ABSTRACT

Title of dissertation: MODULI SPACES OF SHEAVES ON HIRZEBRUCH ORBIFOLDS

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Doctor of Philosophy, 2019

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We provide a stacky fan description of the total space of certain split vector bundles, as well as their projectivization, over toric Deligne-Mumford stacks. We then specialize to the case of Hirzebruch orbifold $\mathcal{H}_r^{ab}$ obtained by projectivizing $\mathcal{O} \oplus \mathcal{O}(r)$ over the weighted projective line $\mathbb{P}(a, b)$. Next, we give a combinatorial description of toric sheaves on $\mathcal{H}_r^{ab}$ and investigate their basic properties. With fixed choice of polarization and a generating sheaf, we describe the fixed point locus of the moduli scheme of μ-stable torsion free sheaves of rank 1 and 2 on $\mathcal{H}_r^{ab}$. Finally, we show that if $\mathcal{X}$ is the total space of the canonical bundle over a Hirzebruch orbifold, then we can obtain generating functions of Donaldson-Thomas invariants.

MODULI SPACES OF SHEAVES ON HIRZEBRUCH ORBIFOLDS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2019

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Acknowledgments

I would like to express my sincere gratitude for people who supported and motivated me during my PhD journey. Doing a PhD is like climbing a mountain, challenging and sometimes even demoralizing. I can't climb up that peak without their help.

First of all, thanks to my advisor Professor Amin Gholampour for bringing me into the world of toric geometry. I deeply appreciate his patience and invaluable guidance throughout my research and writing of this thesis. I am very fortunate to have him as my advisor.

Many thanks to my lovely friends at Maryland. Especially, I would like thank to Jacky Chong for being a great person to talk to. I am truly influenced by his energetic personality and dedication to math. I also thank to Patrick Daniels, Xuesen Na and Yousheng Shi for discussions of algebraic geometry problems.

Most importantly, I am deeply grateful and blessed for my parents' continuous and unconditional support. Their encouragement greatly relieves my anxiety and stress during my grad school life.

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Chapter 1: Introduction

1.1 Background

One of the central topics in algebraic geometry is to study the spaces whose points represent objects of same class. Such objects can be schemes, morphisms and sheaves, and usually come in families. Moduli spaces can be thought of as solutions to the classification problems of certain geometric objects. Many moduli spaces arise as quotients of schemes by reductive algebraic groups. The construction of such spaces are frequently done by utilizing Grothendieck's Quot scheme and Mumford's geometric invariant theory. If we can understand the geometric structure of moduli spaces, we will get a better insight into the geometry of those objects themselves.

The main interest of this thesis is about moduli spaces of stable sheaves and the main tools come from toric geometry. The first formal definition of toric variety was introduced by Demazure [Dem70] in 1970. Back then, it was referred to as the toroidal embedding [KKMS73]. After 1980, the study of toric varieties grew rapidly and gradually became a important part of modern algebraic geometry. A toric variety X is an algebraic variety containing an torus $(\mathbb{C}^*)^n$ as an open dense subset such that the natural action of the torus on itself extends to an action on X. Toric varieties can be constructed by fans [CLS11, Ful93]. A fan is a collection of strongly convex rational polyhedral cones closed under taking intersections and faces. As the geometric properties of toric varieties are translated and encoded in the combinatorial data of fans, they are more computable. Toric geometry creates a deep connection between algebraic geometry and convex geometry, and provides a testing ground for many general theorems.

Toric stacks are stacky generalizations of toric varieties. The word "stack" was first introduced by Deligne and Mumford [DM69] for the original French term "champ" given by Giraud [Gir66]. Roughly speaking, each point in an algebraic stack comes with an automorphism group. A scheme can be thought of as an algebraic stack in which this group is trivial. The major motivation for stacks comes from moduli problems. In many moduli problems, there are no fine moduli schemes, because the geometric objects that we want to parametrize usually have nontrivial automorphism groups. The moduli stacks can remember the automorphisms.

Toric stacks were first introduced by Borisov, Chen and Smith [BCS05]. Just as a toric variety corresponds to a fan, a toric stack is associated with a stacky fan. Fantechi, Mann and Nironi showed that every smooth toric Deligne-Mumford stack $\mathcal{X}$ has an open dense orbit isomorphic to a Deligne-Mumford torus $\mathcal{T} \cong T \times \mathcal{B}G$, where T is a torus and G is a finite abelian group. The natural action of $\mathcal{T}$ on itself extends to an action on $\mathcal{X}$ [FMN10]. Similar to toric varieties, toric stacks serve as an important class to test conjectures about algebraic stacks.

The central topic of this thesis is about moduli spaces of torus-equivariant stable sheaves on Hirzebruch orbifolds, which form an important class of 2-dimensional toric stacks. Every toric variety or orbifold contains an torus as an open dense subset and the regular action of this torus can be lifted to the moduli space. Using the technique of torus localization, one can give a combinatorial description of the fixed point loci of these moduli spaces.

Torus-equivariant vector bundles on toric varieties were first introduced by Klyachko [Kly90] in terms of multi-filtrations of vector spaces satisfying certain compatibility conditions. This combinatorial description was extended to torsion free sheaves and applied to study moduli problems for $\mathbb{P}^2$ [Kly91]. Knutson and Sharpe extended Klyachko's work and made his results accessible for physicists [KS97]. A systematic approach to classify arbitrary coherent sheaves was proposed by Perling [Per04a]. Based on the idea of filtrations, he defined Δ-families and constructed moduli spaces of vector bundles of rank 2 on any smooth toric varieties [Per04b]. Later, Payne showed that the moduli space of toric vector bundles can be constructed as a locally closed subscheme of a product of partial flag varieties [Pay08].

Extending techniques used by Payne, Kool constructed coarse moduli spaces of pure toric sheaves on toric varieties via GIT [Koo11]. He matched μ-stability and Gieseker stability with GIT stability, and gave a combinatorial description of the fixed point loci of moduli spaces of μ-stable reflexive sheaves. Gholampour, Jiang and Kool extended Perling's Δ-families of toric varieties to S-families of toric stacks. For weighted projective stacks $\mathbb{P}(a, b, c)$, they showed that toric sheaves can be locally described by multi-filtrations with both torus grading and fine grading induced by the stabilizer group [GJK17].

One central object of studying moduli problems is to compute invariants associated to the moduli spaces such as the Euler characteristics, Donaldson-Thomas invariants and BPS invariants. Torus localization is an important tool for computing invariants when the underlying algebraic variety or stack is toric. The Euler characteristic of a quasi-projective variety with an regular torus action is the Euler characteristic of the fixed point locus. With the combinatorial description of the fix point loci of moduli spaces, Kool calculated explicitly the generating functions of Euler characteristics of moduli spaces of stable sheaves on $\mathbb{P}^2$ and the Hirzebruch surface [Koo15]. Following Nironi's construction on moduli spaces of sheaves on projective Deligne-Mumford stacks [Nir08b], Gholampour, Jiang and Kool calculated the generating functions for $\mathbb{P}(a, b, c)$ and proved their modularity for $\mathbb{P}(1, 1, 2)$ and $\mathbb{P}(1, 2, 2)$, generalizing Klyachko's result for $\mathbb{P}^2$ [GJK17]. We extended their work to another important class of toric stacks, namely Hirzebruch orbifolds.

The general problem of counting geometric objects is classical in enumerative geometry, among which curve counting has gained most attention. The Donaldson-Thomas invariants, introduced by Thomas and Donaldson [Tho00, DT98], are the virtual counts of stable sheaves on a Calabi-Yau threefold X. Several significant contributions have been made since then. To name a few, Maulik, Nekrasov, Okounkov and Pandharipande flourished the case of ideal sheaves of curves [MNOP06]. Okounkov and Pandharipande studied DT invariants for local curves by localization and degeneration methods [OP05]. Joyce and Song generalized DT invariants for semistable sheaves [JS12]. Recently, Bryan, Oberdieck, Pandharipande and Yin studied reduced DT invariants of abelian threefolds and related them to modular forms [BOPY15]. Toda showed that generating functions of generalized DT invariants of semistable sheaves on local $\mathbb{P}^2$ can be described in terms of modular forms [Tod17].

Other counting theories include Gromov-Witten theory, Gopakumar-Vafa theory and Pandharipande-Thomas theory. They share many properties and have close connections to Donaldson-Thomas theory. Maulik, Nekrasov, Okounkov, Pandharipande related GW theory of curves and DT theory of ideal sheaves in the case of local toric surfaces [MNOP06]. Katz proved the DT/GV correspondence for semistable sheaves on local contractible curves [Kat08]. Toda proved the unweighted Euler characteristic version of DT/PT correspondence for torsion free sheaves of rank one on a Calabi-Yau 3-fold [Tod10] and Bridgeland proved the weighted version [Bri11].

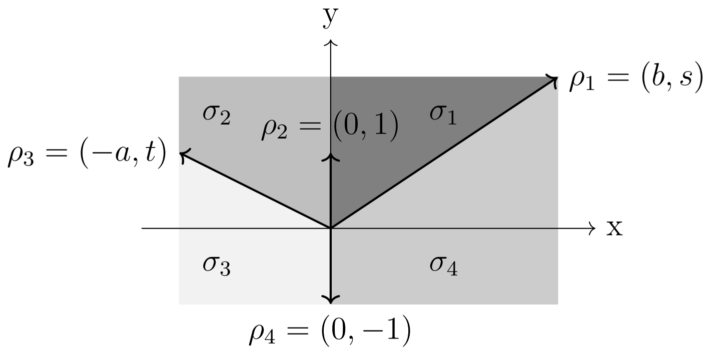
Donaldson-Thomas invariants are constructed based on the perfect obstruction theory on the moduli space of stable sheaves [BF97, LT98]. Consider the moduli space of stable sheaves on a Calabi-Yau threefold X with class α. The coarse moduli space, denote by $\mathcal{M}_s(X;\alpha)$, is a quasi-projective scheme [HL10]. When there are no strictly semistable sheaves, it admits a virtual fundamental class $[\mathcal{M}_s(X;\alpha)]^{vir}$. The Donaldson-Thomas invariant $DT(X;\alpha)$ is defined as $\int_{[\mathcal{M}_s(X;\alpha)]^{vir}} 1$. Behrend proved that DT invariants can also be expressed as the weighted Euler characteristic of some constructible function, called Behrend function [Beh09]. In the presence of strictly semistable sheaves, Joyce and Song introduced generalized DT invariants via PT invariants [JS12].

As an important application of torus-equivariant sheaves, Gholampour and Sheshmani studied the case when X is the total space of the canonical bundle of $\mathbb{P}^2$ [GS15b]. They showed that any compactly supported semistable sheaves are scheme theoretically supported on the zero section and $DT(X;\alpha)$ is the signed Euler characteristic of the moduli space of stable sheaves on $\mathbb{P}^2$ when there are no strictly semistable sheaves. They also adopted Joyce-Song's stable-pair theory [JS12] and tested the integrality property of the generalized DT invariants for certain classes of strictly semistable sheaves. We extend part of their results to the local Hirzebruch orbifold and will continue to investigate in the future work.

1.2 Outline

There is a nice class of vector bundles and projective bundles over toric varieties. They can be constructed from toric fans and hence are also toric varieties. This type of bundles has been well studied in the book [CLS11]. Given a fan, one can construct the line bundle corresponding to a Cartier divisor by extending the fan. Consequently, every vector bundle that can be decomposed into line bundles and its projectivization can be constructed from a fan.

This construction can be naturally generalized to the toric Deligne-Mumford stacks. Such stacks can be described by a stacky fan as in [BCS05]. In the chapter 2, we show that certain types of vector bundles can be constructed from stacky fans. As an application, we first give a general fan description of the weighted projective stacks. Then we construct projective bundles over weighted projective lines $\mathbb{P}(a,b)$ and describe the Hirzebruch stacks, denoted by $\mathcal{H}_r^{ab}$. When $\gcd(a,b)=1$, in which case $\mathcal{H}_r^{ab}$ is an orbifold, the stacky fan can be drawn as below. Here $s,t\in\mathbb{Z}$ are chosen so that $r=sa+bt$. Note that the fiber of the Hirzebruch surface over $\mathbb{P}^1$ is always $\mathbb{P}^1$. But this is not true for Hirzebruch stacks, in which case only the fiber over a non-stacky point is $\mathbb{P}^1$.



Let X be a nonsingular toric variety of dimension d. A. A. Klyachko [Kly90], M. Perling [Per04a] and M. Kool [Koo10] have given a combinatorial description of $\mathbf{T}$-equivariant coherent sheaves on toric varieties. The idea is that every toric variety can be covered by affine $\mathbf{T}$-equivariant subvarieties $U_\sigma \cong \mathbb{C}^d$, corresponding to the maximal cones in the fan. Locally, a sheaf is described by families of vector spaces, called σ-families. Those σ-families agree on the intersection of cones and satisfy some gluing conditions.

The above idea is generalized to smooth toric Deligne-Mumford stacks first by A. Gholampour, Y. Jiang and M. Kool in [GJK17]. Such stacks are covered by open substacks $\mathcal{U}_\sigma \cong [\mathbb{C}^d/N(\sigma)]$ [BCS05, Proposition 4.1]. Hence locally, a sheaf corresponds to a module with both $X(\mathbf{T})$-grading and $X(N(\sigma))$-fine-grading. The local data of such a sheaf consists of families of vector spaces with fine-gradings, called S-families. To obtain a sheaf globally, the gluing conditions are imposed. In the case of weighted projective stacks $\mathbb{P}(a, b, c)$, the gluing conditions are given explicitly in [GJK17].

In the chapter 3, we give the gluing conditions for Hirzebruch orbifolds. To glue the local data for any two substacks U_{σ_i} and $U_{\sigma_{i+1}}$, we pull back the local data to their stack theoretic intersection. Matching S-families over the intersection allows us to describe $\mathbf{T}$-equivariant coherent sheaves on Hirzebruch orbifolds. Then we can study torsion free sheaves and locally free sheaves on $\mathcal{H}_r^{ab}$ and construct the moduli spaces.

In the chapter 4, we investigate some basic properties of $\mathcal{H}_r^{ab}$ including its coarse moduli scheme and modified Hilbert polynomial. From F. Nironi's work [Nir08b], we know that a modified version of Hilbert polynomial is needed to define the Gieseker stability for stacks. Let ϵ be the structure morphism from $\mathcal{H}_r^{ab}$ to its coarse moduli scheme H. With fixed polarization L on H and generating sheaf $\mathcal{E}$ on $\mathcal{H}_r^{ab}$, we define the modified Hilbert polynomial for a sheaf $\mathcal{F}$ as

$$P_{\mathcal{E}}(\mathcal{F}, T) = \chi(\mathcal{H}_r^{ab}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \epsilon^* L^T)$$

and the modified Euler characteristic as

$$\chi_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{F}, 0)$$

In the chapter 5, we consider the moduli scheme of Gieseker stable and μ-stable torsion free sheaves of rank 1 and 2 on Hirzebruch orbifolds. Extending the work of [Koo10], we generalize the characteristic function and match the GIT stability with Gieseker stability. By lifting the action of the torus $\mathbf{T}$ to the moduli scheme $\mathcal{M}^{\mu s}_{P_\mathcal{E}}$ [Section 5.1], we can describe explicitly the fixed point locus $(\mathcal{M}^{\mu s}_{P_\mathcal{E}})^T$ by the GIT quotient $\mathcal{M}^{\mu s}_{\vec{\chi}}$ with gauge-fixed characteristic function $\vec{\chi}$ similar to [Koo10, Theorem 4.15].

In the case of rank 1, it leads to the counting of partitions, which generalizes L. Göttsche's result for nonsingular projective surface in [Göt90]. In the case of higher rank, we express the relation between generating functions of the moduli space of μ-stable torsion free and locally free sheaves, which generalizes L. Göttsche's result in [Göt99].

Theorem 1.2.1. *Suppose* $\gcd(a,b) = 1$*. Let* $P_\mathcal{E}$ *be a choice of modified Hilbert polynomial of a reflexive sheaf of rank* R *on* $\mathcal{H}^{ab}_r$ *and* $\chi_\mathcal{E}$ *be the modified Euler characteristic. Then*

$$\sum_{\chi_\mathcal{E}\in\mathbb{Z}} e(M_{\mathcal{H}^{ab}_r}(R,c_1,\chi_\mathcal{E}))\mathrm{q}^{\chi_\mathcal{E}} = \prod_{k=1}^{\infty} \frac{\sum_{\chi_\mathcal{E}\in\mathbb{Z}} e(M^{vb}_{\mathcal{H}^{ab}_r}(R,c_1,\chi_\mathcal{E}))\mathrm{q}^{\chi_\mathcal{E}}}{(1-\mathrm{q}^{-ak})^{2R}(1-\mathrm{q}^{-bk})^{2R}}.$$

We compute the generating function $H^{\mathrm{vb}}_{c_1}(\mathrm{q}) := \sum e(M^{\mathrm{vb}}_{\mathcal{H}^{ab}_r}(2,c_1,\chi_\mathcal{E}))\mathrm{q}^{\chi_\mathcal{E}}$ for locally free sheaves over Hirzebruch orbifolds $\mathcal{H}^{ab}_r$ with fixed generating sheaf $\mathcal{E}$ and polarization L given in [Section 4.4]. Especially when $r = 0$, we obtain an expression for the orbifold $\mathbb{P}(a,b)\times\mathbb{P}^1$, which is parallel to M. Kool's result for $\mathbb{P}^1\times\mathbb{P}^1$ [Koo10, Corollary 2.3.4].

Theorem 1.2.2. *Suppose* $\gcd(a,b) = 1$*. Let* $f = (\frac{n}{2}+1)(m+C)$ *where* $C = a+b+ab-1$*. Let* $p = \gcd(b,r) = b$ *and* $q = \gcd(a,r) = a$ *as* $r = 0$*. Then for fixed first Chern class* $c_1(\mathcal{F}) = \frac{m}{a}x + ny$ *where* $c_1(\mathcal{D}_{\rho_1}) = x, c_1(\mathcal{D}_{\rho_2}) = y$*,* $\mathcal{D}_{\rho_i}$ *is the* *divisor corresponding to the ray ρ_i, the generating function $H^{vb}_{c_1}(\mathrm{q})$ for the orbifold $\mathbb{P}(a,b)\times\mathbb{P}^1$ is*

$$\left(-\sum_{C_1}+\sum_{C_4}+\sum_{C_5}+2\sum_{C_6}\right)\mathrm{q}^{f-\frac{1}{2}ji}+\left(2\sum_{C_2}+2\sum_{C_3}\right)\mathrm{q}^{f-\frac{1}{4}ij+\frac{1}{4}jk-\frac{1}{4}kl-\frac{1}{4}li}$$

where

$$\begin{aligned}
C_1 &= \{(i,j,k,l)\in\mathbb{Z}^4 : 2\mid m+i, 2\mid n+j, 2\mid j-l, 2b\mid i-k, 2a\mid i+k, i=pqj,\\
&\qquad -j<l<j, -pqj<k<pqj\},\\
C_2 &= \{(i,j,k,l)\in\mathbb{Z}^4 : 2\mid m+i, 2\mid n+j, 2\mid j-l, 2b\mid i-k, 2a\mid i+k,\\
&\qquad -i<k<pql<i, -pqj<k, l<j\},\\
C_3 &= \{(i,j,k,l)\in\mathbb{Z}^4 : 2\mid m+i, 2\mid n+j, 2\mid j-l, 2a\mid i-k, 2b\mid i+k,\\
&\qquad -i<k<pql<i, -pqj<k, l<j\},\\
C_4 &= \{(i,j,k)\in\mathbb{Z}^3 : 2\mid m+i, 2\mid n+j, 2\mid j+k, b\mid i, -\frac{i}{pq}<k<\frac{i}{pq}<j\},\\
C_5 &= \{(i,j,k)\in\mathbb{Z}^3 : 2\mid m+i, 2\mid n+j, 2\mid j+k, a\mid i, -\frac{i}{pq}<k<\frac{i}{pq}<j\},\\
C_6 &= \{(i,j,k)\in\mathbb{Z}^3 : 2\mid m+i, 2\mid n+j, 2a\mid i+k, 2b\mid i-k,\\
&\qquad -pqj<k<pqj<i\}.
\end{aligned}$$

Moreover, in the case of $a=1, b=2$, we can get more explicit expressions [Proposition 5.2.5].

In the last chapter, we study the Donaldson-Thomas invariant $DT(\mathcal{X};\alpha)$ when

$\mathcal{X}$ is the total space of the canonical bundle over a Hirzebruch orbifold $\mathcal{H}_r^{ab}$ and α is the class of a compactly-supported coherent sheaf. If a 2-dimensional coherent sheaf $\mathcal{F}$ is semistable, then we can show that $\mathcal{F}$ is stack theoretically supported on the zero section. Hence we can use the result of chapter 5 to calculate DT invariants.

However, in the orbifold case, sheaves of different K-group classes might have same modified Euler characteristic. We need to count colored partitions to track K-group classes. In the case when $r = 0$, we can obtain an explicit formula for the generating function of Donaldson-Thomas invariants $DT(\mathcal{X};\alpha)$.

Theorem 1.2.3. *Suppose* $\gcd(a,b) = 1$. *Let* $\mathcal{X}$ *be the total space of the canonical bundle over* $\mathbb{P}(a,b)\times\mathbb{P}^1$ *and* $i : \mathcal{S} \cong \mathbb{P}(a,b)\times\mathbb{P}^1 \hookrightarrow \mathcal{X}$ *be the inclusion of the zero section. Denote by* $g := [(-1,0)], h := [(0,-1)]$ *the* K*-group classes of the generators of* $\operatorname{Pic}(\mathbb{P}(a,b)\times\mathbb{P}^1) \cong \mathbb{Z}\oplus\mathbb{Z}$*. Then*

$$\begin{aligned}
&\sum_{\alpha\in K_0(\mathcal{X})} DT\big(\mathcal{X};\alpha) p_{ia}^{\#p_{ia}} q_{jb}^{\#q_{jb}} \\
= \quad & (-1)^{a+b} G^2(-p_0, \underbrace{p_{ia}}_{i\neq 0,b-1}, -p_{(b-1)a}) H'^2(-q_0, \underbrace{q_{jb}}_{j\neq 0}; -p_0, \underbrace{p_{ia}}_{i\neq 0,b-1}, -p_{(b-1)a}),
\end{aligned}$$

where α *is given by*

$$i_*\Big(1 - \sum_{i=0}^{b-1} \#p_{ia}(1-g^a)(1-h)g^i - \sum_{j=0}^{a-2} \#q_{jb}(1-g^b)(1-h)g^j\Big)$$

and $G(p_{ia})$, $H'(q_{jb};p_{ia})$ are defined as

$$\begin{aligned}
G(p_{ia}) = & \quad \frac{1}{\prod_{k\geq 0}\prod_{i=0}^{b-1}\left(1-p_0p_a\cdots p_{ia}(p_0p_a\cdots p_{(b-1)a})^k\right)}, \\
H'(q_{jb};p_{ia}) = & \quad \frac{1}{\prod_{k>0}\left(1-(p_0p_a\cdots p_{(b-1)a})^k\right)} \\
& \cdot \frac{1}{\prod_{k\geq 0}\prod_{j=0}^{a-2}\left(1-q_0q_b\cdots q_{jb}(p_0p_a\cdots p_{(b-1)a})^k\right)}.
\end{aligned}$$

Chapter 2: Toric Stacks

In this section, we will briefly review various definitions of stacky fans and their associated toric Deligne-Mumford stacks. Toric stacks were first introduced in [BCS05] and later in [FMN10]. The theory was further generalized in [GS15a] which encompasses all the notions of toric stacks before. In this paper, we will refer to [BCS05] the notation of toric stacks most of the time, but use [GS15a] when constructing the vector bundles.

Definition 2.0.1. A *stacky fan* [BCS05] is a triple $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$ where

- N is a finitely generated abelian group of rank d, not necessarily free.
- Σ is a rational simplicial fan in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ with n rays, denoted by $\rho_1, ..., \rho_n$.
- $\beta : \mathbb{Z}^n \to N$ is a homomorphism with finite cokernel such that $\beta(e_i) \otimes 1 \in N_{\mathbb{Q}}$ is on the ray ρ_i for $1 \leq i \leq n$.

Given a stacky fan, the way to construct its corresponding toric stack $[Z_\Sigma/G_\beta]$ is as follows:

The variety Z_Σ is defined as $\mathbb{C}^n - V(J_\Sigma)$ where $J_\Sigma = \langle \prod_{\rho_i \not\subset \sigma} z_i \,|\, \sigma \in \Sigma \rangle$ is a reduced monomial ideal. Suppose N is of rank d, then there exists a free resolution $0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to N \to 0$ of N. Let the matrix $B : \mathbb{Z}^n \to \mathbb{Z}^{d+r}$ be a lift of the map

$\beta : \mathbb{Z}^n \to N$. Define the dual group $\mathrm{DG}(\beta) = (\mathbb{Z}^{n+r})^\star/\mathrm{Im}([B\,Q]^\star)$, where $(-)^\star$ is the dual $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{Z})$. Let $\beta^\vee : (\mathbb{Z}^n)^\star \to \mathrm{DG}(\beta)$ be the composition of the inclusion map $(\mathbb{Z}^n)^\star \to (\mathbb{Z}^{n+r})^\star$ and the quotient map $(\mathbb{Z}^{n+r})^\star \to \mathrm{DG}(\beta)$. By applying the functor $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to $\beta^\vee$, we get a homomorphism $G_\beta := \mathrm{Hom}_\mathbb{Z}(DG(\beta), \mathbb{C}^*) \to (\mathbb{C}^*)^n$ which leaves Z_Σ invariant.

The quotient stack $[Z_\Sigma/G_\beta]$ is called the *toric Deligne-Mumford stack* associated to the stacky fan Σ.

Definition 2.0.2. A (non-strict) *stacky fan* [GS15a] is a pair $(\Sigma, \beta : L \to N)$, where Σ is a fan on the lattice L and N is a finitely generated abelian group.

Remark 2.0.3. Since the fan is defined on L instead of N, we are allowed to assume that β is of not finite cokernel. Interested readers can read [GS15a] for more details. In our paper, we will only consider β with the finite cokernel, in which case the construction of G_β in [GS15a] essentially agrees with [BCS05].

Remark 2.0.4. The stacky fan defined in Definition 2.0.1 is a special case of Definition 2.0.2. When N is free, the toric stack arising from such a stacky fan is called a *fantastack* in [GS15a]. When N is not free, the toric stack can be realized as a closed substack of a fantastack, called the *non-strict fantastack*.

Let $\beta : L = \mathbb{Z}^n \to N = \mathbb{Z}^d$ be a homomorphism with the finite cokernel as in Definition 2.0.1. Given a cone $\sigma \in \Sigma$ in N, set $\widehat{\sigma} = \text{cone}\left(\{e_i | \rho_i \in \sigma\}\right)$ where $\{e_i\}_{i=1}^n$ is the standard basis for L. Define $\widehat{\Sigma}$ in L as the fan generated by all the cones $\widehat{\sigma}$. Then the stack defined by a triple $(N, \Sigma, \beta : L \to N)$ [GS15a] is same as the stack defined by a pair $(\widehat{\Sigma}, \beta : L \to N)$ [GS15a]. Conversely, if the rays of $\widehat{\Sigma}$ in L are e_i, the image of $\widehat{\Sigma}$ under β is a stacky fan Σ in N. Since these two definitions agree in the case of the fantastack, we will use them interchangeably when constructing the vector bundle.

Remark 2.0.5. In general, Σ can be a non-complete fan in L. A non-complete fantastack is essentially the *extended* toric Delign-Mumford stack defined in [Jia08].

2.1 Weighted Projective Stack

Let $w_1, w_2, ..., w_{n+1} \in \mathbb{Z}_{>0}$. The weighted projective stack $\mathbb{P}(w_1, ..., w_{n+1})$ is the quotient stack $[\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*]$ where $\mu \in \mathbb{C}^*$ acts by $\mu(x_1, ..., x_{n+1}) = (\mu^{w_1}x_1, ..., \mu^{w_{n+1}}x_{n+1})$. We will give a general description of the stacky fan for the weighted projective stack. Firstly, we assume $\gcd(w_1, ..., w_{n+1}) = 1$, which means $\mathbb{P}(w_1, ..., w_{n+1})$ is an orbifold and the lattice N is free.

