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Abstract

We provide a necessary and sufficient condition for the almost sure convergence and the strong consistency of the sample autocovariance of a discrete spectrum weakly stationary process. This also clarifies the estimation of the autocovariance function of a mixed spectrum weakly stationary processes.

Abbreviated Title: "Autocovariance"

Key words and phrases: Mixed spectrum, SLLN, stationary, sample covariance, almost sure, amplitude, phase, zero-crossing rate.

1 Introduction

The estimation problem of the autocovariance function of a weakly stationary process is of fundamental importance in time series analysis, and has been treated in the past by many authors. Most of the literature deals with the continuous spectrum case where the results depend on a vanishing autocorrelation, and the existence of higher order moments. The more complex mixed spectrum case, however, has received less attention and is less clear. The source of confusion is the fact that in the presence of a discrete spectral component the usual sample estimator *is not necessarily consistent.* For example, for a Gaussian process, the sample autocovariance–the estimator most often used in practice–is inconsistent in mean square sense unless the discrete spectrum is absent (Hannan (1970), p. 210, Koopmans (1974, p. 60)). Moreover, in general consistency results concerning the sample autocovariance are not entirely satisfactory. The purpose of this note is to clarify the general estimation problem in the mixed spectrum case vis-à-vis the almost sure convergence of the sample autocovariance under the assumption of second order stationarity only. *In a nutshell, this is achieved by observing that the product of the process by its shift results in a process which is the Fourier transform of an L^1 random measure.*

The general mixed spectrum case encountered in practice is that of "signal plus noise" where the signal is a stationary process consisting of a sum of random sinusoids, and the noise is a stationary process, with certain ergodic properties, independent of the signal. With this in mind, it is more crucial to consider the discrete spectral component.

To motivate our discussion, consider the strictly stationary real-valued process

$$X_t = \sum_j e^{it\omega_j}\xi_j, \;\; \xi_{-j} = \overline{\xi}_j, \;\; \omega_{-j} = -\omega_j$$

where the ξ_j are orthogonal. It is shown in Hannan (1970, p. 201) that a necessary condition for ergodicity is that the $|\xi_j|$ are almost sure constants. Clearly, under ergodicity the sample autocovariance is strongly consistent. Without addressing ergodicity directly, we shall show that, under a certain assumption on the frequencies, constant amplitudes are necessary and sufficient for the strong consistency at all lags of the sample autocovariance for any complex-valued weakly stationary discrete spectrum process. The arguments, which makes use of a new work on the SLLN by Houdré (1992), involves second order properties only. These arguments are given here only in the discrete parameter case, but the continuous and the multidimensional parameter cases follows similarly (the corresponding strong laws are proved in the above reference.

2 A SLLN

We shall appeal to the following general result which states that the strong law of large numbers (SLLN) holds for processes which are Fourier transforms of random measures provided the latter converges almost surely to zero at punctured dyadic intervals around the origin.

Theorem 2.1 *Suppose* $\{X_t\}$, $t = 0, \pm 1, \cdots$, *admits the Fourier representation*

$$X_t = \int_{-\pi}^{\pi} e^{it\omega} \xi(d\omega)$$

where $\xi : \mathcal{B}(-\pi, \pi] \to L^\alpha(P)$, $1 \leq \alpha \leq 2$, *is* σ*-additive. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} X_t = \xi(\{0\})$$

in $L^\alpha(P)$. *Furthermore,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} X_t = \xi(\{0\}) \quad a.s.$$

if and only if

$$\lim_{n\to\infty} \xi((-2^{-n}, 2^{-n}) \setminus \{0\}) = 0 \quad a.s.$$

This is a generalization of a result by Gaposhkin (1977) who requires weak stationarity. It seems that the precursor to all this is a result of Rajchman (1932) which states that the strong law of large numbers holds for a sequence of uncorrelated random variables whose second moments have a common bound (see Chung (1974, p. 103)). The more general dependent case is discussed in Chapter 10 of Doob (1953).

We note that the dc component $\xi(\{0\})$ may be random or nonrandom, and that we do not require the mean of $\{X_t\}$ to be zero.

