THESIS REPORT

Ph.D.

Simultaneous and Robust Stabilization of Nonlinear Systems by Means of Continuous and Time-Varying Feedback

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Ph.D. 96-8

INSTITUTE FOR SYSTEMS RESEARCH

Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry

Abstract

Title of Dissertation: Simultaneous and Robust Stabilization of Nonlinear Systems by Means of Continuous and Time-Varying Feedback

Bertina Ho-Mock-Qai, Doctor of Philosophy, 1996

Dissertation directed by: Associate Professor W.P. Dayawansa
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In this dissertation, we address the stabilization of uncertain systems described by finite, countably infinite or uncountable families of systems. We adopt an approach that enables us to consider control systems with merely continuous dynamics as well as continuous time-invariant and time-varying feedback laws.

We show that for any countable family of asymptotically stabilizable systems, there exists a continuous nonlinear time-invariant controller that simultaneously stabilizes (not asymptotically) the family. Although these controllers do not achieve simultaneous asymptotic stabilization in the general case, we manage to modify our construction in order to design continuous time-invariant feedback laws that simultaneously asymptotically stabilize certain pairs of systems in the plane.

By introducing continuous time-varying feedback laws, we then prove that any finite family of linear time-invariant (LTI) systems is simultaneously asymptotically stabilizable by means of continuous nonlinear time-varying feedback,

if each system of the family is asymptotically stabilizable by a LTI controller. We also provide sufficient conditions for the simultaneous asymptotic stabilizability of countably infinite families of LTI systems, by means of continuous time-varying feedback.

We then obtain sufficient conditions for the existence of a continuous time-varying feedback law that simultaneously asymptotically stabilizes a finite family of **nonlinear** systems. We illustrate these results by establishing the simultaneous asymptotic stabilizability of the elements of a class of pairs of homogeneous nonlinear systems.

We finally consider a class of parameterized families of systems in the plane [where the parameter lies in an uncountable set] that are not robustly asymptotically stabilizable by means of C^1 feedback. We solve their robust asymptotic stabilization by means of continuous feedback, through a new approach where a robust asymptotic stabilizer is considered as a feedback law that simultaneously robustly asymptotically stabilizes two sub-families of the family under consideration.

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by

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Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1996

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Dedication

To my father, my uncle Philippe and his wife
To the memory of my mother

Acknowledgements

It is with great pleasure that I express my deep gratitude to my advisor Dr W.P. Dayawansa, for his guidance, encouragements and support. His love of science has been a constant inspiration to me. I also owe him very much for providing me the freedom in selecting my research topics.

I would like to thank Dr P.S. Krishnaprasad, Dr A. Tits and Dr E. Abed for their encouragements during my graduate studies.

I thank Dr D.L. Elliott, Dr S.I. Marcus and Dr C.A. Berenstein for serving on my Ph.D. committee, and Dr J.S. Baras for his guidance and support during the first two years of my graduate studies.

I would also like to express my gratitude to J-M. Garot from the Direction Générale de l'Aviation Civile whose valuable help enabled me to come to the United States for doctoral studies. I am also thankful to A. Printemps, D. Colin De Verdière and G. Eydaleine from the CENA for giving me the extra time needed for the completion of my Ph.D. and for their support.

My special thanks are due to Lionel Banège for his patience and constant support during the ups and downs of my graduate life. His comments both on the technical content and on the editing of this dissertation have been invaluable. Many thanks also to all my friends here, within and outside the University, who made my graduate student life more enjoyable.

This research was conducted in the Electrical Engineering Department and the Institute for Systems Research of the University of Maryland at College Park, and was supported partially through NSF Grants ECS 9212126 and EEC 94-02384.

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Chapter 1

Introduction and Definitions

In this dissertation, we discuss the stabilization and asymptotic stabilization of uncertain systems described by parameterized families of nonlinear control systems where the parameter lies in a set that is either finite, countably infinite or uncountable. Given such a family, our goal is to construct a continuous time-invariant or time-varying feedback law which stabilizes or asymptotically stabilizes each one of the systems in the family.

This introductory chapter is organized as follows: In section 1.1, we define the models of uncertainty that we will consider, present the main objective of this dissertation, and point out the motivations. We also briefly review the existing results on the control of uncertain systems. The main contributions of the dissertation are then presented in Section 1.2 while an outline of the dissertation is provided in Section 1.3. Finally, the chapter is completed with a presentation of the main definitions used in the dissertation.

1.1 Introduction

The control of deterministic uncertain systems by non-adaptive controllers, has been a long standing and challenging problem in control theory. The first results on that matter were obtained between the early thirties and the early sixties. During this period, three cornerstone studies were published: In 1932, Nyquist [61] made precise the trade-off between dynamic stability and large-loop gain. Thirteen years later, Bode [13] presented a control strategy for uncertain systems based on the Nyquist frequency domain stability criterion and Black's concept of large-loop gain for system accuracy. Finally, Horowitz [46] generalized Bode's design strategy. These results dealt mainly with single-input single-output systems with uncertainty in the gain and phase, and laid the ground for further work on more general uncertain systems.

However, from the mid-sixties to the mid-seventies, uncertain systems were mainly ignored by the control community that was preoccupied with developing state variable concepts such as controllability, observability and linear quadratic regulator theory. The few results obtained during this period are summarized in Cruz [18].

The focus then shifted to the design of controllers for multivariable systems using the newly developed state space concepts. This led, in particular, to the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR) presented in Doyle and Stein [28]. Around the same time, important results on the analysis of multi-variable systems were reported. For example, the concept of coprime matrix fraction was introduced as a design tool in Youla et al. [80], the Youla parameterization of all stabilizing controllers of a given plant was also derived in Youla et al. [80], and the Nyquist stability criterion was generalized to multi-variable systems by Rosenbrock, MacFarlane and Postlethwaite [58, 66]. These technical results turned out to play key roles in some of the methods used in the control of uncertain systems, which have been subsequently proposed.

Prior to presenting the main objectives of this dissertation, we review the different models of uncertain systems that we will consider.

1.1.1 Uncertainty models

We distinguish four classes of uncertain systems: those with dynamic uncertainty, those with parametric uncertainty, those represented by a finite family of systems, and those described by a countably infinite family of systems.

A system with dynamic uncertainty is described by a nominal system together with an unstructured but bounded perturbation. In the frequency domain approach, an uncertain linear time invariant (LTI) system $P(s)$ is typically represented by a formula of the form

$$P(s) \;=\; P_0(s) + \Delta(s) \qquad \text{or} \qquad P(s) \;=\; P_0(s)[1 + \Delta(s)],$$

where $P_0(s)$ is the nominal system and $\Delta(s)$ is a bounded (in some norm) LTI perturbation with no particular structure. In the state space approach the dynamics of an uncertain linear or nonlinear system with dynamic uncertainty is given by

$$\dot{x} \;=\; f_0(x) \;+\; v(t),$$

where $\dot{x} = f_0(x)$ is the nominal system and $v(\cdot)$ is a bounded perturbation. This description of uncertainty usually accounts for unmodeled high frequency dynamics as well as for the effect of linearization or time variation of a nominal system. Although we do not study this type of uncertain systems in this dissertation, in view of their importance in the control literature, we review the main related results in Subsection 1.1.4.

On the other hand, an uncertain system with parametric uncertainty is represented by a parameterized family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$ where γ is a vector of parameters, and Γ is an uncountable index set. While the structure of the perturbation is not taken into account in systems with dynamic uncertainty, here it is assumed that the structure of each member of the family $S(\gamma)$ is known. Such models are naturally well-suited for representing systems with uncertain physical parameters that are known to lie within certain bounds. Some scientists argue [30] that in most practical cases, the high frequency unmodeled dynamics are negligible once an uncertainty model with the appropriate parameters is derived. In any case, it is plain that parametric models contain more information than models with dynamic uncertainty and should therefore yield finer controllers.

A system with "finite uncertainty" is represented by a finite family of systems $\{S_i,\ i = 1, \ldots, I\}$ [where $I \geq 2$ is an integer]. Such a family may represent a nominal system with its failed modes [2, 29, 69] or the dynamics of a system at several operating points [1] such as a nonlinear system that has been linearized at different operating points.

Finally, a system with "countably infinite uncertainty" is described by a countably infinite family of systems $\{S_i,\ i = 1, 2, \ldots\}$. Such a family may represent a system with an infinite number of failed modes or the dynamics of a system at an infinite number of operating points. As we shall see in Section 1.1.3, if this countable family is a sub-family of a parameterized family of control systems, then the investigation of control problems related to this sub-family may yield important information on the parameterized family.

Although finite and countably infinite families of systems are simply parameterized families where the parameter lies in a countable set, the approaches for solving the related control problems are generally different. We will therefore distinguish the three cases in this dissertation.

The design of a non-adaptive controller that **asymptotically stabilizes** each one of the systems of a family of systems is called robust (resp. simultaneous) asymptotic stabilization if this family represents an uncertain systems with dynamic or parametric (resp. with finite or countably infinite) uncertainty.

The design of a non-adaptive controller which robustly (resp. simultaneously) asymptotically stabilizes an uncertain system with dynamic or parametric (resp. finite or countably infinite) uncertainty and which achieves some **performance** requirements is called robust (resp. simultaneous) control.

1.1.2 Main objective

In this dissertation, we consider finite, countably infinite and parameterized families of systems. Given such a family, our main objective is to investigate the existence of a **non-adaptive** controller that asymptotically stabilizes each one of the systems of the family. In particular, we aim at proposing design procedures that are applicable to families of **nonlinear** systems with possibly **merely continuous** dynamics.

To reach this objective we seek either continuous time-invariant or time-varying controllers.

The practical motivations for addressing these problems are presented in the next subsection, while the need for studying uncertain **nonlinear** systems will be clear from the literature survey in Subsection 1.1.4.

1.1.3 Motivation

While the practical usefulness of robust control and stabilization is now well established (applications can be found in [7, 20, 33, 65]), the possible applications for the control and stabilization of finite families of systems are not as well known. Nevertheless, there are applications that beg for controllers coping with finite uncertainty such as flight control and fault tolerant systems.

Indeed, as noted in Ackermann [1], flight control systems are typically very redundant and complex for two main reasons. First, in order to satisfy the safety requirements imposed by the regulation, the on-board systems must be able to cope with the many failures that can occur for instance in the actuators, sensors, controllers, and electrical systems. Secondly, the models found in flight control problems being highly nonlinear, they are usually linearized at several operating points corresponding to normal regimes, and a controller for each one of these linearized models is then designed. Controllers achieving simultaneous control or stabilization would therefore significantly simplify flight control systems. More generally, such controllers would benefit all plants with different failed or normal modes. In particular, they would be very useful for systems such as aircrafts, public transportation vehicles, nuclear plants, and chemical plants, where failures can lead to "catastrophic" consequences. In such systems a human operator is often present. Thus, although this kind of controllers would not probably yield the same level of performance as a gain scheduling controller, it could be used at the lowest hierarchical level in order to avoid the catastrophic effect of a failure and leave the time to the human operator to react.

Control of fault-prone systems have also been addressed in the context of control of hybrid systems [47]. In this framework, the system is modeled as a hybrid system with autonomous jump, i.e., it is described by a finite family of control systems $\{S_i,\ i = 1, \ldots, I\}$ (the failed and normal modes) together with a continuous time discrete state random process $r(t)$ that takes values in the set $\{1, \ldots, I\}$ (the occurrence of the failures). Whenever $r(t)$ jumps from a value i_0 to a value i_1, the hybrid system switches from dynamics S_{i_0} to S_{i_1}. These models can also represent airplanes subject to exogenous disturbances such as wind, manufacturing processes, and solar thermal receivers (see [59] and references therein). In general, a controller that simultaneously asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$ does not necessarily asymptotically stabilizes a hybrid system that switches between the systems of this family. However, we believe that in some cases this may be true, if for example the switchings are not "too frequent". It would then be of great interest to obtain conditions under which simultaneous asymptotic stabilization of a family of systems yields asymptotic stabilization of a related switched system.

The simultaneous asymptotic stabilization of infinite families has been mentioned in a few papers, but has never been really addressed. In our view, the motivation for studying this problem is two-fold: First, because a countable family of systems $\{S_i,\ i = 1, 2, \ldots\}$ may be a sub-family of a parameterized family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$, if we can can show that the family $\{S_i,\ i = 1, 2, \ldots\}$ is not simultaneously asymptotically stabilizable by some class of feedback laws, then we are ensured that the parameterized family $\{S(\gamma),\ \gamma \in \Gamma\}$ is not robustly asymptotically stabilizable by the same class of feedback laws. In this way, we may be able to obtain useful necessary conditions for the robust asymptotic stabilization of parameterized families of systems. Secondly, in practical implementations, the set Γ where the parameter lies, is always discretized and may be represented by a countably infinite set $\{\gamma_1, \gamma_2, \ldots\}$. In that case, we believe that in some situations, the simultaneous stabilization of the family $\{S(\gamma_1), S(\gamma_2), \ldots\}$ may suffice to provide "practical" robust stabilization [in some sense] to the family $\{S(\gamma),\ \gamma \in \Gamma\}$. We do not investigate further this matter in this dissertation, and leave this topic for further research.

1.1.4 Literature survey

The modern area of robust control of systems with dynamic uncertainty originated from Zames [82], who first discussed the $\mathcal{H}_\infty$ control problem. Two years later, Zames and Francis [83] provided the first $\mathcal{H}_\infty$ design procedure and solved the $\mathcal{H}_\infty$ sensitivity design problem for single-input single-output systems. It was then generalized to multi-input multi-output systems by Doyle [26]. Although these procedures solved the $\mathcal{H}_\infty$ problem analytically, the obtained controllers were very complex and further investigation was conducted, leading to two simpler $\mathcal{H}_\infty$ controller design methodologies. The first one was obtained by Glover and Doyle [34] who used the same approach as that in [26]. The second one was found by Doyle et al. [27] through a more direct approach. Because these $\mathcal{H}_\infty$ controllers have often poor performances, it is now well established that the $\mathcal{H}_\infty$ norm alone cannot yield satisfying robust controllers and that some of the structure of the uncertainty must be accounted for. In order to cope with this problem, several approaches such as μ-analysis and μ-synthesis methods [25, 63], $\mathcal{H}_2/\mathcal{H}_\infty$ theory [70], $\mathcal{H}_2$ theory [27], $\mathcal{L}_1$ theory (see [20] and references therein), and the loop shaping method [65], have been proposed.

We stress that all the aforementioned methodologies address the control of linear time invariant systems with dynamic uncertainty and structured dynamic uncertainty [i.e., systems with dynamic uncertainty where some of the structure of the uncertainty is taken into account]. For nonlinear systems with dynamic uncertainty, little theory is available. The nonlinear $\mathcal{H}_\infty$ control theory [76] is still a new area that has not yet proven to be successful in the design of controllers. Moreover, this method does not provide stabilization of each one of the systems of the family. There exist other approaches such as Lyapunov methods [17, 35, 36]. However, either the resulting controllers produced by these last methods are discontinuous and do not guarantee the existence of solution to the closed-loop differential equation, or stabilization of each one of the systems of the uncertain family is not ensured.

Research on systems with parametric uncertainty has evolved differently from that on systems with dynamic uncertainty, and has focussed on obtaining stability criteria rather than on constructing controllers. In 1983, the western control community discovered the remarkable paper [54] of Kharitonov where it is proved that the stability of certain parameterized families of polynomials is that of four distinguished polynomials of the families. Since then, this theorem has been an inspiration for most of the research on stability of systems with parametric uncertainty, and a great deal of papers presenting extensions, generalization, or new proofs of Kharitonov Theorem, have been published. One of the significant results that has followed is the Edge Theorem introduced by Barlett, Hollot and Lin [6]. This theorem reduces the problem of checking the stability of some polytopes of polynomials to that of checking the stability of their edges. The zero exclusion principle was then developed and several numerical tests [19, 23, 71] based on this concept, were provided. Systems with uncertain parameters lying in spherical bounded sets have also been considered [72], and a theory for linear systems with multilinear parameter perturbation has been initiated [7].

One of the major weakness of the theory on stabilization of uncertain systems with parametric uncertainty lies in the lack of theory for families of **nonlinear** systems. Further, because all the afore-mentioned results have been obtained for families of polynomials, it is not clear how to extend the previous concepts to the nonlinear setting.

Most of the research related to the control of finite families of systems addresses the simultaneous asymptotic stabilization of **linear** systems. This problem was first posed by Saeks and Murray [69] in 1982. The simultaneous asymptotic stabilization of two LTI systems by means of LTI feedback, was completely solved in [69] for single-input single-output systems and in Vidyasagar and Viswanadham [78] for multi-input multi-output systems, by using the Youla parameterization [81]. For more than three systems, the problem is considerably more complex and is still a topic of current interest. Although necessary and sufficient conditions for the simultaneous asymptotic stabilizability of three LTI systems are provided in Blondel et al. [12], Ghosh and Byrnes [31], and Vidyasagar and Viswanadham [78], it turns out that none of them is implementable. It is therefore not clear whether the stabilizability of three LTI systems by means of LTI feedback can be computationally decided. To date, this issue is still open. However, it is now known that a general algorithm for testing whether three LTI systems are simultaneously stabilizable by a LTI controller would require the computation of a transcendental function, which indicates its high computational complexity [9, 11].

To overcome the limitations of LTI controllers, the use of time-varying feedback laws and merely continuous feedback laws for simultaneous stabilization, has been investigated: Kabamba and Yang [48] established the simultaneous asymptotic stabilizability of finite families of LTI systems by means of open loop time-varying feedback laws which involve both the sampled output of the system and a periodic function of time. On the other hand, Zhang and Blondel [84] obtained sufficient conditions for the simultaneous asymptotic stabilizability of such families by controllers based on LTI feedback laws combined with zero-th order hold functions and samplers. While both these design procedures comprise a sampling scheme, Khargonekar et al. [52] introduced a method that does not involve any discretization strategy and proved that any finite family of stabilizable LTI systems can be simultaneously asymptotically stabilized by a periodic linear time varying feedback law which is **piecewise continuous**. Finally, Petersen [64] derived a necessary and sufficient condition for the simultaneous quadratic asymptotic stabilizability of single-input LTI systems by means of continuous feedback. Unfortunately, this condition is not really useful because it relies on a matrix that may be hard to find and for which no design procedure is given.

While families of linear systems have been the main focus of research on simultaneous asymptotic stabilization, the case of families of **nonlinear** systems has never been addressed in the literature. Also, to the best of our knowledge, there exists no published work on the simultaneous asymptotic stabilization of countably infinite families of systems.

It now clearly appears that the control of uncertain **nonlinear** systems is an area that begs for results. On the other hand, because the simultaneous stabilization of LTI systems by means of LTI feedback is a difficult issue that is mainly unsolved, we strongly feel that the investigation of alternative types of simultaneous stabilizers for such families should be pushed further.

1.2 Contributions

In this dissertation, our main focus is the stabilization and asymptotic stabilization of countable and parameterized families of systems. Our contributions are four-fold.

- First, we introduce a new method to interpolate feedback laws, that enables us to prove that given any **finite or countably infinite** family of general nonlinear systems with continuous vector-field, there exists a merely continuous feedback law that simultaneously stabilizes (not asymptotically) the family, if each system of the family is asymptotically stabilizable by means of continuous feedback.

  We actually find two feedback laws that solve the simultaneous stabilization problem. The first one depends on a partition of unity while the second one is simpler and does not involve a partition of unity.

  In case the systems of the family are globally asymptotically stabilizable, we establish the existence of a continuous feedback law which not only simultaneously stabilizes the family but also yields boundedness of the trajectories of the corresponding closed-loop systems starting at any initial state in $\mathbb{R}^n$.

  We stress that these result are new and that our constructions do not provide simultaneous asymptotic stabilization. However, the idea behind these constructions will prove to be useful for the design of controllers that achieve simultaneous asymptotic stabilization.

- Indeed, by introducing time-varying feedback laws and by modifying the construction used to derive the previous results, we are then able to prove that given any finite family of LTI systems that are individually stabilizable by means of LTI feedback, there exists a **continuous** time-varying feedback law which simultaneously globally exponentially stabilizes the family.

Because our approach does not involve any discretization scheme as in [48, 84], our result should be compared to that of Khargonekar et al. [52]. In this paper, it was proved that any finite family of stabilizable LTI systems is simultaneously stabilizable by a periodic linear time varying feedback law which is **piecewise continuous**. Although the controller that we derive is nonlinear it offers the advantage of being **continuous**. Moreover, while the controller proposed in [52] is not explicit and is obtained through an iterative procedure, the controller that we find in this dissertation is given by an explicit formula. Finally, we also obtain a lower bound on the exponential rates of convergence of the closed-loop systems while the authors of [52] do not study the rates of convergence of the closed-loop systems that they obtain.

We then consider countably infinite families of LTI systems that are asymptotically stabilizable by LTI feedback laws and derive sufficient conditions for their asymptotic stabilizability by means of **continuous** time-varying feedback.

- By extending the previous approach to the nonlinear setting, we are able to give sufficient conditions for the simultaneous local and global asymptotic stabilizability of finite families of **nonlinear** systems by means of continuous time-varying feedback laws. We then use these conditions in order to establish the simultaneous asymptotic stabilizability of the elements of a class of pairs of homogeneous nonlinear control systems.

  We point out that the simultaneous asymptotic stabilization of families of **nonlinear** systems has not been previously addressed in the literature.

- Finally, we introduce a new approach to the robust asymptotic stabilization of parameterized families of systems. We view this problem as a "simultaneous design" in the sense that a robust asymptotic stabilizer is considered as a feedback law that **simultaneously robustly asymptotically stabilizes** a finite number of sub-families of the original family. By using this approach and by modifying the construction that enables us to establish the simultaneous stabilizability of countable families of systems, we then design merely continuous robust asymptotic stabilizers for some parameterized families of nonlinear systems in the plane, that do not admit any C^1 robust stabilizer.

  To the best of our knowledge, there exists no previous result on robust asymptotic stabilization (in this sense) of nonlinear systems by means of continuous feedback.

One of the novelties of our work lies in the fact that we consider very general systems with possibly merely continuous dynamics that may be asymptotically stabilizable by merely continuous feedback laws. We also use controllers which are merely continuous (time-varying or time-invariant). The notions of merely continuous control systems and the use of continuous feedback for stabilization purpose is a relatively new concept that has been clarified only since the late eighties [3, 21, 49]. While such notions have become quite popular in the field of nonlinear stabilization, it is not as well-spread in the robust control community, and we hope that this dissertation will help to fill the gap between these two fields.

1.3 Outline of the dissertation

This dissertation is organized as follows:

In Chapter 2, we consider finite families of nonlinear control systems with continuous dynamics, and assume that each one of these systems is asymptotically stabilizable by means of continuous feedback. We then establish the simultaneous stabilizability of such families by means of continuous feedback. The cases of families of globally asymptotically stabilizable systems as well as families of linear systems are also discussed.

By adapting the construction introduced in Chapter 2, we prove in Chapter 3, that any **countably infinite** family of stabilizable systems, is simultaneously stabilizable by means of continuous feedback. We then apply our results to families of globally asymptotically stabilizable systems and to families of LTI systems.

In Chapter 4, we introduce time-varying feedback and consider finite families of LTI systems that are individually asymptotically stabilizable by means of LTI feedback. By modifying the construction of Chapter 2, we prove that given any such family, there exists a continuous time-varying feedback law that simultaneously globally exponentially stabilizes this family. We then derive sufficient conditions for the existence of a time-varying feedback law that simultaneously globally asymptotically stabilizes a countably infinite family of LTI systems.

Chapter 5 contains our discussion on the simultaneous asymptotic stabilization of finite families of nonlinear systems. We derive sufficient conditions for the existence of a time-varying feedback law that simultaneously locally or globally asymptotically stabilizes such a family. By using these conditions we then prove the simultaneous asymptotic stabilizability of the elements of a class of pairs of homogeneous nonlinear control systems.

In Chapter 6, we establish the simultaneous asymptotic stabilizability of certain pairs of systems in the plane by means of continuous **time-invariant** feedback.

Finally, in Chapter 7, we derive a continuous time-invariant feedback law that robustly asymptotically stabilizes the families of a class of parameterized families of nonlinear systems in the plane that do not admit a C^1 robust asymptotic stabilizer.

Our conclusions and suggestions for future work are presented in Chapter 8.

1.4 Definitions

We now present several definitions that are used throughout this dissertation.

1.4.1 Definitions related to stability

For x in $\mathbb{R}^n$, we let $\|x\|$ denote its Euclidean norm, and for $r > 0$ and x_0 in $\mathbb{R}^n$ we let $B_r(x_0)$ denote the open ball $B_r(x_0) \triangleq \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$. Further, given an autonomous (resp. time-varying) dynamical system, we let $x(\cdot, x_0)$ (resp. $x(\cdot, x_0, t_0)$) denote any one of its trajectories that starts at x_0 at time $t = 0$ (resp. $t = t_0$), for each x_0 in $\mathbb{R}^n$ (resp. for each x_0 in $\mathbb{R}^n$ and each $t_0 \geq 0$).

Definition 1.1 (Stability of autonomous systems) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping with $f(0) = 0$. The system $(S) : \dot{x} = f(x)$ is*

i) *stable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t \geq 0$ and each solution $x(., x_0)$ of (S), we have $\|x(t, x_0)\| < \varepsilon$ whenever x_0 lies in $B_\delta(0)$.*

ii) *is locally asymptotically stable if it is stable and if there exists $\bar{\delta} > 0$ such that each solution $x(., x_0)$ of (S) satisfies $x(t, x_0) \to 0$ as $t \to \infty$ whenever $\|x_0\| < \bar{\delta}$.*

iii) *is globally asymptotically stable if* **(ii)** *holds for all $\bar{\delta} > 0$.*

Definition 1.2 (Stability of time-varying systems) *Let $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that $f(t, 0) = 0$ for each $t \geq 0$, and let $x(\cdot, x_0, t_0)$ denote the trajectory of the system $(S) : \dot{x} = f(t, x)$ for any given x_0 in $\mathbb{R}^n$ and $t_0 \geq 0$. The system (S) is*

i) *stable, if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t, x_0, t_0)\| < \varepsilon, \quad t \geq t_0, \quad x_0 \in B_{\delta(\varepsilon, t_0)}(0).$$

ii) *uniformly stable if* **(i)** *holds with* δ *independent of* t_0.

iii) *locally asymptotically stable, if it is stable according to* **(i)** *and for each* $t_0 \geq 0$, *there exists* $\bar{\delta}(t_0) > 0$ *such that*

$$\lim_{t \to +\infty} x(t, x_0, t_0) = 0, \quad x_0 \in B_{\bar{\delta}(t_0)}(0).$$

iv) *locally uniformly asymptotically stable if it is uniformly stable according to* **(ii)**, *and there exists* $\bar{\delta} > 0$ *independent of* t_0 *such that* $x(t, x_0, t_0)$ *converges uniformly in* x_0 *and* t_0 *to the origin as* t *tends to* $+\infty$, *for each* x_0 *in* $B_{\bar{\delta}}(0)$ *and each* $t_0 \geq 0$.

v) *locally exponentially stable if there exist some positive reals* γ, $\bar{\delta}$, *and* L *such that*

$$\|x(t, x_0, t_0)\| \ \leq \ L\|x_0\|e^{-\gamma(t-t_0)}, \quad t \geq t_0,$$

for each $t_0 \geq 0$ *and each* x_0 *in* $B_{\bar{\delta}}(0)$.

vi) *locally uniformly stable with exponential (uniform) convergence, if there exist some positive reals* γ *and* $\bar{\delta}$, *and a mapping* $h : (0, \infty) \to (0, \infty)$ *such that* $\lim_{r \to 0^+} h(r) = 0$ *and*

$$\|x(t, x_0, t_0)\| \ \leq \ h(\|x_0\|)e^{-\gamma(t-t_0)}, \quad t \geq t_0,$$

for each $t_0 \geq 0$ *and each* x_0 *in* $B_{\bar{\delta}}(0)$.

If Definition 1.2 **(iii)**, **(iv)**, **(v)**, **(vi)** respectively hold for all $\bar{\delta} > 0$, the system (S) is said to be respectively globally asymptotically stable, globally uniformly asymptotically stable, globally exponentially stable, and globally uniformly asymptotically stable with exponential convergence.

Both Definition 1.1 and 1.2 can be found in [77] except for Definition 1.2 **(vi)**. This last definition is more general than that of exponential stability but reduces to this concept if the mapping h satisfies $h(r) \leq \alpha r$ for each $r > 0$ close to 0. Here, we allow more general mappings h, but the fundamental idea of uniform (in t_0) exponential convergence is preserved. Furthermore, it is plain that the requirement $\lim_{r \to 0^+} h(r) = 0$ yields uniform stability.

We now let $\mathcal{I}$ be a countable set (finite or infinite) and we let $\{S_i, \ i \in \mathcal{I}\}$ be a collection of control systems

$$S_i : \quad \dot{x} = f_i(x, u),$$

where the state x is in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$ and $f_i(0, 0) = 0$.

Definition 1.3 (Stabilization) *Let i be in $\mathcal{I}$.*

i) *A feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$ such that $u(0) = 0$,* **stabilizes** *(resp.* **asymptotically stabilizes***) the system S_i if the closed-loop system $\dot{x} = f_i(x, u(x))$ is* **stable** *according to Definition 1.1* **(i)** *(resp. locally asymptotically stable according to Definition 1.1* **(ii)***).*

ii) *A feedback law $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$ such that $u(t, 0) = 0$ for all $t \geq 0$, respectively locally asymptotically, locally uniformly asymptotically, and locally exponentially stabilizes S_i if the closed-loop system $\dot{x} = f_i(x, u(t, x))$ is respectively locally asymptotically, locally uniformly asymptotically, and locally exponentially stable according to Definition 1.2* **(iii)**, **(iv)**, *and* **(v)**, *respectively.*

We emphasize that in Definition 1.3 **(i)**, we use the word **"stable"** in the basic sense, i.e, stable according to Definition 1.1 **(i)**, while in the control theory literature it is commonly used to denote "locally asymptotically stable" according to Definition 1.1 **(ii)**. In the sequel, we will often omit the term "locally" and unless otherwise stated "asymptotically stable" will mean "locally asymptotically stable".

Definition 1.4 (Simultaneous stabilization) *A feedback law u simultaneously stabilizes (resp. asymptotically stabilizes) the family of control systems $\{S_i,\ i \in \mathcal{I}\}$, if u stabilizes (resp. asymptotically stabilizes) each one of the system of the family, according to the appropriate definition.*

We now let Γ be an uncountable subset of $\mathbb{R}^k$ and we let $\{S(\gamma),\ \gamma \in \Gamma\}$ be a parameterized family of control system

$$S(\gamma) : \ \dot{x} \ = \ f_\gamma(x, u),$$

where the state x is in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$ and $f_\gamma(0) = 0$ for each γ in Γ.

Definition 1.5 (Robust stabilization) *A feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$ robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$, if it is independent of the parameter γ and if it locally asymptotically stabilizes the system $S(\gamma)$ for each γ in Γ.*

1.4.2 Miscellaneous

Given a positive definite matrix M, we let $\lambda_{min}(M)$ and $\lambda_{max}(M)$, denote respectively its smallest and largest eigenvalues. As usual, the infimum of a mapping over the empty set is taken to be equal to $+\infty$. Further, we let $\mathbb{R}$ and $\mathbb{Z}$ denote the set of reals and integers respectively.

For a given integer $k \geq 1$ and two Banach spaces X and Y, a mapping $f : X \to Y$, is said to be C^k if it is k times continuously differentiable on X. Further f is said to be C^0 if it is continuous on X, while it is said to be "smooth" or equivalently C^∞ if it is C^k for all $k \geq 0$.

Assume that U is a neighborhood of the origin in $\mathbb{R}^n$, and let $k \geq 0$ be an integer. A mapping $f : U \to Y$ is said to be almost C^k if it is C^k on $U \backslash \{0\}$.

Because, we extensively use the concept of Lyapunov functions, we now define them.

Definition 1.6 (Lyapunov function) *Let U be a neighborhood of the origin in $\mathbb{R}^n$. A mapping $V : U \to [0, \infty)$ is said to be a Lyapunov function if it is C^1 and if the following holds :*

- $V(x) = 0 \;\Leftrightarrow x = 0.$
- *There exists a continuous mapping $f : U \to \mathbb{R}^n$ such that $f(0) = 0$ and $\nabla V(x) f(x) < 0$ for each x in $U \backslash \{0\}$.*

If in addition $U = \mathbb{R}^n$ and $V(x)$ converges to $+\infty$ as $\|x\|$ tends to $+\infty$, the Lyapunov function V is said to be radially unbounded.

Throughout, we consider solely time-invariant feedback laws $v : \mathbb{R}^n \to \mathbb{R}^m$ (resp. time-varying feedback laws $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$) such that we have $u(0) = 0$ (resp. $v(t, 0) = 0$ for each $t \geq 0$). This is a usual and natural requirement imposed on feedback laws that are used for stabilization purpose. Indeed, if the feedback law v stabilizes or asymptotically stabilizes a control system $\dot{x} = f(x, u)$ satisfying $f(0, 0) = 0$, then the previous assumptions on v means that no control energy is necessary in order to maintain the corresponding closed-loop system at the the origin once it has reached this state.

Finally, recall that if $f : \mathbb{R}^n \to \mathbb{R}^n$ (resp. $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$) is continuous, then for each x_0 in $\mathbb{R}^n$ and each t_0 in $[0, \infty)$, the differential equation $\dot{x} = f(x)$ (resp. $\dot{x} = f(t, x)$) admits a solution (possibly not unique) that starts from x_0 at time t_0 [38, pp. 10].

Throughout, we always consider control systems $\dot{x} = f(x, u)$ where the mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin, together with time-invariant (resp. time-varying) feedback laws $v : \mathbb{R}^n \to \mathbb{R}^m$ (resp. $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$) such that there exists a neighborhood of the origin U

with v continuous on D (resp. on $[0,\infty)\times D$). Therefore, we are ensured that given x_0 in some neighborhood of the origin and any $t_0 \geq 0$, the closed-loop system $\dot{x} = f(x, v(x))$ (resp. $\dot{x} = f(x, v(t,x))$) admits at least one solution that starts from x_0 (resp. from x_0 at time t_0).

We now define the concept of a partition of unity, which will be the technical key to the derivation of our results. This definition can be found in [79, pp. 8].

Definition 1.7 (Partition of unity) *Let M be a differentiable manifold. A partition of unity on M is a collection $\{p_\alpha,\ \alpha \in A\}$ of C^∞ functions defined from M into $[0,1]$ [where A is an arbitrary index set, not assumed countable] such that*

- *The collection of support $\{support(p_\alpha),\ \alpha \in A\}$ is locally finite.*
- $\sum_{\alpha\in A} f_\alpha \equiv 1$ *on M.*

A partition of unity $\{p_\alpha,\ \alpha \in A\}$ is subordinate to an open cover $\{U_\beta,\ \beta \in B\}$ of a differentiable manifold, if for each α in A there exists β in B such that the support of p_α is included in U_β.

The following theorem can be found in [79, pp. 10].

Theorem 1.1 *Let M be a differentiable manifold, and let $\{U_\alpha,\ \alpha \in A\}$ be an open cover of M. Then, there exists a partition of unity $\{p_\alpha,\ \alpha \in A\}$ subordinate to the cover $\{U_\alpha,\ \alpha \in A\}$ such that the support of p_α is included in U_α, for each α in A.*

We finally introduce a notation that is extensively used throughout this dissertation.