Proposition 2.1.1. *Let* $\gcd(w_i, ..., w_{n+1}) = \lambda_i$ *for* $1 \leq i \leq n$. *Suppose* $\lambda_1 = 1$. *Define the map* $\beta : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ *by the following* $n \times (n+1)$ *matrix* B*:*

$$\begin{bmatrix} \frac{\lambda_2}{\lambda_1} & b_{12} & \cdots & b_{1,i-1} & b_{1i} & b_{1,i+1} & \cdots & b_{1,n-1} & b_{1n} & b_{1,n+1} \\ \vdots & & \ddots & & & & & & & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_i}{\lambda_{i-1}} & b_{i-1,i} & b_{i-1,i+1} & \cdots & b_{i-1,n-1} & b_{i-1,n} & b_{i-1,n+1} \\ 0 & 0 & \cdots & 0 & \frac{\lambda_{i+1}}{\lambda_i} & b_{i,i+1} & \cdots & b_{i,n-1} & b_{i,n} & b_{i,n+1} \\ \vdots & & & & & & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{\lambda_n}{\lambda_{n-1}} & b_{n-1,n} & b_{n-1,n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{w_{n+1}}{\lambda_n} & -\frac{w_n}{\lambda_n} \end{bmatrix}$$

where b_{ij} are chosen so that

$$\frac{\lambda_{i+1}}{\lambda_i} w_i + \sum_{j=i+1}^{n+1} b_{ij} w_j = 0 \text{ for } 1 \leq i \leq n-1, \tag{2.1.1}$$

$$0 \leq b_{1i}, b_{2i}, \cdots, b_{ii} < \frac{\lambda_{i+1}}{\lambda_i} \text{ for } 2 \leq i \leq n-1.$$

Each column represents a ray in the fan Σ. The maximal cones of the fan are given by any n rays. Then the triple $(\mathbb{Z}^n, \Sigma, \beta)$ corresponds to the weighted projective orbifold $\mathbb{P}(w_1, ..., w_{n+1})$.

Note that the choice is not unique.

Proof. The triple induces $\mathbb{P}(w_1, ..., w_{n+1})$ if the following two statements are true:

- $\mathrm{DG}(\beta) = \mathbb{Z}$.
- $\beta^\vee : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is given by $\begin{bmatrix} w_1 & w_2 & ... & w_{n+1} \end{bmatrix}$.

If $\mathrm{DG}(\beta) = \mathbb{Z}^{n+1}/\mathrm{Im}(B^\star) \cong \mathbb{Z}$, then $\begin{bmatrix} w_1 & w_2 & ... & w_{n+1} \end{bmatrix}$ spans the integer null space of the matrix B because b_{ij} are chosen to satisfy (2.1.1). Let B_i denote the minor of B by removing the ith column. If we can show that $\gcd(\det(B_1), ..., \det(B_{n+1})) = 1$, then there exists a matrix $\begin{bmatrix} \boldsymbol{b} \\ B \end{bmatrix}$ with determinant 1. Hence $\mathbb{Z}^{n+1}/\mathrm{Im}(B^\star) = \mathbb{Z}$.

When $i = n, n+1$, we obtain two diagonal matrices and $\det(B_{n+1}) = w_{n+1}, \det(B_n) = -w_n$. For $1 \leq i \leq n-1$, we compute by induction that

$\det(B_i) = (-1)^{n+1-i} w_i$. Denote by

$$C_i = \begin{bmatrix} b_{i,i+1} & \cdots & b_{i,n-1} & b_{i,n} & b_{i,n+1} \\ \frac{\lambda_{i+2}}{\lambda_{i+1}} & \cdots & b_{i+1,n-1} & b_{i+1,n} & b_{i+1,n+1} \\ & \ddots & & & \vdots \\ 0 & \cdots & \frac{\lambda_n}{\lambda_{n-1}} & b_{n-1,n} & b_{n-1,n+1} \\ 0 & \cdots & 0 & \frac{w_{n+1}}{\lambda_n} & -\frac{w_n}{\lambda_n} \end{bmatrix}$$

the bottom-right $(n-i+1) \times (n-i+1)$ submatrix of B, then $\det(B_i) = \lambda_i \cdot \det(C_i)$.

For $i = n-1$, because $\gcd(w_n, w_{n+1}) = \lambda_n$ and $\lambda_{n-1} | w_{n-1}$, integers $b_{n-1,n}$ and $b_{n-1,n+1}$ can be chosen so that

$$\det(C_{n-1}) = -b_{n-1,n} \frac{w_n}{\lambda_n} - b_{n-1,n+1} \frac{w_{n+1}}{\lambda_n} = \frac{w_{n-1}}{\lambda_{n-1}}.$$

Suppose integers $b_{i,i+1}, ..., b_{i,n+1}$ are chosen so that

$$\det(C_i) = (-1)^{n-i} \sum_{j=i+1}^{n+1} b_{i,j} \frac{w_j}{\lambda_{i+1}} = (-1)^{n+1-i} \frac{w_i}{\lambda_i},$$

then we can expand the matrix C_{i-1} by the first column and get

$$\begin{aligned} \det(C_{i-1}) &= b_{i-1,i} \det(C_i) - \frac{\lambda_{i+1}}{\lambda_i} \det(C_i') \\ &= (-1)^{n+1-i} b_{i-1,i} \frac{w_i}{\lambda_i} - (-1)^{n-i} \frac{\lambda_{i+1}}{\lambda_i} \sum_{j=i+1}^{n+1} b_{i-1,j} \frac{w_j}{\lambda_{i+1}} \\ &= (-1)^{n-i} \frac{w_{i-1}}{\lambda_{i-1}}, \end{aligned}$$

where C_i' is the submatrix of C_{i-1} by removing the first column and the second row.

Now we get $\det(B_i) = (-1)^{n-i} w_i$ and $\gcd(\det(B_1), ..., \det(B_{n+1})) = 1$.

If $b_{ji} \geq \frac{\lambda_{i+1}}{\lambda_i}$ or $b_{ji} < 0$, then we can left multiply a elementary matrix and the integer null space will be unchanged. □

Example 2.1.2. Consider the stack $\mathbb{P}(1, 2, 4, 8)$. Since $\gcd(2, 4, 8) = 2$, $\gcd(4, 8) = 4$, the matrix for $\beta : \mathbb{Z}^4 \to \mathbb{Z}^3$ will be

$$\begin{bmatrix} 2 & a & b & c \\ 0 & 2 & d & e \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

such that $4 + 4d + 8e = 0, 2 + 2a + 4b + 8c = 0$. One of the solutions for this system is as follows:

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

When $\lambda_1 \neq 1$, the lattice N is not free and can be identified as $\mathbb{Z}^n \oplus \mathbb{Z}/\lambda_1\mathbb{Z}$. In this case, $\mathbb{P}(w_1, ..., w_{n+1})$ is a μ_{λ_1}-banded gerbe over $\mathbb{P}(\frac{w_1}{\lambda_1}, ..., \frac{w_{n+1}}{\lambda_1})$, which is isomorphic to the root stack [FMN10]

$$\sqrt[\lambda_1]{\mathcal{O}_{\mathbb{P}(\frac{w_1}{\lambda_1}, ..., \frac{w_{n+1}}{\lambda_1})}(1)/\mathbb{P}(\frac{w_1}{\lambda_1}, ..., \frac{w_{n+1}}{\lambda_1})}.$$

Proposition 2.1.3. *Choose $c_1, ..., c_{n+1}$ so that $\sum_{i=1}^{n+1} c_i \frac{w_i}{\lambda_1} \equiv 1 \mod \lambda_1$. Set $\boldsymbol{c} = ([c_1], ..., [c_{n+1}])$ where $[c_i]$ is the class of c_i modulo λ_1. Let B' the matrix corresponding* *to* $\mathbb{P}(\frac{w_1}{\lambda_1}, ..., \frac{w_{n+1}}{\lambda_1})$ *as in Proposition 2.1.1. Define the map* $\beta : \mathbb{Z}^{n+1} \to \mathbb{Z}^n \oplus \mathbb{Z}/\lambda_1\mathbb{Z}$ *by* $B = \begin{bmatrix} B' \\ \\ \boldsymbol{c} \end{bmatrix}$. *Then the triple* $(\mathbb{Z}^n, \Sigma, \beta)$ *corresponds to the weighted projective stack* $\mathbb{P}(w_1, ..., w_{n+1})$.

Proof. The $[B\,Q]$ matrix as in [BCS05] is given by $\begin{bmatrix} B' & \boldsymbol{0} \\ \\ \boldsymbol{c} & \lambda_1 \end{bmatrix}$. Since $\sum_{i=1}^{n+1} c_i \frac{w_i}{\lambda_1} \equiv 1 \mod \lambda_1$, the vector $\begin{bmatrix} w_1 & w_2 & \cdots & w_{n+1} & * \end{bmatrix}$ spans the integer null space of the matrix $[B\,Q]$. $\square$

2.2 Vector Bundles

In [CLS11], it mentions a class of toric morphisms that have a nice local structure. This can be naturally generalized to the morphisms of fantastacks.

Let N_1, N_2 be free abelian groups. Denote the bases of $\mathbb{Z}^{n_1}$ and $\mathbb{Z}^{n_2}$ by $\{e_1, ..., e_{n_1}\}$ and $\{e_{n_1+1}, ..., e_{n_1+n_2}\}$. By abuse of notation, we also assume the basis of $\mathbb{Z}^{n_1+n_2}$ is $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_{n_1+n_2}\}$. Consider the exact sequence of the fantastacks given by a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{n_1} \arrow[r, "{(\mathrm{Id},0)}"] \arrow[d, "\beta_1"] & \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\beta"] & \mathbb{Z}^{n_2} \arrow[r] \arrow[d, "\beta_2"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "f"] & N_1 \oplus N_2 \arrow[r, shift left] & N_2 \arrow[l, shift left, "g"] \arrow[r] & 0
\end{tikzcd} \tag{1.2.1}$$

such that the rows are exact and the column morphisms are of the finite cokernel. Suppose there exists a splitting morphism g satisfying the following conditions:

1. A is a $\mathrm{rk}N_1 \times \mathrm{rk}N_2$ integer matrix such that

$$\beta(e_i) = \begin{cases} f(\beta_1(e_i)) = \begin{bmatrix} \beta_1(e_i) \\ 0 \end{bmatrix} & \text{if } 1 \le i \le n_1 \\ g(\beta_2(e_i)) = \begin{bmatrix} A\beta_2(e_i) \\ \beta_2(e_i) \end{bmatrix} & \text{if } n_1 + 1 \le i \le n_1 + n_2. \end{cases}$$

2. Given cones $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the sum $\sigma_1 + \sigma_2$ lies in Σ, and every cone of Σ arises this way.

Then we say $(\Sigma, \beta : \mathbb{Z}^{n_1+n_2} \to N_1 \oplus N_2)$ is *globally split* by $(\Sigma_1, \beta_1 : \mathbb{Z}^{n_1} \to N_1)$ and $(\Sigma_2, \beta_2 : \mathbb{Z}^{n_2} \to N_2)$.

Theorem 2.2.1. *If $(\Sigma, \beta : \mathbb{Z}^{n_1+n_2} \to N_1 \oplus N_2)$ is globally split by $(\Sigma_1, \beta_1 : \mathbb{Z}^{n_1} \to N_1)$ and $(\Sigma_2, \beta_2 : \mathbb{Z}^{n_2} \to N_2)$, then $\mathcal{X}_{\Sigma,\beta} \cong \mathcal{X}_{\Sigma_1,\beta_1} \times \mathcal{X}_{\Sigma_2,\beta_2}$.*

Proof. Denote the matrices for β_1 and β_2 by

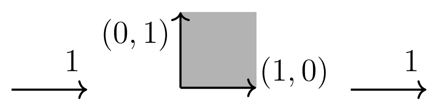
$$B_1 = \begin{bmatrix} \beta_1(e_1) & \beta_1(e_2) & \cdots & \beta_1(e_{n_1}) \end{bmatrix},$$
$$B_2 = \begin{bmatrix} \beta_2(e_{n_1+1}) & \beta_2(e_{n_1+2}) & \cdots & \beta_2(e_{n_1+n_2}) \end{bmatrix}.$$

The matrix for β is given by $B = \begin{bmatrix} B_1 & AB_2 \\ 0 & B_2 \end{bmatrix}$. It is not hard to show that $DG(\beta) \cong DG(\beta_1) \oplus DG(\beta_2)$ and $\beta^\vee \cong \beta_1^\vee \oplus \beta_2^\vee$, which implies $\alpha \cong \alpha_1 \times \alpha_2$, where α, α_1 and α_2 are obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to $\beta^\vee, \beta_1^\vee, \beta_2^\vee$.

It remains to show $Z_\Sigma = Z_{\Sigma_1} \times Z_{\Sigma_2}$. The $\mathbb{C}$-valued points of Z_Σ are $z \in \mathbb{C}^{n_1+n_2}$

such that the cone generated by the set $\{\rho_i : z_i = 0\}$, where ρ_i is the cone generated by b_i in $N_{\mathbb{Q}}$, belongs to Σ. Since every cone of Σ is the sum of cones in Σ_1 and Σ_2, the $\mathbb{C}$-valued points of Z_Σ are exactly the product of $\mathbb{C}$-valued points of Z_{Σ_1} and Z_{Σ_2}. $\square$

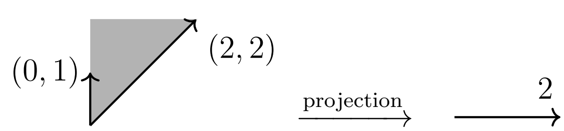
Example 2.2.2. Consider the following exact sequence of fantastacks



$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^1 \arrow[r] \arrow[d, "1"] & \mathbb{Z}^2 \arrow[r] \arrow[d, "\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}"] & \mathbb{Z}^1 \arrow[r] \arrow[d, "2"] & 0 \\
0 \arrow[r] & \mathbb{Z}^1 \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^1 \arrow[r] & 0.
\end{tikzcd}$$

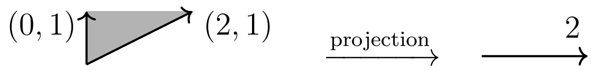
It can be shown that $\mathcal{X}_{\Sigma,\beta} = [\mathbb{C}^2/\mu_2] \cong \mathbb{C} \times [\mathbb{C}/\mu_2] = \mathcal{X}_{\Sigma_1,\beta_1} \times \mathcal{X}_{\Sigma_2,\beta_2}$.

Remark 2.2.3. The above exact sequence of fantastacks can be better understood if we draw the corresponding stacky fans defined in [BCS05]. The morphism from the middle stacky fan to the right can be viewed as the projection of rays from the lattice $\mathbb{Z}^2$ to $\mathbb{Z}$,



which is compatible with $\mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma_2,\beta_2}$ induced from the projection onto the second coordinate.

Remark 2.2.4. The morphism of stacky fans below corresponds to a morphism of stacks $\mathcal{X}_{\Sigma,\beta} \to [\mathbb{C}^1/\mu_2]$. Indeed, $\mathcal{X}_{\Sigma,\beta}$ is a line bundle over $[\mathbb{C}^1/\mu_2]$ whose fiber over the stacky point corresponds to the non-trivial representation of μ_2. Hence the stacky fan of $\mathcal{X}_{\Sigma,\beta}$ is not globally split.



With the above theorem and examples in mind, we can generalize [CLS11, Definition 3.3.18].

Definition 2.2.5. Given an exact sequence like (1.2.1), we say $(\Sigma, \beta : \mathbb{Z}^{n_1+n_2} \to N_1 \oplus N_2)$ is *(locally) split* by $(\Sigma_1, \beta_1 : \mathbb{Z}^{n_1} \to N_1)$ and $(\Sigma_2, \beta_2 : \mathbb{Z}^{n_2} \to N_2)$ if there exists a morphism $g : N_2 \to N_1 \oplus N_2$ satisfying the following conditions:

1. For every maximal cone $\sigma_j \in \Sigma_2$, there exists an $\mathrm{rk}N_1 \times \mathrm{rk}N_2$ integer matrix A_j such that

$$\beta(e_i) = \begin{cases} f(\beta_1(e_i)) = \begin{bmatrix} \beta_1(e_i) \\ 0 \end{bmatrix} & \text{if } 1 \leq i \leq n_1 \\ g(\beta_2(e_i)) = \begin{bmatrix} A_i\beta_2(e_i) \\ \beta_2(e_i) \end{bmatrix} & \text{if } e_i \in \sigma_j. \end{cases}$$

2. Given cones $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the sum $\sigma_1 + \sigma_2$ lies in Σ, and every cone of Σ arises this way.

Remark 2.2.6. The map g here essentially gives the bijection $\sigma' \to \hat{\sigma}$ for the case of toric varieties in [CLS11, Definition 3.3.18].

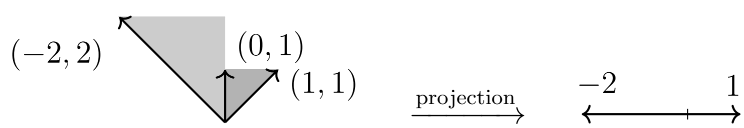
Theorem 2.2.7. *If $(\Sigma, \beta : \mathbb{Z}^{n_1+n_2} \to N_1 \oplus N_2)$ is (locally) split by $(\Sigma_1, \beta_1 : \mathbb{Z}^{n_1} \to N_1)$ and $(\Sigma_2, \beta_2 : \mathbb{Z}^{n_2} \to N_2)$, then $\phi : \mathcal{X}_{\Sigma,\beta} \to \mathcal{X}_{\Sigma_2,\beta_2}$ is a locally trivial fiber bundle with fiber $\mathcal{X}_{\Sigma_1,\beta_1}$, i.e., $\mathcal{X}_{\Sigma_2,\beta_2}$ has a cover by affine open substacks $\mathcal{U}$ satisfying $\phi^{-1}(\mathcal{U}) \cong \mathcal{X}_{\Sigma_1,\beta_1} \times \mathcal{U}$.*

Proof. The proof is similar to that of [CLS11, Theorem 3.3.19]. □

Therefore, if the stacky fan of a vector bundle is locally split, then for every stacky point of the base, the representation of the stabilizer group at that point on the fiber is trivial.

Note that the above theorem can be generalized to the case where N_1 and N_2 are not free.

Example 2.2.8. Consider the following morphism of stacky fans.



The induced morphism $\phi : \mathcal{X}_{\Sigma,\beta} \to \mathbb{P}(2,1)$ corresponds to a line bundle such that its fan is locally split. But it cannot be written globally as the product of one-dimensional toric stacks. Indeed, it represents $\mathcal{O}_{\mathbb{P}(2,1)}(-4)$ by the next theorem.

For a vector bundle over a stack, the fiber over a stacky point might correspond to a non-trivial representation of the stabilizer group. In this case, the corresponding stacky fan is not locally split. To include this type of stacky vector bundles, we generalize [CLS11, Sec. 7.3] to the case of toric stacks.

Let's assume N is free. Given a triple $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$, we define the new stacky fan $(N \times \mathbb{Z}, \widetilde{\Sigma}, \widetilde{\beta} : \mathbb{Z}^{n+1} \to N \times \mathbb{Z})$ as follows:

1. $\widetilde{\beta}(e_i) = (\beta(e_i), -a_i)$ for $1 \leq i \leq n$.

2. $\widetilde{\beta}(e_{n+1}) = (\mathbf{0}, 1)$.

3. Given $\sigma \in \Sigma$, set $\widetilde{\sigma} = \operatorname{Cone}\Big((\mathbf{0}, 1),\, \widetilde{\beta}(e_i) \otimes 1 \,|\, \beta(e_i) \otimes 1 \in \sigma(1)\Big) \in N_{\mathbb{Q}} \times \mathbb{Q}$, and let $\widetilde{\Sigma}$ be the set consisting of $\widetilde{\sigma}$ for all $\sigma \in \Sigma$ and their faces.

The natural projection $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is compatible with $\widetilde{\Sigma}$ and Σ. Therefore it gives a toric morphism $\pi : \mathcal{X}_{\widetilde{\Sigma}, \widetilde{\beta}} \to \mathcal{X}_{\Sigma, \beta}$.

Theorem 2.2.9. *Denote by $\mathcal{D}_{\rho_i}$ the divisor corresponding to the ray ρ_i. Then $\pi : \mathcal{X}_{\widetilde{\Sigma}, \widetilde{\beta}} \to \mathcal{X}_{\Sigma, \beta}$ is a line bundle whose sheaf of sections is*

$$\mathcal{O}_{\mathcal{X}_{\Sigma,\beta}}(\mathcal{D}) = \mathcal{O}_{\mathcal{X}_{\Sigma,\beta}}(\sum_i a_i \mathcal{D}_{\rho_i}).$$

Recall that the category of locally free sheaves on $[Z/G]$ is equivalent to that of G-linearized locally free sheaves on Z. These G-linearized invertible sheaves L_i, without considering the equivariant structure, are all isomorphic to the trivial sheaf $\mathcal{O}_Z$. By the construction of a toric stack, G can be thought of as a subgroup of $(\mathbb{C}^*)^n$. Each $g = (\lambda_1, ..., \lambda_n) \in G$ induces an isomorphism $\mathcal{O}_Z \to g^*\mathcal{O}_Z$ sending 1 to λ_i. The sheaf L_i has a G-invariant global section z_i such that $g^* z_i = \lambda_i z_i$ and corresponds to $\mathcal{O}_{\Sigma,\beta}(\mathcal{D}_{\rho_i})$. [BH06]

Proof of Theorem 2.29. We will use the definition of stacky fan from [GS15a].

Given a triple $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$, we can construct the corresponding fan $\widehat{\Sigma}$ in $\mathbb{Z}^n$, which corresponds to a toric variety $Z_{\widehat{\Sigma}}$. Then by [CLS11], we can construct a new fan $\widehat{\Sigma}' \in \mathbb{Q}^n \times \mathbb{Q}$. Given $\widehat{\sigma} \in \widehat{\Sigma}$, set $\widehat{\sigma}' = \operatorname{Cone}((\mathbf{0}, 1), (e_i, -a_i) | e_i \in \widehat{\sigma})$ and let $\widehat{\Sigma}'$ be the set consisting of cones $\widehat{\sigma}'$ for all $\widehat{\sigma} \in \widehat{\Sigma}$ and their faces. By [CLS11, Proposition 7.3.1], $\pi : Z_{\widehat{\Sigma}'} \to Z_{\widehat{\Sigma}}$ is a line bundle whose sheaf of sections is $\mathcal{O}_{Z_{\widehat{\Sigma}}}(\sum_i a_i D_{e_i})$ where D_{e_i} is the divisor corresponding to the ray generated by e_i in $\widehat{\Sigma}$. Note that $\widehat{\Sigma}$ is not a complete fan, but the proposition still keeps true.

It suffices to show that the G_β-linearizion of this bundle exists and the action of G_β on Z_Σ can be lifted . Define $\widehat{\beta}' : \mathbb{Z}^n \times \mathbb{Z} \to N \times \mathbb{Z}$ by the following matrices

$$\begin{bmatrix} \beta(e_1) & \cdots & \beta(e_n) & \mathbf{0} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then $G_{\widehat{\beta}'} \cong G_\beta$ and its action on the line bundle is compatible with the action of G_β on Z_Σ. The toric stack $\mathcal{X}_{\widehat{\Sigma}', \widehat{\beta}'}$ defined by the stacky fan $(\widehat{\Sigma}', \widehat{\beta}' : \mathbb{Z}^n \times \mathbb{Z} \to N \times \mathbb{Z})$ induces the above line bundle.

However, the rays of $\widehat{\Sigma}'$ do not form a standard basis. Hence $\mathcal{X}_{\widehat{\Sigma}', \widehat{\beta}'}$ is not a fantastack and it is not a stacky fan defined in Definition 2.0.1.

Consider the morphism of stacky fans given by the following commutative diagram

$$\widetilde{\Sigma} \longrightarrow \widehat{\Sigma}'$$

$$\begin{tikzcd} \mathbb{Z}^n \times \mathbb{Z} \arrow[r, "\alpha"] \arrow[d, "\widetilde{\beta} := \widehat{\beta}' \circ \alpha"] & \mathbb{Z}^n \times \mathbb{Z} \arrow[d, "\widehat{\beta}'"] \\ N \times \mathbb{Z} \arrow[r, "\cong"] & N \times \mathbb{Z} \end{tikzcd}$$

where α is defined by the matrix

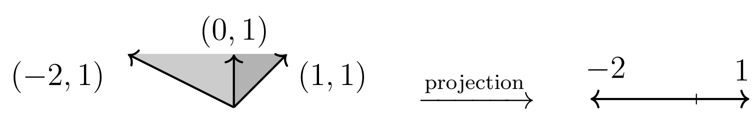
$$\begin{bmatrix} & I_n & & & \mathbf{0} \\ -a_1 & -a_2 & \cdots & -a_n & 1 \end{bmatrix}$$

and I_n is the $n \times n$ identity matrix. Let $\widetilde{\sigma} = \mathrm{Cone}\left(e_i | \alpha(e_i) \in \widehat{\Sigma}'\right)$. The morphism satisfies the conditions mentioned in [GS15a, Theorem B.3]. Thus $\mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}} \to \mathcal{X}_{\widehat{\Sigma}',\widehat{\beta}'}$ is an isomorphism and $\mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}}$ is a fantastack. The matrix of $\widetilde{\beta}$ is given by

$$\begin{bmatrix} \beta(e_1) & \cdots & \beta(e_n) & \mathbf{0} \\ -a_1 & \cdots & -a_n & 1 \end{bmatrix}.$$

$\square$

Example 2.2.10. Consider the morphism of stacky fans as below.



Then $\phi : \mathcal{X}_{\Sigma,\beta} \to \mathbb{P}(2,1)$ is a line bundle whose sheaf of sections is $\mathcal{O}_{\mathbb{P}(2,1)}(-3)$ and its fan is not locally split.

Again this theorem can be generalized to the case where N is not free.

2.3 Projective Bundles

Consider the locally free sheave

$$\mathcal{E} = \mathcal{O}_{\mathcal{X}_{\Sigma,\beta}}(\mathcal{D}_0) \oplus \cdots \oplus \mathcal{O}_{\mathcal{X}_{\Sigma,\beta}}(\mathcal{D}_r)$$

given by the cartier divisors $\mathcal{D}_i = \sum_{j=1}^{n} a_{ij}\mathcal{D}_{\rho_j}$ for $0 \leq i \leq r$, then $\mathbb{P}(\mathcal{E}) \to \mathcal{X}_{\Sigma,\beta}$ is a projective bundle.

Assume N is free. Given a triple $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$, we define the new stacky fan $(N \times \mathbb{Z}^r, \widetilde{\Sigma}, \widetilde{\beta} : \mathbb{Z}^{n+r+1} \to N \times \mathbb{Z}^r)$ as follows:

1. $\widetilde{\beta}(e_j) = (\beta(e_j), a_{1j} - a_{0j}, \cdots, a_{rj} - a_{0j})$ for $1 \leq j \leq n$.
2. $\widetilde{\beta}(e_{n+1+i}) = (\mathbf{0}, e_i) \in N \times \mathbb{Z}^r$ for $0 \leq i \leq r$, where $e_0 = -e_1 - ... - e_r \in \mathbb{Z}^r$.
3. Given $\sigma \in \Sigma$, set $\widetilde{\sigma}_i = \text{Cone}\left(\widetilde{\beta}(e_j) \otimes 1 | \beta(e_j) \otimes 1 \in \sigma(1)\right) + \text{Cone}\left((\mathbf{0}, e_0), ..., (\mathbf{0}, e_{i-1}), (\mathbf{0}, e_{i+1}), ..., (\mathbf{0}, e_r)\right)$ and let $\widetilde{\Sigma}$ be the set consisting of cones $\widetilde{\sigma}_i$ for all $\sigma \in \Sigma$, $1 \leq i \leq r$ and their faces.

Then the natural projection of the fan $\widetilde{\Sigma}$ induces a toric morphism $\pi : \mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$.

