Technical Research Report

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The Performance Of Focused Error Control Codes*

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Abstract

Consider an additive noise channel with inputs and outputs in the field GF(q) where $q > 2$; every time a symbol is transmitted over such a channel, there are $q-1$ different errors that can occur, corresponding to the $q-1$ non-zero elements that the channel can add to the transmitted symbol. In many data communication/storage systems, there are some errors that occur much more frequently than others; however, traditional error correcting codes – designed with respect to the Hamming metric – treat each of these $q-1$ errors the same. Fuja and Heegard have designed a class of codes, called *focused error control codes,* that offer different levels of protection against "common" and "uncommon" errors; the idea is to define the level of protection in a way based not only on the *number* of errors, but the *kind* as well. In this paper, the performance of these codes is analyzed with respect to idealized "skewed" channels as well as realistic non-binary modulation schemes. It is shown that focused codes, used in conjunction with PSK and QAM signaling, can provide more than 1.0 dB of *additional* coding gain when compared with Reed-Solomon codes for small blocklengths.

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1. A graphic interpretation of Lemma 1.
2. P_d versus γ for $t_1 + t_2 = 4$, $n = 50$, and $\epsilon = 10^{-3}$.
3. P_d versus ϵ for $t_1 + t_2 = 4$, $n = 50$, and $\gamma = 10^{-3}$.
4. Decision regions corresponding to 8-PSK.
5. A 4×4 constellation displaying "common" and "uncommon" errors.
6. P_s for 16-ary PSK – uncoded, Reed-Solomon, and focused code. (Blocklength $n = 8$ for both codes.)
7. P_s for 8×8 QAM – uncoded, Reed-Solomon, and focused code. (Blocklength $n = 11$ for both codes.)

1 Introduction and Motivation

When a symbol from GF(q) is sent over a channel with additive noise, there are $q-1$ different non-zero noise symbols that can corrupt the transmitted field element. "Traditional" error control codes, designed with respect to the Hamming metric, treat each of these $q-1$ possibilities the same – as simply representing a generic "error".

However, in many non-binary data transmission and storage channels, there are some errors that occur much more frequently than others. As an example, consider a modulation scheme in which data is mapped onto one of $M = 2^b$ signal points using a Gray code, so that the most likely detection errors cause exactly one bit error per symbol. In such a system the most likely errors will result in a received symbol that differs from the transmitted symbol in exactly one bit of their binary representation; thus, while there are $2^b - 1$ different *possible* errors, there are only b that are *likely.* A similar situation arises in byte-organized memory systems; while a code with "byte wide" symbols may be structurally appropriate for such a system, the dominant error types are often single-bit-per-byte failures. It is obviously inefficient to provide the same degree of protection against the uncommon errors as against the common ones.

It was this observation that led Fuja and Heegard to develop the idea of a *focused* error control code [1]. These codes were designed to give one level of protection against a specific set of common errors while maintaining another (lower) level of protection against uncommon errors. In [1] results are obtained regarding the existence and construction of such codes as well as bounds on their rates.

This paper analyzes the unique rate/performance tradeoffs made possible by focused error control codes. First, the pertinent results from [1] are reviewed; then, the performance of focused codes over an idealized skewed channel is analyzed and that analysis is applied to M-ary PSK and QAM signaling.

2 Background on Focused Codes

In this section the pertinent results from [1] are reviewed.

2.1 Definitions and a Sufficient Condition

For any $\mathbf{x} \in \mathrm{GF}(q)^n$, denote the *Hamming weight* of $\mathbf{x}$ by $\|\mathbf{x}\|$; that is, if $\mathbf{x} = [x_0, x_1, ..., x_{n-1}]$, then

$$\|\mathbf{x}\| \triangleq \sum_{i=0}^{n-1} \mathbf{1}_{\mathrm{GF}(q)^*}(x_i)$$

where $\mathrm{GF}(q)^*$ consists of the non-zero elements of $\mathrm{GF}(q)$ and $\mathbf{1}(\cdot)$ is the indicator function – i.e., $\mathbf{1}_{\mathbf{A}}(x)$ equals one if $x \in \mathbf{A}$ and equals zero otherwise.

More generally, for any $\mathbf{x} \in \mathrm{GF}(q)^n$ and any set $\mathbf{A} \subseteq \mathrm{GF}(q)^*$, define the $\mathbf{A}$-*weight* of $\mathbf{x}$ as the number of components of $\mathbf{x}$ that lie in $\mathbf{A}$; if the $\mathbf{A}$-weight of $\mathbf{x}$ is denoted by $\|\mathbf{x}\|_{\mathbf{A}}$, then

$$\|\mathbf{x}\|_{\mathbf{A}} \triangleq \sum_{i=0}^{n-1} \mathbf{1}_{\mathbf{A}}(x_i)$$

Definition: Let $\mathbf{B} \subset \mathrm{GF}(q)^*$ be a set of non-zero elements of $\mathrm{GF}(q)$. (Here, $\mathbf{B}$ will represent the set of common errors.) A code is (t_1, t_2)-focused on $\mathbf{B}$ if it can correct up to $t_1 + t_2$ errors provided at most t_1 of these errors lie outside $\mathbf{B}$. More precisely, such a code is a set $\mathbf{C}$ of n-tuples over $\mathrm{GF}(q)$ with the following property: There exists a decoding function $f : \mathrm{GF}(q)^n \rightarrow \mathbf{C}$ such that $f(\mathbf{c} + \mathbf{e}) = \mathbf{c}$ for any $\mathbf{c} \in \mathbf{C}$ and any $\mathbf{e} \in \mathrm{GF}(q)^n$ satisfying the following two conditions:

1. $\|\mathbf{e}\| \leq t_1 + t_2$;
2. $\|\mathbf{e}\|_{\mathbf{B}^c} \leq t_1$.

Note that a $(t, 0)$-focused code is a "traditional" t-error correcting code, while a $(0, t)$-focused code is a code that is completely focused on $\mathbf{B}$ – i.e. it can correct up to t errors, provided they are *all* common.

The following result from [1] provides a sufficient condition for the existence of a code that is (t_1, t_2)-focused on a set $\mathbf{B}$.

Lemma 1: Let $\mathbf{C}$ be a set of q-ary n-tuples with the following property. For any $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, *at least one* of the following conditions holds:

- $\|\mathbf{c}_1 - \mathbf{c}_2\| > 2t_1 + 2t_2$;
- $\|\mathbf{c}_1 - \mathbf{c}_2\| + \|\mathbf{c}_1 - \mathbf{c}_2\|_{\mathbf{B^c}} > 4t_1 + 2t_2$.

Then $\mathbf{C}$ is (t_1, t_2)-focused on $\mathbf{B}$.

The implications of Lemma 1 are presented graphically in Figure 1. We can plot every q-ary n-tuple in two dimensions by its Hamming weight and its $\mathbf{B^c}$-weight. As long as no codeword difference lies in the shaded region, the code will be (t_1, t_2)-focused on $\mathbf{B}$. By comparison, to insure correction of *all* error patterns of Hamming weight $t_1 + t_2$ or less would require that all codeword differences have Hamming weight greater than $2t_1 + 2t_2$; by lowering the requirements a "notch" has been cut in the "forbidden zone". This suggests that rate improvements are likely.

2.2 Construction of Combined Focused Codes for Odd-Weight-per-Byte Errors

In [1] a method for constructing focused codes was presented. The codes thus constructed were called *combined focused codes,* and in this section we review that method.

Suppose the goal is to construct a code with blocklength n over GF(2^b) that is (t_1, t_2)-focused on the set of odd-weight symbols; that is, the common error set $\mathbf{B}$ consists of all the elements of GF(2^b) with a binary representation containing an odd number of 1's. (Note that this would include the set of single-bit errors.)