Definition 1.8 *Let $\{x_m,\ m = \ldots,-1,0,1,\ldots\}$ be a sequence of positive integers. Further, for each $i = 1,\ldots,x_n$ and each n in $\mathbb{Z}$, let Q_i^n belong to a given class of mathematical objects. Then, $\{Q_i^n,\ i = 1,\ldots,x_n\}_{n=1}^{\infty}$, $\{Q_i^n,\ i = 1,\ldots,x_n\}_{n=-1}^{-\infty}$, and $\{Q_i^n,\ i = 1,\ldots,x_n\}_{n\in\mathbb{Z}}$ denote respectively the sequences*

$$Q_1^1,\ \ldots,\ Q_{x_1}^1,\ Q_1^2,\ \ldots,\ Q_{x_2}^2,\ Q_1^3,\ \ldots$$
$$\ldots,\ Q_{x_{-3}}^{-3},\ Q_1^{-2},\ \ldots,\ Q_{x_{-2}}^{-2},\ Q_1^{-1},\ \ldots,\ Q_{x_{-1}}^{-1},$$

and

$$\ldots, Q_1^{-1},\ \ldots,\ Q_{x_{-1}}^{-1}, Q_1^0, \ldots, Q_{x_0}^0, Q_1^1,\ \ldots,\ Q_{x_1}^1, \ldots.$$

Chapter 2

Simultaneous Stabilization of Finite Families of Nonlinear Systems

In this chapter, we show that if each system of a finite family of **general nonlinear** systems is asymptotically stabilizable by means of continuous feedback, then there exists a continuous feedback law that simultaneously stabilizes (not asymptotically) the family. For any such family, we construct a simultaneous stabilizer through a new method for interpolating feedback laws. Given a sequence of nested sets $\{U_i^n, i = 1, \ldots, I\}_{n=1}^{\infty}$, and a collection of mappings $\{u_i : \mathbb{R}^n \to \mathbb{R}^m,\ i = 1, \ldots, I\}$, this method enables us to design a **continuous** mapping $v : \mathbb{R}^n \to \mathbb{R}^m$ which is equal to u_i on the boundary ∂U_i^n for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$.

For any finite family of stabilizable systems, we find two feedback laws that solve the simultaneous stabilization problem. The first one is constructed in Section 2.2 and depends on a partition of unity. Because this partition of unity might be difficult to express, we present in Section 2.3 a simpler simultaneous stabilizer that does not involve a partition of unity. In case the systems of the family are globally asymptotically stabilizable, we derive a feedback law that not only simultaneously stabilizes the family, but also yields boundedness of the trajectories of the corresponding closed-loop systems. On the other hand, if the systems of the family are LTI and asymptotically stabilizable by means of LTI feedback, we provide a simple procedure to construct a simultaneous stabilizer. Finally, some technical lemmas are presented in Section 2.4.

2.1 Problem definition

Throughout this chapter, we consider a family $\{S_i,\ i = 1, \ldots, I\}$ of systems

$$S_i : \quad \dot{x} \;=\; f_i(x, u), \quad i = 1, \ldots, I, \tag{2.1}$$

where $I \geq 2$ is an integer, the state x lies in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$, and for each $i = 1, \ldots, I$, the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin with $f_i(0,0) = 0$.

Recall that we use the word "stable" in the basic sense, i.e, stable according to Definition 1.1 **(i)**, while in the control theory literature it is commonly used to denote "locally asymptotically stable" according to Definition 1.1 **(ii)**.

The purpose of this chapter is to discuss the simultaneous stabilization of the family $\{S_i,\ i = 1, \ldots, I\}$, according to Definition 1.4 and under the assumption that for each $i = 1, \ldots, I$, the system S_i is locally asymptotically stabilizable by means of continuous and almost C^k feedback [for some k in $\{0, 1, \ldots\}$].

The originality of this problem is four-fold. First, we discuss the simultaneous stabilization whereas all the studies on finite families of systems usually address their simultaneous asymptotic stabilization. Secondly, we consider families of **nonlinear** systems while all the studies on simultaneous asymptotic stabilization have focussed on families of **linear systems** (see [10] for a recent survey on this topic). Thirdly, we only impose that the dynamics of the systems S_i be continuous and that the systems S_i be asymptotically stabilizable by means of continuous feedback. Fourthly, we use merely continuous controllers.

Although simultaneous stabilization is a topic that has not been addressed in the literature, we feel that studying this problem may yield a new insight into the simultaneous **asymptotic** stabilization problem and that it is an issue worth raising. In this dissertation, this approach proves to be successful, and the method introduced here will turn out to play a key role in the derivation of most of our results on simultaneous asymptotic stabilization. We also believe that simultaneous stabilizers may be useful for "practically" stabilizing families of systems that cannot be simultaneously asymptotically stabilized. For example, in case the systems S_i are globally asymptotically stabilizable, the controller of Section 2.3.2 yields boundedness of the trajectories of the corresponding closed-loop systems and may be used in order to avoid the divergence of a system resulting from a failure.

Because we consider systems that are stabilizable by means of possibly merely continuous feedback, it is natural to seek a merely continuous simultaneous stabilizer. Thus, the results from simultaneous asymptotic stabilization of linear systems by linear feedback laws are of little help in addressing our problem, and we draw our inspiration from recent results obtained in the context of stabilization of nonlinear systems [see [5] for an overview].

The use of merely continuous controllers for stabilization purpose can be traced back to the work of Sussmann [74] and Artstein [4], but it is only since the late eighties that this idea has become clearer throughout the papers by Tsinias [75], Sontag [73], Kawski [49], and Dayawansa et al. [21]. In these papers, surprisingly general results were obtained, namely in the context of stabilization of low dimensional systems. It is therefore natural to hope for interesting results when applying merely continuous feedback laws to simultaneous stabilization problems.

The only published work that addresses the simultaneous **asymptotic** stabilization by means of merely continuous feedback is Petersen [64]. In this paper, a necessary and sufficient condition for the simultaneous quadratic asymptotic stabilization of single-input linear systems by means of continuous feedback is derived. Unfortunately, this condition is not really useful because it depends on a matrix that may be hard to find, and for which no design procedure is given.

The purpose of the next section is to prove the following theorem whose proof is postponed to Subsection 2.2.2.

Theorem 2.1 *Let k be in $\{0, 1, \ldots\}$. Assume that for each $i = 1, \ldots, I$, there exists a feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ which is continuous and almost C^k on some neighborhood of the origin, and which locally asymptotically stabilizes the system S_i. Then, there exists a feedback law $v : \mathbb{R}^n \to \mathbb{R}^m$, which is continuous and almost C^k on a neighborhood of the origin, and which simultaneously stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

This somewhat surprisingly general result is proved through a rather simple proof that nevertheless yields a new insight into the simultaneous stabilization and asymptotic stabilization problems.

2.2 Simultaneous stabilization

In this section we present a proof of Theorem 2.1. The general lines of the proof are as follows: For each $i = 1, \ldots, I$, we let V_i denote a Lyapunov function for the system $\dot{x} = f_i(x, u_i(x))$. We define a sequence of neighborhoods of the origin $\{U_i^n, i = 1, \ldots, I\}_{n=1}^{\infty}$ such that for each $i = 1, \ldots, I$ the boundaries of the sets

U_i^n, $n = 1, 2, \ldots$ are level sets of V_i. Then, we design a continuous feedback law v which is equal to u_i on the boundaries of the sets U_i^n, $n = 1, 2, \ldots$. It follows that for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, the set $\overline{U_i^n}$ is invariant with respect to the system $\dot{x} = f_i(x, v(x))$. We conclude that for each $i = 1, \ldots, I$, the feedback law v stabilizes S_i upon noting that the family $\{U_i^n\}_{n=1}^{\infty}$ is a base at the origin.

The following lemma will be the key for proving the invariance of sets $\overline{U_i^n}$.

2.2.1 Invariance criteria

The next lemma is the main argument to prove the invariance of the set $\overline{U_i^n}$. It is also used in the proofs of Proposition 2.1 and Theorem 2.2, and in the next chapter.

Lemma 2.1 *Let D be a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to [0, \infty)$ be a Lyapunov function. Let $f : \overline{D} \to \mathbb{R}^n$ be a continuous mapping and let (S) denote the system $\dot{x} = f(x)$. Further, let β be in $(0, \inf_{x \in \partial D} V(x))$ and let W^β denote the set $W^\beta \triangleq D \cap V^{-1}([0, \beta))$. Finally, assume that*

$$\nabla V(x) f(x) < 0, \quad x \in \partial W^\beta. \tag{2.2}$$

Then, the set $\overline{W^\beta}$ is invariant with respect to the system (S).

Proof: We prove the lemma by a contradiction argument: Assume that $\overline{W^\beta}$ is not invariant with respect to (S). Then, there exists x_0 in $\overline{W^\beta}$ such that the trajectory $x(\cdot, x_0)$ of (S) starting from x_0 at time $t = 0$, does not remain forever in $\overline{W^\beta}$. By combining Lemma B.3 (ii) (with D, V and W^β) with the fact that $\beta < \inf_{x \in \partial D} V(x)$, we exhibit $\widehat{t} \geq 0$ and $\widehat{h} > 0$ such that

$$x(\widehat{t}, x_0) \in \partial W^\beta, \qquad V(x(\widehat{t}, x_0)) = \beta, \tag{2.3}$$

and for each h in $(0, \widehat{h})$,

$$x(\widehat{t} + h, x_0) \notin V^{-1}([0, \beta]) \qquad \text{with} \qquad x(\widehat{t} + h, x_0) \in D. \tag{2.4}$$

From (2.2) and (2.3) we easily get

$$\frac{\partial V(x(t, x_0))}{\partial t}\Big|_{t=\widehat{t}} = \nabla V(x(\widehat{t}, x_0)) f(x(\widehat{t}, x_0)) < 0.$$

Thus, in view of (2.3), there exists $\widetilde{h}$ in $(0, \widehat{h}]$ such that

$$V(x(\widehat{t} + h, x_0)) < V(x(\widehat{t}, x_0)) = \beta, \quad h \in (0, \widetilde{h}), \tag{2.5}$$

a contradiction with the strict inequality

$$V(x(\widehat{t}+h,x_0)) \;>\; \beta, \quad h \in (0,\widehat{h}),$$

which follows from (2.4). The lemma then follows by contradiction. ■

We note that the proof of this lemma does not hold if we merely assume that

$$\nabla V(x)\, f(x) \leq 0, \quad x \in \partial W^{\beta},$$

instead of (2.2). Indeed, this new assumption does not yield the companion relation of (2.5), namely

$$V(x(\widehat{t}+h,x_0)) \;\leq\; V(x(\widehat{t},x_0)) \;=\; \beta, \quad h \in (0,\widetilde{h}),$$

and we do not obtain the invariance of the set $\overline{W}_{\beta}$.

We finally prove Theorem 2.1 in the next section.

2.2.2 A proof of Theorem 2.1

We are now able to prove Theorem 2.1.

Proof Theorem 2.1 :

Recall that the infimum of a real-valued mapping over the empty set is $+\infty$.

Construction of the simultaneous stabilizer :

Let U_i be a neighborhood of the origin such that the mappings $f_i(\cdot, u_i(\cdot))$ and u_i are continuous on U_i, with u_i almost C^k on U_i for each $i = 1,\ldots,I$. By the Converse Lyapunov Theorem [56] and the local asymptotic stability of $\dot{x} = f_i(x, u_i(x))$, there exist a bounded neighborhood of the origin $D_i \subset U_i$ and a Lyapunov function $V_i : D_i \to [0,\infty)$, satisfying

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in D_i \backslash \{0\},$$

for each $i = 1,\ldots,I$. Let D be a bounded neighborhood of the origin $\overline{D} \subset \bigcap_{i=1}^{I} D_i$, so that we have

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in \overline{D} \backslash \{0\}, \quad i = 1,\ldots,I. \tag{2.6}$$

For each $i = 1, \ldots, I$ and each $\beta > 0$, we define W_i^β by setting

$$W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\},$$

and we construct a sequence of neighborhoods of the origin as follows: By applying Lemma 2.4 (with D, $V_1, \ldots, V_I$), we obtain a sequence of positive reals $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$ satisfying

$$\beta_i^n < \inf_{x \in \partial D} V_i(x), \quad n = 1, 2, \ldots, \quad i = 1, \ldots, I, \tag{2.7}$$

$$\beta_i^n \to 0 \text{ as } n \to \infty, \quad i = 1, \ldots, I, \tag{2.8}$$

with

$$W_I^{\beta_I^n} \supset \overline{W}_1^{\beta_1^{n+1}}, \quad n = 1, 2, \ldots, \tag{2.9}$$

$$W_{i-1}^{\beta_{i-1}^n} \supset \overline{W}_i^{\beta_i^n}, \quad i = 2, \ldots, I, \quad n = 1, 2, \ldots. \tag{2.10}$$

Upon setting

$$U_i^n \triangleq W_i^{\beta_i^n}, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots,$$

the inclusions (2.9) and (2.10) translate to

$$U_I^n \supset \overline{U}_1^{n+1} \quad \text{and} \quad U_{i-1}^n \supset \overline{U}_i^n, \quad i = 2, \ldots, I, \quad n = 1, 2, \ldots. \tag{2.11}$$

From (2.7) and Lemma B.3 (i), we get $\overline{U}_1^1 \subset D$ and we therefore have a sequence of nested neighborhoods

$$\begin{array}{ccccccccc} D & \supset & U_1^1 & \supset & \ldots & \supset & U_I^1 & \supset \\ & & U_1^2 & \supset & \ldots & \supset & U_I^2 & \supset \\ & & \vdots & \vdots & & & \vdots & \vdots \end{array}$$

such that each neighborhood contains the closure of the neighborhood that follows. We now define the set Δ_i^n for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$ by setting

$$\Delta_1^1 \triangleq D \backslash \overline{U}_2^1, \quad \text{and} \quad \Delta_1^n \triangleq U_I^{n-1} \backslash \overline{U}_2^n, \quad n = 2, 3, \ldots$$

$$\Delta_I^n \triangleq U_{I-1}^n \backslash \overline{U}_1^{n+1}, \quad n = 1, 2, \ldots,$$

and

$$\Delta_i^n \triangleq U_{i-1}^n \backslash \overline{U}_{i+1}^n, \quad i = 2, \ldots I - 1, \quad n = 1, 2, \ldots,$$

if $I \geq 3$. From (2.8), it is readily seen that for each $i = 1, \ldots, I$, the family $\{U_i^n\}_{n=1}^\infty$ is a base at the origin so that $\{U_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$ is also a base at the origin. This combined with the inclusions (2.11) and Lemma B.4 implies that $\{\Delta_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$ is an open cover of $D \backslash \{0\}$. Thus, by Theorem 1.1

there exists a partition of unity $\{q_i^n,\ i=1,\ldots,I\}_{n=1}^{\infty}$ subordinate to the open cover $\{\Delta_i^n,\ i=1,\ldots,I\}_{n=1}^{\infty}$ and such that the support of q_i^n is included in Δ_i^n for each $i=1,\ldots,I$ and each $n=1,2,\ldots$. We now define the feedback law $v : D \to \mathbb{R}^m$ by setting

$$v(x) = \begin{cases} 0, & x = 0 \\ \sum_{n=1}^{\infty}\sum_{i=1}^{I} u_i(x)\, q_i^n(x), & x \in D\backslash\{0\} \end{cases} .$$

Before proving that v simultaneously stabilizes $\{S_i,\ i=1,\ldots,I\}$, we study the regularity of v around the origin.

The mapping v is almost C^k and continuous at the origin :

Let x be in $D\backslash\{0\}$ and let r be in $(0, \|x\|)$ satisfying $\overline{B_r(0)} \subset D$. Because $\{U_1^n\}_{n=1}^{\infty}$ is a base at the origin composed of nested neighborhoods, there exists an integer n_r such that

$$U_1^n \subset \overline{B_r(0)}, \quad n = n_r+1, n_r+2, \ldots .$$

It follows from the definition of v together with the fact that the support of q_i^n is included in U_i^n for each $i=1,\ldots,I$, and each $n=1,2,\ldots$, that

$$v(y) = \sum_{n=1}^{n_r}\sum_{i=1}^{I} u_i(y) q_i^n(y), \quad y \in D\backslash\overline{B_r(0)}. \tag{2.12}$$

Because the mappings u_i and q_i^n are C^k on $D\backslash\{0\}$ for each $i=1,\ldots,I$ and each $n=1,2,\ldots$, we obtain from (2.12) that v is C^k on $D\backslash\{0\}$. Furthermore, the mappings q_i^n summing up to 1, we get

$$\|v(x)\| \leq \max(\|u_1(x)\|, \ldots, \|u_I(x)\|), \quad x \in D,$$

and continuity of v at the origin follows from that of the mappings $u_i, i=1,\ldots,I$.

Stability :

From the definitions of the sets U_i^n and Δ_i^n, it is not hard to see that for each $i=1,\ldots,I$, and each $n=1,2,\ldots$, the boundary ∂U_i^n is included in Δ_i^n and does not intersect with any other set Δ_j^m. Thus, because the support of the mapping q_i^n is included in Δ_i^n for each $i=1,\ldots,I$ and each $n=1,2,\ldots$, it follows from the definition of v that

$$v(x) = u_i(x), \quad x \in \partial U_i^n, \quad i=1,\ldots,I, \quad n=1,2,\ldots .$$

This, together with (2.6) and the fact that $\overline{U_i^n}$ is included in D for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, yield

$$\nabla V_i(x)\, f_i(x, v(x)) \;<\; 0, \quad x \in \partial U_i^n, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots . \tag{2.13}$$

For each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, by combining (2.13) with Lemma 2.1 applied with D, V_i, $f_i(\cdot, v(\cdot))$ and β_i^n, we obtain that the set $\overline{U_i^n}$ is invariant with respect to the system $\dot{x} = f_i(x, v(x))$.

We now fix $i = 1, \ldots, I$ and we let $\varepsilon > 0$ be given. Because the family $\{U_i^n\}_{n=1}^{\infty}$ is a base at the origin, there exists a positive integer $\bar{n}$ satisfying

$$\overline{U_i^{\bar{n}}} \;\subset\; B_\varepsilon(0).$$

Let $\delta > 0$ be such that $B_\delta(0) \subset \overline{U_i^{\bar{n}}}$. Then, the invariance of the set $\overline{U_i^{\bar{n}}}$ with respect to the system $\dot{x} = f_i(x, v(x))$ implies that each trajectory of this system starting in $B_\delta(0)$ remains in $B_\varepsilon(0)$ forever. In short the feedback law v stabilizes the system S_i. The proof of the theorem is complete upon noting that the previous argument holds for each $i = 1, \ldots, I$.

■

In view of the comment that follows Lemma 2.1, it is easily seen that for the previous construction to be valid, it is necessary that

$$\nabla V_i(x)\, f_i(x, v(x)) \;<\; 0, \quad x \in \partial U_i^n, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots .$$

Therefore, if the system S_i is merely stabilizable (not asymptotically) for each $i = 1, \ldots, I$, then the previous construction does not yield simultaneous stabilizability of the family $\{S_i,\; i = 1, \ldots, I\}$.

We now consider the feedback law v obtained in the previous proof and we show that we can replace the mappings of the partition of unity $\{q_i^n,\; i = 1, \ldots, I\}_{n=1}^{\infty}$ that appear in the expression of v, by simpler mappings.

2.3 A simple expression for a simultaneous stabilizer

Our goal in this section is to show that we can circumvent the computation of the partition of unity $\{q_i^n,\; i = 1, \ldots, I\}_{n=1}^{\infty}$ that appears in the expression of the simultaneous stabilizer constructed in the proof of Theorem 2.1. The general idea to construct this explicit simultaneous stabilizer is the following.

We first note that in the proof of Theorem 2.1, we do not actually need that the mappings of the partition of unity $\{q_i^n : \; i = 1, \dots, I\}_{n=1}^{\infty}$ sum up to 1. For this proof to carry over it suffices to have a collection of mappings $\{\bar{q}_i^n, \; i = 1, \dots, I\}_{n=1}^{\infty}$ that take non-negative values, that sum up to a real less than 1 and that satisfy

$$\bar{q}_i^n(x) = 1 \quad \text{and} \quad \bar{q}_j^m(x) = 0, \quad x \in \{x \in D : \; V_i(x) = \beta_i^n\}, \quad (j,m) \neq (i,n),$$

for each $i = 1, \dots, I$, and each $n = 1, 2, \dots$. Building upon these comments, we construct a more explicit simultaneous stabilizer for the family $\{S_i : \; i = 1, \dots, I\}$. Then, in case S_i and u_i are linear, we exhibit a simple design procedure that yields this explicit feedback law.

2.3.1 Derivation of the stabilizer

Throughout this subsection, we let k be in $\{0, 1, \dots\}$, and we consider the family of systems $\{S_i, \; i = 1, \dots, I\}$ as defined in (2.1). We assume that for each $i = 1, \dots, I$, there exists a feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ which is continuous and almost C^k on a neighborhood of the origin, and which locally asymptotically stabilizes the system S_i. Our goal is to construct a simultaneous stabilizer $\overline{v} : \mathbb{R}^n \to \mathbb{R}^m$ for the family $\{S_i, \; i = 1, \dots, I\}$, which does not involve a partition of unity.

Let U_i be a neighborhood of the origin such that the mappings $f_i(\cdot, u_i(\cdot))$ and u_i are continuous on U_i, with u_i almost C^k on U_i for each $i = 1, \dots, I$. By the Converse Lyapunov Theorem [56] and the local asymptotic stability of $\dot{x} = f_i(x, u_i(x))$, there exist a neighborhood of the origin $D_i \subset U_i$ and a C^{k_i} Lyapunov function $V_i : D_i \to [0, \infty)$ [where $k_i \geq 1$ is an integer], satisfying

$$\nabla V_i(x) \, f_i(x, u_i(x)) \; < \; 0, \quad x \in D_i \backslash \{0\},$$

for each $i = 1, \dots, I$. Let D be a bounded neighborhood of the origin satisfying $\overline{D} \subset \bigcap_{i=1}^{I} D_i$, so that we have

$$\nabla V_i(x) \, f_i(x, u_i(x)) \; < \; 0, \quad x \in \overline{D} \backslash \{0\}, \quad i = 1, \dots, I. \tag{2.14}$$

We let k' denote the integer

$$k' \triangleq \min(k, k_1, \dots, k_I),$$

and for each $\beta > 0$ and each $i = 1, \ldots, I$, we set $W_i^\beta \triangleq D \cap V_i^{-1}([0, \beta))$. By applying Lemma 2.5 (with D, $V_1, \ldots, V_I$), we obtain three sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$, and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$ such that we have

$$\alpha_i^n,\ \beta_i^n,\ \gamma_i^n \to 0 \ \textit{as}\ n \to \infty, \quad i = 1, \ldots, I, \tag{2.15}$$

and

$$\inf_{x \in \partial D} V_i(x) \ > \ \gamma_i^n \ > \ \beta_i^n \ > \ \alpha_i^n, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots, \tag{2.16}$$

together with

$$W_I^{\alpha_I^n} \ \supset \ \overline{W}_1^{\gamma_1^{n+1}}, \quad n = 1, 2, \ldots, \tag{2.17}$$

and

$$W_{i-1}^{\alpha_{i-1}^n} \ \supset \ \overline{W}_i^{\gamma_i^n}, \quad i = 2, \ldots, I, \quad n = 1, 2, \ldots. \tag{2.18}$$

For each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, we now define the mappings $\overline{q}_i^n : D \to [0, 1]$ by setting:

$$\overline{q}_i^n(x) = \begin{cases} e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\beta_i^n-\alpha_i^n)^2}} & \text{if } V_i(x) \in (\alpha_i^n, \beta_i^n) \\ e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\gamma_i^n-\beta_i^n)^2}} & \text{if } V_i(x) \in [\beta_i^n, \gamma_i^n) \\ 0\,, & \text{otherwise} \end{cases} . \tag{2.19}$$

Finally, we let $\overline{v} : D \to \mathbb{R}^m$ be given by

$$\overline{v}(x) \ = \ \sum_{n=1}^{\infty} \sum_{i=1}^{I} \overline{q}_i^n(x) u_i(x), \quad x \in D,$$

and we prove the following theorem.

Proposition 2.1 *The feedback law $\overline{v}$ is continuous and almost $C^{k'}$ on D, and simultaneously stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

Proof: First we note that (2.16), (2.17), and (2.18) yield a sequence of nested neighborhoods

$$\begin{array}{cccccccccccccc} D & \supset & W_1^{\gamma_1^1} & \supset & W_1^{\beta_1^1} & \supset & W_1^{\alpha_1^1} & \supset & W_2^{\gamma_2^1} & \supset & \ldots & \supset & W_I^{\alpha_I^1} & \supset \\ & & W_1^{\gamma_1^2} & \supset & W_1^{\beta_1^2} & \supset & W_1^{\alpha_1^2} & \supset & W_2^{\gamma_2^2} & \supset & \ldots & \supset & W_I^{\alpha_I^2} & \supset \\ & & \vdots & \vdots & & & & & \vdots & & & & \vdots & \vdots \end{array} \tag{2.20}$$

such that each neighborhood contains the closure of the neighborhood that follows. For each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, we let Π_i^n denote the set

$$\Pi_i^n \triangleq \{x \in D : \overline{q}_i^n(x) \neq 0\}.$$

In view of (2.20), the definition of the mapping $\overline{q}_i^n$ yields

$$\Pi_i^n = W_i^{\gamma_i^n} \backslash \overline{W_i^{\alpha_i^n}}, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots, \tag{2.21}$$

together with

$$\Pi_i^n \cap \Pi_j^m = \emptyset, \quad (i, n) \neq (j, m). \tag{2.22}$$

Let x be in $D \backslash \{0\}$ and let r be in $(0, \|x\|)$ satisfying $\overline{B_r(0)} \subset D$. Because $\{W_1^{\gamma_1^n}\}_{n=1}^{\infty}$ is a base at the origin composed of nested neighborhoods, there exists an integer n_r such that

$$W_1^{\gamma_1^n} \subset \overline{B_r(0)}, \quad n = n_r + 1, n_r + 2, \ldots.$$

It follows from the definition of $\overline{v}$ together with the fact that Π_i^n is included in $W_i^{\gamma_i^n}$ for each $i = 1, \ldots, I$, and each $n = 1, 2, \ldots$, that

$$\overline{v}(y) = \sum_{n=1}^{n_r} \sum_{i=1}^{I} u_i(y) \overline{q}_i^n(y), \quad y \in D \backslash \overline{B_r(0)}. \tag{2.23}$$

Because the mappings u_i and $\overline{q}_i^n$ are $C^{k'}$ on $D \backslash \{0\}$ for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$ [follows from Lemma B.6 applied to $\overline{q}_i^n$] we easily obtain from (2.23) that $\overline{v}$ is $C^{k'}$ on $D \backslash \{0\}$. Furthermore, (2.22) implies that

$$\|\overline{v}(x)\| \leq \max(\|u_1(x)\|, \ldots, \|u_I(x)\|), \quad x \in D,$$

and continuity of $\overline{v}$ at the origin follows from that of the mappings u_i, $i = 1, \ldots, I$.

Stability:

From (2.22) and the definition of the mappings $\overline{q}_i^n$, we deduce that for each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, we have

$$\overline{q}_i^n(x) = 1 \quad \text{with} \quad \overline{q}_j^m(x) = 0, \quad x \in \partial W_i^{\beta_i^n}, \quad (j, m) \neq (i, n),$$

and the definition of $\overline{v}$ yields

$$\overline{v}(x) = u_i(x), \quad x \in \partial W_i^{\beta_i^n}.$$

This, together with (2.14) imply that

$$\nabla V_i(x) f_i(x, \overline{v}(x)) < 0, \quad x \in \partial W_i^{\beta_i^n}, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots. \tag{2.24}$$

We now fix $i = 1, \ldots, I$. For each $n = 1, 2, \ldots$, by combining Lemma 2.1 with (2.24), we obtain that the set $\overline{W}_i^{\beta_i^n}$ is invariant with respect to the system $\dot{x} = f_i(x, \overline{v}(x))$ and stability of this system follows from the fact that $\{W_i^{\beta_i^n}\}_{n=1}^{\infty}$ is a base at the origin. The proof of the theorem is complete upon noting that the previous argument holds for each $i = 1, \ldots, I$. ■

The simultaneous stabilizer $\overline{v}$ does not rely anymore on a partition of unity. In fact, we have replaced the partition of unity that appears in the expression of v by more explicit mappings, that depend on three sequences of positive reals $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$, and $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$. To express these sequences, it is necessary to find, for each $i = 2, \ldots, I$, a condition on the two positive reals α and β, that yields the inclusion

$$D \cap V_{i-1}^{-1}([0, \alpha)) \supset D \cap V_i^{-1}([0, \beta]),$$

as well as a condition for the inclusion

$$D \cap V_I^{-1}([0, \alpha)) \supset D \cap V_1^{-1}([0, \beta])$$

to be satisfied. In case the system S_i and the feedback law u_i are linear for each $i = 1, \ldots, I$, as we shall see in Subsection 2.3.3, these conditions are well known and the feedback law $\overline{v}$ is therefore entirely explicit.

On the other hand if the feedback law u_i globally asymptotically stabilizes the system S_i for each $i = 1, \ldots, I$, the previous construction can be slightly modified in order to yield a feedback law $\bar{v}$ that simultaneously stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$ and such that for each $i = 1, \ldots, I$, all the trajectories of the closed loop system $\dot{x} = f_i(x, \bar{v}(x))$ are bounded.

2.3.2 Application to globally asymptotically stabilizable systems

We now consider the family $\{S_i,\ i = 1, \ldots, I\}$ of nonlinear systems as defined in (2.1) and we let k be in $\{0, 1, \ldots\}$. We assume that for each $i = 1, \ldots, I$, there exists a continuous and almost C^k feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ which globally asymptotically stabilizes S_i. Further, we assume that the mapping $f_i(\cdot, u_i(\cdot))$ is continuous on $\mathbb{R}^n$ for each $i = 1, \ldots, I$. Thus, by the Converse Lyapunov Theorem [56], there exists a C^k radially unbounded Lyapunov function $V_i : \mathbb{R}^n \to [0, \infty)$ for the system $\dot{x} = f_i(x, u_i(x))$. It is easy to adapt Lemma 2.5 (applied with D, $V_1, \cdots, V_I$) in order to obtain the following lemma.

Lemma 2.2 *Let $I \geq 2$ be an integer. For each $i = 1, \ldots, I$, let $V_i : \mathbb{R}^n \to [0, \infty)$ be a radially unbounded Lyapunov function and let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\}, \quad \beta > 0.$$

Then, there exist sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$ converging to 0 and $+\infty$ as n tends to $+\infty$ and $-\infty$ respectively and satisfying

$$\begin{array}{lll} W_I^{\alpha_I^n} & \supset \overline{W_1^{\gamma_1^{n+1}}}, & n \in \mathbb{Z}, \\ W_{i-1}^{\alpha_{i-1}^n} & \supset \overline{W_i^{\gamma_i^n}} & i = 2, \ldots, I, \quad n \in \mathbb{Z}. \end{array}$$

This lemma produces three two-sided sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$ that yield a two-sided sequence of neighborhoods

$$\begin{array}{ccccccccccccc} \vdots & & \vdots & & & & & & & & \vdots & & \\ W_1^{\gamma_1^0} & \supset & W_1^{\beta_1^0} & \supset & W_1^{\alpha_1^0} & \supset & W_2^{\gamma_2^0} & \supset & \ldots & \supset & W_I^{\alpha_I^0} & \supset & \\ W_1^{\gamma_1^1} & \supset & W_1^{\beta_1^1} & \supset & W_1^{\alpha_1^1} & \supset & W_2^{\gamma_2^1} & \supset & \ldots & \supset & W_I^{\alpha_I^1} & \supset & \\ W_1^{\gamma_1^2} & \supset & W_1^{\beta_1^2} & \supset & W_1^{\alpha_1^2} & \supset & W_2^{\gamma_2^2} & \supset & \ldots & \supset & W_I^{\alpha_I^2} & \supset & \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & \vdots \end{array}$$

such that each neighborhood contains the closure of the neighborhood that follows. For each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, we let the mapping $\overline{q}_i^n : \mathbb{R}^n \to [0, 1]$ be given by the formula (2.19) and we define the mapping $\overline{v} : \mathbb{R}^n \to \mathbb{R}^m$ by setting

$$\overline{v}(x) = \sum_{n \in \mathbf{Z}} \sum_{i=1}^{I} \overline{q}_i^n(x) u_i(x), \quad x \in \mathbb{R}^n.$$

We now let x be in $\mathbb{R}^n \backslash \{0\}$ and we note that

$$\{x \in \mathbb{R}^n : \overline{q}_i^n(x) \neq 0\} \subset W_i^{\gamma_i^n} \backslash \overline{W_i^{\alpha_i^n}},$$

for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$. Thus, because $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$ converges to 0 and $+\infty$ as n tends to $+\infty$ and $-\infty$ respectively, there exist a neighborhood U_x of x and a positive integer N satisfying

$$\overline{v}(y) = \sum_{n=-N}^{N} \sum_{i=1}^{I} \overline{q}_i^n(y) u_i(y), \quad y \in U_x,$$

and we easily conclude that $\overline{v}$ is C^k on $\mathbb{R}^n \backslash \{0\}$. Further, because the mappings of the collection $\{\overline{q}_i^n,\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$ sum up to a real less than 1, we get

$$\|\overline{v}(x)\| \leq \min\left(\|u_1(x)\|, \cdots, \|u_I(x)\|\right), \quad x \in \mathbb{R}^n,$$

and continuity of $\bar{v}$ follows from that of the mappings u_i and $\overline{q}_i^n$ for each $i = 1, \ldots, I$ and n in $\mathbb{Z}$.

Fix $i = 1, \ldots, I$. By an argument similar to that used in the proof of Theorem 2.1, we obtain that for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, the trajectories of $\dot{x} = f_i(x, \bar{v}(x))$ starting in $\overline{W_i^{\beta_i^n}}$ remain in this set forever. It follows that $\bar{v}$ stabilizes S_i. Next, for each x_0 in $\mathbb{R}^n$, because $\lim_{n \to -\infty} \beta_i^n = +\infty$, there exists n in $\mathbb{Z}$ such that x_0 lies in $\overline{W_i^{\beta_i^n}}$. Therefore, for each x_0 in $\mathbb{R}^n$, there exists n in $\mathbb{Z}$ such that the trajectories of $\dot{x} = f_i(x, \bar{v}(x))$ starting from x_0, remain forever in the bounded set $\overline{W_i^{\beta_i^n}}$. We summarize these results in the following corollary.

Theorem 2.2 *Let k be in $\{0, 1, \ldots\}$, and assume that for each $i = 1, \ldots, I$, there exists a continuous and almost C^k feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$, which globally asymptotically stabilizes S_i. Further, assume that the mapping $f_i(\cdot, u_i(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for each $i = 1, \ldots, I$. Then, there exists a continuous and almost C^k feedback law $\bar{v} : \mathbb{R}^n \to \mathbb{R}^m$, such that the following holds.*

i) *$\bar{v}$ simultaneously stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

ii) *The trajectories of the system $\dot{x} = f_i(x, \bar{v}(x))$, starting at x_0, are bounded for each x_0 in $\mathbb{R}^n$ and each $i = 1, \ldots, I$.*

2.3.3 Application to families of linear systems

We now assume that for each $i = 1, \ldots, I$, the system S_i and the feedback law u_i are linear and we let $V_i : \mathbb{R}^n \to [0, \infty)$ be a Lyapunov function for the system $\dot{x} = f_i(x, u_i(x))$ given by $V_i(x) = x^t P_i x$, where P_i is a positive definite matrix. It is well known that in this case u_i globally asymptotically stabilizes the system S_i, for each $i = 1, \ldots, I$, so that we can use the construction of Subsection 2.3.2 in order to obtain a simultaneous stabilizer.

