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Optimal Stochastic Scheduling Of Systems With Poisson Noises

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Abstract: We consider the problem of optimal stochastic scheduling for nonlinear systems with Poisson noise disturbances and a performance index including both operating costs and costs for sceduling changes. In general, the value functions of the dynamic programming, quasi-variational inequalities which define the optimality conditions for such problems are not differentiable. However, we can treat them as "viscosity solutions" as introduced by Crandall and Lions. Existence and uniqueness questions are studied from this point of view.

Key Words: Dynamic programming, quasi-variational inequalities, scheduling problems, viscosity solutions.

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Contents

1. Introduction

Optimal scheduling problems arise in many contexts, including inventory control systems and unit commitment scheduling in electric power systems. These problems typically involve stochastic dynamical systems, admitting discrete state transitions at random times as control actions, and incurring both switching costs and continuous running costs. Using the dynamic programming principle, one can show that the optimality conditions for these problems are expressed mathematically by *quasi-variational inequalities* (QVI). It is difficult to treat QVI's explicitly, and most of the work has focussed on proving existence, uniqueness, and regularity of solutions.

In our case, the state system is forced by Poisson noises. Since the infinitesimal generator of the state process is first order and has a translation in the argument, the associated QVI is first order and fully nonlinear; and so, the standard existence and uniqueness theory developed for diffusion - parabolic systems does not apply. To treat the problem, we use the method of *viscosity solutions* introduced by M. G. Crandall and P. L. Lions [1]; see also P. L. Lions [2]. Various properties of viscosity solutions are developed in Crandall - Evans - Lions [3]. We use the approach in Capuzzo Dolcetta - Evans [4] developed for deterministic systems[1]. We prove that the value function u associated with the optimization problem is a viscosity solution of the corresponding (QVI). Existence of solutions to the (QVI) is shown by using a discrete approximation to an associated penalized system and then using results for accretive operators as in [7]. On the other hand, we use dynamic programming to obtain a decreasing sequence of value functions u_l optimal for controls with at most l switches, which converges uniformly. This approach was used to obtain a maximum solution of certain (QVI) in Menaldi [10-11] without nondegeneracy assumptions. In Blankenship - Menaldi [12], related problems were treated involving the application of (QVI) to power generation

[1]Cases with white noise models are treated in [5] and [6], while control problems for diffusion processes with jumps are treated in Bensoussan [7]. See also [8] for an introduction to the subject.

systems with scheduling delays. See also [13][14] for a survey of viscosity methods for the control of diffusions.

The optimal stochastic control of linear regulator systems with Poisson noise disturbances is considered in [15]; stochastic stability properties of linear systems with multiplicative Poisson noises are derived in [16]. See also [17].

1.1 Problem Statement.

Let (Ω, F, P) be a probability space and $\{F_t,\ t \geq 0\}$ a non-decreasing, right-continuous family of completed sub σ-fields of F such that $F_t \uparrow F_\infty \triangleq F$, t ≥ 0. Consider the general nonlinear dynamical system

$$\begin{cases} dy_x(t) = g(y_x(t), \alpha(t))dt \ + \ h(y_x(t), \alpha(t))dN_{\alpha(t)}(t) \\ y_x(0) = x \end{cases} \tag{1.1}$$

where $N_i(t)$, $i = 1, \ldots, m$, are independent Poisson processes with intensities λ_i, $i = 1, \ldots, m$. $\alpha(t)$ is a right continuous, piecewise constant random function with finite range $1, \ldots, m$, and is measurable with respect to F_t, $t \geq 0$. Actually, α is an admissible control consisting of random switching times θ_i and random switching decisions d_i such that θ_i are adapted to $\{F_t\}$ and d_i are F_{θ_i} - measurable so that

$$\begin{gathered} 0 \equiv \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{i-1} \leq \theta_i \leq \theta_{i+1}, \quad \theta_i \to +\infty \qquad a.s. \\ d_i \in \{1, \ldots, m\}, \ d_i \neq d_{i-1} \text{ if } \theta_i < \infty \end{gathered} \tag{1.2}$$

And so

$$\alpha(t) \triangleq d_i \quad \text{if } \theta_i \leq t < \theta_{i+1}, \ i \geq 0$$

is indeed F_t - measurable.

Let the set of all admissible controls with initial setting d be

$$A^d \triangleq \{\alpha \mid \alpha = \{\theta_i, d_i\} \text{ satisfies the above properties with initial setting } d_0 = d\}. \tag{1.3}$$

We take the performance index to be

$$J_x^d(\alpha) \triangleq E_{x,d}\{\int_0^\infty f(y_x(t),\alpha(t))e^{-\beta t}\,dt + \sum_{i=1}^\infty k(d_{i-1},d_i)e^{-\beta\theta_i}\}$$

$$= E_{x,d}\{\sum_{i=1}^\infty [\int_{\theta_{i-1}}^{\theta_i} f(y_x(t),d_{i-1})e^{-\beta t}\,dt + k(d_{i-1},d_i)e^{-\beta\theta_i}]\} \tag{1.4}$$

where $\beta > 0$ is a discount factor and $k(d,\hat{d})$ is the cost of switching[2] from d to $\hat{d}$ such that

$$k(d,\hat{d}) > 0 \text{ if } d \neq \hat{d};\ k(d,d) = 0 \tag{1.5}$$

$$k(d,\hat{d}) < k(d,\tilde{d}) + k(\tilde{d},\hat{d}) \text{ if } d \neq \tilde{d} \neq \hat{d}.$$

Without loss of generality, we can define $k_0 \triangleq \min\{k(d,\hat{d}),\ d \neq \hat{d}\}$. We assume $f \geq 0$, g and h are bounded and Lipschitz continuous

$$|q(x,d)| \leq ||q|| < \infty$$

$$|q(x,d) - q(\hat{x},d)| \leq L\,|x-\hat{x}| \tag{1.6}$$

with $q = f$, g and h, for all $x, \hat{x} \in \mathbb{R}^n$, $d \in 1,\ldots,m$.

Under these assumptions, (1.1) has a unique solution. Defining the value function

$$u^d(x) \triangleq \inf_{\alpha \in A^d} J_x^d(\alpha),\quad x \in \mathbb{R}^n,\ d \in \{1,\ldots,m\} \tag{1.7}$$

we want to design an optimal control α^* such that

$$u^d(x) = J_x^d(\alpha^*) = \inf_{\alpha \in A^d} J_x^d(\alpha). \tag{1.8}$$

[2]The case when the switching costs can be zero is treated in section 5.

Remark. $N_{\alpha(t)}(t)$ is an inhomogeneous Poisson process with intensity function $\lambda_{\alpha(t)}$.

1.2 Summary of Results.

In section 2 we show that the optimal value function $u^d(x)$ in (1.6) maybe defined as the limit of the value functions $u_l^d(x)$ of systems with a finite number l of switches as $l \to \infty$ (Theorem 2.3). We show that the convergence is uniform (Theorem 2.5); and we derive two representations of $u^d(x)$ as the *optimal* value function (Theorems 2.6 and 2.7). We describe the associated optimal (control) switching policy (Theorem 2.8), and we use it to obtain an additional estimate on the convergence of u_l^d to u^d.

In section 3 we derive the QVI which must be satisfied by the optimal value function (equation (3.3)). We show that the optimal value function is a viscosity solution of the QVI (Theorem 3.1). Then we show that the solution is unique.

In section 4 we prove that the QVI has a viscosity solution by constructing a sequence of solutions to a penalized system (equation (4.2)) and proving that hese solutions are uniformly bounded and uniformly Hölder continuous (Theorem 4.4). We show that the limit of the sequence of solutions to the penalized system is a viscosity solution of the QVI.

In section 5 we consider the case when the switching costs vanish ($k(d,\hat{d}) = 0$ for $d \neq \hat{d}$ in (1.4)). In this case the optimal value function u is independent of the initial control configuration d (since we can switch for "free" at any time), and it (formally) satisfies a Hamiliton - Jacobi - Bellman equation which is fully nonlinear in ∇u. The method of viscosity solutions is required to treat this case. We show that the optimal value function corresponding to non-zero switching costs will converge to u as the switching costs tend to zero, and that u is the unique viscosity solution of the Hamiliton - Jacobi - Bellman equation. The result is analogous to those in Capuzzo Dolcetta - Evans [4]. Thus, the method of viscosity solutions provides a *complete* framework for the treatment of the optimal control problem (1.1) - (1.8) over the full range of parameter values and operating regimes.

2. Dynamic Programming and Some Preliminary Results.

Before using dynamic programming to investigate the properties of the value function $u^d(x)$, we need some preliminary results.

Lemma 2.1. *For any stopping time τ which is adapted to $\{F_t\}$ and any measurable bounded function q, we have*

$$E\ [q(y_x(t+\tau)) \mid F_\tau] = E_{y_x(\tau)} q(y_{y_x(\tau)}(t)). \tag{2.1}$$

Proof. Since

$$\begin{aligned} y_x(t+\tau) &= y_x(\tau) + \int_\tau^{\tau+t} g(y_x(s),\alpha(s))ds + \int_\tau^{\tau+t} h(y_x(s),\alpha(s))dN_{\alpha(s)}(s) \\ &= y_x(\tau) + \int_0^t g(y_x(s+\tau),\hat{\alpha}(s))ds + \int_0^t h(y_x(s+\tau),\hat{\alpha}(s))d\hat{N}_{\alpha(s+\tau)}(s) \end{aligned} \tag{2.2}$$

where

$$\hat{\alpha}(s) = \alpha(s+\tau)$$

$$\hat{N}_i(s) = N_i(s+\tau) - N_i(\tau).$$

We claim $\hat{N}_i$(s) is a Poisson process which is independent of τ. Since τ is adapted to F_t and

$$F_\tau = \{ B \in F \mid B \cap \{\tau \le t\} \in F_t \},$$

we know τ is F_τ-measurable. Thus

$$P[\hat{N}_i(s)=k] = P[N_i(s+\tau)-N_i(\tau)=k]$$
$$= E\{P[N_i(s+\tau)-N_i(\tau)=k \mid F_\tau]\}.$$

Since a right continuous Poisson process is a Feller process,[3] then N_i is a strong Markov process. Thus, for $t \geq s$,

$$P[\hat{N}_i(t) - \hat{N}_i(s)=k] = P[N_i(t+\tau)-N_i(s+\tau)=k]$$
$$= E\{P[N_i(t+\tau)-N_i(s+\tau)=k \mid F_{s+\tau}]\}$$
$$= E\{P[N_i(t+\tau)-N_i(s+\tau)=k \mid N_i(s+\tau), \tau]\}$$
$$= \frac{[\lambda_i(t-s)]^k}{k!} e^{-\lambda_i(t-s)}$$
$$= P[N_i(t+\tau)-N_i(s+\tau)=k \mid N_i(s+\tau), \tau] \tag{2.3}$$

Moreover, if $0 \leq t_0 \leq t_1 \leq \cdots \leq t_l < \infty$, and

$$\hat{N}_i(t_j) - \hat{N}_i(t_{j-1}) = N_i(t_j+\tau) - N_i(t_{j-1}+\tau),$$

then by the strong Markov property and (2.3)

$$P\{\hat{N}_i(t_0)=k_0, \hat{N}_i(t_1)-\hat{N}_i(t_0)=k_1, \ldots, \hat{N}_i(t_l)-\hat{N}_i(t_{l-1})=k_l\}$$
$$= E\{I_{N_i(t_0+\tau)-N_i(\tau)=k_0} I_{N_i(t_1+\tau)-N_i(t_0+\tau)=k_1} \cdots I_{N_i(t_l+\tau)-N_i(t_{l-1}+\tau)=k_l}\}$$
$$= E\{\underbrace{E\{\cdots E}_{l+1\ times}\{I_{N_i(t_0+\tau)-N_i(\tau)=k_0} \cdots I_{N_i(t_l+\tau)-N_i(t_{l-1}+\tau)=k_l} \mid F_{t_{l-1}+\tau}\} \cdots \mid F_{t_0+\tau}\}\}$$
$$= E\{\underbrace{E\{\cdots}_{l\ times} E\{I_{N_i(t_0+\tau)-N_i(\tau)=k_0} \cdots I_{N_i(t_{l-1}+\tau)-N_i(t_{l-2}+\tau)=k_{l-1}}$$
$$\cdot E\{I_{N_i(t_l+\tau)-N_i(t_{l-1}+\tau)=k_l} \mid N(t_{l-1}+\tau),\tau\} \mid F_{t_{l-2}+\tau}\} \cdots \mid F_{t_0+\tau}\}\}$$
$$= E\{\underbrace{E\{\cdots}_{l\ times} E\{I_{N_i(t_0+\tau)-N_i(\tau)=k_0} \cdots I_{N_i(t_{l-1}+\tau)-N_i(t_{l-2}+\tau)=k_{l-1}} \mid F_{t_{l-2}+\tau}\} \cdots \mid F_{t_0+\tau}\}\}$$
$$\cdot P\{\hat{N}_i(t_l)-\hat{N}_i(t_{l-1})=k_l\} \tag{2.4}$$

[3]See E. B. Dynkin, *Markov Processes*, Vol. 1, Springer-Verlag, Berlin, 1965 (section 2.18, pp. 69).

where I denotes the indicator function. By induction, (2.4) is equal to

$$P\{\hat{N}_i(t_0)=k_0\} \cdots P\{\hat{N}_i(t_l)-\hat{N}_i(t_{l-1})=k_l\},$$

so that $\hat{N}_i$ has independent increments. Thus, $\hat{N}_i$ is the Poisson process with intensity λ_i. Since $\hat{y}_{y_x(\tau)}(t) \triangleq y_x(t+\tau)$ is a unique solution of (2.2), the statistics of $\hat{y}_x(t)$ and $y_x(t)$ are the same if $\hat{\alpha} = \alpha$. From (2.2), we know that

$$\begin{aligned} E\ \{q(y_x(t+\tau)) \mid F_\tau\} &= E_{y_x(\tau)}\, q(y_x(t+\tau)) \\ &= E_{y_x(\tau)}\, q(\hat{y}_{y_x(\tau)}(t)) \\ &= E_{y_x(\tau)}\, q(y_{y_x(\tau)}(t)). \end{aligned}$$

QED

Lemma 2.2. *For each* $d \in \{1, \ldots, m\}$ *and* $x \in \mathbb{R}^n$,

$$\text{(i) } u^d(x) \leq \min_{\hat{d} \neq d} \{u^{\hat{d}}(x) + k(d, \hat{d})\} \tag{2.5}$$