The construction from [1] is described as follows. For any n-tuple $\mathbf{x}$ over GF(2^b), let $\mathbf{b}(\mathbf{x})$ be the binary n-tuple obtained by taking the mod-two sum of each component of $\mathbf{x}$. For example, if $\mathbf{x}$=[0011,0100,1101,1010,1111], then $\mathbf{b}(\mathbf{x})$=[01100]. Let C_1 be an (n, nR_1) binary *inner* code with minimum distance $d_1 = 2t_1 + 2t_2 + 1$; let C_2 be an (n, nR_2) *outer* code over GF(2^{b-1}) with minimum distance $d_2 = 2t_1 + t_2 + 1$. To construct a codeword from the focused code, first take a codeword c_1 from C_1 and a codeword c_2 from C_2. Then add one bit to each symbol from c_2 such that $\mathbf{c}$, the resulting n-tuple over GF(2^b), satisfies $\mathbf{b}(\mathbf{c}) = c_1$.

It can be seen that the code thus constructed is (t_1, t_2)-focused on $\mathbf{B}$ by considering the following decoding algorithm. Given a received 2^b-ary n-tuple $\mathbf{r}$, compute $\mathbf{b}(\mathbf{r})$; find the codeword $\mathbf{x} \in C_1$ that is closest to $\mathbf{b}(\mathbf{r})$. As long as at most $t_1 + t_2$ odd-weight errors have occurred, $\mathbf{x}$ will be equal to $\mathbf{b}(\mathbf{c})$, where $\mathbf{c}$ is the codeword that was actually transmitted. Mark the locations where $\mathbf{x}$ differs from $\mathbf{b}(\mathbf{r})$ as erasures; strip off the last bit in each code symbol and pass the resulting 2^{b-1}-ary n-tuple plus erasure locations to a decoder for C_2.

Suppose ℓ_1 common and ℓ_2 uncommon errors occur during transmission; then as long as

$$\ell_1 \leq \lfloor (d_1 - 1)/2 \rfloor$$

and

$$\ell_2 \leq \lfloor (d_2 - \ell_1 - 1)/2 \rfloor$$

the above algorithm will correctly estimate the transmitted codeword. It is trivial to show that as long as $\ell_1 + \ell_2 \leq t_1 + t_2$ and $\ell_2 \leq t_1$ the above inequalities are met; thus, the code described above *is* (t_1, t_2)-focused.

Indeed, there are *other* error patterns – other values of ℓ_1 and ℓ_2 – that satisfy the above inequalities. As an example, suppose we wish to construct a $(0, 2)$-focused code over GF(2^b) using the above technique. Then we would need a binary inner code with minimum distance $d_1 = 5$ and an outer code over GF(2^{b-1}) with minimum distance $d_2 = 3$. Such a code would be able to correct any single *uncommon* error – as long as there were no common errors – in addition to the error patterns described by the "$(0, 2)$-focused" designation.

Note that the overall rate of this code is $(1/b)R_1 + (b - 1)R_2/b$. Furthermore, this technique can be generalized to cover a variety of common error sets; for details, refer to [1].

3 Performance of Focused Codes on an Idealized Channel

Now consider the performance of a (t_1, t_2)-focused code over an idealized "skewed" channel. One of the fundamental questions to be considered is: Under what conditions does a (t_1, t_2)-focused code perform identically to a $t_1 + t_2$-error correcting code ? Since a (t_1, t_2)-focused code can generally be constructed at a higher rate than a $t_1 + t_2$-error correcting code, answering this question will give insight into when focused codes might be appropriate for a particular application.

3.1 Block Error Probability Over a Skewed Symmetric Channel

Consider the following model for a communication/storage channel. A character $X \in \mathrm{GF}(q)$ is transmitted and the character $Y = X + Z \in \mathrm{GF}(q)$ is received. Here, the noise Z is assumed to be i.i.d. and independent of the input X and is distributed according to

$$\mathrm{P}(Z = z) = \begin{cases} 1 - \epsilon, & \text{if } z = 0; \\ \epsilon(1-\gamma)/|\mathbf{B}|, & \text{if } z \in \mathbf{B}; \\ \epsilon\gamma/|\mathbf{B}^c|, & \text{if } z \in \mathbf{B}^c, \end{cases}$$

where $\mathbf{B}$ is a set of non-zero field elements, $\epsilon = \mathrm{P}(Z \neq 0)$ is the probability of symbol channel error, and $\gamma = \mathrm{P}(Z \notin \mathbf{B} | Z \neq 0)$ is the probability that Z lies outside $\mathbf{B}$, given that $Z \neq 0$. This channel – called the *skewed symmetric channel* (SSC) for the focus set $\mathbf{B}$ – was introduced in [1] as an idealized model of a channel that exhibits the "skewing" property that focused codes were designed to address. Note that in the case of interest, $\mathbf{B}$ represents the class of common errors and so $\gamma \ll 1$; further, within each class of errors, a *uniform* distribution on the errors is assumed.

We now proceed to analyze the performance of a (t_1, t_2)-focused code operating over a SSC as described above. A (t_1, t_2)-focused code can correct up to $t_1 + t_2$ errors provided at *most* t_1 errors are uncommon. The decoder block error probability of a (t_1, t_2)-focused code on a skewed symmetric channel (SSC) can thus be written as

$$\begin{aligned} P_d \;=\; & \sum_{i=t_1+1}^{t_1+t_2} \sum_{j=t_1+1}^{i} \binom{n}{i} \binom{i}{j} \epsilon^i (1-\epsilon)^{n-i} \gamma^j (1-\gamma)^{i-j} \\ & + \sum_{i=t_1+t_2+1}^{n} \binom{n}{i} \epsilon^i (1-\epsilon)^{n-i} \end{aligned} \tag{1}$$

The first sum in (1) is the probability that there are at most $t_1 + t_2$ errors in a block of n transmitted symbols, but more than t_1 of them are uncommon; the second sum is the probability that there are more than $t_1 + t_2$ errors. Thus, the second sum is the probability of decoder error for a "traditional" $t_1 + t_2$-error correcting code.

If $\epsilon \ll 1$, we can approximate P_d by taking only the first terms in the sums in equation (1):

$$\begin{aligned} P_d &\approx \binom{n}{t_1+1} \epsilon^{t_1+1}(1-\epsilon)^{n-t_1-1}\gamma^{t_1+1} \\ &+ \binom{n}{t_1+t_2+1} \epsilon^{t_1+t_2+1}(1-\epsilon)^{n-t_1-t_2-1} \end{aligned} \tag{2}$$

Figure 2 shows P_d versus γ on a log-log scale for fixed $\epsilon = 10^{-3}$, blocklength $n = 50$ and for values of t_1 and t_2 such that $t_1 + t_2 = 4$. Each curve can be broken up into two distinct regions; for large values of γ, the graph is a straight line with slope $t_1 + 1$, whereas for small values of γ the graph has slope zero and coincides with the graph of P_d for a $(4, 0)$-focused code – i.e., a four-error correcting code. This is because, for large values of γ, the channel is not very focused – i.e., the uncommon errors are not *that* uncommon – and so the dominant cause of decoder error is the occurrence of $t_1 + 1$ uncommon errors; to put it more simply, when $\gamma \not\ll 1$, the first term in equation (2) is dominant. Similarly, when $\gamma \ll 1$ then the uncommon errors are *so* uncommon that the primary source of decoder error is the occurrence of $t_1 + t_2 + 1$ errors, which means that a (t_1, t_2)-focused code performs identically to a $t_1 + t_2$-error correcting code – i.e., the *second* term in equation (2) dominates.

So, partially answering the question that was posed at the beginning of this section: For fixed ϵ, a (t_1, t_2)-focused code has the same decoder error probability as a $t_1 + t_2$-error correcting code when the second term in equation (2) is much larger than the first term. If we define (for fixed ϵ) γ_{crit} to be the value of γ for which the two terms in equation (2) are equal to one another, then

$$\log_{10}\gamma_{crit} = \frac{1}{t_1+1}\left[t_2\log_{10}\left(\frac{\epsilon}{1-\epsilon}\right) + \log_{10}\left[\frac{\binom{n}{t_1+t_2+1}}{\binom{n}{t_1+1}}\right]\right]. \tag{3}$$

It is observed from Figure 2 that γ_{crit} is a good approximation to the point at which a (t_1, t_2)-focused code begins to "match" the decoder error probability of a $t_1 + t_2$-error correcting code;

that is, for $\gamma < \gamma_{crit}$ the two codes perform equivalently.