However, in this particular case, it turns out that the sequence of reals $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n \in \mathbb{Z}}$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n \in \mathbb{Z}}$ and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n \in \mathbb{Z}}$ used in Subsection 2.3.2 for the construction of the stabilizer, may be obtained through the lemma below instead of Lemma 2.2.

Indeed, the sequences produced by Lemma 2.3 satisfy each one of the assertions of Lemma 2.2, and have the advantage of being more explicit than those produced by Lemma 2.2.

Lemma 2.3 *Let $I \geq 2$ be an integer. For each $i = 1, \ldots, I$, let P_i be a positive definite matrix, let $\frac{1}{M_i}$ and $\frac{1}{m_i}$ denote respectively its smallest and largest eigenvalue, let $V_i : \mathbb{R}^n \to [0, \infty)$ denote the mapping given by $V_i(x) = x^t P_i x$, and* *let θ_i be in $(0,1)$. Furthermore, let π_1 be in $(0,\frac{m_I}{M_1})$ and let π_i be in $(0,\frac{m_{i-1}}{M_i})$ for each $i=2,\dots,I$. Assume that*

$$(\pi_1\cdots\pi_I)\,(\theta_1^2\cdots\theta_I^2) \;<\; 1 \tag{2.25}$$

and let $\widehat{\gamma}_1$ be an arbitrary positive real. Finally, let the sequences of positive reals $\{\gamma_i^n,\ i=1,\dots,I\}_{n\in\mathbf{Z}}$, $\{\beta_i^n,\ i=1,\dots,I\}_{n\in\mathbf{Z}}$ and $\{\alpha_i^n,\ i=1,\dots,I\}_{n\in\mathbf{Z}}$ be defined by setting on one hand

$$\gamma_1^0=\widehat{\gamma}_1,\quad \beta_i^n\;=\;\theta_i\gamma_i^n,\quad \alpha_i^n\;=\;\theta_i\beta_i^n,\quad i=1,\dots,I,\quad n=0,1,\dots\,, \tag{2.26}$$

with

$$\gamma_1^{n+1}\;=\;\pi_1\alpha_I^n\quad \textit{and}\quad \gamma_i^n=\pi_i\alpha_{i-1}^n,\quad i=2,\dots,I,\quad n=0,1,\dots, \tag{2.27}$$

and on the other hand

$$\alpha_I^n=\frac{\gamma_1^{n+1}}{\pi_1}\quad \textit{and}\quad \alpha_i^n=\frac{\gamma_{i+1}^n}{\pi_{i+1}},\quad i=I-1,\dots,1,\quad n=-1,-2,\dots \tag{2.28}$$

with

$$\beta_i^n=\frac{\alpha_i^n}{\theta_i}\quad \textit{and}\quad \gamma_i^n=\frac{\beta_i^n}{\theta_i},\quad i=I,\dots,1,\quad n=-1,-2,\dots. \tag{2.29}$$

Then, for each n in $\mathbb{Z}$ we have

$$V_{i-1}^{-1}([0,\alpha_{i-1}^n))\quad\supset\;V_i^{-1}(\,[0,\gamma_i^n]\,),\quad i=2,\dots,I\,, \tag{2.30}$$

$$V_I^{-1}([0,\alpha_I^n))\quad\supset\;V_1^{-1}(\,[0,\gamma_1^{n+1}]\,). \tag{2.31}$$

and for each $i=1,\dots,I$ we have

$$\beta_i^n\to 0\ \textit{as}\ n\to\infty,$$
$$\beta_i^n\to\infty\ \textit{as}\ n\to-\infty.$$

Proof:

In what follows we fix $n=1,2,\dots$. Let $\delta>0$. It is well known that for each $i=1,\dots,I$, the set $V_i^{-1}([0,\delta])$ is the volume bounded by an ellipsoid centered at the origin with smallest axis $\sqrt{m_i\,\delta}$ and largest axis $\sqrt{M_i\,\delta}$. Thus, (2.30) and (2.31) will hold if we have

$$\gamma_i^n\;<\;\frac{m_{i-1}}{M_i}\,\alpha_{i-1}^n\quad\text{and}\quad\gamma_1^{n+1}\;<\;\frac{m_I}{M_1}\,\alpha_I^n$$

Because π_1 is in $(0,\dfrac{m_I}{M_1})$ and the real π_i is in $(0,\dfrac{m_{i-1}}{M_i})$ for each $i=2,\dots,I$, we obtain (2.30) and (2.31).

Further, it is easily checked from the assumptions that for each $i = 1, \ldots, I$ we have

$$\beta_i^{n+1} = (\pi_1 \cdots \pi_I)(\theta_1^2 \cdots \theta_I^2)\beta_i^n, \quad n \in \mathbb{Z},$$

and it follows from (2.25) combined with the definition of α_i^n and γ_i^n that for each $i = 1, \ldots, I$, the sequences $\{\gamma_i^n\}_{n\in\mathbb{Z}}$, $\{\beta_i^n\}_{n\in\mathbb{Z}}$ and $\{\alpha_i^n\}_{n\in\mathbb{Z}}$ converge to 0 and $+\infty$ as n tends to $+\infty$ and $-\infty$ respectively. ∎

In the particular case $I = 2$, Theorem 2.1 and Proposition 2.1, are also proved in Ho-Mock-Qai and Dayawansa [41] and [43] respectively. The constructions presented in these two papers are similar to those introduced in this chapter, but may seem simpler because $I = 2$.

Finally, we present in the next section several technical lemmas that were used in the proof of the Theorem 2.1 and in Subsection 2.3.1.

2.4 Technical lemmas

The following lemma was used in the proof of Theorem 2.1, for the construction of the sequence of neighborhoods $\{U_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$.

Lemma 2.4 *Let $I \geq 2$ be an integer and let D be a bounded neighborhood of the origin (resp. let $D = \mathbb{R}^n$). For each $i = 1, \ldots, I$, let $V_i : \overline{D} \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function). Further, for each $i = 1, \ldots, I$ and each $\beta > 0$, let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\}.$$

Then, there exists a sequence of positive reals $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$ satisfying:

$$\begin{aligned}
\beta_i^1 &< \inf_{x\in\partial D} V_i(x), \quad i = 1, \ldots, I, \\
\beta_i^n &\to 0 \ \text{as}\ n \to \infty, \quad i = 1, \ldots, I, \\
W_I^{\beta_I^n} &\supset \overline{W_1^{\beta_1^{n+1}}}, \quad n = 1, 2, \ldots, \\
W_{i-1}^{\beta_{i-1}^n} &\supset \overline{W_i^{\beta_i^n}}, \quad i = 2, \ldots, I, \quad n = 1, 2, \ldots.
\end{aligned}$$

Proof: We define the sequence $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^{\infty}$ by induction on n and i.

For $n = 1$, we first pick β_1^1 in $(0, \inf_{x\in\partial D} V_1(x))$. Then, for each $i = 2, \ldots, I$, we define β_i^1 from β_{i-1}^1 as described hereafter: By Lemma B.1, for each $i = 2, \ldots, I$

the family $\{W_i^\beta\}_{\beta>0}$ is a neighborhood base at the origin with $W_i^\alpha \subset W_i^\beta$ for all $\alpha < \beta$, so that we can pick β_i^1 in the interval $(0, \inf_{x\in\partial D} V_i(x))$ such that

$$W_{i-1}^{\beta_{i-1}^1} \supset \overline{W}_i^{\beta_i^1}.$$

For $n = 2, 3, \ldots$, we define the sequence $\{\beta_i^n,\ i = 1, \ldots, I\}$ from the sequence $\{\beta_i^{n-1},\ i = 1, \ldots, I\}$ as follows: By Lemma B.1 applied to $\{W_1^\beta\}_{\beta>0}$, there exists β_1^n in the interval $(0, \frac{\beta_1^{n-1}}{2}]$ such that

$$W_I^{\beta_I^{n-1}} \supset \overline{W}_1^{\beta_1^n}.$$

Similarly, for each $i = 2, \ldots, I$, we define β_i^n from β_{i-1}^n: By using Lemma B.1, we select β_i^n in $(0, \frac{\beta_i^{n-1}}{2}]$ such that

$$W_{i-1}^{\beta_{i-1}^n} \supset \overline{W}_i^{\beta_i^n}.$$

It is plain from this construction that for each $i = 1, \ldots, I$, we have

$$\lim_{n\to\infty} \beta_i^n = 0.$$

The result is then proved upon noting that by construction $\beta_i^1 < \inf_{x\in\partial D} V_i(x)$ for each $i = 1, \ldots, I$. ■

The next lemma is used in Subsection 2.3.1 to obtain an explicit simultaneous stabilizer.

Lemma 2.5 *Let $I \geq 2$ be an integer, and let D be a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$). For each $i = 1, \ldots, I$, let $V_i : \overline{D} \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function) and let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\}, \quad \beta > 0.$$

Then, there exist sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$, and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n=1}^\infty$ converging to the origin as n tends to ∞ and satisfying

$$\inf_{x\in\partial D} V_i(x) > \gamma_i^n > \beta_i^n > \alpha_i^n, \quad i = 1, \ldots, I, \quad n = 1, 2, \ldots,$$

with

$$\begin{aligned} W_I^{\alpha_I^n} &\supset \overline{W}_1^{\gamma_1^{n+1}}, \quad n = 1, 2, \ldots, \\ W_{i-1}^{\alpha_{i-1}^n} &\supset \overline{W}_i^{\gamma_i^n} \quad i = 2, \ldots, I, \quad n = 1, 2, \ldots . \end{aligned}$$

Proof: We define the sequence $\{\gamma_i^n,\ i=1,\dots,I\}_{n=1}^{\infty}$, $\{\beta_i^n,\ i=1,\dots,I\}_{n=1}^{\infty}$, and $\{\alpha_i^n,\ i=1,\dots,I\}_{n=1}^{\infty}$ by induction on n and i.

For $n=1$, we pick γ_1^1 in $(0, \inf_{x\in\partial D} V_1(x))$ and we choose β_1^1 and α_1^1 such that $\gamma_1^1 > \beta_1^1 > \alpha_1^1 > 0$. Then, for each $i=2,\dots,I$, we define γ_i^1, β_i^1 and α_i^1 as follows: By using Lemma B.1 (with D and V_1), we pick γ_i^1 in $(0, \inf_{x\in\partial D} V_i(x))$ such that

$$W_{i-1}^{\alpha_{i-1}^1} \supset \overline{W}_i^{\gamma_i^1},$$

and we choose β_i^1 and α_i^1 satisfying $\gamma_i^1 > \beta_i^1 > \alpha_i^1 > 0$.

For $n=2,3,\dots$, we define the sequence $\{\gamma_i^n, \beta_i^n, \alpha_i^n\ i=1,\dots,I\}$ from the sequence $\{\gamma_i^{n-1}, \beta_i^{n-1}, \alpha_i^{n-1},\ i=1,\dots,I\}$ as described below:

For $i=1$, we select γ_1^n in $(0, \frac{\gamma_1^{n-1}}{2}]$ (Lemma B.1) satisfying

$$W_I^{\alpha_I^{n-1}} \supset \overline{W}_1^{\gamma_1^n}.$$

We then choose β_1^n and α_1^n such that $\gamma_1^n > \beta_1^n > \alpha_1^n > 0$.

Next, for each $i=2,\dots,I$, we pick γ_i^n in $(0, \frac{\gamma_i^{n-1}}{2}]$ (Lemma B.1) such that

$$W_{i-1}^{\alpha_{i-1}^n} \supset \overline{W}_i^{\gamma_i^n},$$

and we choose β_i^n and α_i^n such that $\gamma_i^n > \beta_i^n > \alpha_i^n > 0$.

It is then easily seen that the obtained sequences satisfy the assertions of the lemma. ■

Chapter 3

Simultaneous Stabilization of Infinite Families of Nonlinear Systems

By generalizing the construction introduced in the proof of Theorem 2.1, we now establish the simultaneous stabilizability of any **countably infinite** family of asymptotically stabilizable systems.

We follow the organization of the previous chapter and start in Section 3.2 by constructing a simultaneous stabilizer based on a partition of unity. Then, we show in Section 3.3 how we can circumvent the computation of this partition of unity and exhibit a simpler simultaneous stabilizer. We then apply our results to families of globally asymptotically stabilizable systems and to families of linear systems. Finally, we present in Section 3.4 some technical results.

3.1 Problem definition

Throughout this chapter, we consider the countably infinite family $\{S_i,\ i = 1, 2, \ldots\}$ of systems

$$S_i : \quad \dot{x} \;=\; f_i(x, u), \quad i = 1, 2, \ldots\,, \tag{3.1}$$

where the state x lies in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$, and for each $i = 1, 2, \ldots$, the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin with $f_i(0, 0) = 0$.

We let k be in $\{0, 1, \ldots\}$, and we assume that for each $i = 1, 2, \ldots$, there exists a continuous and almost C^k feedback law u_i that locally asymptotically stabilizes the system S_i. Under these assumptions, we establish the simultaneous stabilizability of the family $\{S_i,\ i = 1, 2, \ldots\}$ by means of continuous feedback.

The motivations for studying the simultaneous stabilization and asymptotic stabilization of **infinite** families of systems are the following.

A countably infinite family may be a sub-family of a larger parameterized family $\{S(\gamma),\ \gamma \in \Gamma\}$, where the set Γ is uncountable. In this case, if the countable family is not simultaneously stabilizable by some class of feedback laws, then it follows that the family $\{S(\gamma),\ \gamma \in \Gamma\}$ cannot be robustly stabilizable by the feedback laws of this class. In this way, we may be able to obtain useful necessary conditions for the robust stabilizability and asymptotic stabilizability of the family $\{S(\gamma),\ \gamma \in \Gamma\}$. Further, in practical implementations, the set of parameter values Γ is always represented by a countable set $\{\gamma_1, \gamma_2, \ldots\}$, and the simultaneously stabilizability or asymptotic stabilizability of the family of systems $\{S(\gamma_1), S(\gamma_2), \ldots\}$ may suffice to ensure "practical" robust stability (in some sense) of the family $\{S(\gamma),\ \gamma \in \Gamma\}$. It is not the purpose of this dissertation to go into this matter more closely, but it would be certainly of great practical interest to clarify this idea and find conditions under which the robust stability of a parameterized family of systems is "practically" equivalent to the simultaneous stability of one of its countable sub-families.

We present in the next section the main result of this chapter.

3.2 Simultaneous stabilization

The purpose of this section is to prove Theorem 3.1. Although, the main idea of the proof is similar to that of the proof of Theorem 2.1, because we have an infinite number of systems, the construction is technically more involved and additional care is needed in order to ensure that the obtained simultaneous stabilizer is continuous and almost C^k on a neighborhood of the origin.

The main lines of the proof of this theorem are as follows: For each $i = 1, 2, \ldots$, we let V_i denote a Lyapunov function for the asymptotically stable system $\dot{x} = f_i(x, u_i(x))$. We construct a base at the origin $\{U_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$ composed of nested neighborhoods, such that the boundary of the set U_i^n is a Lyapunov level set of V_i for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$. We then design a continuous feedback law v which is equal to u_i on the boundary of the set U_i^n for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$. For each $i = 1, 2, \ldots$, we conclude that v stabilizes S_i upon noting that the sets of the family $\{\overline{U_i^n}\}_{n=i}^{\infty}$ are invariant with respect to $\dot{x} = f_i(x, v(x))$ and that $\{U_i^n\}_{n=i}^{\infty}$ is a base at the origin.

Theorem 3.1 *Let k be in $\{0, 1, \ldots\}$. Further, assume that for each $i = 1, 2, \ldots$, there exists a feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ which is continuous and almost C^k on a* *neighborhood of the origin, and which locally asymptotically stabilizes the system S_i. Then, there exists a feedback law $v : \mathbb{R}^n \to \mathbb{R}^m$ which is continuous and almost C^k on a neighborhood of the origin, and which simultaneously stabilizes the family $\{S_i,\ i = 1, 2, \ldots\}$.*

Proof:

Construction of the simultaneous stabilizer :

Let U_i be a neighborhood of the origin such that the mappings $f_i(\cdot, u_i(\cdot))$ and u_i are continuous on U_i, with u_i almost C^k on U_i for each $i = 1, 2, \ldots$. By the Converse Lyapunov Theorem [56] and the local asymptotic stability of $\dot{x} = f_i(x, u_i(x))$, there exists a bounded neighborhood of the origin $D_i \subset U_i$ and a Lyapunov function $V_i : D_i \to [0, \infty)$, satisfying

$$\nabla V_i(x)\, f_i(x, u_i(x)) \ < \ 0, \quad x \in \overline{D}_i \backslash \{0\}, \tag{3.2}$$

for each $i = 1, 2, \ldots$. For each $i = 1, 2, \ldots$, we let W_i^β denote the set

$$W_i^\beta \ \triangleq \ \{x \in D_i : V_i(x) \ < \ \beta\}, \quad \beta > 0,$$

and we define the mapping $\bar{u}_i : D_1 \to \mathbb{R}^m$ by setting

$$\bar{u}_i(x) \ = \begin{cases} u_i(x), & x \in D_1 \cap D_i \\ 0, & x \in D_1 \backslash D_i \end{cases} .$$

Let $\theta > 0$. By applying Lemma 3.3 with θ and the families $\{D_i,\ i = 1, 2, \ldots\}$, $\{V_i,\ i = 1, 2, \ldots\}$ and $\{u_i,\ i = 1, 2, \ldots\}$, we obtain a sequence of positive reals $\{\beta_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$ satisfying

$$\beta_n^n \ < \ \inf_{x \in \partial D_n} V_n(x), \quad n = 1, 2, \ldots, \tag{3.3}$$

$$\beta_i^n \ \to \ 0 \ \text{ as } \ n \to \infty, \quad i = 1, 2, \ldots. \tag{3.4}$$

Upon setting

$$U_i^n \triangleq W_i^{\beta_i^n}, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots \ ,$$

the remaining assertions of Lemma 3.3 translate to

$$U_{n-1}^{n-1} \ \supset \ \overline{U}_1^n \quad \text{and} \quad U_{i-1}^n \ \supset \ \overline{U}_i^n, \quad i = 2, \ldots, n\,, \quad n = 2, 3, \ldots, \tag{3.5}$$

with

$$\overline{U}_{n-1}^n \ \subset \ D_n, \quad n = 2, 3, \ldots \ , \tag{3.6}$$

and

$$\|u_k(x)\| \ < \ \frac{\theta}{n}, \quad x \in D_k \cap U_n^n, \quad k = 1, \ldots, n+2\,, \quad n = 1, 2, \ldots. \tag{3.7}$$

From Lemma B.3 (i) combined with (3.3) we get $D_1 \supset \overline{U}_1^1$ and in view of (3.5), we have a sequence of nested neighborhoods

$$\begin{array}{cccccccc} D_1 & \supset & U_1^1 & \supset & & & & \\ & & U_1^2 & \supset & U_2^2 & \supset & & \\ & & U_1^3 & \supset & U_2^3 & \supset & U_3^3 & \supset \\ & & U_1^4 & \supset & \dots & \dots & & \dots \\ & & \vdots & \vdots & & \vdots & & \end{array} \tag{3.8}$$

such that each neighborhood contains the closure of the neighborhood that follows.

For each $n = 1, 2, \dots$ and each $i = 1, \dots, n$, we now define the sets Δ_i^n by setting

$$\begin{array}{lll} \Delta_1^1 & \triangleq & D_1 \backslash \overline{U}_1^2, \\ \Delta_1^n & \triangleq & U_{n-1}^{n-1} \backslash \overline{U}_2^n, \quad n = 2, 3, \dots , \\ \Delta_i^n & \triangleq & U_{i-1}^n \backslash \overline{U}_{i+1}^n, \quad i = 2, \dots, n-1, \quad n = 3, 4, \dots , \\ \Delta_n^n & \triangleq & U_{n-1}^n \backslash \overline{U}_1^{n+1}, \quad n = 2, 3, \dots . \end{array}$$

Because the family $\{U_i^n,\ i = 1, \dots, n\}_{n=1}^{\infty}$ is a base at the origin [follows from (3.4)] composed of nested neighborhoods, Lemma B.4 implies that the family $\{\Delta_i^n,\ i = 1, \dots, n\}_{n=1}^{\infty}$ is an open cover of $D_1 \backslash \{0\}$.

By Theorem 1.1 there exists a partition of unity $\{q_i^n,\ i = 1, \dots, n\}_{n=1}^{\infty}$ subordinate to the cover $\{\Delta_i^n,\ i = 1, \dots, n\}_{n=1}^{\infty}\}$ such that the support of q_i^n is included in Δ_i^n for each $i = 1, \dots, n$ and each $n = 1, 2, \dots$. We now define the feedback law $v : D_1 \to \mathbb{R}^m$ by setting

$$v(x) = \begin{cases} 0, & x = 0 \\ \sum_{n=1}^{\infty} \sum_{i=1}^{n} \bar{u}_i(x)\, q_i^n(x), & x \in D_1 \backslash \{0\} \end{cases},$$

and we prove that v is almost C^k and continuous at the origin.

v is almost C^k :

Let x be in $D_1 \backslash \{0\}$ and let r be in $(0, \|x\|)$ with $\overline{B_r(0)} \subset D_1$. Because $\{U_i^n,\ i = 1, \dots, n\}_{n=1}^{\infty}$ is a base at the origin composed of nested neighborhoods, there exists an integer N such that

$$U_N^N \cup \{U_i^n,\ i = 1, \dots, n\}_{n=N+1}^{\infty} \subset \overline{B_r(0)}. \tag{3.9}$$

This together with the fact that the support of each mapping q_i^n is included in Δ_i^n imply that

$$v(y) = \sum_{n=1}^{N} \sum_{i=1}^{n} \bar{u}_i(y) q_i^n(y), \quad y \in D_1 \backslash \overline{B_r(0)}. \tag{3.10}$$

Note that $D_1 \backslash \overline{B_r(0)}$ is a neighborhood of x.

We now show that v is C^k on $D_1 \backslash \overline{B_r(0)}$, by proving that the mapping $\bar{u}_i\, q_i^n : D_1 \backslash \{0\} \to \mathbb{R}^m$ is C^k on $D_1 \backslash \{0\}$, for each $i = 1, \ldots, n$ and each $n = 1, 2, \ldots$.

We fix $n = 2, 3, \ldots$ and $i = 1, \ldots, n$. By definition of $\bar{u}_i$ we have $\bar{u}_i = u_i$ on D_i, so that

$$\bar{u}_i(y)\, q_i^n(y) \;=\; u_i(y)\, q_i^n(y), \quad y \in (D_1 \cap D_i) \backslash \{0\}. \tag{3.11}$$

Further, as the support of q_i^n is included in Δ_i^n, we get

$$\bar{u}_i(y)\, q_i^n(y) \;=\; 0, \quad y \in \left(D_1 \backslash \overline{\Delta_i^n}\right) \backslash \{0\}. \tag{3.12}$$

Because q_i^n is smooth on $D_1 \backslash \{0\}$ and u_i is C^k on $D_i \backslash \{0\}$, it follows from (3.11) and (3.12) that $\bar{u}_i\, q_i^n$ is C^k on

$$\left((D_1 \backslash \overline{\Delta_i^n}) \cup (D_1 \cap D_i)\right) \backslash \{0\}. \tag{3.13}$$

Next, by proving that $\overline{\Delta_i^n} \subset D_i$, for each $i = 1, \ldots, n$ and each $n = 2, 3, \ldots$, we show that the set in (3.13) is equal to $D_1 \backslash \{0\}$. By combining the definition of the sets Δ_i^n with (3.5) and (3.6), we obtain for each $n = 2, 3, \ldots$ the inclusions

$$\begin{array}{lllll}
\overline{\Delta_1^n} & \subset \overline{U_{n-1}^{n-1}} & \subset D_1, & & \\
\overline{\Delta_i^n} & \subset \overline{U_{i-1}^n} & \subset U_i^{n-1} & \subset D_i, & i = 1, \ldots, n-1\,, \\
\overline{\Delta_n^n} & \subset \overline{U_{n-1}^n} & \subset D_n & & \text{[follows from (3.6)]},
\end{array}$$

and because we also have $\overline{\Delta_i^n} \subset D_1 \backslash \{0\}$ from (3.8), we get

$$\overline{\Delta_i^n} \subset (D_1 \cap D_i) \backslash \{0\}.$$

This implies that the set in (3.13) is equal to $D_1 \backslash \{0\}$, and it follows that the mapping $\bar{u}_i\, q_i^n$ is C^k on $D_1 \backslash \{0\}$ for each $i = 1, \ldots, n$ and each $n = 2, 3, \ldots$. Moreover, the mapping $\bar{u}_1 q_1^1$ is C^k on $D_1 \backslash \{0\}$, since by definition of $\bar{u}_1$, we have $\bar{u}_1 q_1^1 = u_1 q_1^1$ on $D_1 \backslash \{0\}$. In view of (3.10), we conclude that for each x in $D_1 \backslash \{0\}$, there exists a neighborhood U_x of x included in $D_1 \backslash \{0\}$ such that the mapping v is C^k on U_x. In short, the mapping v is C^k on $D_1 \backslash \{0\}$.

Next, we prove that v is continuous at the origin.

Continuity of v :

We fix $n = 2, 3, \ldots$. From the definition of the sets Δ_j^m, it is easily checked that for each $m = n+2, n+3, \ldots$, we have

$$\left(U_{n-1}^{n-1} \backslash \overline{U}_{n+1}^{n+1}\right) \cap \Delta_j^m = \emptyset, \quad j = 1, \ldots, m,$$

and because the support of each function q_j^m is included in Δ_j^m, we get

$$v(x) \;=\; \sum_{m=1}^{n+1} \sum_{j=1}^{m} \bar{u}_j(x)\, q_j^m(x), \quad x \in U_{n-1}^{n-1} \backslash \overline{U}_{n+1}^{n+1}. \tag{3.14}$$

As the functions q_j^m sum up to 1, we obtain from (3.14) that

$$\|v(x)\| \;\leq\; \max(\|\bar{u}_1(x)\|, \ldots, \|\bar{u}_{n+1}(x)\|), \quad x \in U_{n-1}^{n-1} \backslash \overline{U}_{n+1}^{n+1},$$

so that (3.7) combined with the definition of the mappings $\bar{u}_i$, $i = 1, 2, \ldots$ yield

$$\|v(x)\| \;<\; \frac{\theta}{n-1}, \quad x \in U_{n-1}^{n-1} \backslash \overline{U}_{n+1}^{n+1}. \tag{3.15}$$

Further, upon noting that the family $\{U_l^l\}_{l=1}^{\infty}$ is a base at the origin composed of nested neighborhoods such that each neighborhood contains the closure of the neighborhood that follows, we deduce from Lemma B.4 that

$$U_{l-1}^{l-1} \backslash \{0\} \;=\; \bigcup_{n=l}^{\infty} \left(U_{n-1}^{n-1} \backslash \overline{U}_{n+1}^{n+1}\right), \quad l = 2, 3, \ldots,$$

and (3.15) implies that

$$\|v(x)\| \;\leq\; \frac{\theta}{l-1}, \quad x \in U_{l-1}^{l-1} \backslash \{0\}, \quad l = 2, 3, \ldots.$$

As $\frac{\theta}{l-1} \to 0$ *as* $l \to \infty$, continuity of v at the origin follows from the fact that $\{U_l^l\}_{l=1}^{\infty}$ is a base at the origin.

Finally, we show that the feedback law v simultaneously stabilizes the family of systems $\{S_i,\ i = 1, 2, \ldots\}$.

Stability :

The following argument is almost the same as that used in the proof of Theorem 2.1 to prove that v simultaneously stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$. The main difference is that here, to show that v stabilizes the system S_i for each $i = 1, 2, \ldots$, we need to consider the family $\{U_i^n\}_{n=i}^{\infty}$ instead of the family $\{U_i^n\}_{n=1}^{\infty}$ because of the "pyramidal" structure of the sequence of neighborhoods $\{U_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$.

From the definitions of the sets U_i^n and Δ_i^n, it is not hard to see that for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$, the boundary ∂U_i^n is included in Δ_i^n and does not intersect with any other set Δ_j^m. Thus, because the support of the mapping q_i^n is included in Δ_i^n for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$, it follows from the definition of v that

$$v(x) = u_i(x), \quad x \in \partial U_i^n, \quad n = i, i+1, \ldots, \quad i = 1, 2, \ldots.$$

This, together with (3.2) and the fact that $\overline{U}_i^n$ is included in $\overline{D}_i$ for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$, yield

$$\nabla V_i(x)\, f_i(x, v(x)) \ < \ 0, \quad x \in \partial U_i^n, \quad n = i, i+1, \ldots, \quad i = 1, 2, \ldots. \qquad (3.16)$$

For each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$, by combining (3.16) with Lemma 2.1 applied with D, V_i, $f_i(\cdot, v(\cdot))$ and β_i^n we obtain that the set $\overline{U}_i^n$ is invariant with respect to the system $\dot{x} = f_i(x, v(x))$.

For each $i = 1, 2, \ldots$, because the family $\{U_i^n\}_{n=i}^{\infty}$ is a base at the origin, by using the invariance of the sets $\overline{U}_i^n$, $n = i, i+1, \ldots$, it is easily checked that v stabilizes the system S_i, which completes the proof of the theorem. ∎

We now show how to circumvent the computation of the partition of unity that appears in the expression of the simultaneous stabilizer, just obtained.

3.3 A simple expression for a simultaneous stabilizer

Following the construction of an explicit simultaneous stabilizer for a finite families of nonlinear systems, we obtain in this section a more explicit feedback law $\bar{v}$ that simultaneously stabilizes the family $\{S_i,\ i = 1, 2, \ldots\}$ and that involves no partition of unity.

Our design follows the same lines as that of section 2.3: In the expression of the feedback law v obtained through the proof of Theorem 3.1, we replace the mappings of the partition of unity $\{q_i^n, i = 1, \ldots, n\}_{n=1}^{\infty}$ by the mappings of a new family $\{\bar{q}_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$ that sum up to a real less than 1 and such that for each $n = 1, 2, \ldots$ and each $i = 1, \ldots, n$ we have

$$\bar{q}_i^n(x) = 1 \qquad \text{with} \qquad \bar{q}_j^m(x) = 0, \quad x \in \partial U_i^n, \quad (j, m) \neq (i, n).$$

In Subsection 3.3.1, we study the case of a family $\{S_i,\ i = 1,2,\ldots\}$ of general nonlinear systems. Then, in Subsection 3.3.2, we look at the case where S_i is globally asymptotically stabilizable for each $i = 1,2,\ldots$. Finally, in Subsection 3.3.3, we present a simple design procedure that produces the explicit simultaneous stabilizer in case the system S_i and the feedback law u_i are linear for each $i = 1,2,\ldots$.

3.3.1 Derivation of the stabilizer

We let k be in $\{0,1,\ldots\}$. Further, we consider the family of systems $\{S_i,\ i = 1,2,\ldots\}$ as defined in (3.1) and we assume that for each $i = 1,2,\ldots$, there exists a feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ which is continuous and almost C^k on a neighborhood of the origin, and which locally asymptotically stabilizes the system S_i.

We begin with constructing the explicit stabilizer. Let U_i be a neighborhood of the origin such that the mappings $f_i(\cdot, u_i(\cdot))$ and u_i are continuous on U_i, with u_i almost C^k on U_i for each $i = 1,2,\ldots$. By the Converse Lyapunov Theorem [56] and the local asymptotic stability of $\dot{x} = f_i(x, u_i(x))$, there exists a bounded neighborhood of the origin $D_i \subset U_i$ and a C^{k_i} Lyapunov function $V_i : D_i \to [0,\infty)$ [where $k_i \geq 1$ is an integer], satisfying

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in \overline{D}_i\backslash\{0\}, \tag{3.17}$$

for each $i = 1,2,\ldots$. We define k' by setting

$$k' \;\triangleq\; \min\,[\,k,\ \inf_{i=1,2,\ldots}(k_i)\,].$$

For each $i = 1,2,\ldots$, we let W_i^β denote the set

$$W_i^\beta \;\triangleq\; \{x \in D_i : V_i(x) \;<\; \beta\}, \quad \beta > 0,$$

and we define the mapping $\bar{u}_i : D_1 \to \mathbb{R}^m$ by setting

$$\bar{u}_i(x) \;=\begin{cases} u_i(x), & x \in D_1 \cap D_i \\ 0, & x \in D_1\backslash D_i \end{cases}.$$

Let $\theta > 0$. By applying Lemma 3.4 with θ and the families $\{D_i,\ i = 1,2,\ldots\}$, $\{V_i,\ i = 1,2,\ldots\}$ and $\{u_i,\ i = 1,2,\ldots\}$, we obtain three sequence of positive reals $\{\gamma_i^n,\ i = 1,\ldots,n\}_{n=1}^\infty$, $\{\beta_i^n,\ i = 1,\ldots,n\}_{n=1}^\infty$ and $\{\alpha_i^n,\ i = 1,\ldots,n\}_{n=1}^\infty$ converging to 0 as n tends to ∞ and satisfying

$$\inf_{x\in\partial D_i} V_i(x) \;>\; \gamma_i^n \;>\; \beta_i^n \;>\; \alpha_i^n, \quad i = 1,\ldots,n, \quad n = 1,2,\ldots, \tag{3.18}$$

with

$$\begin{aligned} W_{n-1}^{\alpha_{n-1}^{n-1}} &\supset \overline{W_1^{\gamma_1^n}}\,, \quad n = 2, 3, \ldots \\ W_{i-1}^{\alpha_{i-1}^{n}} &\supset \overline{W_i^{\gamma_i^n}}, \quad i = 2, \ldots, n\,, \quad n = 2, 3, \ldots . \end{aligned}$$

and

$$\|u_k(x)\| \;<\; \frac{\theta}{n}, \quad x \in D_k \cap W_n^{\beta_n^n}, \quad k = 1, \ldots, n+2\,.$$

We note that the sequences obtained through Lemma 3.4 yield a sequence of nested neighborhoods

$$\begin{array}{cccccccccccccc} D_1 & \supset & W_1^{\gamma_1^1} & \supset & W_1^{\beta_1^1} & \supset & W_1^{\alpha_1^1} & \supset & & & & & & \\ & & W_1^{\gamma_1^2} & \supset & W_1^{\beta_1^2} & \supset & W_1^{\alpha_1^2} & \supset & W_2^{\gamma_2^2} & \supset & W_2^{\beta_2^2} & \supset & W_2^{\alpha_2^2} & \supset \\ & & W_1^{\gamma_1^3} & \vdots & & \vdots & & & & & & & & \end{array} \tag{3.19}$$

such that each neighborhood contains the closure of the neighborhood that follows.