(ii) For any stopping time $\theta \geq 0$,

$$u^d(x) \leq E\ \{\int_0^\theta f(y_x(s), d)e^{-\beta s}\, ds + u^d(y_x(\theta))e^{-\beta\theta}\} \tag{2.6}$$

Proof. (i) Let d, $\hat{d} \in \{1, \ldots, m\}$ be such that $\hat{d} \neq d$. Set $\alpha = \{\theta_i, d_i\} \in A^{\hat{d}}$. Define $\hat{\alpha} = \{\hat{\theta}_i, \hat{d}_i\}_{i=0}^\infty$ by $\hat{\theta}_0 = 0$, $\hat{d}_0 = d$, $\hat{\theta}_i = \theta_{i-1}$, $\hat{d}_i = d_{i-1}$ for $i \geq 1$. Then $\hat{\theta}_i = \theta_{i-1}$ is adapted to $\{F_t\}$ and $\hat{d}_i = d_{i-1}$ is $F_{\hat{\theta}_i}$ - measurable, so that $\hat{\alpha} \in A^d$. Thus,

$$\begin{aligned} u^d(x) &\leq J_x^d(\hat{\alpha}) \\ &= E\ \{\sum_{i=1}^\infty \int_{\hat{\theta}_{i-1}}^{\hat{\theta}_i} f(y_x(s), \hat{d}_{i-1})e^{-\beta s}\, ds + k(\hat{d}_{i-1}, \hat{d}_i)e^{-\beta\hat{\theta}_i}\} \end{aligned}$$

$$= E \left\{ k(d,\hat{d}) + \sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} f(y_x(s), d_{i-1}) e^{-\beta s} ds + k(d_{i-1}, d_i) e^{-\beta\theta_i} \right\}$$

$$= k(d,\hat{d}) + J_x^{\hat{d}}(\alpha).$$

Since $\alpha \in A^{\hat{d}}$ and $\hat{d}$ are arbitrary, we know that

$$u^d(x) \leq \min_{\hat{d} \neq d} \{ k(d,\hat{d}) + u^{\hat{d}}(x) \}.$$

(ii) Let $\alpha = \{\theta_i, d_i\} \in A^d$ and fix a stopping time $\theta \geq 0$. Define $\hat{\alpha} = \{\hat{\theta}_i, \hat{d}_i\}$ by

$$\hat{\theta}_0 = 0, \quad \hat{d}_0 = d;$$

$$\hat{\theta}_i = \theta_i + \theta, \quad \hat{d}_i = d_i, \quad i \geq 1.$$

Then $\hat{\theta}_i = \theta_i + \theta$ is adapted to $\{F_t\}$ and $\hat{d}_i = d_i$ is $F_{\hat{\theta}_i}$ - measurable, so that $\hat{\alpha} \in A^d$. Thus, from Lemma 2.1,

$$u^d(x) \leq J_x^d(\hat{\alpha})$$

$$= E \left\{ \sum_{i=1}^{\infty} \int_{\theta_{i-1}+\theta}^{\theta_i+\theta} f(y_x(s), d_{i-1}) e^{-\beta s} ds + k(d_{i-1}, d_i) e^{-\beta(\theta_i+\theta)} \right\}$$

$$= E \left\{ \int_0^{\theta} f(y_x(s), d) e^{-\beta s} ds + E \left\{ \left[\sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} f(y_x(s+\theta), d_{i-1}) e^{-\beta s} ds \right.\right.\right.$$

$$\left.\left.\left. + k(d_{i-1}, d_i) e^{-\beta\theta_i} \right] e^{-\beta\theta} \,\middle|\, F_\theta \right\} \right\}$$

$$= E \left\{ \int_0^{\theta} f(y_x(s), d) e^{-\beta s} ds + E_{y_x(\theta)} \left[\sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} f(y_{y_x(\theta)}(s), d_{i-1}) e^{-\beta s} ds \right.\right.$$

$$+ \; k(d_{i-1}, d_i) e^{-\beta\theta_i} \Big] e^{-\beta\theta} \Big\}$$

$$= E \left\{ \int_0^\theta f(y_x(s), d) e^{-\beta s} \, ds \; + \; J^d_{y_x(\theta)}(\alpha) \, e^{-\beta\theta} \right\}.$$

Since $\alpha \in A^d$ is arbitrary, we have

$$u^d(x) \le E \left\{ \int_0^\theta f(y_x(s), d) e^{-\beta s} \, ds \; + \; u^d(y_x(\theta)) \, e^{-\beta\theta} \right\}.$$

QED

Notation. For $x \in \mathbb{R}^n$, $d \in 1,\ldots,m$,

$$M^d[u](x) \triangleq \min_{\hat{d} \neq d} \{ u^{\hat{d}}(x) + k(d, \hat{d}) \}. \tag{2.7}$$

Now, we want to use the dynamic programming principle to show there exists a convergent sequence $\{u_l^d\}$ of optimal solutions of the problem with respect to controls which have at most l switches.

For each $x \in \mathbb{R}^n$, $d \in 1,\ldots,m$, let

$$u_0^d(x) \triangleq \int_0^\infty f(y_x(s), d) e^{-\beta s} \, ds. \tag{2.8}$$

Notation. If $u, v \in C(\mathbb{R}^n)^m$, then we say $u \ge v$ if $u^d \ge v^d$, $\forall$ $d = 1, \ldots, m$.

Define an operator $\Gamma_d : C(\mathbb{R}^n)^m \to C(\mathbb{R}^n)$ by

$$\Gamma_d u(x) \triangleq \inf_{\theta \geq 0} E \left\{ \int_0^{\theta} f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d [u](y_x(\theta)) \right\}. \tag{2.9}$$

Here we understand the infimum is taken for all stopping times $\theta \geq 0$ adapted to $\{F_t\}$. If $u \geq v$, then for each $\epsilon > 0$, there exists a stopping time $\theta_\epsilon \geq 0$ and d_ϵ F_{θ_ϵ} - measurable such that

$$\Gamma_d u(x) > E \left\{ \int_0^{\theta_\epsilon} f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta_\epsilon}[u^{d_\epsilon}(y_x(\theta_\epsilon)) + k(d, d_\epsilon)] \right\} - \epsilon$$

$$\geq E \left\{ \int_0^{\theta_\epsilon} f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta_\epsilon}[v^{d_\epsilon}(y_x(\theta_\epsilon)) + k(d, d_\epsilon)] \right\} - \epsilon$$

$$\geq \Gamma_d v(x) - \epsilon.$$

Let $\epsilon \downarrow 0$, we have $\Gamma_d u \geq \Gamma_d v$. Let $0 \leq \eta \leq 1$, then

$$\Gamma_d [(1-\eta)u + \eta v]$$

$$= \inf_{\theta \geq 0} E \left\{ \int_0^{\theta} f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d [(1-\eta)u + \eta v](y_x(\theta)) \right\}$$

$$\geq \inf_{\theta \geq 0} E \left\{ \int_0^{\theta} f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} \{(1-\eta) M^d [u](y_x(\theta)) + \eta M^d [v](y_x(\theta))\} \right\}$$

$$\geq (1 - \eta)\Gamma_d u(x) + \eta \Gamma_d v(x).$$

Thus, Γ_d is a non-decreasing, concave function.

Suppose we are given u_{l-1}. We can define

$$u_l^d(x) \triangleq \Gamma_d u_{l-1}(x). \tag{2.10}$$

Since $u_1^d(x) = \Gamma_d u_0(x) \leq u_0^d(x)$, then by the non-decreasing property of Γ_d, we have $u_2^d(x) = \Gamma_d^2 u_0(x) \leq \Gamma_d u_0(x)$, and so

$$0 \leq u_l^d(x) \leq u_{l-1}^d(x) \leq \cdots \leq u_0^d(x) \leq \frac{\|f\|}{\beta}. \tag{2.11}$$

Thus, $u_l^d(x)$ converges. We can define

$$u_\infty^d(x) \triangleq \lim_{l \to \infty} u_l^d(x). \tag{2.12}$$

Theorem 2.3.

$$u_l^d(x) = inf\{ J_x^d(\alpha_l) \mid \alpha_l \in A^d \ \ has \ at \ most \ l \ switches \} \tag{2.13}$$

and thus

$$u_\infty^d(x) = u^d(x) \triangleq \inf_{\alpha \in A^d} J_x^d(\alpha). \tag{2.14}$$

Proof. Clearly, (2.13) is satisfied for $l=0$. Suppose (2.13) is true for $l-1$ and the switching policy $\alpha_{l-1}^* = \{\theta_i^{l-1,d}, d_i^{l-1,d}\} \in A^d, \forall\ d = 1, \ldots, m$. Then let

$$\overline{\theta} = inf\{stopping \ time \ \theta \geq 0 \mid u_{l-1}^d(y_x(\theta)) = M^d[u_{l-1}](y_x(\theta)) \ a.s. \ \} \tag{2.15}$$

and

$$\overline{d} = any \ F_{\overline{\theta}} - measurable \ randam \ variable \ \tilde{d} \neq d \ such \ that$$
$$M^d[u_{l-1}](y_x(\overline{\theta})) = u_{l-1}^{\tilde{d}}(y_x(\overline{\theta})) + k(d, \tilde{d}) \ a.s. \tag{2.16}$$

Now, we can define $\alpha_l^* = \{\theta_i^{l,d}, d_i^{l,d}\}$ by

$$\theta_0^{l,d} = 0, \quad d_0^{l,d} = d$$

$$\theta_i^{l,d} = \theta_{i-1}^{l-1,\bar d} + \bar\theta, \quad i=1,\ldots,l \tag{2.17}$$
$$d_1^{l,d} = \bar d, \quad d_i^{l,d} = d_{i-1}^{l-1,\bar d}, \quad i=2,\ldots,l.$$

Then $\theta_1^{l,d} = \bar\theta$ and by induction $\theta_i^{l,d} = \theta_{i-1}^{l-1,\bar d} + \bar\theta$, $2 \le i \le l$, are adapted to $\{F_t\}$ while $d_1^{l,d} = \bar d$ and $d_i^{l,d} = d_{i-1}^{l-1,\bar d}$, $2 \le i \le l$, are $F_{\theta_i^{l,d}}$ - measurable. Thus, $\alpha_l^* \in A^d$. By the principle of dynamic programming,

$$u_l^d(x) \le J_x^d(\alpha_l), \quad \alpha_l \text{ has at most } l \text{ switches.}$$

On the other hand, by induction and Lemma 2.1,

$$u_l^d(x) = E\left\{\int_0^{\bar\theta} f(y_x(s),d)e^{-\beta s}\,ds + k(d,\bar d)e^{-\beta\bar\theta} + u_{l-1}^{\bar d}(y_x(\bar\theta))e^{-\beta\bar\theta}\right\}$$

$$= E\left\{\int_0^{\bar\theta} f(y_x(s),d)e^{-\beta s}\,ds + k(d,\bar d)e^{-\beta\bar\theta}\right.$$

$$+ E_{y_x(\bar\theta),\bar d}\left\{e^{-\beta\bar\theta}\left[\sum_{i=1}^{l-1}\int_{\theta_{i-1}^{l-1,\bar d}}^{\theta_i^{l-1,\bar d}} f(y_{y_x(\bar\theta)}(s),d_{i-1}^{l-1,\bar d})e^{-\beta s}\,ds\right.\right.$$

$$\left.\left.\left.+ k(d_{i-1}^{l-1,\bar d},d_i^{l-1,\bar d})e^{-\beta\theta_i^{l-1,\bar d}}\right]\right\}\right\}$$

$$= E\left\{\int_0^{\bar\theta} f(y_x(s),d)e^{-\beta s}\,ds + k(d,\bar d)e^{-\beta\bar\theta}\right.$$

$$\left.+ \sum_{i=1}^{l-1}\left[\int_{\theta_{i-1}^{l-1,\bar d}+\bar\theta}^{\theta_i^{l-1,\bar d}+\bar\theta} f(y_x(s),d_{i-1}^{l-1,\bar d})e^{-\beta s}\,ds + k(d_{i-1}^{l-1,\bar d},d_i^{l-1,\bar d})e^{-\beta(\theta_i^{l-1,\bar d}+\bar\theta)}\right]\right\}$$

$$= E \left\{ \sum_{i=1}^{l} \int_{\theta_{i-1}^{l,d}}^{\theta_i^{l,d}} f\,(y_x(s), d_{i-1}^{l,d}) e^{-\beta s}\, ds \;+\; k\,(d_{i-1}^{l,d}, d_i^{l,d}) e^{-\beta \theta_i^{l,d}} \right\} \qquad \textbf{(2.18)}$$

$$= J_x^d(\alpha_l^*).$$

Thus (2.13) and (2.14) follow.

QED

Lemma 2.4. *For each* $0 < \gamma < \min \left\{ 1, \dfrac{\beta}{L\,(1+\lambda_{\max})} \right\}$,

$$|\, u_l^d(x) - u_l^d(\hat{x}) \,| \leq C_\gamma \,|\, x - \hat{x} \,|^\gamma \qquad (2.19)$$

for all $1 \leq l \leq \infty$ *and* $x, \hat{x} \in \mathbb{R}^n$ *with*

$$C_\gamma = \frac{||f\,||^{1-\gamma} L^\gamma}{\beta - \gamma L\,(1+\lambda_{\max})} \qquad (2.20)$$

where

$$\lambda_{\max} = \max\,\{\lambda_1, \ldots, \lambda_m\}.$$

If $\beta > L(1 + \lambda_{\max})$, *then* γ *can be taken to be* 1.