A similar analysis can be performed if we assume that γ is held constant and ϵ is varied. Figure 3 shows P_d versus ϵ for fixed $\gamma = 10^{-3}$, blocklength $n = 50$, and values of t_1 and t_2 such that $t_1 + t_2 = 4$. Analogous to the case described above, there is a critical value of ϵ – call it ϵ_{crit} – such that for $\epsilon > \epsilon_{crit}$ the decoder error probability for a (t_1, t_2)-focused code is identical to that of a $t_1 + t_2$-error correcting code. Here, ϵ_{crit} is given by

$$\log_{10} \epsilon_{crit} = \frac{1}{t_2} \left[(t_1 + 1) \log_{10} \gamma + \log_{10} \left[\frac{\binom{n}{t_1 + 1}}{\binom{n}{t_1 + t_2 + 1}} \right] \right]. \tag{4}$$

(Note: In obtaining equation (4) we have made the simplifying assumption that $\epsilon/(1-\epsilon) \approx \epsilon$.)

Finally, we note that the above analysis implicitly assumes that ϵ and γ can be specified independently of one another. As Section 4 shows, ϵ and γ are often both dependent on a third parameter – e.g., signal-to-noise ratio. If this is the case, then we can guarantee "performance matching" by insuring that the second term in equation (2) is much larger than the first term; if we define a "benchmark" β by

$$\beta \triangleq \left[\frac{\gamma^{t_1+1}}{[\epsilon/(1-\epsilon)]^{t_2}} \right] \left[\frac{\binom{n}{t_1 + 1}}{\binom{n}{t_1 + t_2 + 1}} \right], \tag{5}$$

then as long as $\beta \ll 1$ the decoder error probability of a (t_1, t_2)-focused code will be the same as that of a $t_1 + t_2$-error correcting code.

3.2 Symbol Error Probability Over a Skewed Symmetric Channel

It is often more important to consider the *symbol* error probability of a coding scheme rather than the block error probability – i.e., the probability that any particular symbol is decoded incorrectly, as opposed to the probability that a codeword is decoded incorrectly.

Let N denote the expected number of symbol errors in a decoded block of length n; then the symbol error rate P_s is given by $P_s = N/n$. By definition, as long as no more than $t_1 + t_2$ channel errors occur and no more than t_1 of them are uncommon, then there will be no symbol errors in the decoder output. Furthermore, we make the pessimistic assumption that, when more than $t_1 + t_2$ error occur *or* more than t_1 uncommon error occur, the decoder *adds* an additional $t_1 + t_2$ errors to its output; thus,

$$\begin{aligned} P_s &= \frac{1}{n} \sum_{i=t_1+1}^{t_1+t_2} \sum_{j=t_1+1}^{i} \min(t_1 + t_2 + i, n) \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \epsilon^i (1-\epsilon)^{n-i} \gamma^j (1-\gamma)^{i-j} \\ &+ \frac{1}{n} \sum_{i=t_1+t_2+1}^{n} \min(t_1 + t_2 + i, n) \begin{pmatrix} n \\ i \end{pmatrix} \epsilon^i (1-\epsilon)^{n-i}. \end{aligned}$$

Note that for $\epsilon \ll 1$ this can be approximated by

$$\begin{aligned} P_s \approx & \frac{2t_1 + t_2 + 1}{n} \begin{pmatrix} n \\ t_1 + 1 \end{pmatrix} \epsilon^{t_1+1} (1-\epsilon)^{n-t_1-1} \gamma^{t_1+1} \\ & + \frac{2t_1 + 2t_2 + 1}{n} \begin{pmatrix} n \\ t_1 + t_2 + 1 \end{pmatrix} \epsilon^{t_1+t_2+1} (1-\epsilon)^{n-t_1-t_2-1}. \end{aligned} \tag{6}$$

To guarantee that the *symbol* error probability of a (t_1, t_2)-focused code matches that of a $t_1 + t_2$-error correcting code requires that the second term in equation (6) be dominant. Furthermore, note the similarity between equation (6) above and the formula for the decoder error probability in equation (2); the two terms in (6) are the terms from (2) weighted by (slightly) different constants. For reasonable values of t_1 and t_2 these constants are close enough that matching the decoder error probability of a (t_1, t_2)-focused code to that of a $t_1 + t_2$-error correcting code is equivalent to doing the same for the symbol error probability.

3.3 Performance of "Combined" Focused Codes

Recall the technique for constructing "combined" (t_1, t_2)-focused codes that was described in Section 2.2. Specifically, recall that the construction sometimes yields codes and decoding algorithms that are capable of correcting *more* errors than those indicated by the "(t_1, t_2)-focused" label – for instance, some error patterns that contain *more* than t_1 uncommon errors. In this section we show briefly how this enhanced capability can be taken into account when computing the performance of such a code.

Suppose that ℓ_1 common and ℓ_2 uncommon errors occur in a codeword during transmission. It was noted that the decoding algorithm described in Section 2.2 will yield a correct estimate provided $\ell_1 \leq \lfloor (d_1-1)/2 \rfloor$ and $\ell_2 \leq \lfloor (d_2-\ell_1-1)/2 \rfloor$, where $d_1 = 2(t_1+t_2)+1$ and $d_2 = 2t_1+t_2+1$. If we assume that such a combined code is employed over a SSC with parameters ϵ and γ, then the decoder error probability is given by

$$
\begin{aligned}
P_d{}' \; &= \sum_{\ell_1=0}^{\lfloor (d_1-1)/2 \rfloor} \; \sum_{\substack{\ell_2= \\ \lfloor (d_2-\ell_1-1)/2 \rfloor+1}}^{n-\ell_1} \binom{n}{\ell_1+\ell_2} \binom{\ell_1+\ell_2}{\ell_1} \epsilon^{\ell_1+\ell_2} (1-\epsilon)^{n-\ell_1-\ell_2} \gamma^{\ell_2} (1-\gamma)^{\ell_1} \\
&+ \sum_{\substack{\ell_1= \\ \lfloor (d_1-1)/2 \rfloor+1}}^{n} \binom{n}{\ell_1} [\epsilon(1-\gamma)]^{\ell_1} [1-\epsilon(1-\gamma)]^{n-\ell_1}.
\end{aligned}
$$

Similarly, the symbol error probability can be pessimistically approximated by

$$
\begin{aligned}
P_s{}' \; &= \frac{1}{n} \sum_{\ell_1=0}^{\lfloor (d_1-1)/2 \rfloor} \; \sum_{\substack{\ell_2= \\ \lfloor (d_2-\ell_1-1)/2 \rfloor+1}}^{n-\ell_1} \min(\ell_1+\ell_2+\lfloor (d_2-\ell_1-1)/2 \rfloor, n) \\
&\qquad \cdot \binom{n}{\ell_1+\ell_2} \binom{\ell_1+\ell_2}{\ell_1} \epsilon^{\ell_1+\ell_2} (1-\epsilon)^{n-\ell_1-\ell_2} \gamma^{\ell_2} (1-\gamma)^{\ell_1} \\
&+ \frac{1}{n} \sum_{\substack{\ell_1= \\ \lfloor (d_1-1)/2 \rfloor+1}}^{n} \; \sum_{\ell_2=0}^{n-\ell_1} \min(\ell_1+\ell_2+\lfloor (d_1-1)/2 \rfloor + \lfloor (d_2-\lfloor (d_1-1)/2 \rfloor -1)/2 \rfloor, n) \\
&\qquad \cdot \binom{n}{\ell_1+\ell_2} \binom{\ell_1+\ell_2}{\ell_1} \epsilon^{\ell_1+\ell_2} (1-\epsilon)^{n-\ell_1-\ell_2} \gamma^{\ell_2} (1-\gamma)^{\ell_1}. \qquad (7)
\end{aligned}
$$

4 Performance of Focused Codes with Non-Binary Modulation Schemes

In this section it will be shown how common non-binary modulation techniques can be approximated by the skewed symmetric channel with appropriate choice of parameters. Specifically, the results from Section 3 will be used to analyze the effectiveness of focused codes operating over an additive white Gaussian noise channel in conjunction with PSK and QAM modulation.