Next, for each $i = 1, 2, \ldots$ and each $n = i, i+1, \ldots$, we define the mappings $\overline{q}_i^n : D_1 \to [0, 1]$ by setting

$$\overline{q}_i^n(x) = \begin{cases} e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\beta_i^n-\alpha_i^n)^2}} & \text{if } V_i(x) \in (\alpha_i^n, \beta_i^n) \\ e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\gamma_i^n-\beta_i^n)^2}} & \text{if } V_i(x) \in [\beta_i^n, \gamma_i^n) \\ 0\,, & \text{otherwise} \end{cases},$$

and we let $\overline{v} : D_1 \to \mathbb{R}^m$ be given by

$$\overline{v}(x) \;=\; \sum_{n=1}^{\infty} \sum_{i=1}^{n} \overline{q}_i^n(x) \bar{u}_i(x), \quad x \in D_1.$$

We now prove the following theorem.

Proposition 3.1 *The feedback law $\overline{v}$ is continuous and almost $C^{k'}$ on D_1. Moreover $\overline{v}$ simultaneously stabilizes the family $\{S_i,\ i = 1, 2, \ldots\}$.*

Proof:

The mapping $\overline{v}$ is almost $C^{k'}$:

For each $n = 1, 2, \ldots$ and each $i = 1, \ldots, n$, we let Π_i^n denote the set

$$\Pi_i^n \;\triangleq\; \{x \in D_1 : \overline{q}_i^n(x) \neq 0\},$$

so that (3.18) combined with Lemma B.3 **(i)** and (3.19) yield

$$\Pi_i^n = W_i^{\gamma_i^n} \backslash \overline{W}_i^{\alpha_i^n}. \tag{3.20}$$

Let x be in $D_1 \backslash \{0\}$ and let r be in $(0, \|x\|)$ with $B_r(0) \subset D_1$. Because $\{W_i^{\gamma_i^n},\ i = 1, \ldots, n\}_{n=1}^{\infty}$ is a base at the origin composed of nested neighborhoods, there exists an integer N such that

$$\{W_i^{\gamma_i^n},\ i = 1, \ldots, n\}_{n=N+1}^{\infty} \subset \overline{B_r(0)}.$$

This together with (3.20) imply that

$$\bar{v}(y) = \sum_{n=1}^{N} \sum_{i=1}^{n} \bar{u}_i(y) q_i^n(y), \quad y \in D_1 \backslash \overline{B_r(0)}. \tag{3.21}$$

We note that $D_1 \backslash \overline{B_r(0)}$ is a neighborhood of x and we show that $\overline{v}$ is $C^{k'}$ on $D_1 \backslash \overline{B_r(0)}$, by proving that the mapping $\bar{u}_i \overline{q}_i^n$ is $C^{k'}$ on $D_1 \backslash \{0\}$ for each $i = 1, \ldots, n$ and each $n = 1, 2, \ldots$.

We fix n in $\{1, 2, \ldots\}$ and i in $\{1, \ldots, n\}$. By definition of $\bar{u}_i$ we have $\bar{u}_i = u_i$ on $D_i \cap D_1$, so that

$$\bar{u}_i(y)\, \overline{q}_i^n(y) = u_i(y)\, \overline{q}_i^n(y), \quad y \in (D_1 \cap D_i) \backslash \{0\}. \tag{3.22}$$

Further, the definition of Π_i^n yields

$$\bar{u}_i(y)\, \overline{q}_i^n(y) = 0, \quad y \in \left(D_1 \backslash \overline{\Pi}_i^n\right) \backslash \{0\}. \tag{3.23}$$

Because $\overline{q}_i^n$ is $C^{k'}$ on $D_1 \backslash \{0\}$ [follows from Lemma B.6] and u_i is $C^{k'}$ on $D_i \backslash \{0\}$, it follows from (3.22) and (3.23) that $\bar{u}_i \overline{q}_i^n$ is $C^{k'}$ on

$$\left((D_1 \backslash \overline{\Pi}_i^n) \cup (D_1 \cap D_i)\right) \backslash \{0\}. \tag{3.24}$$

Next, by combining (3.18) with (3.20) and Lemma B.3 **(i)**, we get

$$\overline{\Pi}_i^n \subset \overline{W}_i^{\gamma_i^n} \subset D_i.$$

Thus the set in (3.24) is equal to $D_1 \backslash \{0\}$ and it follows that $\overline{u}_i \overline{q}_i^n$ is $C^{k'}$ on $D_1 \backslash \{0\}$, for each $n = 1, 2, \ldots$ and each $i = 1, \ldots, n$. We easily conclude from (3.21) that the mapping v is $C^{k'}$ on $D_1 \backslash \{0\}$.

We now show that the mapping $\overline{v}$ is continuous at the origin.

Continuity of $\overline{v}$:

We fix n in $\{2,3,\ldots\}$. Because the set Π_j^m is included in $W_j^{\gamma_j^m}$ for each $m = n+2, n+3, \ldots$ and each $j = 1, \ldots, m$, we obtain from (3.19) that

$$\left(W_{n-1}^{\beta_{n-1}^{n-1}} \backslash \overline{W}_{n+1}^{\beta_{n+1}^{n+1}}\right) \cap \Pi_j^m \;\subset\; \left(W_{n-1}^{\beta_{n-1}^{n-1}} \backslash \overline{W}_{n+1}^{\beta_{n+1}^{n+1}}\right) \cap W_j^{\gamma_j^m} \;=\; \emptyset,$$

and it follows that

$$\bar{v}(x) \;=\; \sum_{m=1}^{n+1} \sum_{j=1}^{m} \bar{u}_j(x)\, \overline{q}_j^m(x), \qquad x \in W_{n-1}^{\beta_{n-1}^{n-1}} \backslash \overline{W}_{n+1}^{\beta_{n+1}^{n+1}}. \tag{3.25}$$

Upon setting

$$U_i^n \;\triangleq\; W_i^{\beta_i^n}, \quad i = 1, \ldots, n \quad n = 1, 2, \ldots,$$

one can easily adapt the argument used in the proof of Theorem 3.1 to show that $\bar{v}$ is continuous at the origin in order to prove that $\overline{v}$ (as defined here) is continuous at the origin.

Stability :

We fix $n = 1, 2, \ldots$ and $i = 1, \ldots, n$. Because $\beta_i^n \in (0, \inf_{x \in \partial D_i} V_i(x))$, Lemma B.3 **(i)** together with (3.19) yield

$$\partial U_i^n \;=\; D_i \cap V_i^{-1}(\beta_i^n).$$

It is then easily seen that for each x in ∂U_i^n, we have $\overline{q}_i^n(x) = 1$ with $\overline{q}_j^m(x) = 0$ for all $(j,m) \neq (i,n)$. It follows that

$$\bar{v}(x) \;=\; u_i(x), \quad x \in \partial U_i^n,$$

and from (3.17) we obtain

$$\nabla V_i(x)\, f_i(x, \bar{v}(x)) \;<\; 0, \quad x \in \partial U_i^n. \tag{3.26}$$

We now fix $i = 1, 2, \ldots$. In view of (3.26), Lemma 2.1 yields the invariance of the set $\overline{U}_i^n$ with respect to the system $\dot{x} = f_i(x, \bar{v}(x))$, for each $n = i, i+1, \ldots$. This, combined with the fact that $\{U_i^n\}_{n=i}^{\infty}$ is a base at the origin, imply that $\bar{v}$ stabilizes S_i. The proof of the theorem is complete upon noting that this last argument holds for each $i = 1, 2, \ldots$. ■

We now discuss the case where the feedback law u_i globally asymptotically stabilizes S_i for each $i = 1, 2, \ldots$.

3.3.2 Application to globally asymptotically stabilizable systems

Throughout this subsection, we consider the family of systems $\{S_i,\ i=1,2,\ldots\}$ as defined in (3.1). We assume that for each $i=1,2,\ldots$, there exists a continuous and almost C^k feedback law u_i that globally asymptotically stabilizes the system S_i. We then construct a feedback law $\overline{v}$ that simultaneously stabilizes the family $\{S_i,\ i=1,2,\ldots\}$ and such that for each $i=1,2,\ldots$, the trajectories of $\dot{x}=f_i(x,\bar{v}(x))$ are bounded.

The stabilizing feedback law is based on a sequence of neighborhoods that has a different structure from that introduced in the previous subsection. This sequence if the union of a sequence similar to the one represented in (3.19) together with another sequence that has the "shape of an inverted pyramid". To construct this sequence, we use both Lemma 3.4 and Lemma 3.5.

Theorem 3.2 *Let k be in $\{0,1,\ldots\}$. Assume that for each $i=1,2,\ldots$ there exists a continuous and almost C^k feedback law $u_i:\mathbb{R}^n\to\mathbb{R}^m$, which globally asymptotically stabilizes S_i. Further, assume that the mapping $f_i(\cdot,u_i(\cdot)):\mathbb{R}^n\to\mathbb{R}^n$ is continuous for each $i=1,2,\ldots$. Then, there exists a continuous and almost C^k feedback law $\bar{v}:\mathbb{R}^n\to\mathbb{R}^m$, such that the following holds.*

i) *$\bar{v}$ simultaneously stabilizes the family $\{S_i,\ i=1,2,\ldots\}$.*

ii) *The trajectories of the system $\dot{x}=f_i(x,\bar{v}(x))$, starting at x_0, are bounded for each x_0 in $\mathbb{R}^n$ and each $i=1,2,\ldots$.*

Proof:

Construction of $\bar{v}$:
By the Converse Lyapunov Theorem [56], there exists a C^k radially unbounded Lyapunov function $V_i:\mathbb{R}^n\to[0,\infty)$ such that

$$\nabla V_i(x)\, f_i(x,u_i(x)) \;<\; 0, \quad x\in\mathbb{R}^n\setminus\{0\}. \tag{3.27}$$

For each $i=1,2,\ldots$, we let W_i^β denote the set

$$W_i^\beta \;\triangleq\; V_i^{-1}(\,[0,\beta)\,), \quad \beta>0.$$

Let $\theta>0$. By applying Lemma 3.4 with θ and the families $\{D_i=\mathbb{R}^n,\ i=1,2,\ldots\}$, $\{V_i,\ i=1,2,\ldots\}$ and $\{u_i,\ i=1,2,\ldots\}$, we obtain three sequence of positive reals $\{\gamma_i^n,\ i=1,\ldots,n\}_{n=1}^\infty$, $\{\beta_i^n,\ i=1,\ldots,n\}_{n=1}^\infty$ and $\{\alpha_i^n,\ i=1,\ldots,n\}_{n=1}^\infty$ converging to 0 as n tends to ∞ and satisfying the assertions of the lemma.

Now by applying Lemma 3.5 with γ_1^1 as defined in the sequence $\{\gamma_i^n,\ i = 1,\ldots,n\}_{n=1}^{\infty}$, we obtain three sequences $\{\gamma_i^n,\ i = 1,\ldots,|n|\}_{n=-1}^{-\infty}$, $\{\beta_i^n,\ i = 1,\ldots,|n|\}_{n=-1}^{-\infty}$ and $\{\alpha_i^n,\ i = 1,\ldots,|n|\}_{n=1}^{-\infty}$ converging to $+\infty$ as n tends to $-\infty$ and satisfying the assertions of the lemma.

It is not hard to see from the assertion of Lemmas 3.4 and 3.5 that we have

$$\gamma_i^n > \beta_i^n > \alpha_i^n, \quad i = 1,\ldots,|n|, \quad n \in \mathbb{Z}\backslash\{0\},$$

with

$$\|u_k(x)\| < \frac{\theta}{n}, \quad x \in W_n^{\beta_n^n}, \quad k = 1,\ldots,n+2, \quad n = 1,2,\ldots.$$

Moreover, the obtained sequences yield a sequence of nested neighborhoods

$$\begin{array}{llllllllllll}
W_1^{\gamma_1^{-3}} & \vdots & \vdots & & & & & & & & & \\
W_1^{\gamma_1^{-2}} & \supset & W_1^{\beta_1^{-2}} & \supset & W_1^{\alpha_1^{-2}} & \supset & W_2^{\gamma_2^{-2}} & \supset & W_2^{\beta_2^{-2}} & \supset & W_2^{\alpha_2^{-2}} & \supset \\
W_1^{\gamma_1^{-1}} & \supset & W_1^{\beta_1^{-1}} & \supset & W_1^{\alpha_1^{-1}} & \supset & & & & & & \\
W_1^{\gamma_1^{1}} & \supset & W_1^{\beta_1^{1}} & \supset & W_1^{\alpha_1^{1}} & \supset & & & & & & \\
W_1^{\gamma_1^{2}} & \supset & W_1^{\beta_1^{2}} & \supset & W_1^{\alpha_1^{2}} & \supset & W_2^{\gamma_2^{2}} & \supset & W_2^{\beta_2^{2}} & \supset & W_2^{\alpha_2^{2}} & \supset \\
W_1^{\gamma_1^{3}} & \vdots & \vdots & & & & & & & & &
\end{array} \tag{3.28}$$

such that each neighborhood contains the closure of the neighborhood that follows.

For each n in $\mathbb{Z}\backslash\{0\}$ and each $i = 1,\ldots,|n|$, we define the mappings $\overline{q}_i^n : \mathbb{R}^n \to [0,1]$ by setting

$$\overline{q}_i^n(x) = \begin{cases} e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\beta_i^n-\alpha_i^n)^2}} & \text{if } V_i(x) \in (\alpha_i^n, \beta_i^n) \\ e^{\frac{(V_i(x)-\beta_i^n)^2}{(V_i(x)-\beta_i^n)^2-(\gamma_i^n-\beta_i^n)^2}} & \text{if } V_i(x) \in [\beta_i^n, \gamma_i^n) \\ 0\,, & \text{otherwise} \end{cases},$$

and we let $\overline{v} : \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$\overline{v}(x) = \sum_{n=1}^{+\infty}\sum_{i=1}^{n} \overline{q}_i^n(x)u_i(x) + \sum_{n=-1}^{-\infty}\sum_{i=1}^{|n|} \overline{q}_i^n(x)u_i(x), \quad x \in \mathbb{R}^n.$$

We now study the regularity of $\bar{v}$.

$\bar{v}$ is almost C^k and continuous on $\mathbb{R}^n$:

Let x be in $\mathbb{R}^n\backslash\{0\}$ and let r and R be in $(0,\|x\|)$ and $(\|x\|,+\infty)$ respectively. Because $\{\gamma_1^n\}_{n\in\mathbf{Z}\backslash\{0\}}$ converges to 0 and $+\infty$ as n tends to $+\infty$ and $-\infty$ respectively, it follows from the fact that V_1 is radially unbounded that there exists a positive integer n such that

$$W_1^{\gamma_1^n} \supset \overline{B_r(0)} \qquad \text{with} \qquad B_R(0) \subset W_1^{\gamma_1^{-n}}.$$

Therefore, we have

$$\overline{v}(y) = \sum_{m=1}^{n}\sum_{i=1}^{m}\overline{q}_i^m(y)u_i(y) + \sum_{m=-1}^{-n}\sum_{i=1}^{|m|}\overline{q}_i^m(y)u_i(y), \quad y \in B_R(0)\backslash\overline{B_r(0)},$$

and it follows that $\overline{v}$ is C^k on $\mathbb{R}^n\backslash\{0\}$.

Upon noting that

$$\overline{v}(x) = \sum_{n=1}^{+\infty}\sum_{i=1}^{n}\overline{q}_i^n(x)u_i(x), \quad x \in W_1^{\gamma_1},$$

and by using the argument given in the proof of Theorem 3.1 to prove that $\bar{v}$ is continuous at the origin, we obtain the continuity of $\bar{v}$ (as defined here) at the origin.

Stability :

It is easily seen that

$$\overline{v}(x) = u_i(x), \quad x \in V_i^{-1}(\beta_i^n), \quad i = 1,\ldots,|n|, \quad n = \ldots,-2,-1,1,2,\ldots.$$

Thus, by Lemma 2.1, the set $\overline{W_i^{\beta_i^n}}$ is invariant with respect to $\dot{x} = f_i(x,\bar{v}(x))$, for each $i = 1,2,\ldots$ and each n in $\mathbb{Z}\backslash\{0\}$. For each $i = 1,2,\ldots$, because $\{W_i^{\beta_i^n}\}_{n=i}^{\infty}$ is a base at the origin, the previous comment implies that $\bar{v}$ stabilizes S_i. Further, let $i = 1,2,\ldots$ and let x_0 be in $\mathbb{R}^n$. Because $\{\beta_i^n\}_{n=-1}^{-\infty}$ converges to $+\infty$ there exists an integer n such that $x_0 \in W_i^{\beta_i^n}$ and it follows that each trajectory of $\dot{x} = f_i(x,\bar{v}(x))$ starting from x_0 remains in the bounded set $W_i^{\beta_i^n}$. Hence the theorem.

■

In the next subsection, we show that the design procedure presented in the proof of Theorem 3.2 reduces to a rather simple procedure if the system S_i and the feedback law u_i are linear for each $i = 1,2,\ldots$.

3.3.3 Application to families of linear systems

We now assume that for each $i = 1, 2, \ldots$, the system S_i and the feedback law u_i are linear and we let $V_i : \mathbb{R}^n \to [0, \infty)$ be a Lyapunov function for the system $\dot{x} = f_i(x, u_i(x))$ given by $V_i(x) = x^t P_i x$, where P_i is a positive definite matrix. In this case, the two lemmas below yield more explicit and simple design procedures that produce sequences of positive reals having the same properties as those obtained through Lemmas 3.4 and 3.5. We can therefore use these new sequences to define a simultaneous stabilizer $\bar{v}$ for the family $\{S_i,\ i = 1, 2, \ldots\}$, exactly as we did in the proof of Theorem 3.2, and all the arguments used in this proof carry over to this case.

Lemma 3.1 *For each $i = 1, 2, \ldots$, let P_i be a positive definite matrix, let $\frac{1}{M_i}$ and $\frac{1}{m_i}$ denote respectively its smallest and largest eigenvalue, let $V_i : \mathbb{R}^n \to [0, \infty)$ denote the mapping given by $V_i(x) = x^t P_i x$, and let the mapping $u_i : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on a neighborhood of the origin with $u_i(0) = 0$. Further, for each $i = 1, 2, \ldots$, let θ_i be in $(0, 1)$, and let the sequence $\{\pi_i\}_{i=2}^{\infty}$ be such that*

$$0 < \pi_i < \min\left(\frac{m_{i-1}}{M_i}, \frac{1}{\theta_i^2}\right), \quad i = 2, 3, \ldots. \tag{3.29}$$

Next, let $\theta > 0$ and let $\{k_i\}_{i=1}^{\infty}$ be a sequence of positive reals such that

$$k_i > 1 \qquad \textit{with} \qquad \frac{m_i}{k_i M_1}\theta_1^2 < 1, \quad i = 1, 2, \ldots. \tag{3.30}$$

Finally, let the sequences of positive reals $\{\gamma_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$, $\{\beta_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$ and $\{\alpha_i^n,\ i = 1, \ldots, n\}_{n=1}^{\infty}$ be defined by choosing $\gamma_1^1 > 0$ and γ_n^n in $(0, \pi_n \alpha_{n-1}^n]$ for each $n = 2, 3, \ldots$ such that

$$\|u_k(x)\| < \frac{\theta}{n}, \quad x \in V_n^{-1}([0, \gamma_n^n)), \quad k = 1, \ldots, n+2, \quad n = 1, 2, \ldots,$$

and by setting

$$\beta_i^n = \theta_i \gamma_i^n, \quad \alpha_i^n = \theta_i \beta_i^n, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots,$$

with

$$\gamma_1^n = \frac{m_{n-1}}{k_{n-1} M_1}\alpha_{n-1}^{n-1}, \qquad \gamma_i^n = \pi_i \alpha_{i-1}^n, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots.$$

Then, for each $n = 2, 3, \ldots$ we have

$$V_{n-1}^{-1}([0, \alpha_{n-1}^{n-1})) \supset V_1^{-1}([0, \gamma_1^n]), \tag{3.31}$$
$$V_{i-1}^{-1}([0, \alpha_{i-1}^n)) \supset V_i^{-1}([0, \gamma_i^n]), \quad i = 2, \ldots, n, \tag{3.32}$$

together with

$$\gamma_i^n,\ \beta_i^n,\ \alpha_i^n \to 0 \ \textit{as}\ n \to \infty, \quad i = 1, 2, \ldots.$$

Proof: In what follows we fix $n = 1, 2, \ldots$. Let $\delta > 0$. It is well known that for each $i = 1, 2, \ldots$, the set $V_i^{-1}([0, \delta])$ is the volume bounded by an ellipsoid centered at the origin with smallest axis $\sqrt{m_i \delta}$ and largest axis $\sqrt{M_i \delta}$. Thus, (3.31) and (3.32) will hold if we have

$$\gamma_1^n < \frac{m_{n-1}}{M_1} \alpha_{n-1}^{n-1} \quad \text{and} \quad \gamma_i^n < \frac{m_{i-1}}{M_i} \alpha_{i-1}^n, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots$$

Because $\frac{m_n}{k_n M_1} < \frac{m_n}{M_1}$ and the real π_i is in $(0, \frac{m_{i-1}}{M_i})$ for each $i = 2, \ldots, n$ and each $n = 2, 3, \ldots$, we obtain (3.31) and (3.32).

Next, we set

$$\begin{aligned} y_1 &\triangleq \ln(\frac{m_1}{k_1 M_1} \theta_1^2) \\ y_n &\triangleq \ln(\frac{m_n}{k_n M_1} \theta_1^2) + \ln(\pi_2 \theta_2^2) + \ldots + \ln(\pi_n \theta_n^2), \quad n = 2, 3, \ldots, \end{aligned}$$

and from (3.29) together with (3.30), we obtain that

$$y_n \leq \ln(\pi_2 \theta_2^2), \quad n = 2, 3, \ldots. \tag{3.33}$$

We now fix $i = 1, 2, \ldots$. It is not hard to check that

$$\ln(\gamma_i^{i+l}) \leq y_i + y_{i+1} + \ldots + y_{i+l-1} + \ln(\gamma_i^i), \quad l = 1, 2, \ldots.$$

Therefore, (3.33) combined with the fact that $\ln(\pi_2 \theta_2^2) < 0$, imply that γ_i^{i+l} converges to 0 as l tends to ∞, and the proof of the lemma is complete. ■

Lemma 3.2 *For each $i = 1, 2, \ldots$, let P_i be a positive definite matrix, let $\frac{1}{M_i}$ and $\frac{1}{m_i}$ denote respectively its smallest and largest eigenvalue, let $V_i : \mathbb{R}^n \to [0, \infty)$ denote the mapping given by $V_i(x) = x^t P_i x$, and let θ_i be in $(0, 1)$. Further, let γ_1^1 be a given positive real and let $\{\eta_i\}_{i=1}^\infty$ and $\{r_i\}_{i=1}^\infty$ be sequences of positive reals such that*

$$r_i > \max(1, \frac{m_i}{M_1} \theta_1^2) \quad \textit{with} \quad \eta_i > \max(\frac{M_{i+1}}{m_i}, \theta_i^2), \quad i = 1, 2, \ldots. \tag{3.34}$$

Finally, let the sequences of positive reals $\{\gamma_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$, $\{\beta_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$ and $\{\alpha_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$ be defined by setting

$$\alpha_1^{-1} = 2\gamma_1^1, \quad \beta_i^n = \frac{\alpha_i^n}{\theta_i}, \quad \gamma_i^n = \frac{\beta_i^n}{\theta_i}, \quad i = 1, \ldots, |n|, \quad n = -1, -2, \ldots,$$

with

$$\begin{aligned} \alpha_{|n-1|}^{n-1} &= \frac{M_1 r_{|n-1|}}{m_{|n-1|}}\gamma_1^n, \quad n=-1,-2,\ldots \\ \alpha_i^n &= \eta_i\,\gamma_{i+1}^n, \quad i=1,\ldots,|n+1|, \quad n=-2,-3,\ldots. \end{aligned}$$

Then, for each $n=-2,-3,\ldots$ *we have*

$$V_i^{-1}([0,\alpha_i^n)) \supset V_i^{-1}(\,[0,\gamma_{i+1}^n]\,), \quad i=1,\ldots,|n+1|, \tag{3.35}$$

together with

$$V_{|n-1|}^{-1}([0,\alpha_{|n-1|}^{n-1})) \supset V_1^{-1}(\,[0,\gamma_1^n]\,), \quad n=-1,-2,\ldots, \tag{3.36}$$

and

$$\gamma_i^n \to +\infty \text{ as } n \to -\infty, \quad i=1,2,\ldots. \tag{3.37}$$

Proof: By using the fact that for each $i=1,2,\ldots$ and each $\delta>0$, the set $V^{-1}([0,\delta)$ is the volume of an ellipsoid, it follows that the inclusions (3.35) and (3.36) will hold if

$$\alpha_{|n-1|}^{n-1} > \frac{M_1}{m_{|n-1|}}\gamma_1^n \qquad \text{with} \qquad \alpha_i^n > \frac{M_{i+1}}{m_i}\gamma_{i+1}^n.$$

Thus, because $r_i>1$ and $\eta_i > \frac{M_{i+1}}{m_i}$ for each $i=1,2,\ldots$, we obtain that (3.35) and (3.36) hold.

Next, we set

$$y_l = \ln(\frac{M_1 r_{l+1}}{\theta_{l+1}^2 m_{l+1}}) + \ln(\frac{\eta_l}{\theta_l^2}) + \ldots + \ln(\frac{\eta_1}{\theta_1^2}), \quad l=1,2,\ldots.$$

It is not hard to check that for each $i=1,2,\ldots$, we have

$$\ln(\gamma_i^{-i-l}) = y_i + y_{i+1} + \cdots + y_{i+l-1} + \ln(\gamma_i^{-i}), \quad l=1,2,\ldots,$$

and because the definition of y_l, $l=1,2,\ldots$ together with (3.34) yield

$$y_l \geq \ln(\frac{\eta_1}{\theta_1^2}) > 0\,, \quad l=1,2,\ldots,$$

we obtain that $\ln(\gamma_i^{-i-l})$ converges to $+\infty$ as l tends to $+\infty$. Hence, (3.37) and the lemma. ■

3.4 Technical lemmas

We now present a lemma that is needed for the proof of Theorem 3.1

Lemma 3.3 *For each $i = 1, 2, \ldots$, let D_i be a bounded neighborhood of the origin (resp. $D_i = \mathbb{R}^n$), let $V_i : \overline{D}_i \to [0, \infty)$ be a Lyapunov function (resp. radially unbounded Lyapunov function), and let the mapping $u_i : D_i \to \mathbb{R}^m$ be continuous on a neighborhood of the origin with $u_i(0) = 0$. Let $\theta > 0$, and for each $i = 1, 2, \ldots$ let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D_i : \ V_i(x) < \beta\}, \quad \beta > 0.$$

Then, there exists a sequence of positive reals $\{\beta_i^n, \ i = 1, \ldots, n\}_{n=1}^\infty$ such that

$$\begin{aligned}
\beta_n^n &< \inf_{x \in \partial D_n} V_n(x), \quad n = 1, 2, \ldots, \\
\beta_i^n &\to 0 \ \ as \ \ n \to \infty, \quad i = 1, 2, \ldots, \\
W_{n-1}^{\beta_{n-1}^{n-1}} &\supset \overline{W}_1^{\beta_1^n}, \quad n = 2, 3, \ldots, \\
W_{i-1}^{\beta_{i-1}^n} &\supset \overline{W}_i^{\beta_i^n}, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots,
\end{aligned}$$

with

$$\overline{W}_{n-1}^{\beta_{n-1}^n} \subset D_n, \quad n = 2, 3, \ldots,$$

and

$$\|u_k(x)\| < \frac{\theta}{n}, \quad x \in D_k \cap W_n^{\beta_n^n}, \quad k = 1, \ldots, n+2, \quad n = 1, 2, \ldots.$$

Proof: We define the sequence $\{\beta_i^n, \ i = 1, \ldots, n\}_{n=1}^\infty$ by induction on n and i. Because $\{W_1^\beta\}_{\beta>0}$ is a base at the origin with $W_1^\alpha \subset W_1^\beta$ for all $\alpha < \beta$ (Lemma B.1), continuity at the origin of u_k combined with the fact that $u_k(0) = 0$ for each $k = 1, 2, 3$, yields the existence of β_1^1 in $(0, \inf_{x \in \partial D_1} V_1(x))$ such that

$$\|u_k(x)\| < \theta, \quad x \in D_k \cap W_1^{\beta_1^1}, \quad k = 1, 2, 3.$$

For $n = 2, 3, \ldots$, we define the sequence $\{\beta_i^n, \ i = 1, \ldots, n\}$ from the sequence $\{\beta_i^{n-1}, \ i = 1, \ldots, n-1\}$ as described hereafter: First, by using Lemma B.1 applied to $\{W_1^\beta\}_{\beta>0}$ we choose β_1^n in $(0, \frac{\beta_1^{n-1}}{2}]$ such that

$$W_{n-1}^{\beta_{n-1}^{n-1}} \supset \overline{W}_1^{\beta_1^n}.$$

Similarly, for each $i = 2, \ldots, n-1$, Lemma B.1 considered with $\{W_i^\beta\}_{\beta>0}$ enables us to define β_i^n from β_{i-1}^n and β_i^{n-1} by selecting β_i^n in $(0, \frac{\beta_i^{n-1}}{2}]$ such that

$$W_{i-1}^{\beta_{i-1}^n} \supset \overline{W}_i^{\beta_i^n}.$$

When $i = n-1$, we choose β_i^n such that we have the additional inclusion

$$\overline{W}_{n-1}^{\beta_{n-1}^n} \subset D_n.$$

Finally, by combining Lemma B.1 with the continuity of u_k at the origin for each $k = 1, \ldots, n+2$, we pick β_n^n in $(0, \inf_{x\in\partial D_n} V_n(x))$ such that

$$W_n^{\beta_{n-1}^n} \supset \overline{W}_n^{\beta_n^n},$$

and

$$\|u_k(x)\| < \frac{\theta}{n}, \quad x \in D_k \cap W_n^{\beta_n^n}, \quad k = 1, \ldots, n+2.$$

It is plain from this construction that for each $i = 1, 2, \ldots$, the sequence $\{\beta_i^n\}_{n=i}^\infty$ converges to 0 as n tends to ∞, and that for each $n = 1, 2, \ldots$, we have $\beta_n^n < \inf_{x\in\partial D_n} V_n(x)$, so that the lemma follows. ■

The next lemma is used in order to construct an explicit simultaneous stabilizer.

Lemma 3.4 *For each $i = 1, 2, \ldots$, let D_i be a bounded neighborhood of the origin (resp. $D_i = \mathbb{R}^n$), let $V_i : \overline{D}_i \to [0, \infty)$ be a Lyapunov function (resp. radially unbounded Lyapunov function), and let the mapping $u_i : D_i \to \mathbb{R}^m$ be continuous on a neighborhood of the origin with $u_i(0) = 0$. For each $i = 1, 2, \ldots$ let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D_i : V_i(x) < \beta\}, \quad \beta > 0,$$

and let $\theta > 0$. Then, there exists sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$, $\{\beta_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$ and $\{\gamma_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$, converging to the origin as n tends to ∞ and such that

$$\inf_{x\in\partial D_i} > \gamma_i^n > \beta_i^n > \alpha_i^n, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots,$$

with

$$\begin{aligned} W_{n-1}^{\alpha_{n-1}^{n-1}} &\supset \overline{W}_1^{\gamma_1^n}, \quad n = 2, 3, \ldots \\ W_{i-1}^{\alpha_{i-1}^n} &\supset \overline{W}_i^{\gamma_i^n}, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots, \end{aligned}$$

and

$$\|u_k(x)\| < \frac{\theta}{n}, \quad x \in D_k \cap W_n^{\beta_n^n}, \quad k = 1, \ldots, n+2, \quad n = 1, 2, \ldots.$$

Proof: We define the sequence $\{\gamma_i^n,\ i=1,\dots,n\}_{n=1}^{\infty}$, $\{\beta_i^n,\ i=1,\dots,n\}_{n=1}^{\infty}$ and $\{\alpha_i^n,\ i=1,\dots,n\}_{n=1}^{\infty}$ by induction on n and i. Because $\{W_1^\beta\}_{\beta>0}$ is a base at the origin with $W_1^\alpha \subset W_1^\beta$ for all $\alpha<\beta$ (Lemma B.1), continuity at the origin of u_k combined with the fact that $u_k(0)=0$ for each $k=1,2,3$, yields the existence of γ_1^1 in $(0, \inf_{x\in\partial D_1} V_1(x))$ such that

$$\|u_k(x)\| \;<\; \theta, \quad x\in W_1^{\gamma_1^1}, \quad k=1,2,3.$$

We then choose β_1^1 and α_1^1 satisfying $\gamma_1^1>\beta_1^1>\alpha_1^1>0$.

For $n=2,3,\dots$, we define the sequence $\{\gamma_i^n,\beta_i^n,\alpha_i^n\ i=1,\dots,n\}$ from the sequence $\{\gamma_i^{n-1},\ i=1,\dots,n-1\}$ as described hereafter: First, by using Lemma B.1 applied to $\{W_1^\gamma\}_{\gamma>0}$ we choose γ_1^n in $(0,\frac{\gamma_1^{n-1}}{2}]$ satisfying

$$W_{n-1}^{\alpha_{n-1}^{n-1}} \;\supset\; \overline{W_1^{\gamma_1^n}},$$

and we pick β_1^n and α_1^n such that $\gamma_1^n>\beta_1^n>\alpha_1^n>0$.

Similarly, for each $i=2,\dots,n-1$, Lemma B.1 considered with $\{W_i^\gamma\}_{\gamma>0}$ enables us to define γ_i^n from γ_{i-1}^n by selecting γ_i^n in $(0,\frac{\gamma_i^{n-1}}{2}]$ satisfying

$$W_{i-1}^{\alpha_{i-1}^n} \;\supset\; \overline{W_i^{\gamma_i^n}}.$$

We then select β_i^n and α_i^n such that $\gamma_i^n>\beta_i^n>\alpha_i^n>0$. When $i=n-1$, we choose β_i^n in order to have the additional inclusion

$$\overline{W_{n-1}^{\beta_{n-1}^n}} \;\subset D_n.$$

For $i=n$, Lemma B.1 yields the existence of γ_n^n in $(0, \inf_{x\in\partial D_n} V_n(x))$ satisfying

$$W_n^{\alpha_{n-1}^n} \;\supset\; \overline{W_n^{\gamma_n^n}},$$

and in view of the continuity of u_k at the origin for each $k=1,\dots,n+1$ we select β_n^n and α_n^n such that $\gamma_n^n>\beta_n^n>\alpha_n^n>0$ with

$$\|u_k(x)\| \;<\; \frac{\theta}{n}, \quad x\in D_k\cap W_n^{\beta_n^n}, \quad k=1,\dots,n+2.$$

It is then easily seen from this construction that for each $i=1,2,\dots$, the reals γ_i^n, β_i^n and α_i^n tend to 0 as n goes to ∞ and that the remaining assertions of the lemma hold. ∎

Lemma 3.5 *For each $i = 1, 2, \ldots$, let $V_i : \mathbb{R}^n \to [0, \infty)$ be a radially unbounded Lyapunov function and let W_i^β denote the set*

$$W_i^\beta \triangleq \{x \in D_i : V_i(x) < \beta\}, \quad \beta > 0.$$

Let γ_1^1 be a given positive reals. Then, there exists sequences of positive reals $\{\alpha_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$, $\{\beta_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$ and $\{\gamma_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$, converging to $+\infty$ as n tends to $-\infty$ and satisfying

$$\alpha_1^{-1} > \gamma_1^1 \qquad \textit{with} \qquad \gamma_i^n > \beta_i^n > \alpha_i^n, \quad i = 1, \ldots, |n|, \quad n = -1, -2, \ldots,$$

and

$$\begin{aligned} W_{|n-1|}^{\alpha_{|n-1|}^{n-1}} &\supset \overline{W_1^{\gamma_1^n}}, \quad n = -1, -2, \ldots \\ W_i^{\alpha_i^n} &\supset \overline{W_{i+1}^{\gamma_{i+1}^n}}, \quad i = 1, \ldots, |n+1|, \quad n = -2, -3, \ldots . \end{aligned}$$

Proof: We define the sequence $\{\gamma_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$, $\{\beta_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$ and $\{\alpha_i^n,\ i = 1, \ldots, |n|\}_{n=-1}^{-\infty}$ by induction on n and i.