Theorem 2.3.1. *$\mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}}$ is the projective bundle $\mathbb{P}(\mathcal{E})$.*

Proof. The proof is similar to that of [CLS11, Theorem 7.3.3]. □

Suppose $\gcd(a, b) = 1$, then by Propostion 2.1.1, the fan of $\mathbb{P}(a, b)$ is given by $\beta(e_1) = b$ and $\beta(e_2) = -a$. Suppose $r = sa + tb$, then consider

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}(a,b)} \oplus \mathcal{O}_{\mathbb{P}(a,b)}(s\mathcal{D}_{e_1} + t\mathcal{D}_{e_2})$$

where $\mathcal{D}_{e_i}$ is the divisor corresponding to the ray generated by e_i. Hence $\widetilde{\beta} : \mathbb{Z}^4 \to \mathbb{Z}^2$ is given by

$$\begin{aligned} \widetilde{\beta}(e_1) &= (b, s), & \widetilde{\beta}(e_2) &= (-a, t), \\ \widetilde{\beta}(e_3) &= (0, -1), & \widetilde{\beta}(e_4) &= (0, 1). \end{aligned} \tag{2.3.1}$$

If $\gcd(a, b) = d \neq 1$ and $c_1\frac{a}{d} + c_2\frac{b}{d} \equiv 1 \mod d$, then by Proposition 2.1.3, the fan of $\mathbb{P}(a, b)$ is given by $\beta' : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ such that

$$\beta'(e_1) = (\frac{b}{d}, c_1 \bmod d), \quad \beta'(e_2) = (-\frac{a}{d}, c_2 \bmod d).$$

Suppose $d \mid r$ and $r = s\frac{a}{d} + t\frac{b}{d}$, then $\widetilde{\beta} : \mathbb{Z}^4 \to \mathbb{Z}^2 \oplus \mathbb{Z}/d\mathbb{Z}$ is given by

$$\begin{aligned} \widetilde{\beta}(e_1) &= (\frac{b}{d}, s, c_1 \bmod d), & \widetilde{\beta}(e_2) &= (-\frac{a}{d}, t, c_2 \bmod d), \\ \widetilde{\beta}(e_3) &= (0, -1, 0), & \widetilde{\beta}(e_4) &= (0, 1, 0). \end{aligned} \tag{2.3.2}$$

Definition 2.3.2. The *Hirzebruch stack* $\mathcal{H}_r^{ab}$ is defined as

$$\mathcal{H}_r^{ab} = \mathbb{P}(\mathcal{O}_{\mathbb{P}(a,b)} \oplus \mathcal{O}_{\mathbb{P}(a,b)}(r))$$

and its fan is given by (2.3.1) when $\gcd(a, b) = 1$ and by (2.3.2) when $\gcd(a, b) = d \neq 1$ and $d \mid r$.

From now on, to simplify the notation, we assume $\gcd(a, b) = 1$[1]. In this case,

[1]Our method still works without this assumption.

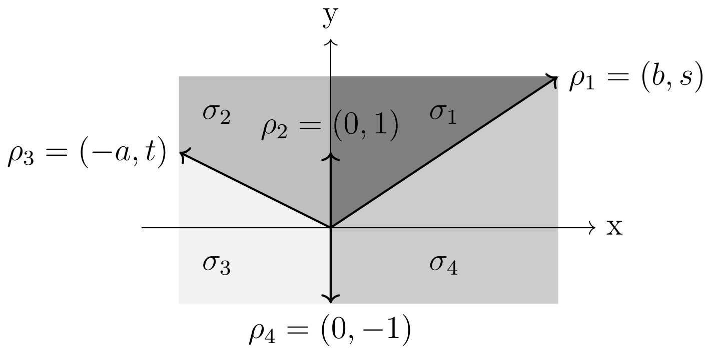


Figure 2.1: $\mathcal{H}_r^{ab}$

the matrix for $\beta : \mathbb{Z}^4 \to \mathbb{Z}^2$ is given by

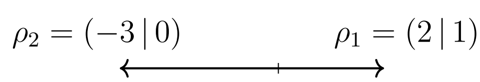
$$B = \begin{bmatrix} b & 0 & -a & 0 \\ s & 1 & t & -1 \end{bmatrix} \tag{2.3.3}$$

where $r = sa + bt$. The stacky fan can be drawn as below and $\mathcal{H}_r^{ab}$ is called the *Hirzebruch orbifold*.

2.4 Bundles over Gerbes

In (2.3.2), we require the condition that $d \mid r$. The reason is that when N is not free [Definition 2.0.1], $\mathcal{X}_\Sigma$ is a banded gerbe over $\mathcal{X}_\Sigma^{\mathrm{rig}}$ [FMN10]. Hence the line bundles $\mathcal{O}_{\mathcal{X}_\Sigma}(\mathcal{D}_{\rho_i})$ do not generate the K-theory any more.

Example 2.4.1. The stacky fan for the weighted projective line $\mathbb{P}(6, 4)$ is given by



where the last coordinate comes from the torsion part $\mathbb{Z}/2\mathbb{Z}$. However $\mathcal{O}_{\mathbb{P}(6,4)}(\mathcal{D}_{\rho_1}) \cong$

$\mathcal{O}_{\mathbb{P}(6,4)}(6)$ and $\mathcal{O}_{\mathbb{P}(6,4)}(\mathcal{D}_{\rho_2}) \cong \mathcal{O}_{\mathbb{P}(6,4)}(4)$. The construction in the previous two sections won't give us the total space of $\mathcal{O}_{\mathbb{P}(6,4)}(1)$.

In general, given a line bundle $\mathcal{L}$ on a toric Deligne-Mumford stack $\mathcal{X}_\Sigma$, if there exists a minimal positive integer n such that

$$\mathcal{L}^n \cong \mathcal{O}_{\mathcal{X}_\Sigma}(\Sigma_i a_i \mathcal{D}_{\rho_i}),$$

then we can modify the stacky fan construction in the last section to show that the total space of $\mathcal{L}$ is toric.

Consider a toric stack given by the fan $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$, N not necessarily free. We define the new stacky fan $(N \times \mathbb{Z}, \widetilde{\Sigma}, \widetilde{\beta} : \mathbb{Z}^{n+1} \to N \times \mathbb{Z})$ as follows:

1. $\widetilde{\beta}(e_i) = (\beta(e_i), -a_i)$ for $1 \le i \le n$.

2. $\widetilde{\beta}(e_{n+1}) = (\mathbf{0}, n)$.

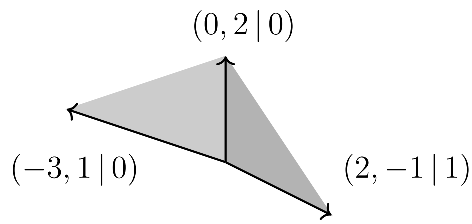
3. Given $\sigma \in \Sigma$, set $\widetilde{\sigma} = \text{Cone}\Big((\mathbf{0}, n),\, \widetilde{\beta}(e_i) \otimes 1 \,|\, \beta(e_i) \otimes 1 \in \sigma(1)\Big) \in N_{\mathbb{Q}} \times \mathbb{Q}$, and let $\widetilde{\Sigma}$ be the set consisting of $\widetilde{\sigma}$ for all $\sigma \in \Sigma$ and their faces.

Theorem 2.4.2. *$\mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}}$ is the total space of the line bundle $\mathcal{L}$ over $\mathcal{X}_{\Sigma,\beta}$*

Proof. The proof is similar to that of Theorem 2.2.9. □

Example 2.4.3. Consider the line bundle $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}(6,4)}(1)$. Since $\mathcal{L}^2 \cong \mathcal{O}_{\mathbb{P}(6,4)}(\mathcal{D}_{\rho_1} -$

$\mathcal{D}_{\rho_2}$), the stacky fan for the total space of $\mathcal{L}$ is given by



where the last coordinate comes from the torsion part $\mathbb{Z}/2\mathbb{Z}$. One can show that $\mathcal{X}_{\widetilde{\Sigma}} \cong [\mathbb{C}^3 - V(x, z)/\mathbb{C}^*]$ where the action is given by

$$\tau \in \mathbb{C}^* : (x, y, z) \to (\tau^6 x, \tau y, \tau^4 z).$$

Now we can extend the construction to the projective bundles. Given a triple $(N, \Sigma, \beta : \mathbb{Z}^n \to N)$, consider the locally free sheaf

$$\mathcal{E} = \mathcal{O}_{\mathcal{X}_\Sigma} \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$$

over the corresponding toric stack $\mathcal{X}_\Sigma$. Suppose there exist minimal positive integers n_i such that

$$\mathcal{L}_i^{n_i} \cong \mathcal{O}_{\mathcal{X}_\Sigma}(\sum_{j=1}^{n} a_{ij} \mathcal{D}_j)$$

for $1 \leq i \leq r$. We define the new stacky fan $(N \times \mathbb{Z}^r, \widetilde{\Sigma}, \widetilde{\beta} : \mathbb{Z}^{n+r+1} \to N \times \mathbb{Z}^r)$ as follows:

1. $\widetilde{\beta}(e_j) = (\beta(e_j), a_{1j}, \cdots, a_{rj})$ for $1 \leq j \leq n$.

2. $\widetilde{\beta}(e_{n+i}) = (\mathbf{0}, n_i e_i) \in N \times \mathbb{Z}^r$ for $1 \leq i \leq r$.

3. $\widetilde{\beta}(e_{n+r+1}) = (\mathbf{0}, e_0) \in N \times \mathbb{Z}^r$ where $e_0 = -n_1e_1 - ... - n_re_r$.

4. Given $\sigma \in \Sigma$, set $\widetilde{\sigma}_i = \text{Cone}\left(\widetilde{\beta}(e_j) \otimes 1 | \beta(e_j) \otimes 1 \in \sigma(1)\right) + \text{Cone}\left((\mathbf{0}, e_0), ..., (\mathbf{0}, n_{i-1}e_{i-1}), (\mathbf{0}, n_{i+1}e_{i+1}), ..., (\mathbf{0}, n_re_r)\right)$ and let $\widetilde{\Sigma}$ be the set consisting of cones $\widetilde{\sigma}_i$ for all $\sigma \in \Sigma$, $1 \leq i \leq r$ and their faces.

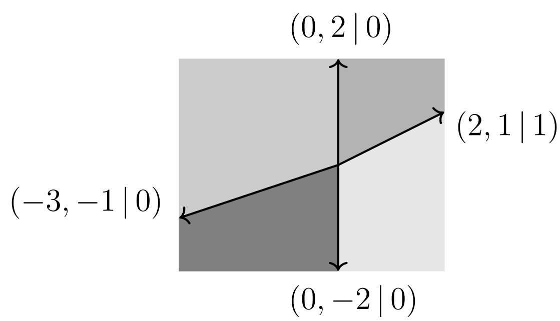
Theorem 2.4.4. *$\mathcal{X}_{\widetilde{\Sigma},\widetilde{\beta}}$ is the projective bundle $\mathbb{P}(\mathcal{E})$.*

Now we can finally give the fan description of the Hirzebruch stack $\mathcal{H}_r^{ab}$ when $\gcd(a, b) = d \neq 1$ and $d \nmid r$.

Suppose there exists a minimal positive integer n such that $nr = sa + tb$, then the stacky fan of $\mathcal{H}_r^{ab}$ is given by $\widetilde{\beta} : \mathbb{Z}^4 \to \mathbb{Z}^2 \oplus \mathbb{Z}/d\mathbb{Z}$ as follows:

$$\begin{aligned} \widetilde{\beta}(e_1) &= (\frac{b}{d}, s, c_1 \bmod d), & \widetilde{\beta}(e_2) &= (-\frac{a}{d}, t, c_2 \bmod d), \\ \widetilde{\beta}(e_3) &= (0, n, 0), & \widetilde{\beta}(e_4) &= (0, -n, 0). \end{aligned}$$

Example 2.4.5. Let $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}(6,4)} \oplus \mathcal{O}_{\mathbb{P}(6,4)}(1)$. Then the stacky fan for $\mathbb{P}(\mathcal{E})$ is given by



where the last coordinate comes from the torsion part $\mathbb{Z}/2\mathbb{Z}$. One can show that

$\mathcal{X}_{\widetilde{\Sigma}} \cong [\mathbb{C}^4 - V(xy, yz, zw, wx)/(\mathbb{C}^*)^2]$ where the action is given by

$$(\tau, \lambda) \in \mathbb{C}^* : (x, y, z, w) \to (\tau^6 x, \lambda y, \tau^4 z, \tau\lambda w).$$

Chapter 3: Sheaves on Hirzebruch Orbifolds

The Hirzebruch orbifold can be covered by open substacks of the form $[\mathbb{C}^2/H]$ where H is a finite group. Hence, to describe a sheaf on the Hirzebruch Orbifold, we can define it locally over each substack and then glue each part together.

Let $\mathbf{T} \cong (\mathbb{C}^*)^2$ act linearly on Spec $\mathbb{C}[x_1, x_2]$ and suppose the action is given as $t \cdot x_i = \chi(m_i)(t)(x_i)$ for some m_i in the character lattice $X(\mathbf{T})$. Given a $\mathbf{T}$-equivariant sheaf $\mathcal{F}$ on $[\mathbb{C}^2/H]$, the corresponding module can be decomposed into $X(\mathbf{T})$-graded weight spaces:

$$H^0(\mathbb{C}^2, \mathcal{F}) = \bigoplus_{m \in X(\mathbf{T})} F(m).$$

Suppose H acts by $h \cdot x_i = \chi(n_i)(h)(x_i)$, then $F(m)$ can be further decomposed into $X(H)$-graded weight spaces:

$$F(m) = \bigoplus_{n \in X(H)} F(m)_n.$$

Hence the category of $\mathbf{T}$-equivariant sheaves on $[\mathbb{C}^2/H]$, by [GJK17], is equivalent to the category of stacky S-families. A object $\hat{F}$ in this category consists of the following data:

- A collection of vector spaces $\{F(m)_n\}_{m \in X(\mathbf{T}), n \in X(H)}$.

- A collection of linear maps

$$\{\chi_i(m) : F(m) \to F(m+m_i)\}_{i=1,2,m\in X(\mathbf{T})}.$$

induced by multiplication by x_i satisfying

$$\chi_i(m) : F(m)_n \to F(m+m_i)_{n+n_i}, \chi_j(m+m_i) \cdot \chi_i(m) = \chi_i(m+m_j) \cdot \chi_j(m)$$

for $i, j = 1, 2$, $m \in X(\mathbf{T})$ and $n \in X(H)$.

3.1 Open Affine Covers

Let N_{σ_i} be the subgroup of $N \cong \mathbb{Z}^2$ generated by the rays of σ_i and $N(\sigma_i)$ be the quotient group N/N_{σ_i}. By [BCS05], each 2-dimensional cone σ_i defines an open substack $\mathcal{U}_i \cong [\mathbb{C}^2/N(\sigma_i)]$ of $\mathcal{H}_r^{ab}$. One can show that

$$\mathcal{U}_1 \cong \mathcal{U}_4 \cong [\mathbb{C}^2/(\mathbb{Z}/b\mathbb{Z})], \quad \mathcal{U}_2 \cong \mathcal{U}_3 \cong [\mathbb{C}^2/(\mathbb{Z}/a\mathbb{Z})].$$

The integer null space of the matrix B (2.3.3) is spanned by $\begin{bmatrix} a & 0 & b & r \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$. Hence $(\tau, \lambda) \in G_\beta \cong (\mathbb{C}^*)^2$ acts on $Z_\Sigma = \operatorname{Spec} \mathbb{C}[x, y, z, w] - V(xy, yz, zw, wx)$ by

$$(\tau, \lambda) : (x, y, z, w) \to (\tau^a x, \lambda y, \tau^b z, \tau^r \lambda w)$$

and $\mathcal{H}_r^{ab} = [Z_\Sigma/G_\beta]$.

Let β_1 be the morphism given by the first two columns of the matrix B. It induces a stacky fan with two rays and the corresponding toric stack $[Z_1/G_1]$ is exactly $[\mathbb{C}^2/(\mathbb{Z}/b\mathbb{Z})]$. Consider the open subvariety U_1 of Z_Σ defined as the complement of the vanishing locus of the monomial zw. There is a natural closed embedding $\phi_1 : Z_1 \to U_1$ given by

$$\phi_1(Z_1) = \mathbb{C}^2 \times \mathbf{1} = \{(x, y, 1, 1)\} \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cong U_1.$$

By [BCS05], an element $g \in G_\beta$ belongs to G_1 if and only if $\phi_1(Z_1) \cdot g \cap \phi_1(Z_1) \neq \emptyset$. In this case,

$$\tau^b = 1, \tau^r \lambda = 1 \Longrightarrow \lambda = \tau^{-r}.$$

Let μ_b be the group of bth roots of unity, then

$$\mathcal{U}_1 \cong [\mathbb{C}^2/\mu_b], \quad \tau \in \mu_b : (x, y) \to (\tau^a x, \tau^{-r} y).$$

Similarly, one can show that

$$\mathcal{U}_2 \cong [\mathbb{C}^2/\mu_a], \quad \tau \in \mu_a : (y, z) \to (\tau^{-r} y, \tau^b z),$$
$$\mathcal{U}_3 \cong [\mathbb{C}^2/\mu_a], \quad \tau \in \mu_a : (z, w) \to (\tau^b z, \tau^r w),$$
$$\mathcal{U}_4 \cong [\mathbb{C}^2/\mu_b], \quad \tau \in \mu_b : (w, x) \to (\tau^r w, \tau^a x).$$

Consider the morphism $\widetilde{\phi_i} : \mathcal{U}_i \hookrightarrow \mathcal{H}_r^{ab}$ induced by $Z_i \xrightarrow{\phi_i} U_i \hookrightarrow Z_\Sigma$. We can compute stack theoretic intersections via the fiber product of U_i and U_j over $\mathcal{H}_r^{ab}$.

$$\begin{tikzcd}
\mathcal{U}_{12} := \mathcal{U}_1 \times_{\mathcal{H}_r^{ab}} \mathcal{U}_2 \arrow[r] \arrow[d] & {[\text{spec } \mathbb{C}[x,y]/\mu_b]} \arrow[d, "\widetilde{\phi_1}"] \\
{[\text{spec } \mathbb{C}[y,z]/\mu_a]} \arrow[r, "\widetilde{\phi_2}"] & \mathcal{H}_r^{ab}
\end{tikzcd}$$

By calculating the fiber product of the corresponding groupoids [ALR07], one can show that

$$\mathcal{U}_{12} \cong [\mathbb{C} \times \mathbb{C}^*/\mu_b \times \mu_a], (\mu, \nu) \in \mu_b \times \mu_a : (y, \tau) \to (\mu^{-r} y, \nu\mu^{-1}\tau).$$

Similarly, the fiber products of other open substacks are given as follows:

$$\begin{aligned}
\mathcal{U}_{23} &\cong [\mathbb{C} \times \mathbb{C}^* \times \mu_a/\mu_a \times \mu_a], && (\mu, \nu) \in \mu_a \times \mu_a : (z, \lambda, \tau) \to (\mu^b z, \mu^r \lambda, \nu\mu^{-1}\tau). \\
\mathcal{U}_{34} &\cong [\mathbb{C} \times \mathbb{C}^*/\mu_a \times \mu_b], && (\mu, \nu) \in \mu_a \times \mu_b : (w, \tau) \to (\mu^r w, \nu\mu^{-1}\tau). \\
\mathcal{U}_{41} &\cong [\mathbb{C} \times \mathbb{C}^* \times \mu_b/\mu_b \times \mu_b], && (\mu, \nu) \in \mu_b \times \mu_b : (x, \lambda, \tau) \to (\mu^a x, \nu^{-r}\lambda, \nu\mu^{-1}\tau).
\end{aligned}$$

Actually $\mathcal{U}_{23}$ can be further simplified. Consider the groupoid morphism

$$(\psi_1 \times \psi_0, \psi_0) : (\mu_a \times \mathbb{C} \times \mathbb{C}^* \rightrightarrows \mathbb{C} \times \mathbb{C}^*) \longrightarrow (\mu_a \times \mu_a \times \mathbb{C} \times \mathbb{C}^* \times \mu_a \rightrightarrows \mathbb{C} \times \mathbb{C}^* \times \mu_a)$$

defined by

$$\psi_1(\mu) = (\mu, \mu), \quad \psi_0(z, \lambda) = (z, \lambda, 1).$$

One can show that it is a Morita equivalence and hence

$$\mathcal{U}_{23} \cong [\mathbb{C} \times \mathbb{C}^*/\mu_a], \quad \mu \in \mu_a : (z, \lambda) \to (\mu^b z, \mu^r \lambda).$$

Similarly,

$$\mathcal{U}_{41} \cong [\mathbb{C} \times \mathbb{C}^*/\mu_b], \quad \mu \in \mu_b : (x, \lambda) \to (\mu^a x, \mu^{-r}\lambda).$$

The open immersions $\widetilde{\phi}_{ij} : \mathcal{U}_{ij} = [Z_{ij}/G_{ij}] \hookrightarrow \mathcal{U}_i = [Z_i/G_i]$ and $\widetilde{\phi}_{ji} : \mathcal{U}_{ij} \hookrightarrow \mathcal{U}_j$ are induced from $\phi_{ij} : Z_{ij} \to Z_i$ and $\phi_{ji} : Z_{ji} = Z_{ij} \to Z_j$.

$$\phi_{12} : (y, \tau) \to (\tau^{-a}, y) \qquad \phi_{21} : (y, \tau) \to (y\tau^{-r}, \tau^b).$$

$$\phi_{23} : (z, \lambda) \to (\lambda^{-1}, z) \qquad \phi_{32} : (z, \lambda) \to (z, \lambda).$$

$$\phi_{34} : (w, \tau) \to (\tau^{-b}, w) \qquad \phi_{43} : (w, \tau) \to (\tau^r w, \tau^a).$$

$$\phi_{41} : (x, \lambda) \to (\lambda^{-1}, x) \qquad \phi_{14} : (x, \lambda) \to (x, \lambda).$$

To find $X(T)$-grading on each open substack $\mathcal{U}_i$, we need to determine how the torus $\mathbf{T}$ is embedded in $\mathcal{H}_r^{ab}$. One can show that

$$\mathcal{U}_{1234} := \mathcal{U}_{12} \times_{\mathcal{H}_r^{ab}} \mathcal{U}_{34} \cong [(\mathbb{C}^*)^2/\mu_b \times \mu_a]$$

$$(\mu, \mu') \in \mu_b \times \mu_a : (\alpha, \beta) \to (\mu(\mu')^{-1}\alpha, \mu^{-r}\beta).$$

Hence $\mathcal{U}_{1234} \cong (\mathbb{C}^*)^2$. Suppose (α, β) acts on itself by multiplication, then we can extend this action to the orbifold $\mathcal{H}_r^{ab}$ by requiring all the open immersions to be $\mathbf{T}$-equivariant.

For example, from the following commutative diagram

$$\begin{tikzcd}
Z_{1234} \arrow[r] & Z_{12} \cong \mathbb{C} \times \mathbb{C}^* \arrow[r] \arrow[rr, bend left] & Z_1 = \text{spec } \mathbb{C}[x,y] & Z_2 = \text{spec } \mathbb{C}[y,z] \\
(\alpha, \beta) \arrow[r] \arrow[d, "(1{,}0)"', "(0{,}1)"] & (\beta, \alpha^{-1}) \arrow[r] \arrow[rr, bend left] \arrow[d, "(0{,}1)"', "(-1{,}0)"] & (\alpha^a, \beta) \arrow[d, "(a{,}0)"', "(0{,}1)"] & (\beta\alpha^r, \alpha^{-b}) \arrow[d, "(r{,}1)"', "(-b{,}0)"] \\
(t_1\alpha, t_2\beta) \arrow[r] & (t_2\beta, t_1^{-1}\alpha^{-1}) \arrow[r] \arrow[rr, bend right] & (t_1^a\alpha^a, t_2 y) & (t_2\beta t_1^r\alpha^r, t_1^{-b}\alpha^{-b})
\end{tikzcd}$$

we see that $\mathbf{T}$-weights are $(0,1)$ and $(-1,0)$ on Z_{12}, $(a,0)$ and $(0,1)$ on Z_1, $(r,1)$ and $(-b,0)$ on Z_2.

Similarly, one can show that $\mathbf{T}$-weights are given by the following tables:

	$\mathbf{T}$-weights on Z_i
$\mathcal{U}_1$	$(a,0),(0,1)$
$\mathcal{U}_2$	$(r,1),(-b,0)$
$\mathcal{U}_3$	$(-b,0),(-r,-1)$
$\mathcal{U}_4$	$(0,-1),(a,0)$

	$\mathbf{T}$-weights on Z_{ij}
$\mathcal{U}_{12}$	$(0,1),(-1,0)$
$\mathcal{U}_{23}$	$(-b,0),(-r,-1)$
$\mathcal{U}_{34}$	$(-r,-1),(1,0)$
$\mathcal{U}_{41}$	$(a,0),(0,1)$

3.2 Gluing Conditions

To describe $\mathbf{T}$-equivariant torsion free sheaves on $\mathcal{H}_r^{ab}$, we first determine the stacky S-family $\hat{F}_i$ of the sheaf $\mathcal{F}_i$ on each open cover $\mathcal{U}_i$. Then we pull back those families to the intersection $\mathcal{U}_{ij}$ and match them for all i, j. This allows us to glue those sheaves $\mathcal{F}_i$ to get a sheaf $\mathcal{F}$ on $\mathcal{H}_r^{ab}$. Note that this gluing approach follows closely the work of [GJK17].

Let's first compute the family $\hat{F}_{1,12}$, which is the pullback of $\hat{F}_1$.

Given a torus action $t \cdot x_i = \chi(m_i)(t)(x_i)$, the associated box $\mathcal{B}_{\mathbf{T}}$ [GJK17] is defined as the subset of $X(\mathbf{T})$ of the form $\sum_i q_i m_i$ with $0 \leq q_i \leq 1$. By the above table, the $\mathbf{T}$-weights on U_1 are $(a, 0)$ and $(0, 1)$. Hence $q_1 = \frac{k}{a}$ for $0 \leq k \leq a-1$ and $q_2 = 0$. Note that the box $\mathcal{B}_{\mathbf{T}}$ of $\mathcal{U}_1$ can also be viewed as $[0, a-1] \times 0$ and the size of this box is a.

For the stacky family $\hat{F}_1$, denote by

$$_{(k/a,0)}F_1(l_1, l_2)$$

the vector space whose $\mathbf{T}$-weight is $(k/a + l_1)(a, 0) + (0 + l_2)(0, 1)$.

Consider the inclusion $\mathcal{U}_{12} \hookrightarrow \mathcal{U}_1$ induced from

$$\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^2, \quad \phi_{12} : (y, \tau) \to (\tau^{-a}, y) = (x, y).$$

We first restrict the sheaf $\mathcal{F}_1$ to $\mathrm{Im}(\phi_{12}) \cong \mathbb{C}^* \times \mathbb{C}$ and then pull it back along the morphism ϕ_{12}.

The sheaf $\mathcal{F}$ is torsion free, hence the vector spaces $_{(k/a,0)}F_1(l_1, l_2)$ stabilize for $l_1 \gg 0$, l_2 fixed. It means that they are isomorphic for $l_1 \gg 0$ [Koo11]. We denote this limit by

$$_{(k/a,0)}F_1(\infty, l_2).$$

The sheaf $\mathcal{F}_1|_{\mathbb{C}^*\times\mathbb{C}}$ corresponds to a S-family $\hat{G}_1$ and

$$_{(k/a,0)}G(l_1, l_2) = {}_{(k/a,0)}F_1(\infty, l_2)$$

is independent of l_1 because G_1 is a $\mathbb{C}[x^{\pm}, y]$-module and multiplication by x induces an isomorphism of vector spaces.

Pulling back the family $\hat{G}_1$ to Z_{12} along the étale morphism ϕ_{12}, we get a $\mathbb{C}[\tau^{\pm}, y]$-module. An element of $\hat{F}_{1,12}$ at the weight $(k/a + l_1)(a, 0) + (0 + l_2)(0, 1)$ can be uniquely written as

$$\bigoplus_{0\leq k'\leq a-1} v_{k'} \otimes \tau^{k'-k}$$

where $v_{k'} \in {}_{(k'/a,0)}G_1(l_1, l_2)$, since the $\mathbf{T}$-weight of τ is $(-1, 0)$ on U_{12}.

Next, we set the fine-grading on the limit space $_{(k/a,0)}F_1(\infty, l_2)$ by

$$_{(k/a,0)}F_1(\infty, l_2)_m = {}_{(k/a,0)}G_1(0, l_2)_m.$$

Thus the fine-grading of S-family $\hat{G}_1$ for any l_1 will be

$$_{(k/a,0)}G_1(l_1, l_2)_m = {}_{(k/a,0)}G_1(0, l_2)_{m-al_1} \otimes \hat{\mu}_b^{al_1}.$$

Here $\otimes\hat{\mu}_b$ means tensoring with the 1-dimensional representation of the group μ_b of weight $1 \in \mathbb{Z}/b\mathbb{Z}$.