The proof of Theorem 2.1 depends on a careful analysis in a neighborhood of the origin of the kernels $h_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\exp(it\theta)$. Since $h_N(\theta)$ is periodic with period 2π, Theorem 2.1 has to be modified when the limits of integration extend beyond $(-\pi, \pi]$ and cover multiples of $\pm 2\pi$. In this case, in addition to the origin, we must examine neighborhoods of multiples of $\pm 2\pi$. In particular, when the limits extend from -2π to 2π, the modified claim is

Theorem 2.2 *Suppose $\{X_t\}$, $t = 0, \pm 1, \cdots$, admits the Fourier representation*

$$X_t = \int_{-2\pi}^{2\pi} e^{it\omega}\xi(d\omega)$$

where $\xi : \mathcal{B}(-2\pi, 2\pi] \to L^\alpha(P)$, $1 \le \alpha \le 2$, is σ-additive. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} X_t = \xi(\{0\}) + \xi(\{2\pi\})$$

in $L^\alpha(P)$. Furthermore,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} X_t = \xi(\{0\}) + \xi(\{2\pi\}) \quad a.s.$$

if and only if

$$\lim_{n\to\infty}\{\xi((-2^{-n}, 2^{-n}) \setminus \{0\}) + \xi(-2\pi, -2\pi + 2^{-n}) + \xi(2\pi - 2^{-n}, 2\pi)\} = 0$$

almost surely.

This consideration becomes relevant in the proof of strong consistency of the sample covariance. The proof of Theorem 2.2 follows as in Houdré (1992) where the case of a spectrum supported on $(-\pi, \pi]$ is tackled. It is also a special case of a general a.s. principle developed in Houdré (1993).

There are various applications in spectral analysis where the almost sure result concerning the sample autocovariance is of use. For example, in the mixed spectrum case, when precise autocovariance estimates are available, it is possible to detect the discrete frequencies from the sample covariance evaluated at sufficiently high lags (Priestley (1981), p. 626). More recently, it has been shown in Li and Kedem (1993) that a certain parametrization of the sample autocorrelation forms a contraction mapping whose set of fixed points contains approximations to the cosines of the true frequencies. The method, which leads to very precise frequency estimates, requires however, among other things, the almost sure convergence of the sample autocovariance.

3 The Discrete Spectrum Case

3.1 Finitely Many Frequencies

Let $\{X_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, be a weakly stationary discrete spectrum process with mean $E[X_t] = m$, autocovariance $R(k) = E[(X_{t+k} - m)\overline{(X_t - m)}]$, $k = 0, \pm 1, \pm 2, \cdots$, and spectral distribution $F(d\omega)$ supported at $p+1$ distinct atoms in $\{\omega_0, \omega_1, \cdots, \omega_p\} \in (-\pi, \pi]$, such that $\omega_0 = 0$. Then,

$$X_t - m = \int_{-\pi}^{\pi} e^{it\omega} \xi(d\omega)$$

where,

$$\xi(d\omega) \equiv \sum_{j=0}^{p} \xi_j \delta_{\omega_j}(d\omega), \qquad F(d\omega) = \sum_{j=0}^{p} E|\xi_j|^2 \delta_{\omega_j}(d\omega)$$

and the

$$\xi(\{\omega_j\}) \equiv \xi_j$$

are orthogonal with mean zero. Clearly $\xi : \mathcal{B}(-\pi, \pi] \to L^2(P)$ is σ-additive.

3.1.1 Estimation of the Mean

As an illustration of the use of Theorem 2.1, consider the estimation problem of the mean $E[X_t] = m$. Write

$$X_t = \int_{-\pi}^{\pi} e^{it\omega} (\xi(d\omega) + m\delta_0(d\omega))$$

Since the ω_j are distinct, for sufficiently large n,

$$\xi((-2^{-n}, 2^{-n}) \setminus \{0\}) + m\delta_0((-2^{-n}, 2^{-n}) \setminus \{0\}) = 0 \quad a.s.$$

Therefore,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} X_t = \xi(\{0\}) + m \quad a.s. \tag{1}$$

and we obtain the well know result that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} X_t = m$$

if and only if $\xi(\{0\}) = 0$ almost surely. Observe that the dc component $\xi(\{0\}) + m$ in the second order stationary case is a sum of a random part $\xi(\{0\})$ plus a nonrandom constant m. Clearly, "$\xi(\{0\}) + m$" here plays the role of "$\xi(\{0\})$" in Theorem 2.1.