Proof. Without loss of generality, we can assume $u_l^d(x) \geq u_l^d(\hat{x})$. Let $\hat{\alpha}_l = \{\hat{\theta}_i, \hat{d}_i\} \in A^d$ be the optimal policies with at most l switches at state $\hat{x}$. Then,

$$|\, u_l^d(x) - u_l^d(\hat{x}) \,| \leq u_l^d(x) - u_l^d(\hat{x}) \leq J_x^d(\hat{\alpha}_l) - J_{\hat{x}}^d(\hat{\alpha}_l)$$

$$\leq E \int_0^\infty |\, f\,(y_x^l(t), \hat{\alpha}_l(t)) - f\,(y_{\hat{x}}^l(t), \hat{\alpha}_l(t)) \,|\, e^{-\beta t}\, dt \qquad (2.21)$$

Since

$$y_x^l(t) - y_{\hat{x}}(t) = x - \hat{x} + \int_0^t [g(y_x^l(s),\hat{\alpha}_l(s)) - g(y_{\hat{x}}^l(s),\hat{\alpha}_l(s))]ds$$

$$+ \int_0^t [h(y_x^l(s),\hat{\alpha}_l(s)) - h(y_{\hat{x}}^l(s),\hat{\alpha}_l(s))]dN_{\hat{\alpha}_l(s)}(s),$$

then taking the expectation, we have

$$E \; | y_x^l(t) - y_{\hat{x}}^l(t) | \leq | x - \hat{x} | + L(1+\lambda_{\max})\int_0^t E \; | y_x^l(s) - y_{\hat{x}}^l(s) | \, ds.$$

By Gronwall's inequality

$$E \; | y_x^l(t) - y_{\hat{x}}^l(t) | \leq | x - \hat{x} | \; e^{L(1+\lambda_{\max})t},$$

so that

$$\begin{aligned} & E \; | f(y_x^l(t),\hat{\alpha}_l(t)) - f(y_{\hat{x}}^l(t),\hat{\alpha}_l(t)) | \\ \leq \; & ||f||^{1-\gamma} [E \; | f(y_x^l(t),\hat{\alpha}_l(t)) - f(y_{\hat{x}}^l(t),\hat{\alpha}_l(t)) |]^\gamma \\ \leq \; & ||f||^{1-\gamma} L^\gamma | x - \hat{x} |^\gamma e^{\gamma L(1+\lambda_{\max})t}. \end{aligned} \tag{2.22}$$

Thus, from (2.21),

$$\begin{aligned} | u_l^d(x) - u_l^d(\hat{x}) | &\leq ||f||^{1-\gamma} L^\gamma | x - \hat{x} |^\gamma \int_0^\infty e^{-[\beta - \gamma L(1+\lambda_{\max})]t} \, dt \\ &= \frac{||f||^{1-\gamma} L^\gamma}{\beta - \gamma L(1+\lambda_{\max})} \; | x - \hat{x} |^\gamma \\ &\triangleq C_\gamma | x - \hat{x} |^\gamma. \end{aligned}$$

For $l = \infty$,

$$\begin{aligned} | u_\infty^d(x) - u_\infty^d(\hat{x}) | &\leq | u_\infty^d(x) - u_l^d(x) | \\ &+ | u_l^d(x) - u_l^d(\hat{x}) | + | u_l^d(\hat{x}) - u_\infty^d(\hat{x}) | \\ &\leq C_\gamma | x - \hat{x} |^\gamma + | u_\infty^d(x) - u_l^d(x) | \end{aligned}$$

$$+ \ | u_l^d(\hat{x}) - u_\infty^d(\hat{x}) |$$

As $l \to \infty$, the last two terms tend to zero. We have (2.19).

QED

Remark. Since N_i has independent increments, then F_s is independent of any sub σ-field generated by $\{N_i(t) - N_i(s),\ s \le t,\ i = 1, \ldots, m\}$, so that for t $\geq$ s,

$$E\,[\,|\, y_x^l(t) - y_x^J(t) \,|\ \ |\ F_s\,] \le |\, y_x^l(s) - y_x^J(s) \,|\ e^{L(1+\lambda_{\max})(t-s)}.$$

Thus,

$$|\, u^d(y_x^l(s)) - u^d(y_x^J(s)) \,| \le C_\gamma\, |\, y_x^l(s) - y_x^J(s) \,|^\gamma \qquad a.s.$$

Remark. If $k_0 \geq \dfrac{||f\,||}{\beta}$, then $u_0(x)$ is the optimal solution, i.e., no switching occurs.

We can obtain the following estimate by the method in [10][11].

Theorem 2.5. *If* $0 < k_0 < \dfrac{||f\,||}{\beta}$, *then*

$$||u_l^d - u_\infty^d|| \le ||u_0^d||\left(1 - \frac{\beta k_0}{||f\,||}\right)^l. \tag{2.23}$$

Thus, $u_l \downarrow u_\infty$ *uniformly.*

Proof. Using the fact that $\dfrac{\sigma + w}{\sigma + z} \geq \dfrac{w}{z}$ if $\sigma \geq 0$ and $w \le z$, we have

$$\Gamma_d(0)(x) = \inf_{\theta \geq 0} E \left\{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds \;+\; e^{-\beta\theta} M^d[0](y_x(\theta)) \right\}$$

$$\geq \inf_{\theta \geq 0} E \left\{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds \;+\; k_0 e^{-\beta\theta} \right\}$$

$$\geq \inf_{\theta \geq 0} \left[\frac{E \int_0^\theta f(y_x(s), d) e^{-\beta s} ds \;+\; k_0 e^{-\beta\theta}}{E \int_0^\infty f(y_x(s), d) e^{-\beta s} ds} \right] u_0^d(x)$$

$$\geq \inf_{\theta \geq 0} \left[\frac{E\ k_0 e^{-\beta\theta}}{E \int_\theta^\infty f(y_x(s), d) e^{-\beta s} ds} \right] u_0^d(x)$$

$$\geq \inf_{\theta \geq 0} \left[\frac{E\ k_0 e^{-\beta\theta}}{E\ \frac{||f||}{\beta} e^{-\beta\theta}} \right] u_0^d(x)$$

$$\geq \frac{\beta k_0}{||f||} u_0^d(x)$$

$$\triangleq q_0 u_0^d(x).$$

Thus, if $-p_2 v \leq u - v \leq p_1 u$, $0 \leq p_1, p_2 \leq 1$, and $0 \leq u, v \leq u_0$, then

$$\Gamma_d v(x) \geq \Gamma_d(0)(x) \geq q_0 u_0^d(x) \geq q_0 u^d(x) \geq q_0 \Gamma_d u(x) \tag{2.25}$$

so that by the property of Γ_d,

$$\Gamma_d v(x) \geq \Gamma_d [(1-p_1)u](x)$$

$$= \Gamma_d [(1-p_1)u + p_1 0](x)$$

$$\geq (1-p_1)\Gamma_d u(x) + p_1 \Gamma_d (0)(x).$$

Thus,

$$\Gamma_d u(x) - \Gamma_d v(x) \leq p_1[\Gamma_d u(x) - \Gamma_d (0)(x)]$$

$$\leq p_1(1-q_0)\Gamma_d u(x).$$

On the other hand,

$$\Gamma_d u(x) \geq \Gamma_d [(1-p_2)v](x)$$

$$= \Gamma_d [(1-p_2)v + p_2 0](x)$$

$$\geq (1-p_2)\Gamma_d v(x) + p_2 \Gamma_d (0)(x),$$

so that

$$\Gamma_d u(x) - \Gamma_d v(x) \geq -p_2[\Gamma_d v(x) - \Gamma_d (0)(x)]$$

$$\geq -p_2(1-q_0)\Gamma_d v(x).$$

Hence, by induction,

$$-p_2(1-q_0)^l \Gamma_d^l v(x) \leq \Gamma_d^l u(x) - \Gamma_d^l v(x) \leq p_1(1-q_0)^l \Gamma_d^l u(x), \quad \forall\ l \geq 1. \quad (2.26)$$

Note that

$$u_0^d(x) \geq \Gamma_d u_0(x) \geq \cdots \geq \Gamma_d^l u_0(x) \geq \Gamma_d^l u(x)$$

and with $p_1 = p_2 = 1$, $i \geq l$,

$$||u_l^d - u_i^d|| = ||\Gamma_d^l u_0 - \Gamma_d^l u_{i-l}|| \leq (1-q_0)^l \, ||u_0^d||.$$

Let $i \to \infty$ and the result follows.

QED

Theorem 2.6.

$$u_\infty^d(x) = \inf_{\theta \geq 0} \{ E \int_0^\theta f(y_x(s), d) e^{-\beta s} ds + M^d[u_\infty](y_x(\theta)) e^{-\beta\theta} \}. \tag{2.27}$$

Proof. Since

$$u_\infty^d(x) = \lim_{l \to \infty} u_l^d(x)$$

$$= \lim_{l \to \infty} \inf_{\theta \geq 0} E \{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d[u_{l-1}](y_x(\theta)) \}$$

and $u_\infty \leq u_l \leq u_{l-1} \leq u_0$, then

$$M^d[u_\infty] \leq M^d[u_l] \leq M^d[u_{l-1}]$$

so that

$$u_\infty^d(x) \geq \inf_{\theta \geq 0} E \{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d[u_\infty](y_x(\theta)) \}.$$

On the other hand, from Theorem 2.5, $u_l \downarrow u_\infty$ uniformly. Thus, for each small $\epsilon > 0$ and stopping time $\theta \geq 0$, there exists l sufficiently large so that

$$M^d[u_\infty](y_x(\theta)) > M^d[u_{l-1}](y_x(\theta)) - \epsilon$$

and so

$$E \{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d[u_\infty](y_x(\theta)) \}$$

$$\geq E \{ \int_0^\theta f(y_x(s), d) e^{-\beta s} ds + e^{-\beta\theta} M^d[u_{l-1}](y_x(\theta)) \} - \epsilon$$

$$\geq u_l^d(x) - \epsilon.$$

Letting $l \to \infty$, taking infimum for all stopping time $\theta \geq 0$ and dropping $\epsilon \downarrow 0$, we know

$$\inf_{\theta\geq 0} E \{\int_0^\theta f(y_x(s),d)e^{-\beta s}\,ds + e^{-\beta\theta}M^d[u_\infty](y_x(\theta))\} \geq u_\infty^d(x).$$

Thus, we have the desired result (2.27).

QED

Theorem 2.7. *If* $\exists\ x_0$ *such that* $u^d(x_0) < M^d[u](x_0)$, *then* $\theta_1 > 0$ *a.s. and*

$$u^d(x_0) = E\{\int_0^\theta f(y_{x_0}(s),d)e^{-\beta s}\,ds + u^d(y_{x_0}(\theta))e^{-\beta\theta}\} \tag{2.28}$$

for all $0 \leq \theta \leq \theta_1$.

Proof. Suppose not, $P(\theta_1 = 0) = a > 0$. Let $\Omega_1 = \{\omega \in \Omega \mid \theta_1(\omega) = 0\}$. If

$$\sigma \triangleq \min_{\tilde{d}\neq d}\{u^{\tilde{d}}(x_0) + k(d,\tilde{d})\} - u^d(x_0) > 0,$$

then from (2.27), $\theta_1 \neq 0$ a.s., so that $a < 1$. Since $y_{x_0}(t)$ is right continuous, and f is bounded, then for t small enough,

$$u^d(y_{x_0}(t,\omega))\,e^{-\beta t} < u^d(x_0) + \frac{\sigma}{3} \tag{2.29}$$

for all $\omega \in \Omega_2$ with $P(\Omega_2) \geq 1 - \frac{a}{2}$ and

$$\int_0^t f(y_{x_0}(s),d)\,e^{-\beta s} < \frac{\sigma}{3}. \tag{2.30}$$

Thus, $P(\Omega_1 \cap \Omega_2) \geq \frac{a}{2}$. Let $\Omega_3 = \Omega \setminus (\Omega_1 \cap \Omega_2)$. Then, from (2.27), (2.29) and (2.30), we have

$$u^d(x_0) > E\{\int_0^{\theta_1} f(y_{x_0}(s),d)e^{-\beta s}\,ds + u^d(y_{x_0}(\theta_1))\,e^{-\beta\theta_1}\}\,I_{\Omega_3}$$

$$+ \; E \; \{ \int_0^t f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; u^d(y_{x_0}(t))\; e^{-\beta t} \}\; I_{\Omega_1 \bigcap \Omega_2}.$$

Let

$$\hat{\theta}(\omega) = \begin{cases} \theta_1(\omega), & \omega \in \Omega_3 \\ \\ t\,, & \omega \in \Omega_1 \bigcap \Omega_2\,. \end{cases}$$

Then it is easy to check that $\hat{\theta}$ is adapted to $\{F_t\}$ and

$$u^d(x_0) > E\; \{ \int_0^{\hat{\theta}} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; u^d(y_{x_0}(\hat{\theta}))\; e^{-\beta\hat{\theta}} \}$$

which contradicts (2.6). Hence, $a = 0$. By (2.27) and Lemma 2.1, we have for any $0 \leq \theta \leq \theta_1$,

$$u^d(x_0) = E\; \{ \int_0^{\theta_1} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; M^d[u](y_{x_0}(\theta_1))e^{-\beta\theta_1} \}$$

$$= E\, \{ \int_0^{\theta} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; E\,[\int_{\theta}^{\theta_1} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds + M_d[u](y_{x_0}(\theta_1)) \mid F_\theta] \}$$

$$= E\; \left\{ \int_0^{\theta} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; E_{y_{x_0}(\theta)} \left[\int_0^{\theta_1 - \theta} f\,(y_{y_{x_0}(\theta)}(s),d)e^{-\beta s}\, ds \right.\right.$$

$$\left.\left. + \; M^d[u](y_{y_{x_0}(\theta)}(\theta_1 - \theta)\; e^{-\beta(\theta_1 - \theta)} \right]\; e^{-\beta\theta} \right\}$$

$$\geq E\; \{ \int_0^{\theta} f\,(y_{x_0}(s),d)e^{-\beta s}\, ds \;\; + \; u^d(y_{x_0}(\theta))\; e^{-\beta\theta} \}.$$

By (2.6), we have (2.28).