First, we choose α_1^{-1}, β_1^{-1} and γ_1^{-1} such that

$$\gamma_1^{-1} > \beta_1^{-1} > \alpha_1^{-1} > \gamma_1^1.$$

For $n = -2, -3, \ldots$, we define the sequence $\{\alpha_i^n, \beta_i^n, \gamma_i^n,\ i = 1, \ldots, |n|\}$ from the sequence $\{\alpha_i^{n+1},\ i = 1, \ldots, |n+1|\}$ as described hereafter: First, by using Lemma B.1 applied to $\{W_{|n|}^\alpha\}_{\alpha > 0}$ we choose $\alpha_{|n|}^n$ satisfying

$$W_{|n|}^{\alpha_{|n|}^n} \supset \overline{W_1^{\gamma_1^{n+1}}},$$

and we pick $\beta_{|n|}^n$ and $\gamma_{|n|}^n$ such that $\gamma_{|n|}^n > \beta_{|n|}^n > \alpha_{|n|}^n$.

Similarly, for each $i = |n+1|, \ldots, 1$, Lemma B.1 considered with $\{W_i^\alpha\}_{\alpha > 0}$ enables us to define α_i^n from γ_{i+1}^n by selecting $\alpha_i^n \geq 2\alpha_i^{n+1}$ such that

$$W_i^{\alpha_i^n} \supset \overline{W_{i+1}^{\gamma_{i+1}^n}}.$$

We then select β_i^n and α_i^n such that $\gamma_i^n > \beta_i^n > \alpha_i^n$. It is easily seen from this construction that for each $i = 1, 2, \ldots$, the reals γ_i^n, β_i^n and α_i^n tend to $+\infty$ as n goes to $-\infty$ and that the remaining assertions of the lemma hold. ∎

Chapter 4

Time-Varying Simultaneous Asymptotic Stabilization of Linear Systems

We now introduce time-varying feedback laws and enrich the ideas introduced in Chapter 2 in order to prove that given any finite family LTI systems that are asymptotically stabilizable by means of LTI feedback, there exists a continuous time-varying feedback law that simultaneously globally exponentially stabilizes this family. We then derive sufficient conditions for the simultaneous asymptotic stabilizability of countably infinite families of stabilizable LTI systems. In both cases we provide simple design procedures as well as explicit controllers.

The construction of the simultaneous asymptotic stabilizer for finite families, differs significantly from the methods proposed in the literature in that we do not use any discretizing method and the controller that we provide is nonlinear and continuous. On the other hand, we do not know of any work that addresses the simultaneous asymptotic stabilization of **countably infinite** families of systems.

The case of finite families is discussed in Section 4.2 while that of countably infinite families is presented in Section 4.3. In Section 4.4 we illustrate these results with some examples. Finally, Section 4.5 contains some technical results.

4.1 Introduction

The design of time-varying feedback laws that simultaneous stabilize or simultaneously steer to the origin each one of the systems of a finite family of systems has been investigated in several papers. In [53], Khargonekar, Poolla and Tannenbaum consider finite collections of discrete time linear systems that individually admits a dead-beat controller. They show that given any such family, there exists a periodic linear time-varying (LTV) controller that simultaneously steers the systems of the family to the origin. In fact, the constructed feedback law periodically switches between the dead-beat controller of each system in order to steer each one of them to the origin. Along the same lines, Olbrot [62] proves that given any finite collection of continuous time or discrete time systems which are controllable in finite time to the origin, there exists a periodically time-varying feedback law that simultaneously steers each one of the systems of the family to the origin. Besides, discrete time LTV systems described by autoregressive moving average are considered in [32] and a procedure for the design of a simultaneous asymptotic stabilizer is proposed.

In the context of simultaneous stabilization of finite families of LTI systems by means of time-varying feedback, there exist three main results in the literature: In [48], Kabamba and Yang establish the simultaneous asymptotic stabilizability of such families by means of open loop periodically time-varying feedback. The controller that they find involves both the sampled output of the system and a periodic function of time. The resulting closed loop systems are therefore periodic linear systems whose asymptotic stability can be studied by means of Floquet Theory. On the other hand, Zhang and Blondel [84] propose a sufficient condition for the simultaneous asymptotic stabilizability of finite families of LTI systems by controllers based on LTI feedback laws together with zero-th order hold functions and samplers. While both of the two aforementioned design procedures comprise a sampling scheme, Khargonekar et al. [52] adopt a method that does not involve any discretization strategy and prove that any finite family of stabilizable LTI systems can be simultaneously asymptotically stabilized by a periodic LTV controller which is **piecewise continuous** with respect to the time. Although a design procedure can be deduced from [52], neither the simultaneous stabilizer nor the rates of convergence of the closed-loop systems are explicit.

In the following section we consider finite families of LTI systems that can be asymptotically stabilized by means of LTI feedback. Given any such family, we establish the existence of a continuous time-varying feedback law that simultaneously **globally exponentially** stabilizes the family. To prove this result, we adapt the ideas introduced in Chapter 2 to the use of time-varying feedback laws. Because our approach does not comprise a discretizing scheme, it should be compared to that used in Khargonekar et al. [52]. There are actually two main differences between the results derived in [52] by Khargonekar et al. and ours: First their controller is **discontinuous and linear** while ours is **continuous and nonlinear**. Secondly, we obtain explicit controllers as well as a lower bound on the exponential rate of convergence, while neither the controller nor the rates of convergence are explicit in [52].

4.2 Finite families

Throughout this section, we consider a finite family $\{S_i,\ i = 1, \ldots, I\}$ of **linear** systems

$$S_i: \quad \dot{x} = A_i x + B_i u, \quad i = 1, \ldots, I,$$

where $I \geq 2$ is a positive integer, the state x lies in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$ and for each $i = 1, \ldots, I$, the matrices A_i and B_i belong to $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ respectively. Finally, for each $i = 1, \ldots, I$, we assume that there exists K_i in $\mathbb{R}^{n\times m}$ such that the linear feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$ given by $u_i(x) = K_i x$ asymptotically stabilizes S_i.

Our goal is to prove the following theorem.

Theorem 4.1 *Assume that for each $i = 1, \ldots, I$, there exists a linear feedback law u_i that asymptotically stabilizes the linear system S_i. Then, there exists a time-varying feedback law $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, continuous on $[0, \infty) \times \mathbb{R}^n$, C^∞ on $[0, \infty) \times (\mathbb{R}^n \setminus \{0\})$, which simultaneously globally exponentially stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

The general lines of the proof of this theorem are as follows: For each $i = 1, \ldots, I$, we let V_i denote a Lyapunov function for the system $\dot{x} = (A_i + B_i K_i)x$. We introduce a sequence $\{b_i^n(\cdot),\ i = 1, \ldots, I\}_{n\in\mathbb{Z}}$ of mappings defined from $[0, \infty)$ into $(0, +\infty)$, decreasing to 0 as t tends to $+\infty$, and such that for each $t \geq 0$, the sequence of neighborhoods $\{V_i^{-1}(\,[0, b_i^n(t))\,)\}_{n\in\mathbb{Z}}$ is a base at the origin. We then design a time-varying feedback law $v(t, x)$ such that for each $i = 1, \ldots, I$, each n in $\mathbb{Z}$ and each $t \geq 0$, we have $v(t, x) = u_i(x)$ for all x in $V_i^{-1}(b_i^n(t))$. Finally, we show that for each $i = 1, \ldots, I$, each n in $\mathbb{Z}$ and each $t_0 \geq 0$, each trajectory of the system $\dot{x} = f_i(x, v(t, x))$, that starts in the set $V_i^{-1}([0, b_i^n(t_0)])$ at time $t = t_0$, remains in the set $V_i^{-1}([0, b_i^n(t)])$ for all $t \geq t_0$. For each $i = 1, \ldots, I$, we conclude that v asymptotically stabilizes the system S_i, upon noting that the mapping b_i^n converges to 0 as t tends to $+\infty$ for each n in $\mathbb{Z}$.

We now present a technical lemma which is used to prove that the trajectory $x(\cdot, x_0, t_0)$ of the system S_i lies in $V_i^{-1}(\,[0, b_i^n(t)]\,)$, for each $t \geq t_0$, whenever x_0 belongs to $V_i^{-1}(\,[0, b_i^n(t_0)]\,)$.

4.2.1 Invariance criteria

The following lemma is the key to prove the invariance of the set $V_i([0, b_i^n(t)])$ for each $t \geq 0$. It is also used in Section 4.3 and in Chapter 5.

Lemma 4.1 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to (0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function). Further, let the mapping $f : [0, \infty) \times \overline{D} \to \mathbb{R}^n$ be continuous, and let the mapping $b : [0, \infty) \to (0, \inf_{x \in \partial D} V(x))$ be C^1. Finally, for each $\beta > 0$ set*

$$W^\beta \triangleq \{x \in D : \ V(x) < \beta\},$$

and assume that

$$\nabla V(x) \, f(t, x) \ < \ \dot{b}(t), \quad x \in \partial W^{b(t)}, \quad t \geq 0. \tag{4.1}$$

Then, for each $t_0 \geq 0$ and each x_0 in $\overline{W}^{b(t_0)}$, the trajectory $x(\cdot, x_0, t_0)$ of $\dot{x} = f(t, x)$ starting from x_0 at time t_0 satisfies

$$x(t, x_0, t_0) \ \in \ \overline{W}^{b(t)}, \quad t \geq t_0.$$

Proof: Fix $t_0 \geq 0$ and x_0 in $\overline{W}^{b(t_0)}$, and let $x(\cdot, x_0, t_0)$ denote the trajectory of $\dot{x} = f(t, x)$ that starts from x_0 at time t_0.

For the sake of clarity, we split the proof into that of two claims.

Claim 1: *Let $t_3 > t_0$ be such that $x(t, x_0, t_0)$ lies in D for each t in $[t_0, t_3)$. Then, $V(x(t, x_0, t_0)) \leq b(t), \ t \in [t_0, t_3)$*

Assume that Claim 1 does not hold. Then, there exists t_2 in $[t_0, t_3)$ such that

$$V(x(t_2, x_0, t_0)) > b(t_2)$$

Because $V(x(t_0, x_0, t_0)) - b(t_0) \leq 0$, continuity of the mapping $V(x(\cdot, x_0, t_0)) - b(\cdot)$ yields the existence of t_1 in $[t_0, t_2)$ and h_1 in $(0, t_3 - t_1)$ such that

$$V(x(t_1, x_0, t_0)) - b(t_1) = 0, \tag{4.2}$$

and

$$V(x(t_1 + h, x_0, t_0)) - b(t_1 + h) > 0, \quad h \in (0, h_1). \tag{4.3}$$

By assumption $x(t_1, x_0, t_0)$ belongs to D, and we obtain from (4.2) that

$$x(t_1, x_0, t_0) \ \in \ \partial W^{b(t_1)}. \tag{4.4}$$

This, together with Assumption (4.1) yields

$$\frac{d}{dt}|_{t=t_1} V(x(t, x_0, t_0)) \ = \ \nabla V(x(t_1, x_0, t_0)) \, f(\, t_1, x(t_1, x_0, t_0) \,) \ < \ \dot{b}(t_1),$$

and continuity of the mappings $\nabla V(\cdot)$, $f(\cdot, \cdot)$ and $\dot{b}(\cdot)$, combined with (4.2) yields

$$V(x(t_1 + h, x_0, t_0)) \ < \ b(t_1 + h), \quad \text{for } \ h > 0 \ \text{ small enough,}$$

a contradiction with (4.3). The proof of Claim 1 is thus complete.

Claim 2: $x(t, x_0, t_0)$ *lies in* D, *for each* $t \geq t_0$.

Because Claim 2 clearly holds if $D = \mathbb{R}^n$, we assume that D is bounded.

Suppose that the claim does not hold. Because x_0 is in D, Lemma B.2 **(iii)** yields the existence of $t_1 > t_0$ such that

$$x(t_1, x_0, t_0) \in \partial D, \tag{4.5}$$

with

$$x(t, x_0, t_0) \in D, \quad t \in [t_0, t_1). \tag{4.6}$$

This last relation combined with Claim 1, implies that

$$V(x(t, x_0, t_0)) \leq b(t), \quad t \in [t_0, t_1). \tag{4.7}$$

On the other hand, from (4.5) and the definition of b, we get

$$V(x(t_1, x_0, t_0)) \geq \inf_{x \in \partial D} V(x) > b(t_1),$$

and continuity of the mapping $V(x(\cdot, x_0, t_0)) - b(\cdot)$ at t_1 yields the existence of h_1 in $(0, t_1 - t_0)$ satisfying

$$V(x(t_1 - h, x_0, t_0)) > b(t_1 - h), \quad h \in (0, h_1),$$

which contradicts (4.7). Hence Claim 2.

Let $t_3 > t_0$. By Claim 2, the point $x(t, x_0, t_0)$ lies in D for each $t \geq t_0$, and it follows from Claim 1 applied with t_3 that

$$V(x(t, x_0, t_0)) \leq b(t), \quad t \in [t_0, t_3).$$

The proof of the lemma is completed upon noting that this last argument holds for all $t_3 > t_0$, all x_0 in $\overline{W}^{b(t_0)}$ and all $t_0 \geq 0$. ■

We are now able to prove Theorem 4.1.

4.2.2 A proof of Theorem 4.1

Proof of Theorem 4.1 :

For each $i = 1, \ldots, I$, because the system $\dot{x} = A_i x + B_i u_i(x)$ is linear and asymptotically stable, it admits a Lyapunov function $V_i : \mathbb{R}^n \to (0, \infty)$ such that $V_i(x) = x^t P_i x$ where P_i is a positive definite matrix. For each $i = 1, \ldots, I$, let Q_i be the positive definite matrix defined by

$$\nabla V_i(x)\,(A_i\,x + B_i\,u_i(x)) \;=\; -x^t Q_i x, \quad x \in \mathbb{R}^n .$$

For each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, we let W_i^β denote the set

$$W_i^\beta \;\triangleq\; V_i^{-1}(\,[0,\beta)\,).$$

For each $i = 1, \ldots, I$, let θ_i and π_i satisfy the assumptions of Lemma 2.3. Further let $\{\alpha_i^n,\ i = 1, \ldots, I\}_{n\in\mathbf{Z}}$, $\{\beta_i^n,\ i = 1, \ldots, I\}_{n\in\mathbf{Z}}$ and $\{\gamma_i^n,\ i = 1, \ldots, I\}_{n\in\mathbf{Z}}$ be defined by the formulas (2.26)-(2.29) given in the statement of Lemma 2.3. Thus, by Lemma 2.3, for each n in $\mathbb{Z}$ we have

$$V_{i-1}^{-1}([0,\alpha_{i-1}^n)) \;\supset\; V_i^{-1}(\,[0,\gamma_i^n]\,), \quad i = 2, \ldots, I, \tag{4.8}$$

$$V_I^{-1}([0,\alpha_I^n)) \;\supset\; V_1^{-1}(\,[0,\gamma_1^{n+1}]\,), \tag{4.9}$$

and for each $i = 1, \ldots, I$ we have

$$\beta_i^n \to 0 \;\; as \;\; n \to \infty \qquad \text{with} \qquad \beta_i^n \to \infty \;\; as \;\; n \to -\infty.$$

Construction of the simultaneous stabilizer :

We now seek a C^1 mapping $h : [0,\infty) \to (0,\infty)$ such that, the mapping $b_i^n : [0,\infty) \to (0,\infty)$ given by

$$b_i^n(t) \;=\; \beta_i^n\, h(t), \quad t \geq 0,$$

satisfy

$$\nabla V_i(x)\,(\,A_i x + B_i u_i(x)\,) \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0, \tag{4.10}$$

or equivalently

$$-\,x^t Q_i\, x \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0, \tag{4.11}$$

for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$. In what follows, we fix $i = 1, \ldots, I$, n in $\mathbb{Z}$ and $t \geq 0$. Let x be such that $x^t P_i\, x = b_i^n(t)$. Then, by elementary linear algebra, we get

$$-x^t x \;\leq\; -\frac{b_i^n(t)}{\lambda_{max}(P_i)}$$

and because we also have $-x^t Q_i\, x \leq -\lambda_{min}(Q_i)\, x^t x$, inequality (4.11) will be satisfied if

$$-\,\frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)}\, b_i^n(t) \;<\; \dot{b}_i^n(t). \tag{4.12}$$

Because we require that $h(t) > 0$ and $b_i^n(t) = \beta_i^n h(t)$, inequality (4.12) will hold if

$$\frac{\dot{h}(t)}{h(t)} > -\frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)}. \tag{4.13}$$

We now set $\rho \triangleq \min_{i=1,\ldots,I} \left(\frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} \right)$ and we deduce from (4.13) that the desired assertion (4.10) will be satisfied for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, if

$$\frac{\dot{h}(t)}{h(t)} > -\rho, \quad t \geq 0. \tag{4.14}$$

Let the mapping $h : [0, \infty) \to (0, \infty)$ be given by

$$h(t) = e^{-\frac{\rho}{\eta}t}, \quad t \geq 0,$$

where η is a fixed constant in $(1, \infty)$. It is plain that h satisfies (4.14) so that for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, the mapping $b_i^n : [0, \infty) \to (0, \infty)$ defined by

$$b_i^n(t) = \beta_i^n h(t) = \beta_i^n e^{-\frac{\rho}{\eta}t}, \quad t \geq 0,$$

satisfies the desired assertion (4.10).

Next, for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, we define the mappings a_i^n, c_i^n : $[0, \infty) \to (0, \infty)$ by setting

$$c_i^n(t) = \gamma_i^n h(t) \quad \text{and} \quad a_i^n(t) = \alpha_i^n h(t), \quad t \geq 0.$$

For each $t \geq 0$, because we have

$$b_i^n(t) = \theta_i c_i^n(t) \quad \text{with} \quad a_i^n(t) = \theta_i b_i^n(t), \quad i = 1, \ldots, I, \quad n \in \mathbb{Z},$$

and

$$c_1^{n+1}(t) = \pi_1 a_I^n(t) \quad \text{with} \quad c_i^n(t) = \pi_i a_{i-1}^n(t), \quad i = 2, \ldots, I, \quad n \in \mathbb{Z},$$

it is easily checked that the sequences $\{c_i^n(t),\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, $\{b_i^n(t),\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$ and $\{a_i^n(t),\ i = 1, \ldots, I\}_{n \in \mathbf{Z}}$, satisfy the assertions of Lemma 2.3. This together with the fact that $h(\cdot)$ decreases to 0 as t tends to ∞, yield for each $t_0 \geq 0$

$$\sup_{t \geq t_0} b_i^n(t) \to 0 \quad \text{as} \quad n \to +\infty, \quad i = 1, \ldots, I, \tag{4.15}$$

$$b_i^n(t_0) \to +\infty \quad \text{as} \quad n \to -\infty, \quad i = 1, \ldots, I. \tag{4.16}$$

It also follows from Lemma 2.3 that for each $t \geq 0$, we have a double-sided sequence of neighborhoods

$$\begin{array}{cccccccccccccc}
\vdots & & & & & & & & & & & \vdots & & \\
W_1^{c_1^0(t)} & \supset & & & & \dots & & & & & & W_I^{a_I^0(t)} & \supset & \\
W_1^{c_1^1(t)} & \supset & W_1^{b_1^1(t)} & \supset & W_1^{a_1^1(t)} & \supset & W_2^{c_2^1(t)} & \supset & \dots & \supset & & W_I^{a_I^1(t)} & \supset & \\
W_1^{c_1^2(t)} & \supset & W_1^{b_1^2(t)} & \supset & W_1^{a_1^2(t)} & \supset & W_2^{c_2^2(t)} & \supset & \dots & \supset & & W_I^{a_I^2(t)} & \supset & \\
\vdots & & & & & & \vdots & & & & & \vdots & &
\end{array} \tag{4.17}$$

such that each neighborhood contains the closure of the neighborhood that follows. In view of this comment, for each $i = 1, \dots, I$ and each n in $\mathbb{Z}$, we define the mapping $q_i^n : [0, \infty) \times \mathbb{R}^n \to [0, 1]$ by setting

$$q_i^n(t,x) = \begin{cases} e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(b_i^n(t)-a_i^n(t))^2}} & \text{if } V_i(x) \in (a_i^n(t), b_i^n(t)] \\ e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(c_i^n(t)-b_i^n(t))^2}} & \text{if } V_i(x) \in (b_i^n(t), c_i^n(t)) \\ 0\,, & \text{otherwise} \end{cases}$$

and we let the mapping $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$v(t,x) = \sum_{i=1}^{I} \sum_{n \in \mathbb{Z}} u_i(x) q_i^n(t,x), \quad (t,x) \in [0, \infty) \times \mathbb{R}^n .$$

We note that $v(t, 0) = 0$ for each $t \geq 0$ and we show that v is continuous on $[0, \infty) \times \mathbb{R}^n$ and C^∞ on $[0, \infty) \times (\mathbb{R}^n \backslash \{0\})$.

The feedback law v is continuous on $[0, \infty) \times \mathbb{R}^n$ and C^∞ on $[0, \infty) \times (\mathbb{R}^n \backslash \{0\})$:

Let (t, x) be in $[0, \infty) \times \mathbb{R}^n \backslash \{0\}$. It is easily checked from (4.17) that there exists a unique pair of integer (i, n) in $\{1, \dots, I\} \times \mathbb{Z}$ such that either one of the following two assertions holds:

- We have $V_i(x) \in [a_i^n(t), c_i^n(t)]$. In that case (4.17) together with the continuity of the mappings V_j, a_j^m and c_j^m for each $j = 1, \dots, I$ and each m in $\mathbb{Z}$, yield the existence of a neighborhood U of (t, x) in $[0, \infty) \times (\mathbb{R}^n \backslash \{0\})$ such that
$$v(\tau, y) \;=\; u_i(y)\, q_i^n(\tau, y), \quad (\tau, y) \in U. \tag{4.18}$$
- We have $V_i(x) \in (a_i^n(t), c(t))$ where the mapping c denotes either c_1^{n+1} if $i = I$ or otherwise c_{i+1}^n. In that case the continuity of the mappings

V_i, a_j^m, c_j^m for each $j = 1, \ldots, I$ and each $m = 1, 2, \ldots$, yields the existence a neighborhood U of (t, x) in $[0, \infty) \times (\mathbb{R}^n \backslash \{0\})$ such that $V_i(y) \in (a_i^n(\tau), c(\tau))$, $(\tau, y) \in U$ and it follows that

$$v(\tau, y) \;=\; 0, \quad (\tau, y) \in U. \tag{4.19}$$

Because q_i^n is C^∞ on $[0, \infty) \times \mathbb{R}^n$ [follows from Lemma B.6] and u_i is C^∞ on $\mathbb{R}^n$ for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, (4.18) and (4.19) imply that v is C^∞ on $[0, \infty) \times \mathbb{R}^n \backslash \{0\}$.

Moreover, because the mappings q_i^n take values in $[0, 1]$, the equalities (4.18) and (4.19) together with the fact that $v(t, 0) = 0$, $t \geq 0$ yield

$$\|v(t, x)\| \;\leq\; \max\left(\|u_1(x)\|, \ldots, \|u_I(x)\|\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

and continuity of u_i for each $i = 1, \ldots, I$, implies that v is continuous at each point $(t, 0)$, $t \geq 0$. Therefore, v is continuous on $[0, \infty) \times \mathbb{R}^n$.

Global exponential stability :

Throughout the rest of the proof, we fix $i = 1, \ldots, I$. From the definition of v, it is not hard to see that

$$v(t, x) = u_i(x), \quad t \geq 0, \quad x \in V_i^{-1}(\, b_i^n(t)\,), \quad n \in \mathbb{Z}.$$

Therefore, from (4.10) we get

$$\nabla V_i(x)\,(A_i x + B_i v(t, x)) \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0, \quad n \in \mathbb{Z},$$

and upon recalling that $\partial W_i^\beta = V_i^{-1}(\beta)$ and $\overline{W}_i^\beta = V_i^{-1}([0, \beta])$ for each $\beta > 0$ [follows from Lemma B.3 **(i)**], Lemma 4.1 implies that for each $t_0 \geq 0$ and each n in $\mathbb{Z}$, the trajectory $x(\cdot, x_0, t_0)$ of $\dot{x} = A_i x + B_i v(t, x)$ starting from x_0 at time $t = t_0$, satisfies

$$V_i(x(t, x_0, t_0)) \;\leq\; b_i^n(t), \quad t \geq t_0, \quad x_0 \in \overline{W}_i^{b_i^n(t_0)}. \tag{4.20}$$

Recall that from the definition of the sequence $\{\beta_i^n\}_{n \in \mathbf{Z}}$, we have

$$\beta_i^{n+k} \;=\; (\pi_1 \cdots \pi_I\, \theta_1^2 \cdots \theta_I^2)^k \beta_i^n, \quad n \in \mathbb{Z}, \quad k = 0, 1, \ldots.$$

Upon setting $T \triangleq -\frac{\eta}{\rho} \log(\pi_1 \ldots \pi_I\, \theta_1^2 \ldots \theta_I^2)$, the previous equality translates to

$$\beta_i^n e^{-\frac{\rho}{\eta} kT} \;=\; \beta_i^{n+k}, \quad n \in \mathbb{Z}, \quad k = 0, 1, \ldots, \tag{4.21}$$

which in turn easily yields

$$b_i^n(t+kT) \;=\; b_i^{n+k}(t), \quad t \geq 0, \quad n \in \mathbb{Z}, \quad k = 0, 1, \ldots. \tag{4.22}$$

Let x_0 be in $\mathbb{R}^n$. Because the sequence $\{b_i^n(0) = \beta_i^n\}_{n\in\mathbb{Z}}$ is strictly decreasing and converges to 0 and $+\infty$ as n tends to $+\infty$ and $-\infty$ respectively, there exists an integer $\bar{n}$ such that

$$b_i^{\bar{n}+1}(0) \;<\; V_i(x_0) \;\leq\; b_i^{\bar{n}}(0). \tag{4.23}$$

Let t_0 be in $[0,\infty)$. Then, there exist an integer k and t_0' in $[0,T)$ such that $t_0 = kT + t_0'$. By combining (4.22) with the fact that the mapping b_i^n is decreasing for each n in $\mathbb{Z}$, we get

$$b_i^{\bar{n}}(0) \;=\; b_i^{\bar{n}-k-1}((k+1)T) \;\leq\; b_i^{\bar{n}-k-1}(t_0),$$

so that (4.23) yields $x_0 \in V_i^{-1}([0, b_i^{\bar{n}-k-1}(t_0)]$. Thus, assertion (4.20), implies that

$$V_i(x(t,x_0,t_0)) \;\leq\; b_i^{\bar{n}-k-1}(t), \quad t \geq t_0,$$

and from the expression of $b_i^{\bar{n}-k-1}(t)$ we obtain that

$$V_i(x(t,x_0,t_0)) \;\leq\; \beta_i^{\bar{n}-k-1}\, e^{-\frac{\varrho}{\eta}(t-t_0+t_0'+kT)}, \quad t \geq t_0.$$

Next, using (4.21), we can rewrite this last inequality as

$$V_i(x(t,x_0,t_0)) \;\leq\; \beta_i^{\bar{n}-1}\, e^{-\frac{\varrho}{\eta}(t-t_0+t_0')}, \quad t \geq t_0,$$

and from the non-negativeness of t_0', we get

$$V_i(x(t,x_0,t_0)) \;\leq\; \beta_i^{\bar{n}-1}\, e^{-\frac{\varrho}{\eta}(t-t_0)}, \quad t \geq t_0. \tag{4.24}$$

The identity (4.21) yields $\beta_i^{\bar{n}-1} = e^{\frac{2\varrho}{\eta}T}\beta_i^{\bar{n}+1}$, so that the inequality $\beta_i^{\bar{n}-1} < e^{\frac{2\varrho}{\eta}T}V_i(x_0)$ follows from (4.23). Thus, (4.24) implies that

$$V_i(x(t,x_0,t_0)) \;\leq\; \left(e^{\frac{2\varrho}{\eta}T}V_i(x_0)\right) e^{-\frac{\varrho}{\eta}(t-t_0)}, \quad t \geq t_0,$$

or equivalently

$$\sqrt{V_i(x(t,x_0,t_0))} \;\leq\; e^{\frac{\varrho}{\eta}T}\sqrt{V_i(x_0)}\; e^{-\frac{\varrho}{2\eta}(t-t_0)}, \quad t \geq t_0.$$

Because $e^{\frac{\varrho}{\eta}T}$ is a constant and the mapping $x \mapsto \sqrt{V_i(x)}$ is a norm on $\mathbb{R}^n$, we obtain from the equivalence of all norms on $\mathbb{R}^n$ that v globally exponentially stabilizes S_i [according to Definition 1.2 (**v**)]. The proof of the theorem is complete upon noting that the previous argument holds for each $i = 1, \ldots, I$.

We note the rates of convergence of the closed-loop systems corresponding to S_i is greater than $\frac{\varrho}{2\eta}$, for each $i = 1, \ldots, I$.

■

We now extend this result and we establish the simultaneous asymptotic stabilizability of a class of countably infinite family of stabilizable LTI systems.

4.3 Infinite families

Throughout this section, we consider a **countably infinite** family $\{S_i,\ i = 1, 2, \ldots\}$ of linear systems

$$S_i : \quad \dot{x} \; = \; A_i x \; + \; B_i u, \quad i = 1, 2, \ldots ,$$

where the state x lies in $\mathbb{R}^n$, the input u is in $\mathbb{R}^m$ and for each $i = 1, 2, \ldots$, the matrices A_i and B_i belong to $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$ respectively. We assume that for each $i = 1, 2, \ldots$, there exists K_i in $\mathbb{R}^{m\times n}$ such that the linear feedback $u_i : \mathbb{R}^n \to \mathbb{R}^m$ given by $u_i(x) = K_i x$ for each x in $\mathbb{R}^n$, asymptotically stabilizes S_i. For each $i = 1, 2, \ldots$, we let P_i be a positive definite matrix in $\mathbb{R}^{n\times n}$ and we let $V_i : \mathbb{R}^n \to [0, \infty)$ be a Lyapunov function for the system $\dot{x} = (A_i + B_i K_i)\, x$ given by $V_i(x) = x^t P_i x$ for each x in $\mathbb{R}^n$. Further, we let Q_i be the positive definite matrix defined by

$$\nabla V_i(x)\left((A_i + B_i K_i)\, x\right) \; = \; -x^t Q_i x, \quad x \in \mathbb{R}^n .$$

The purpose of this section is to prove the following theorem. The proof is based on the same ideas as those of the proof of Theorem 4.1. The main difference lies in the structure of the sequence of mappings $\{b_i^n\}$ that must be considered here.

Theorem 4.2 *Assume that, in addition to the enforced assumptions, the following holds:*

i) *There exists a positive real M such that $\|K_i\| \leq M,\ i = 1, 2, \ldots$.*

ii) *The real $\rho \triangleq \inf_{i=1,2,\ldots} \left(\dfrac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} \right)$ is strictly positive.*

Then, there exists a time-varying feedback law $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$, continuous on $[0, \infty) \times \mathbb{R}^n$, C^∞ on $[0, \infty) \times (\mathbb{R}^n \backslash \{0\})$, and which simultaneously globally asymptotically stabilizes the family $\{S_i,\ i = 1, 2, \ldots\}$.

Proof:

Throughout, we let W_i^β denote the set

$$W_i^\beta \triangleq V_i^{-1}([0,\beta)), \quad \beta > 0, \quad i = 1,2,\dots.$$

Let $\widehat{\gamma}_1^1$ be a given positive real. By applying Lemma 4.2 with $\widehat{\gamma}_1^1$ and the family $\{V_i,\ i=1,2,\dots\}$, we obtain three sequence of positive reals $\{\gamma_i^n,\ i = 1,\dots,n\}_{n=1}^\infty$, $\{\beta_i^n,\ i=1,\dots,n\}_{n=1}^\infty$ and $\{\alpha_i^n,\ i=1,\dots,n\}_{n=1}^\infty$ converging to 0 as n tends to ∞ and satisfying the assertions of the lemma.

Now by applying Lemma 3.2 with γ_1^1 as defined in the sequence $\{\gamma_i^n,\ i = 1,\dots,n\}_{n=1}^\infty$, we obtain three sequences $\{\gamma_i^n,\ i = 1,\dots,|n|\}_{n=-1}^{-\infty}$, $\{\beta_i^n,\ i = 1,\dots,|n|\}_{n=-1}^{-\infty}$ and $\{\alpha_i^n,\ i = 1,\dots,|n|\}_{n=1}^{-\infty}$ converging to $+\infty$ as n tends to $-\infty$ and satisfying the assertions of the lemma.