4.1 Parameters for PSK Modulation

If M-ary PSK modulation is used with a Gray code so that the difference between the binary representation of any two adjacent signals is one bit, then an additive Gaussian noise channel can be approximated by an M-ary SSC for the focus set $\mathbf{B}$ consisting of all elements of GF(M) with a binary representation containing exactly one "1".

Assume that every T seconds a signal $s_i(t)$ is transmitted for some i, $0 \leq i \leq M-1$. Here, $s_i(t)$ is given by

$$s_i(t) = \begin{cases} a_i\phi_1(t) + b_i\phi_2(t), & \text{if } 0 \leq t \leq T; \\ 0, & \text{otherwise,} \end{cases}$$

where $a_i^2 + b_i^2 = E_s$ and $b_i/a_i = \tan(2\pi i/M)$. Here, $\phi_1(t)$ and $\phi_2(t)$ are a pair of orthonormal real-valued signals – i.e.,

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Thus, a_i and b_i represent the coordinates of the signal $s_i(t)$ with respect to the basis signals $\phi_1(t)$ and $\phi_2(t)$, and E_s is the symbol energy. (See Figure 4.) We assume further that the channel is zero-mean additive white Gaussian with power spectral density $S_z(f) = N_0/2$. Finally, it is assumed that hard-decisions are made at the demodulator, meaning that the received signal $r(t)$ is mapped onto the signal $s_i(t)$ that minimizes the Euclidean distance.

In order to compute the symbol error probability P_s for a focused code used in conjunction with such a system, one must determine ϵ and γ, the parameters of the associated SSC. Recall that ϵ is the probability of channel error while γ is the probability of an uncommon error, given that an error has occurred; in the PSK context, γ is the probability that the received signal lies outside of the decision regions *adjacent to* the one containing the transmitted signal, given that the received signal lies outside the region containing the transmitted signal. (See Figure 4.)

Given the symmetry of M-ary PSK, it is permissible to assume that any signal $s_i(t)$ was transmitted. Then ϵ is just the probability that the received signal lies outside $s_i(t)$'s decision region, given $s_i(t)$ was transmitted. This is well known [2] to be approximated by

$$\epsilon \approx 2\,Q\left(\sqrt{2\,E_s/N_0}\,\sin\frac{\pi}{M}\right) \tag{8}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp[-t^2/2]\,dt.$$

We now turn our attention to computing γ. For $M = 4$ the problem is trivial; the only way an "uncommon" error can occur is if a transmitted signal is perturbed by the noise into the decision region of the signal directly opposite. Thus it can be shown that

$$P(\text{Uncommon error}) = \mathrm{P}(\sqrt{E_s/2} + n_1 < 0, \sqrt{E_s/2} + n_2 < 0),$$

where n_1 and n_2 are independent zero-mean Gaussian random variable with variance $N_0/2$. Therefore

$$P(\text{Uncommon error}) = Q^2\left(\sqrt{\frac{E_s}{N_0}}\right),$$

and so

$$\begin{aligned}\gamma &= \frac{P(\text{Uncommon error})}{P(\text{Error})} = \frac{Q^2\left(\sqrt{E_s/N_0}\right)}{2\,Q\left(\sqrt{E_s/N_0}\right)} \\ &= \frac{1}{2}Q\left(\sqrt{\frac{E_s}{N_0}}\right).\end{aligned} \tag{9}$$

To obtain γ for $M \geq 8$ the following approximation for the pdf of the angle of the received signal [3] is useful:

$$f_\Theta(\theta) = \sqrt{E_s/(\pi N_0)}\cos\theta\exp[-(E_s/N_0)\sin^2\theta].$$

(This approximation is valid for $E_s/N_0 \gg 1$ and for small angle θ.) An uncommon error is made if, for any j, the noise causes a phase displacement greater than $3\pi/M$ in absolute value. Using the above approximation for $f_\Theta(\theta)$, it can be shown that

$$P(\text{Uncommon error}) = 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{3\pi}{M}\right),$$

and so

$$\begin{aligned}\gamma &= \frac{P(\text{Uncommon error})}{P(\text{Error})} \\ &= \frac{Q\left(\sqrt{2E_s/N_0}\,\sin(3\pi/M)\right)}{Q\left(\sqrt{2E_s/N_0}\,\sin(\pi/M)\right)}. \end{aligned} \tag{10}$$

4.2 Parameters for a Square Constellation (QAM)

An additive white Gaussian noise (AWGN) channel and an M-ary square signal constellation can be approximated by an M-ary SSC channel with parameters ϵ and γ to be determined. For the square constellation, we assume that a two-dimensional Gray code is used so that the binary representation of two signals immediately adjacent to each other either horizontally or vertically differ in only one bit; thus, our set of common errors is (once again) the elements of GF(M) containing a single "1" in its binary representation. For example, when $M = 16$ as shown in Figure 5, if signal s_1 is sent, then a common error will be made if the demodulator estimates the transmitted signal to be s_2, s_3, s_4 or s_5.

For a square ($\sqrt{M} \times \sqrt{M}$ where $\log_2 M$ is even) constellation of QAM signals, the coordinates of the signals with respect to the basis signals $\phi_1(t)$ and $\phi_2(t)$ are:

$$a_i = (2i + 1 - \sqrt{M})\frac{d}{2}$$

and

$$b_j = (2j + 1 - \sqrt{M})\frac{d}{2}$$

where i and j take on the values $0, 1, 2, \ldots, \sqrt{M} - 1$, and d is the constant horizontal or vertical distance between any two neighbors. The value of d is determined by the average symbol energy E_s and is given by

$$d = \sqrt{\frac{6\,E_s}{M-1}}.$$

In computing the parameters ϵ and γ a pessimistic approach will be used. It will be assumed that an error occurs whenever the received signal lies outside of a square with sides d in length centered on the transmitted signal; furthermore, an *uncommon* error occurs whenever the received signal lies outside of such a square *and* outside of the four squares immediately adjacent (horizontally and vertically) to it. (Such an assumption is pessimistic because the points on the *exterior* of the signal constellation will actually have lower probabilities of error than those indicated.) It is well known [3] that ϵ – the probability of channel error – is given by

$$\epsilon = 2q - q^2 \tag{11}$$

where

$$q = 2\,Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right).$$

Now consider the derivation of γ. Using the same pessimistic assumption used in deriving ϵ,

$$\begin{aligned} P(\text{Uncommon Error}) &= P(|n_1| > d/2, |n_2| > d/2) + P(|n_1| > 3d/2, |n_2| \leq d/2) \\ &\quad + P(|n_2| > 3d/2, |n_1| \leq d/2), \end{aligned}$$

where n_1 and n_2 are independent zero-mean Gaussian random variables with variance $N_0/2$. If p_1 and p_2 are defined as

$$\begin{aligned} p_1 &\triangleq Q\left(d/\sqrt{2N_0}\right) = Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right) \\ &= \frac{1}{2}P(|n_1| > d/2) \end{aligned}$$

and

$$\begin{aligned} p_2 &\triangleq Q\left(3d/\sqrt{2N_0}\right) = Q\left(3\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right) \\ &= \frac{1}{2}P(|n_1| > 3d/2) \end{aligned}$$

then

$$P(\text{Uncommon error}) = 4p_1^2 + 4p_2(1 - 2p_1).$$

This in turn implies

$$\gamma = \frac{p_1^2 + p_2(1-2p_1)}{p_1(1-p_1)}, \tag{12}$$

where p_1 and p_2 are as above.