Since the $\mu_b \times \mu_a$-weight of τ is $(-1,1)$ on $\mathcal{U}_{12}$, the S-family of $\hat{F}_{1,12}$ at the $\mathbf{T}$-weight $(k/a + l_1)(a,0) + (0 + l_2)(0,1)$ with the fine grading is

$$\begin{aligned} & \bigoplus_{\substack{0 \leq k' \leq a-1 \\ m \in \mathbb{Z}_b}} {}_{(k'/a,0)} G_1(l_1, l_2)_m \otimes \hat{\mu}_b^{k-k'} \otimes \hat{\mu}_a^{k'-k} \\ = & \bigoplus_{\substack{0 \leq k' \leq a-1 \\ m \in \mathbb{Z}_b}} {}_{(k'/a,0)} F_1(\infty, l_2)_m \otimes \hat{\mu}_b^{k-k'+al_1} \otimes \hat{\mu}_a^{k'-k}. \end{aligned}$$

Similarly, one can show that the S-family of $\hat{F}_{2,12}$ at the $\mathbf{T}$-weight $(0 + l_1)(r, -r) + (j/b + l_2)(-b, 0)$ is

$$\bigoplus_{\substack{0 \leq j' \leq b-1 \\ n \in \mathbb{Z}_a}} {}_{(0,j'/b)} F_2(l_1, \infty)_n \otimes \hat{\mu}_a^{j-j'} \otimes \hat{\mu}_b^{j'-j+bl_2}$$

Since multiplication by τ is an isomorphism, the S-family $\hat{F}_{1,12}$ is determined by its elements at the weight $(0/a + 0)(a,0) + (0 + l)(0,1) = (0,l)$ for all $l \in \mathbb{Z}$. Therefore it suffices to compute the S-family $\hat{F}_{1,12}$ at the above weight, which is given by

$$\bigoplus_{\substack{0 \leq k' \leq a-1 \\ m \in \mathbb{Z}_b}} {}_{(k'/a,0)} F_1(\infty, l_2)_m \otimes \hat{\mu}_b^{-k'} \otimes \hat{\mu}_a^{k'}.$$

Similarly, we only compute the S-family $\hat{F}_{2,12}$ at the weight $(0 + l)(r, 1) + (0/b + 0)(-b, 0) = (lr, l)$, which is given by

$$\bigoplus_{\substack{0 \leq j' \leq b-1 \\ n \in \mathbb{Z}_a}} {}_{(0,j'/b)} F_2(l, \infty)_n \otimes \hat{\mu}_a^{-j'} \otimes \hat{\mu}_b^{j'}.$$

We can't equate them since they are at different weights. To jump from the weight (lr, l) to $(0, l)$, we multiply the second family by τ^{lr} as the $\mathbf{T}$-weight of τ is $(-1, 0)$. As a result, the fine grading is changed to

$$\bigoplus_{\substack{0 \le j' \le b-1 \\ n \in \mathbb{Z}_a}} {}_{(0,j'/b)}F_2(l, \infty)_n \otimes \hat{\mu}_a^{-j'+lr} \otimes \hat{\mu}_b^{j'-lr}.$$

Hence the gluing conditions on the substack $\mathcal{U}_{12}$ are given by:

$$\bigoplus_{0 \le k \le a-1}^{m \in \mathbb{Z}_b} {}_{(k/a,0)}F_1(\infty, l)_m \otimes \hat{\mu}_b^{-k} \otimes \hat{\mu}_a^{k} = \bigoplus_{0 \le j \le b-1}^{n \in \mathbb{Z}_a} {}_{(0,j/b)}F_2(l, \infty)_n \otimes \hat{\mu}_a^{-j+lr} \otimes \hat{\mu}_b^{j-lr}$$

for all $l \in \mathbb{Z}$. Here $\otimes \hat{\mu}_b^k$ means tensoring with the 1-dimensional representation of the group μ_b of weight $k \in \mathbb{Z}/b\mathbb{Z}$ and $\otimes \hat{\mu}_a^j$ means tensoring with the 1-dimensional representation of the group μ_a of weight $j \in \mathbb{Z}/a\mathbb{Z}$.

Similarly, we can get gluing conditions for other substacks.

Proposition 3.2.1. *The category of $\mathbf{T}$-equivariant torsion free sheaves on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is equivalent to the category of finite stacky S-families $\{\hat{F}_i\}_{i=1,2,3,4}$ on $\mathcal{U}_i$ satisfying the gluing conditions given by the following equalities of $\mu_a \times \mu_b$ rep-*

resentations:

$$\begin{aligned}
\bigoplus_{\substack{0\le k\le a-1\\ m\in\mathbb{Z}_b}} {}_{(k/a,0)}F_1(\infty,l)_m\otimes\hat{\mu}_b^{-k}\otimes\hat{\mu}_a^{k} &= \bigoplus_{\substack{0\le j\le b-1\\ n\in\mathbb{Z}_a}} {}_{(0,j/b)}F_2(l,\infty)_n\otimes\hat{\mu}_a^{-j+lr}\otimes\hat{\mu}_b^{j-lr}\\
\bigoplus_{m\in\mathbb{Z}_a} {}_{(0,j/b)}F_2(\infty,l)_m &= \bigoplus_{n\in\mathbb{Z}_a} {}_{(j/b,0)}F_3(l,\infty)_n\\
\bigoplus_{\substack{0\le j\le b-1\\ m\in\mathbb{Z}_a}} {}_{(j/b,0)}F_3(\infty,l)_m\otimes\hat{\mu}_a^{-j}\otimes\hat{\mu}_b^{j} &= \bigoplus_{\substack{0\le k\le a-1\\ n\in\mathbb{Z}_b}} {}_{(0,k/a)}F_4(l,\infty)_n\otimes\hat{\mu}_b^{-k-lr}\otimes\hat{\mu}_a^{k+lr}\\
\bigoplus_{m\in\mathbb{Z}_b} {}_{(0,k/a)}F_4(\infty,l)_m &= \bigoplus_{n\in\mathbb{Z}_b} {}_{(k/a,0)}F_1(l,\infty)_n
\end{aligned}$$

for all $l\in\mathbb{Z}$ and similar gluing conditions between the corresponding inclusions.

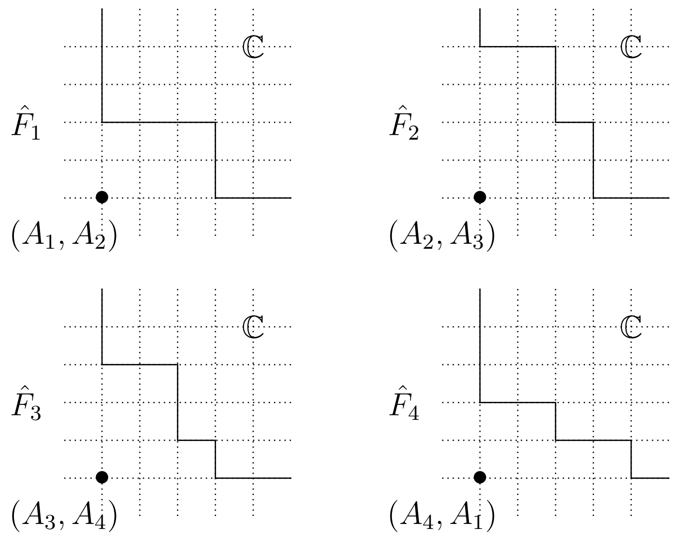
3.3 Examples

In this section, we will give some examples of torsion free sheaves of rank 1 and 2 on $\mathcal{H}_r^{ab}$.

Example 3.3.1. Let F be a torsion free sheaf of rank 1 on the Hirzebruch surface $\mathcal{H}_r^{11}$. Then the gluing conditions are

$$\begin{aligned}
{}_{(0,0)}F_1(\infty,l) = {}_{(0,0)}F_2(l,\infty),\quad {}_{(0,0)}F_2(\infty,l) = {}_{(0,0)}F_3(l,\infty),\\
{}_{(0,0)}F_3(\infty,l) = {}_{(0,0)}F_4(l,\infty),\quad {}_{(0,0)}F_4(\infty,l) = {}_{(0,0)}F_1(l,\infty).
\end{aligned}$$

On each chart, ${}_{(0,0)}\hat{F}_i$ can be described as follows:



Example 3.3.2. Let $\mathcal{F}$ be a locally free sheaf of rank 1 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. The charts $\mathcal{U}_1$ and $\mathcal{U}_4$ has a box of size a, while the charts $\mathcal{U}_2$ and $\mathcal{U}_3$ has a box of size b. Since the rank is 1, the only possible choice for nonzero ${}_{b_i}\hat{F}_i$ is

$$b_1 = (k/a, 0), b_2 = (0, j/b), b_3 = (j/b, 0), b_4 = (0, k/a).$$

For fixed i, j, The $\mathbf{T}$-weights of the generator on each chart are given by

$$\mathcal{U}_1 : (k/a + A_1)(a, 0) + A_2(0, 1), \qquad \mathcal{U}_2 : A_2(r, 1) + (j/b + A_3)(-b, 0),$$
$$\mathcal{U}_3 : (j/b + A_3)(-b, 0) + A_4(-r, -1), \quad \mathcal{U}_4 : A_4(0, -1) + (k/a + A_1)(a, 0).$$

Set

$$B_1 = k + aA_1, B_2 = A_2, B_3 = j + bA_3, B_4 = A_4.$$

The sheaf $\mathcal{F}$ is uniquely determined by B_i. We will show below that the fine grading is also determined.

Suppose the μ_b-weight of the generator is m_1 on chart $\mathcal{U}_1$, then

$$_{(k/a,0)}F_1(A_1, A_2)_{m_1} =_{(k/a,0)} G_1(0, A_2)_{m_1 - aA_1} =_{(k/a,0)} F_1(\infty, A_2)_{m_1 - aA_1}.$$

The first equation of the gluing conditions implies that

$$m_1 \equiv k + aA_1 + j - rA_2 \equiv B_1 + B_3 - rB_2 \mod b.$$

Similarly, one can show that the fine gradings of all the generators are determined as follows:

$$B_1 + B_3 - rB_2 \mod b \text{ on } \mathcal{U}_1, \quad B_1 + B_3 - rB_2 \mod a \text{ on } \mathcal{U}_2,$$
$$B_1 + B_3 + rB_4 \mod a \text{ on } \mathcal{U}_3, \quad B_1 + B_3 + rB_4 \mod b \text{ on } \mathcal{U}_4.$$

Denote by $L_{(B_1,B_2,B_3,B_4)}$ the $\mathbf{T}$-equivariant locally free sheaf of rank 1 corresponding to $(B_1, B_2, B_3, B_4) \in \mathbb{Z}^4$.

Proposition 3.3.3. *Let $Pic^T(\mathcal{H}_r^{ab})$ be the $\mathbf{T}$-equivariant Picard group of the Hirzebruch orbifold. Then*

$$(B_1, B_2, B_3, B_4) \in \mathbb{Z}^4 \longmapsto L_{(B_1,B_2,B_3,B_4)} \in Pic^T(\mathcal{H}_r^{ab})$$

is a group isomorphism.

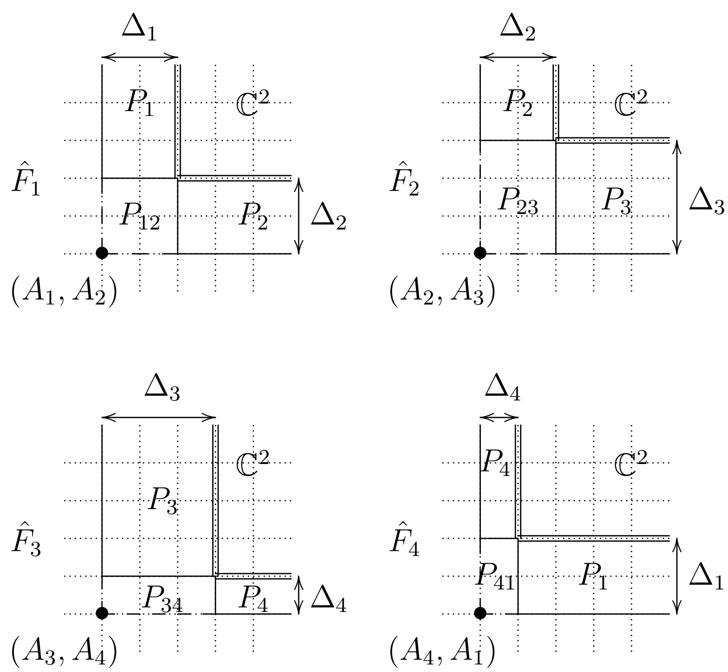
Remark 3.3.4. The non-equivariant Picard group of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$

is $\mathbb{Z} \oplus \mathbb{Z}$ and

$$L_{(1,0,0,0)} = (-1,0) \quad L_{(0,0,1,0)} = (-1,0)$$

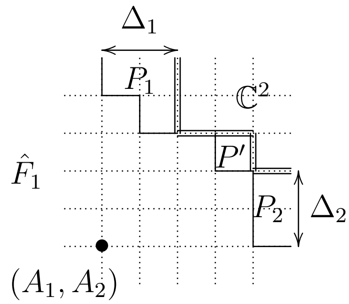
$$L_{(0,1,0,0)} = (0,-1) \quad L_{(0,0,0,1)} = (-r,-1)$$

Example 3.3.5. Let $\mathcal{F}$ be a locally free sheaf of rank 2 on the Hirzebruch surface $\mathcal{H}_r^{11}$. On each chart, $\hat{F}_i$ can be described by a double filtration of $\mathbb{C}^2$:



Hence $\mathcal{F}$ is fully determined by $A_1, A_2, A_3, A_4 \in \mathbb{Z}$, $\Delta_1, \Delta_2, \Delta_3, \Delta_4 \in \mathbb{Z}_{\geq 0}$ and $P_1, P_2, P_3, P_4 \subset \mathbb{C}^2$, which can also be viewed as a point $(P_1, P_2, P_3, P_4) \in (\mathbb{P}^1)^4$. The label P_{ij} stands for the vector space $P_i \cap P_j$.

Generally, for torsion free sheaves, the double filtrations may not have strict corners [Koo10].



Example 3.3.6. Let $\mathcal{F}$ be a locally free sheaf of rank 2 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. Since the rank is 2, either 1 or 2 box summands are nonempty. There are 4 possible choices for ${}_{b_i}\hat{F}_i$ to be nonzero.

1. ${}_{b_i}\hat{F}_i \neq 0$ for $b_1 = (k/a, 0), b_2 = (0, j/b), b_3 = (j/b, 0), b_4 = (0, k/a)$.

   On each chart, it is described by a double filtration as for $\mathcal{H}_r^{11}$.

   Since we will later work on stable sheaves and the decomposable sheaves are not stable, we'd like to classify all the types of indecomposable sheaves. They are listed below:

   (a) P_i's are mutually distinct and $\Delta_i > 0$ for all i.

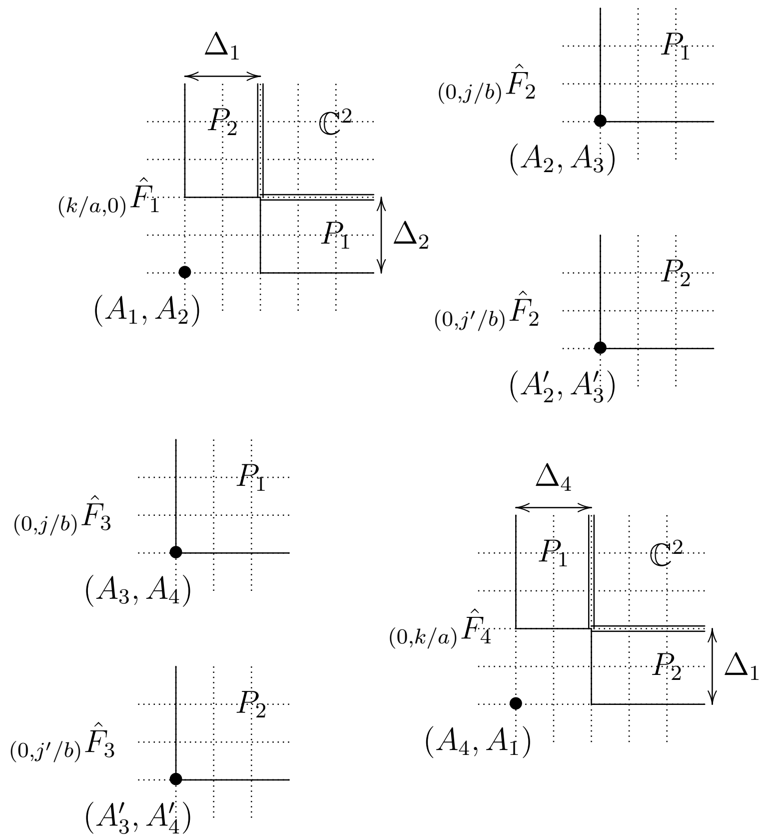
   (b) P_i's are mutually distinct and $\Delta_i = 0$ for only one i.

   (c) Only two of P_i's are same and $\Delta_i > 0$ for all i.

2. ${}_{b_i}\hat{F}_i \neq 0$ for $b_1 = (k/a, 0), b_2 = (0, j/b), b_2 = (0, j'/b), b_3 = (j/b, 0), b_3 = (j'/b, 0), b_4 = (0, k/a)$.

   Suppose $A_2' - A_2 = \Delta_2 \geq 0$ and $A_4' - A_4 = \Delta_4 \geq 0$. Denote $\Delta_3 = A_3' - A_3$. Sheaves of this type are fully determined by $A_1, A_2, A_3, A_4, b \nmid \Delta_3 \in \mathbb{Z}$,

$\Delta_1, \Delta_2, \Delta_4 \in \mathbb{Z}_{\geq 0}$, and $P_1 \neq P_2 \subset \mathbb{C}^2$. They are decomposable and can be described as follows:



3. ${}_{b_i}\hat{F}_i \neq 0$ for $b_1 = (k/a, 0), b_1 = (k'/a, 0), b_2 = (0, j/b), b_3 = (j/b, 0), b_4 = (0, k/a), b_4 = (0, k'/a)$.

   It's similar to the second case and all the sheaves of this type are decomposable.

4. Two box summands are nonzero for all the charts.

   It can be easily seen that $\mathcal{F}$ is decomposable in this case.

Chapter 4: Hilbert Polynomial

4.1 K-Group

Let $K_0(\mathcal{H}_r^{ab})$ be the Grothendieck group of coherent sheaves on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. By [BH06], $K_0(\mathcal{H}_r^{ab})_{\mathbb{Q}}$ is isomorphic to the quotient of the Laurent polynomial ring $\mathbb{Q}[x^{\pm}, y^{\pm}, z^{\pm}, w^{\pm}]$ by the ideal generated by the relations

$$\begin{cases} x^b z^{-a} = 1 \\ x^s y z^t w^{-1} = 1 \\ (1-x)(1-y)(1-z) = 0 \\ (1-x)(1-y)(1-w) = 0 \\ (1-x)(1-z)(1-w) = 0 \\ (1-y)(1-z)(1-w) = 0. \end{cases}$$

It is isomorphic to the quotient ring $\mathbb{Q}[g^{\pm}, h^{\pm}]/I$ where I is generated by

$$\begin{cases} (1-g^a)(1-g^b)(1-h) \\ (1-g^a)(1-g^b)(1-g^r h) \\ (1-g^a)(1-h)(1-g^r h) \\ (1-g^b)(1-h)(1-g^r h). \end{cases}$$

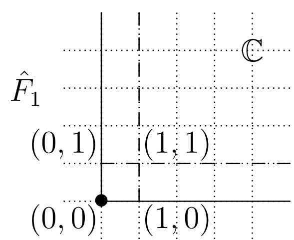
Here $g := [(-1,0)], h := [(0,-1)]$ are K-group classes of the generators of $\mathrm{Pic}(\mathcal{H}_r^{ab}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Recall that the $\mathbf{T}$-action on $\mathcal{H}_r^{ab}$ has four fixed points corresponding to the origin of each chart. Denote them by P_1, P_2, P_3, P_4.

Proposition 4.1.1. *In $K_0(\mathcal{H}_r^{ab})$, we have*

$$[\mathcal{O}_{P_1} \otimes \hat{\mu}_b^i] = (1-g^a)(1-h)g^i, \qquad [\mathcal{O}_{P_2} \otimes \hat{\mu}_a^i] = (1-g^b)(1-h)g^i,$$
$$[\mathcal{O}_{P_3} \otimes \hat{\mu}_a^i] = (1-g^b)(1-g^r h)g^i, \quad [\mathcal{O}_{P_4} \otimes \hat{\mu}_b^i] = (1-g^a)(1-g^r h)g^i.$$

Proof. The sheaf $\mathcal{O}_{P_1} \otimes \hat{\mu}_b^i$ is described by a S-family where $\hat{F}_2 = \hat{F}_3 = \hat{F}_4 = 0$ and $\hat{F}_1$ only consists of a single vector space $\mathbb{C}$ with μ_b-weight i at the position $(0,0)$.



Using the description of the line bundle introduced in Proposition 3.3.3, we can construct the exact sequence:

$$0 \longrightarrow L_{(a\cdot 1,1,0,0)} \longrightarrow L_{(a\cdot 1,0,0,0)} \oplus L_{(0,1,0,0)} \longrightarrow L_{(0,0,0,0)} \longrightarrow \mathcal{O}_{P_1} \longrightarrow 0.$$

Hence

$$[\mathcal{O}_{P_1}] = 1 + g^a h - g^a - h = (1-g^a)(1-h).$$

Since $B_1 = aA_1 = 0, B_2 = A_2 = 0$, the fine grading of $\mathcal{O}_{P_1} \otimes \hat{\mu}_b^i$ is equal to B_3 mod b

on $\mathcal{U}_1$. As a result,

$$[\mathcal{O}_{P_1} \otimes \hat{\mu}_b^i] = [\mathcal{O}_{P_1} \otimes L_{(0,0,i,0)}] = (1-g^a)(1-h)g^i.$$

The calculation for other charts is similar. □

Now let's consider the general case. Suppose there is a S-family such that $\hat{F}_2 = \hat{F}_3 = \hat{F}_4 = 0$ and $\hat{F}_1$ consists of a single space $\mathbb{C}$ with μ_b-grading i at the position $(k/a+A_1)(a,0)+A_2(0,1)$. The corresponding sheaf is $\mathcal{O}_{P_1} \otimes L_{(k+aA_1,A_2,i-k-aA_1+rA_2,0)}$. Therefore its class in $K_0(\mathcal{H}_r^{ab})$ is

$$(1-g^a)(1-h)g^{i+rA_2}h^{A_2} = (1-g^a)(1-h)g^i.$$

As a result, the class of such a sheaf in $K_0(\mathcal{H}_r^{ab})$ only depends on the fine grading. This is quite useful when we calculate the Hilbert polynomial later.

4.2 Riemann-Roch

Riemann-Roch on Deligne-Mumford stacks was first proved in [Toe99]. Later, [Edi12] gives a simpler proof based on the equivariant localization theorem. In our paper, we will follow the notation of inertia stacks used in the appendix of [Tse10], which is essentially same as [Edi12, Section 4].

Recall from [BCS05] that for each d-dimensional cone in the fan Σ, $\mathrm{Box}(\sigma)$ is the set of elements $v \in N \cong \mathbb{Z}^2$ such that $v = \sum_{\rho_i \in \sigma} q_i b_i$ where b_i is the ith column of the matrix B (2.3.3). Denote by $\mathrm{Box}(\Sigma)$ the union of $\mathrm{Box}(\sigma)$ for all d-dimensional cones.

Since $\mathcal{H}_r^{ab} \cong [Z/G]$ is a quotient stack, each component of its inertia stack is isomorphic to $[Z^g/G]$ where Z^g denotes the locus of points fixed in Z by g. By [BCS05], the elements $v \in \mathrm{Box}(\Sigma)$ are in one-to one correspondence with elements $g \in G$ that fix a point of Z.

Suppose $\gcd(a,b) = 1$. A box element for the stacky fan of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ can be in $\rho_1, \rho_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4$. Hence to find all the components of the inertia stack, we classify all the substacks which correspond to the minimal cones that contain the box elements.

If a box element is on ρ_1, then $x = 0$ and the corresponding stabilizer $g = (\tau, \lambda)$ must satisfy $\lambda = 1, \tau^b = 1, \tau^r\lambda = 1$. Suppose $\gcd(r,b) = p$, then

$$g = (e^{2\pi\sqrt{-1}\frac{l}{p}}, 1), l = 1, ..., p-1.$$

Hence the corresponding component of the inertia stack is

$$\mathcal{X}_{\rho_1} \cong [Z^g/G] \cong [\mathbb{C}^3 - V(yz, zw)/(\mathbb{C}^*)^2], \quad (\tau, \lambda) : (y, z, w) \to (\lambda y, \tau^b z, \tau^r \lambda w).$$

Let $\gcd(r,a) = q$. We summarize all the components in the table below:

	stablizer g	substack $[Z^g/G]$	$(\tau,\lambda)\in(\mathbb{C}^*)^2$-action
ρ_1	$(e^{2\pi\sqrt{-1}\frac{l}{p}},1)$ $l=1,...,p-1$	$[\mathbb{C}^3-V(yz,zw)/(\mathbb{C}^*)^2]$	$(y,z,w)\to(\lambda y,\tau^b z,\tau^r\lambda w)$
ρ_3	$(e^{2\pi\sqrt{-1}\frac{l}{q}},1)$ $l=1,...,q-1$	$[\mathbb{C}^3-V(xy,wx)/(\mathbb{C}^*)^2]$	$(x,y,w)\to(\tau^a x,\lambda y,\tau^r\lambda w)$
σ_1	$(e^{2\pi\sqrt{-1}\frac{l}{b}},e^{-2\pi\sqrt{-1}\frac{sal}{b}})$ $\frac{b}{p}\nmid l, l=1,...,b-1$	$[\mathbb{C}^2-V(zw)/(\mathbb{C}^*)^2]$	$(z,w)\to(\tau^b z,\tau^r\lambda w)$
σ_2	$(e^{2\pi\sqrt{-1}\frac{l}{a}},e^{-2\pi\sqrt{-1}\frac{tbl}{a}})$ $\frac{a}{q}\nmid l, l=1,...,a-1$	$[\mathbb{C}^2-V(xw)/(\mathbb{C}^*)^2]$	$(x,w)\to(\tau^a x,\tau^r\lambda w)$
σ_3	$(e^{2\pi\sqrt{-1}\frac{l}{a}},1)$ $\frac{a}{q}\nmid l, l=1,...,a-1$	$[\mathbb{C}^2-V(xy)/(\mathbb{C}^*)^2]$	$(x,y)\to(\tau^a x,\lambda y)$
σ_4	$(e^{2\pi\sqrt{-1}\frac{l}{b}},1)$ $\frac{b}{p}\nmid l, l=1,...,b-1$	$[\mathbb{C}^2-V(yz)/(\mathbb{C}^*)^2]$	$(y,z)\to(\lambda y,\tau^b z)$

Write $I\mathcal{H}_r^{ab}$ for the inertia stack of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. Let $\pi: I\mathcal{H}_r^{ab}\to\mathcal{H}_r^{ab}$ be the natural projection. Suppose a vector bundle V on $I\mathcal{H}_r^{ab}$ is decomposed into a direct sum $\oplus_{\zeta_i}V_i$ of eigenbundles with eigenvalue ζ_i. Let μ_∞ be the group of all roots of unity, then we define $\rho(V):=\sum_{\zeta_i}\zeta_i V^i$ and $\widetilde{ch}: K^0(\mathcal{H}_r^{ab})\to A^*(I\mathcal{H}_r^{ab})\otimes\mu_\infty$

as the composition

$$K^0(\mathcal{H}_r^{ab}) \xrightarrow{\pi^*} K^0(I\mathcal{H}_r^{ab}) \xrightarrow{\rho} K^0(I\mathcal{H}_r^{ab}) \otimes \mu_\infty \xrightarrow{ch} A^*(I\mathcal{H}_r^{ab}) \otimes \mu_\infty.$$

For a line bundle L on $\mathcal{H}_r^{ab}$, define $\widetilde{Td} : \mathrm{Pic}(\mathcal{H}_r^{ab}) \to A^*(I\mathcal{H}_r^{ab}) \otimes \mu_\infty$ as

$$\widetilde{Td}(L) = \begin{cases} Td(\pi^* L) & \text{if the eigenvalue of } \pi^* L \text{ is } 1 \\ \dfrac{1}{ch(1 - \zeta^{-1} \cdot \pi^* L^\vee)} & \text{if the eigenvalue of } \pi^* L \text{ is } \zeta \neq 1. \end{cases}$$

Then by Riemann-Roch, the Euler characteristic of a coherent sheaf $\mathcal{F}$ on $\mathcal{H}_r^{ab}$ is given by

$$\chi(\mathcal{F}) = \int_{I\mathcal{H}_r^{ab}} \widetilde{ch}(\mathcal{F}) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_1})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_2})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_3})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_4}))$$

where $\mathcal{D}_{\rho_i}$ is the divisor corresponding to the ray ρ_i in Figure 2.1.