3.1.2 Estimation of the Autocovariance

We shall treat the general case which takes into account the presence of a possible dc component. That is, we do not necessarily require $m = 0$ and/or $\xi(\{0\}) = 0$ a.s. . In practice however, the dc component is often removed by subtracting the sample average in accordance with the SLLN (1) (see Koopmans (1974), p.58). As an estimator for $R(k)$ we choose the (simplified) sample autocovariance $(1/N)\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)}$.

If we define

$$\xi^{(k)}(d\omega) \equiv \left\{ \sum_{j=0}^{p} |\xi(\{\omega_j\})|^2 e^{ik\omega_j} \right\} \delta_0(d\omega) + \sum_{j \neq l} \xi(\{\omega_j\})\overline{\xi(\{\omega_l\})} e^{ik\omega_j} \delta_{\omega_j - \omega_l}(d\omega)$$

Then

$$(X_{t+k} - m)\overline{(X_t - m)} = \int_{-2\pi}^{2\pi} e^{it\omega} \xi^{(k)}(d\omega)$$

and we can see that for each fixed k, the lag process $(X_{t+k} - m)\overline{(X_t - m)}$ admits a Fourier representation with respect to $\xi^{(k)} : \mathcal{B}(-2\pi, 2\pi] \to L^1(P)$, and a dc component

$$\xi^{(k)}(\{0\}) = \sum_{j=0}^{p} |\xi(\{\omega_j\})|^2 e^{ik\omega_j}$$

Notice the emergence of a dc component in the lag process $(X_{t+k} - m)\overline{(X_t - m)}$ regardless of whether it is absent or present in the original process $\{X_t\}$.

Since the ω_j are distinct, there is no $\xi^{(k)}(\cdot)$ mass in sufficiently small punctured neighborhoods of 0 and of $\pm 2\pi$. In addition, $\xi^{(k)}(\{2\pi\}) = 0$. Therefore, by Theorem 2.2

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)} \overset{a.s}{\to} \xi^{(k)}(\{0\}), \quad \forall k$$

We need conditions to ensure that $\xi^{(k)}(\{0\})$ is a nonrandom constant for all k, in which case

$$\xi^{(k)}(\{0\}) = E[\xi^{(k)}(\{0\})] \text{ a.s.}, \qquad \forall k$$

Clearly, $E[\xi^{(k)}(\{0\})] = R(k)$.

So, assume $\xi^{(k)}(\{0\}) = E[\xi^{(k)}(\{0\})]$, $k = 0, \pm 1, \cdots$. Then for all k,

$$\sum_{j=0}^{p}(|\xi_j|^2 - E|\xi_j|^2)e^{ik\omega_j} = 0$$

and this implies that

$$|\xi_j|^2 = E|\xi_j|^2, \qquad j = 0, 1, \cdots, p$$

or,

$$\xi_j = \sqrt{E|\xi_j|^2}e^{i\phi_j}, \qquad j = 0, 1, \cdots, p$$

where the ϕ_j are random such that the ξ_j are orthogonal with mean 0.

On the other hand, when $\xi_j = \sqrt{E|\xi_j|^2}e^{i\phi_j}$, then $|\xi_j|^2 = E|\xi_j|^2$ and $\xi^{(k)}(\{0\}) = E[\xi^{(k)}(\{0\})] = R(k)$ is a.s. nonrandom for all k. We thus have.