4.3 Performance Matching for Focused Codes Used with PSK and QAM Modulation

Recall the question posed at the beginning of of Section 3: Under what conditions does a (t_1, t_2)-focused code perform identically to a $t_1 + t_2$-error correcting code ? In Sections 3.1 and 3.2 this question was answered for a skewed symmetric channel; it was shown that the two have the same block error rate as long as the second term in equation (2) was dominant – or, equivalently, if $\beta \ll 1$, where β is defined in (5).

Having shown in Sections 4.1 and 4.2 how M-ary PSK and QAM can be approximated by a SSC, we are now prepared to determine when a (t_1, t_2)-focused code operating in conjunction with these modulation schemes perform identically to a $t_1 + t_2$-error correcting code.

PSK: Equations (8)-(10) give the values of ϵ and γ that approximate M-ary PSK at a given signal-to-noise ratio; substituting these values into equation (5) and determining when $\beta \ll 1$ yields the following results.

- **M=8:** For octal PSK, a $(0, t)$-focused code performs identically to a t-error correcting code for $t = 1, 2, 3$ for all values of E_s/N_0 and all blocklengths $n \geq 7$.
- **M=16:** For 16-ary PSK, a $(0, t)$-focused code performs identically to a t-error correcting code for $t = 1, 2, 3, 4, 5, 6$ for all values of E_s/N_0 and all blocklengths $n \geq 10$.

The above results indicate that a code capable of correcting t adjacent-region errors will perform identically to a code capable of correcting any t errors for many blocklengths and many values of t. It is interesting at this point to compare the "focused approach" to PSK modulation with that taken by Lee-metric codes. A t-error correcting Lee-metric code will provide error correction to a transmitted word provided that the total Lee distance between what was transmitted and what was received is no more than t; a received signal that is i regions away from the transmitted signal contributes a value i to the Lee distance between the transmitted and received words. Thus, a 2-error correcting code with respect to the Lee metric, when used with M-ary PSK, can correct any two adjacent-region errors *and* it can correct any single error where the received symbol is *two* regions away from what was transmitted. The above results suggest that the added capability of Lee-metric codes – the ability to correct a (reduced) number of non-adjacent errors – often provides negligible performance improvement to PSK modulation.

Square Signal Sets: Equations (11) and (12) give the values of ϵ and γ that approximate M-ary QAM at a given signal-to-noise ratio; substituting these values into equation (5) and determining when $\beta \ll 1$ yields the following results.

- **M=64:** For an 8×8 constellation at all values of E_s/N_0 and all blocklengths $n \geq 7$:
  - A $(0,1)$-focused code performs identically to a 1-error correcting code.

    A $(1,1)$-focused code performs identically to a 2-error correcting code.
  - A $(1,2)$-focused code performs identically to a 3-error correcting code.
  - A $(2,2)$-focused code performs identically to a 4-error correcting code.
  - A $(2,3)$-focused code performs identically to a 5-error correcting code.
- **M=256:** For a 16×16 constellation at all values of E_s/N_0 and all blocklengths $n \geq 8$:
  - A $(0,1)$-focused code performs identically to a 1-error correcting code.

    A $(1,1)$-focused code performs identically to a 2-error correcting code.
  - A $(1,2)$-focused code performs identically to a 3-error correcting code.
  - A $(2,2)$-focused code performs identically to a 4-error correcting code.
  - A $(2,3)$-focused code performs identically to a 5-error correcting code.

4.4 Coding Gain of Focused Codes Used with PSK and QAM Modulations

In this section the performance of some (t_1, t_2)-focused codes is compared with the performance of $t_1 + t_2$-error correcting codes. The focused codes under consideration are constructed using the technique outlined in Section 2.2; the channel under consideration is PSK and QAM signaling over additive white Gaussian noise.

The approach taken is to compare a (t_1, t_2)-focused code with a $t_1 + t_2$-error correcting code such that the two codes have identical symbol error rate; because the focused code can (in general) be constructed at a higher rate than the "traditional" code, there will be some coding gain enjoyed by the focused code. For example, Section 4.3 shows that a $(0,1)$-focused code over GF(8) with blocklength $n = 7$ will perform identically to a single-error correcting code of the same blocklength when used with octal PSK; however, using the procedure described in Section 2.2 it's possible to construct a $(0,1)$-focused code at a rate of $16/21 \approx 0.762$, whereas a single-error correcting Reed-Solomon code over GF(8) with blocklength $n = 7$ has rate $5/7 \approx 0.714$. The 6.7% rate improvement offered by the focused code translates into a constant coding gain of 0.28 dB.

Equation (6) gives the symbol error rate for a (t_1, t_2)-focused code operating over a skewed symmetric channel with parameters ϵ and γ; equation (7) gives the same when the code is one of the "combined" codes from Section 2.2. Equations (8), (9), and (10) give ϵ and γ for an AWGN channel employing PSK modulation; equations (11) and (12) give ϵ and γ for an AWGN channel when the modulation is QAM with a square signal set. Thus, for a given signal-to-noise ratio we can use equations (8)-(12) to compute ϵ and γ and then use (6)-(7) to yield P_s. To make a fair comparison between codes with different rates, P_s is computed as a function of E_b/N_0, where E_b is the energy per information bit – i.e., for a code with rate R,

$$E_b = \frac{E_s}{R \log_2 M}.$$

Figures 6 shows P_s versus E_b/N_0 for 16-ary PSK used in conjunction with various coding schemes. The solid line in Figure 6 gives the performance of uncoded 16-ary PSK, while the dotted line shows the performance of an (8,4) two-error correcting shortened Reed-Solomon code over GF(16). The two dashed lines display the performance of a $(0,2)$-focused code over GF(16) with blocklength $n = 8$ that is constructed using the technique described in Section 2.2; the short-dashed line shows P_s as computed by equation (6), while the long-dashed line gives P_s as determined by equation (7). (Since the code is capable of correcting any single uncommon error

– a fact that equation (7) takes into account, while (6) does *not* – we find that equation (7) shows the performance to be slightly improved over that suggested by (6).)

Figure 6 shows that, at a symbol error rate of $P_s = 10^{-6}$, the focused code provides approximately 1.13 dB of coding gain *above* that of the Reed-Solomon code; comparing the focused code with uncoded 16-ary PSK, we find a coding gain of approximately 2.48 dB.

Similarly, Figure 7 compares a blocklength $n = 11$ $(1, 2)$-focused code with a 3-error correcting shortened Reed-Solomon code when used with a 8×8 square signal constellation. In this case the performance given by equations (6) and (7) were identical; the focused code provided a constant 0.91 dB of coding gain over the Reed-Solomon code and provided 2.38 dB of gain above uncoded 64-QAM at $P_s = 10^{-6}$.

In each of the two figures, the focused code is constructed according to Section 2.2 with the inner code being the highest-known-rate [4] binary code with minimum distance $d_{min} = 2(t_1+t_2)+1$ and the outer code being a shortened Reed-Solomon code. The gains are significant primarily for codes with blocklengths that are short relative to the field size; for instance, if the blocklength goes above $n = 20$ for 64-QAM, then the focused code construction of Section 2.2 provides a coding gain that is no more than 0.25 dB better than a shortened Reed-Solomon code. It would appear that this limitation is more a function of the particular construction than of focused codes in general.

5 Conclusions

This paper analyzed the performance of focused error control codes – both over idealized channels and in conjunction with non-binary signaling. It was shown that construction techniques currently available can provide more than 1.0 dB of coding gain *above* that provided by shortened Reed-Solomon codes at small blocklengths when used in conjunction with PSK and QAM signaling.

References

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[2] Bernard Sklar, *Digital Communications,* Prentice Hall, 1988.

[3] S. Benedetto, E. Biglieri, V. Castellani, *Digital Transmission Theory,* Prentice Hall, 1987.

[4] T. Verhoff, "An Updated Table of Minimum-Distance Bounds for Binary Linear Codes", *IEEE Transactions on Information Theory,* Vol. 33, No.5, pp. 665-680, September 1987.