Proposition 4.2.1. *Suppose* $\gcd(a,b) = 1$. *Consider the line bundle* $(m,n) \in \mathrm{Pic}(\mathcal{H}_r^{ab})$ *(Remark 3.3.4). The Euler characteristic is given as follows:*

$$\chi((m,n)) = \frac{1+n}{2a} + \frac{1+n}{2b} + \frac{(1+n)m}{ab} - \frac{n(n+1)r}{2ab} + \sum_{l=1}^{p-1} \frac{\omega_p^{ml}}{1-\omega_p^{-al}} \frac{n+1}{b}$$

$$+\sum_{l=1}^{q-1}\frac{\omega_q^{ml}}{1-\omega_q^{-bl}}\frac{n+1}{a}+\sum_{\substack{l=1\\ \frac{b}{p}\nmid l}}^{b-1}\frac{\omega_b^{ml}}{1-\omega_b^{-al}}\left(\frac{1-\omega_b^{-(n+1)sal}}{1-\omega_b^{-sal}}\right)\frac{1}{b}$$
$$+\sum_{\substack{l=1\\ \frac{a}{q}\nmid l}}^{a-1}\frac{\omega_a^{ml}}{1-\omega_a^{-bl}}\left(\frac{1-\omega_a^{-(n+1)tbl}}{1-\omega_a^{-tbl}}\right)\frac{1}{a}.$$

where $p=\gcd(b,r)$, $q=\gcd(a,r)$ *and* $\omega_k=e^{\frac{2\pi\sqrt{-1}}{k}}$ *for* $k=a,b,p,q$. *Especially,* $\chi(\mathcal{O}_{\mathcal{H}_r^{ab}})=\chi(\mathcal{O}(\mathcal{D}_{\rho_1}))=\chi(\mathcal{O}(\mathcal{D}_{\rho_3}))=1$. *Suppose* $r=uab-v_1a-v_2b$ *such that* $0\le v_1<b, 0\le v_2<a$, *then* $\chi(\mathcal{O}(\mathcal{D}_{\rho_2}))=2-u$.

Proof. The only 2-dimensional component of $I\mathcal{H}_r^{ab}$ is $\mathcal{H}_r^{ab}$ itself. By [EM13] and [CLS11], the orbifold Chow ring is

$$\mathbb{Q}[x,y,z,w]/(xz,yw,bx-az,sx+y+tz-w)\cong\mathbb{Q}[x,y]/(x^2,ay^2+rxy)$$

and $\int_{\mathcal{H}_r^{ab}}xy=\frac{1}{b}$.

The 1-dimensional components come from ρ_1 and ρ_3. By [BCS05], the substack $[Z^g/G]$ for ρ_1 is isomorphic to the substack constructed from the quotient stacky fan Σ/ρ_1 [BCS05]. One can show that $Z(\rho_1)\cong\mathbb{C}^2-V(y,w)$ and the action of $G(\rho_1)\cong\mathbb{C}^*\times\mu_b$ on $Z(\rho_1)$ is given by $(\lambda,\zeta)(y,w)=(\lambda y,\lambda\zeta^s w)$. Hence the Chow ring is $\mathbb{Q}[y]/(y^2)$ and $\int_{\mathcal{X}_{\rho_1}}y=\frac{1}{b}$.

Similarly, the Chow ring is $\mathbb{Q}[y]/(y^2)$ for another type of 1-dimensional components and $\int_{\mathcal{X}_{\rho_3}}y=\frac{1}{a}$.

There are 4 types of 0-dimensional components induced by σ_i. Two of them are isomorphic to $B\mu_b$, and others $B\mu_a$. The Chow ring is $\mathbb{Q}$ and $\int_{B\mu_b}1=\frac{1}{b}$, $\int_{B\mu_a}1=\frac{1}{a}$.

Thus $I\mathcal{H}_r^{ab}$ is the disjoint union of 7 types of components in general. Depending on the relations among a, b and r, there may be fewer types.

On each type of components, the Chern character of a line bundle $\widetilde{ch}((m,n))$ is given by

$$\Big(1+(\frac{m}{a}x+ny)+\frac{1}{2}(\frac{m}{a}x+ny)^2, (1+ny)\omega_p^{ml}, (1+ny)\omega_q^{ml},$$
$$\omega_b^{(m-nsa)l}, \omega_a^{(m-ntb)l}, \omega_a^{ml}, \omega_b^{ml}\Big).$$

Note that l runs over $\{1, ..., p-1\}$ for the 2nd type, $\{1, ..., q-1\}$ for the 3rd type, $\{1, ..., b-1; \frac{b}{p} \nmid l\}$ for the 4th and 7th types, $\{1, ..., a-1; \frac{a}{q} \nmid l\}$ for the 5th and 6th types.

One can also show that $\widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_1})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_2})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_3})) \cdot \widetilde{Td}(\mathcal{O}(\mathcal{D}_{\rho_4}))$ on each type of components is

$$\Big(1+y+(\frac{b}{2a}+\frac{r}{2a}+\frac{1}{2})x+(\frac{b}{2a}+\frac{1}{2})xy, \frac{1+y}{1-\omega_p^{-al}}, \frac{1+y}{1-\omega_q^{-bl}},$$
$$\frac{1}{(1-\omega_b^{-al})(1-\omega_b^{sal})}, \frac{1}{(1-\omega_a^{-bl})(1-\omega_a^{tbl})},$$
$$\frac{1}{(1-\omega_a^{-bl})(1-\omega_a^{-tbl})}, \frac{1}{(1-\omega_b^{-al})(1-\omega_b^{-sal})}\Big).$$

Adding all the integrals together, we get the desired result.

To prove $\chi(\mathcal{O}_{\mathcal{H}_r^{ab}}) = \chi(\mathcal{O}(\mathcal{D}_{\rho_1})) = \chi(\mathcal{O}(\mathcal{D}_{\rho_3})) = 1$, we repeatedly use the following two facts:

- If a, p are coprime, $\displaystyle\sum_{l=1}^{p-1} \frac{1}{1-\omega_p^{-al}} = \sum_{l=1}^{p-1} \frac{1}{1-\omega_p^{l}}$.
- $\displaystyle\frac{1}{1-\omega_p^{l}} + \frac{1}{1-\omega_p^{-l}} = 1.$

Adding all the integrals together, we get the desired result.

To show $\chi(\mathcal{O}_{\mathcal{H}_r^{ab}}) = \chi(\mathcal{O}(\mathcal{D}_{\rho_1})) = \chi(\mathcal{O}(\mathcal{D}_{\rho_3})) = 1$, we repeatedly use the following two facts:

- If a, p are coprime, $\displaystyle\sum_{l=1}^{p-1} \frac{1}{1-\omega_p^{-al}} = \sum_{l=1}^{p-1} \frac{1}{1-\omega_p^{l}}$.
- $\displaystyle\frac{1}{1-\omega_p^{l}} + \frac{1}{1-\omega_p^{-l}} = 1.$

Since $\mathcal{O}(\mathcal{D}_{\rho_2}) \cong (0,1) \in \mathrm{Pic}(\mathcal{H}_r^{ab})$,

$$\chi(\mathcal{O}(\mathcal{D}_{\rho_2})) = -\frac{s}{b} - \frac{t}{a} + 2 - \sum_{\substack{l\,=\,1 \\ \frac{b}{p} \nmid l}}^{b-1} \frac{1-\omega_b^{sl}}{1-\omega_b^{l}} \frac{1}{b} - \sum_{\substack{l\,=\,1 \\ \frac{a}{q} \nmid l}}^{a-1} \frac{1-\omega_a^{tl}}{1-\omega_a^{l}} \frac{1}{a}.$$

Now suppose $r = sa + bt$ is chosen so that $s \geq 0$ and $t \leq 0$. Assume $s = u_1 b - v_1$ and $t = -u_2 a - v_2$ where $0 \leq v_1 < b, 0 \leq v_2 < a, u_1 \geq 0, u_2 \geq 0$. Then $r = (u_1 - u_2)ab - v_1 a - v_2 b$. Set $u = u_1 - u_2$. Hence r can be written uniquely as $r = uab - v_1 a - v_2 b$ satisfying the conditions of the proposition.

One can show that

$$\sum_{\substack{l\,=\,1 \\ \frac{b}{p} \nmid l}}^{b-1} \omega_b^{jl} = \begin{cases} -p & \text{if } p \mid j \text{ and } b \nmid j \\ b-p & \text{if } b \mid j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\substack{l=1 \\ \frac{b}{p} \nmid l}}^{b-1} \frac{1-\omega_b^{sl}}{1-\omega_b^l} = \sum_{\substack{l=1 \\ \frac{b}{p} \nmid l}}^{b-1} \sum_{j=0}^{s-1} \omega_b^{jl} = (\frac{s}{p} - 1 - (u_1 - 1))(-p) + (b-p)u_1 = -s + bu_1.$$

Similarly,

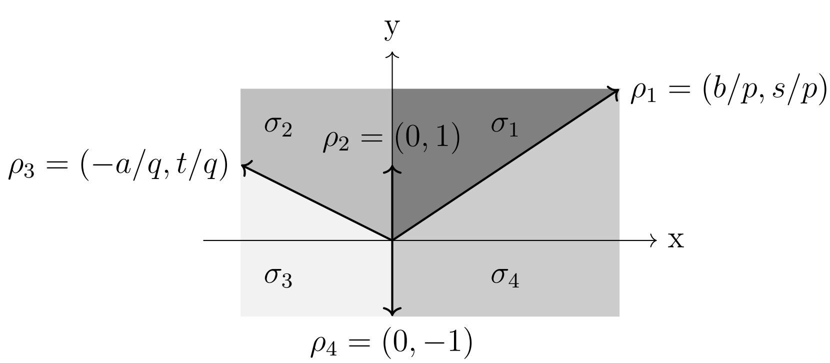
$$\sum_{\substack{l=1 \\ \frac{a}{q} \nmid l}}^{a-1} \frac{1-\omega_a^{tl}}{1-\omega_a^l} = -t - au_2.$$

Hence $\chi(\mathcal{O}(\mathcal{D}_{\rho_2})) = -u_1 + u_2 + 2 = 2 - u.$

□

4.3 Coarse Moduli Space

Suppose gcd(a,b)=1. Then the coarse moduli space of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is a toric variety H given by the following fan



where $r = sa + bt, p = \gcd(b, r)$ and $q = \gcd(a, r)$. Since $\gcd(a, b) = 1$, $\frac{s}{p}$ and $\frac{t}{q}$ are integers. Let D_i be the divisor corresponding to the ray ρ_i. Then

$$\begin{cases} \dfrac{b}{p}D_1 \sim \dfrac{a}{q}D_3 \\ \dfrac{s}{p}D_1 + D_2 + \dfrac{t}{q}D_3 \sim D_4. \end{cases}$$

To find the Picard group, we need to determine when a Weil divisor is Cartier. Suppose $D = t_1D_1 + t_2D_2$ is Cartier. Denote by n_ρ the primitive generator of the ray ρ. Then for each σ_i, there exists $m_{\sigma_i} = (x_i, y_i)$ such that $\langle m_{\sigma_i}, n_\rho\rangle = -t_\rho$ for all $\rho \in \sigma_i(1)$, where $\sigma_i(1)$ denotes the collection of rays of σ_i [CLS11].

For σ_1, it implies

$$\begin{cases} \dfrac{b}{p}x_1 + \dfrac{s}{p}y_1 = -t_1 \\ y_1 = -t_2 \end{cases}$$

from which we get $\frac{b}{p} \mid -t_1 + \frac{s}{p}t_2$. By checking each σ_i, one can show that the conditions for D to be Cartier are $\frac{b}{p} \mid t_1, \frac{ba}{pq} \mid t_2$. Therefore

$$\operatorname{Pic}(\mathrm{H}) \cong \{t_1\frac{b}{p}D_1 + t_2\frac{ba}{pq}D_2\} \cong \{t_1\frac{b}{p}D_1 + t_4\frac{ba}{pq}D_4\} \cong \mathbb{Z}^2.$$

The Cartier divisor $t_1\frac{b}{p}D_1 + t_4\frac{ba}{pq}D_4$ is ample if and only if for each σ_i, there exists $m_{\sigma_i} = (x_i, y_i)$ such that

$$\begin{cases} \langle m_{\sigma_i}, n_\rho\rangle = -t_\rho \text{ for all } \rho \in \sigma_i(1) \\ \langle m_{\sigma_i}, n_\rho\rangle > -t_\rho \text{ for all } \rho \in \Sigma(1)/\sigma_i(1) \end{cases}$$

One can compute that

$$m_{\sigma_1} = (-t_1, 0), m_{\sigma_2} = (0,0), m_{\sigma_3} = (\frac{bt}{pq}t_4, \frac{ba}{pq}t_4), m_{\sigma_4} = (-\frac{sa}{pq}t_4 - t_1, \frac{ba}{pq}t_4)$$

is a solution. We get several inequalities which reduce to $t_1 > 0$, $t_4 > 0$. Thus

$$\mathcal{O}_{\mathrm{H}}(t_1\frac{b}{p}D_1 + t_4\frac{ba}{pq}D_4) \text{ is ample if and only if } t_1, t_4 > 0.$$

Consider the ample line bundle $L = \mathcal{O}_{\mathrm{H}}(\frac{b}{p}D_1 + \frac{ba}{pq}D_4)$. By the property of the root stack [FMN10], we see that $\epsilon : \mathcal{H}_r^{ab} \to \mathrm{H}$ is a morphism with divisor multiplicities $(p, 1, q, 1)$. Hence

$$\epsilon^* L = \mathcal{O}_{\mathcal{H}_r^{ab}}(b\mathcal{D}_{\rho_1} + \frac{ba}{pq}\mathcal{D}_{\rho_4}) \cong (ba(1 + \frac{r}{pq}), \frac{ba}{pq}) \in \mathrm{Pic}(\mathcal{H}_r^{ab}).$$

For any coherent sheaf $\mathcal{F}$ on $\mathcal{H}_r^{ab}$, we can then define the Hilbert polynomial of $\mathcal{F}$ with respect to $\epsilon^* L$ as

$$P(\mathcal{F}, T) := \chi(\mathcal{F} \otimes \epsilon^* L^T).$$

Proposition 4.3.1. *Suppose* $\gcd(a,b) = 1$. *Consider the line bundle* $(m,n) \in \mathrm{Pic}(\mathcal{H}_r^{ab})$, *then*

$$\begin{aligned} P((m,n),T) &= \left(\frac{bar}{2p^2q^2} + \frac{ba}{pq}\right)T^2 + \left(\frac{a+b+2m+r}{2pq} + n + 1 + \right. \\ &\quad \left. + \sum_{l=1}^{p-1} \frac{\omega_p^{ml}}{1-\omega_p^{-al}}\frac{a}{pq} + \sum_{l=1}^{q-1} \frac{\omega_q^{ml}}{1-\omega_q^{-bl}}\frac{b}{pq}\right)T + \chi((m,n)). \end{aligned}$$

Proof. To calculate $\chi((m + ba(1 + \frac{r}{pq})T, n + \frac{ba}{pq}T))$, we note that

$$\omega_b^{\frac{ba}{pq}sa} = \omega_b^{\frac{ba}{pq}(r-tb)} = 1, \omega_a^{\frac{ba}{pq}tb} = \omega_a^{\frac{ba}{pq}(r-sa)} = 1.$$

Then the result follows. □

4.4 Modified Hilbert Polynomial

By [OS03] [Nir08b], A locally free sheaf $\mathcal{E}$ on Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is a generating sheaf if for every geometric point x of $\mathcal{H}_r^{ab}$, the representation $\mathcal{E}_x$ of the stabilizer group at that point contains every irreducible representation.

One can show that $\mathcal{E} = \bigoplus_{k=0}^{ab-1}(-k, 0)$ is a generating sheaf, although is not of minimal rank usually. Let $\epsilon : \mathcal{H}_r^{ab} \to \mathrm{H}$ be the structure morphism. Fix the generating sheaf $\mathcal{E}$ as above and the ample invertible sheaf $L = \mathcal{O}(\frac{b}{p}D_1 + \frac{ba}{pq}D_4)$. We define the modified Hilbert polynomial for a sheaf $\mathcal{F}$ on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ as

$$P_{\mathcal{E}}(\mathcal{F}, T) = \chi(\mathcal{H}_r^{ab}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \epsilon^* L^T)$$

and the modified Euler characteristic as

$$\chi_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{F}, 0).$$

Proposition 4.4.1. *Suppose* $\gcd(a,b)=1$. *Then*

$$\begin{aligned} P_{\mathcal{E}}((m,n),T) &= \left(\frac{b^2a^2r}{2p^2q^2}+\frac{b^2a^2}{pq}\right)T^2+\left(\frac{ab}{2pq}(a+b+r+2m-1+ab)\right. \\ &\left.+ab(n+1)\right)T+\frac{1+n}{2}(a+b+2m+ab-1-nr). \end{aligned}$$

Proof. To prove the proposition, we note that:

- $\displaystyle\sum_{k=0}^{ab-1}\sum_{l=1}^{q-1}\frac{\omega_p^{(m+k)l}}{1-\omega_p^{-al}}=\sum_{l=1}^{q-1}\frac{\sum_{k=0}^{ab-1}\omega_p^{(m+k)l}}{1-\omega_p^{-al}}=0$, since $p\mid ab$.
- $\displaystyle\sum_{k=0}^{ab-1}\sum_{\substack{l=1\\ \frac{b}{p}\nmid l}}^{b-1}\frac{\omega_b^{(m+k)l}}{1-\omega_b^{-al}}=0.$

Then the result follows. $\square$

Proposition 4.4.2.

$$P_{\mathcal{E}}([\mathcal{O}_{P_1}\otimes\hat{\mu}_b^i],T)=P_{\mathcal{E}}([\mathcal{O}_{P_4}\otimes\hat{\mu}_b^i],T)=a,$$

$$P_{\mathcal{E}}([\mathcal{O}_{P_2}\otimes\hat{\mu}_a^i],T)=P_{\mathcal{E}}([\mathcal{O}_{P_3}\otimes\hat{\mu}_a^i],T)=b.$$

Proof. Recall from Proposition 4.1.1 that $[\mathcal{O}_{P_1}\otimes\hat{\mu}_b^i]=g^i+g^{a+i}h-g^{a+i}-g^ih$. Hence

$$\begin{aligned} P_{\mathcal{E}}([\mathcal{O}_{P_1}\otimes\hat{\mu}_b^i],T) &= P_{\mathcal{E}}((-i,0),T)+P_{\mathcal{E}}((-a-i,-1),T) \\ &\quad - P_{\mathcal{E}}((-a-i,0),T)-P_{\mathcal{E}}((-i,-1),T)=a. \end{aligned}$$

Similarly, we can obtain the other results. $\square$

Generally, if there is a S-family such that $\hat{F}_2=\hat{F}_3=\hat{F}_4=0$ and $\hat{F}_1$ consists of a single space $\mathbb{C}$ with μ_b-weight i at the position $(k/a+A_1)(a,0)+A_2(0,1)$, then the K-group class of the corresponding sheaf is $(1-g^a)(1-h)g^i$ and $P_{\mathcal{E}}(\mathcal{O}_{P_1} \otimes L_{(k+aA_1,A_2,i-k-aA_1+rA_2,0)}, T) = a$.

Thus the modified Hilbert polynomial of a sheaf corresponding to a single space $\mathbb{C}$ in one chart only depends on the chart itself.

We will now look at the modified Hilbert polynomial of indecomposable locally free sheaves of rank 2 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$.

Recall that a necessary condition for such a sheaf to be indecomposable is exactly one nonzero box summand for each chart. In this case, we set

$$B_1 = k + aA_1, B_2 = A_2, B_3 = j + bA_3, B_4 = A_4$$

$$\Lambda_1 = a\Delta_1, \Lambda_2 = \Delta_2, \Lambda_3 = b\Delta_3, \Lambda_4 = \Delta_4$$

A locally free sheaf of such kind is entirely determined by $B_1, B_2, B_3, B_4 \in \mathbb{Z}$, $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in \mathbb{Z}_{\geq 0}$ such that $a \mid \Lambda_1, b \mid \Lambda_3$ and $P_1, P_2, P_3, P_4 \subset \mathbb{C}^2$. It is indecomposable if and only if it satisfies one of the conditions in the first case of Example 3.3.6

Proposition 4.4.3. *Let $\mathcal{F}$ be a sheaf with exactly one nonzero box summand for each chart. Then the modified Hilbert polynomial of $\mathcal{F}$ is given by*

$$\begin{aligned} & P_{\mathcal{E}}\left((-B_1 - B_3 - B_4 r, -B_2 - B_4)\right) \\ + \quad & P_{\mathcal{E}}\left((-B_1 - \Lambda_1 - B_3 - \Lambda_3 - B_4 r - \Lambda_4 r, -B_2 - \Lambda_2 - B_4 - \Lambda_4)\right) \\ - \quad & (1-\delta_{P_1P_2})\Lambda_1\Lambda_2 - (1-\delta_{P_2P_3})\Lambda_2\Lambda_3 - (1-\delta_{P_3P_4})\Lambda_3\Lambda_4 - (1-\delta_{P_4P_1})\Lambda_4\Lambda_1. \end{aligned}$$

where $\delta_{P_iP_i}$ is 1 if $P_i = P_j$ and 0 if $P_i \neq P_j$.

Proof. We can define another toric sheaf $\mathcal{G}$ such that its S-family $\hat{G}$ satisfies

$$\dim({}_bG_i(l_1, l_2)_m) = \dim({}_bF_i(l_1, l_2)_m)$$

for all charts. Then according to [GJK17, Lemma 7.7], $[\mathcal{F}] = [\mathcal{G}] \in K_0(\mathcal{H}_r^{ab})$.

To define the S-family $\hat{G}$, we set

$${}_bG_i(l_1, l_2) := {}_bL_{(B_1,B_2,B_3,B_4),i}(l_1, l_2) \oplus {}_bL_{(B_1+\Lambda_1,B_2+\Lambda_2,B_3+\Lambda_3,B_4+\Lambda_4),i}(l_1, l_2)$$

in the following regions

$$l_1 \geq A_i + \Delta_i \text{ or } l_2 \geq A_{i+1} + \Delta_{i+1},$$
$$l_1 < A_i + \Delta_i \text{ and } l_2 < A_{i+1} + \Delta_{i+1}, \text{ if } P_i = P_{i+1}$$

for $1 \leq i \leq 4$. Note that if $P_i \neq P_{i+1}$, then a rectangle of size $\Delta_i\Delta_{i+1}$ is removed. Hence the modified Hilbert polynomial is decreased by $\Lambda_i\Lambda_{i+1}$. □

Chapter 5: Moduli Space of Torsion Free Sheaves

5.1 Moduli Functor

Suppose the modified Hilbert polynomial of a pure coherent sheaf $\mathcal{F}$ of dimension d is

$$P_{\mathcal{E}}(\mathcal{F}, T) = \sum_{i=0}^{d} \alpha_{\mathcal{E},i}(\mathcal{F}) \frac{T^i}{i!}.$$

Then the reduced modified Hilbert polynomial is defined as

$$p_{\mathcal{E}}(\mathcal{F}, T) = \frac{P_{\mathcal{E}}(\mathcal{F}, T)}{\alpha_{\mathcal{E},d}(\mathcal{F})}$$

and the slope of $\mathcal{F}$ is defined as

$$\mu_{\mathcal{E}}(\mathcal{F}) = \frac{\alpha_{\mathcal{E},d-1}}{\alpha_{\mathcal{E},d}}.$$

Definition 5.1.1. $\mathcal{F}$ is Gieseker-stable if $p_{\mathcal{E}}(\mathcal{F}') < p_{\mathcal{E}}(\mathcal{F})$ for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$. [Nir08b]

Definition 5.1.2. $\mathcal{F}$ is μ-stable if $\mu_{\mathcal{E}}(\mathcal{F}') < \mu_{\mathcal{E}}(\mathcal{F})$ for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$.

For toric varieties or orbifolds, we only need to check all the equivariant subsheaves for stability. It's proved for reflexive sheaves in [Koo11] and recently for torsion-free sheaves in [BDGP18].

We can then define a moduli functor $\underline{\mathcal{M}}^s_{P_\mathcal{E}}$, where $\underline{\mathcal{M}}^s_{P_\mathcal{E}}(S)$ is the set of equivalent classes of S-flat families of Gieseker stable torsion-free sheaves on the Hirzebruch orbifold $\mathcal{H}^{ab}_r$ with the modified Hilbert polynomial $P_\mathcal{E}$. It's shown in [Nir08b] that there exists a quasi-projective scheme $\mathcal{M}^s_{P_\mathcal{E}}$ that corepresents $\underline{\mathcal{M}}^s_{P_\mathcal{E}}$ and is indeed a coarse moduli space. The closed points of $\mathcal{M}^s_{P_\mathcal{E}}$ are therefore in bijection with isomorphism classes of Gieseker stable torsion free sheaves on $\mathcal{H}^{ab}_r$ with the modified Hilbert polynomial $P_\mathcal{E}$.

We also define a moduli functor $\underline{\mathcal{M}}^{\mu s}_{P_\mathcal{E}} \subset \underline{\mathcal{M}}^s_{P_\mathcal{E}}$ which only consists of μ-stable locally free shaves. The coarse moduli space is an open subset $\mathcal{M}^{\mu s}_{P_\mathcal{E}} \subset \mathcal{M}^s_{P_\mathcal{E}}$.

To get similar results of [Koo11, Theorem 4.15], we need to modify the definition of the characteristic function for $\mathcal{H}^{ab}_r$ and match the GIT stability with the Gieseker stability.

Definition 5.1.3. Suppose $\gcd(a,b) = 1$. Let $\mathcal{F}$ be a torsion free sheaf on the Hirzebruch orbifold $\mathcal{H}^{ab}_r$. The characteristic function $\vec{\chi}_\mathcal{F}$ is defined as the disjoint union

$$\vec{\chi}_\mathcal{F} = \coprod_{k=0}^{a-1} \coprod_{j=0}^{b-1} {}_{(k,j)}\vec{\chi}_\mathcal{F}$$

where ${}_{(k,j)}\vec{\chi}_{\mathcal{F}} : (\mathbb{Z}^2)^4 \to \mathbb{Z}^4$ is the characteristic function

$$\begin{aligned} & \big({}_{(k,j)}\chi^{\sigma_1}_{\mathcal{F}}(m_1),\ {}_{(k,j)}\chi^{\sigma_2}_{\mathcal{F}}(m_2),\ {}_{(k,j)}\chi^{\sigma_3}_{\mathcal{F}}(m_3),\ {}_{(k,j)}\chi^{\sigma_4}_{\mathcal{F}}(m_4)\big) \\ = \ & \big(\dim_{\mathbb{C}}({}_{(k/a,0)}F^{\sigma_1}_{m_1}), \dim_{\mathbb{C}}({}_{(0,j/b)}F^{\sigma_2}_{m_2}), \dim_{\mathbb{C}}({}_{(j/b,0)}F^{\sigma_3}_{m_3}), \dim_{\mathbb{C}}({}_{(0,k/a)}F^{\sigma_4}_{m_4})\big). \end{aligned}$$

restricted to the following box summand

$$b_1 = (k/a, 0), b_2 = (0, j/b), b_3 = (j/b, 0), b_4 = (0, k/a).$$

Let $\mathcal{F}$ be a torsion free sheaf of rank 2, then $\mathcal{F}$ is μ-stable if and only if $\mathcal{F}^{**}$ is μ-stable. Since $\mathcal{F}^{**}$ is locally free, indecomposability of $\mathcal{F}^{**}$ implies that the S-family ${}_{b_i}\hat{F}^{**}_i \neq 0$ for only one box element by Example 3.3.6. Hence ${}_{b_i}\hat{F}_i \neq 0$ for the same b_i. As a result, the characteristic function of a stable sheaf $\mathcal{F}$ must be of the form

$$\vec{\chi}_{\mathcal{F}} = {}_{(k,j)}\vec{\chi}_{\mathcal{F}}.$$

Denote by $Gr(m, n)$ the Grassmannian of m-dimensional subspaces of $\mathbb{C}^n$. We define the following ambient quasi-projective variety:

$$\mathcal{A} = \coprod_{k=0}^{a-1}\coprod_{j=0}^{b-1}\left(\prod_{i=1}^{4}\prod_{m_i\in\mathbb{Z}^2} Gr({}_{(k,j)}\chi^{\sigma_i}_{\mathcal{F}}(m_i), 2)\right).$$

Then there is a locally closed subcheme $\mathcal{N}_{\vec{\chi}}$ of $\mathcal{A}$ whose closed points are framed [Koo10] torsion-free S-families with characteristic function $\vec{\chi}$. Consider the special linear group $G = SL(2, \mathbb{C})$. Then G acts regularly on $\mathcal{A}$ leaving $\mathcal{N}_{\vec{\chi}}$ invariant. For any G-equivariant line bundle $\mathcal{L} \in \mathrm{Pic}^G(\mathcal{N}_{\vec{\chi}})$, we can define the GIT stability with respect to $\mathcal{L}$ [Dol03]. Denote by $\mathcal{N}^s_{\vec{\chi}}$ the G-invariant open subset of GIT stable points. We obtain a geometric quotient $\pi : \mathcal{N}^s_{\vec{\chi}} \to \mathcal{M}^s_{\vec{\chi}} = \mathcal{N}^s_{\vec{\chi}}/G$.

