and $E[S_2^{eff}] = 11.18$. Note that although $E[S_1 + R_1] > E[S_2 + R_2]$, the average effective service time of the second node server is greater then the average effective service time of the first node server since the second node server is more likely to breakdown. The effect of breakdowns can easily be seen from the queue size probabilities in Table 5.1. As expected, in the first case, there are fewer than three jobs in the buffer for most of the time whereas in cases II and III there are more than three jobs in the buffer for most of the time. On the other hand, although the second server has a higher probability of failure when it is idle, the probability of having zero or one job in the buffer is sligthly less in case II then in case III. The reason for this is that the idling probability of the first node server is high enough to offset this difference, so that, relative to the second node server, the first node server is slightly slower in case II.

Steady state queue size probabilities

Queue size	Case 1	Case 2	Case 3
0	0.3042	0.0619	0.0672
1	0.3950	0.1131	0.1143
2	0.1904	0.1495	0.1483
3	0.0758	0.1952	0.1936
4	0.0286	0.2513	0.2492
5	0.0059	0.2290	0.2275

Table 5.1.

6. THE CONTINUOUS-TIME FORMULATION

In this section, the solution to the continuous-time formulation of the model of Section 2 is briefly discussed. Although the same solution techniques can be used, the continuous-time formulation has two advantages: (i) The matrix C_1 is irreducible and so is, by Theorem 3.2.1, the matrix P given by equations (3.1.1) and (2.2), respectively, are both *irreducible.* The reader is referred to [6, p. 51] for a proof of the irreducibility of the matrix C_1 in this case. (ii) No invertibility assumptions are needed for all values of λ. For the case $\lambda = \infty$, both matrices M and N are invertible [9]. For the case $\lambda < \infty$, $A_0 = \lambda I_r$, and no matrix inversion is required to obtain the R matrix. Similar extensions to systems with failure servers are available [6]. Feedback from the second node server to the first buffer can also be incorporated into the solution since the underlying QBD process will have a similar structure, while for the discrete-time formulation this type of feedback does not yield a QBD process.

APPENDIX

In this Appendix, several definitions and results from the theory of nonnegative matrices are presented for sake of easy reference. The reader is referred to Berman and Plemmons [3] for a general reference on the theory of nonnegative matrices. Throughout the discussion, all the matrices have real entries unless otherwise mentioned.

Definition A.1. *An $n \times n$ nonnegative matrix A is* **cogredient** *to an $n \times n$ matrix E if for some permutation matrix P, $PAP^T = E$. The matrix A is said to be* **reducible** *if it is cogredient to*

$$E = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices, otherwise, A is said to be **irreducible.**

Definition A.2. *The* **directed graph** *$G(A)$ associated with an $n \times n$ nonnegative matrix A is the graph made up of n vertices, say $P_1, P_2, \ldots P_n$, with an edge leading from P_i to P_j if and only if $A_{ij} > 0$, $1 \le i, j \le n$.*

It is well known [3, p. 30] that a matrix A is *irreducible* if and only if $G(A)$ is *strongly connected*, that is, for every ordered pair (P_i, P_j) of vertices of $G(A)$, there exists a path (i. e., a sequence of edges) which leads from P_i to P_j. The following Lemma is a simple application of the Definition A.2 and will be useful in the proofs of Theorems 3.2.1 and Lemma 4.1.1.

Lemma A.3. *For any two nonnegative square matrices A and B, the matrix $(A+B)$ is irreducible if and only if the matrix $(c_1A + c_2B)$ is irreducible for all scalars $c_1, c_2 > 0$.*

Proof. Owing to the nonnegativity of the matrices A and B, $(A + B)_{ij} > 0$ if and only if $(c_1 A + c_2 B)_{ij} > 0$, whenever $c_1, c_2 > 0$, while $(A + B)_{ij} = 0$ if and only if $A_{ij} = B_{ij} = c_1 A_{ij} + c_2 B_{ij} = (c_1 A + c_2 B)_{ij} = 0$. The directed graphs $G(A + B)$ and $G(c_1 A + c_2 B)$, c_1, $c_2 > 0$, thus have exactly the same topology, and the result follows by Definition A.1. □

When a nonnegative matrix is *stochastic*, the notion of irreducibility in the Definition A.1 is related to the probabilistic one in that every stochastic matrix can be viewed as the one-step transition matrix of a Markov chain. In order to clarify the terminology, it is necessary to classify the states of a Markov chain. Thus consider a finite state Markov chain on the state space $S = \{s_1, \ldots, s_n\}$.