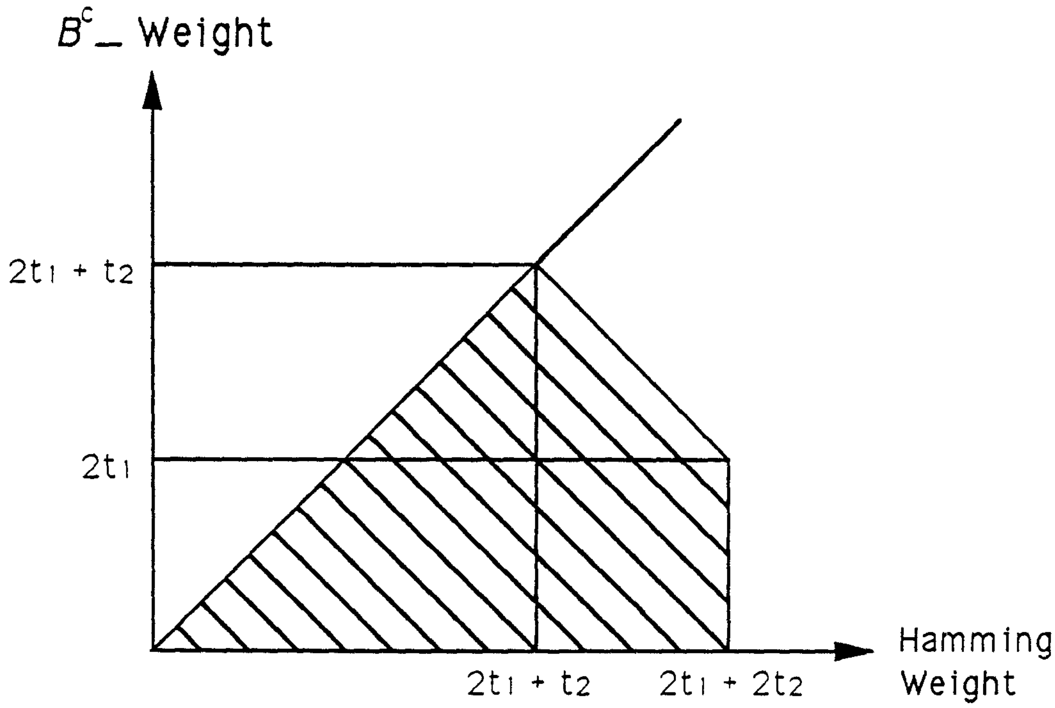

Figure 1: A graphic interpretation of Lemma 1.