Proposition 5.1.4. *Let $\vec{\chi}$ be the characteristic function of a torsion free sheaf of rank 2 on the Hirzebruch orbifold $\mathcal{H}^{ab}_r$. Let $P_{\mathcal{E}}$ be the modified Hilbert polynomial with respect to the ample sheaf $L = \mathcal{O}(\frac{b}{p}D_1 + \frac{ba}{pq}D_4)$ and the generating sheaf $\mathcal{E} = \bigoplus_{k=0}^{ab-1}(-k, 0)$. Then there exists an ample equivariant line bundle $\mathcal{L}_{\vec{\chi}} \in Pic^G(\mathcal{N}_{\vec{\chi}})$ such that any torsion free sheaf $\mathcal{F}$ on $\mathcal{H}^{ab}_r$ with characteristic function $\vec{\chi}$ is Gieseker stable if and only if it is GIT stable w.r.t. $\mathcal{L}_{\vec{\chi}}$.*

Proof. If $\vec{\chi}_{\mathcal{F}} = {}_{(k,j)}\vec{\chi}_{\mathcal{F}}$, then the S-family has exactly one nonzero box summand for each chart. Hence the double filtrations are similar to the cases of toric varieties as in [Koo11] and the proof carries over without any difficulties. □

Remark 5.1.5. For locally free sheaves of rank 2, we can also match the μ-stability with the GIT stability w.r.t some line bundle $\mathcal{L}^{\mu}_{\vec{\chi}}$. But in general, the line bundle $\mathcal{L}^{\mu}_{\vec{\chi}}$ is different from $\mathcal{L}_{\vec{\chi}}$. We denote the GIT quotient w.r.t this line bundle by $\mathcal{M}^{\mu s}_{\vec{\chi}}$.

Suppose $\mathcal{F}$ is a $\mathbf{T}$-equivariant sheaf on the Hirzebruch orbifold $\mathcal{H}^{ab}_r$. By tensoring a character of $\mathbf{T}$, the equivariant structure is changed, but not the underlying sheaf. This degree of freedom can be fixed by requiring $B_3 = B_4 = 0$. In this case, we call $\vec{\chi}_{\mathcal{F}}$ *gauge-fixed*. Note that our definition is slightly different from [Koo11] as we choose B_3, B_4 from σ_4, which has the largest index, to make the calculation easier.

By [Koo11], the Hilbert polynomial of a torsion free sheaf on a smooth toric variety is fully determined by the characteristic function of that sheaf. The result also applies to the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. Therefore, we can write $\mathcal{X}_{P_{\mathcal{E}}}$ for the set of characteristic functions with the modified Hilbert polynomial $P_{\mathcal{E}}$.

Since the $\mathbf{T}$-action lifts naturally to $M^s_{P_{\mathcal{E}}}$, we get the following two theorems similar to [Koo11].

Theorem 5.1.6. *For any choice of a generating sheaf $\mathcal{E}$ on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$, there is a canonical isomorphism*

$$(\mathcal{M}^s_{P_{\mathcal{E}}})^T \cong \coprod_{\vec{\chi} \in (\mathcal{X}_{P_{\mathcal{E}}})^{gf}} \mathcal{M}^s_{\vec{\chi}}.$$

Since (geometrically) μ-stability and locally freeness are open properties for the moduli functor $\underline{\mathcal{M}}^s_{P_{\mathcal{E}}}$ [Koo11] [HL10], we obtain

Theorem 5.1.7. *For any choice of a generating sheaf $\mathcal{E}$ on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$, there is a canonical isomorphism*

$$(\mathcal{M}^{\mu s}_{P_{\mathcal{E}}})^T \cong \coprod_{\vec{\chi} \in (\mathcal{X}_{P_{\mathcal{E}}})^{gf}} \mathcal{M}^{\mu s}_{\vec{\chi}}.$$

5.2 Generating Functions

Denote the moduli scheme of μ-stable torsion free, resp. locally free, sheaves of rank R with first Chern class c_1 and modified Euler characteristic $\mathcal{X}_{\mathcal{E}}$ by $M_{\mathcal{H}_r^{ab}}(R, c_1, \chi_{\mathcal{E}})$, resp. $M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(R, c_1, \chi_{\mathcal{E}})$. Our goal is to use the idea of fixed point loci to compute the following two generating functions:

$$\sum_{\chi_{\mathcal{E}}\in\mathbb{Z}} e(M_{\mathcal{H}_r^{ab}}(R, c_1, \chi_{\mathcal{E}}))\mathrm{q}^{\chi_{\mathcal{E}}}$$

$$\sum_{\chi_{\mathcal{E}}\in\mathbb{Z}} e(M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(R, c_1, \chi_{\mathcal{E}}))\mathrm{q}^{\chi_{\mathcal{E}}}$$

for $R = 1, 2$ with fixed c_1.

5.2.1 Rank 1

Consider μ-stable torsion free sheaves of rank 1 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ with fixed first Chern class $c_1 = m\frac{x}{a} + ny$ where $x = c_1(\mathcal{D}_{\rho_1}), y = c_1(\mathcal{D}_{\rho_2})$. Let

$$\mathsf{G}_{c_1}(\mathrm{q}) = \sum_{\chi_{\mathcal{E}}\in\mathbb{Z}} e(M_{\mathcal{H}_r^{ab}}(1, c_1, \chi_{\mathcal{E}}))\mathrm{q}^{\chi_{\mathcal{E}}}$$

be the generating function. Note that $e(M_{\mathcal{H}_r^{ab}}(1, c_1, \chi_{\mathcal{E}}) = e(M_{\mathcal{H}_r^{ab}}(1, c_1, \chi_{\mathcal{E}})^{\mathbf{T}})$ by torus localization.

Proposition 5.2.1.

$$\mathsf{G}_{m\frac{x}{a}+ny}(\mathrm{q}) = \mathrm{q}^{\frac{1+n}{2}(a+b+2m+ab-1-nr)}\prod_{k=1}^{\infty}\frac{1}{(1-\mathrm{q}^{-ak})^2(1-\mathrm{q}^{-bk})^2}.$$

Proof. An equivariant line bundle $L_{(B_1,B_2,B_3,B_4)}$ is non-equivariantly trivial if and only if

$$B_1 + B_3 + rB_4 = 0; B_2 + B_4 = 0.$$

If $\mathcal{F}$ is a torsion free sheaf of rank 1, then $\mathcal{F} \otimes L_{(B_3+B_4r, B_4, -B_3, -B_4)}$ is gauge-fixed. Therefore, we only consider torsion free sheaves of rank 1 with reflexive hulls $L_{(B_1,B_2,0,0)}$.

For fixed c_1, the reflexive hull is uniquely determined as $L_{(-m,-n,0,0)} \cong (m,n)$. The modified Euler characteristic is given by

$$\chi_{\mathcal{E}}((m,n)) = \frac{1+n}{2}(a+b+2m+ab-1-nr).$$

For a torsion free sheaf $\mathcal{F}$ with the reflexive hull $L_{(-m,-n,0,0)}$, the cokernel sheaf $\mathcal{Q}$ of the exact sequence

$$0 \to \mathcal{F} \to L_{(-m,-n,0,0)} \to \mathcal{Q} \to 0$$

can be described by young diagrams. By Proposition 4.4.2, the modified Euler characteristic of $\mathcal{Q}$ increases by a, resp. by b for each cell in the young diagrams on charts $\mathcal{U}_1$ and $\mathcal{U}_4$, resp. $\mathcal{U}_2$ and $\mathcal{U}_3$. Hence the closed points of $M_{\mathcal{H}_r^{ab}}(1, c_1, \chi_{\mathcal{E}})^T$ are in bijection with four partitions $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ such that

$$\frac{1+n}{2}(a+b+2m+ab-1-nr) - a(\#\lambda_1 + \#\lambda_4) - b(\#\lambda_2 + \#\lambda_3) = \chi_{\mathcal{E}}.$$

$\square$

Remark 5.2.2. By Proposition 4.4.2 and Proposition 4.1.1, the modified Euler characteristic of $\mathcal{Q}$ is independent of the fine grading, whereas the K-group class is not. Hence we do not need to consider the colored Young diagrams as in [GJK17].

5.2.2 Rank 2

For a toric surface, there is a nice expression that relates the generating functions of torsion free and locally free sheaves given by [Göt99]. We also derive a similar relation for the Hirzebruch orbifold $\mathcal{H}_r^{ab}$, which is given in Theorem 1.2.1.

Proof of Theorem 1.0.1. The proof is similar to that of [GJK17, lemma 7.4] except in our case the moduli scheme is stratified by the modified Euler characteristics. □

Let $\mathcal{F}$ be a locally free sheaf of rank 2 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. By tensoring with $L_{(B_3+rB_4,B_4,-B_3,-B_4)}$, we only consider sheaves with $B_3 = B_4 = 0$, which are gauge-fixed.

From Example 3.3.6, we know that there are three types of indecomposable sheaves. Hence, the connected components of the fixed locus $M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(R, c_1, \chi_{\mathcal{E}})^{\mathbf{T}}$ can be explicitly classified as follows:

1. P_i are mutually distinct and Λ_i are all positive. [1]

   Consider four equivariant line bundles $L_1, L_2, L_3, L_4 \subset \mathcal{F}$ generated by P_1, P_2, P_3, P_4 respectively.

$$L_1 = L_{B_1,B_2+\Lambda_2,\Lambda_3,\Lambda_4}, \qquad L_2 = L_{B_1+\Lambda_1,B_2,\Lambda_3,\Lambda_4},$$
$$L_3 = L_{B_1+\Lambda_1,B_2+\Lambda_2,0,\Lambda_4}, \quad L_4 = L_{B_1+\Lambda_1,B_2+\Lambda_2,\Lambda_3,0}.$$

[1] For notation, see Section 4.4 and Example 3.3.6

Any equivariant subsheaf of $\mathcal{F}$ is contained in one of L_i and does not have bigger slope. Hence it suffices to test $\mu_{\mathcal{E}}(L_i) < \mu_{\mathcal{E}}(\mathcal{F})$ for all L_i. The stability conditions are given by

$$\Lambda_1 < pq\Lambda_2 + \Lambda_3 + (r+pq)\Lambda_4, \quad pq\Lambda_2 < \Lambda_1 + \Lambda_3 + (r+pq)\Lambda_4,$$
$$\Lambda_3 < \Lambda_1 + pq\Lambda_2 + (r+pq)\Lambda_4, \quad (r+pq)\Lambda_4 < \Lambda_1 + pq\Lambda_2 + \Lambda_3.$$

Denote by D the set of points $(P_1, P_2, P_3, P_4) \in (\mathbb{P}^1)^4$ where P_1, P_2, P_3, P_4 are mutually distinct. Then the connected component of the fixed locus is given by $D/\mathrm{SL}(2,\mathbb{C})$ and $e(D/\mathrm{SL}(2,\mathbb{C})) = e(\mathbb{P}^1 - \{0, 1, \infty\}) = -1$.

2. P_i are mutually distinct and one of Λ_i is 0.

   Suppose Λ_1 is 0, then the above inequalities are reduced to

   $$pq\Lambda_2 < \Lambda_3 + (r+pq)\Lambda_4, \quad \Lambda_3 < pq\Lambda_2 + (r+pq)\Lambda_4,$$
   $$(r+pq)\Lambda_4 < pq\Lambda_2 + \Lambda_3.$$

   Hence the connected component is $D/\mathrm{SL}(2,\mathbb{C})$, where D is the set of points $(P_2, P_3, P_4) \in (\mathbb{P}^1)^3$ where P_2, P_3, P_4 are mutually distinct, and $e(D/\mathrm{SL}(2,\mathbb{C})) = 1$.

3. $P_i = P_j$ for some i, j and Λ_i are all positive.

   Suppose $P_1 = P_2$, P_3, P_4 are mutually distinct. Then we need to consider line bundles L_1', L_3, L_4 where $L_1' = L_{(B_1, B_2, \Lambda_3, \Lambda_4)})$. The stability conditions are are given by

$$\Lambda_1 + pq\Lambda_2 < \Lambda_3 + (r+pq)\Lambda_4, \quad \Lambda_3 < \Lambda_1 + pq\Lambda_2 + (r+pq)\Lambda_4,$$
$$(r+pq)\Lambda_4 < \Lambda_1 + pq\Lambda_2 + \Lambda_3.$$

Similar to the case 2, the topological Euler number of this component is 1.

Thus there are 11 types of incidence spaces contributing to the generating function similar to the case of Hirzebruch surface in [Koo15].

Consider locally free sheaves of rank 2 with fixed first Chern class $c_1 = \frac{m}{a}x + ny$ where $c_1(\mathcal{D}_{\rho_1}) = x, c_1(\mathcal{D}_{\rho_2}) = y$. By Proposition 4.4.3, one can show that

$$c_1 = -(2B_1 + \Lambda_1 + \Lambda_3 + \Lambda_4 r)\frac{x}{a} - (2B_2 + \Lambda_2 + \Lambda_4)y.$$

Hence

$$2B_1 + \Lambda_1 + \Lambda_3 + \Lambda_4 r = -m, \quad 2B_2 + \Lambda_2 + \Lambda_4 = -n.$$

If $\mathcal{F}$ is of the first type mentioned above, then the modified Euler characteristic is given by

$$\begin{aligned} &P_{\mathcal{E}}\left((-B_1, -B_2), 0\right) + P_{\mathcal{E}}\left((-B_1 - \Lambda_1 - \Lambda_3 - \Lambda_4 r, -B_2 - \Lambda_2 - \Lambda_4), 0\right) \\ &-\Lambda_1\Lambda_2 - \Lambda_2\Lambda_3 - \Lambda_3\Lambda_4 - \Lambda_4\Lambda_1 \\ = &\frac{1}{2}(C-r)n + C + m + \frac{mn}{2} - \frac{n^2 r}{4} - \frac{1}{2}(\Lambda_2 + \Lambda_4)(\Lambda_1 + \frac{r}{2}\Lambda_2 + \Lambda_3 - \frac{r}{2}\Lambda_4) \end{aligned}$$

where $C = a + b + ab - 1$. Similarly, we can obtain the modified Euler characteristics for other types.

Let

$$\mathsf{H}^{\mathrm{vb}}_{c_1}(\mathrm{q}) = \sum e(M^{\mathrm{vb}}_{\mathcal{H}^{ab}_r}(2, c_1, \chi\varepsilon))\mathrm{q}^{\chi\varepsilon}$$

be the generating function. Define $f = \frac{1}{2}(C-r)n + C + m + \frac{mn}{2} - \frac{n^2 r}{4}$. Then

$$\begin{aligned}
\mathsf{H}^{\mathrm{vb}}_{\frac{m}{a}x+ny}(\mathrm{q}) = &- \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\in\mathbb{Z}_{>0},\, a\mid\Lambda_1,\, b\mid\Lambda_3 \\ 2\mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2\mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_1 < pq\Lambda_2+\Lambda_3+(r+pq)\Lambda_4 \\ pq\Lambda_2 < \Lambda_1+\Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < \Lambda_1+pq\Lambda_2+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < \Lambda_1+pq\Lambda_2+\Lambda_3}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)} \\
&+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\in\mathbb{Z}_{>0},\, a\mid\Lambda_1,\, b\mid\Lambda_3 \\ 2\mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2\mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_1+\Lambda_3 < pq\Lambda_2+(r+pq)\Lambda_4 \\ pq\Lambda_2 < \Lambda_1+\Lambda_3+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < \Lambda_1+pq\Lambda_2+\Lambda_3}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)} \\
&+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\in\mathbb{Z}_{>0},\, a\mid\Lambda_1,\, b\mid\Lambda_3 \\ 2\mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2\mid -n-\Lambda_2-\Lambda_4 \\ pq\Lambda_2+(r+pq)\Lambda_4 < \Lambda_1+\Lambda_3 \\ \Lambda_1 < pq\Lambda_2+\Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < \Lambda_1+pq\Lambda_2+(r+pq)\Lambda_4}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)} \\
&+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\in\mathbb{Z}_{>0},\, a\mid\Lambda_1,\, b\mid\Lambda_3 \\ 2\mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2\mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_1+pq\Lambda_2 < \Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < \Lambda_1+pq\Lambda_2+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < pq\Lambda_2+\Lambda_3+\Lambda_1}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)+\Lambda_1\Lambda_2} \\
&+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\in\mathbb{Z}_{>0},\, a\mid\Lambda_1,\, b\mid\Lambda_3 \\ 2\mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2\mid -n-\Lambda_2-\Lambda_4 \\ pq\Lambda_2+\Lambda_3 < \Lambda_1+(r+pq)\Lambda_4 \\ \Lambda_1 < pq\Lambda_2+\Lambda_3+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < pq\Lambda_2+\Lambda_3+\Lambda_1}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)+\Lambda_2\Lambda_3}
\end{aligned}$$

$$+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4 \in \mathbb{Z}_{>0},\, a \mid \Lambda_1,\, b \mid \Lambda_3 \\ 2 \mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2 \mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_3+(r+pq)\Lambda_4 < \Lambda_1+pq\Lambda_2 \\ \Lambda_1 < pq\Lambda_2+\Lambda_3+(r+pq)\Lambda_4 \\ pq\Lambda_2 < \Lambda_1+\Lambda_3+(r+pq)\Lambda_4}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)+\Lambda_3\Lambda_4}$$

$$+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4 \in \mathbb{Z}_{>0},\, a \mid \Lambda_1,\, b \mid \Lambda_3 \\ 2 \mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2 \mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_1+(r+pq)\Lambda_4 < pq\Lambda_2+\Lambda_3 \\ pq\Lambda_2 < \Lambda_1+\Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < \Lambda_1+pq\Lambda_2+(r+pq)\Lambda_4}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)+\Lambda_4\Lambda_1}$$

$$+ \sum_{\substack{\Lambda_2,\Lambda_3,\Lambda_4 \in \mathbb{Z}_{>0},\, b \mid \Lambda_3 \\ 2 \mid -m-\Lambda_3-r\Lambda_4,\, 2 \mid -n-\Lambda_2-\Lambda_4 \\ pq\Lambda_2 < \Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < pq\Lambda_2+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < pq\Lambda_2+\Lambda_3}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\frac{r}{2}\Lambda_2+\Lambda_3-\frac{r}{2}\Lambda_4)}$$

$$+ \sum_{\substack{\Lambda_1,\Lambda_3,\Lambda_4 \in \mathbb{Z}_{>0},\, a \mid \Lambda_1,\, b \mid \Lambda_3 \\ 2 \mid -m-\Lambda_1-\Lambda_3-r\Lambda_4,\, 2 \mid -n-\Lambda_4 \\ \Lambda_1 < \Lambda_3+(r+pq)\Lambda_4 \\ \Lambda_3 < \Lambda_1+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < \Lambda_1+\Lambda_3}} \mathrm{q}^{f-\frac{1}{2}\Lambda_4(\Lambda_1+\Lambda_3-\frac{r}{2}\Lambda_4)}$$

$$+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_4 \in \mathbb{Z}_{>0},\, a \mid \Lambda_1 \\ 2 \mid -m-\Lambda_1-r\Lambda_4,\, 2 \mid -n-\Lambda_2-\Lambda_4 \\ \Lambda_1 < pq\Lambda_2+(r+pq)\Lambda_4 \\ pq\Lambda_2 < \Lambda_1+(r+pq)\Lambda_4 \\ (r+pq)\Lambda_4 < \Lambda_1+pq\Lambda_2}} \mathrm{q}^{f-\frac{1}{2}(\Lambda_2+\Lambda_4)(\Lambda_1+\frac{r}{2}\Lambda_2-\frac{r}{2}\Lambda_4)}$$

$$+ \sum_{\substack{\Lambda_1,\Lambda_2,\Lambda_3 \in \mathbb{Z}_{>0},\, a \mid \Lambda_1,\, b \mid \Lambda_3 \\ 2 \mid -m-\Lambda_1-\Lambda_3,\, 2 \mid -n-\Lambda_2 \\ \Lambda_1 < pq\Lambda_2+\Lambda_3 \\ pq\Lambda_2 < \Lambda_1+\Lambda_3 \\ \Lambda_3 < \Lambda_1+pq\Lambda_2}} \mathrm{q}^{f-\frac{1}{2}\Lambda_2(\Lambda_1+\frac{r}{2}\Lambda_2+\Lambda_3)}$$

Note that the first term corresponds to the component of the first type and the negative sign comes from $e(\mathbb{P}^1 - \{0,1,\infty\}) = -1$. The signs for the remaining terms are positive because the topological Euler number is 1 for the other components. Using proper substitutions, we can simplify this generating function further.

Proposition 5.2.3. *Suppose* $\gcd(a,b) = 1$. *Let* $f = \frac{1}{2}(C-r)n + C + m + \frac{mn}{2} - \frac{n^2 r}{4}$ *where* $C = a + b + ab - 1$. *If* $r \geq 0$, *the generating function* $\mathsf{H}^{vb}_{\frac{m}{a}x+ny}(\mathrm{q})$ *equals*

$$\begin{aligned} \mathsf{H}^{vb}_{\frac{m}{a}x+ny}(\mathrm{q}) = & \Bigg(-\sum_{C_1} + \sum_{C_6} + \sum_{C_7} + \sum_{C_8} + \sum_{C_9} \Bigg) \mathrm{q}^{f - \frac{1}{2}j(i + \frac{r}{2}j)} \\ & + \Bigg(\sum_{C_2} + \sum_{C_3} + \sum_{C_4} + \sum_{C_5} \Bigg) \mathrm{q}^{f - \frac{1}{4}ij + \frac{1}{4}jk - \frac{1}{4}kl - \frac{1}{4}li - \frac{r}{4}l^2} \end{aligned}$$

where

$$C_1 = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid m+i, 2 \mid n+j, 2 \mid j-l, 2a \mid i+k+r(j-l),$$
$$2b \mid i-k, i = pqj, -j < l < j, -pqj - r(j-l) < k < pqj\},$$
$$C_2 = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid m+i, 2 \mid n+j, 2 \mid j-l, 2a \mid i+k+r(j+l),$$
$$2b \mid i-k, k < pql < i, l < j, -i - r(j+l) < k, -pqj - r(j+l) < k\},$$
$$C_3 = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid m+i, 2 \mid n+j, 2 \mid j-l, 2b \mid i+k+r(j-l),$$
$$2a \mid i-k, k < pql < i, l < j, -i - r(j+l) < k, -pqj - r(j+l) < k\},$$
$$C_4 = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid m+i, 2 \mid n+j, 2 \mid j-l, 2b \mid i+k-r(j-l),$$
$$2a \mid i-k, k < pql < i, l < j, -i + r(j-l) < k, -pqj < k\},$$
$$C_5 = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid m+i, 2 \mid n+j, 2 \mid j-l, 2a \mid i+k-r(j-l),$$
$$2b \mid i-k, k < pql < i, l < j, -i + r(j-l) < k, -pqj < k\},$$

$$C_6 = \{(i,j,k) \in \mathbb{Z}^3 : 2 \mid m+i, 2 \mid n+j, 2 \mid j+k, 2b \mid 2i + r(j+k),$$

$$-\frac{r}{2}(j+k) < i, i < pqj - \frac{i}{r+pq} - \frac{rj}{r+pq} < k < \frac{i}{pq}\},$$

$$C_7 = \{(i,j,k) \in \mathbb{Z}^3 : 2 \mid m+i, 2 \mid n+j, 2 \mid j+k, 2a \mid 2i + r(j+k),$$

$$-\frac{r}{2}(j+k) < i, i < pqj - \frac{i}{r+pq} - \frac{rj}{r+pq} < k < \frac{i}{pq}\},$$

$$C_8 = \{(i,j,k) \in \mathbb{Z}^3 : 2 \mid m+i, 2 \mid n+j, 2a \mid i+k+2rj, 2b \mid i-k,$$

$$-pqj - 2rj < k < pqj < i\},$$

$$C_9 = \{(i,j,k) \in \mathbb{Z}^3 : 2 \mid m+i, 2 \mid n+j, 2a \mid i+k, 2b \mid i-k,$$

$$-pqj < k < pqj < i\},$$

Proof. Set $i = \Lambda_1 + \Lambda_3 - r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = \Lambda_1 - \Lambda_3 - r\Lambda_4, l = \Lambda_2 - \Lambda_4$. The first term is split into two

$$-\sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid m+i, 2 \mid n+j, 2 \mid j-l \\ 2b \mid i-k, 2a \mid i+k+r(j-l) \\ pqj \leq i, -j < l < j \\ -pqj - r(j-l) < k < pqj}} \mathrm{q}^{f - \frac{1}{2}j(i + \frac{r}{2}j)} - \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ 2 \mid m+i, 2 \mid n+j, 2 \mid j-l \\ 2b \mid i-k, 2a \mid i+k+r(j-l) \\ i < pqj, -i < pql < i + r(j-l) \\ -i - r(j-l) < k < i}} \mathrm{q}^{f - \frac{1}{2}j(i + \frac{r}{2}j)}$$

based on whether $pqj \leq i$ or $pqj > j$. By same substitutions, the first three terms can be combined into one. The remaining terms can be obtained by the following substitution.

Term	Substitutions
4th	$i = \Lambda_1 + \Lambda_3 - r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = \Lambda_1 - \Lambda_3 - r\Lambda_4, l = \Lambda_4 - \Lambda_2$
5th	$i = \Lambda_1 + \Lambda_3 - r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = -\Lambda_1 + \Lambda_3 - r\Lambda_4, l = \Lambda_4 - \Lambda_2$
6th	$i = \Lambda_1 + \Lambda_3 + r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = -\Lambda_1 + \Lambda_3 + r\Lambda_4, l = \Lambda_2 - \Lambda_4$
7th	$i = \Lambda_1 + \Lambda_3 + r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = \Lambda_1 - \Lambda_3 + r\Lambda_4, l = \Lambda_2 - \Lambda_4$
8th	$i = \Lambda_3 - r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = \Lambda_4 - \Lambda_2$
9th	$i = \Lambda_1 + \Lambda_3 - r\Lambda_4, j = \Lambda_4, k = \Lambda_1 - \Lambda_3 - r\Lambda_4$
10th	$i = \Lambda_1 - r\Lambda_4, j = \Lambda_2 + \Lambda_4, k = \Lambda_4 - \Lambda_2$
11th	$i = \Lambda_1 + \Lambda_3, j = \Lambda_2, k = \Lambda_1 - \Lambda_3$

$\square$

If $r = 0$, the above result yields the Theorem 1.2.2 for the orbifold $\mathbb{P}(a, b) \times \mathbb{P}^1$.

Remark 5.2.4. If $a = b = 1$, the orbifold becomes the variety $\mathbb{P}^1 \times \mathbb{P}^1$ and $f = \frac{mn}{2} + m + n + 2$. Consider a torsion free sheaf $\mathcal{F}$ of rank 2 with $c_1 = mx + ny$ where $c_1(\mathcal{D}_{\rho_1}) = x, c_1(\mathcal{D}_{\rho_2}) = y$. Suppose $c_2(\mathcal{F}) = cxy$. One can show that $\mathcal{X}(\mathcal{F}) = -c + mn + m + n + 2$. Hence the above generating function agrees with the one given in [Koo10, Corollary 2.3.4] when $\lambda = 1$. Note that the divisor D_4 in [Koo10] is really D_2 in our paper, but $D_2 \sim D_4$ in the case of $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $(i, j) \in \mathrm{Pic}(\mathcal{H}_r^{ab})$. One can show that tensoring $- \otimes (i, j)$ preserves μ-

stability. Suppose $\mathcal{F}$ is a locally free sheaf of rank 2 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ with $c_1(\mathcal{F}) = \frac{m}{a}x + ny$. Then

$$\chi_{\mathcal{E}}(\mathcal{F} \otimes (i,j)) = \chi_{\mathcal{E}}(\mathcal{F}) + i(2+n+2j) + j(ab+a+b-1-r+m-nr-rj).$$

Let $g(i,j) = i(2+n+2j) + j(ab+a+b-1-r+m-nr-rj)$. We obtain an isomorphism

$$M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(2, c_1, \chi_{\mathcal{E}}) \cong M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(2, c_1 + \frac{2i}{a}x + 2jy, \chi_{\mathcal{E}} + g(i,j)),$$

which induces

$$\sum_{\chi_{\mathcal{E}} \in \mathbb{Z}} e(M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(2, c_1 + \frac{2i}{a}x + 2jy, \chi_{\mathcal{E}}))\mathrm{q}^{\chi_{\mathcal{E}}} = \mathrm{q}^{g(i,j)} \sum_{\chi_{\mathcal{E}} \in \mathbb{Z}} e(M^{\mathrm{vb}}_{\mathcal{H}_r^{ab}}(2, c_1, \chi_{\mathcal{E}}))\mathrm{q}^{\chi_{\mathcal{E}}}.$$

Thus for the Hirzebruch orbifold, the only interesting cases for the generating functions are $(m,n) = (0,0), (0,1), (1,0)$ and $(1,1)$.