QED

Now, suppose we have a Hölder continuous function u^d satisfying (1.6). We can define an optimal policy $\alpha^* = \{\theta_i, d_i\} \in A^d$ as follows.

$$\theta_0 = 0, \quad d_0 = d,$$

If we are given θ_{i-1}, d_{i-1}, then set

$$\theta_i \triangleq inf\{ \; stopping \;\; time \;\; \theta \geq \theta_{i-1} \;\mid\; u^{d_{i-1}}(y_x(\theta)) = M^{d_{i-1}}[u]\,(y_x(\theta)) \;\; a.s. \;\} \tag{2.31}$$

If $\theta_i < \infty$, set

$$d_i = any \;\; F_{\theta_i} - measurable \;\; random \;\; variable \;\; \tilde{d} \in \{1, \ldots, m\}, \; \tilde{d} \neq d_{i-1}$$

$$such \;\; that \;\; M^{d_{i-1}}[u]\,(y_x(\theta_i)) = u^{\tilde{d}}(y_x(\theta_i)) + k(d, \tilde{d}) \;\; a.s. \tag{2.32}$$

and

$$y_x(t) \;\; controlled \;\; by \;\; decision \;\; d_{i-1} \;\; when \;\; \theta_{i-1} \leq t < \theta_i .$$

Theorem 2.8. *The control policy α^* defined by (2.31) and (2.32) is optimal, i.e., $u^d(x) = J_x^d(\alpha^*) = \min_{\alpha \in A^d} J_x^d(\alpha)$. In addition, $\theta_i \to \infty$ a.s. as $i \to \infty$.*

Proof. Set $y_{i-1} = y_x(\theta_{i-1})$. By Theorem 2.6 and Lemma 2.2,

$$u^{d_{i-1}}(y_{i-1}) = E_{y_{i-1}, d_{i-1}} \left\{ \int_0^{\theta_i - \theta_{i-1}} f\,(y_{y_{i-1}}(s), d_{i-1}) e^{-\beta s}\, ds \;+\; u^{d_{i-1}}(y_i)\, e^{-\beta(\theta_i - \theta_{i-1})} \right\}$$

$$= E \left\{ \int_{\theta_{i-1}}^{\theta_i} f\,(y_x(s), d_{i-1}) e^{-\beta(s - \theta_{i-1})} ds \;+\; u^{d_{i-1}}(y_i)\, e^{-\beta(\theta_i - \theta_{i-1})} \;\middle|\; F_{\theta_{i-1}} \right\}.$$

Since

$$u^{d_{i-1}}(y_i) = M^{d_{i-1}}[u](y_i) = u^{d_i}(y_i) \;+\; k(d_{i-1},d_i) \; a.s.\,,$$

we have by Lemma 2.1,

$$e^{-\beta\theta_{i-1}} u^{d_{i-1}}(y_{i-1})$$

$$= \left\{ \int_{\theta_{i-1}}^{\theta_i} f(y_x(s), d_{i-1}) e^{-\beta s}\, ds \;+\; k(d_{i-1},d_i) e^{-\beta\theta_i} \;+\; u^{d_i}(y_i)\, e^{-\beta\theta_i} \;\middle|\; F_{\theta_{i-1}} \right\}.$$

Summing up all i until some $\theta_l = \infty$ and taking expectation, we have

$$u^d(x) = E\left\{ \sum_{i\geq 1} \int_{\theta_{i-1}}^{\theta_i} f(y_x(s), d_{i-1}) e^{-\beta s}\, ds \;+\; k(d_{i-1},d_i) e^{-\beta\theta_i} \right\} \tag{2.33}$$

$$= J_x^d(\alpha^*).$$

As from (1.6), we have proved $\alpha*$ is optimal. We claim $\theta_i \rightarrow \infty$ a.s. as $i \rightarrow \infty$. Suppose not, then $\exists$ T > 0 such that $\theta_i \leq T$ for all i with positive probability δ. Then from (2.11) and (2.33),

$$\frac{||f\,||}{\beta} \geq \sum_{i\geq 1} E\; k(d_{i-1},d_i) e^{-\beta\theta_i} \geq \sum_{i\geq 1} k_0 e^{-\beta T}\delta$$

which is unbounded, a contradiction.

QED

Corollary 2.9. *We have the additional estimate*

$$||u_l^d - u_\infty^d|| \leq \frac{||f\,||^2}{\beta^2 k_0(l+1)}\,. \tag{2.34}$$

Proof. From (2.31) and (2.32), we obtain an optimal policy $\alpha^* = \{\theta_i, d_i\} \in A^d$. Let $\alpha_l = \{\theta_i^l, d_i^l\} \in A^d$ be the first l switches such that $\theta_i^l = \theta_i$, $d_i^l = d_i$ for i $\leq l$ and $\theta_i^l = \infty$ for i $> l$. Denote by $y_x^l(t)$ and $y_x(t)$ the trajectories corresponding to the controls α_l and α^*, respectively. Then $y_x^l(t) \equiv y_x(t)$ for $0 \leq t < \theta_{l+1}$. Hence,

$$\begin{aligned} 0 \leq u_l^d(x) - u_\infty^d(x) &\leq J_x^d(\alpha_l) - J_x^d(\alpha^*) \\ &\leq E \int_{\theta_{l+1}}^{\infty} f(y_x^l(s), d_l) e^{-\beta s}\, ds \\ &\leq \frac{\|f\|}{\beta} E\, e^{-\beta\theta_{l+1}}. \end{aligned} \tag{2.35}$$

Since $k(d_i, d_j) \geq k_0$, $i \neq j$ a.s., and $\theta_i \leq \theta_{l+1}$, $i \leq l+1$ a.s., we have

$$e^{-\beta\theta_{l+1}} \leq \frac{1}{k_0(l+1)} \sum_{i=1}^{l+1} k(d_{i-1}, d_i) e^{-\beta\theta_i} \quad a.s. \tag{2.36}$$

Since

$$u_\infty^d(x) = J_x^d(\alpha^*) = E\, \{\sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} f(y_x(s), d_{i-1}) e^{-\beta s}\, ds + k(d_{i-1}, d_i) e^{-\beta\theta_i}\}$$

$$\leq u_0^d(x) \leq \frac{\|f\|}{\beta},$$

we have

$$E \sum_{i=1}^{\infty} k(d_{i-1}, d_i) e^{-\beta\theta_i} \leq \frac{\|f\|}{\beta}, \tag{2.37}$$

so that from (2.34) - (2.36),

$$0 \leq u_l^d(x) - u_\infty^d(x) \leq \frac{\|f\|^2}{\beta^2 k_0(l+1)}$$

for all $x \in \mathbb{R}^n$. Thus (3.18) follows.

QED

3. Viscosity Solutions of the Quasi-Variational Inequality (QVI).

We want to derive necessary and sufficient conditions for the optimal solution $u^d(x)$, $x \in \mathbb{R}^n$, $d \in \{1,...,m\}$. Assume for the moment that the value functions $u^1, \ldots, u^m$ belong to $C^1(\mathbb{R}^n)$. Then by the necessary condition in Lemma 2.2, we have

$$E\left\{\frac{u^d(x) - u^d(y_x(t))}{t}\right\} \leq E\left\{\frac{1}{t}\int_0^t f(y_x(s),d)e^{-\beta s}\,ds + \left(\frac{e^{-\beta t} - 1}{t}\right) u^d(y_x(t))\right\} \tag{3.1}$$

and so, we obtain a differential form as $t \downarrow 0$,

$$-g(x,d)\cdot\nabla u^d(x) - \lambda_d[u^d(x+h(x,d)) - u^d(x)] \leq f(x,d) - \beta u^d(x) \tag{3.2}$$

for all $x \in \mathbb{R}^n$ and $d \in 1,...,m$. Combining (2.5), (2.28) and (3.2), we obtain a quasi-variational inequality (QVI)

$$\max\{\beta u^d - g^d\cdot\nabla u^d - \lambda_d[u^d(\cdot+h^d) - u^d] - f^d,\ u^d - M^d[u]\} = 0 \tag{3.3}$$

on $\mathbb{R}^n$, where

$$f^d(\cdot) \triangleq f(\cdot, d), \quad g^d(\cdot) \triangleq g(\cdot, d), \quad h^d(\cdot) \triangleq h(\cdot, d). \tag{3.4}$$

Note that (3.3) is a fully nonlinear first order partial differential equation which does not admit a differentiable solution in general. But, we can treat it using the method of viscosity solutions, which was introduced by M. G. Crandall and P. L. Lions [1], and which was used for deterministic switching problems by I. Capuzzo Dolcetta and L. C. Evans [4].

We denote by $BUC(\mathbb{R}^n)^m$, the *space of bounded, uniformly continuous* $\mathbb{R}^m$*-valued functions on* $\mathbb{R}^n$.

Definition. A function $u = (u^d, \ldots, u^m) \in C(\mathbb{R}^n)^m$ is said to be a *viscosity solution* of the (QVI) if for each $d \in \{1,\ldots,m\}$ and each $\phi \in C^1(\mathbb{R}^n)$ such that

(i) if $u^d - \phi$ attains a local maximum at $x_0 \in \mathbb{R}^n$, then

$$\max\{\beta u^d(x_0) - g^d(x_0)\cdot\nabla\phi(x_0) - \lambda_d[u^d(x_0+h^d(x_0)) - u^d(x_0)] - f^d(x_0),\; u^d(x_0) - M^d[u](x_0)\} \leq 0 \tag{3.5}$$

and

(ii) if $u^d - \phi$ attains a local minimum at $z_0 \in \mathbb{R}^n$, then

$$\max\{\beta u^d(z_0) - g^d(z_0)\cdot\nabla\phi(z_0) - \lambda_d[u^d(z_0+h^d(z_0)) - u^d(z_0)] - f^d(z_0),\; u^d(z_0) - M^d[u](z_0)\} \geq 0. \tag{3.6}$$

Theorem 3.1. *Under the previous assumptions, the value function* $u = (u^1, \ldots, u^m)$ *with*

$$u^d(x) \triangleq \inf_{\alpha \in A^d} J_x^d(\alpha)$$

is a viscosity solution of the (QVI) (3.3).

Proof. By Lemma 2.4, $u \in BUC(\mathbb{R}^n)^m$. Now, let $\phi \in C^1(\mathbb{R}^n)$. If $u^d - \phi$ attains a local maximum at $x_0 \in \mathbb{R}^n$ for some d, then

$$u^d(x_0) - \phi(x_0) \geq u^d(x) - \phi(x) \tag{3.7}$$

for all x in some ball $B(x_0,\epsilon)$, with ϵ sufficiently small. By Lemma 2.2 (ii), we have

$$E\left\{ \frac{u^d(x_0) - u^d(y_{x_0}(t))}{t} \right\} \leq E\left\{ \frac{1}{t}\int_0^t f(y_{x_0}(s),d)e^{-\beta s}\,ds \right.$$

$$\left. + \left(\frac{e^{-\beta t} - 1}{t} \right) u^d(y_{x_0}(t)) \right\} \tag{3.8}$$

Since g is bounded, Lipschitz continuous and y_x is right continuous, then for sufficiently small $t > 0$

$$x_1(t) \triangleq x_0 + \int_0^t g(y_{x_0}(s),d)ds \tag{3.9}$$

belongs to some ball $B(x_0,\epsilon)$ a.s., so that by the mean value theorem,

$$E\left\{ \frac{u^d(x_0) - u^d(x_1(t))}{t} \right\} \geq E\left\{ \frac{\phi(x_0) - \phi(x_1(t))}{t} \right\} \to -g^d(x_0)\cdot\nabla\phi(x_0) \tag{3.10}$$

as $t \to 0$. Since t is small,

$$N_d(t) = \begin{cases} 0 \text{ with probability } 1-\lambda_d t + o(t) \\ 1 \text{ with probability } \lambda_d t + o(t) \\ \geq 2 \text{ with probability } o(t). \end{cases} \tag{3.11}$$

Thus,

$$E\left\{ \frac{u^d(x_1(t)) - u^d(y_{x_0}(t))}{t} \right\}$$

$$= \frac{\lambda_d t + o(t)}{t} E\left\{ u^d(x_1(t)) - u^d(x_1(t) + h^d(y_{x_0}(\tau_1^-))) \mid \tau_1 \leq t \right\} + \frac{o(t)}{t}$$

$$\to \lambda_d [u^d(x_0) - u^d(x_0 + h^d(x_0))] \tag{3.12}$$

as $t \downarrow 0$. In addition,

$$E\ f^{\,d}(y_{x_0}(s)) = (1-\lambda_d\, s + o(s))\, E\ \{f^{\,d}(x_1(s)) \mid \tau_1 > s\}$$

$$+\ (\lambda_d\, s + o(s)) \bullet E\ \{f^{\,d}(x_1(s) + h^d(y_{x_0}(\tau_1^-))) \mid \tau_1 \leq s\}$$

$$v + o(s) \bullet (\text{high order jumps}).$$

$$(3.13)$$

Thus,

$$\lim_{t \to 0} E\ \frac{1}{t} \int_0^t f\ (y_{x_0}(s), d)e^{-\beta s}\, ds = f\ (x_0, d), \tag{3.14}$$

so that the right hand side of (3.8) tends to

$$f^{\,d}(x_0) - \beta u^d(x_0). \tag{3.15}$$

From (2.5), (3.8) - (3.15), we know that (3.5) holds.

On the other hand, if $u^d - \phi$ attains a local minimum at $z_0 \in \mathbb{R}^n$ for some d, then

$$u^d(z_0) - \phi(z_0) \leq u^d(z) - \phi(z) \tag{3.16}$$

for all z in some ball $B(z_0, \epsilon)$ with ϵ small. If

$$u^d(z_0) = M^d[u](z_0),$$

(3.6) follows, otherwise, by Theorem 2.7 for t small enough,

$$E\left\{\frac{u^d(z_0) - u^d(y_{z_0}(t))}{t}\right\} = E\left\{\frac{1}{t}\int_0^t f\ (y_{z_0}(s), d)e^{-\beta s}\, ds\right.$$

$$\left. + \left(\frac{e^{-\beta t} - 1}{t}\right) u^d(y_{z_0}(t))\right\}$$

$$(3.17)$$

In the same way, let

$$z_1(t) \triangleq z_0 + \int_0^t g(y_{z_0}(s), d)\, ds$$

then the right hand side of (3.17) tends to

$$f^d(z_0) - \beta u^d(z_0) \tag{3.18}$$

as $t \downarrow 0$. In the same manner,

$$E\left\{ \frac{u^d(z_0) - u^d(z_1(t))}{t} \right\} \leq E\left\{ \frac{\phi(z_0) - \phi(z_1(t))}{t} \right\} \tag{3.19}$$

$$\rightarrow - g^d(z_0) \cdot \nabla \phi(z_0)$$

and

$$E\left\{ \frac{u^d(z_1(t)) - u^d(y_{z_0}(t))}{t} \right\} \rightarrow \lambda_d [u^d(z_0) - u^d(z_0 + h^d(z_0))] \tag{3.20}$$

as $t \downarrow 0$. In view of (3.17) - (3.20), we know that (3.6) holds. Thus, u is a viscosity solution of the (QVI).

QED

Before showing the existence of a solution to the (QVI), we show that (3.3) admits a unique solution, so that any functions constructed to satisfy (3.3) must be the optimal solution.