Theorem 3.1 *Let*

$$X_t = m + \sum_{j=0}^{p} e^{i\omega_j t}\xi_j, \qquad t = 0, \pm 1, \pm 2, \cdots \tag{2}$$

where $E[\xi_j] = 0$, and for $j \neq l$ $E[\xi_j\overline{\xi_l}] = 0$, be a complex valued weakly stationary process with mean m, autocovariance $R(\cdot)$, and a spectrum supported at $p+1$ distinct atoms $\{\omega_0 = 0, \omega_1, \cdots, \omega_p\} \in (-\pi, \pi]$. Then as $N \to \infty$,

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)} \overset{L^1, a.s}{\to} \xi^{(k)}(\{0\}), \qquad k = 0, \pm 1, \pm 2, \cdots \tag{3}$$

Furthermore,

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)} \overset{L^1, a.s.}{\to} R(k), \qquad k = 0, \pm 1, \pm 2, \cdots \tag{4}$$

if and only if

$$\xi_j = \sqrt{E|\xi_j|^2}e^{i\phi_j}, \qquad j = 0, 1, \cdots, p \tag{5}$$

where the ϕ_j are random phases.

As a corollary of Theorem 3.1 we can see that if $\{X_t\}$ in (2) is a real-valued Gaussian process, its sample autocovariance is not consistent. This is so because of the requirement of constant amplitudes $|\xi_j|$. This is in line with the well known fact, mentioned earlier, that the presence of a discrete spectrum renders the sample autocovariance of a Gaussian process inconsistent in $L^2(P)$ sense.

Example 1. Consider the model of random phases

$$X_t = \sum_{j=1}^{p} A_j \cos(\omega_j t + \phi_j), \qquad t = 0, \pm 1, \pm 2 \cdots$$

where $E[X_t] = m = 0$, $0 < \omega_1 < \omega_2 < \cdots < \omega_p \leq \pi$, A_j are positive constants, and the ϕ_j are independent random variables uniformly distributed in $(-\pi, \pi]$. Define

$$\phi_{-j} = -\phi_j, \ \omega_{-j} = -\omega_j, \ A_{-j} = A_j, \ A_0 = 0$$

Then we have the representation

$$X_t = \sum_{j=-p}^{p} \frac{1}{2} A_j e^{i\phi_j} e^{i\omega_j t}$$

so that $(1/N)\sum_{t=1}^{N} X_{t+k} X_t \stackrel{a.s.}{\to} R(k)$. From this, the sample autocorrelation is a.s. consistent as well. □

3.2 Infinitely Many Frequencies

This case takes after the case of finitely many frequencies with the added assumption that for $j \neq l$, $|\omega_j - \omega_l|$ cannot be too close to 0 and $\pm 2\pi$. At this point it might be worthwhile to mention that the case of infinitely many frequencies is also important in practice: Simple nonlinear operations on a sum of pure random sinusoids, e.g., clipping, lead to a discrete spectrum process with infinitely many frequencies (see [7]).

Suppose that for $\omega_j \in (-\pi, \pi]$,

$$X_t = m + \sum_{j=0}^{\infty} e^{i\omega_j t}\xi_j \tag{6}$$

is weakly stationary with mean m. Then by stationarity the ξ_j are orthogonal and

$$E|X_t|^2 = \sum_{j=0}^{\infty} |\xi_j|^2 < \infty$$

It follows that

$$(X_{t+k} - m)\overline{(X_t - m)} = \int_{-2\pi}^{2\pi} e^{it\omega}\xi^{(k)}(d\omega)$$

where

$$\xi^{(k)}(d\omega) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \xi_j\bar{\xi}_l e^{i\omega_j k}\delta_{\omega_j - \omega_l}(d\omega)$$

is σ-additive from $\mathcal{B}(-2\pi, 2\pi)$ to $L^1(P)$ since $\sum_j |\xi_j|^2 < \infty$. Thus for each fixed k, the lag process $\{(X_{t+k} - m)\overline{(X_t - m)}\}$, $t = 0, \pm 1, \pm 2, \cdots$, admits a Fourier representation with respect to $\xi^{(k)}(d\omega)$, with a dc component

$$\xi^{(k)}(\{0\}) = \sum_{j=0}^{\infty} |\xi_j|^2 e^{ik\omega_j}$$

Note that since $-\pi < \omega_j \leq \pi$, we always have $-2\pi < \omega_j - \omega_l < 2\pi$ for distinct ω_j, ω_l, so that $\xi^{(k)}(\{2\pi\}) = 0$. Therefore, by Theorem 2.2 and the previous discussion on equality between $\xi^{(k)}(\{0\})$ and $R(k)$, we have.