Definition A.4. *If in the state transition diagram of the Markov chain S there exists a path from state s_i in S to state s_j in S, then the state s_i is said to have* **access** *to state s_j, written $s_i \to s_j$. If s_i has access to s_j and s_j has access to s_i, then s_i and s_j are said to* **communicate,** *written as $s_i \leftrightarrow s_j$.*

Implicit in Definition A.4 is that every state in S communicates with itself, and with this convention, the communication relation is an *equivalence* relation on the set of states and thus partitions S into equivalence classes. With this in mind, the following definition is given.

Definition A.5. *A state s_i in S is called* **transient** *if there exists some $s_j \neq s_i$ in S with the property that $s_i \to s_j$ but $s_j \not\to s_i$, i. e., s_i has access to some other state which does not have access to s_i. Otherwise, the state s_i is called* **ergodic.**

Thus, s_i is ergodic if and only if $s_i \rightarrow s_j$ implies $s_j \rightarrow s_i$ for some $s_j \neq s_i$ in S. It follows that if one state in an equivalence class of states associated with a Markov chain is transient (resp. ergodic), then each state in that class is transient (resp. ergodic). This leads to the next definition.

Definition A.6. *A class induced by the communication relation on the set S is called* **transient** *if it contains a transient state and* **ergodic** *otherwise.*

Definition A.7. *A Markov chain is called* **irreducible** *if it consists of a single ergodic class.*

In view of the above definitions, the following Lemma can easily be proved.

Lemma A.8. *A finite state Markov chain is irreducible if and only if the corresponding one-step transition matrix is irreducible.*

Therefore, these two concepts of irreducibility are used interchangeably in this paper. The following notion of *reachability* is used repeateadly.

Definition A.9 *Let U and V be subsets of the state space of a Markov chain with corresponding one-step probability transition matrix T. The set U is* **reachable** *from the set V if there is a path from some state in V to a state in U in the directed graph $G(T)$ of the matrix T.*

The following theorem is a basic result in the Perron-Frobenius theory of nonnegative matrices.

Theorem A.10. *If A is an $n \times n$ nonnegative matrix, then*

(i) *The spectral radius $\rho(A)$ of A is an eigenvalue of A, and*

(ii) *There always exists left and right eigenvectors with nonnegative components which corresponds to $\rho(A)$.*

The *invariant* probability vector π of an n-state Markov chain with one-step probability transition matrix A is defined as the $1 \times n$ vector π that satisfies

$$\pi A = \pi \quad , \quad \pi e_n = 1 \ .$$

The following results investigate the existence and uniqueness of the invariant probability vector of a finite state Markov chain.

Theorem A.11. *Every finite state Markov chain has an invariant probability vector.*

Proof. Let the $n \times n$ matrix A be the state transition matrix associated with the chain. It is well known [3] that for any $n \times n$ nonnegative matrix A, the following bounds

$$\min_i \{\sum_{j=1}^{n} A_{ij}\} \leq \rho(A) \leq \max_i \{\sum_{j=1}^{n} A_{ij}\}$$

hold true for its spectral radius. Since the matrix A is stochastic, each one of its rows sum to 1, whence $\rho(A) = 1$. Therefore by Theorem A.10 there exists a row vector $x > 0$ with $xA = x$. Normalizing x gives $\pi = (xe_n)^{-1} x$ with $\pi A = \pi$ and $\pi e_n = 1$, i.e., π is an invariant probability vector of the chain. $\square$

Although Theorem A.11 guarantees the existence of an invariant probability vector, it is *not* unique in general. The next result gives the general form of an invariant probability vector.

Theorem A.13. *Let S_i, $1 \leq i \leq r$, be the ergodic classes of a finite state Markov chain. For each S_i there is a unique invariant probability vector $\pi(i)$ with the property that the entries of $\pi(i)$ corresponding to the states of S_i are positive whereas all other entries are zero. Moreover, any invariant probability vector π of the chain can be expressed as a linear convex combination of the vectors $\pi(i)$, $1 \leq i \leq r$, i. e.,*

$$\pi = \sum_{i=1}^{r} \lambda_i \pi(i) , \quad \lambda_i \geq 0 , \quad \sum_{i=1}^{r} \lambda_i = 1 .$$

Proof. See Berman and Plemnons [3, pp. 224-225].

In view of Theorem A.13, even when the Markov chain is not irreducible, if it has a single ergodic class, then the invariant probability vector is unique with positive entries for positions corresponding to the ergodic class and zero entries for positions corresponding to the transient states. Note that the ergodic classes are defined as equivalence classes induced by the communication relation as defined above, with no assumptions made on the *periodicity* of the Markov chain. Therefore, in cases where there is a single ergodic class, the unique invariant probability vector will coincide with the *long-run* average probability vector of the Markov chain (defined in the *Cesaro* sense).

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