Lemma 3.2. *If $u = (u^1, \ldots, u^m)$ is any viscosity solution of (3.3), then*

$$u^d(x) \leq M^d[u](x), \quad \forall x \in \mathbb{R}^n, \quad d \in \{1, \ldots, m\}. \tag{3.21}$$

Proof. Suppose $\exists\, x_0 \in \mathbb{R}^n$ such that (3.21) does not hold. Then for $u \in C(\mathbb{R}^n)^m$,

$$u^d(x) > u^{\hat{d}}(x) + k(d, \hat{d}) \tag{3.22}$$

for all $x \in B(x_0, \epsilon)$, ϵ small and some $\hat{d} \neq d$. It is not hard to show that there exists a smooth function $\phi \in C^1(\mathbb{R}^n)$ such that $u^d - \phi$ attains a local maximum at some point $x_1 \in B(x_0, \epsilon)$. Thus, by the definition of viscosity solution,

$$\max \{ \beta u^d(x_1) - g^d(x_1) \cdot \nabla\phi(x_1) - \lambda_d [u^d(x_1 + h^d(x_1)) - u^d(x_1)] - f^d(x_1),$$
$$u^d(x_1) - M^d[u](x_1) \} \leq 0.$$

In particular, $u^d(x_1) \leq M^d[u](x_1)$ which contradicts (3.22).

QED

Theorem 3.3. *If $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m)$ are viscosity solutions of* (3.3). *Then $u \equiv v$.*

Proof. Since $u, v \in BUC(\mathbb{R}^n)^m$, let

$$K = \max \{ ||u||, ||v||, 1 \} < \infty.$$

Choose $\gamma \in C^1(\mathbb{R}^n)$ such that

$$\begin{cases} \gamma(0) = 5K, \quad |\nabla\gamma| \leq 10K \\ 0 \leq \gamma(x) < 5K \quad \text{if } x \neq 0 \\ \gamma(x) = 0 \quad \text{if } |x| \geq 1. \end{cases} \tag{3.23}$$

and let $\gamma_\epsilon(x) = \gamma(\frac{x}{\epsilon})$, $\epsilon > 0$, $x \in \mathbb{R}^n$. Consider the auxiliary function

$$\Phi^d(x, y) = u^d(x) - v^d(y) + \gamma_\epsilon(x - y). \tag{3.24}$$

Since u, v and γ_ϵ are bounded, then for each $\epsilon > 0$, $\exists\, (x_1, y_1) \in \mathbb{R}^{2n}$ such that

$$\max_{1 \le d \le m} \Phi^d(x_1,y_1) \ge \sup_{x,y} \max_{1 \le d \le m} \Phi^d(x,y) - \epsilon. \tag{3.25}$$

Since the supremum in (3.25) may not be attained at some point, we have to add an additional small function to make it occur. Fix $\epsilon > 0$ and (x_1,y_1), choose $\xi \in C^1(\mathbb{R}^{2n})$ such that

$$\begin{cases} \xi(x_1,y_1) = 1, \ 0 \le \xi \le 1, \ |\nabla \xi| \le 2 \\ \xi(x,y) < 1 \quad \text{if } (x,y) \ne (x_1,y_1) \\ \xi(x,y) = 0 \quad \text{if } |x-x_1|^2 + |y-y_1|^2 \ge 1. \end{cases} \tag{3.26}$$

Define

$$\begin{aligned} \Psi^d(x,y) &= \Phi^d(x,y) + 2\epsilon\xi(x,y) \\ &= u^d(x) - v^d(y) + \gamma_\epsilon(x-y) + 2\epsilon\xi(x,y). \end{aligned}$$

Then for any $\overline{d}$, (x_2,y_2) such that $|x_2-x_1|^2 + |y_2-y_1|^2 \ge 1$, we have $\xi(x_2,y_2) = 0$ and

$$\Psi^{\overline{d}}(x_2,y_2) = \Phi^{\overline{d}}(x_2,y_2) \le \sup_{x,y} \max_{1 \le d \le m} \Phi^d(x,y).$$

But by (3.25),

$$\max_{1 \le d \le m} \Psi^d(x_1,y_1) = \max_{1 \le d \le m} \Phi^d(x_1,y_1) + 2\epsilon \ge \sup_{x,y} \max_{1 \le d \le m} \Phi^d(x,y) + \epsilon,$$

so that $\exists\ (x_0,y_0) \in \mathbb{R}^{2n}$ with

$$|x_0-x_1|^2 + |y_0-y_1|^2 < 1 \tag{3.27}$$

and $\hat{d}$ such that

$$\Psi^{\hat{d}}(x_0,y_0) = \max_{x,y} \max_{1 \le d \le m} \Psi^d(x,y). \tag{3.28}$$

We claim that $|x_0 - y_0| = o(\epsilon)$ as $\epsilon \downarrow 0$. Note that suppose $|x_0 - y_0| \ge \epsilon$. Then,

$$\hat{\Psi}^d(x_0,y_0) = \hat{u}^d(x_0) - \hat{v}^d(y_0) + \gamma_\epsilon(x_0-y_0) + 2\epsilon\xi(x_0,y_0)$$
$$\leq 2K + 2\epsilon < 3K$$

where we can assume $2\epsilon < K$ without loss of generality. But,

$$\hat{\Psi}^d(x,x) = \hat{u}^d(x) - \hat{v}^d(x) + \gamma_\epsilon(0) + 2\epsilon\xi(x,x)$$
$$\geq -K - K + 5K = 3K$$

which is a contradiction, so that $|x_0 - y_0| < \epsilon$. We will refine the above estimate as follows. Since $\hat{\Psi}^d$ attains a maximum at (x_0,y_0), then

$$\begin{aligned}
\hat{u}^d(x_0) - \hat{v}^d(y_0) + \gamma_\epsilon(x_0-y_0) + 2\epsilon\xi(x_0,y_0) \\
= \hat{\Psi}^d(x_0,y_0) \\
\geq \hat{\Psi}^d(x_0,x_0) \\
= \hat{u}^d(x_0) - \hat{v}^d(x_0) + \gamma_\epsilon(0) + 2\epsilon\xi(x_0,x_0)
\end{aligned}$$

which implies

$$\gamma_\epsilon(x_0-y_0) \geq \gamma_\epsilon(0) + \hat{v}^d(y_0) - \hat{v}^d(x_0) + 2\epsilon[\xi(x_0,x_0) - \xi(x_0,y_0)]$$
$$= \gamma_\epsilon(0) + o(1) \quad as \ \epsilon \downarrow 0 \tag{3.29}$$

for v continuous, $|x_0 - y_0| < \epsilon$ and $0 \leq \xi \leq 1$. Thus, from (3.29)

$$\lim_{\epsilon \to 0} \gamma\left(\frac{x_0-y_0}{\epsilon}\right) = \lim_{\epsilon \to 0} \gamma_\epsilon(x_0-y_0) \geq 5K$$

and so

$$\lim_{\epsilon \downarrow 0} \frac{|x_0-y_0|}{\epsilon} = 0$$

i.e., $|x_0 - y_0| = o(\epsilon)$ as $\epsilon \downarrow 0$. For fixed y_0, let

$$\phi_1(x) \triangleq \hat{v}^d(y_0) - \gamma_\epsilon(x-y_0) - 2\epsilon\xi(x,y_0). \tag{3.30}$$

Then $\phi_1 \in C^1(\mathbb{R}^n)$ and $\Psi^{\hat{d}}(x, y_0) = u^{\hat{d}}(x) - \phi_1(x)$ attains its maximum at x_0, so that

$$\max\{\beta u^{\hat{d}}(x_0) - g^{\hat{d}}(x_0)\cdot\nabla\phi_1(x_0) - \lambda_{\hat{d}}[u^{\hat{d}}(x_0 + h^{\hat{d}}(x_0)) - u^{\hat{d}}(x_0)] - f^{\hat{d}}(x_0),$$
$$u^{\hat{d}}(x_0) - M^{\hat{d}}[u](x_0)\} \leq 0. \tag{3.31}$$

In the same way, for fixed x_0, let

$$\phi_2(y) \triangleq u^{\hat{d}}(x_0) + \gamma_\epsilon(x_0 - y) + 2\epsilon\xi(x_0, y). \tag{3.32}$$

Then $\phi_2 \in C^1(\mathbb{R}^n)$ and $-\Psi^{\hat{d}}(x_0, y) = v^{\hat{d}}(y) - \phi_2(y)$ attains its minimum at y_0. Thus,

$$\max\{\beta v^{\hat{d}}(y_0) - g^{\hat{d}}(y_0)\cdot\nabla\phi_2(y_0) - \lambda_{\hat{d}}[v^{\hat{d}}(y_0 + h^{\hat{d}}(y_0)) - v^{\hat{d}}(y_0)] - f^{\hat{d}}(y_0),$$
$$v^{\hat{d}}(y_0) - M^{\hat{d}}[v](y_0)\} \geq 0. \tag{3.33}$$

We consider the following two cases.

Case (i) $v^{\hat{d}}(y_0) < M^{\hat{d}}[v](y_0)$. Then from (3.31)

$$\begin{aligned}\beta u^{\hat{d}}(x_0) &\leq g^{\hat{d}}(x_0)\cdot\nabla\phi_1(x_0) + \lambda_{\hat{d}}[u^{\hat{d}}(x_0+h^{\hat{d}}(x_0)) - u^{\hat{d}}(x_0)] + f^{\hat{d}}(x_0)\\ &= g^{\hat{d}}(x_0)\cdot\left[-\frac{1}{\epsilon}\nabla\gamma\left(\frac{x_0-y_0}{\epsilon}\right) - 2\epsilon\nabla_x\xi(x_0,y_0)\right]\\ &\quad + \lambda_{\hat{d}}[u^{\hat{d}}(x_0+h^{\hat{d}}(x_0)) - u^{\hat{d}}(x_0)] + f^{\hat{d}}(x_0).\end{aligned} \tag{3.34}$$

By (3.33), we have

$$\begin{aligned}\beta v^{\hat{d}}(y_0) &\geq g^{\hat{d}}(y_0)\cdot\nabla\phi_2(y_0) + \lambda_{\hat{d}}[v^{\hat{d}}(y_0+h^{\hat{d}}(y_0)) - v^{\hat{d}}(y_0)] + f^{\hat{d}}(y_0)\\ &= g^{\hat{d}}(y_0)\cdot\left[-\frac{1}{\epsilon}\nabla\gamma\left(\frac{x_0-y_0}{\epsilon}\right) + 2\epsilon\nabla_y\xi(x_0,y_0)\right]\\ &\quad + \lambda_{\hat{d}}[v^{\hat{d}}(y_0+h^{\hat{d}}(y_0)) - v^{\hat{d}}(y_0)] + f^{\hat{d}}(y_0).\end{aligned} \tag{3.35}$$

From (3.34) and (3.35), we get

$$\beta[u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0)]$$

$$\leq -\frac{1}{\epsilon}[g^{\hat{d}}(x_0) - g^{\hat{d}}(y_0)]\cdot\nabla\gamma\left(\frac{x_0-y_0}{\epsilon}\right) - 2\epsilon[\nabla_x\xi(x_0,y_0) + \nabla_y\xi(x_0,y_0)]$$

$$+ \lambda_{\hat{d}}[u^{\hat{d}}(x_0+h^{\hat{d}}(x_0)) - u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0+h^{\hat{d}}(y_0)) + v^{\hat{d}}(y_0)]$$

$$+ f^{\hat{d}}(x_0) - f^{\hat{d}}(y_0). \tag{3.36}$$

Note that, since $|x_0-y_0| = o(\epsilon)$, f and g are Lipschitz continuous, we have

$$|f^{\hat{d}}(x_0) - f^{\hat{d}}(y_0)| = o(1) \tag{3.37}$$

$$\frac{1}{\epsilon}\left|\nabla\gamma\left(\frac{x_0-y_0}{\epsilon}\right)\right|\left|g^{\hat{d}}(x_0) - g^{\hat{d}}(y_0)\right| \leq 10KL\frac{|x_0-y_0|}{\epsilon} = o(1) \tag{3.38}$$

as $\epsilon \downarrow 0$ and

$$2\epsilon(|\nabla_x\xi(x_0,y_0)| + |\nabla_y\xi(x_0,y_0)|) \leq 8\epsilon. \tag{3.39}$$

Moreover,

$$u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0) + \gamma_\epsilon(x_0-y_0) + 2\epsilon\xi(x_0,y_0)$$

$$= \Psi^{\hat{d}}(x_0,y_0)$$

$$\geq \Psi^{\hat{d}}(x_0+h^{\hat{d}}(x_0), y_0 + h^{\hat{d}}(x_0))$$

$$= u^{\hat{d}}(x_0+h^{\hat{d}}(x_0)) - v^{\hat{d}}(y_0+h^{\hat{d}}(x_0)) + \gamma_\epsilon(x_0-y_0) + 2\epsilon\xi(x_0+h^{\hat{d}}(x_0), y_0+h^{\hat{d}}(x_0)).$$

Thus,

$$u^{\hat{d}}(x_0) - u^{\hat{d}}(x_0+h^{\hat{d}}(x_0)) - v^{\hat{d}}(y_0) + v^{\hat{d}}(y_0+h^{\hat{d}}(y_0))$$

$$\geq v^{\hat{d}}(y_0+h^{\hat{d}}(y_0)) - v^{\hat{d}}(y_0+h^{\hat{d}}(x_0)) + 2\epsilon[\xi(x_0+h^{\hat{d}}(x_0), y_0 + h^{\hat{d}}(x_0)) - \xi(x_0,y_0)]$$

$$= o(1) \quad as\ \epsilon \downarrow 0. \tag{3.40}$$

From (3.36) - (3.40), we know that

$$u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0) \leq o(1) \tag{3.41}$$

Again,

$$u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0) + \gamma_\epsilon(x_0 - y_0) + 2\epsilon\xi(x_0, y_0)$$
$$= \Psi^{\hat{d}}(x_0, y_0)$$
$$\geq \Psi^{d}(x, x)$$
$$= u^{d}(x) - v^{d}(x) + \gamma_\epsilon(0) + 2\epsilon\xi(x, x)$$

which implies

$$u^{d}(x) - v^{d}(x)$$
$$\leq [u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0)] + [\gamma_\epsilon(x_0 - y_0) - \gamma_\epsilon(0)] + 2\epsilon[\xi(x_0, y_0) - \xi(x, x)]$$
$$= o(1) \qquad as \ \epsilon \downarrow 0.$$

Thus, $u^{d}(x) \leq v^{d}(x)$ for all $x \in \mathbb{R}^n$ and $d \in \{1,\ldots,m\}$. If we change the role of u and v in our argument, we have $v^{d}(x) \leq u^{d}(x)$. Hence, $u \equiv v$.