Theorem 3.2 *Let $\{X_t\}$, $t = 0, \pm 1, \cdots$, be a complex valued weakly stationary process with mean m, autocovariance $R(\cdot)$, and a spectrum supported at infinitely many distinct atoms $\omega_j \in (-\pi, \pi]$ as in (6). Then as $N \to \infty$,*

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)} \xrightarrow{L^1} \xi^{(k)}(\{0\}) \tag{7}$$

and the convergence holds almost surely if and only if as $n \to \infty$

$$\begin{aligned}[\xi^{(k)}((-2^{-n}, 2^{-n}) \setminus \{0\}) &+ \xi^{(k)}(-2\pi, -2\pi + 2^{-n}) \\ &+ \xi^{(k)}(2\pi - 2^{-n}, 2\pi)] \xrightarrow{a.s.} 0\end{aligned} \tag{8}$$

Furthermore, when (8) holds,

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k}-m)\overline{(X_t-m)} \overset{a.s.}{\to} R(k) \tag{9}$$

for all k $(=0,\pm1,\pm2,\cdots)$*, if and only if*

$$\xi_j = \sqrt{E|\xi_j|^2}e^{i\phi_j}, \qquad j=0,1,2,\cdots \tag{10}$$

where the ϕ_j *are random phases.*

What Theorem 3.2 says in effect is that if the frequency differences cluster in the critical intervals $(-2^{-n},2^{-n})\setminus\{0\})$, $(-2\pi,-2\pi+2^{-n})$, $(2\pi-2^{-n},2\pi)$, the almost sure convergence might not hold. Evidently, a.s. convergence holds when $\inf_{j\neq l}|\omega_j-\omega_l|>0$ and $\sup_{j\neq l}|\omega_j-\omega_l|<2\pi$. Once the almost sure convergence occurs, the result also provides a necessary and sufficient condition for the limit to be a constant.

3.2.1 Connection With the Zero-Crossing Rate

Theorem 3.2 clarifies the asymptotic behavior of the observed zero-crossing rate.

Let $\{Z_t\}$ be a real-valued stationary Gaussian process with mean 0, and a purely discrete spectrum containing a finite number of jumps (atoms). Also, let $X_t=1$ when $Z_t\geq 0$ and $X_t=0$ otherwise. We observe that the spectrum of $\{X_t\}$ is an infinite sum of convolutions in terms of the spectrum of $\{Z_t\}$, so that $\{X_t\}$ admits the representation (6) with $E[X_t]=m=1/2$.

The zero-crossing rate (ZCR) $\hat{\gamma}$ (in discrete time) observed in a time series $Z_1,Z_2,\cdots,Z_N$ is defined in terms of the indicators $d_t\equiv(X_t-X_{t-1})^2$,

$$\hat{\gamma}=\frac{D}{N-1}=\frac{1}{N-1}\sum_{t=2}^{N}d_t$$

Thus, asymptotically the ZCR is the same as

$$\hat{\gamma}=\frac{1}{2}-\frac{2}{N}\sum_{t=2}^{N}(X_t-m)(X_{t-1}-m)$$

where $E[X_t] = m = 1/2$, and we can study the asymptotic ZCR through the first order sample autocovariance of $\{X_t\}$. So, does the ZCR converge almost surely to a constant ? This is addressed in Kedem and Slud (1993). Let $\sigma_d(\cdot)$ denote the spectral measure of the binary process $\{d_t\}$.

Theorem 3.3 *(Kedem and Slud (1993).) Let $\{Z_t;\ t = 0, \pm 1, \pm 2, \cdots\}$ be a real-valued zero-mean stationary Gaussian process with normalized spectral measure $\sigma_z(\cdot)$. If σ_z has at least two atoms in $[0, \pi]$, then $\sigma_d(\{0\}) > 0$. That is, the spectrum of $\{d_t\}$ has a jump at* 0.