Let $\rho \triangleq \inf_{i=1,2,\dots}\left(\dfrac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)}\right)$ and let $\zeta > 1$. We note that by Assumption **(i)** of Theorem 4.2, the real ρ is positive and we let the mapping $h : [0,\infty) \to (0,\infty)$ be given by

$$h(t) \;=\; e^{-\frac{\rho}{\zeta}t}, \quad t \geq 0.$$

For each $i = 1,2,\dots$ and each n in $\mathbb{Z}\setminus\{0\}$, we now define the mappings $c_i^n, b_i^n, a_i^n :$ $[0,\infty) \to (0,\infty)$ by setting

$$a_i^n(t) \;\triangleq\; \alpha_i^n h(t), \quad c_i^n(t) \;\triangleq\; \beta_i^n\, h(t) \;\text{ and }\; c_i^n(t) \;\triangleq\; \gamma_i^n\, h(t), \quad t \geq 0.$$

It is not hard to see from Lemmas 4.2 and 3.2 that for each $t \geq 0$ we have

$$a_1^{-1}(t) > c_1^1(t) \quad \text{and} \quad a_{|n-1|}^{n-1}(t) \;>\; \frac{M_1}{m_{|n-1|}}c_1^n(t), \qquad n = \dots,-2,-1,2,3,\dots$$

together with

$$a_i^n(t) \;>\; \frac{M_{i+1}}{m_i}c_{i+1}^n(t), \quad i = 1,\dots,|n|-1, \quad n = \dots,-3,-2,2,3,\dots,$$

and

$$c_i^n(t) \;>\; b_i^n(t) \;>\; a_i^n(t), \quad i = 1,\dots,|n|, \quad n \in \mathbb{Z}\setminus\{0\}.$$

Thus, for each $t \geq 0$, we have a sequence of nested neighborhoods

$$\begin{array}{lclclclclclcl}
W_1^{c_1^{-3}(t)} & \vdots & \vdots & & & & & & & & & & \\
W_1^{c_1^{-2}(t)} & \supset & W_1^{b_1^{-2}(t)} & \supset & W_1^{a_1^{-2}(t)} & \supset & W_2^{c_2^{-2}(t)} & \supset & W_2^{b_2^{-2}(t)} & \supset & W_2^{a_2^{-2}(t)} & \supset & \\
W_1^{c_1^{-1}(t)} & \supset & W_1^{b_1^{-1}(t)} & \supset & W_1^{a_1^{-1}(t)} & \supset & & & & & & & \\
W_1^{c_1^{1}(t)} & \supset & W_1^{b_1^{1}(t)} & \supset & W_1^{a_1^{1}(t)} & \supset & & & & & & & \\
W_1^{c_1^{2}(t)} & \supset & W_1^{b_1^{2}(t)} & \supset & W_1^{a_1^{2}(t)} & \supset & W_2^{c_2^{2}(t)} & \supset & W_2^{b_2^{2}(t)} & \supset & W_2^{a_2^{2}(t)} & \supset & \\
W_1^{c_1^{3}(t)} & \vdots & \vdots & & & & & & & & & &
\end{array} \tag{4.25}$$

such that each neighborhood contains the closure of the neighborhood that follows. Moreover, for each $i = 1, 2, \ldots$, because α_i^n, β_i^n and γ_i^n tend to 0 and $+\infty$ as n tends respectively to $+\infty$ and $-\infty$, we obtain

$$\begin{aligned} a_i^n(t),\ b_i^n(t),\ c_i^n(t) &\to 0 \quad as \quad n \to +\infty, \quad t \geq 0, \\ a_i^n(t),\ b_i^n(t),\ c_i^n(t) &\to +\infty \quad as \quad n \to -\infty, \quad t \geq 0. \end{aligned}$$

In view of (4.25), we define the mapping $q_i^n : [0,\infty) \times \mathbb{R}^n \to [0,1]$ by setting

$$q_i^n(t,x) = \begin{cases} e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(b_i^n(t)-a_i^n(t))^2}} & \text{if } V_i(x) \in (a_i^n(t), b_i^n(t)] \\ e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(c_i^n(t)-b_i^n(t))^2}} & \text{if } V_i(x) \in (b_i^n(t), c_i^n(t)) \\ 0\,, & \text{otherwise} \end{cases}, \tag{4.26}$$

for each $i = 1, 2, \ldots$ and each n in $\mathbb{Z}\backslash\{0\}$. Finally, we let the mapping $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$v(t,x) = \sum_{n=1}^{+\infty} \sum_{i=1}^{n} u_i(x) q_i^n(t,x) \ + \sum_{n=-1}^{-\infty} \sum_{i=1}^{|n|} u_i(x) q_i^n(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n .$$

Because the supports of the mappings of the collection $\{q_i^n,\ i = 1, \ldots, n\}_{n \in \mathbf{Z}\backslash\{0\}}$ are disjoint [follows from (4.25)], we have

$$\|v(t,x)\| \ \leq \ \inf_{i=1,2,\ldots} \|u_i(x)\| \ \leq \ \left(\inf_{i=1,2,\ldots} \|K_i\|\right) \|x\|, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n,$$

so that Assumption **(ii)** of Theorem 4.2 yields the continuity of v at any point $(t,0),\ t \geq 0$.

Further, in view of (4.25) and by using exactly the same argument as that used in the proof of Theorem 4.1 to show that v is C^∞ on $[0,\infty) \times (\mathbb{R}^n \backslash\{0\})$, we obtain that v (as defined here) is C^∞ on $[0,\infty) \times (\mathbb{R}^n \backslash\{0\})$.

Stability :

We now fix $i = 1, 2, \ldots$ and n in $\mathbb{Z}\backslash\{0\}$. Recall that by definition, we have

$$b_i^n(t) \ = \ \beta_i^n e^{-\frac{\rho}{\zeta} t}, \quad t \geq 0,$$

with $\zeta > 1$ and $\rho = \inf_{i=1,2,\ldots} \left(\frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)}\right)$. Using the inequalities

$$x^t Q_i x \geq \lambda_{min}(Q_i) x^t x \qquad \text{and} \qquad x^t P_i x \leq \lambda_{max}(P_i) x^t x,$$

together with an argument similar to that used in the proof of Theorem 4.1 to construct the mapping h, we obtain

$$\nabla V_i(x)\,(A_i x + B_i u_i(x)) \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0,$$

and because $v(t,x) = u_i(x)$ whenever $x \in V_i^{-1}(b_i^n(t))$, we get

$$\nabla V_i(x)\,(A_i x + B_i v(t,x)) \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0. \tag{4.27}$$

Next, we fix $i = 1, 2, \ldots$ and we show that v stabilizes S_i: Let $\varepsilon > 0$ and $t_0 \geq 0$ be given. Because $b_i^n(t_0)$ converges to 0 as n tends to $+\infty$, there exists an integer n such that $\overline{W}_i^{b_i^n(t_0)} \subset B_\varepsilon(0)$. Let $\delta > 0$ be such that $B_\delta(0) \subset \overline{W}_i^{b_i^n(t_0)}$ and let x_0 be in $B_\delta(0)$. In view of (4.27) and Lemma 4.1, for each $t \geq t_0$, the trajectory $x(t, x_0, t_0)$ of $\dot{x} = A_i x + B_i v(t,x)$ lies in the set $\overline{W}_i^{b_i^n(t)}$. Thus, because $b_i^n(\cdot)$ is a decreasing function of time, it follows that this trajectory remains in $B_\varepsilon(0)$. In short v stabilizes S_i.

Convergence to the origin :

First, we fix $i = 1, 2, \ldots$. Let x_0 be in $\mathbb{R}^n \backslash \{0\}$ and let $t_0 \geq 0$. As $b_i^n(t_0)$ converges to $+\infty$ as n tends to $-\infty$, there exists an integer n such that $x_0 \in \overline{W}_i^{b_i^n(t_0)}$, so that (4.27) yields

$$x(t, x_0, t_0) \in \overline{W}_i^{b_i^n(t)} \quad \text{i.e.} \quad V_i(x(t, x_0, t_0)) \;\leq\; b_i^n(t), \quad t \geq 0.$$

Thus, it follows from the convergence to 0 of the mapping $b_i^n(t)$ as t tends to $+\infty$ together with the positive definiteness of V_i, that $x(t, x_0, t_0)$ converges to 0 as t tends to $+\infty$. The proof of Theorem 4.2 is complete upon noting that the previous argument holds for each $i = 1, 2, \ldots$. ■

We stress that in order to extend to countably infinite families of LTI systems, the method introduced in Theorem 4.1 for finite families of LTI systems, two additional assumptions are needed: Assumption **(i)** of Theorem 4.2 ensures that the simultaneous stabilizer is continuous at any point $(t, 0)$, $t \geq 0$, while Assumption **(ii)** is necessary for the construction of the sequence of mappings $\{b_i^n\}$.

To the contrary of the finite case, our construction does not yield uniform stability and uniform asymptotic stability in general. On the other hand, the stabilizing feedback law v that we obtain through the previous construction depends on the sequences $\{\pi_i\}_{i=2}^{+\infty}$, $\{k_i\}_{i=1}^{+\infty}$, $\{\eta_i\}_{i=1}^{+\infty}$ and $\{r_i\}_{i=1}^{+\infty}$. Because these sequences are not uniquely defined we may consider that they are design parameters. It would be interesting to find what conditions should be imposed on these design parameters and on the systems S_i, $i = 1, 2, \ldots$, in order that the system $\dot{x} = f_i(x, v(t, x))$ be uniformly asymptotically stable for each $i = 1, 2, \ldots$. This is an issue that we do not investigate in the context of this dissertation and that we leave for further research.

4.4 Examples

We now present a few examples that illustrate Theorem 4.1 and 4.2.

In the following example, we consider a linear system in the plane S that is asymptotically stabilizable by a linear feedback law. Given a Lyapunov function for the corresponding closed-loop system, we follow the construction of the proof of Theorem 4.1 and we produce a mapping $b : [0, \infty) \to (0, \infty)$ and a feedback law $v : [0, \infty) \times \mathbb{R}^2$ such the following holds: For each $t_0 \geq 0$ and each x_0 such that $V(x_0) \leq b(t_0)$, the trajectory of S satisfies $V(x(t, x_0, t_0)) \leq b(t)$ for each $t \geq t_0$.

Example 1 :

We consider the system

$$S : \quad \begin{cases} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 + \frac{u}{2} \end{cases},$$

where u is a scalar input and we let $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denote the vector-field of S. Further, we let P and Q denote the matrices

$$P = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easily checked that the feedback law $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x) = -2x_2$, asymptotically stabilizes the system S. Moreover, the mapping $V : \mathbb{R}^2 \to [0, \infty)$ defined by setting $V(x) = x^t P x$ for each x in $\mathbb{R}^n$ is a Lyapunov function for the corresponding closed-loop system S and we have $\nabla V(x) f(x, u(x)) = -x^t Q x$ for each x in $\mathbb{R}^2$. We note that the largest eigenvalue of P is equal to 1.8090 and we set $\rho = \frac{1}{1.8090}$. Next, we define the mappings $a, b, c : [0, \infty) \to (0, \infty)$ by setting

$$a(t) = 9\, e^{\frac{\rho}{2} t}, \quad b(t) = 10\, e^{\frac{\rho}{2} t}, \quad \text{and} \quad c(t) = 11\, e^{\frac{\rho}{2} t},$$

for each $t \geq 0$ and we let $x_0 = (1, 1)$. Finally, we let the mapping $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$v(t, x) = -2x_2\, q(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

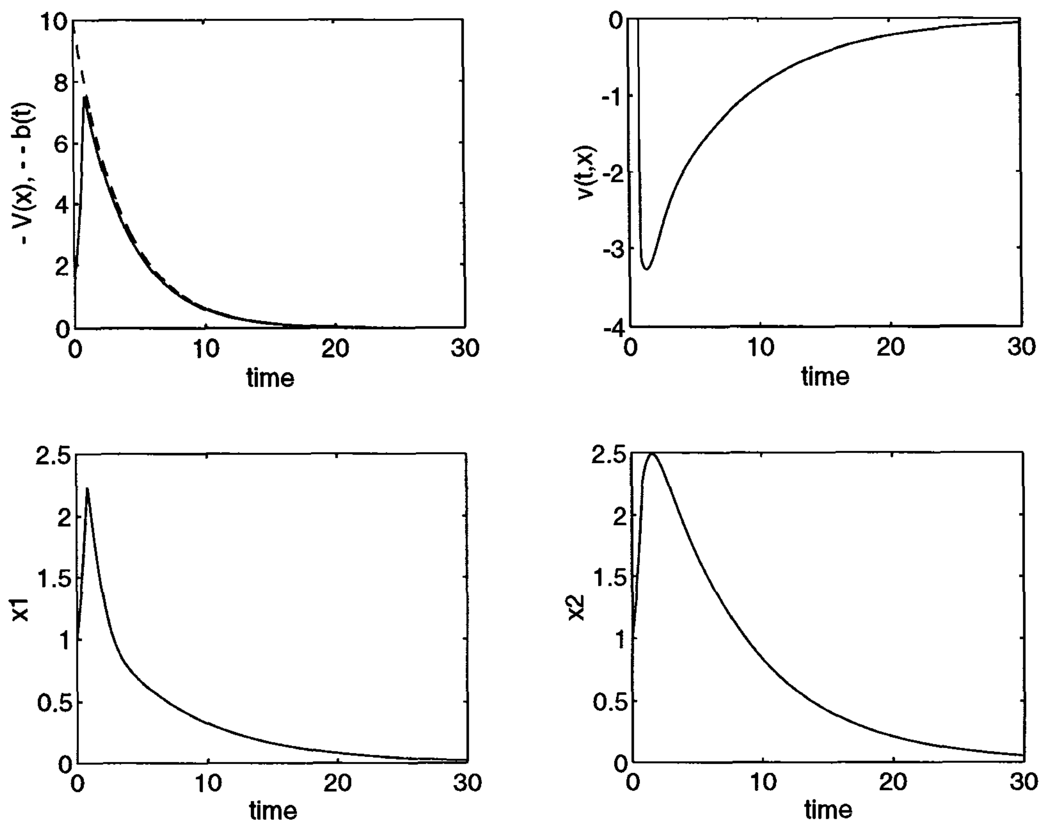

Figure 4.1: Linear system in the plane

where $q(t,x)$ is obtained by replacing V_i, a_i^n, b_i^n, c_i^n by a, b, c respectively, in the formula (4.26). From the proof of Theorem 4.1, it should be clear that for each $t \geq 0$, the trajectory $x(\cdot, x_0, 0)$ of $\dot{x} = f(x, v(t,x))$ satisfies

$$V(x(t, x_0, t_0)) \;\leq\; b(t), \quad t \geq 0,$$

since $V(x_0) \leq b(0)$. Because $b(t)$ converges to 0 as t tends to $+\infty$ it follows that the state (x_1, x_2) converges to the origin as t tends to $+\infty$. The simulation results in Fig. 4.1 confirm these facts.

We now give an example where we consider two stabilizable linear systems S_1 and S_2 with Lyapunov functions V_1 and V_2 respectively. Using the construction of Theorem 4.1, we then construct two mappings b_1, $b_2 : [0, \infty) \to (0, \infty)$ and a feedback law $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that for each $t_0 \geq 0$ and each x_0 in $V_1^{-1}([0, b_1(t_0))$ (resp. $V_2^{-1}([0, b_2(t_0)))$ the trajectory $x(\cdot, x_0, t_0)$ of S_1 (resp. S_2)

satisfies

$$V_1(x(t,x_0,t_0)) \leq b_1(t), \quad (\text{resp. } V_2(x(t,x_0,t_0)) \leq b_2(t)\,), \quad t \geq t_0.$$

The feedback law v will actually represent two terms of the infinite sum that appears in the expression of the stabilizing feedback law v constructed in the proof of Theorem 4.1.

Example 2 :

We consider the pair of scalar systems

$$S_1: \quad \dot{x} \;=\; x - u \qquad \text{and} \qquad S_2: \quad \dot{x} \;=\; x + u\;,$$

where u is a scalar input, and we let $f_1, f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the dynamics of S_1 and S_2 respectively. It is plain that the two linear feedback laws $u_1(x) = 2x$ and $u_2(x) = -2x$ stabilizes S_1 and S_2 respectively, and that there exists no continuous feedback law that simultaneously asymptotically stabilizes S_1 and S_2. However, by Theorem 4.1, there exists a time-varying feedback law that simultaneously stabilizes S_1 and S_2.

We let the Lyapunov function V_1 and V_2 be both equal to the mapping V given by $V(x) = \frac{1}{2}x^2$, and we note that

$$\nabla V(x) f_1(x, u_1(x)) \;=\; \nabla V(x) f_2(x, u_2(x)) \;=\; -x^2, \quad x \in \mathbb{R}.$$

Further, we let the mappings $a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2$ be given by

$$a_1(t) = 3e^{-t}, \quad b_1(t) = (3 + \frac{1}{3})e^{-t}, \quad c_1(t) = (3 + \frac{2}{3})e^{-t}, \quad t \geq 0,$$
$$a_2(t) = 2e^{-t}, \quad b_2(t) = (2 + \frac{1}{3})e^{-t}, \quad c_2(t) = (2 + \frac{2}{3})e^{-t}, \quad t \geq 0,$$

and we define the mapping $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by setting

$$v(t,x) \;=\; 2x\, q_1(t,x) \;-\; 2x\, q_2(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

where $q_1(t,x)$ and $q_2(t,x)$ are obtained by replacing $V_i,\ a_i^n,\ b_i^n,\ c_i^n$ by respectively $a_1,\ b_1,\ c_1$ and $a_2,\ b_2,\ c_2$ in the formula (4.26). Set $x_0 \triangleq 2$. For each $i = 1, 2$, because $V(x_0)$ is less than $b_i(0)$, the trajectory $x(t, x_0, 0)$ of the closed-loop system obtained once v is fed-back into S_i satisfies

$$V_i(x(t,x_0,0)) \;\leq\; b_i(t), \quad t \geq 0.$$

The curves in Fig. 4.2 show the evolution of the Lyapunov function V_1, the state x and the control $v(t,x)$. Although the curve $\{(V_1(x(t,x_0,0),t),\ t \geq 0\}$ crosses the curve $\{(b_2(t),t),\ t \geq 0\}$, it remains below the curve $\{(b_1(t),t),\ t \geq 0\}$ as desired. In Fig. 4.3 are presented the simulation results for the system S_2.

We present in the next section, a technical lemma that was used in the proof of Theorem 4.2.

4.5 Technical lemma

The following technical lemma was used in the proof of Theorem 4.2, and is a modified version of Lemma 3.1 which was introduced in the previous chapter to prove the simultaneous stabilizability of infinite families of linear systems. The difference between the following lemma and Lemma 3.1 lies in the fact that here we do not introduce the constant θ.

Lemma 4.2 *For each $i = 1, 2, \ldots$, let P_i be a positive definite matrix, let $\frac{1}{M_i}$ and $\frac{1}{m_i}$ denote respectively its smallest and largest eigenvalue, let $V_i : \mathbb{R}^n \to [0, \infty)$ denote the mapping given by $V_i(x) = x^t P_i x$, and let θ_i be in $(0,1)$. Further, let $\hat{\gamma}_1^1$ be a given positive real and let the sequences of positive reals $\{\pi_i\}_{i=2}^\infty$ and $\{k_i\}_{i=1}^\infty$ be such that*

$$0 \;<\; \pi_i \;<\; \min(\frac{m_{i-1}}{M_i}, \frac{1}{\theta_i^2}), \quad i = 2, 3, \ldots, \tag{4.28}$$

with

$$k_i \;>\; 1 \qquad \text{and} \qquad \frac{m_i}{k_i M_1}\theta_1^2 \;<\; 1, \quad i = 1, 2, \ldots. \tag{4.29}$$

Finally, let the sequences of positive reals $\{\gamma_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$, $\{\beta_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$ and $\{\alpha_i^n,\ i = 1, \ldots, n\}_{n=1}^\infty$ be defined by setting

$$\gamma_1^1 \triangleq \hat{\gamma}_1^1, \quad \beta_i^n \;=\; \theta_i \gamma_i^n, \quad \alpha_i^n \;=\; \theta_i \beta_i^n, \quad i = 1, \ldots, n, \quad n = 1, 2, \ldots,$$

with

$$\gamma_1^n \;=\; \frac{m_{n-1}}{k_{n-1} M_1}\alpha_{n-1}^{n-1}, \qquad \gamma_i^n = \pi_i \alpha_{i-1}^n, \quad i = 2, \ldots, n, \quad n = 2, 3, \ldots.$$

Then, for each $n = 2, 3, \ldots$ we have

$$V_{n-1}^{-1}([0, \alpha_{n-1}^{n-1}]) \;\supset\; V_1^{-1}([0, \gamma_1^n]), \tag{4.30}$$

$$V_{i-1}^{-1}([0, \alpha_{i-1}^n]) \;\supset\; V_i^{-1}([0, \gamma_i^n]), \quad i = 2, \ldots, n, \tag{4.31}$$

together with

$$\gamma_i^n,\ \beta_i^n,\ \alpha_i^n \to 0 \ \text{as}\ n \to \infty, \quad i = 1, 2, \ldots.$$

Proof: The proof of this lemma is similar to that of Lemma 3.1. However, for the sake of completeness we produce it below.

In what follows we fix $n = 1, 2, \ldots$ and we let $\delta > 0$. It is well known [8, p. 44] that for each $i = 1, 2, \ldots$, the set $V_i^{-1}([0, \delta])$ is the volume bounded by an ellipsoid centered at the origin with smallest axis $\sqrt{m_i\,\delta}$ and largest axis $\sqrt{M_i\,\delta}$. Thus, (4.30) and (4.31) will hold if we have

$$\gamma_1^n \;<\; \frac{m_{n-1}}{M_1}\,\alpha_{n-1}^{n-1} \quad\text{and}\quad \gamma_i^n \;<\; \frac{m_{i-1}}{M_i}\,\alpha_{i-1}^n, \quad i=2,\ldots,n, \quad n=2,3,\ldots.$$

Because we have $\frac{m_n}{k_n M_1} < \frac{m_n}{M_1}$ for each $n=1,2,\ldots$, and the real π_i lies in $(0,\dfrac{m_{i-1}}{M_i})$ for each $i=2,\ldots,n$ and each $n=2,3,\ldots$, we already obtain the inclusions (4.30) and (4.31).

Next, we set

$$\begin{aligned} y_1 &\triangleq \ln\left(\frac{m_1}{k_1 M_1}\theta_1^2\right) \\ y_n &\triangleq \ln\left(\frac{m_n}{k_n M_1}\theta_1^2\right)+\ln(\pi_2\theta_2^2)+\ldots+\ln(\pi_n\theta_n^2), \quad n=2,3,\ldots. \end{aligned}$$

It follows from (4.28) together with (4.29) that the reals $\ln(\pi_i\theta_i^2)$ and $\ln(\frac{m_i}{k_i M_1}\theta_1^2)$ are negative for each $i=2,3,\ldots$ and each $i=1,2,\ldots$ respectively. Thus, we have

$$y_n \;\le\; \ln(\pi_2\theta_2^2), \quad n=2,3,\ldots. \tag{4.32}$$

We now fix $i=1,2,\ldots$. It is not hard to check from the definition of γ_i^n, $n=i,i+1,\ldots$, that

$$\ln(\gamma_i^{i+l}) \;=\; y_i+y_{i+1}+\ldots+y_{i+l-1}+\ln(\gamma_i^i), \quad l=1,2,\ldots.$$

Therefore, (4.32) combined with the fact that $\ln(\pi_2\theta_2^2)<0$ yield the convergence to 0 of γ_i^{i+l} as l tends to ∞, which completes the proof of the lemma. ■

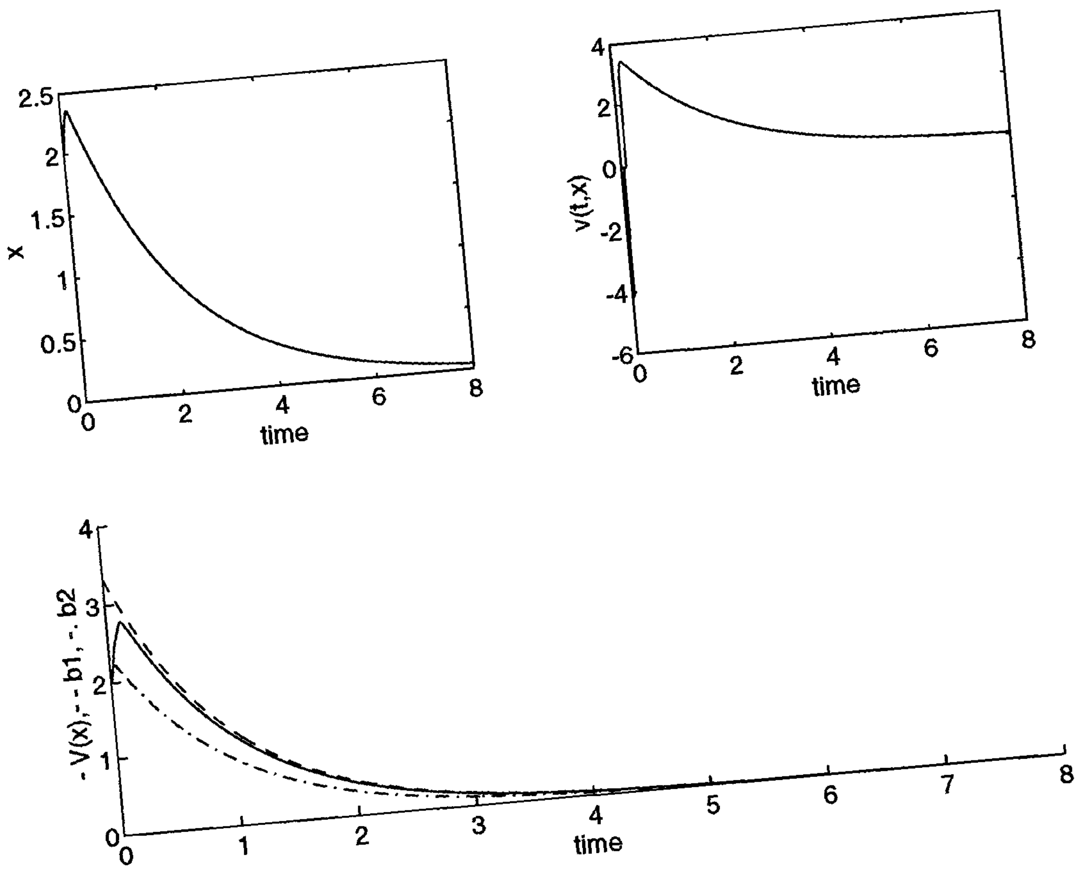

Figure 4.2: Scalar system S_1

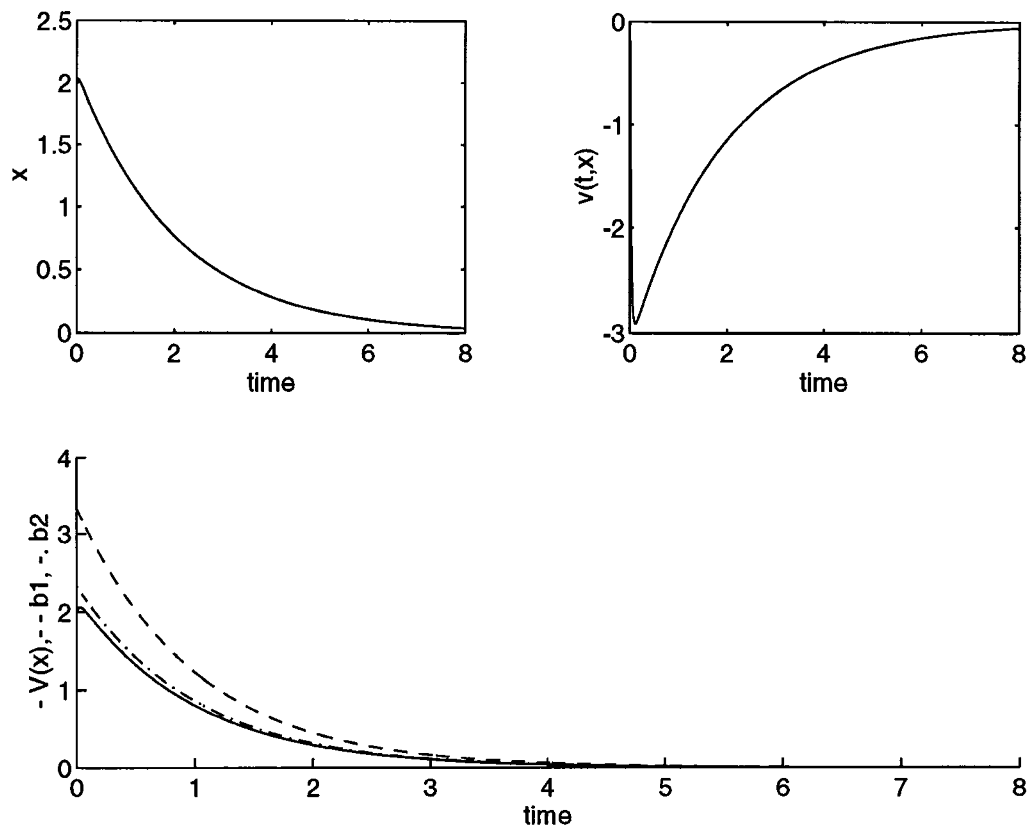

Figure 4.3: Scalar system S_2

Chapter 5

Time-Varying Simultaneous Asymptotic Stabilization of Nonlinear Systems

In this chapter, we discuss the simultaneous asymptotic stabilization of finite families of **nonlinear** systems that are individually asymptotically stabilizable by continuous feedback laws. By using the approach of Chapter 4, we are able to provide sufficient conditions for the existence of a continuous time-varying feedback law that simultaneously locally or globally **asymptotically** stabilizes such a family. We then focus on a class of pairs of homogeneous nonlinear systems, and by using the previous sufficient conditions, we establish their asymptotic stabilizability by means of time-varying feedback.

The chapter is organized as follows: We precisely state the problem under consideration in Section 5.1 while we provide the desired sufficient conditions in Section 5.2. In section 5.3, we establish the simultaneous asymptotic stabilizability of certain pairs of homogeneous systems. Finally, Section 5.4 contains a technical yet important lemma.

5.1 Problem definition

Throughout this chapter, we let I be in $\{2, 3, \ldots\}$, and we consider a family $\{S_i,\ i = 1, \ldots, I\}$ of systems

$$S_i : \quad \dot{x} \;=\; f_i(x, u), \quad i = 1, \ldots, I, \tag{5.1}$$

where the state x lies in $\mathbb{R}^n$, the input u takes value in $\mathbb{R}^m$, and for each $i = 1, \ldots, I$, the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin with $f_i(0, 0) = 0$. We assume that for each $i = 1, \ldots, I$, the system

S_i is asymptotically stabilizable by means of continuous feedback and we discuss the simultaneous asymptotic stabilization of the family $\{S_i,\ i = 1,\ldots,I\}$ by means of time-varying feedback.

We stress that the simultaneous asymptotic stabilization of nonlinear systems has not been addressed in the literature. Indeed, all the studies on simultaneous stabilization have focussed on linear systems. By proving in Chapter 2 that any finite family of stabilizable nonlinear systems is simultaneously stabilizable by means of continuous feedback, we have accomplished a first step towards the understanding of this problem. Here, we continue our investigation and we aim at constructing time-varying feedback laws that achieves simultaneous **asymptotic** stabilization. We adopt the approach of Chapter 4 that has been successful in producing time-varying simultaneous asymptotic stabilizer for families of linear systems.

5.2 Simultaneous asymptotic stabilization

We distinguish two cases based on whether the systems S_i, $i = 1,\ldots,I$ are locally or globally asymptotically stabilizable.

5.2.1 Families of locally asymptotically stabilizable systems

The purpose of this subsection is to prove the following theorem.

Theorem 5.1 *Let $k \geq 0$ and $k' \geq 1$ be two integers. Let D be a bounded neighborhood of the origin, and assume that there exists a continuous and almost C^k feedback law $u_i : D \to \mathbb{R}^m$ which locally asymptotically stabilizes S_i for each $i = 1,\ldots,I$. Further, assume that the mapping $f_i(\cdot, u_i(\cdot))$ is continuous on D and let $V_i : \overline{D} \to [0,\infty)$ be a $C^{k'}$ Lyapunov function, satisfying*

$$\nabla V_i(x)\, f_i(x, u_i(x)) \ < \ 0, \quad x \in D\backslash\{0\},$$

for each $i = 1,\ldots,I$. Finally, assume that there exists a sequence $\{b_i^n,\ i = 1,\ldots,I\}_{n=1}^{\infty}$ of $C^{k'}$ mappings $b_i^n : [0,\infty) \to (0,\infty)$, such that the following assertions hold.

i) $\sup\limits_{t \geq t_0} b_i^n(t) \to 0$ *as* $n \to \infty$, $\quad i = 1,\ldots,I, \quad t_0 \geq 0.$

ii) $b_i^n(t) < \inf_{x\in\partial D} V_i(x), \quad t \geq 0, \quad i = 1,\ldots,I, \quad n = 1,2,\ldots.$

iii) $b_i^n(t) \to 0$ *as* $t \to \infty, \quad i = 1,\ldots,I, \quad n = 1,2,\ldots.$

iv) *For each* $n = 1,2,\ldots$, *we have*

$$D \cap V_I^{-1}(\,[0, b_I^n(t))\,) \;\supset\; D \cap V_1^{-1}(\,[0, b_1^{n+1}(t)]\,), \quad t \geq 0, \tag{5.2}$$

and for each $n = 1,2,\ldots$ *and each* $i = 1,\ldots,I-1$ *we have*

$$D \cap V_i^{-1}(\,[0, b_i^n(t)]\,) \;\supset\; D \cap V_{i+1}^{-1}(\,[0, b_{i+1}^n(t))\,), \quad t \geq 0. \tag{5.3}$$

v) *For each each* $n = 1,2,\ldots$ *and each* $i = 1,\ldots,I$, *we have*

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; \dot{b}_i^n(t), \quad x \in D \cap V_i^{-1}(b_i^n(t)), \quad t \geq 0.$$

Let $k'' \triangleq \min(k, k')$. *Then, there exists a time-varying feedback law* $v : [0,\infty) \times D \to \mathbb{R}^m$, *continuous on* $[0,\infty) \times D$, $C^{k''}$ *on* $[0,\infty) \times (D\backslash\{0\})$, *which simultaneously locally asymptotically stabilizes the family* $\{S_i,\ i = 1,\ldots,I\}$.

Proof: For each $i = 1,\ldots,I$ and each $\beta > 0$, we set

$$W_i^\beta \;\triangleq\; \{x \in D :\ V_i(x) < \beta\},$$

and immediately see from **(ii)** and Lemma B.3 **(i)** that

$$\overline{W}_i^{b_i^n(t)} \;=\; D \cap V_i^{-1}(\,[0, b_i^n(t)]\,), \quad i = 1,\ldots,I, \quad n = 1,2,\ldots, \quad t \geq 0.$$

In the sequel, we repeatedly use this equality without further reference.

Construction of the simultaneous stabilizer v :

We first construct two sequences $\{a_i^n,\ i = 1,\ldots,I\}_{n=1}^\infty$ and $\{c_i^n,\ i = 1,\ldots,I\}_{n=1}^\infty$ of mappings $a_i^n, b_i^n : [0,\infty) \to (0,\infty)$, satisfying for each $n = 1,2,\ldots$

$$\inf_{x\in\partial D} V_i(x) \;>\; c_i^n(t) \;>\; b_i^n(t) \;>\; a_i^n(t) \;>\; 0, \quad t \geq 0, \quad i = 1,\ldots,I, \tag{5.4}$$

with

$$W_I^{a_I^n(t)} \;\supset\; \overline{W}_1^{c_1^{n+1}(t)}, \quad t \geq 0, \tag{5.5}$$

and

$$W_i^{a_i^n(t)} \;\supset\; \overline{W}_{i+1}^{c_{i+1}^n(t)}, \quad t \geq 0, \quad i = 1,\ldots,I-1. \tag{5.6}$$

We define $c_1^1 : [0,\infty) \to (0,\infty)$ by setting

$$c_1^1(t) = \frac{b_1^1(t) + \inf_{x\in\partial D} V_1(x)}{2}, \quad t \geq 0.$$

Then, for each $n = 1,2,\ldots$ and each $i = 1,\ldots,I-1$, we define two C^∞ mappings $a_i^n, c_{i+1}^n : [0,\infty) \to (0,\infty)$ from b_i^n and b_{i+1}^n by applying Lemma 5.2 with $b_1 = b_i^n$, $b_2 = b_{i+1}^n$ and (5.3). The obtained mappings satisfy

$$b_i^n(t) > a_i^n(t) > 0, \qquad \text{with} \qquad \inf_{x\in\partial D} V_{i+1}(x) > c_{i+1}^n(t) > b_{i+1}^n(t), \quad t \geq 0,$$

and

$$W_i^{a_i^n(t)} \supset \overline{W}_{i+1}^{c_{i+1}^n(t)}, \quad t \geq 0.$$

Finally, for each $n = 1,2,\ldots$, we define two C^∞ mappings $a_I^n, c_1^{n+1} : [0,\infty) \to (0,\infty)$ by applying Lemma 5.2 with $b_1 = b_I^n$, $b_2 = b_1^{n+1}$ and the inequality (5.2). The resulting mappings satisfy

$$b_I^n(t) > a_I^n(t) > 0, \qquad \text{with} \qquad \inf_{x\in\partial D} V_1(x) > c_1^{n+1}(t) > b_1^{n+1}(t), \quad t \geq 0,$$

and

$$W_I^{a_I^n(t)} \supset \overline{W}_1^{c_1^{n+1}(t)}, \quad t \geq 0.$$

It is easily seen that the sequences $\{a_i^n,\ i = 1,\ldots,I\}_{n=1}^\infty$, $\{b_i^n,\ i = 1,\ldots,I\}_{n=1}^\infty$ and $\{c_i^n,\ i = 1,\ldots,I\}_{n=1}^\infty$ satisfy (5.4), (5.5) and (5.6).