Case (ii) $v^{\hat{d}}(y_0) = M^{\hat{d}}[v](y_0)$. Then $\exists \ \tilde{d} \neq \hat{d}$ such that

$$v^{\hat{d}}(y_0) = v^{\tilde{d}}(y_0) + k(\hat{d}, \tilde{d}). \tag{3.42}$$

Then,

$$u^{\hat{d}}(x_0) - v^{\hat{d}}(y_0) + \gamma_\epsilon(x_0 - y_0) + 2\epsilon\xi(x_0, y_0)$$
$$= \Psi^{\hat{d}}(x_0, y_0)$$
$$\geq \Psi^{\tilde{d}}(x_0, y_0)$$
$$= u^{\tilde{d}}(x_0) - v^{\tilde{d}}(y_0) + \gamma_\epsilon(x_0, y_0) + 2\epsilon\xi(x_0, y_0)$$

which implies

$$u^{\hat{d}}(x_0) - u^{\tilde{d}}(x_0) \geq v^{\hat{d}}(y_0) - v^{\tilde{d}}(y_0) = k(\hat{d}, \tilde{d}).$$

But in general,

$$u^{\hat{d}}(x_0) \leq u^{\tilde{d}}(x_0) + k(\hat{d}, \tilde{d}),$$

so that

$$u^{\hat{d}}(x_0) - u^{\tilde{d}}(x_0) = v^{\hat{d}}(y_0) - v^{\tilde{d}}(y_0)$$

and

$$\Psi^{\tilde{d}}(x_0,y_0) = \Psi^{\hat{d}}(x_0,y_0).$$

Now, we can consider the same situation with index $\tilde{d}$ instead of index $\hat{d}$. If there exists $\bar{d} \neq \tilde{d}$ such that

$$v^{\tilde{d}}(y_0) = v^{\bar{d}}(y_0) + k(\tilde{d},\bar{d}).$$

From (3.42),

$$\begin{aligned} v^{\hat{d}}(y_0) &= v^{\bar{d}}(y_0) + k(\hat{d},\tilde{d}) + k(\tilde{d},\bar{d}) \\ &> v^{\bar{d}}(y_0) + k(\hat{d},\bar{d}) \end{aligned}$$

which contradicts

$$v^{\hat{d}}(y_0) \leq v^{d}(y_0) + k(\hat{d},d), \ \forall \ d \neq \hat{d}.$$

Hence, we are in case (i) with index $\tilde{d}$ instead of index $\hat{d}$ and the proof is completed.

QED

4. Existence of Viscosity Solutions.

Now, we use a finite difference approximation to construct a sequence of solutions which converges to the solution of (3.3).

Let $\rho \in C^2(\mathbb{R}^n)$ such that

$$\begin{cases} \rho(x) = 0, & x \leq 0 \\ \rho(x) > 0, & x > 0 \\ 0 < \rho'(x) \leq 1, & \rho''(x) > 0 \text{ for } x > 0 \end{cases} \tag{4.1}$$

and $\rho_\epsilon(x) = \rho(\frac{x}{\epsilon})$, $x \in \mathbb{R}^n$, $\epsilon > 0$.

Consider the penalized system of approximation.

$$\beta u_\epsilon^d(x) - \frac{1}{\epsilon}[u_\epsilon^d(x+\epsilon g^d(x)) - u_\epsilon^d(x)] - \lambda_d[u_\epsilon^d(x+h^d(x)) - u_\epsilon^d(x)]$$

$$+ \sum_{\hat{d} \neq d} \rho_\epsilon(u_\epsilon^d(x) - u_\epsilon^{\hat{d}}(x) - k(d,\hat{d})) = f^d(x) \tag{4.2}$$

or

$$u_\epsilon^d(x) - \frac{1}{\beta\epsilon}[u_\epsilon^d(x+\epsilon g^d(x)) - u_\epsilon^d(x)] - \frac{\lambda_d}{\beta}[u_\epsilon^d(x+h^d(x)) - u_\epsilon^d(x)]$$

$$+ \frac{1}{\beta}\sum_{\hat{d} \neq d} \rho_\epsilon(u_\epsilon^d(x) - u_\epsilon^{\hat{d}}(x) - k(d,\hat{d})) = \frac{1}{\beta} f^d(x). \tag{4.3}$$

We define operators Λ, Π_1, Π_2: $C(\mathbb{R}^n)^m \to C(\mathbb{R}^n)^m$ such that $\Lambda u = (\Lambda^1 u, \ldots, \Lambda^m u)$, $\Pi_1 u = (\Pi_1^1 u, \ldots, \Pi_1^m u)$ and $\Pi_2 u = (\Pi_2^1 u, \ldots, \Pi_2^m u)$ where

$$\Lambda^d u(x) \triangleq -\frac{1}{\beta\epsilon}[u^d(x+\epsilon g^d(x)) - u^d(x)] \tag{4.4}$$

$$\Pi_1^d u(x) \triangleq -\frac{\lambda_d}{\beta}[u^d(x+h^d(x)) - u^d(x)] \tag{4.5}$$

$$\Pi_2^d u(x) \triangleq \frac{1}{\beta}\sum_{\hat{d} \neq d} \rho_\epsilon(u^d(x) - u^{\hat{d}}(x) - k(d,\hat{d})). \tag{4.6}$$

Definition. (i) An operator $S: X \to X$ with domain $D(S)$ is said to be *accretive* on the real Banach space X if

$$||x - \hat{x} + \gamma[S(x) - S(\hat{x})]|| \geq ||x - \hat{x}|| \tag{4.7}$$

for all $x, \hat{x} \in D(S)$, $\forall\ \gamma > 0$.

(ii) An operator S is said to be *m-accretive* on X if S is accretive on X and the range $R(I + \gamma S) = X$ for all $\gamma > 0$ (or equivalently for some $\gamma > 0$).

The following lemma is from Evans [9, pp. 242].

Perturbation Lemma 4.1. *If S is m-accretive on $X = C(\mathbb{R}^n)^m$ and T is accretive, Lipschitz continuous everywhere defined on X, then $(S+T)$ is m-accretive on X, in particular, the range $R(I+S+T) = C(\mathbb{R}^n)^m$.*

Lemma 4.2. *Λ is m-accretive on $C(\mathbb{R}^n)^m$.*

Proof. Suppose there exist x_0 and $\hat{d}$ such that

$$u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0) = \max_{d} \sup_{x \in \mathbb{R}^n} |u^d(x) - \hat{u}^d(x)| \triangleq ||u - \hat{u}||, \tag{4.8}$$

then

$$\Lambda^{\hat{d}} u(x_0) - \Lambda^{\hat{d}} \hat{u}(x_0)$$

$$= -\frac{1}{\beta\epsilon}[u^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0)) - u^{\hat{d}}(x_0)] + \frac{1}{\beta\epsilon}[\hat{u}^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0)) - \hat{u}^{\hat{d}}(x_0)]$$

$$= \frac{1}{\beta\epsilon}\{[u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0)] - [u^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0)) - \hat{u}^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0))]\}$$

$$= \frac{1}{\beta\epsilon}\{||u - \hat{u}|| - [u^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0)) - \hat{u}^{\hat{d}}(x_0 + \epsilon g^{\hat{d}}(x_0))]\}$$

$$\geq 0$$

so that

$$||u - \hat{u} + \gamma(\Lambda u - \Lambda \hat{u})|| \geq u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0) + \gamma[\Lambda^{\hat{d}} u^{\hat{d}}(x_0) - \Lambda^{\hat{d}} \hat{u}^{\hat{d}}(x_0)]$$

$$\geq u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0)$$

$$= ||u - \hat{u}||.$$

If there exists no x_0 such that (4.8) holds, then for each $\epsilon > 0$, let x_ϵ, d_ϵ be such that

$$u^{d_\epsilon}(x_\epsilon) - \hat{u}^{d_\epsilon}(x_\epsilon) > ||u - \hat{u}|| - \epsilon$$

Choose $\varsigma \in C^1(\mathbb{R}^n)$ such that

$$\begin{cases} \varsigma(x_\epsilon) = 1 \\ 0 \leq \varsigma(x) < 1 \quad \text{for } x \neq x_\epsilon \\ \varsigma(x) = 0 \quad \text{for } ||x - x_\epsilon|| \geq 1. \end{cases} \tag{4.9}$$

Consider the auxiliary function

$$\Phi^d(x) \triangleq u^d(x) - \hat{u}^d(x) + 2\epsilon\varsigma(x). \tag{4.10}$$

Then,

$$\Phi^{d_\epsilon}(x_\epsilon) = u^{d_\epsilon}(x_\epsilon) - \hat{u}^{d_\epsilon}(x_\epsilon) + 2\epsilon > ||u - \hat{u}|| + \epsilon$$

and

$$\Phi^d(x) = u^d(x) - \hat{u}^d(x) \leq ||u - \hat{u}||$$

if $x \notin B(x_\epsilon, 1)$, so that there is a d_1 such that Φ^{d_1} attains a maximum at x_1, say, in the ball $B(x_\epsilon, 1)$. Thus,

$$\begin{aligned} &\Lambda^{d_1}u(x_1) - \Lambda^{d_1}\hat{u}(x_1) + \frac{2}{\beta}[\varsigma(x_1) - \varsigma(x_1 + \epsilon g^{d_1}(x_1))] \\ &= \frac{1}{\beta\epsilon}\{[u^{d_1}(x_1) - \hat{u}^{d_1}(x_1) + 2\epsilon\varsigma(x_1)] \\ &- [u^{d_1}(x_1 + \epsilon g^{d_1}(x_1)) - \hat{u}^{d_1}(x_1 + \epsilon g^{d_1}(x_1)) + 2\epsilon\varsigma(x_1 + \epsilon g^{d_1}(x_1))]\} \\ &\geq 0 \end{aligned}$$

so that

$$\begin{aligned} &||u - \hat{u} + \gamma(\Lambda u - \Lambda\hat{u})|| \\ &\geq u^{d_1}(x_1) - \hat{u}^{d_1}(x_1) + \gamma[\Lambda^{d_1}u(x_1) - \Lambda^{d_1}\hat{u}(x_1)] \\ &\geq \Phi^{d_1}(x_1) - 2\epsilon\varsigma(x_1) + \frac{2\gamma}{\beta}[\varsigma(x_1 + \epsilon g^{d_1}(x_1)) - \varsigma(x_1)] \end{aligned}$$

$$\geq ||\Phi|| + o(1) \quad \text{as } \epsilon \downarrow 0 \text{ since } \varsigma \in C^1$$

$$\geq u^{d_\epsilon}(x_\epsilon) - \hat{u}^{d_\epsilon}(x_\epsilon) + 2\epsilon\varsigma(x_\epsilon) + o(1) \quad as\ \epsilon \downarrow 0$$

$$> ||u - \hat{u}|| - \epsilon + 2\epsilon\varsigma(x_\epsilon) + o(1) \quad as\ \epsilon \downarrow 0$$

$$= ||u - \hat{u}|| + o(1) \quad as\ \epsilon \downarrow 0.$$

Letting $\epsilon \downarrow 0$, we show that Λ is accretive. To show $R(\mathrm{I} + \Lambda) = C(\mathbb{R}^n)^m$, i.e., the equation $u + \Lambda u = q$ is solvable for all $q \in C(\mathbb{R}^n)^m$. Then,

$$u^d(x) - \frac{1}{\beta\epsilon}[u^d(x + \epsilon g^d(x)) - u^d(x)] = q^d(x)$$

or

$$u^d(x) = \frac{1}{1+\beta\epsilon}[u^d(x + \epsilon g^d(x)) + \beta\epsilon q^d(x)]$$

$$\triangleq Tu^d(x),$$

and

$$Tu^d(x) - Tv^d(x) = \frac{1}{1+\beta\epsilon}[u^d(x + \epsilon g^d(x)) - v^d(x + \epsilon g^d(x))]$$

$$\leq \frac{1}{1+\beta\epsilon}||u^d - v^d||,$$

which implies

$$||Tu^d - Tv^d|| \leq \frac{1}{1+\beta\epsilon}||u^d - v^d||$$

so T is a contraction. By the contraction mapping theorem, there exists a unique fixed solution $u^d \in C(\mathbb{R}^n)$. Hence, Λ is m-accretive.

QED

Remark. In the same manner, we can show Π_1 is m-accretive.

Lemma 4.2. *Π_2 is accretive.*

Proof. Define

$$\Delta^{d,\hat{d}} u(x) = u^d(x) - u^{\hat{d}}(x) - k(d,\hat{d}).$$

Suppose $\exists$ x_0 and $\hat{d}$ such that

$$u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0) = \max_{d} \sup_{x \in \mathbb{R}^n} |u^d(x) - \hat{u}^d(x)| \triangleq ||u - \hat{u}||. \tag{4.11}$$

Then, for any $\tilde{d} \neq \hat{d}$,

$$\begin{aligned} &\Delta^{\hat{d},\tilde{d}} u(x_0) - \Delta^{\hat{d},\tilde{d}} \hat{u}(x_0) \\ &= [u^{\hat{d}}(x_0) - u^{\tilde{d}}(x_0) - k(\hat{d},\tilde{d})] - [\hat{u}^{\hat{d}}(x_0) - \hat{u}^{\tilde{d}}(x_0) - k(\hat{d},\tilde{d})] \\ &= [u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0)] - [u^{\tilde{d}}(x_0) - \hat{u}^{\tilde{d}}(x_0)] \\ &\geq 0. \end{aligned}$$

Since ρ_ϵ is continuously increasing, we have

$$\rho_\epsilon(\Delta^{\hat{d},\tilde{d}} u(x_0)) \geq \rho_\epsilon(\Delta^{\hat{d},\tilde{d}} \hat{u}(x_0))$$

and so

$$\Pi_2^{\hat{d}} u(x_0) \geq \Pi_2^{\hat{d}} \hat{u}(x_0).$$

Thus,

$$\begin{aligned} ||u - \hat{u} + \gamma(\Pi_2 u - \Pi_2 \hat{u})|| &\geq u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0) + \gamma[\Pi_2^{\hat{d}} u(x_0) - \Pi_2^{\hat{d}} \hat{u}(x_0)] \\ &\geq u^{\hat{d}}(x_0) - \hat{u}^{\hat{d}}(x_0) \\ &= ||u - \hat{u}|| \end{aligned} \tag{4.12}$$

which shows that Π_2 is accretive. Again, if there exists no x_0 such that (4.11) holds, we can consider a similar auxiliary function as (4.10) to show (4.12).