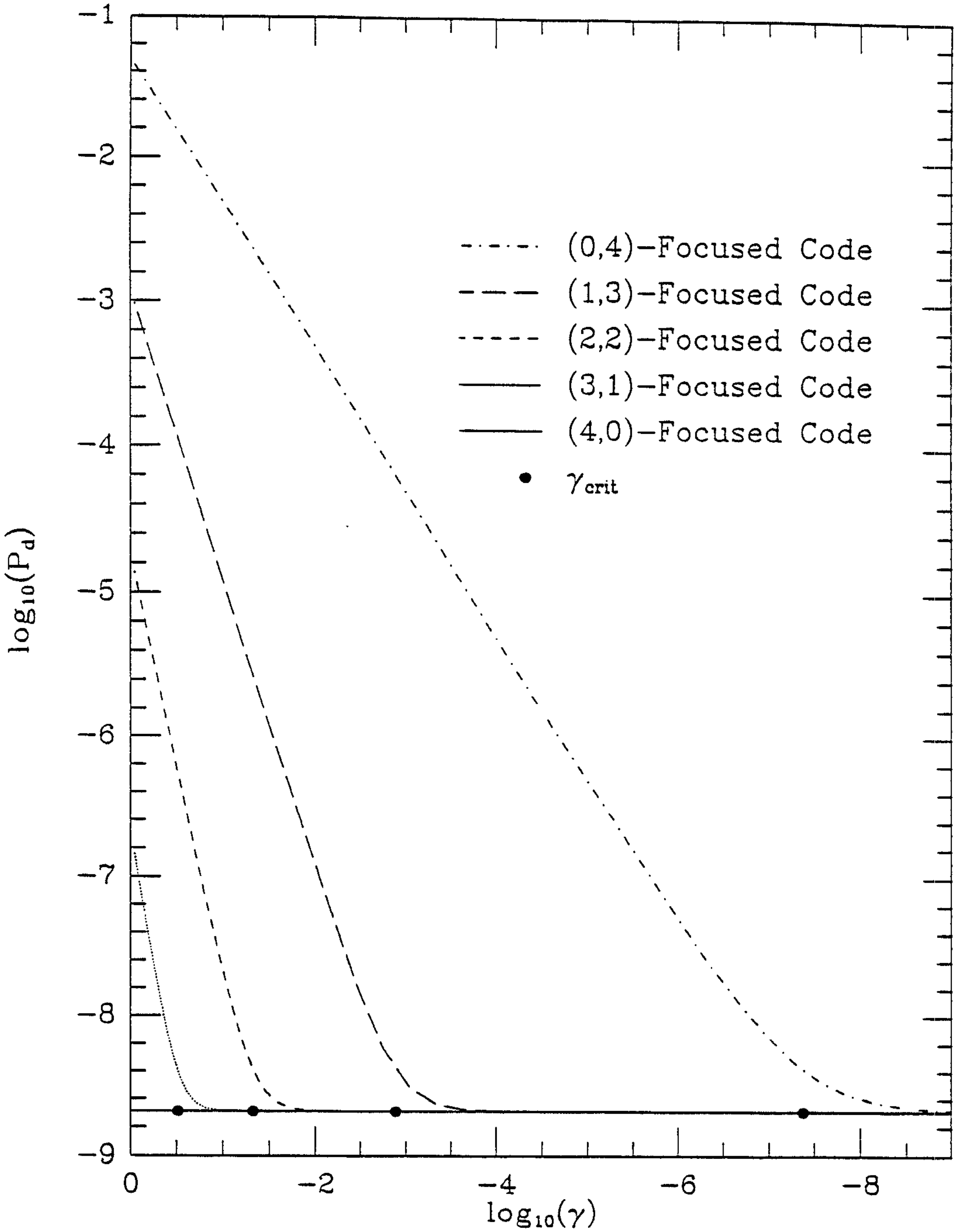

Figure 2: P_d versus γ for $t_1 + t_2 = 4$, n=50, and $\varepsilon = 10^{-3}$

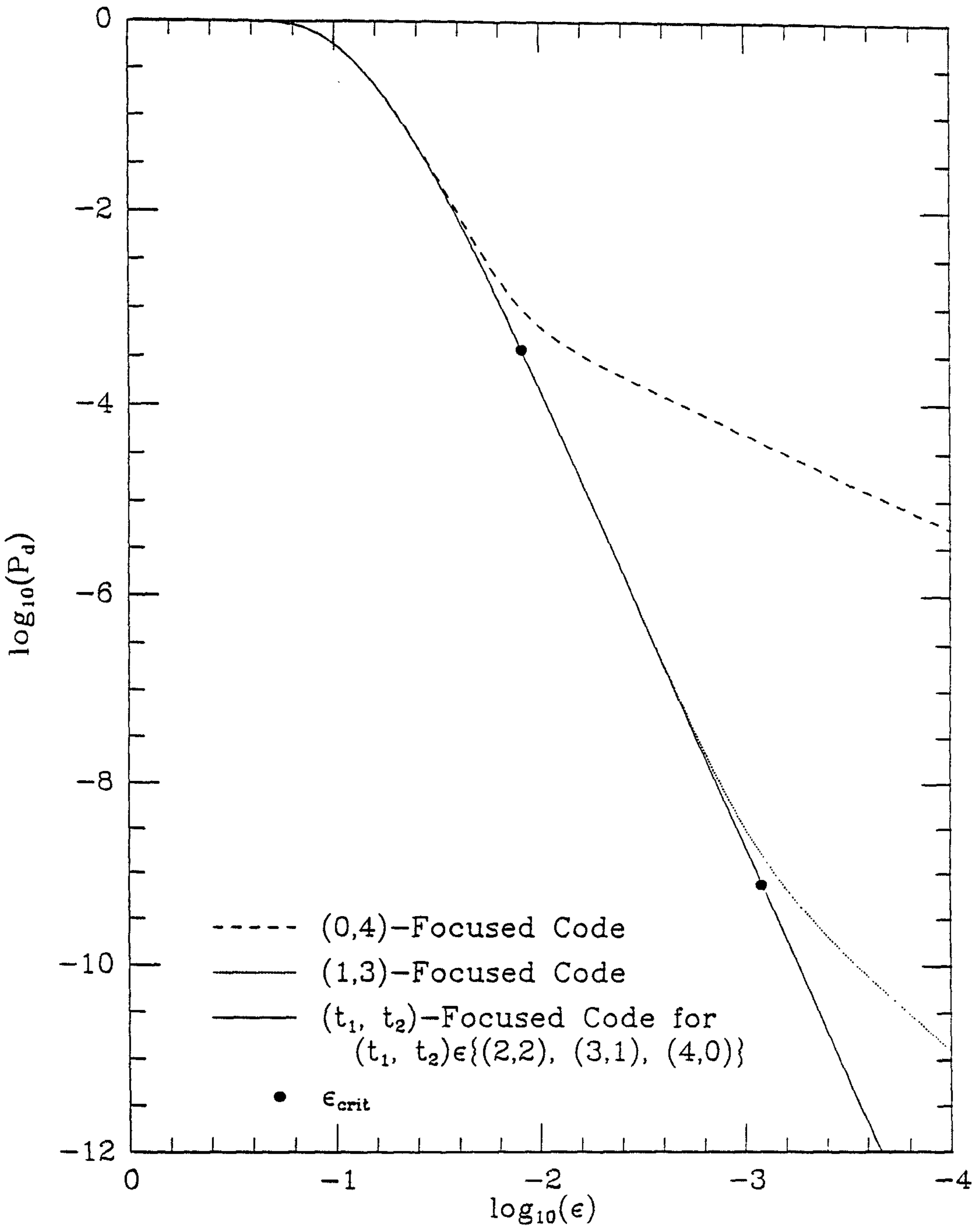

Figure 3: P_d versus ϵ for $t_1 + t_2 = 4$, n=50, and $\gamma = 10^{-3}$.

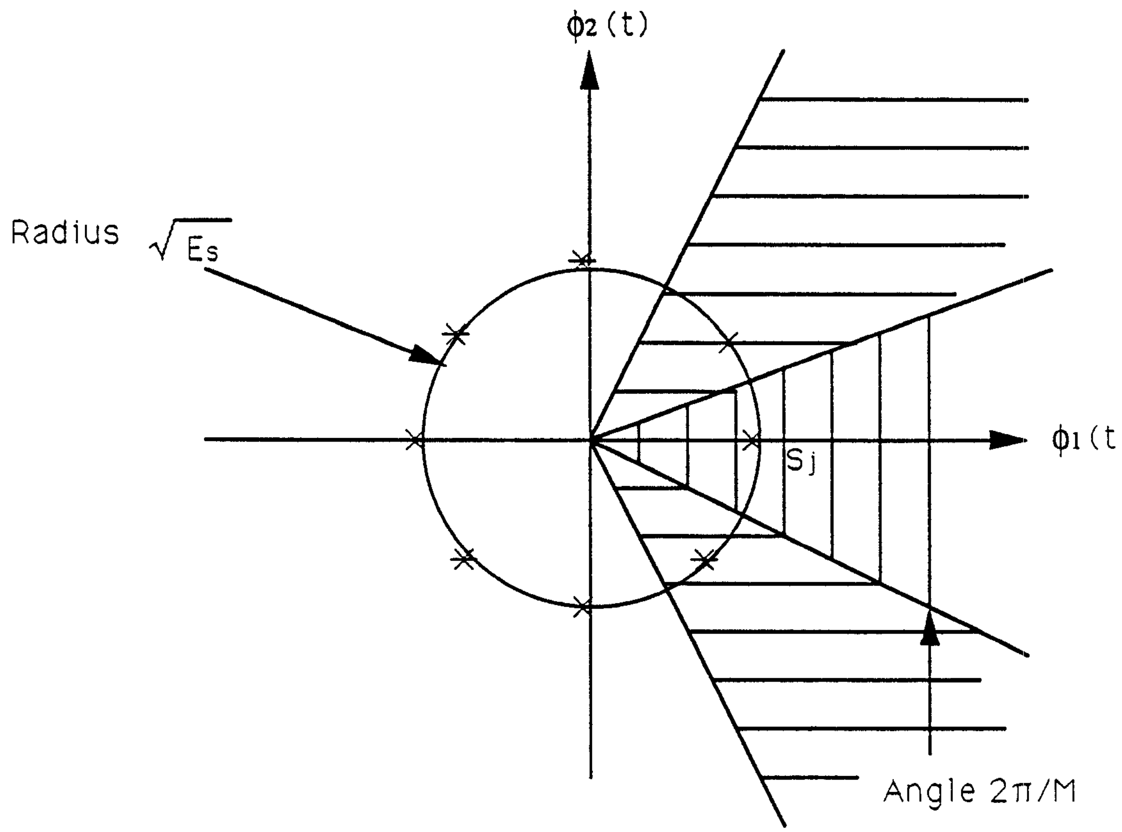

Figure 4: Decision regions corresponding to 8-PSK.

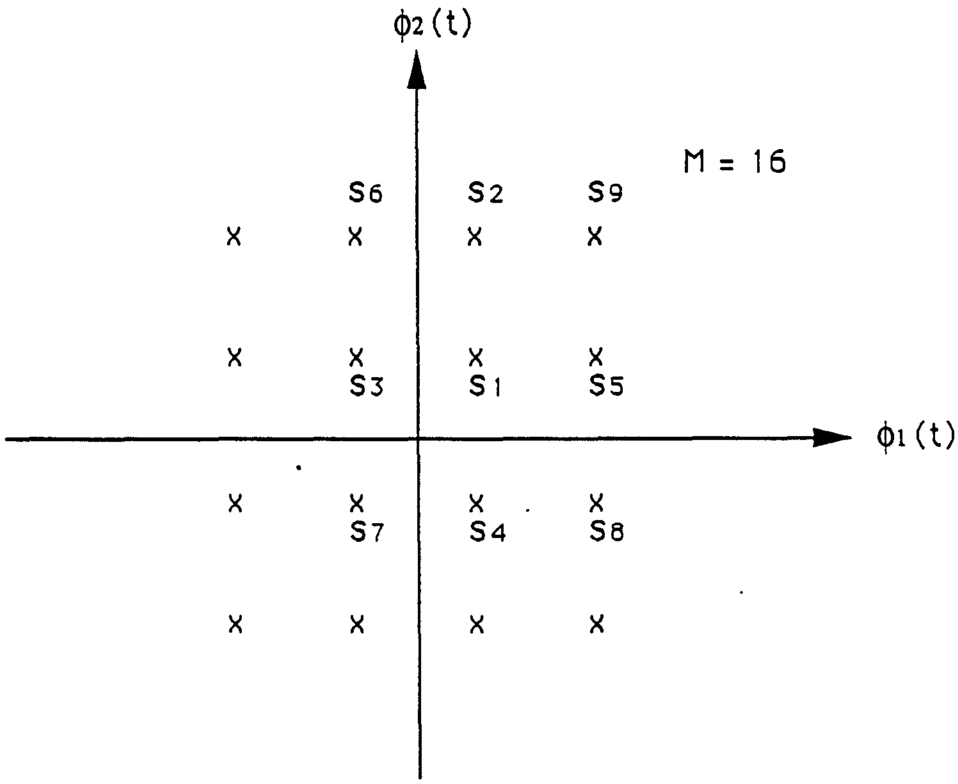

Figure 5: A 4 x 4 constellation displaying "common" and "uncommon" errors.