We therefore have, for each $t \geq 0$, a sequence of nested neighborhoods

$$\begin{array}{ccccccccccccc} D & \supset & W_1^{c_1^1(t)} & \supset & W_1^{b_1^1(t)} & \supset & W_1^{a_1^1(t)} & \supset & W_2^{c_2^1(t)} & \supset & \cdots & \supset & W_I^{a_I^1(t)} & \supset \\ & & W_1^{c_1^2(t)} & \supset & W_1^{b_1^2(t)} & \supset & W_1^{a_1^2(t)} & \supset & W_2^{c_2^2(t)} & \supset & \cdots & \supset & W_I^{a_I^2(t)} & \supset \\ & & \vdots & & & & & & \vdots & & & & \vdots & \end{array} \tag{5.7}$$

such that each neighborhood contains the closure of the neighborhood that follows.

Next, for each $n = 1,2,\ldots$ and each $i = 1,\ldots,I$, we define the mapping $q_i^n : [0,\infty) \times D \to [0,1]$ by setting

$$q_i^n(t,x) = \begin{cases} e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(b_i^n(t)-a_i^n(t))^2}}, & \text{if } V_i(x) \in (a_i^n(t), b_i^n(t)] \\ e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(c_i^n(t)-b_i^n(t))^2}}, & \text{if } V_i(x) \in (b_i^n(t), c_i^n(t)) \\ 0\,, & \text{otherwise} \end{cases} \tag{5.8}$$

For each $i = 1, \ldots, I$ and each $n = 1, 2, \ldots$, it follows from Lemma B.6 combined with the fact that the mappings a_i^n and c_i^n are C^∞ and the mappings b_i^n and V_i are $C^{k'}$, that the mapping q_i^n is $C^{k'}$ on $[0, \infty) \times D$.

We now define the feedback law $v : [0, \infty) \times D \to \mathbb{R}^m$ by setting

$$v(t, x) = \sum_{i=1}^{I} \sum_{n=1}^{\infty} u_i(x) q_i^n(t, x), \quad (t, x) \in [0, \infty) \times D. \tag{5.9}$$

From the definition of the mappings q_i^n, we easily obtain that $v(t, 0) = 0$ for each $t \geq 0$. We next show that v is well-defined, C^k on $[0, \infty) \times (D \backslash \{0\})$ and continuous at any point $(t, 0)$.

Regularity of the feedback law v :

Let (t, x) be in $[0, \infty) \times D \backslash \{0\}$. It should be clear from (5.7) that there exists a unique pair of integer (i, n) in $\{1, \ldots, I\} \times \{1, 2, \ldots\}$ such that either one of the following two assertions holds:

- We have $V_i(x) \in [a_i^n(t), c_i^n(t)]$. In that case (5.7) combined with the continuity of the mappings V_i, a_i^n and c_i^n, yields the existence of a neighborhood U of (t, x) in $[0, \infty) \times (D \backslash \{0\})$ such that

$$v(\tau, y) \; = \; u_i(y)\, q_i^n(\tau, y), \quad (\tau, y) \in U. \tag{5.10}$$

- We have $x \in W_i^{a_i^n(t)} \backslash \overline{W}$ where W denotes either the set $W_1^{c_1^{n+1}(t)}$ if $i = I$ or the set $W_{i+1}^{c_{i+1}^n(t)}$ otherwise. In this case, by continuity of the mappings a_j^m, c_j^m for each $j = 1, \ldots, I$ and each $m = 1, 2, \ldots$, there exists a neighborhood U of (t, x) in $[0, \infty) \times (D \backslash \{0\})$ such that

$$v(\tau, y) \; = \; 0, \quad (\tau, y) \in U. \tag{5.11}$$

Recall that $k'' \triangleq \min(k, k')$. Thus, because the mappings q_i^n and u_i are $C^{k''}$ on $[0, \infty) \times D$ and $D \backslash \{0\}$ respectively, (5.10) and (5.11) imply that v is $C^{k''}$ on $[0, \infty) \times (D \backslash \{0\})$.

Moreover, because $v(t, 0) = 0$ for each $t \geq 0$, (5.10) together with (5.11) yield

$$\|v(t, x)\| \; \leq \; \max(\|u_1(x)\|, \ldots, \|u_I(x)\|), \quad (t, x) \in [0, \infty) \times D,$$

so that v is continuous at any point $(t, 0)$, $t \geq 0$. Thus v is continuous on $[0, \infty) \times D$.

Asymptotic Stability :

In what follows, we fix $i = 1, \ldots, I$ and $n = 1, 2, \ldots$. Because $\partial W_i^{b_i^n(t)} = D \cap V_i^{-1}(b_i^n(t))$ and $q_i^n(t, x) = 1$ for each x in $V_i^{-1}(b_i^n(t))$ and each $t \geq 0$, we obtain from (5.10) that

$$v(t,x) \;=\; u_i(x), \quad x \in \partial W_i^{b_i^n(t)}, \quad t \geq 0,$$

and it follows from the assumption **(v)** that

$$\begin{aligned} \nabla V_i(x)\, f_i(x, v(t,x)) \;&=\; \nabla V_i(x)\, f_i(x, u_i(x)) \\ &<\; \dot{b}_i^n(t), \quad x \in \partial W_i^{b_i^n(t)}, \quad t \geq 0. \end{aligned}$$

From this last inequality combined with Lemma 4.1, we obtain that the point $x(t, x_0, t_0)$ of the trajectory of $\dot{x} = f_i(x, v(t,x))$ starting from x_0 at time t_0 satisfies

$$x(t, x_0, t_0) \;\in\; \overline{W}_i^{b_i^n(t)}, \quad t \geq t_0, \tag{5.12}$$

for each $t_0 \geq 0$, each x_0 in $\overline{W}^{b_i^n(t_0)}$. Using this last result, we now prove that v asymptotically stabilizes the system S_i, for each $i = 1, \ldots, I$.

We fix $i = 1, \ldots, I$ and we let $\varepsilon > 0$ and $t_0 \geq 0$ be arbitrary reals. By Assumption **(i)**, there exists an integer $\bar{n}$ satisfying

$$\overline{W}_i^{\sup\{b_i^{\bar{n}}(t):\ t \geq t_0\}} \;\subset\; B_\varepsilon(0),$$

so that

$$\overline{W}_i^{b_i^{\bar{n}}(t)} \;\subset\; B_\varepsilon(0), \quad t \geq t_0. \tag{5.13}$$

Let $\delta > 0$ be such that $B_\delta(0) \subset \overline{W}_i^{b_i^{\bar{n}}(t_0)}$ and let x_0 be in $B_\delta(0)$. By (5.12), we have

$$x(t, x_0, t_0) \;\in\; \overline{W}_i^{b_i^{\bar{n}}(t)}, \quad t \geq t_0, \tag{5.14}$$

and in view of (5.13), we find that $x(t, x_0, t_0) \in B_\varepsilon(0)$ for each $t \geq t_0$. In short for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exist $\delta(t_0, \varepsilon) > 0$ such that whenever $\|x_0\| < \delta$ we have $\|x(t, x_0, t_0)\| < \varepsilon,\ t \geq t_0$. Thus, the origin is a stable equilibrium point for the system $\dot{x} = f_i(x, v(t,x))$.

We now arbitrarily pick $n = 1, 2, \ldots$ and we let δ_0 be such that

$$B_{\delta_0}(0) \;\subset\; W_i^{b_i^n(t_0)}.$$

Recall that i is fixed in $\{1, \ldots, I\}$ and let x_0 be in $B_{\delta_0}(0)$. From (5.12) and Assumption **(iii)**, we easily deduce that $V_i(x(t, x_0, t_0)) \to 0 \;\; as \;\; t \to \infty$ and that

$x(t, x_0, t_0)$ lies in D for each $t \geq t_0$. We conclude from the positive-definiteness of V_i on D that

$$x(t, x_0, t_0) \to 0 \ \ as \ \ t \to \infty.$$

In view of the previous stability results, this last relation implies that the system $\dot{x} = f_i(x, v(t, x))$ is locally asymptotically stable. The proof of the theorem is complete upon noting that this last results holds for each $i = 1, \ldots, I$. ∎

To the contrary of the case where the system S_i and the stabilizing feedback law u_i are linear for each $i = 1, \ldots, I$, the feedback law v obtained through the proof of Theorem 5.1 does not yield uniform asymptotic stability. Indeed, as the initial time t_0 increases, it follows from the assumption **(iii)** of Theorem 5.1 that the domain of attraction of the closed-loop systems obtained once v is fed-back into the system $S_i, i = 1, \ldots, I$, becomes smaller and smaller.

We believe that in some cases, it may be possible to slightly modify the construction of the proof of Theorem 5.1 in order to obtain a feedback law v that yields a larger domain of attraction. The following example gives an idea on how one may proceed. Assume that in addition to the assumptions of Theorem 5.1, there exists $T > 0$ satisfying

$$b_I^2(t - T) \ > \ b_1^1(t), \quad t \geq T,$$

so that for each $t \geq T$, we have

$$b_1^1(t - T) > \cdots > b_I^1(t - T) > b_1^2(t - T) > \cdots > b_I^2(t - T) \ > b_1^1(t) \ > \cdots > b_I^1(t).$$

For each $n = -1, -3, \ldots$ and each $i = 1, \ldots, I$, we define the mappings $b_i^n, b_i^{n-1} : [0, \infty) \to [0, \infty)$ by setting

$$b_i^n(t) \ = \ \begin{cases} 0, & t < \frac{|n| + 1}{2} T \\ b_i^2(t - \frac{|n| + 1}{2} T), & t \geq \frac{|n| + 1}{2} T \end{cases},$$

and

$$b_i^{n-1}(t) \ = \ \begin{cases} 0, & t < \frac{|n| + 1}{2} T \\ b_i^1(t - \frac{|n| + 1}{2} T), & t \geq \frac{|n| + 1}{2} T \end{cases}.$$

For each $i = 1, \ldots, I$ and each $n = -1, -3, \ldots$, we define $q_i^n, q_i^{n-1} : [0, \infty) \times D \to [0, 1]$ by letting $q_i^n(t, x)$ and $q_i^{n-1}(t, x)$ be given by the formula (5.8) for $t \geq \frac{|n| + 1}{2} T$ and by setting $q_i^n(t, x) = q_i^{n-1}(t, x) = 0$ for $t < \frac{|n| + 1}{2} T$. Finally, we let the feedback law $v : [0, \infty) \times D \to \mathbb{R}^m$ be given by

$$v(t, x) = \sum_{i=1}^{I} \sum_{n \in \mathbf{Z}} u_i(x) q_i^n(t, x), \quad (t, x) \in [0, \infty) \times D.$$

By slightly adapting the proof of Theorem 5.1, it is not hard to check that v simultaneously asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$. Furthermore, for each $t \geq 0$, the domain of attraction of the corresponding closed loop systems is larger than $V_1^{-1}([0, b_1^2(T)))$ instead of becoming smaller and smaller as t goes to $+\infty$.

We next consider the case where the system S_i is globally asymptotically stabilizable for each $i = 1, \ldots, I$, and obtain a global version of Theorem 5.2.

5.2.2 Families of globally asymptotically stabilizable systems

In case the system S_i is globally asymptotically stabilizable for each $i = 1, \ldots, I$, then under some assumptions we establish the existence of a time-varying feedback law that simultaneously **globally** asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.

Theorem 5.2 *Let $k \geq 0$ and $k' \geq 1$ be two integers. Assume that there exists a continuous and almost C^k feedback law $u_i : \mathbb{R}^n \to \mathbb{R}^m$, which globally asymptotically stabilizes S_i for each $i = 1, \ldots, I$. Further, assume that the mapping $f_i(\cdot, u_i(\cdot))$ is continuous on $\mathbb{R}^n$ and let $V_i : \mathbb{R}^n \to [0, \infty)$ be a $C^{k'}$ radially unbounded Lyapunov function, satisfying*

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in \mathbb{R}^n \backslash \{0\},$$

for each $i = 1, \ldots, I$. Finally, assume that there exists a sequence $\{b_i^n,\ i = 1, \ldots, I\}_{n \in \mathbb{Z}}$ of $C^{k'}$ mappings $b_i^n : [0, \infty) \to (0, \infty)$, such that the following assertions hold.

i) $\sup\limits_{t \geq t_0} b_i^n(t) \to 0$ *as* $n \to +\infty, \quad i = 1, \ldots, I, \quad t_0 \geq 0,$

ii) $b_i^n(t_0) \to +\infty$ *as* $n \to -\infty, \quad i = 1, \ldots, I, \quad t_0 \geq 0,$

iii) $b_i^n(t) \to 0$ *as* $t \to \infty, \quad i = 1, \ldots, I, \quad n \in \mathbb{Z}.$

iv) *For each n in $\mathbb{Z}$, we have*

$$V_I^{-1}([0, b_I^n(t))) \;\supset\; V_1^{-1}([0, b_1^{n+1}(t)]), \quad t \geq 0, \tag{5.15}$$

and for each n in $\mathbb{Z}$ and each $i = 1, \ldots, I$, we have

$$V_i^{-1}([0, b_i^n(t)]) \;\supset\; V_{i+1}^{-1}([0, b_{i+1}^n(t))), \quad t \geq 0. \tag{5.16}$$

v) *For each n in $\mathbb{Z}$ and each $i = 1, \ldots, I$, we have*

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; \dot{b}_i^n(t), \quad x \in V_i^{-1}(b_i^n(t)), \quad t \geq 0.$$

Let $k'' \triangleq \min(k, k')$. Then, there exists a time-varying feedback law $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$, continuous on $[0,\infty) \times \mathbb{R}^n$, $C^{k''}$ on $[0,\infty) \times (\mathbb{R}^n \setminus \{0\})$, which simultaneously globally asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.

Proof: The proof of Theorem 5.1 can be easily transposed to this case. However, for the sake of completeness, we sketch the proof below.

As usual for each $i = 1, \ldots, I$, we set

$$W_i^{\beta} \;\triangleq\; V_i^{-1}(\,[0,\beta)\,), \quad \beta > 0.$$

By combining Lemma 5.2 with (5.15) and (5.16), we obtain two sequences $\{a_i^n,\ i = 1, \ldots, I\}_{n\in\mathbb{Z}}$ and $\{c_i^n,\ i = 1, \ldots, I\}_{n\in\mathbb{Z}}$ of C^∞ mappings a_i^n, c_i^n : $[0,\infty) \to (0,\infty)$, satisfying

$$c_i^n(t) \;>\; b_i^n(t) \;>\; a_i^n(t) \;>\; 0, \quad t \geq 0, \quad i = 1, \ldots, I, \quad n \in \mathbb{Z},$$

with

$$W_I^{a_I^n(t)} \;\supset\; \overline{W_1^{c_1^{n+1}(t)}}, \quad t \geq 0,$$

and

$$W_i^{a_i^n(t)} \;\supset\; \overline{W_{i+1}^{c_{i+1}^n(t)}}, \quad t \geq 0, \quad i = 1, \ldots, I-1, \quad n \in \mathbb{Z}.$$

Then, for each $i = 1, \ldots, I$ and each n in $\mathbb{Z}$, we define the $C^{k'}$ mapping q_i^n : $[0,\infty) \times \mathbb{R}^n \to [0,1]$ by setting

$$q_i^n(t,x) = \begin{cases} e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(b_i^n(t)-a_i^n(t))^2}}, & \text{if } V_i(x) \in (a_i^n(t), b_i^n(t)] \\ e^{\frac{(V_i(x)-b_i^n(t))^2}{(V_i(x)-b_i^n(t))^2-(c_i^n(t)-b_i^n(t))^2}}, & \text{if } V_i(x) \in (b_i^n(t), c_i^n(t)) \\ 0\,, & \text{otherwise} \end{cases}, \tag{5.17}$$

and we let $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$v(t,x) = \sum_{i=1}^{I} \sum_{n\in\mathbb{Z}} u_i(x) q_i^n(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n\,.$$

By using the same argument as that used in the proof of Theorem 5.1, it can be checked that the mapping v is continuous on $[0,\infty) \times \mathbb{R}^n$ and $C^{k''}$ on $[0,\infty) \times (\mathbb{R}^n \setminus \{0\})$.

We now fix $i = 1, \dots, I$ and we show that v globally asymptotically stabilizes S_i. From the definition of v we get

$$v(t,x) \;=\; u_i(x), \quad x \in \partial W_i^{b_i^n(t)}, \quad t \geq 0, \quad n \in \mathbb{Z},$$

and it follows from Assumption (v) that

$$\begin{aligned} \nabla V_i(x)\, f_i(x, v(t,x)) &= \nabla V_i(x)\, f_i(x, u_i(x)) \\ &< \dot{b}_i^n(t), \quad t \geq 0, \quad x \in \partial W_i^{b_i^n(t)}, \quad n \in \mathbb{Z}. \end{aligned}$$

This last inequality combined with Lemma 4.1, implies that for each $t_0 \geq 0$, each $t \geq t_0$, each x_0 in $\overline{W}_i^{b_i^n(t_0)}$ and each n in $\mathbb{Z}$, the point $x(t, x_0, t_0)$ of the trajectory of $\dot{x} = f_i(x, v(t,x))$ lies in $\overline{W}_i^{b_i^n(t)}$.

This together with Assumption (i) yields stability of $\dot{x} = f_i(x, v(t,x))$. Furthermore, for each x_0 in $\mathbb{R}^n$ and each $t_0 \geq 0$, by Assumption (ii) there exists an integer $\bar{n}$ such that $x_0 \in \overline{W}_i^{b_i^{\bar{n}}(t_0)}$. Thus, $x(t, x_0, t_0)$ lies in $\overline{W}_i^{b_i^{\bar{n}}(t)}$ for each $t \geq t_0$, and in view of Assumption (iii), convergence of $x(t, x_0, t_0)$ to 0 as t tends to $+\infty$ follows, which completes the proof of the theorem. ■

Finally, we illustrate Theorem 5.1 and 5.2, by solving in the next section the simultaneous asymptotic stabilization of a class of pairs of stabilizable systems whose corresponding asymptotically stable closed-loop systems are homogeneous.

5.3 Simultaneous stabilization of homogeneous systems

Throughout this section, we consider a pair of control systems

$$S_1 : \; \dot{x} = f(x) - g(x)u \qquad \text{and} \qquad S_2 : \; \dot{x} = f(x) + g(x)u, \tag{5.18}$$

where the state x lies in $\mathbb{R}^n$ and the input u lies in $\mathbb{R}^m$. We assume that the mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous with $f(0) = 0$, and that there exists a continuous feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$, that asymptotically stabilizes S_2, so that $-u$ locally asymptotically stabilizes S_1. We define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$F(x) \;=\; f(x) \;+\; g(x)\, u(x), \quad x \in \mathbb{R}^n, \tag{5.19}$$

and we assume that F is homogeneous i.e., there exists s in $\mathbb{R}$ and $(r_1, \ldots, r_n)$ in $(0, +\infty)^n$ such that

$$F(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \begin{pmatrix} \lambda^{s+r_1} F_1(x) \\ \vdots \\ \lambda^{s+r_n} F_n(x) \end{pmatrix}, \quad x \in \mathbb{R}^n \backslash \{0\}, \quad \lambda > 0, \tag{5.20}$$

with F_i being the i-th coordinate mapping of F for each $i = 1, \ldots, n$.

The purpose of this section is to establish the existence of a continuous time-varying feedback law that simultaneously asymptotically stabilizes S_1 and S_2, regardless of the homogeneity degree s of F.

Before, presenting our results, we need some definitions. For a given mapping $\phi : \mathbb{R}^j \to \mathbb{R}^l$, we let ϕ_i denote its i-th coordinate mapping for each $i = 1, \ldots, l$. Further, the mapping ϕ is said to be homogeneous if there exists s in $\mathbb{R}$ [s is called the homogeneity degree of ϕ] and $(r_1, \ldots, r_j)$ in $(0, +\infty)^j$ such that

$$\phi(\lambda^{r_1} x_1, \ldots, \lambda^{r_j} x_j) = \begin{pmatrix} \lambda^{s+r_1} \phi_1(x) \\ \vdots \\ \lambda^{s+r_n} \phi_l(x) \end{pmatrix}, \quad x \in \mathbb{R}^n \backslash \{0\}, \quad \lambda > 0.$$

We present in the next section, a few preliminary results.

5.3.1 Preliminary results

The stabilization of homogeneous control systems has been recently addressed by several authors, e.g., Andreini et al. [3], Kawski [50], and Dayawansa et al. [22]. The main reason for this attention lies in the fact that homogeneity properties play an important role in the theory on the stabilization of linear systems by linear feedback laws. Thus, it is natural to expect similar results for the stabilization of homogeneous systems by homogeneous feedback laws. Given an homogeneous system if an homogeneous stabilizing feedback law exists then the corresponding closed-loop system may be homogeneous as assumed here.

It is therefore necessary to investigate the properties of asymptotically stable homogeneous systems. Studies on that matter encompasses the work of Hahn [37] and Rosier [67]. In this last paper, Rosier extended Hahn's results and established the existence of an homogeneous radially unbounded Lyapunov function for any given asymptotically stable homogeneous system. More precisely, he obtained the following theorem.

Theorem 5.3 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping with $F(0) = 0$. Assume that F is homogeneous or equivalently that there exist $(r_1, \ldots, r_n)$ in $(0, +\infty)^n$ and s in $\mathbb{R}$ such that*

$$F_i(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^{s+r_i}F_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \quad \lambda > 0, \quad x \in \mathbb{R}.$$

Further, assume that the system $(S): \dot{x} = F(x)$ is asymptotically stable. Let p be in $\{1, 2, \ldots\}$, and let $k \geq p \max_{i=1,\ldots,n} r_i$. Then, (S) admits a C^p radially unbounded homogeneous Lyapunov function $V : \mathbb{R}^n \to [0, \infty)$, which satisfies

$$\nabla V(x)\, F(x) < 0, \quad x \in \mathbb{R}^n \backslash \{0\}, \tag{5.21}$$

and

$$V(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^k V(x), \quad x \in \mathbb{R}^n \backslash \{0\}, \quad \lambda > 0. \tag{5.22}$$

We now let $p = 1$ and we choose k satisfying

$$k \geq \max_{i=1,\ldots,n} (r_i) \quad \text{and} \quad 1 + \frac{s}{k} > 0.$$

Let V be a radially unbounded Lyapunov function for the system $\dot{x} = F(x)$, obtained through Theorem 5.3 [applied with $p = 1$, k and F as defined here]. We stress that V satisfies (5.22), and that (5.21) together with the radial unboundedness of V implies that the system $\dot{x} = F(x)$ is globally asymptotically stable. Further, as mentioned in [67, p. 470], the identities (5.20) and (5.22) yield

$$\nabla V(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)\, F(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^{s+k}\, \nabla V(x)\, F(x), \tag{5.23}$$

for each x in $\mathbb{R}^n \backslash \{0\}$ and each $\lambda > 0$.

In order, to prove the simultaneous asymptotic stabilizability of S_1 and S_2, the idea is to use the Lyapunov function V to define a sequence of mappings $\{b_i^n, i = 1, 2\}_{n=1}^{\infty}$ or $\{b_i^n, i = 1, 2\}_{n \in \mathbf{Z}}$ satisfying the assumptions of Theorem 5.1 or 5.2. The following simple yet important lemma will be the key for constructing these mappings.

Lemma 5.1 *There exists $\theta > 0$ such that we have*

$$\nabla V(x)\, F(x) < -\theta \rho^{1+\frac{s}{k}}, \quad x \in V^{-1}(\rho), \quad \rho > 0. \tag{5.24}$$

Proof: Let $\rho > 0$ and let x be in $V^{-1}(\rho)$. By Lemma 1 in [67], there exists a unique positive real α and a unique y in $S^{n-1} \triangleq \{y \in \mathbb{R}^n : \|y\| = 1\}$ satisfying

$$x = (\alpha^{r_1}y_1, \ldots, \alpha^{r_n}y_n), \tag{5.25}$$

so that

$$\nabla V(x) F(x) \;=\; \alpha^{s+k} \nabla V(y)\, F(y). \tag{5.26}$$

Because $V(x) = \rho$, we obtain from the homogeneity of V combined with (5.25) that $\alpha = \left(\frac{\rho}{V(y)}\right)^{\frac{1}{k}}$, and it follows from (5.26) that

$$\nabla V(x)\, F(x) \;=\; \left(\frac{\rho}{V(y)}\right)^{1+\frac{s}{k}} \nabla V(y)\, F(y). \tag{5.27}$$

Upon setting

$$-m \;\triangleq\; \max_{y \in S^{n-1}} \nabla V(y)\, F(y), \quad M \;\triangleq\; \max_{y \in S^{n-1}} V(y), \qquad \text{and} \qquad \theta \triangleq \frac{m}{2M^{1+\frac{s}{k}}},$$

equality (5.27) yields

$$\nabla V(x)\, F(x) \;\leq\; \frac{-m}{M^{1+\frac{s}{k}}} \rho^{1+\frac{s}{k}} \;<\; -\theta \rho^{1+\frac{s}{k}},$$

which completes the proof of the lemma. ■

Throughout, we let θ satisfy (5.24), and we seek a parameterized family of mappings $h_\beta : [0, \infty) \to (0, \infty)$ such that for each β in some interval included in $(0, \infty)$, we have

$$-\,\theta h_\beta(t)^{1+\frac{s}{k}} \;\leq\; \dot{h}_\beta(t), \quad t \geq 0, \tag{5.28}$$

with $h_\beta(0) = \beta$. We distinguish three cases based on the the sign of s.

1) If $s = 0$, then for each $\beta > 0$, we define $h_\beta : [0, \infty) \to (0, \infty)$ by setting

$$h_\beta(t) \;=\; \beta e^{-\theta t}, \quad t \geq 0. \tag{5.29}$$

It is plain that $h_\beta(0) = \beta$ and that (5.28) holds.

2) If $s > 0$, then for each $\beta > 0$, we define $h_\beta : [0, \infty) \to (0, \infty)$ by setting

$$h_\beta(t) \;=\; \frac{1}{\left(\beta^{-\frac{s}{k}} + \frac{s}{k}\theta t\right)^{\frac{k}{s}}}, \quad t \geq 0. \tag{5.30}$$

By direct computation, we get $\dot{h}_\beta = -\theta h_\beta(t)^{1+\frac{s}{k}}$ for each $t \geq 0$, so that (5.28) holds. Furthermore, we clearly have $h_\beta(0) = \beta$.

3) If $s < 0$, we set

$$\bar{\beta} \triangleq \left[\theta \left(\frac{-s}{2k}\right)^{\frac{1}{2}} e^{\frac{1}{2}}\right]^{\frac{1}{\frac{s}{k}}}, \tag{5.31}$$

and for each β in $(0, \bar{\beta}]$, we define $h_\beta : [0, \infty) \to (0, \infty)$ by setting

$$h_\beta(t) = \begin{cases} \beta, & t \leq \frac{\beta^{\frac{-s}{k}}}{\theta} \\ \beta e^{-\left(t - \frac{\beta^{\frac{-s}{k}}}{\theta}\right)^2}, & t > \frac{\beta^{\frac{-s}{k}}}{\theta} \end{cases}. \tag{5.32}$$

By Lemma B.13 applied with $r = 1 + \frac{s}{k}$, $\delta = 1$ and θ as defined here, we obtain (5.28).

We now fix β in $(0, \infty)$ if $s \geq 0$ (resp. β in $(0, \bar{\beta}]$ if $s < 0$). By definition, the mapping h_β given by either one of the formulas (5.29) and (5.30) (resp. (5.32)), satisfies the inequality (5.28), and because (5.24) holds with θ as defined here, it follows that

$$\nabla V(x)\, F(x) < \dot{h}_\beta(t), \quad x \in V^{-1}(h_\beta(t)), \quad t \geq 0. \tag{5.33}$$

Next, it is easily seen that h_β is non increasing and converges to 0 as t tends to $+\infty$, in all three cases. By direct inspection of the formulas (5.29) and (5.30) if $s \geq 0$ (resp. by Lemma B.13 if $s < 0$), we obtain

$$h_\beta(t) < h_\gamma(t), \quad t \geq 0,$$

for each β and each γ in $(0, \infty)$ (resp. $(0, \bar{\beta}]$) with $\beta < \gamma$.

Using the mapping h_β, we finally prove the simultaneous asymptotic stabilizability of S_1 and S_2.

5.3.2 Simultaneous asymptotic stabilization

The purpose of this subsection is to prove the following proposition.

Proposition 5.1 *Let the systems S_1 and S_2, and the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the assumptions (5.18), (5.19) and (5.20). Then, the following holds:*

i) *If $s = 0$, then there exists a continuous time-varying feedback law $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$, which simultaneously globally uniformly asymptotically stabilizes S_1 and S_2, with uniform exponential convergence of the corresponding closed-loop systems [according to Definition 1.2* **(vi)***].*

ii) *If $s > 0$, then there exists a continuous time-varying feedback law $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$, which simultaneously globally asymptotically stabilizes S_1 and S_2.*

iii) *If $s < 0$, then there there exists a continuous time-varying feedback law which simultaneously locally asymptotically stabilizes S_1 and S_2*

Proof:

Simultaneous asymptotic stabilization in the case $s \geq 0$

Throughout this paragraph, we assume $s \geq 0$. We let β be in $(0,1)$ and we define the sequences $\{\beta_i^n,\ i = 1,2\}_{n\in\mathbb{Z}}$ by setting

$$\beta_1^n \triangleq (\beta)^{2n-1} \qquad \text{and} \qquad \beta_2^n \triangleq (\beta)^{2n}, \quad n \in \mathbb{Z}\,.$$

Further, for each $i = 1, 2$, and each n in $\mathbb{Z}$, we let α_i^n and γ_i^n be such that

$$\gamma_1^n > \beta_1^n > \alpha_1^n > \gamma_2^n > \beta_2^n > \alpha_2^n > \gamma_1^{n+1}, \quad n \in \mathbb{Z},$$

and we define the mappings $a_i^n,\ b_i^n,\ c_i^n : [0,\infty) \to (0,\infty)$ by setting

$$a_i^n(t) = h_{\alpha_i^n}(t), \quad b_i^n(t) = h_{\beta_i^n}(t) \qquad \text{and} \qquad c_i^n(t) = h_{\gamma_i^n}(t), \quad t \geq 0,$$

with h_β defined by (5.29) if $s = 0$ (resp. by (5.30) if $s > 0$). Because h_β satisfies (5.33), it follows from the definition of the mappings b_i^n, that

$$\nabla V(x)\, F(x) < \dot{b}_i^n(t), \quad x \in V^{-1}(h_\beta(t)), \quad t \geq 0, \tag{5.34}$$

for each $i = 1, 2$ and each n in $\mathbb{Z}$, which yields Assumption **(v)** of Theorem 5.2. Further, by combining the equality $b_i^n(0) = \beta_i^n$ with the fact that the mapping b_i^n is decreasing and converges to 0 as t tends to $+\infty$, it is easily checked that Assumption **(i)-(iv)** of Theorem 5.2 also hold. Therefore, by Theorem 5.2, there exists a continuous time-varying feedback law $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ that simultaneously **globally asymptotically stabilizes S_1 and S_2**.

For each $i = 1, 2$, and each n in $\mathbb{Z}$, we let the mapping $q_i^n : [0,\infty) \times \mathbb{R}^n \to [0,1]$ be defined by the formula (5.17), with a_i^n, b_i^n and c_i^n as given here, after substituting V for V_i.

Finally, we define the mapping $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ by setting

$$v(t,x) = -\sum_{n\in\mathbb{Z}} u(x)\, q_1^n(t,x) + \sum_{n\in\mathbb{Z}} u(x)\, q_2^n(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^n\,.$$

It is easily seen from the proof of Theorem 5.2, that v simultaneously globally asymptotically stabilizes S_1 and S_2 if $s = 0$ (resp. $s > 0$), and that v is continuous on $[0, \infty) \times \mathbb{R}^n$.

If $s = 0$, we now show that the feedback law v also yields uniform stability and uniform exponential convergence, by using an argument similar to that used in the proof of Theorem 4.1 to prove exponential stability.

Global uniform asymptotic stability and exponential convergence in the case $s = 0$:

Throughout, we assume that $s = 0$, so that we have

$$b_1^n(t) \;=\; (\beta)^{2n-1}\, e^{-\theta t} \qquad \text{and} \qquad b_2^n(t) \;=\; (\beta)^{2n}\, e^{-\theta t}, \quad t \geq 0, \quad n \in \mathbb{Z},$$

and upon setting $T \triangleq -\frac{1}{\theta}\ln(\beta^2)$, these definitions yield

$$b_i^n(t + kT) \;=\; b_i^{n+k}(t), \quad t \geq 0, \quad n \in \mathbb{Z}, \quad k = 0, 1, \ldots, \quad i = 1, 2. \tag{5.35}$$

We fix $i = 1, 2$ and we let x_0 be in $\mathbb{R}^n$. Because the sequence $\{b_i^n(0)\}_{n\in\mathbb{Z}}$ is strictly decreasing, and converges to $+\infty$ and 0 as n tends to $-\infty$ and $+\infty$ respectively, there exists $\bar{n}$ in $\mathbb{Z}$ such that

$$b_i^{\bar{n}+1}(0) \;<\; V(x_0) \;\leq\; b_i^{\bar{n}}(0). \tag{5.36}$$

Further, let t_0 be in $[0, \infty)$. Then, there exists t_0' in $[0, T)$ and an integer k satisfying $t_0 = kT + t_0'$. By combining (5.35) with the fact that the mapping b_i^n is decreasing for each n in $\mathbb{Z}$, we get

$$b_i^{\bar{n}}(0) \;=\; b_i^{\bar{n}-k-1}((k+1)T) \;\leq\; b_i^{\bar{n}-k-1}(t_0),$$

so that (5.36) yields $V(x_0) \leq b_i^{\bar{n}-k-1}(t_0)$. Therefore, Lemma 4.1 combined with (5.34), imply that

$$V(x(t, x_0, t_0)) \;\leq\; b_i^{\bar{n}-k-1}(t), \quad t \geq t_0. \tag{5.37}$$

Next, for x in $\mathbb{R}^n$, Lemma 1 in [67] yields the existence of $(\lambda_x) > 0$ and y in $S^{n-1} \triangleq \{x \in \mathbb{R}^n : \|x\| = 1\}$ satisfying

$$x \;=\; (\,(\lambda_x)^{r_1} y_1, \ldots, (\lambda_x)^{r_n} y_n\,),$$

and upon setting $m_V = \min\limits_{y\in S^{n-1}} V(y)$, we get

$$V(x) \;=\; (\lambda_x)^k\, V(y) \;\geq\; m_V (\lambda_x)^k. \tag{5.38}$$

We now define r and R by setting $r \triangleq \min_{i=1,\ldots,n} r_i$ and $R \triangleq \max_{i=1,\ldots,n} r_i$, and we obtain

$$\|x\|^2 = (\lambda_x)^{2r_1} y_1^2 + \ldots + (\lambda_x)^{2r_n} y_n^2 \le \begin{cases} (\lambda_x)^{2R}, & \text{if } (\lambda_x) \ge 1 \\ (\lambda_x)^{2r}, & \text{if } (\lambda_x) < 1 \end{cases} .$$

Thus, for each $t \ge t_0$, the inequalities (5.37) and (5.38) together with the expression of $b_i^{\bar{n}-k-1}(t)$ yield

$$\begin{aligned} \|x(t, x_0, t_0)\|^{\frac{k}{R}} &\le \frac{\beta_i^{\bar{n}-1-k}}{m_V} e^{-\theta t}, \quad \text{if } (\lambda_{x(t,x_0,t_0)}) \ge 1, \\ \|x(t, x_0, t_0)\|^{\frac{k}{r}} &\le \frac{\beta_i^{\bar{n}-1-k}}{m_V} e^{-\theta t}, \quad \text{if } (\lambda_{x(t,x_0,t_0)}) < 1, \end{aligned}$$

Because $t_0 = kT + t_0'$ with $t_0' \ge 0$, it is easily seen from (5.35) that

$$\begin{aligned} \|x(t, x_0, t_0)\| &\le \left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{R}{k}} e^{-\theta\frac{R}{k}(t-t_0)}, \quad \text{if } (\lambda_{x(t,x_0,t_0)}) \ge 1, \\ \|x(t, x_0, t_0)\| &\le \left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{r}{k}} e^{-\theta\frac{r}{k}(t-t_0)}, \quad \text{if } (\lambda_{x(t,x_0,t_0)}) < 1, \end{aligned}$$

for each $t \ge t_0$, whence

$$\|x(t, x_0, t_0)\| \le \max\left[\left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{R}{k}}, \left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{r}{k}}\right] e^{-\theta\frac{r}{k}(t-t_0)}, \quad t \ge t_0. \tag{5.39}$$

Note that the real

$$\max\left[\left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{R}{k}}, \left(\frac{\beta_i^{\bar{n}-1}}{m_V}\right)^{\frac{r}{k}}\right]$$

depends solely on $\|x_0\|$ (uniformly in t_0), and because $b_i^{\bar{n}}(0)$ converges to 0 as $\|x_0\|$ tends to 0 [follows from the definition of $\bar{n}$], global uniform asymptotic stability with exponential convergence follows from (5.39) [according to Definition 1.2 **(vi)**].