Lemma 4.3. $\Pi = \Pi_1 + \Pi_2$ *is Lipschitz continuous on* $C(\mathbb{R}^n)^m$.

Proof. Since

$$|\Pi_1^d u(x) - \Pi_1^d v(x)|$$

$$= \frac{\lambda_d}{\beta} \,|\, -u^d(x+h^d(x)) + u^d(x) + v^d(x+h^d(x)) - v^d(x)|$$

$$\leq \frac{\lambda_d}{\beta} [\,|u^d(x) - v^d(x)| + |u^d(x+h^d(x)) - v^d(x+h^d(x))|\,]$$

$$\leq 2\frac{\lambda_d}{\beta} ||u-v||,$$

we have

$$||\Pi_1 u - \Pi_1 v|| \leq 2\frac{\lambda_d}{\beta} ||u-v||.$$

Again,

$$|\Pi_2^d u(x) - \Pi_2^d v(x)|$$

$$= \frac{1}{\beta} \,|\sum_{\hat{d}\neq d} \rho_\epsilon(u^d(x)-u^{\hat{d}}(x)-k(d,\hat{d})) - \rho_\epsilon(v^d(x)-v^{\hat{d}}(x)-k(d,\hat{d}))|$$

$$\leq \frac{1}{\beta} \sum_{\hat{d}\neq d} ||\rho'_\epsilon|| \,|u^d(x) - v^d(x) - u^{\hat{d}}(x) + v^{\hat{d}}(x)|$$

$$\leq \frac{2(m-1)}{\beta\epsilon} ||u-v||,$$

so that

$$||\Pi_2 u - \Pi_2 v|| \leq \frac{2(m-1)}{\beta\epsilon} ||u-v||.$$

Thus,

$$||\Pi u - \Pi v|| \leq \frac{2}{\beta} [\lambda_d + \frac{(m-1)}{\epsilon}] ||u-v||$$

which shows that Π is Lipschitz continuous on $C(\mathbb{R}^n)^m$.

QED

By the Perturbation Lemma 4.1, $\Lambda + \Pi$ is m-accretive and so, for each $\epsilon > 0$, we have a solution $u_\epsilon \in C(\mathbb{R}^n)^m$ of (4.3).

Theorem 4.4.

(i) $0 \leq u_\epsilon^d(x) \leq \dfrac{||f||}{\beta}$, $\epsilon > 0$, $d \in 1,\ldots,m$.

(ii) *For each* $0 < \gamma < \min\left\{\dfrac{\beta}{L(1+\lambda_{\max})}, 1\right\}$,

$$| u_\epsilon^d(x) - u_\epsilon^d(\hat{x}) | \leq C_\gamma | x - \hat{x} |^\gamma, \quad x \in \mathbb{R}^n, \ \epsilon > 0, \ d \in \{1, \ldots, m\}$$

with the same constant C_γ *in* (2.20). *If* $\beta > L(1+\lambda_{\max})$, *we can take* $\gamma = 1$.

Proof. (i) Suppose $\exists$ x_0 and $\hat{d}$ such that

$$u_\epsilon^{\hat{d}}(x_0) = \min_d \min_{x \in \mathbb{R}^n} u_\epsilon^d(x) \tag{4.13}$$

then

$$u_\epsilon^{\hat{d}}(x_0) < u_\epsilon^{\tilde{d}}(x_0) + k(\hat{d}, \tilde{d}), \quad \tilde{d} \neq \hat{d},$$

so that

$$\frac{1}{\beta} \sum_{\tilde{d} \neq \hat{d}} \rho_\epsilon(u_\epsilon^{\hat{d}}(x_0) - u_\epsilon^{\tilde{d}}(x_0) - k(\hat{d}, \tilde{d})) = 0.$$

Thus from (4.3), we have

$$u_\epsilon^{\hat{d}}(x_0) \geq \frac{1}{\beta} f^{\hat{d}}(x_0) \geq 0,$$

so that

$$u_\epsilon^d(x) \geq u_\epsilon^{\hat{d}}(x_0) \geq 0. \tag{4.14}$$

If there exist no x_0, $\hat{d}$ such that the minimum in (4.13) occurs, we can consider $u_\epsilon^d(x) - 2\epsilon\varsigma(x)$ instead of $u_\epsilon^d(x)$ to force the minimum in (4.13) to occur. The same analysis can be adapted from the proof of Lemma 4.2 to show (4.14).

Without loss of generality, we can assume there exists x_1, $\tilde{d}$ such that

$$u_\epsilon^{\tilde{d}}(x_1) = \max_d \max_{x \in \mathbb{R}^n} u_\epsilon^d(x).$$

Then from (4.3), we have

$$u_\epsilon^{\tilde{d}}(x_1) \leq \frac{1}{\beta} f^{\tilde{d}}(x_1) \leq \frac{\|f\|}{\beta}.$$

Thus,

$$0 \leq u_\epsilon^d(x) \leq u_\epsilon^{\tilde{d}}(x_1) \leq \frac{\|f\|}{\beta}.$$

(ii) Let

$$\Phi^d(x,y) \triangleq \frac{|u^d(x+y) - u^d(x)|}{|y|^\gamma}, \quad |y| \neq 0,$$

where γ will be determined later.

Again, without loss of generality, we can suppose $\exists\ \hat{d}$, x_0, y_0 such that

$$\Phi^{\hat{d}}(x_0,y_0) = \max_d \max_{x,y} \Phi^d(x,y)$$

and $u^{\hat{d}}(x_0+y_0) - u^{\hat{d}}(x_0) > 0$, $y_0 \neq 0$. Then

$$u_\epsilon^{\hat{d}}(x_0+y_0) - u_\epsilon^{\hat{d}}(x_0) \geq u_\epsilon^{\tilde{d}}(x_0+y_0) - u_\epsilon^{\tilde{d}}(x_0), \quad \tilde{d} \neq \hat{d},$$

so that

$$\rho_\epsilon(u_\epsilon^{\hat{d}}(x_0+y_0) - u_\epsilon^{\tilde{d}}(x_0+y_0) - k(\hat{d},\tilde{d})) \geq \rho_\epsilon(u_\epsilon^{\hat{d}}(x_0) - u_\epsilon^{\tilde{d}}(x_0) - k(\hat{d},\tilde{d})). \tag{4.15}$$

Since

$$u_\epsilon^{\hat d}(x_0+y_0) - \frac{1}{\beta\epsilon}[u_\epsilon^{\hat d}(x_0+y_0+\epsilon g^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+y_0)]$$

$$- \frac{\lambda_{\hat d}}{\beta}[u_\epsilon^{\hat d}(x_0+y_0+h^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+y_0))]$$

$$+ \frac{1}{\beta}\sum_{\tilde d \neq \hat d} \rho_\epsilon(u_\epsilon^{\hat d}(x_0+y_0) - u_\epsilon^{\tilde d}(x_0+y_0) - k(\hat d,\tilde d)) = \frac{1}{\beta} f^{\hat d}(x_0+y_0) \tag{4.16}$$

and

$$u_\epsilon^{\hat d}(x_0) - \frac{1}{\beta\epsilon}[u_\epsilon^{\hat d}(x_0+\epsilon g^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0)]$$

$$- \frac{\lambda_{\hat d}}{\beta}[u_\epsilon^{\hat d}(x_0+h^{\hat d}(x_0)) - u_\epsilon^{\hat d}(x_0)]$$

$$+ \frac{1}{\beta}\sum_{\tilde d \neq \hat d} \rho_\epsilon(u_\epsilon^{\hat d}(x_0) - u_\epsilon^{\tilde d}(x_0) - k(\hat d,\tilde d)) = \frac{1}{\beta} f^{\hat d}(x_0). \tag{4.17}$$

Substracting (4.17) from (4.16), we have

$$(1+\frac{1}{\beta\epsilon}+\frac{\lambda_{\hat d}}{\beta})[u_\epsilon^{\hat d}(x_0+y_0) - u_\epsilon^{\hat d}(x_0)]$$

$$= \frac{1}{\beta}[f^{\hat d}(x_0+y_0) - f^{\hat d}(x_0)] + \frac{1}{\beta\epsilon}[u_\epsilon^{\hat d}(x_0+y_0+\epsilon g^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+\epsilon g^{\hat d}(x_0))]$$

$$+ \frac{\lambda_{\hat d}}{\beta}[u_\epsilon^{\hat d}(x_0+y_0+h^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+h^{\hat d}(x_0))]$$

$$- \frac{1}{\beta}\sum_{\tilde d \neq \hat d}[\rho_\epsilon(u_\epsilon^{\hat d}(x_0+y_0) - u_\epsilon^{\tilde d}(x_0+y_0) - k(\hat d,\tilde d)) - \rho_\epsilon(u_\epsilon^{\hat d}(x_0) - u_\epsilon^{\tilde d}(x_0) - k(\hat d,\tilde d))]$$

$$\leq \frac{1}{\beta}[f^{\hat d}(x_0+y_0) - f^{\hat d}(x_0)] + \frac{1}{\beta\epsilon}[u_\epsilon^{\hat d}(x_0+y_0+\epsilon g^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+\epsilon g^{\hat d}(x_0))]$$

$$+ \frac{\lambda_{\hat d}}{\beta}[u_\epsilon^{\hat d}(x_0+y_0+h^{\hat d}(x_0+y_0)) - u_\epsilon^{\hat d}(x_0+h^{\hat d}(x_0))]. \tag{4.18}$$

Dividing (4.18) by $|y_0|^\gamma$, we have

$$(1+\frac{1}{\beta\epsilon}+\frac{\lambda_{\hat d}}{\beta})\left[\frac{u_\epsilon^{\hat d}(x_0+y_0) - u_\epsilon^{\hat d}(x_0)}{|y_0|^\gamma}\right]$$

$$\leq \frac{1}{\beta} \frac{|\hat{f}^d(x_0+y_0) - \hat{f}^d(x_0)|}{|y_0|^\gamma}$$

$$+ \frac{|\hat{u}^d_\epsilon(x_0+y_0+\epsilon\hat{g}^d(x_0+y_0)) - \hat{u}^d_\epsilon(x_0+\epsilon\hat{g}^d(x_0))|}{\beta\epsilon\ |y_0+\epsilon\hat{g}^d(x_0+y_0)-\epsilon\hat{g}^d(x_0)|^\gamma} \frac{|y_0+\epsilon\hat{g}^d(x_0+y_0)-\epsilon\hat{g}^d(x_0)|^\gamma}{|y_0|^\gamma}$$

$$+ \frac{\lambda_{\hat{d}}}{\beta} \frac{|\hat{u}^d_\epsilon(x_0+y_0+\hat{h}^d(x_0+y_0)) - \hat{u}^d_\epsilon(x_0+\hat{h}^d(x_0))|}{|y_0+\hat{h}^d(x_0+y_0) - \hat{h}^d(x_0)|^\gamma} \cdot \frac{|y_0+\hat{h}^d(x_0+y_0) - \hat{h}^d(x_0)|^\gamma}{|y_0|^\gamma}$$

$$\leq \frac{1}{\beta} ||f||^{1-\gamma} L^\gamma + \frac{1}{\beta\epsilon} \Phi^{\hat{d}}(x_0,y_0)(1+\epsilon L)^\gamma + \frac{\lambda_{\hat{d}}}{\beta} \Phi^{\hat{d}}(x_0,y_0)(1+L)^\gamma.$$

Thus,

$$\left\{ 1 + \frac{1}{\beta\epsilon}[1-(1+\epsilon L)^\gamma] + \frac{\lambda_{\hat{d}}}{\beta}[1-(1+L)^\gamma] \right\} \Phi^{\hat{d}}(x_0,y_0) \leq \frac{L^\gamma}{\beta} ||f||^{1-\gamma}. \quad (4.19)$$

Let the constant be

$$a_\gamma \triangleq 1 + \frac{1}{\beta\epsilon}[1-(1+\epsilon L)^\gamma] + \frac{\lambda_{\hat{d}}}{\beta}[1-(1+L)^\gamma].$$

Then a_γ is a decreasing function of γ with $a_0 = 1$. We want to find the range of γ such that $a_\gamma > 0$. We claim $1+x\gamma \geq (1+x)^\gamma$ for x ≥ 0, $0 \leq \gamma \leq 1$. Let $\phi_1(\gamma) \triangleq (1+x)^\gamma$ and $\phi_2(\gamma) \triangleq 1+x\gamma$. Since ϕ_1 is a convex increasing function such that $\phi_1(0) = 1 = \phi_2(0)$, and $\phi_1(1) = 1+x = \phi_2(1)$, $\phi_1(\gamma) \leq \phi_2(\gamma)$, $0 \leq \gamma \leq 1$. Thus, $1 - (1+x)^\gamma \geq -x\gamma$, $\forall\ x \geq 0$, $0 \leq \gamma \leq 1$. Hence

$$a_\gamma \geq 1 + \frac{1}{\beta\epsilon}(-\epsilon L\gamma) + \frac{\lambda_{\hat{d}}}{\beta}(-L\gamma)$$

$$= 1 - \frac{L\gamma}{\beta}(1+\lambda_{\hat{d}})$$

$$> 0 \quad \text{if } \gamma < \min\left(\frac{\beta}{L(1+\lambda_{\max})}, 1 \right).$$

With this choice of γ, we have

$$|\Phi^d(x,y)| \leq \Phi^{\hat{d}}(x_0,y_0) \leq \frac{L^\gamma}{\beta a_\gamma} ||f||^{1-\gamma} = C_\gamma .$$

QED

From the above lemma, $\{u_\epsilon^d\}$ are uniformly bounded and uniformly Hölder continuous. Then by the Arzela-Ascoli Theorem, there exists a subsequence ϵ_l such that $u_{\epsilon_l}^d \to u^d \in C(\mathbb{R}^n)$ for all $d \in 1,...,m$. The convergence is uniform on each compact subset of $\mathbb{R}^n$. In fact, u is bounded and Hölder continuous with the same Hölder exponent γ. We shall prove that u solves the (QVI) in the viscosity sense.