Proposition 5.2.5. *Consider the orbifold $\mathcal{H}_0^{12}$, which is $\mathbb{P}(1,2) \times \mathbb{P}^1$. In this case, $r = 0, a = 1, b = 2, p = 1, q = 2, C = 4$. Let $c_1(\mathcal{F}) = mx + ny$ where $c_1(\mathcal{D}_{\rho_1}) = x$ and $c_1(\mathcal{D}_{\rho_2}) = y$.*

1. If $(m,n)=(0,0)$*, then* $f=4$*.*

$$
\begin{aligned}
\mathsf{H}_0^{vb}(\mathrm{q}) &= -\sum_{t=1}^{\infty}(2t-1)^2\mathrm{q}^{4-4t^2} \\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t}4\mathrm{q}^{4-(4t+4)(t-p+1)-2p-2u}\,\frac{\mathrm{q}^{-(2u+2p)p}-\mathrm{q}^{-(2u+2p)(2t+1)}}{1-\mathrm{q}^{-(2u+2p)}} \\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t}4\mathrm{q}^{4-(4t+2)(t-p+1)}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-(2u+2p-2)(2t+1)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t}4\mathrm{q}^{4-(4t+4)(t-p+1)-2p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-(2t+1)(2p)}}{(1-\mathrm{q}^{-(4t+4-2p)})(1-\mathrm{q}^{-2p})} \\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1}4\mathrm{q}^{4-2t(2t-2p+1)}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t-2p)})(1-\mathrm{q}^{-2p})} \\
&+\sum_{t=1}^{\infty}2(2t-1)\frac{\mathrm{q}^{4-4t(t+1)}}{1-\mathrm{q}^{-4t}}+\sum_{t=1}^{\infty}2(2t-1)\frac{\mathrm{q}^{4-(4t-2)t}}{1-\mathrm{q}^{-(4t-2)}} \\
&+\sum_{t=1}^{\infty}2(2t-1)\frac{\mathrm{q}^{4-4t(t+1)}}{1-\mathrm{q}^{-4t}}+\sum_{t=1}^{\infty}2(2t-1)\frac{\mathrm{q}^{4-2t(2t+1)}}{1-\mathrm{q}^{-4t}} \\
&= 2\mathrm{q}^2+5+\frac{8}{\mathrm{q}^2}+\frac{18}{\mathrm{q}^4}+O\left[\frac{1}{\mathrm{q}}\right]^5.
\end{aligned}
$$

2. If $(m,n)=(1,0)$*, then* $f=5$*.*

$$
\begin{aligned}
\mathsf{H}_x^{vb}(\mathrm{q}) &= \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t}2\mathrm{q}^{5-(4t+1)(t-p+1)}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-(2u+2p-2)(2t+1)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t}2\mathrm{q}^{5-(4t+2)(t-p+1)+t+u}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-(2u+2p-2)(2t+1)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1}2\mathrm{q}^{5-(4t-1)(t-p)}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-2t(2u+2p-2)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t}2\mathrm{q}^{5-(4t+2)(t-p+1)-t-u}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-(2u+2p-2)(2t+1)}}{1-\mathrm{q}^{-(2u+2p-2)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{5-(4t+1)(t-p)-2p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{5-(4t+1)(t-p)-p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{5-(4t+3)(t-p)-2p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{5-(4t+3)(t-p)-3p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty} 2t\frac{\mathrm{q}^{5-(4t+1)(t+1)}}{1-\mathrm{q}^{-(4t+1)}}+\sum_{t=1}^{\infty} 2t\,\frac{\mathrm{q}^{5-(4t-1)t}}{1-\mathrm{q}^{-(4t-1)}}+\sum_{t=1}^{\infty} 4t\frac{\mathrm{q}^{5-(4t+1)t}}{1-\mathrm{q}^{-2t}}\\
&=\ 2\mathrm{q}^3+4\mathrm{q}^2+6\mathrm{q}+8+\frac{12}{\mathrm{q}}+\frac{12}{\mathrm{q}^2}+\frac{14}{\mathrm{q}^3}+\frac{20}{\mathrm{q}^4}+O\left[\frac{1}{\mathrm{q}}\right]^5.
\end{aligned}
$$

3. If $(m,n)=(0,1)$, then $f=6$.

$$
\begin{aligned}
\mathsf{H}_y^{vb}(\mathrm{q}) &= -\sum_{t=1}^{\infty} 4t^2\mathrm{q}^{6-(2t+1)^2}\\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1} 4\mathrm{q}^{6-2t(2t-2p+1)}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-2t(2u+2p-2)}}{1-\mathrm{q}^{-(2u+2p-2)}}\\
&+\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1} 4\mathrm{q}^{5-4t(t-p+1)-2u}\,\frac{\mathrm{q}^{-(2u+2p)p}-\mathrm{q}^{-2t(2u+2p)}}{1-\mathrm{q}^{-(2u+2p)}}\\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t} 4\mathrm{q}^{6-(4t+2)(t-p+1)}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-2p(2t+1)}}{(1-\mathrm{q}^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty}\sum_{p=1}^{2t-1} 4\mathrm{q}^{5-4t(t-p+1)}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-4pt}}{(1-\mathrm{q}^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})}\\
&+\sum_{t=1}^{\infty}(4t-1)\frac{\mathrm{q}^{6-2t(2t+1)}}{1-\mathrm{q}^{-4t}}+\sum_{t=1}^{\infty}(4t-3)\frac{\mathrm{q}^{6-(2t-1)(2t+1)}}{1-\mathrm{q}^{-(4t-2)}}\\
&+\sum_{t=1}^{\infty} 2(2t-1)\frac{\mathrm{q}^{6-2t(2t-1)}}{1-\mathrm{q}^{-(4t-2)}}+\sum_{t=1}^{\infty} 4t\,\frac{\mathrm{q}^{6-(2t+1)(2t+3)}}{1-\mathrm{q}^{-(4t+2)}}\\
&=\ 2\mathrm{q}^4+\mathrm{q}^3+6\mathrm{q}^2+\mathrm{q}+9+\frac{5}{\mathrm{q}}+\frac{14}{\mathrm{q}^2}-\frac{3}{\mathrm{q}^3}+\frac{17}{\mathrm{q}^4}+O\left[\frac{1}{\mathrm{q}}\right]^5.
\end{aligned}
$$

4. If $(m,n)=(1,1)$*, then* $f=\dfrac{15}{2}$.

$$
\begin{aligned}
&\mathsf{H}^{vb}_{x+y}(\mathrm{q}) = \\
&\quad \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t} 2\mathrm{q}^{7-(4t+3)(t-p)-2p}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-(2u+2p-2)(2t+1)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&\quad + \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{7-(4t-1)(t-p+1)-u+p}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-2t(2u+2p-2)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&\quad + \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{8-(4t-1)(t-p)-2p}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-2t(2u+2p-2)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&\quad + \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}\sum_{p=1}^{2t-1} 2\mathrm{q}^{7-(4t+1)(t-p)-p+u}\,\frac{\mathrm{q}^{-(2u+2p-2)p}-\mathrm{q}^{-2t(2u+2p-2)}}{1-\mathrm{q}^{-(2u+2p-2)}} \\
&\quad + \sum_{t=1}^{\infty}\sum_{p=1}^{2t} 2\mathrm{q}^{8-(4t+3)(t-p+1)}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-2p(2t+1)}}{(1-\mathrm{q}^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})} \\
&\quad + \sum_{t=1}^{\infty}\sum_{p=1}^{2t} 2\mathrm{q}^{8-(4t+3)(t-p+1)-p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-2p(2t+1)}}{(1-\mathrm{q}^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})} \\
&\quad + \sum_{t=1}^{\infty}\sum_{p=1}^{2t} 2\mathrm{q}^{7-(4t+1)(t-p+1)}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-2p(2t+1)}}{(1-q^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})} \\
&\quad + \sum_{t=1}^{\infty}\sum_{p=1}^{2t} 2\mathrm{q}^{7-(4t+1)(t-p+1)+p}\,\frac{\mathrm{q}^{-2p^2}-\mathrm{q}^{-2p(2t+1)}}{(1-\mathrm{q}^{-(4t+2-2p)})(1-\mathrm{q}^{-2p})} \\
&\quad + \sum_{t=1}^{\infty} 2t\,\frac{\mathrm{q}^{\frac{15}{2}-\frac{1}{2}(2t+1)(4t+1)}}{1-\mathrm{q}^{-(4t+1)}} + \sum_{t=1}^{\infty} 2t\,\frac{\mathrm{q}^{\frac{15}{2}-\frac{1}{2}(2t+1)(4t-1)}}{1-\mathrm{q}^{-(4t-1)}} \\
&\quad + \sum_{t=1}^{\infty} 2(2t-1)\,\frac{\mathrm{q}^{\frac{15}{2}-\frac{1}{2}(2t-1)(4t+1)}}{1-\mathrm{q}^{-(4t-2)}} + \sum_{t=1}^{\infty} 2(2t-1)\,\frac{\mathrm{q}^{\frac{15}{2}-\frac{1}{2}(2t-1)(4t-1)}}{1-\mathrm{q}^{-(4t-2)}} \\
&\quad = 2\mathrm{q}^6+4\mathrm{q}^5+6\mathrm{q}^4+8\mathrm{q}^3+10\mathrm{q}^2+14+\frac{14}{\mathrm{q}}+\frac{18}{\mathrm{q}^2}+\frac{24}{\mathrm{q}^3}+\frac{22}{\mathrm{q}^4}+O\left[\frac{1}{\mathrm{q}}\right]^5.
\end{aligned}
$$

Proof. We will show how to rewrite the sums over C_2 and C_3 in the case of $(m,n)=(0,0)$. The calculation of other parts is similar.

The second and third terms can be combined into one

$$\sum_{C_2'} 4\mathrm{q}^{4-\frac{1}{4}ij+\frac{1}{4}jk-\frac{1}{4}kl-\frac{1}{4}li}$$

where

$$C_2' = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid i, 2 \mid j, 2 \mid l, 2 \mid k, 4 \mid i-k, -i < k < 2l < i, -2j < k, l < j\}.$$

It can be then split into two terms by either $i < 2j$ or $2j \leq i$.

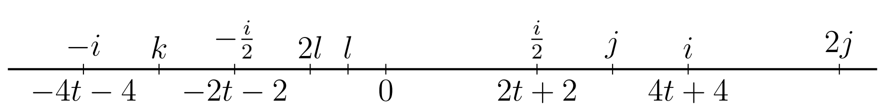
$$\Big(\sum_{C_2''} + \sum_{C_3''}\Big) 4\mathrm{q}^{4-\frac{1}{4}ij+\frac{1}{4}jk-\frac{1}{4}kl-\frac{1}{4}li}$$

where

$$C_2'' = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid i, 2 \mid j, 2 \mid l, 2 \mid k, 4 \mid i-k, \quad -2j < -i < k < 2l < i < 2j\},$$

$$C_3'' = \{(i,j,k,l) \in \mathbb{Z}^4 : 2 \mid i, 2 \mid j, 2 \mid l, 2 \mid k, 4 \mid i-k, \quad -i < -2j < k < 2l < 2j < i\}.$$

Suppose $i = 4t+4$, then we have the following picture for the case $i < 2j$.



Hence $j = 2t + 2 + 2u, l = 2t + 2 - 2p, k = 4t + 4 - 4p - 4s$ and

$$\sum_{C_2''} \mathrm{q}^{\frac{1}{4}ij - \frac{1}{4}jk + \frac{1}{4}kl + \frac{1}{4}li} = \sum_{t=1}^{\infty} \sum_{u=1}^{\infty} \sum_{p=1}^{2t} \sum_{s=1}^{2t+1-p} \mathrm{q}^{4t^2+8t-4pt-4p+2up+2p^2+4+2s(u+p)}$$

$$= \sum_{t=1}^{\infty} \sum_{u=1}^{\infty} \sum_{p=1}^{2t} \mathrm{q}^{(4t+4)(t-p+1)+2p+2u} \frac{\mathrm{q}^{(2u+2p)p} - \mathrm{q}^{(2u+2p)(2t+1)}}{1 - \mathrm{q}^{2u+2p}}.$$

This is the second term of the generating function in the case of $(m, n) = (0, 0)$.

Suppose $i = 4t + 2$, we will obtain the third term. The fourth and fifth terms come from the case when $2j \leq i$.

Basically, we split the terms by $4 \mid i$ or $4 \mid i+2$ when i is even, and by $4 \mid i+1$ or $4 \mid i+3$ when i is odd. Then the result follows from tedious calculation. $\square$

Chapter 6: Donaldson-Thomas Invariants

The Donaldson-Thomas invariant $DT(X;\alpha)$ of a Calabi-Yau manifold X, constructed by [Tho00], is the virtual count of stable sheaves on X with Chern character α. It is originally defined in the case when there are no strictly semistable sheaves.

Let X be a Calabi-Yau threefold over $\mathbb{C}$. Denote by $K_0(X) = K_0(\mathrm{coh}(X))$ the Grothendieck group of the abelian category $\mathrm{coh}(X)$ of coherent sheaves on X. The Euler form [JS12] is an anti-symmetric bilinear map:

$$\bar{\chi} : K_0(X) \times K_0(X) \to \mathbb{Z}$$
$$\bar{\chi}([\mathcal{E}], [\mathcal{F}]) = \textstyle\sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$$

for all $\mathcal{E}, \mathcal{F} \in \mathrm{coh}(X)$. The numerical Grothendieck group $K(X)$ is the quotient of $K_0(X)$ by the two-sided kernel of $\bar{\chi}$. The Euler form descends to a non-degenerate anti-symmetric bilinear form on $K(X)$.

Define the positive cone $C(X)$ to be

$$\{[\mathcal{F}] \in K(X) \,|\, 0 \ncong \mathcal{F} \in \mathrm{coh}(X)\}.$$

Fix an ample line bundle $\mathcal{O}_X(1)$ on X. For any $\alpha \in C(X)$, write $\mathcal{M}_s(X;\alpha)$ (resp.

$\mathcal{M}_{ss}(X;\alpha)$) for the moduli space of Gieseker-(semi)stable sheaves on X with class α. It is shown in [HL10] that $\mathcal{M}_{ss}(X;\alpha)$ is a quasi-projective scheme of finite type and $\mathcal{M}_s(X;\alpha)$ is an open subscheme.

In the case $\mathcal{M}_s(X;\alpha) = \mathcal{M}_{ss}(X;\alpha)$, i.e. there are no strictly semistable sheaves, $\mathcal{M}_s(X;\alpha)$ is proper and admits a virtual class $[\mathcal{M}_s(X;\alpha)]^{\mathrm{vir}} \in A_0(\mathcal{M}_s(X;\alpha))$. The Donaldson-Thomas invariant is defined as

$$DT(X;\alpha) = \int_{[\mathcal{M}_s(X;\alpha)]^{\mathrm{vir}}} 1.$$

One can also define $DT(X;\alpha)$ via a constructible function $v_{\mathcal{M}_s(X;\alpha)} : \mathcal{M}_s(X;\alpha) \to \mathbb{Z}$, called the Behrend function [Beh09]. Then the Donaldson-Thomas invariant can be expressed as the weighted Euler characteristic:

$$DT(X;\alpha) = \chi(\mathcal{M}_s(X;\alpha), v_{\mathcal{M}_s(X;\alpha)}).$$

In addition, if $\mathcal{M}_s(X;\alpha)$ is smooth, then

$$DT(X;\alpha) = (-1)^{\dim(\mathcal{M}_s(X;\alpha))} e(\mathcal{M}_s(X;\alpha)). \tag{6.1}$$

If the moduli space $\mathcal{M}_{ss}(X;\alpha)$ contains strictly semistable sheaves, then $DT(X;\alpha)$ cannot be defined via the virtual cycle. It is also not a good idea to use (6.1) as the definition because it is not unchanged under deformations of X.

Instead, Joyce and Song defined a $\mathbb{Q}$-valued invariant $\bar{DT}(X;\alpha)$ for $\mathcal{M}_{ss}(X;\alpha)$, called the generalized Donaldson-Thomas invariant [JS12], which is given by the naïve Euler characteristic [Joy06] weighted by the Brehend function. Their version of stable-pair theory provides us a concrete tool to compute $\bar{DT}(X;\alpha)$.

$\bar{DT}(X;\alpha)$ is deformation-invariant. When $\mathcal{M}_s(X;\alpha) = \mathcal{M}_{ss}(X;\alpha)$, $\bar{DT}(X;\alpha)$ coincides with the original invariant $DT(X;\alpha)$.

In the previous discussion, the Calabi-Yau 3-fold X is assumed to be compact. If X is non-compact, the Euler form on $\mathrm{coh}(X)$ may not be defined. Hence we need to consider the abelian category $\mathrm{coh}_{\mathrm{cs}}(X)$ of compactly-support coherent sheaves on X [JS12, Section 6.7].

By [Ful13], $K_0(\mathrm{coh}_{\mathrm{cs}}(X))$ can be identified with the image of the compactly-supported Chern characteristic $ch_{\mathrm{cs}} : K_0(\mathrm{coh}_{\mathrm{cs}}(X)) \to H^{\mathrm{even}}_{\mathrm{cs}}(X;\mathbb{Q})$. Hence the Euler form

$$\bar{\chi} : K_0(\mathrm{coh}(X)) \times K_0(\mathrm{coh}_{\mathrm{cs}}(X)) \to \mathbb{Z}$$

is well-defined and mapped to the pairing

$$H^{\mathrm{even}}(X;\mathbb{Q}) \times H^{\mathrm{even}}_{\mathrm{cs}}(X;\mathbb{Q}) \to \mathbb{Q}$$

$$(\alpha, \beta) = \deg(\alpha^{\vee} \cdot \beta \cdot td(X))_3$$

given by the Poincaré duality.

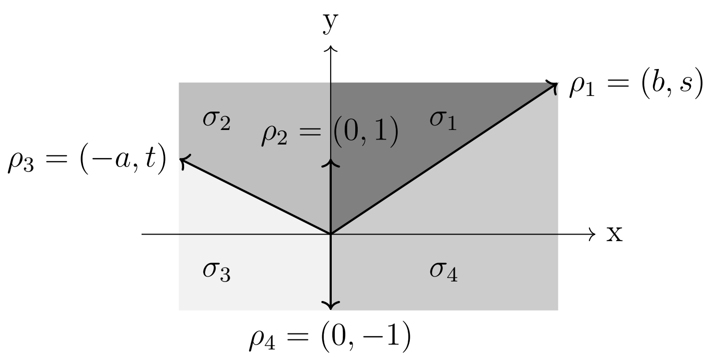
For any coherent sheaf $\mathcal{F}$ with compact support, the Hilbert polynomial is defined as $\bar{\chi}(\mathcal{O}_X(-t), \mathcal{F})$.

If X is compactly-embeddable [JS12], one can still define $\bar{DT}(X;\alpha)$ via the Behrend function and stable-pair theory. But the moduli space is not necessarily proper if X is not compact. As a result, the Donaldson-Thomas invariant may not be deformation-invariant.

6.1 Local Hirzebruch Orbifolds

Now we study the Donaldson-Thomas invariant $DT(\mathcal{X};\alpha)$ when $\mathcal{X}$ is the total space of the canonical bundle over a Hirzebruch orbifold $\mathcal{H}_r^{ab}$ and α is the class of a compactly-supported coherent sheaf. In the case when $r=0$ and α is the class of a 2-dimensional sheaf such that $c_1(\alpha)$ is the class of the zero section, we find an explicit formula for the generating function of Donaldson-Thomas invariants.

Suppose $\gcd(a,b)=1$. Recall that the stacky fan of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is given by



Hence $b\mathcal{D}_{\rho_1} \sim a\mathcal{D}_{\rho_3}$, $\mathcal{D}_{\rho_4} \sim s\mathcal{D}_{\rho_1} + \mathcal{D}_{\rho_2} + t\mathcal{D}_{\rho_3}$ where $\mathcal{D}_{\rho_i}$ is the divisor corresponding to the ray ρ_i. The canonical bundle of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is

$$\omega_{\mathcal{H}_r^{ab}} \cong \mathcal{O}_{\mathcal{H}_r^{ab}}(\sum_i -\mathcal{D}_{\rho_i}) \cong \mathcal{O}_{\mathcal{H}_r^{ab}}(-\frac{a+b-r}{a}\mathcal{D}_{\rho_1} - 2\mathcal{D}_{\rho_4}),$$

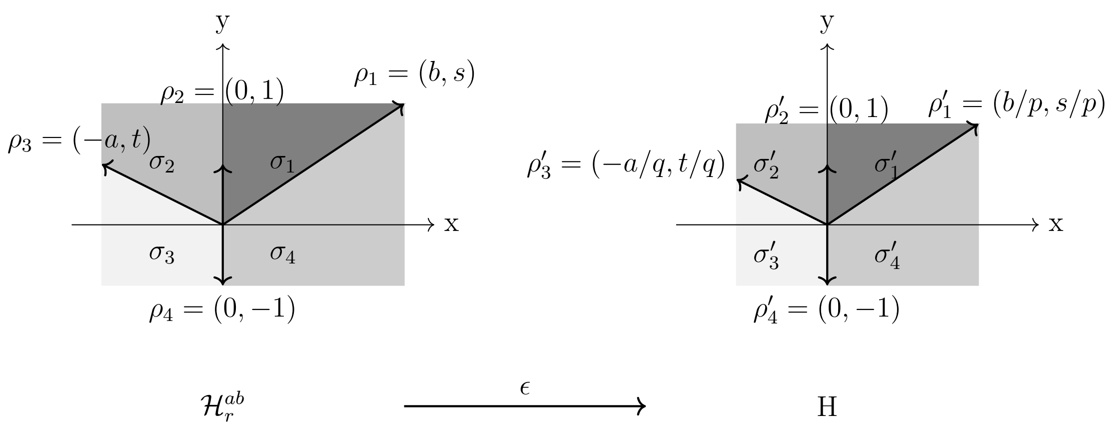
Let $\mathcal{X}$ be the total space of the canonical bundle $\omega_{\mathcal{H}_r^{ab}}$ over $\mathcal{H}_r^{ab}$. Then $\mathcal{X}$ is a

Calabi-Yau stack [PS06] of dimension 3, called the local Hirzebruch orbifold.

Denote by $\mathcal{S} \cong \mathcal{H}_r^{ab}$ the zero section of $\omega_{\mathcal{H}_r^{ab}}$. By [Beh04], the compactly-supported cohomology groups of $\mathcal{X}$ can be identified with the cohomology groups of $\mathcal{H}_r^{ab}$:

$$H^k_{\text{cs}}(\mathcal{X}, \mathbb{Q}) \cong H^{k-2}(\mathcal{H}_r^{ab}, \mathbb{Q}).$$

Let $\epsilon : \mathcal{H}_r^{ab} \to \mathrm{H}$ be the morphism from $\mathcal{H}_r^{ab}$ to its coarse moduli scheme.



Recall that $p = \gcd(b, r), q = \gcd(a, r), r = sa + bt$ and $\mathrm{Pic}(\mathcal{H}_r^{ab}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Fix the ample sheaf $L \cong \mathcal{O}_{\mathrm{H}}(\frac{b}{p}D_1 + \frac{ba}{pq}D_4)$ on H and the generating sheaf $\mathcal{E} = \bigoplus_{k=0}^{ab-1}(-k, 0)$ on $\mathcal{H}_r^{ab}$ [Section 4.4]. Denote by $\pi : \mathcal{X} \to \mathcal{H}_r^{ab}$ the projection map. Let $\mathcal{F}$ be a compactly-supported coherent sheaf of dimension 2 on $\mathcal{X}$ such that $c_1(\mathcal{F}) = k[\mathcal{S}]$ for some $k > 0$. Its modified Hilbert polynomial is defined as

$$P_{\pi^*\mathcal{E}} = \bar{\chi}(\pi^*\mathcal{E} \otimes \pi^*\epsilon^* L^{-1}, \mathcal{F}).$$

Since $\mathcal{O}(\mathcal{D}_{\rho_1}) \cong (a,0)$ and $\mathcal{O}(\mathcal{D}_{\rho_4}) \cong (r,1)$, we have

$$\omega^k_{\mathcal{H}^{ab}_r} \cong \mathcal{O}_{\mathcal{H}^{ab}_r}(-\frac{k(a+b-r)}{a}\mathcal{D}_{\rho_1} - 2k\mathcal{D}_{\rho_4}) \cong (-k(a+b+r), -2k)$$

By Proposition 4.4.1, $P_{\mathcal{E}}(\mathcal{O}_{\mathcal{H}^{ab}_r}, T) > P_{\mathcal{E}}(\omega^k_{\mathcal{H}^{ab}_r}, T)$ for any $k > 0$. It implies that $H^0(\omega^k_{\mathcal{H}^{ab}_r}) \cong \mathrm{Hom}(\mathcal{O}_{\mathcal{H}^{ab}_r}, \omega^k_{\mathcal{H}^{ab}_r}) = 0$. Hence $\mathcal{F}$ is set theoretically supported on $\mathcal{S}$.

Proposition 6.1.1. *If $\mathcal{F}$ as above is semistable, then the stack theoretical support of $\mathcal{F}$ is $\mathcal{S}$. Denote by $\mathcal{M}_{ss}(\mathcal{X}; P)$ (resp. $\mathcal{M}_{ss}(\mathcal{H}^{ab}_r; P)$) the moduli space of compactly-supported semistable torsion-free sheaves of dimension 2 on $\mathcal{X}$ (resp. $\mathcal{H}^{ab}_r$) with modified Hilbert polynomial P. Then*

$$\mathcal{M}_{ss}(\mathcal{X}; P) \cong \mathcal{M}_{ss}(\mathcal{H}^{ab}_r; P).$$

Proof. The ideal sheaf of $\mathcal{S}$ in $\mathcal{X}$ is isomorphic to $\omega^{-1}_{\mathcal{H}^{ab}_r} \cong \mathcal{O}_{\mathcal{H}^{ab}_r}(\frac{a+b-r}{a}\mathcal{D}_{\rho_1} + 2\mathcal{D}_{\rho_4})$, hence there exists an exact sequence

$$\mathcal{F} \otimes \omega^{-1}_{\mathcal{H}^{ab}_r} \to \mathcal{F} \to \mathcal{F}|_{\mathcal{S}} \to 0.$$

Since $\mathcal{F}$ is semistable, to prove $\mathrm{Hom}(\mathcal{F} \otimes \omega^{-1}_{\mathcal{H}^{ab}_r}, \mathcal{F}) = 0$, it suffices to show that $P(\mathcal{F} \otimes \omega^{-1}_{\mathcal{H}^{ab}_r}) > P(\mathcal{F})$ by [HL10], where $P(\mathcal{F}) := P_{\mathcal{E}}(\mathcal{F}, T) = \chi(\mathcal{H}^{ab}_r, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \epsilon^* L^T)$ denotes the modified Hilbert polynomial.

Because of the generating sheaf $\mathcal{E} = \bigoplus_{k=0}^{ab-1}(-k, 0)$, the only new contribution to the linear term of the modified Hilbert polynomial comes from the 2-dimensional component of the inertial stack $I\mathcal{H}_r^{ab}$, which is $\mathcal{H}_r^{ab}$ itself.

Recall that the Chow ring of the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ is $\mathbb{Q}[x,w]/(x^2, aw^2 - rxw)$. Then $P(\mathcal{F}\otimes\omega_{\mathcal{H}_r^{ab}}^{-1}) > P(\mathcal{F})$ because

$$\begin{aligned}&\int_{\mathcal{H}_r^{ab}} ch_1(\epsilon^* L^T)\cdot ch_1(\omega_{\mathcal{H}_r^{ab}})\\ =\ &\int_{\mathcal{H}_r^{ab}} (bx+\frac{ba}{pq}w)T\cdot\left(\frac{a+b-r}{a}x+2w\right)\\ =\ &(\frac{a+b+r}{pq}+2)T>0.\end{aligned}$$

Hence $\mathcal{F}\cong\mathcal{F}|_{\mathcal{S}}$. □

By Serre duality for Deligne-Mumford stacks [Nir08a], $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{F}\otimes\omega_{\mathcal{H}_r^{ab}}) = 0$ for any torsion-free semistable sheaf $\mathcal{F}$ on $\mathcal{H}_r^{ab}$ as shown in the proof of the previous proposition. Hence in the case when there are no strictly semistable sheaves, $\mathcal{M}_s(\mathcal{X};P_{\mathcal{E}})$ and $\mathcal{M}_s(\mathcal{X};\alpha)$ are unobstructed and smooth. Hence the Donaldson-Thomas invariant $DT(\mathcal{X};\alpha)$ is the signed Euler characteristic:

$$DT(\mathcal{X};\alpha) = (-1)^{\dim\mathcal{M}_s(\mathcal{X};\alpha)} e(\mathcal{M}_s(\mathcal{X};\alpha))$$

In the variety case, i.e. $a=b=1$, we also have

$$DT(\mathcal{X};P_{\mathcal{E}}) = (-1)^{\dim\mathcal{M}_s(\mathcal{X};P_{\mathcal{E}})} e(\mathcal{M}_s(\mathcal{X};P_{\mathcal{E}})).$$

But it does not hold when $\mathcal{X}$ is an orbifold. Sheaves of different K-group classes might have same modified Hilbert polynomial. The moduli space $\mathcal{M}_s(\mathcal{X};P_{\mathcal{E}})$ will have components of different dimensions.