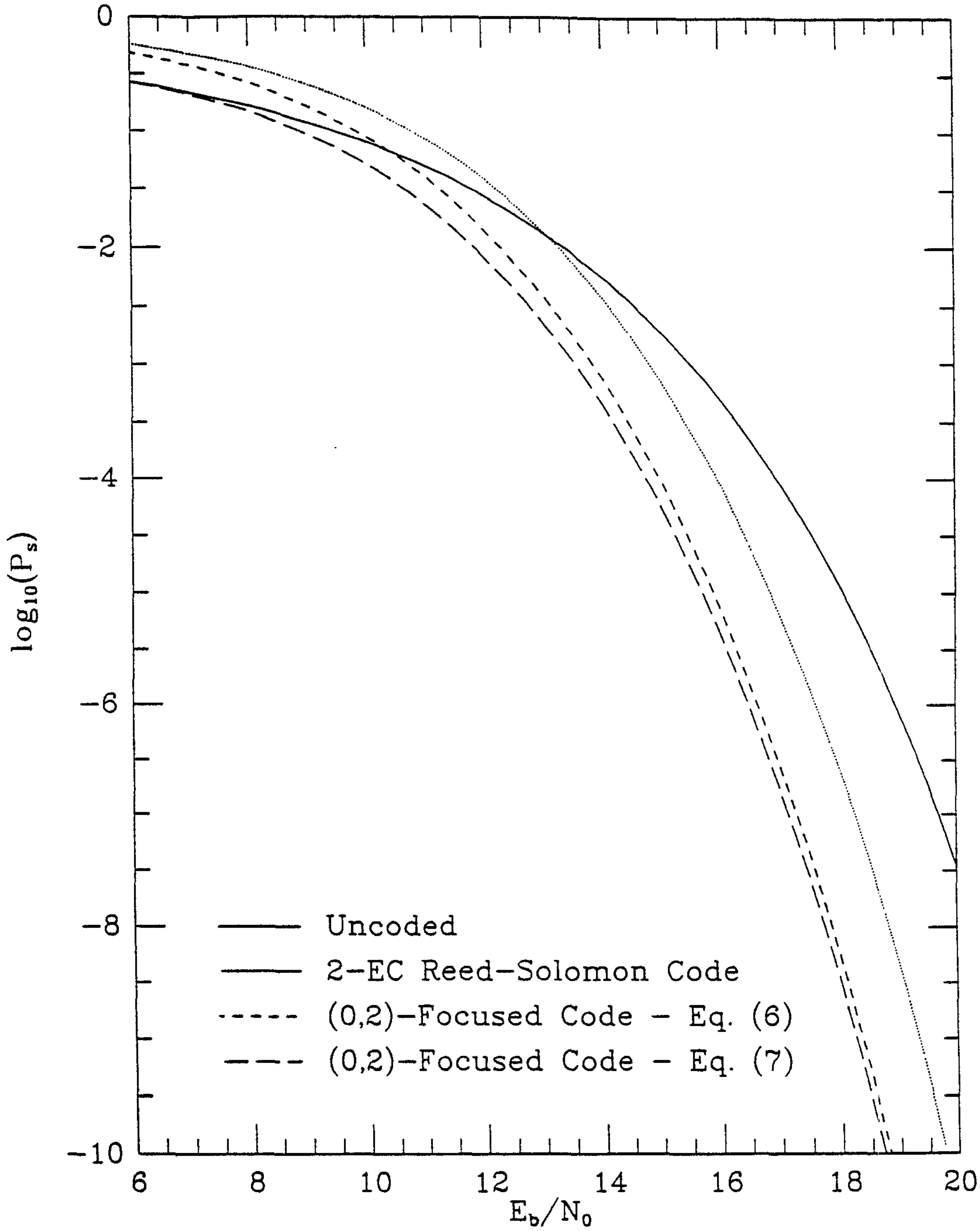

Figure 6: P_s for 16-ary PSK -- uncoded, Reed-SOlomon, and focused code. (Blocklength n=8 for both codes.)

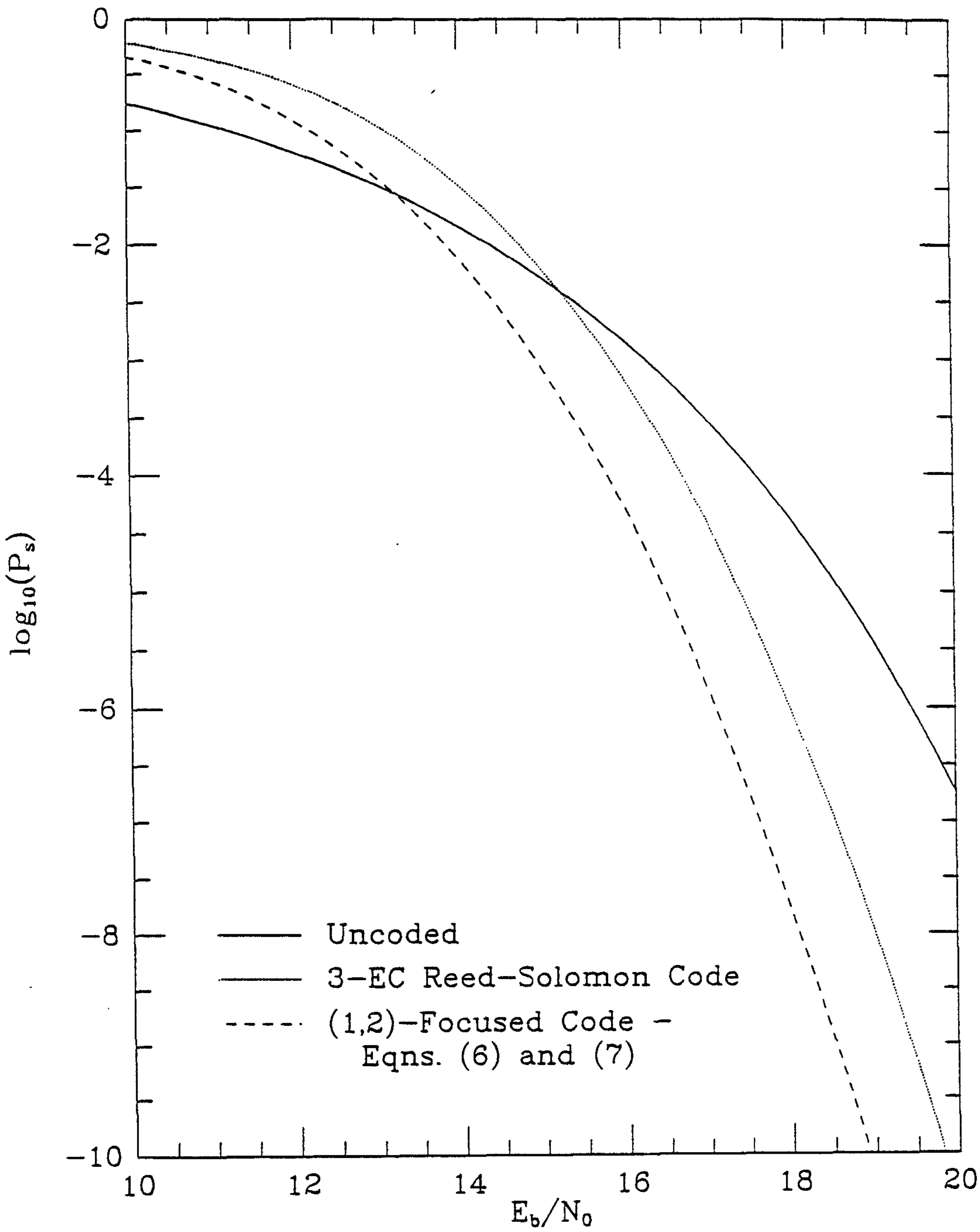

Figure 7: P_s for 8 x 8 QAM -- unooded, Reed-Solomon, and focused code.
(Blocklength n=11 for both codes.)