Theorem 4.5. $u_{\epsilon_l} \to$ u *locally uniformly in* $\mathbb{R}^n$ *and* u *solves* (3.3) *in the viscosity sense.*

Proof. We first claim that $u^d \leq M^d[u]$. Suppose not, that is, $\exists\ \hat{d} \neq d$, $\delta > 0$ and an open ball B such that

$$u^d(x) \geq u^{\hat{d}}(x) + k(d,\hat{d}) + 2\delta, \quad x \in \overline{B}.$$

As $u_{\epsilon_l}^d \to u^d$, $u_{\epsilon_l}^{\hat{d}} \to u^{\hat{d}}$ uniformly on $\overline{B}$, we get

$$u_{\epsilon_l}^d(x) \geq u_{\epsilon_l}^{\hat{d}}(x) + k(d,\hat{d}) + \delta, \quad x \in \overline{B}$$

for all sufficiently large l. Now, let $\phi \in C^1(\mathbb{R}^n)$ and without loss of generality, we can assume $x_0 \in B$ is a strict local maximum of $u^d - \phi$; otherwise we can add a small function to force this to occur. Since $u_{\epsilon_l}^d \to u^d$ uniformly on $\overline{B}$, $u_{\epsilon_l}^d - \phi$ also attains a strict local maximum at $x_l \in B$ for l large. Since

$$u_{\epsilon_l}^d(x_l) - \frac{1}{\beta\epsilon_l}[u_{\epsilon_l}^d(x_l + \epsilon_l g^d(x_l)) - u_{\epsilon_l}^d(x_l)] - \frac{\lambda_d}{\beta}[u_{\epsilon_l}^d(x_l + h^d(x_l)) - u_{\epsilon_l}^d(x_l)].$$

$$+ \frac{1}{\beta} \sum_{\hat{d} \neq d} \rho_{\epsilon_l} (u_{\epsilon_l}^d (x_l) - u_{\epsilon_l}^{\hat{d}} (x_l) - k(d, \hat{d})) = \frac{1}{\beta} f^d (x_l),$$

then

$$\rho_{\epsilon_l}(\delta) \leq \sum_{\hat{d} \neq d} \rho_{\epsilon_l} (u_{\epsilon_l}^d (x_l) - u_{\epsilon_l}^{\hat{d}} (x_l) - k(d, \hat{d}))$$

$$\leq - \beta u_{\epsilon_l}^d (x_l) + \frac{1}{\epsilon_l} [u_{\epsilon_l}^d (x_l + \epsilon_l g^d (x_l)) - u_{\epsilon_l}^d (x_l)]$$

$$+ \lambda_d [u_{\epsilon_l}^d (x_l + h^d (x_l)) - u_{\epsilon_l}^d (x_l)] + f^d (x_l)$$

$$\leq ||f|| + \frac{1}{\epsilon_l} [\phi(x_l + \epsilon_l g^d (x_l)) - \phi(x_l)] + \lambda_d C_\gamma | h^d (x_l) |^\gamma + ||f||$$

$$\leq 2||f|| + ||\nabla \phi|| \, ||g|| + \lambda_d C_\gamma ||h||^\gamma < \infty. \tag{4.20}$$

But the left hand side of (4.20) tends to infinity as $\epsilon_l \to 0$, which is a contradiction. Thus $u^d \leq M^d$[u].

(i) As above, suppose $u^d - \phi$ attains a strict local maximum at $x_0 \in B$ while $u_{\epsilon_l}^d - \phi$ attains a strict local maximum at $x_l \in B$ and $x_l \to x_0$. Then, by (4.2),

$$\beta u_{\epsilon_l}^d (x_l) - \frac{1}{\epsilon_l} [u_{\epsilon_l}^d (x_l + \epsilon_l g^d (x_l)) - u_{\epsilon_l}^d (x_l)]$$

$$- \lambda_d [u_{\epsilon_l}^d (x_l + h^d (x_l)) - u_{\epsilon_l}^d (x_l)] \leq f^d (x_l)$$

so that

$$\beta u_{\epsilon_l}^d (x_l) - \frac{1}{\epsilon_l} [\phi(x_l + \epsilon_l g^d (x_l)) - \phi(x_l)]$$

$$- \lambda_d [u_{\epsilon_l}^d (x_l + h^d (x_l)) - u_{\epsilon_l}^d (x_l)] \leq f^d (x_l).$$

Letting $l \to \infty$, we have

$$\beta u^d(x_0) - g^d(x_0)\cdot\nabla\phi(x_0) - \lambda_d\,[u^d(x_0+h^d(x_0)) - u^d(x_0)] \leq f^d(x_0)$$

so that (3.5) holds.

(ii) Let $\psi \in C^1(\mathrm{IR}^n)$ and $u^d - \psi$ attains a strict local minimum at z_0. If $u^d(z_0) = M^d[u](z_0)$ then (3.6) holds. Otherwise $u^d(z_0) < M^d[u](z_0)$, so that

$$u_{\epsilon_l}^d(z) < M^d[u_{\epsilon_l}](z) \quad \text{for } z \in \text{ some ball } B$$

and $u_{\epsilon_l}^d - \psi$ attains a strict local minimum at z_l. Then we can assume $z_l \rightarrow z_0$ without loss of generality. Since $u_{\epsilon_l}^d(z_l) < M^d[u_{\epsilon_l}](z_l)$,

$$\rho_{\epsilon_l}(u_{\epsilon_l}^d(z_l) - M^d[u_{\epsilon_l}](z_l)) = 0.$$

By (4.2),

$$\beta u_{\epsilon_l}^d(z_l) - \frac{1}{\epsilon_l}[u_{\epsilon_l}^d(z_l + \epsilon_l g^d(z_l)) - u_{\epsilon_l}^d(z_l)]$$

$$- \lambda_d\,[u_{\epsilon_l}^d(z_l+h^d(z_l)) - u_{\epsilon_l}^d(z_l)] = f^d(z_l)$$

so that

$$\beta u_{\epsilon_l}^d(z_l) - \frac{1}{\epsilon_l}[\psi(z_l + \epsilon_l g^d(z_l)) - \psi(z_l)]$$

$$- \lambda_d\,[u_{\epsilon_l}^d(z_l+h^d(z_l)) - u_{\epsilon_l}^d(z_l)] \geq f^d(z_l).$$

As $l \rightarrow \infty$, we have

$$\beta u^d(z_0) - g^d(z_0)\cdot\nabla\psi(z_0) - \lambda_d\,[u^d(z_0+h^d(z_0)) - u^d(z_0)] \geq f^d(z_0).$$

Thus, (3.6) holds at z_0. Consequently, u is a viscosity solution of (QVI).

QED

Remark. In general, u is only Hölder continuous. If we know u has some regularity properties, say u' exists in some neighborhood, then one can show u satisfies (3.3) in the ordinary sense. The point is that the derivative of u is not continuous across characteristic curves.

5. The Case of Vanishing Switching Costs.

In the case when the switching costs vanish ($k(d,\hat{d}) = 0$ for some $\hat{d} \neq d$ in (1.4)), then the dynamics may be switched at any time without incurring a cost; hence, the minimum cost does not depend on the initial control. That is,

$$u^1 = u^2 = \cdots = u^m \triangleq u \tag{5.1}$$

If we follow the arguments used in the previous sections, we can show that u is bounded and Hölder continuous with the same Hölder constant C_γ used in Lemma 2.4. In u were continuously differentiable on $\mathbb{R}^n$, then by the principle of dynamic programming, u would be (formally) a solution of the Hamiltonian - Jacobi - Bellman equation

$$\max_{d=1,\ldots,m} \{ \beta u - g^d \cdot \nabla u - \lambda_d \, [u(\cdot + h^d) - u] - f^d \} = 0 \tag{5.2}$$

on $\mathbb{R}^n$. However, u is not always C^1. By invoking the same arguments used in section 4, we can show that u is the unique viscosity solution of (5.2) in the following sense:

Definition 5.1. A bounded and continuous function u on $\mathbb{R}^n$ is a *viscosity solution* of (5.2) if for each $\phi \in C^1(\mathbb{R}^n)$ such that

(i) if $u - \phi$ attains a local maximum at $x_0 \in \mathbb{R}^n$, then

$$\max_{d=1,\ldots,m} \{ \beta u(x_0) - g(x_0)^d \cdot \nabla u(x_0) \tag{5.3}$$
$$- \lambda_d \, [u(x_0 + h^d(x_0)) - u(x_0)] - f^d(x_0) \} \leq 0$$

and

(ii) if $u - \phi$ attains a local minimum at $x_0 \in \mathbb{R}^n$, then

$$\max_{d=1,\ldots,m} \{ \beta u(z_0) - g(z_0)^d \cdot \nabla u(z_0) \tag{5.4}$$

$$- \lambda_d [u(z_0 + h^d(z_0)) - u(z_0)] - f^d(z_0) \} \geq 0$$

We now prove that the optimality system is *closed*; that is, each value function corresponding to non-zero switching costs will converge to u as the switching costs tend to zero. The result corresponds to a similar result in Capuzzo Dolcetta - Evans [4].

Theorem 5.1. *Suppose we have a set of switching costs* $\{ k_\epsilon(d,\hat{d}) \}$ *such that*

$$k_\epsilon(d,\hat{d}) > 0 \quad \forall\ d \neq \hat{d} \in \{ 1, \cdots, m \} \tag{5.5}$$

$$k_\epsilon(d,\hat{d}) < k_\epsilon(d,\tilde{d}) + k_\epsilon(\tilde{d},\hat{d}), \quad d \neq \tilde{d} \neq \hat{d}$$

For each $\epsilon > 0$ *let* $u_\epsilon = (u_\epsilon^1, \cdots, u_\epsilon^m)$ *be the unique viscosity solution of the corresponding QVI with switching costs* $\{ k_\epsilon(d,\hat{d}) \}$ *and let* u *be the unique viscosity solution of* (5.2). *If* $k_\epsilon(d,\hat{d}) \to 0$ *as* $\epsilon \to 0$ *for all* $d, \hat{d} \in \{ 1, \cdots, m \}$, *then* $u_\epsilon^d \to u$ *as* $\epsilon \to 0$ *for all* $d \in \{ 1, \cdots, m \}$.

Proof. Since the $\{ u_\epsilon^d \}$ are bounded and uniformly Hölder continuous with the same Hölder constant as in Lemma 2.4, there exists a subsequence $\{ \epsilon_i \}$ such that $u_{\epsilon_i}^d \to u^d$ (say) locally, uniformly on $\mathbb{R}^n$, $\forall\ d$. By Lemma 3.2

$$u_{\epsilon_i}^d \leq M^d[u_{\epsilon_i}] \leq u_{\epsilon_i}^{\tilde{d}} + k_{\epsilon_i}(d,\tilde{d}), \quad \tilde{d} \neq d \tag{5.6}$$

and so, $u^d \leq u^{\tilde{d}}$ as $\epsilon_i \to 0$. Since d and $\tilde{d}$ are arbitrary, we can obtain the reverse inequality by reversing the roles of d and $\tilde{d}$. Thus,

$$u^1 = u^2 = \cdots = u^m \triangleq u \tag{5.7}$$

It remains to show that u is the viscosity solution.

(i) Suppose $\phi \in C^1(!Rn)$ and $u - \phi$ attains a strict local maximum at x_0. Then for each $d \in \{1,...,m\}$ and each sufficiently small $\epsilon_i > 0$, $u^d_{\epsilon_i} - \phi$ attains a local maximum at $x^d_{\epsilon_i}$ near x_0, and $x^d_{\epsilon_i} \rightarrow x_0$ an $\epsilon_i \rightarrow 0$. Since u_{ϵ_i} is the viscosity solution corresponding to the switching costs $\{k_\epsilon(d, \hat{d})\}$, we have

$$\beta u^d_{\epsilon_i}(x^d_{\epsilon_i}) - g^d(x^d_{\epsilon_i}) \cdot \nabla\phi(x^d_{\epsilon_i}) \tag{5.8}$$
$$- \lambda_d \; [u \epsilon_i{}^d(x^d_{\epsilon_i} + h^d(x^d_{\epsilon_i})) - u \epsilon_i{}^d(x^d_{\epsilon_i})] - f^d(x^d_{\epsilon_i})\} \leq 0$$

Letting $\epsilon_i \rightarrow 0$

$$\beta u(x_0) - g^d(x_0) \cdot \nabla\phi(x_0) - \lambda_d \; [u(x_0 + h^d(x_0)) - u(x_0)] - f^d(x_0)\} \leq 0 \tag{5.9}$$

Thus, (5.3) holds.

(ii) If $u - \phi$ has a strict local minimum at some point z_0, then for each d and each ϵ_i small enough, $u^d_{\epsilon_i} - \phi$ attains a local minimum near z_0. So we can choose d_i and z_i such that

$$(u^{d_i}_{\epsilon_i} - \phi)(z_i) = \min_{z \in \mathbb{R}^n} \min_{\hat{d}} (u^{\hat{d}}_{\epsilon_i} - \phi)(z) \tag{5.10}$$

and $z_i \rightarrow z_0$ as $\epsilon_i \rightarrow 0$. Hence,

$$(u^{d_i}_{\epsilon_i} - \phi)(z_i) \leq (u^{\tilde{d}}_{\epsilon_i} - \phi)(z_i) \tag{5.11}$$

We have

$$u^{d_i}_{\epsilon_i}(z_i) < u^{\tilde{d}}_{\epsilon_i}(z_i) + k_{\epsilon_i}(d_i, \tilde{d}) \quad \forall \; \tilde{d} \neq d_i \tag{5.12}$$

and so

$$u^{d_i}_{\epsilon_i}(z_i) < M^{d_i}[u_{\epsilon_i}](z_i) \tag{5.13}$$

Thus,

$$\beta u_{\epsilon_i}^{d_i}(z_i) - g^{d_i}(z_i) \cdot \nabla\phi(z_i) \tag{5.14}$$

$$- \lambda_{d_i} [u \epsilon_i{}^{d_i}(z_i + h^{d_i}(z_i)) - u \epsilon_i{}^{d_i}(z_i)] - f^{d_i}(z_i) \} \leq 0$$

Passing to the limit as $\epsilon_i \rightarrow 0$,

$$\beta u(z_0) - g^{d_i}(z_0) \cdot \nabla\phi(z_0) - \lambda_{d_i} [u(z_0 + h^{d_i}(z_0)) - u(z_0)] - f^{d_i}(z_0) \} \geq 0 \tag{5.15}$$

Thus, (5.4) holds.

Since (5.2) has a unique viscosity solution, the limit u is the required solution.

QED

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