Thus, two or more atoms (jumps) in the spectrum of $\{Z_t\}$ prevent the ZCR from converging to a *constant* a.s., though it always converges a.s. to a random variable because of the assumed Gaussianity and hence strict stationarity. Hence, by Theorem 3.2 the spectrum of $\{(X_t - .5)(X_{t-1} - .5)\}$ *does not* contain a clustered difference in the critical intervals $(-2^{-n}, 2^{-n}) \setminus \{0\})$, $(-2\pi, -2\pi + 2^{-n})$, $(2\pi - 2^{-n}, 2\pi)$. On the other hand, in the representation of $\{X_t\}$ the ξ_j do not have constant amplitudes as in (10).

In the case of a single pure atom, the ZCR converges almost surely to a constant, so that by Theorem 3.2, (10) holds.

4 Mixed Spectrum Case

The case of practical interest is a finite sum of complex exponentials plus continuous spectrum noise. The argument proceeds in exactly the same way as before. Suppose, for $\omega_j \in (-\pi, \pi]$, , $j = 0, 1, \cdots, p$,

$$X_t - m = \sum_{j=0}^{p} e^{i\omega_j t}\xi_j + \epsilon_t$$

is weakly stationary. Further, suppose the noise ϵ_t is zero mean ergodic independent of the zero mean ξ_j. When forming the product $(X_{t+k} - m)\overline{(X_t - m)}$, the sample covariance of the cross terms (and their conjugates) converges to zero with probability one (by zero mean ergodicity). Since the noise ϵ is ergodic, its sample autocovariance also converges almost surely to its autocovariance. Hence, a simple consequence of the results of the previous section as well as of independence, we have.

Theorem 4.1 *Suppose, for* $\omega_j \in (-\pi, \pi]$, $j = 0, 1, \cdots, p$,

$$X_t - m = \sum_{j=0}^{p} e^{i\omega_j t}\xi_j + \epsilon_t$$

is weakly stationary. Let the $\xi_j = \sqrt{E|\xi_j|^2}e^{i\phi_j}$, $j = 0, 1, \cdots, p$, *be zero mean and independent of the zero mean ergodic process* ϵ. *Then,*

$$\frac{1}{N}\sum_{t=1}^{N}(X_{t+k} - m)\overline{(X_t - m)} \xrightarrow{a.s.} R(k), \qquad k = 0, \pm 1, \cdots$$

Example 2. Let $\{X_t\}$ be as in Example 1, and consider the model of signal plus noise

$$Y_t = X_t + \epsilon_t, \qquad t = 0, \pm 1, \cdots$$

where $\{\epsilon_t\}$ is a linear process,

$$\epsilon_t = \sum_{j=-\infty}^{\infty} a_j u_{t-j}, \qquad \{u_t\} \sim \mathrm{IID}(0, \sigma_u^2), \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty$$

independent of $\{X_t\}$. Then $\{\epsilon_t\}$ is a continuous spectrum strictly stationary ergodic process (Hannan (1970), p. 204). By Theorem 4.1 then

$$\frac{1}{N}\sum_{t=1}^{N} Y_{t+k}Y_t \xrightarrow{a.s.} R(k), \qquad \text{for all } k$$

Filtering $\{Y_t\}$ by a time invariant linear filter gives again a discrete spectrum signal plus an independent ergodic stationary noise component, so that again the sample autocovariance converges a.s. to the true autocovariance for all lags k. In particular, let $\mathcal{L}_\alpha$ be such a (parametric) filter with an impulse response $h_j(\alpha)$ depending on a parameter $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an open set. Define,

$$Y_t(\alpha) = \mathcal{L}_\alpha Y_t = \sum_{j=0}^{\infty} h_j(\alpha) Y_{t-j}$$

If there are constants $c_j > 0$ such that $\sum jc_j < \infty$ and $|h_j(\alpha)| \leq c_j$ for all j and uniformly in α, then the almost sure convergence of the sample autocovariance is also uniform in $\alpha \in \mathcal{A}$ (Li and Kedem (1993)).

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