Denote by $\mathcal{M}_{\text{even}}$ (resp. $\mathcal{M}_{\text{odd}}$) the components of the moduli space $\mathcal{M}_s(\mathcal{X}; P_{\mathcal{E}})$ with even (resp. odd) dimension. Then the Donaldson-Thomas invariant $DT(\mathcal{X}; P_{\mathcal{E}})$ can be expressed as

$$DT(\mathcal{X}; P_{\mathcal{E}}) = e(\mathcal{M}_{\text{even}}) - e(\mathcal{M}_{\text{odd}}).$$

Hence the coefficients of generating functions we obtained in Chapter 5, for example Proposition 5.2.1, might not be Donaldson-Thomas invariants. We need to modify those generating functions to track both K-group classes and Euler characteristics. This process involves colored partitions.

Let α be the class of a 2-dimensional compactly-supported semistable sheaf on $\mathcal{X}$ with $c_1(\alpha) = [\mathcal{S}]$. Since this sheaf is stack theoretically supported on $S \cong \mathcal{H}_r^{ab}$, it corresponds to a semistable sheaf $\mathcal{F}$ of rank 1 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. By tensoring $- \otimes (i, j)$, we assume that the reflexive hull of $\mathcal{F}$ is $L_{(0,0,0,0)}$, in which case $c_1(\mathcal{F}) = 0$. The cokernel sheaf $\mathcal{Q}$ of the exact sequence

$$0 \to \mathcal{F} \to L_{(0,0,0,0)} \to \mathcal{Q} \to 0$$

is 0-dimensional.

By Example 3.3.2, in each chart $\mathcal{U}_i$, the stacky family ${}_{b_i}\hat{\mathcal{F}}_i$ is nonzero only for

$j = k = 0$. The set

$$\{(l_1, l_2\} \in \mathbb{Z}^2_{\geq 0} | \, {}_{(0,0)}\mathcal{Q}_i(l_1, l_2) \neq 0 \text{ i.e. } {}_{(0,0)}F_i(l_1, l_2) = 0\}$$

corresponds to the stacky family ${}_{b_i}\hat{\mathcal{Q}}_i$ and defines a colored Young diagram for each chart.

For example, in the chart $\mathcal{U}_1$, the μ_b-weight of ${}_{(0/a,0)}\mathcal{Q}_1(0,0)$ is 0 by Example 3.3.2, because $B_i = 0$ for all i. Since the μ_b-action on $\mathcal{U}_1$ is given by

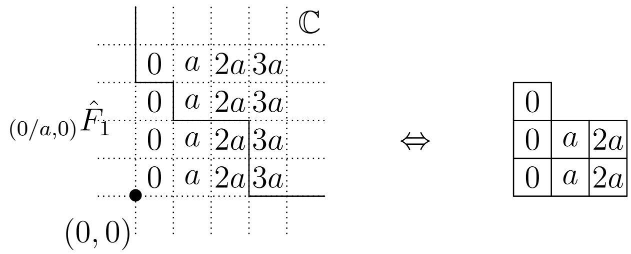
$$\tau \in \mu_b : (x, y) \to (\tau^a x, \tau^{-r} y),$$

the μ_b-weight of ${}_{(0/a,0)}\mathcal{Q}_1(l_1, l_2)$ is

$$l_1 a - l_2 r \mod b.$$

The Young diagram associated to $\mathcal{Q}|_{\mathcal{U}_1}$ is nonempty if and only if ${}_{(0/a,0)}\mathcal{Q}_1(l_1, l_2) \neq 0$. The color assigned to each block is determined by the μ_b-weight. Similarly, Young diagrams for other charts are also colored based on the fining gradings.

Example 6.1.2. Suppose $\gcd(a, b) = 1$ and $r = 0$, then the orbifold is $\mathbb{P}(a, b) \times \mathbb{P}^1$. The colored Young diagram associated to $\mathcal{F}$ for the first chart is illustrated as follows:



Note that ia might be bigger than b, so the color is given by $ia \mod b$.

Proposition 6.1.3. *Let $\mathcal{F}$ be a torsion free sheaf of rank 1 on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$ such that the reflexive hull of $\mathcal{F}$ is $L_{(0,0,0,0)}$. Denote by $\lambda_i(\mathcal{F})$ the corresponding colored Young diagram for chart $\mathcal{U}_i$ and $\#_l\lambda_i(\mathcal{F})$ the number of boxes with color l in the diagram $\lambda_i(\mathcal{F})$. The K-group class of $\mathcal{F}$ is*

$$\begin{aligned}1 &- \sum_{l=0}^{b-1} \#_l\lambda_1(\mathcal{F})(1-g^a)(1-h)g^l - \sum_{l=0}^{a-1} \#_l\lambda_2(\mathcal{F})(1-g^b)(1-h)g^l \\ &- \sum_{l=0}^{a-1} \#_l\lambda_3(\mathcal{F})(1-g^b)(1-g^r h)g^l - \sum_{l=0}^{b-1} \#_l\lambda_4(\mathcal{F})(1-g^a)(1-g^r h)g^l\end{aligned}$$

where $g := [(-1,0)], h := [(0,-1)]$ are K-group classes of the generators of $\mathrm{Pic}(\mathcal{H}_r^{ab}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The result follows from Proposition 4.1.1 □

Let $M_{\mathcal{H}_r^{ab}}(1, c_1 = 0, \chi_{\mathcal{E}})$ be the moduli scheme of stable torsion free sheaves of rank 1 on $\mathcal{H}_r^{ab}$ with first Chern class $c_1 = 0$ and modified Euler characteristic $\chi_{\mathcal{E}}$. Our goal is to stratify this moduli scheme by K-group classes and determine the dimension of each component.

By the above proposition, we know that the K-group class of a torsion-free sheaf of rank 1 is fully determined by $\#_l\lambda_i(\mathcal{F})$. So the problem of determining the dimension is reduced to that of counting colored partitions.

When $r = 0$, i.e. the orbifold is $\mathbb{P}(a, b) \times \mathbb{P}^1$, the group action on each chart always fixes one variable. For example, the action on the first chart is given by $\tau \in \mu_b : (x, y) \to (\tau^a x, y)$. Hence the Young diagram is colored by layers as illustrated in Example 6.1.2. In this case, there is a simple formula for the generating function.

Proposition 6.1.4. *Suppose a Young diagram is colored based on the weight of μ_b-action on $\mathbb{C}^2$ given by $\tau \in \mu_b : (x, y) \to (\tau^a x, y)$. Let p_i be the variable that tracks the color $i \mod b$. Then the generating function for the colored partition is given by*

$$G(p_{ia}) = \frac{1}{\prod_{k\geq 0}\prod_{i=0}^{b-1}\left(1 - p_0 p_a \cdots p_{ia}(p_0 p_a \cdots p_{(b-1)a})^k\right)}.$$

Proof. The result follows from the observation that for each horizontal layer of a Young diagram, the number of boxes with color $(i+1)a$ is always equal to or one less than that of ia. For example, the Young diagram in Example 6.1.2 corresponds to the term $(p_0 p_a p_{2a}) \cdot (p_0 p_a p_{2a}) \cdot p_0$. □

As a result, the generating functions for charts $\mathcal{U}_1$ and $\mathcal{U}_4$ are both $G(p_i)$. Similarly, the generating functions for charts $\mathcal{U}_2$ and $\mathcal{U}_3$ are

$$H(q_{jb}) = \frac{1}{\prod_{k\geq 0}\prod_{j=0}^{a-1}\left(1 - q_0 q_b \cdots q_{jb}(q_0 q_b \cdots q_{(a-1)b})^k\right)}.$$

Hence the partition function for stable torsion free sheaves of rank 1 with

$c_1 = 0$ on $\mathbb{P}(a,b) \times \mathbb{P}^1$ is given by

$$\mathrm{q}^{\frac{1}{2}(a+b+ab-1)} G^2(\mathrm{q}^{-a} p_{ia}) H^2(\mathrm{q}^{-b} q_{jb}),$$

where q tracks the modified Euler chracteristic and p_{ia} (resp. q_{jb}) tracks the number of boxes with color $ia \mod b$ (resp. $jb \mod a$) in charts $\mathcal{U}_1$ and $\mathcal{U}_4$ (resp. $\mathcal{U}_2$ and $\mathcal{U}_3$).

Note that by setting $p_i = q_i = 1$, we will get back the generating function

$$\mathsf{G}_0(\mathrm{q}) = \mathrm{q}^{\frac{1}{2}(a+b+ab-1)} \prod_{k=1}^{\infty} \frac{1}{(1-\mathrm{q}^{-ak})^2(1-\mathrm{q}^{-bk})^2}$$

in Proposition 5.2.1

However the coefficients of the above partition function are still not Donaldson-Thomas invariants. We need to determine $\chi(\mathcal{F}, \mathcal{F})$ for any torsion free sheaf $\mathcal{F}$ of rank 1 with fixed K-group class.

When $\mathcal{X}$ is an orbifold, it is not easy to calculate $\chi(\mathcal{F}, \mathcal{F})$ directly. Given a sheaf $\mathcal{F}$ on a variety X, if $ch(\mathcal{F}) = \oplus_i v_i \in \oplus_i H^{2i}(X, \mathbb{Q})$, then the dual class of $\mathcal{F}$ is defined as $ch^{\vee}(\mathcal{F}) = \oplus_i (-1)^i v_i$. But this is not true for orbifolds because of the existence of ρ in the definition of $\widetilde{ch}$ [Section 4.2]. The eigenvalues are assigned as the weights for the decomposition of eigenbundles.

However, we are only interested in $(-1)^{\dim \mathcal{M}_s(\mathcal{X};\alpha)}$. So we carry out the following steps to determine whether $\chi(\mathcal{F}, \mathcal{F})$ is even or odd.

Proposition 6.1.5. *Given a sheaf $\mathcal{F}$ on the Hirzebruch orbifold $\mathcal{H}_r^{ab}$. Suppose there* *exists a locally-free resolution*

$$0 \to \bigoplus_{j=1}^{n_2} \mathcal{L}_{2j} \to \bigoplus_{i=1}^{n_1} \mathcal{L}_{1i} \to \mathcal{F},$$

then

$$\chi(\mathcal{F},\mathcal{F}) = (n_1+n_2) + \sum_{i\neq i'} \Big(\chi(\mathcal{L}_{1i}\otimes\mathcal{L}_{1i'}^\vee) + \chi(\mathcal{L}_{1i'}\otimes\mathcal{L}_{1i}^\vee)\Big)$$
$$+\sum_{j\neq j'} \Big(\chi(\mathcal{L}_{2j}\otimes\mathcal{L}_{2j'}^\vee) + \chi(\mathcal{L}_{2j'}\otimes\mathcal{L}_{2j}^\vee)\Big) - \sum_{i,j} \Big(\chi(\mathcal{L}_{1i}\otimes\mathcal{L}_{2j}^\vee) + \chi(\mathcal{L}_{2j}\otimes\mathcal{L}_{1i}^\vee)\Big).$$

The result also holds when the K-group class of $\mathcal{F}$ is $[\bigoplus_{i=1}^{n_1}\mathcal{L}_{1i}] - [\bigoplus_{j=1}^{n_2}\mathcal{L}_{2j}]$.

Proof. The dual sheaf of $\mathcal{F}$ is defined as the derived dual. The Euler form is bilinear and $\chi(\mathcal{O}_{\mathcal{H}_r^{ab}}) = 1$ by Proposition 4.2.1. $\square$

Therefore, to determine whether $\mathcal{X}(\mathcal{F},\mathcal{F})$ is even or odd, we need to find the pattern for $\chi(\mathcal{L}) + \chi(\mathcal{L}^\vee)$ for any line bundle $\mathcal{L}$ on $\mathbb{P}(a,b)\times\mathbb{P}^1$.

Proposition 6.1.6. *Define $\widetilde{\chi}[\mathcal{L}] := \chi(\mathcal{L}) + \chi(\mathcal{L}^\vee)$ and*

$$\delta_{i,j} := \begin{cases} 0 & \text{if } i \nmid j \\ 1 & \text{if } i \mid j. \end{cases}$$

Suppose gcd(a,b)=1. Consider the line bundle $(m,n) \in \mathrm{Pic}(\mathbb{P}(a,b)\times\mathbb{P}^1)$. Then

$$\widetilde{\chi}[(m,n)] \equiv (\delta_{b,m} + \delta_{a,m})(n-1) \mod 2.$$

Proof. By Proposition 4.2.1, when $r = 0$, the Euler characteristic is given by

$$\chi((m,n)) = \frac{1+n}{2a} + \frac{1+n}{2b} + \frac{(1+n)m}{ab} + \sum_{l=1}^{b-1} \frac{\omega_b^{ml}}{1-\omega_b^{-al}} \frac{n+1}{b} + \sum_{l=1}^{a-1} \frac{\omega_a^{ml}}{1-\omega_a^{-bl}} \frac{n+1}{a}.$$

We first consider the case when $n = 0$. Using the fact that $\sum_{l=1}^{b-1} \frac{\omega_b^{ml}}{1-\omega_b^{-al}} = \sum_{l=1}^{b-1} \frac{\omega_b^{-ml}}{1-\omega_b^{al}}$ and $\frac{\omega_b^{-ml}}{1-\omega_b^{al}} + \frac{\omega_b^{-ml}}{1-\omega_b^{-al}} = \omega_b^{-ml}$, we get

$$\widetilde{\chi}[(m,0)] = \frac{1}{a} + \frac{1}{b} + \sum_{l=1}^{b-1} \omega_b^{ml} \frac{1}{b} + \sum_{l=1}^{a-1} \omega_a^{ml} \frac{1}{a}.$$

Note that

$$\sum_{l=1}^{b-1} \omega_b^{ml} = \begin{cases} -1 & \text{if } b \nmid m \\ b-1 & \text{if } b \mid m. \end{cases}$$

Therefore

$$\widetilde{\chi}[(m,0)] = \delta_{b,m} + \delta_{a,m}.$$

When $n = 1$,

$$\widetilde{\chi}[(m,1)] = 2\chi((m,0)) \equiv 0 \mod 2.$$

When $n \geq 2$,

$$\chi((m,n)) - \chi((-m,-n)) = 2\chi((m,0)) + (\delta_{b,m} + \delta_{a,m})(n-1).$$

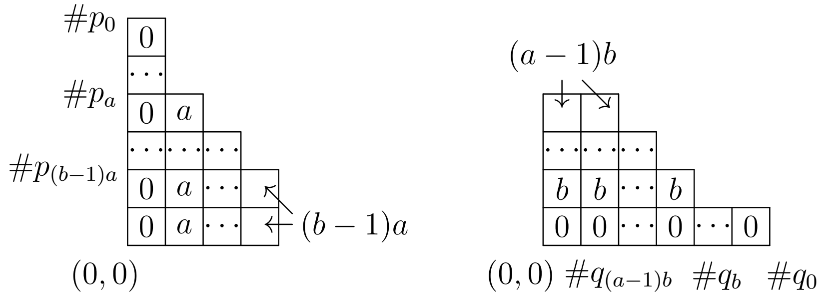
Hence

$$\widetilde{\chi}[(m,n)] \equiv (\delta_{b,m} + \delta_{a,m})(n-1) \mod 2.$$

$\square$

Denote by $\#p_{ia}$ (resp. $\#q_{jb}$) the number of boxes with color $ia \mod b$ (resp. $jb \mod a$) in charts $\mathcal{U}_1$ and $\mathcal{U}_4$ (resp. $\mathcal{U}_2$ and $\mathcal{U}_3$).

From Proposition 6.1.3 , we know that changing the position of a colored box won't change the K-group class as long as the coloring is kept unchanged. Hence we can assume that corresponding Young diagrams for $\mathcal{F}$ are empty for charts $\mathcal{U}_3$ and $\mathcal{U}_4$ and have the following shapes for charts $\mathcal{U}_1$ and $\mathcal{U}_2$.



Consider the following sheaves:

$$\begin{aligned}
\mathcal{F}_0 &= \bigoplus_{0\le i\le b-1} L_{(ia,\#p_{ia},0,0)} \bigoplus L_{(ba,0,0,0)} \bigoplus_{0\le j\le a-1} L_{(jb,\#q_{jb},0,0)} \bigoplus L_{(0,0,ab,0)} \\
&= \bigoplus_{0\le i\le b-1} (-ia,-\#p_{ia}) \bigoplus (-ba,0) \bigoplus_{0\le j\le a-1} (-jb,-\#q_{jb}) \bigoplus (-ab,0), \\
\mathcal{F}_1 &= \bigoplus_{0\le i\le b-1} L_{((i+1)a,\#p_{ia},0,0)} \bigoplus_{0\le j\le a-1} L_{((j+1)b,\#q_{jb},0,0)} \bigoplus L_{(0,0,0,0)} \\
&= \bigoplus_{0\le i\le b-1} (-(i+1)a,-\#p_{ia}) \bigoplus_{0\le j\le a-1} (-(j+1)b,-\#q_{jb}) \bigoplus (0,0).
\end{aligned}$$

Then the K-Group class of $\mathcal{F}$ is $[\mathcal{F}_0]-[\mathcal{F}_1]$.

Proposition 6.1.7. *Suppose* $\gcd(a,b)$=1. Given a sheaf $\mathcal{F}$ on $\mathbb{P}(a,b)\times\mathbb{P}^1$ as above, then

$$\chi(\mathcal{F},\mathcal{F}) \equiv 1 + a + b + \#p_0 + \#p_{(b-1)a} + \#q_0 + \#q_{(a-1)b} \mod 2.$$

Proof. By Proposition 6.1.5, we need to decide whether $\widetilde{\chi}[L \otimes L'^{\vee}] := \chi(L \otimes L'^{\vee}) + \chi(L^{\vee} \otimes L')$ is even or odd for any summands L, L' of $\mathcal{F}_0$ and $\mathcal{F}_1$.

Notice that

$$\begin{aligned}
&\widetilde{\chi}[(i-j)a, \#p_{ia} - \#p_{ja}] = \widetilde{\chi}[((i+1)-(j+1))a, \#p_{ia} - \#p_{ja}],\\
&\widetilde{\chi}[(i-j)b, \#q_{ib} - \#q_{jb}] = \widetilde{\chi}[((i+1)-(j+1))b, \#q_{ib} - \#q_{jb}],\\
&\widetilde{\chi}[(i-b)a, \#p_{ia}] = \widetilde{\chi}[ia, \#p_{ia}], \quad \widetilde{\chi}[(i+1-b)a, \#p_{ia}] = \widetilde{\chi}[(i+1)a, \#p_{ia}],\\
&\widetilde{\chi}[(j-a)b, \#q_{ib}] = \widetilde{\chi}[jb, \#q_{ib}], \quad \widetilde{\chi}[(j+1-a)b, \#q_{ib}] = \widetilde{\chi}[(j+1)b, \#q_{ib}].
\end{aligned}$$

So they can be grouped into pairs and won't influence the evenness of $\chi(\mathcal{F},\mathcal{F})$.

Since

$$\delta_{a,ia-(j+1)b} + \delta_{a,ia-jb} + \delta_{b,ia-(j+1)b} + \delta_{b,ia-jb} = \begin{cases} \text{odd} & \text{if } j = 0, a-1 \\ \text{even} & \text{otherwise,} \end{cases}$$

one can show that

$$\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} \widetilde{\chi}[ia-(j+1)b, \#p_{ia} - \#q_{jb}] + \widetilde{\chi}[ia-jb, \#p_{ia} - \#q_{jb}] \equiv b(n_0 + n_{a-1}) \mod 2.$$

Similarly,

$$\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\widetilde{\chi}[(i+1)a-(j+1)b,\#p_{ia}-\#q_{jb}]+\widetilde{\chi}[(i+1)a-jb,\#p_{ia}-\#q_{jb}]$$
$$\equiv \quad b(n_0+n_{a-1}) \mod 2.$$

So the sum of above terms is even.

Next, one can show that

$$\begin{aligned}
&\sum_{i=0}^{b-1}\sum_{j=0}^{b-1}\widetilde{\chi}[ia-(j+1)a,\#p_{ia}-\#p_{ja}]\\
=&\sum_{i=0}^{b-1}\widetilde{\chi}[-a,0]+\sum_{\substack{i,j=0\\ i<j}}^{b-1}\Big(\widetilde{\chi}[ia-(j+1)a,\#p_{ia}-\#p_{ja}]+\widetilde{\chi}[(i+1)a-ja,\#p_{ia}-\#p_{ja}]\Big)\\
\equiv& \; b \mod 2.
\end{aligned}$$

Similarly,

$$\sum_{i=0}^{a-1}\sum_{j=0}^{a-1}\widetilde{\chi}[ib-(j+1)b,\#q_{ib}-\#q_{jb}]\equiv a \mod 2.$$

Lastly, one can also show that

$$\sum_{i=0}^{b-1}\widetilde{\chi}[ia-ab,\#p_{ia}]+\widetilde{\chi}[(i+1)a-ab,\#p_{ia}]\equiv \#p_0+\#p_{(b-1)a}.$$
$$\sum_{j=0}^{a-1}\widetilde{\chi}[jb-ab,\#q_{jb}]+\widetilde{\chi}[(j+1)b-ab,\#q_{jb}]\equiv \#q_0+\#q_{(a-1)b}.$$

Since there are $2a+2b+3$ summations in $\mathcal{F}_0$ and $\mathcal{F}_1$ in total, the result follows from adding all the above terms together. □

Proposition 6.1.8. *Suppose* $\gcd(a,b)=1$. *When* $\mathcal{X}$ *is the total space of the canonical bundle over* $\mathbb{P}(a,b)\times\mathbb{P}^1$ *and* α *is the class of a 2-dimensional compactly-*

supported sheaf over $\mathbb{P}(a,b)\times\mathbb{P}^1$ with $c_1(\alpha)=[\mathcal{S}]$, the Donaldson-Thomas partition function is given by

$$(-1)^{a+b}\mathrm{q}^{\frac{a+b+ab-1}{2}}G^2(-\mathrm{q}^{-a}p_0,\underbrace{\mathrm{q}^{-a}p_{ia}}_{i\neq 0,b-1},-\mathrm{q}^{-a}p_{(b-1)a})H^2(-\mathrm{q}^{-b}q_0,\underbrace{\mathrm{q}^{-b}q_{jb}}_{j\neq 0,a-1},-\mathrm{q}^{-b}q_{(a-1)b}).$$

Proof. The dimension of the moduli space is

$$1-\chi(\mathcal{F},\mathcal{F})\equiv a+b+\#p_0+\#p_{(b-1)a}+\#q_0+\#q_{(a-1)b}\mod 2.$$

□

By setting $p_i=q_i=1$, we get a generating function such that the coefficients are $DT(\mathcal{X};P_{\mathcal{E}})$.

Proposition 6.1.9. *Suppose* $\gcd(a,b)=1$. Let $\mathcal{X}$ be the total space of the canonical bundle over $\mathbb{P}(a,b)\times\mathbb{P}^1$, then

$$\begin{aligned}&\sum DT\big(\mathcal{X};P_{\mathcal{E}}=abT^2+(\frac{1}{2}(a+b+ab-1)+ab)T+\chi_{\mathcal{E}}\big)\mathrm{q}^{\chi_{\mathcal{E}}}\\&=(-1)^{a+b}\mathrm{q}^{\frac{a+b+ab-1}{2}}G^2(-\mathrm{q}^{-a},\underbrace{\mathrm{q}^{-a},\cdots,\mathrm{q}^{-a}}_{i\neq 0,b-1},-\mathrm{q}^{-a})H^2(-\mathrm{q}^{-b},\underbrace{\mathrm{q}^{-b},\cdots,\mathrm{q}^{-b}}_{j\neq 0,a-1},-\mathrm{q}^{-b}),\end{aligned}$$

where $G(p_{ia})$ and $H(q_{jb})$ are given in Proposition 6.1.4.

Example 6.1.10. Suppose $a=1$ and $b=2$. Then $\mathcal{X}$ is the total space of the canonical bundle over $\mathbb{P}(1,2)\times\mathbb{P}^1$. The DT partition function is given by

$$-\mathrm{q}^2\frac{1}{\prod_{k>0}(1-(p_0p_1\mathrm{q}^{-2})^k)^2}\frac{1}{\prod_{k\geq 0}(1+p_0\mathrm{q}^{-1}(p_0p_1\mathrm{q}^{-2})^k)^2}\frac{1}{\prod_{k>0}(1-(q_0\mathrm{q}^{-2})^k)^2}.$$

By setting $p_0 = p_1 = q_0 = 1$, we get

$$\sum DT\big(\mathcal{X}; P_{\mathcal{E}} = 2T^2 + 4T + \chi_{\mathcal{E}}\big)\mathrm{q}^{\chi_{\mathcal{E}}}$$
$$= -\mathrm{q}^2 \frac{1}{\prod_{k>0}\left(1-(\mathrm{q}^{-2})^k\right)^2} \frac{1}{\prod_{k\geq 0}\left(1+\mathrm{q}^{-1}(\mathrm{q}^{-2})^k\right)^2} \frac{1}{\prod_{k>0}\left(1-(\mathrm{q}^{-2})^k\right)^2}.$$

Next, we are interested in finding a generating function for $DT(\mathcal{X};\alpha)$.

Recall that in Proposition 6.1.4, we introduced p_{ia} and q_{jb} to track colored boxes. Indeed, variables p_{ia} (resp. q_{jb}) are keeping track of the K-group classes of $\mathcal{O}_{P_1} \otimes \hat{\mu}_b^{ia}$ and $\mathcal{O}_{P_3} \otimes \hat{\mu}_b^{ia}$ (resp. $\mathcal{O}_{P_2} \otimes \hat{\mu}_a^{jb}$ and $\mathcal{O}_{P_2} \otimes \hat{\mu}_a^{jb}$) defined in Proposition 4.1.1, where P_k is the origin of chart $\mathcal{U}_k$. By Proposition 4.1.1, we need to impose the following relation among these variables:

$$p_0 p_a \cdots p_{(b-1)a} = q_0 q_b \cdots q_{(a-1)b}.$$

Our goal is to combine terms of the DT partition function in Proposition 6.1.8 that represent the same K-group class based on this relation. We modify the function $H(q_{jb})$ in Proposition 6.1.4 into

$$H'(q_{jb}; p_{ia}) = \frac{1}{\prod_{k>0}\big(1-(p_0p_a\cdots p_{(b-1)a})^k\big)} \cdot \frac{1}{\prod_{k\geq 0}\prod_{j=0}^{a-2}\big(1-q_0q_b\cdots q_{jb}(p_0p_a\cdots p_{(b-1)a})^k\big)}.$$

Then we get a generating function for $DT(\mathcal{X};\alpha)$, which is given by Theorem 1.2.3.

Example 6.1.11. Suppose $a = 1$ and $b = 2$. Let $\mathcal{X}$ be the total space of the canonical bundle over $\mathbb{P}(1,2)\times\mathbb{P}^1$ and $i : \mathcal{S} \cong \mathbb{P}(1,2)\times\mathbb{P}^1 \hookrightarrow \mathcal{X}$ be the inclusion of the zero section. In this case $q_0 = p_0 p_1$. Hence

$$\sum_{\alpha \in K_0(\mathcal{X})} DT\big(\mathcal{X};\alpha\big) p_0^{\#p_0} p_1^{\#p_1} = -\frac{1}{\prod_{k>0}\left(1-(p_0p_1)^k\right)^4} \frac{1}{\prod_{k\geq 0}\left(1+p_0(p_0p_1)^k\right)^2},$$

where α is given by

$$i_*\Big(1 - \#p_0(1-g)(1-h) - \#p_1(1-g)(1-h)g\Big).$$

Remark 6.1.12. Let α be the class of a 2-dimensional compactly supported semistable sheaf on $\mathcal{X}$ with $c_1(\alpha) = k[\mathcal{S}]$ for $k > 1$. This sheaf is stack theoretically supported on $\mathcal{S}$ and hence corresponds to a semistable sheaf $\mathcal{F}$ of rank k on $\mathcal{H}_r^{ab}$. If $\mathcal{F}$ is strictly semistable, the Donaldson-Thomas invariant $DT(\mathcal{X};\alpha)$ is not the signed Euler characteristic any more. We need to adopt the stable-pair theory from [JS12] and generalize the method of [GS15b] to the orbifold case.

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