Weakly-Mixing Systems with Dense Prime Orbits

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The Main Result

We provide the first examples of smooth, weak mixing dynamical systems for which all points have dense orbits along primes.

The Main Ingredients

There will be a two main ingredients in the proof, namely:

Theorem (Siegel-Walfisz)

Uniformly in all primes $q < (\log x)^2$ *and* $0 < a < q$,

$$\pi(x; q, a) \sim \frac{Li(x)}{q-1}$$

Theorem

Let X *be a compact metric space.* $T : X \to X$ *is uniquely ergodic if and only if,* $\forall g \in C(X)$,

$$\frac{1}{n}\sum_{i=0}^{n-1} g(T^i x) \to \int_X g d\mu$$

uniformly for all $x \in X$.

Birkhoff Ergodic Theorem

Denote the ergodic sums by $S_n(f)(x) = \sum_{i=0}^{n-1} f(T^i(x))$

Theorem (Birkhoff Ergodic Theorem)

Let $(X, \mathcal{B}, \mu)$ be a standard probability Borel space. Let $T : X \to X$ be a measure-preserving transformation, and f be an L^1 function. Then for a.e. $x \in X$,

$$\frac{1}{n} S_n(f)(x) \to \int_X f d\mu$$

Bourgain's Addition

Theorem (Bourgain)

Let $(X, \mathcal{B}, \mu)$ *be a standard probability Borel space. Let* $T : X \to X$ *be a measure-preserving transformation, and* f *be an* L^1 *function. Then for a.e.* $x \in X$*,*

$$\frac{1}{\pi(n)} \sum_{p<n} f(T^i(x)) \to \int_X f d\mu$$

Results for All Points

Vinogradov

Showed every point's orbit along primes is equidistributed for an irrational rotation of the circle.

Green-Tao (2012)

Showed every point's orbit along primes is equidistributed for nilmanifolds.

Bourgain (2013)

Showed every point's orbit along primes is equidistributed for some 3 IETs.

The Flow we Consider

We will consider the linear flow on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:
Let $\alpha \in (0,1) \setminus \mathbb{Q}$, and $r : \mathbb{T}^2 \to \mathbb{R}^+$, and set

$$\frac{dx}{dt} = \alpha r(x,y), \text{ and } \frac{dy}{dt} = r(x,y)$$

essentially, given any point in $\mathbb{T}^2$, we flow at a slope $\frac{1}{\alpha}$ from the horizontal with speed scaled by the function r.

The Main Theorem

Theorem

For uncountably many α, there is an analytic $r : \mathbb{T}^2 \to \mathbb{R}^+$ such that the associted linear flow on the torus is weakly-mixing and every point has a dense orbit along primes.

Fayad's Result and The Set of Irrationals

Theorem (Fayad)

For any $\alpha \in (0,1) \setminus \mathbb{Q}$, there exists an anayltic $r : \mathbb{T}^2 \to \mathbb{R}^+$ such that the associated linear flow on the torus is weak-mixing.

We will consider α in the following set: Let c_o, δ be positive constants, and define

$$D = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \forall n \in \mathbb{N},\ q_n \text{ is prime, and } q_{n+1} \geq c_o e^{\delta q_n}\}$$

where (q_n) is the sequence of denominators given by α's continued fraction expansion.

Representing the Flow Differently

The linear flows defined above can be represented as suspension flows (suspending an irrational rotation of the circle).
Let C_x be the path given by flowing from the point $(x, 0) \in \mathbb{T}^2$ to the point $(x + \alpha, 1) \in \mathbb{T}^2$ in the linear flow given by α, r.
Define $f : \mathbb{T} \to \mathbb{R}^+$ by

$$f(x) = \int_{C_x} r(x(t), y(t))dt$$

and then we have a conjugacy between the suspension flow on the space $\mathbb{T}_f$ (over an underlying rotation by α) and the linear flow on $\mathbb{T}^2$. Note that f is analytic since r is.
We will denote the flow on $\mathbb{T}_f$ by (T_t)

Equidistribution of Orbits Along $\mathbb{N}$

The linear flow on $\mathbb{T}^2$ is uniquely ergodic (hence so is the time-1 map) and so the same is true for (T_t) and T_1.
This, crucially, implies that for any $g \in C(\mathbb{T}_f)$,

$$\frac{1}{n}\sum_{i=0}^{n-1} g(T^i(x,s)) \to \int_{\mathbb{T}_f} g d\nu$$

uniformly in $(x,s) \in \mathbb{T}_f$.

Technical Estimates

Using exponential decay of Fourier coefficients and $q_{n+1} \geq c_o e^{\delta q_n}$,

Lemma

For sufficiently large n and some positive constants C, c,

$$\left| S_{q_n}(f)(x) - q_n \int_{\mathbb{T}} f \, d\mu \right| \leq C e^{-c q_n}$$

which immediately implies

Lemma

$\exists d > 0$ *such that* $\forall K \in \mathbb{N}$ *with* $K < e^{d q_n}$, *there are positive constants* C', c' *such that for sufficiently large* n,

$$\left| S_{K q_n}(f)(x) - K q_n \int_{\mathbb{T}} f \, d\mu \right| \leq C' e^{-c' q_n}$$

Return Times

Assuming now that $\int_{\mathbb{T}} f d\mu = 1$, this implies that for $\forall K \in [0, e^{dq_n}] \cap \mathbb{N}$

$$d(T_{Kq_n}(x,s),(x,s)) \to 0$$

uniformly for all $(x,s) \in \mathbb{T}_f$.

The Proof

(We will show equidistribution along a subsequence, something slightly stronger than the stated result at the beginning)

Let $d < \delta$ be given. Define the sequence (K_n) by $K_n = e^{dq_n}$.

Let $g \in C(\mathbb{T}_f)$ be given. We must show that

$$\frac{1}{\pi(K_n q_n)} \sum_{p < K_n q_n} g(T_p(x, s)) \to \int_{\mathbb{T}_f} g d\nu$$

First, we will rewrite this sum as a double sum along residue classes modulo q_n; for $p \equiv a \pmod{q_n}$, we will write $p = k_p q_n + a$:

$$\sum_{p < K_n q_n} g(T_p(x, s)) = \sum_{a < q_n} \left(\sum_{\substack{p = k_p q_n + a \\ p < K_n q_n}} g(T_{k_p q_n + a}(x, s)) \right)$$

(Continued)

But flowing for $k_p q_n$ time returns roughly to the initial point, so since

$$\sum_{a<q_n}\left(\sum_{\substack{p=k_p q_n+a\\ p<K_n q_n}} g(T_a(x,s))\right) = \sum_{a<q_n} g(T_a(x,s))\cdot\pi(K_n q_n; q_n, a)$$

we can apply the Siegel-Walfisz theorem to obtain

$$\left|\frac{1}{\pi(K_n q_n)}\sum_{p<K_n q_n} g(T_p(x,s)) - \frac{1}{q_n-1}\sum_{a<q_n} g(T_a(x,s))\right| \to 0$$

The Punchline

So by unique ergodicity, we have

$$\frac{1}{q_n - 1} \sum_{a<q_n} g(T_a(x,s)) \to \int_{\mathbb{T}_f} g d\nu$$

and therefore

$$\frac{1}{\pi(K_n q_n)} \sum_{p<K_n q_n} g(T_p(x,s)) \to \int_{\mathbb{T}_f} g d\nu$$

Next Steps

- Possibly holds for any irrational α?
- Relax the assumptions of the ceiling function?
- Equidistribution along primes?
- More general theorems entirely?

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The End