Simultaneous local asymptotic stabilization in the case $s < 0$:

We let

$$\beta_1^n \triangleq \frac{\bar{\beta}}{2n-1} \quad \text{and} \quad \beta_2^n \triangleq \frac{\bar{\beta}}{2n}, \quad n = 1, 2, \ldots,$$

with $\bar{\beta}$ given by (5.31), and we let $b_i^n : [0, \infty) \to (0, \infty)$ be defined by

$$b_i^n(t) = h_{\beta_i^n}(t), \quad t \ge 0,$$

for each $i = 1, 2$, and each $n = 1, 2, \ldots$, with $h_\beta : [0, \infty) \to (0, \infty)$ given by (5.32). Because the mappings b_i^n are decreasing and converge to 0 as t tends to $+\infty$, it is easily checked from the equality $b_i^n(0) = \beta_i^n$ that Assumptions **(i)-(iv)** of Theorem 5.1 hold. Moreover, because h_β satisfies (5.33), Assumption **(v)** follows from the definition of b_i^n for each $i = 1, 2$ and each $n = 1, 2, \ldots$. Therefore, Theorem 5.1 yields the existence of a continuous time-varying feedback law $v : [0, \infty) \times V^{-1}([0, \bar{\beta} + 1)) \to \mathbb{R}^m$ that simultaneously locally asymptotically stabilizes S_1 and S_2. ■

We note that the previous construction does not depend directly on the control systems S_1 and S_2. What really matters is that the application of $-u$ and u to S_1 and S_2 respectively, yields an homogeneous system $\dot{x} = F(x)$. In fact, the previous construction can be easily modified in order to prove the following corollary.

Corollary 5.4 *Let $I \geq 2$ be an integer. For each $i = 1, \ldots, I$, let the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuous with $f_i(0,0) = 0$, and assume that there exists a continuous mapping $u_i : \mathbb{R}^n \to \mathbb{R}^m$, such that $\dot{x} = f_i(x, u_i(x))$ is asymptotically stable. Further, assume that there exists an homogeneous mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f_i(x, u_i(x)) = F(x), \quad x \in \mathbb{R}^n, \quad i = 1, \ldots, I,$$

and let s be the homogeneity degree of F. Then, the following holds :

i) *If $s = 0$, then there exists a continuous time-varying feedback law $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$, which simultaneously globally uniformly stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$, with uniform exponential convergence of the corresponding closed-loop systems [according to Definition 1.2* **(vi)***].*

ii) *If $s > 0$, then there exists a continuous time-varying feedback law $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$, which simultaneously globally asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

iii) *If $s < 0$, then there there exists a continuous time-varying feedback law which simultaneously locally asymptotically stabilizes the family $\{S_i,\ i = 1, \ldots, I\}$.*

We complete this chapter by providing the technical lemma that was used in the proof of Theorem 5.1 and 5.2.

5.4 Technical lemma

In this section, we present a technical lemma that was used in the proof of Theorem 5.1 and 5.2.

Lemma 5.2 *Let D_1 and D_2 be two bounded neighborhoods of the origin (resp. $D_1 = D_2 = \mathbb{R}^n$), and let $V_1 : \overline{D}_1 \to [0,\infty)$ and $V_2 : \overline{D}_2 \to [0,\infty)$ be two Lyapunov functions (resp. two radially unbounded Lyapunov functions). Further, let b_1 and b_2 be two continuous mappings from $[0,\infty)$ into $(0, \inf_{x\in\partial D_1} V_1(x))$ and $(0, \inf_{x\in\partial D_2} V_2(x))$ respectively, such that*

$$D_1 \cap V_1^{-1}(\,[0, b_1(t))\,) \supset D_2 \cap V_2^{-1}(\,[0, b_2(t)]\,), \quad t \geq 0. \tag{5.40}$$

Then, there exist two C^∞ mappings $a_1 : [0,\infty) \to (0, \inf_{x\in\partial D_1} V_1(x))$ and $c_2 : [0,\infty) \to (0, \inf_{x\in\partial D_2} V_2(x))$, such that for each $t \geq 0$, we have

$$b_1(t) > a_1(t) \qquad \text{and} \qquad c_2(t) > b_2(t), \tag{5.41}$$

together with

$$D_1 \cap V_1^{-1}(\,[0, a_1(t))\,) \supset D_2 \cap V_2^{-1}(\,[0, c_2(t)]\,). \tag{5.42}$$

Proof: For each $t \geq 0$ and each δ_t either in $(0,t)$ if $t > 0$ or in $(0,\infty)$ if $t = 0$, we let $I(t,\delta_t)$ denote the set

$$I(t,\delta_t) \triangleq \begin{cases} (t-\delta_t, t+\delta_t) & \text{if} \quad t > 0 \\ [0,\delta_t) & \text{if} \quad t = 0 \end{cases}.$$

As usual $\overline{I}(t,\delta_t)$ denotes the closure of $I(t,\delta_t)$.

We first construct a mapping a_1 satisfying $b_1(t) > a_1(t)$ and

$$D_1 \cap V_1^{-1}(\,[0, a_1(t))\,) \supset D_2 \cap V_2^{-1}(\,[0, b_2(t)]\,),$$

for each $t \geq 0$.

Construction of a_1 :

Fix t in $[0,\infty)$. In view of (5.40), Lemma B.12 (applied with D_1, D_2, V_1, V_2, b_1 and b_2) yields the existence of $\delta_t > 0$ (with δ_t in $(0,t)$ if $t > 0$) such that

$$D_1 \cap V_1^{-1}(\,[0, \min_{\tau\in\overline{I}(t,\delta_t)} b_1(\tau))\,) \supset D_2 \cap V_2^{-1}(\,[0, \max_{\tau\in\overline{I}(t,\delta_t)} b_2(\tau)]\,) \tag{5.43}$$

In view of this last inclusion, it follows from Lemma B.11 that there exists α_t such that we have

$$\alpha_t \in (0, \min_{\tau \in \bar{I}(t,\delta_t)} b_1(\tau)), \tag{5.44}$$

and

$$D_1 \cap V_1^{-1}([0, \alpha_t)) \supset D_2 \cap V_2^{-1}([0, \max_{\tau \in \bar{I}(t,\delta_t)} b_2(\tau)]). \tag{5.45}$$

For each t in $[0, \infty)$ let δ_t and α_t be such that (5.43), (5.44) and (5.45) are satisfied.

Because the family $\{I(t, \delta_t)\}_{t \in [0,\infty)}$ is an open cover of $[0, \infty)$ (in its subspace topology), it follows from the fact that $[0, \infty)$ is Lindelöf that we can extract from this cover a countable sub-cover $\{I(t_k, \delta_{t_k})\}_{k=0}^{\infty}$ of $[0, \infty)$ with $t_k < t_{k+1}$, $k = 0, 1, \ldots$. If necessary we choose δ'_{t_k} and δ''_{t_k} in $(0, \delta_{t_k}]$, for each $k = 0, 1, \ldots$, such that the sets

$$I_0 \triangleq [0, \delta_0) \qquad \text{and} \qquad I_k \triangleq (t_k - \delta'_{t_k}, t_k + \delta''_{t_k}), \quad k = 1, 2, \ldots,$$

form an open cover $\{I_k\}_{k=0}^{\infty}$ of $[0, \infty)$ and any t in $[0, \infty)$ lies in at most two successive sets I_k and I_{k+1}.

By Theorem 1.1 there exists a partition of unity $\{\bar{p}_k\}_{k=0}^{\infty}$ subordinate to $\{I_k\}_{k=0}^{\infty} \backslash \{0\}$ such that for each $k = 1, 2, \ldots$, the support of $\bar{p}_k$ is included in I_k and the support of $\bar{p}_0$ is included in $(0, \delta t_0)$.

We now define the mapping $p_k : [0, \infty) \to [0, 1]$, for each $k = 0, 1, \ldots$, by setting

$$p_k(t) = \bar{p}_k(t), \quad t > 0, \quad k = 0, 1, \ldots,$$

and

$$p_k(0) = 0, \quad k = 1, 2, \ldots; \qquad p_0(0) = 1. \tag{5.46}$$

Because the support of $\bar{p}_k$ is included in I_k for each $k = 1, 2, \ldots$ and in $I_0 \backslash \{0\}$ for $k = 0$, it is easily seen from (5.46) that for each $k = 0, 1, \ldots$, the mapping p_k is C^∞ on $[0, \infty)$, and its support is included in I_k. Also note that the sum $\sum_{k=0}^{\infty} p_k$ is identically equal to 1 on $[0, \infty)$.

We now define the mapping $a_1 : [0, \infty) \to (0, \infty)$ by setting

$$a_1(t) = \sum_{k=0}^{\infty} \alpha_{t_k} p_k(t), \quad t \geq 0.$$

It is plain from this definition that $a_1(t)$ is positive for each $t \geq 0$, since α_{t_k} is positive for each $k = 0, 1, \ldots$ and the mapping p_k sum up to 1.

Smoothness and properties of a_1 :

We first prove that a_1 is C^∞ on $[0,\infty)$. Let t be in $[0,\infty)$. Then, there exists an open neighborhood U_t of t in $[0,\infty)$ that intersects with at most two sets I_k and I_{k+1} of the family $\{I_k\}_{k=0}^\infty$. Therefore, we have

$$a_1(\tau) \;=\; \alpha_{t_k}\, p_k(\tau) \;+\; \alpha_{t_{k+1}}\, p_{k+1}(\tau), \quad \tau \in U_t,$$

and it follows that a_1 is well-defined and C^∞ on $[0,\infty)$.

We now show that we have

$$b_1(t) \;>\; a_1(t), \quad t \geq 0, \tag{5.47}$$

and

$$D_1 \cap V_1^{-1}(\,[0,a_1(t))\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0,b_2(t)]\,), \quad t \geq 0. \tag{5.48}$$

Let t be in $[0,\infty)$. Then, as previously noted, t lies either in a unique set I_k, or in two successive sets I_k and I_{k+1} of the family $\{I_k\}_{k=0}^\infty$. If t lies in a unique set I_k, then we get

$$a_1(t) \;=\; \alpha_{t_k}. \tag{5.49}$$

By definition of I_k we have

$$\overline{I}_k \;\subset\; \overline{I}(t_k,\delta_{t_k}), \quad k = 0,1,\ldots \tag{5.50}$$

so that $\min\limits_{\tau\in\overline{I}(t_k,\delta_{t_k})} b_1(\tau) \leq \min\limits_{\tau\in\overline{I}_k} b_1(\tau)$ and (5.44) combined with (5.49) yield $b_1(t) > \alpha_{t_k}$ or equivalently $b_1(t) > a_1(t)$. On the other hand, the inclusion (5.48) follows easily from (5.45).

If t lies in two sets of the family $\{I_k\}_{k=0}^\infty$, then there exists k in $\{0,1,\ldots\}$ such that

$$a_1(t) \;=\; \alpha_{t_k}\, p_k(t) \;+\; \alpha_{t_{k+1}}\, p_{k+1}(t). \tag{5.51}$$

We first note that $a_1(t)$ is a convex combination of α_{t_k} and $\alpha_{t_{k+1}}$ since the mappings of the family $\{p_k\}_{k=0}^\infty$ sum up to 1. In view of (5.44) we have

$$b_1(t) > \alpha_{t_k} \quad \text{and} \quad b_1(t) > \alpha_{t_{k+1}},$$

so that (5.51) yields $b_1(t) > a_1(t)$. Furthermore, from (5.45) we obtain

$$D_1 \cap V_1^{-1}(\,[0,\alpha_{t_j})\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0,b_2(t)]\,), \quad j = k, k+1,$$

and since we have $a_1(t) \geq \min\,(\alpha_{t_k}, \alpha_{t_{k+1}})$ [follows from (5.51)], the desired inclusion (5.48) follows. Therefore, the mapping a_1 satisfies the assertions (5.47) and (5.48).

Construction of c_2 :

Using the mappings a_1, b_1, and the inclusion (5.48), we now produce a mapping $c_2 : [0,\infty) \to (0,\infty)$ such that for all $t \geq 0$, we have

$$c_2(t) \;\in\; (b_2(t), \inf_{x\in\partial D_2} V_2(x)\,), \quad t \geq 0, \tag{5.52}$$

and

$$D_1 \cap V_1^{-1}(\,[0, a_1(t))\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, c_2(t)]\,), \quad t \geq 0. \tag{5.53}$$

The construction is similar to that of a_1. For each $t \geq 0$, by combining Lemma B.12 (with D_1, D_2, V_1, V_2, a_1 and b_2) with (5.48), we obtain $\delta_t > 0$ such that

$$D_1 \cap V_1^{-1}(\,[0, \min_{\tau\in\overline{I}(t,\delta_t)} a_1(\tau))\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \max_{\tau\in\overline{I}(t,\delta_t)} b_2(\tau)]\,)$$

and by Lemma B.11 we get γ_t satisfying

$$\gamma_t \;\in\; (\max_{\tau\in\overline{I}(t,\delta_t)} b_2(\tau), \inf_{x\in\partial D_2} V_2(x)\,) \tag{5.54}$$

with

$$D_1 \cap V_1^{-1}(\,[0, \min_{\tau\in\overline{I}(t,\delta_t)} a(\tau))\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \gamma_t]\,). \tag{5.55}$$

Let $\{I(t_k, \delta_{t_k})\}_{k=0}^{\infty}$ be a countable open sub-cover of $\{I(t,\delta_t)\}_{t\geq 0}$ such that $t_k < t_{k+1}, k = 0, 1, \ldots$. If necessary, we choose δ'_{t_k} and δ''_{t_k} in $(0, \delta_{t_k}]$ such that the sets

$$I_0 \triangleq [0, \delta t_0) \qquad \text{and} \qquad I_k \triangleq (t_k - \delta'_{t_k}, t_k + \delta''_{t_k}), \quad k = 1, 2, \ldots,$$

form an open cover $\{I_k\}_{k=0}^{\infty}$ of $[0,\infty)$ and any t in $[0,\infty)$ lies in at most two successive sets I_k and I_{k+1}.

We then let $\{\bar{p}_k\}_{k=0}^{\infty}$ be a partition of unity subordinate to $\{I_k\}_{k=0}^{\infty}\backslash\{0\}$. For each $k = 0, 1, \ldots$, by extending $\bar{p}_k$ exactly as we did in the construction of a_1, we obtain a C^∞ mapping $p_k : [0,\infty) \to [0,1]$ whose support is included in I_k. In addition, the summation $\sum_{k=0}^{\infty} p_k$ is identically equal to 1 on $[0,\infty)$.

We finally define the mapping $c_2 : [0,\infty) \to (0,\infty)$, by setting

$$c_2(t) = \sum_{k=0}^{\infty} \gamma_{t_k}\, p_k(t), \quad t \geq 0.$$

Because γ_{t_k} is positive for each $k = 0, 1, \ldots$ and the mappings p_k sum up to 1, we have $c_2(t) > 0$ for each $t \geq 0$. For each $t \geq 0$, it is plain that there exist a neighborhood of t that intersects with at most two sets of the family $\{I_k\}_{k=0}^{\infty}$.

Because the support of the mapping p_k is included in I_k, for each $k = 0, 1, \ldots$, it follows that the mapping c_2 is well-defined and C^∞ on $[0, \infty)$.

We now fix $t \geq 0$. If t lies in a unique set I_k, then $c_2(t) = \gamma_{t_k}$, and (5.52) follows from (5.54) while (5.53) is obtained from (5.55).

If t lies in two sets I_k and I_{k+1}, the real $c_2(t)$ is a convex combination of γ_{t_k} and $\gamma_{t_{k+1}}$. In view of (5.54), the reals γ_{t_k} and $\gamma_{t_{k+1}}$ both lie in $(\, b_2(t), \inf_{x \in \partial D_2} V_2(x)\,)$, and (5.52) follows. Further, because the inclusion (5.55) holds for both γ_{t_k} and $\gamma_{t_{k+1}}$, we obtain

$$D_1 \cap V_1^{-1}(\,[0, a_1(t))\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \max(\gamma_{t_k}, \gamma_{t_{k+1}}))\,),$$

and (5.53) is finally obtained from the inequality $\max(\gamma_{t_k}, \gamma_{t_{k+1}}) \geq c_2(t)$. Hence the lemma. ■

Chapter 6

Time-Invariant Simultaneous Asymptotic Stabilization in the Plane

In Chapter 4 and 5, we enriched the approach introduced in Chapter 2 in order to design time-varying feedback laws that simultaneously asymptotically stabilize families of systems of arbitrary dimension. Here, we show that the ideas introduced in Chapter 2, can be refined differently in order to provide **time-invariant asymptotic** stabilizers for a class of systems in the plane.

The chapter is organized as follows: We begin with stating the problem under consideration, and we then prove the simultaneous asymptotic stabilizability of the considered pairs of systems in Section 6.2. We finally give some concluding remarks in Section 6.3 and present some technical lemmas in section 6.4.

6.1 Introduction

Throughout this chapter, we consider the pair of systems

$$S_-: \begin{cases} \dot{x}_1 &= a_- x_1 + b_- x_2 \\ \dot{x}_2 &= u \end{cases} \quad \text{and} \quad S_+: \begin{cases} \dot{x}_1 &= a_+ x_1 + b_+ x_2 \\ \dot{x}_2 &= u \end{cases},$$

where a_-, a_+, b_+ are positive, b_- is negative and u is a scalar input. Using elementary algebra, it is easily seen that these two systems are not simultaneously asymptotically stabilizable by means of C^1 feedback. To design a merely continuous simultaneous asymptotic stabilizer for S_- and S_+, we modify the construction introduced in the proof of Theorem 2.1.

Given two control systems S_1 and S_2, that are globally asymptotically stabilizable by the continuous time-invariant feedback laws u_1 and u_2 respectively, it follows from Theorem 2.2 that we can construct a simultaneous stabilizer (not asymptotic) for S_1 and S_2 as follows: We let V_1 (resp. V_2) denote a Lyapunov function for the closed-loop system obtained once u_1 (resp. u_2) is fed-back into S_1 (resp. S_2). We introduce a base at the origin $\{U_j\}_{j\in\mathbf{Z}}$ such that the boundaries of the odd (resp. even) sets U_{2n+1} (resp. U_{2n}) are Lyapunov level sets of V_1 (resp. V_2). We design a feedback law v which is equal to u_1 (resp. u_2) on the boundaries of the odd (resp. even) sets U_{2n+1} (resp. U_{2n}), and we conclude that v stabilizes S_1 (resp. S_2) upon noting that the family $\{U_{2n+1}\}_{n\in\mathbf{Z}}$ (resp. $\{U_{2n}\}_{n\in\mathbf{Z}}$) is a base at the origin.

In general, the obtained feedback law v does not provide asymptotic stability to S_1 (resp. S_2), because it is not guaranteed that the positive limit set in $\overline{U}_{2n+1}$ (resp. $\overline{U}_{2n}$) of the closed-loop system corresponding to S_1 (resp. S_2), is the trivial set $\{0\}$.

In the particular setup of this chapter, it turns out that we can actually construct a base at the origin $\{W_j\}_{j\in\mathbf{Z}}$ (whose boundaries are not level sets of Lyapunov functions) and a feedback law u_{k_0} such that the odd sets $\overline{W}_{2n+1}$ (resp. $\overline{W}_{2n}$) are invariant with respect to the closed-loop systems corresponding to the application of u_{k_0} to S_- (resp. S_+). Because the considered systems are in the plane, we can use the Poincaré-Bendixson Theory in order to guarantee that the positive limit set of S_- (resp. S_+) in the compact sets $\overline{W}_{2n+1}$ (resp. $\overline{W}_{2n}$), is the trivial set $\{0\}$. In this way, we obtain both stability and asymptotic convergence to the origin.

6.2 Simultaneous asymptotic stabilization

The purpose of this section is to establish the existence of a merely continuous and time-invariant feedback law that simultaneously globally asymptotically stabilizes S_- and S_+.

This result is contained in the following theorem. The general line of the proof is to construct two feedback laws $u_{k_0}^-$ and $u_{k_0}^+$ that globally asymptotically stabilize S_- and S_+ respectively. We introduce two bases at the origin $\{W_\beta^-\}_{\beta>0}$ and $\{W_\beta^+\}_{\beta>0}$ such that for each $\beta > 0$, the neighborhoods W_β^- and W_β^+ are invariant with respect to the systems S_- (with $u = u_{k_0}^-$) and S_+ (with $u = u_{k_0}^+$) respectively. We then construct a new base at the origin $\{W_j\}_{j\in\mathbf{Z}}$ such that the odd (resp. even) sets W_{2n+1} (resp. W_{2n}) belong to the family $\{W_\beta^-\}_{\beta>0}$ (resp. $\{W_\beta^+\}_{\beta>0}$). Finally, we define a continuous feedback law u_{k_0} which is equal to $u_{k_0}^-$ (resp. $u_{k_0}^+$) on the boundary of the odd sets W_{2n+1} (resp. even sets W_{2n}). It follows that the closure of each neighborhood of the base at the origin $\{W_{2n+1}\}_{n\in\mathbf{Z}}$ (resp. $\{W_{2n}\}_{n\in\mathbf{Z}}$) is invariant with respect to the closed-loop system obtained once u_{k_0} is fed back into S_- (resp. S_+). This implies that u_{k_0} simultaneously stabilizes S_- and S_+. Asymptotic stability is then established by proving that the only positive limit set of the system S_- (resp. S_+) with $u = u_{k_0}$, in the sets $\overline{W}_{2n+1}$ (resp. $\overline{W}_{2n}$), is the origin.

Theorem 6.1 *Assume that a_-, a_+ and b_+ are positive, and that b_- is negative. Then, there exists a continuous and almost C^∞ feedback law $v : \mathbb{R}^2 \to \mathbb{R}$, that simultaneously globally asymptotically stabilizes the systems S_- and S_+.*

Proof: Throughout the proof, we use the following notation: For each x in $\mathbb{R}^2$, we denote by x_1 and x_2 its coordinates, and we define the mappings, $f_-, f_+ : \mathbb{R}^2 \to \mathbb{R}$ by setting $f_-(x) = a_-x_1 + b_-x_2$ and $f_+(x) = a_+x_1 + b_+x_2$ respectively. For a subset Y of $\mathbb{R}^2$, we denote by $\widehat{Y}$ its symmetric with respect to the origin, i.e., $\widehat{Y} \triangleq \{-y : \ y \in Y\}$. Finally, for each real α, we let Σ_α denote the half-line

$$\Sigma_\alpha \ \triangleq \ \{x \in \mathbb{R}^2 : \ x_1 = \alpha x_2, \ x_1 > 0\}.$$

Construction of u_k^- and u_k^+:

Let θ, μ and δ be fixed positive reals with $\theta > \max\left(\frac{b_+}{a_+}, \frac{-b_-}{a_-}\right)$, $\delta > 2\theta$ and $\mu < \min\left(\frac{b_+}{a_+}, \frac{-b_-}{a_-}\right)$. Consider Fig. 6.1, and for each $\beta > 0$, let W_β^- and W_β^+ be the **open** subsets of $\mathbb{R}^2$ bounded by the closed curves in bold. The sets W_β^- and W_β^+ are symmetric with respect to the origin. The segments $[\widehat{A}_5, A_1]$ and $[A_2, A_3]$ are respectively horizontal and vertical, while the segments $[A_5, A_4]$ and $[A_4, A_3]$ have respective slopes $\frac{dx_1}{dx_2} = -\delta$ and $\frac{dx_1}{dx_2} = \mu$. Furthermore, the segments $[\widehat{B}_5, B_1]$ and $[B_3, B_4]$ are respectively horizontal and vertical, while the segments $[B_1, B_2]$ and $[B_2, B_3]$ have respective slopes $\frac{dx_1}{dx_2} = \delta$ and $\frac{dx_1}{dx_2} = -\mu$. From the assumptions made on θ, δ and μ, it is easily checked that W_β^- and W_β^+ are well-defined for each $\beta > 0$. We now define the following open subsets of $\mathbb{R}^2 \backslash \{0\}$:

R_1 : region between the half-lines $\widehat{\Sigma}_{-\frac{b_+}{a_+}}$ and $\Sigma_{-\frac{b_-}{a_-}}$,

R_2 : region between the half-lines $\Sigma_{-\frac{b_-}{2a_-}}$ and $\Sigma_{2\theta}$,

R_3 : region between the half-lines Σ_θ and $\Sigma_{-\theta}$,

R_4 : region between the half-lines $\Sigma_{-2\theta}$ and $\Sigma_{-\frac{b_+}{2a_+}}$,

Q_β : region delimited by $\Sigma_{-\frac{b_-}{a_-}}$, $\Sigma_{-\frac{b_-}{2a_-}}$ and the segment $[A_2, A_3]$, (6.1)

T_β : region delimited by $\Sigma_{-\frac{b_+}{a_+}}$, $\Sigma_{-\frac{b_+}{2a_+}}$ and the segment $[B_3, B_4]$. (6.2)

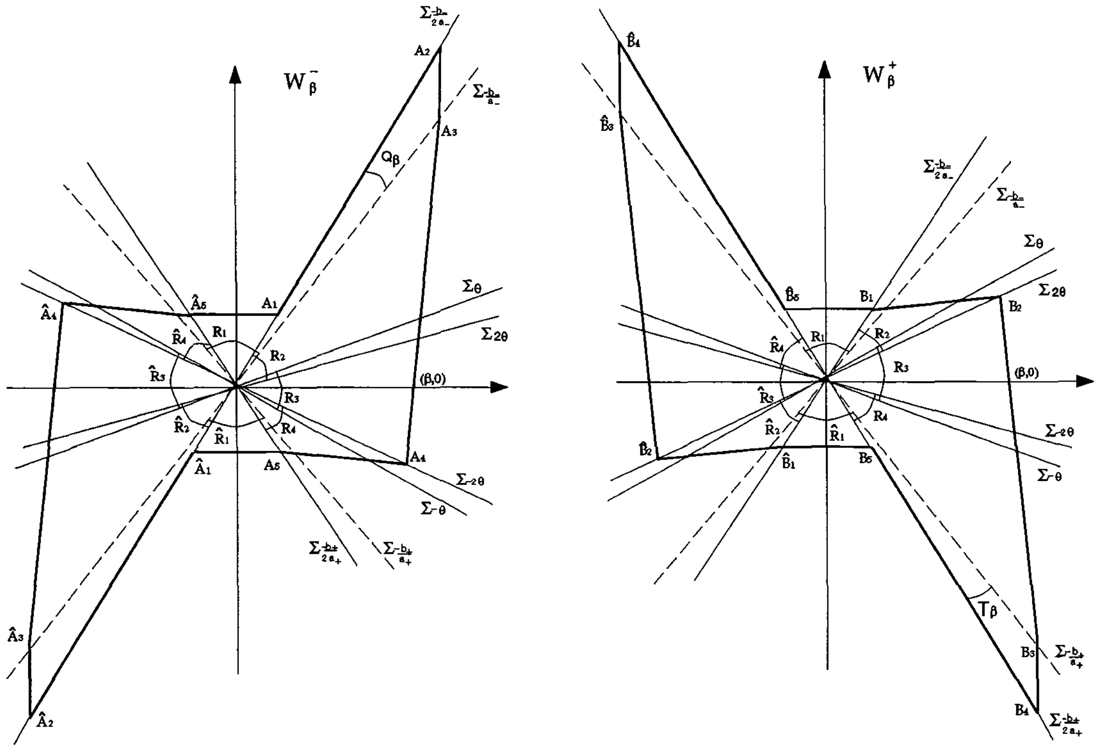

Figure 6.1: Neighborhoods W_β^- and W_β^+

Because $\{R_1,..,R_4,\widehat{R}_1,..,\widehat{R}_4\}$ [where $\widehat{R}_i$ is the symmetric set of R_i with respect to the origin for each $i=1,2,3,4$] is an open cover of $\mathbb{R}^2\setminus\{0\}$, by Theorem 1.1, there exists a C^∞ partition of unity $\{p_1,..,p_4,\widehat{p}_1,..,\widehat{p}_4\}$ subordinate to it such that the support of p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) for each $i=1,2,3,4$.

For each $k>0$, we now define the mappings $u_k^-, u_k^+ : \mathbb{R}^2 \to \mathbb{R}$, by setting

$$u_k^-(x) = \begin{cases} 0 & \text{if } x = 0 \\ -kx_2\,(\,p_1(x)+\widehat{p}_1(x)\,) & \\ +\frac{1}{\mu}(2a_-x_1+b_-x_2)\,(\,p_2(x)+p_3(x)+\widehat{p}_2(x)+\widehat{p}_3(x)\,) & \\ -\frac{1}{\delta}(\frac{a_-}{2}x_1+b_-x_2)\,(\,p_4(x)+\widehat{p}_4(x)\,) & \text{otherwise}, \end{cases}$$

and

$$u_k^+(x) = \begin{cases} 0 & \text{if } x = 0 \\ -kx_2\,(\,p_1(x)+\widehat{p}_1(x)\,)\;+\;\frac{1}{\delta}(\frac{a_+}{2}x_1+b_+x_2)\,(\,p_2(x)+\widehat{p}_2(x)\,) & \\ -\frac{1}{\mu}(2a_+x_1+b_+x_2)\,(\,p_3(x)+p_4(x)+\widehat{p}_3(x)+\widehat{p}_4(x)\,) & \text{otherwise}. \end{cases}$$

Because the mapping p_i (resp. $\widehat{p}_i$) is C^∞ on $\mathbb{R}^2\setminus\{0\}$ for each $i=1,2,3,4$, it is plain that u_k^- and u_k^+ are C^∞ on $\mathbb{R}^2\setminus\{0\}$ for each $k>0$. Furthermore, the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k^- and u_k^+ that

$$|u_k^-(x)| \leq \max\left(|kx_2|, \frac{1}{\mu}|2a_-x_1 + b_-x_2|, \frac{1}{\delta}|\frac{a_-}{2}x_1 + b_-x_2|\right), \quad x \in \mathbb{R}^2,$$

and

$$|u_k^+(x)| \leq \max\left(|kx_2|, \frac{1}{\mu}|2a_+x_1 + b_+x_2|, \frac{1}{\delta}|\frac{a_+}{2}x_1 + b_+x_2|\right), \quad x \in \mathbb{R}^2,$$

and continuity of u_k^- and u_k^+ at the origin follows for each $k > 0$.

Construction of u_k:

We first note that the families $\{W_\beta^-\}_{\beta>0}$ and $\{W_\beta^+\}_{\beta>0}$ are bases at the origin such that $W_\beta^- \subset W_{\beta'}^-$ and $W_\beta^+ \subset W_{\beta'}^+$ for all $\beta < \beta'$. This, together with the fact that for each bounded subset U of $\mathbb{R}^2$ there exists $\beta > 0$ such that $U \subset W_\beta^-$ and $U \subset W_\beta^+$, yield the existence of a sequence of positive reals $\{\beta_j\}_{j\in\mathbb{Z}}$ satisfying

$$\beta_j \to 0 \;\; as \;\; j \to +\infty \qquad \text{and} \qquad \beta_j \to +\infty \;\; as \;\; j \to -\infty, \tag{6.3}$$

with

$$W_{\beta_{2n}}^+ \supset \overline{W}_{\beta_{2n+1}}^- \qquad \text{and} \qquad W_{\beta_{2n-1}}^- \supset \overline{W}_{\beta_{2n}}^+, \quad n \in \mathbb{Z}. \tag{6.4}$$

Using the notation

$$W_{2n} \triangleq W_{\beta_{2n}}^+ \qquad \text{and} \qquad W_{2n+1} \triangleq W_{\beta_{2n+1}}^-, \quad n \in \mathbb{Z},$$

the inclusions (6.4) translate to

$$W_j \supset \overline{W}_{j+1}, \quad j \in \mathbb{Z}. \tag{6.5}$$

By Lemma B.4 combined with (6.3) and (6.5), we obtain that $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$ is an open cover of $\mathbb{R}^2\backslash\{0\}$. Thus, Theorem 1.1 yields the existence of a partition of unity $\{q_j\}_{j\in\mathbb{Z}}$ subordinate to $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$ such that the support of q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$, for each j in $\mathbb{Z}$.

Finally, for each $k > 0$, we define the feedback law $u_k : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$u_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ u_k^+(x) \sum\limits_{n\in\mathbb{Z}} q_{2n}(x) \; + \; u_k^-(x) \sum\limits_{n\in\mathbb{Z}} q_{2n+1}(x) & \text{otherwise.} \end{cases}$$

Fix $k > 0$ and let x be in $\mathbb{R}^2\backslash\{0\}$. It is easily checked that there exists a neighborhood U_x of x such that U_x intersects with at most three sets of the collection $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$. As the support of the mapping q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$ for each j in $\mathbb{Z}$, it follows that on U_x, the infinite sums in the expressions of $u_k(x)$ reduce to the sum of at most three terms. This last comment combined with the smoothness on $\mathbb{R}^2\backslash\{0\}$ of the mappings u_k^-, u_k^+ and q_j, $j\in\mathbb{Z}$ imply that u_k is C^∞ on $\mathbb{R}^2\backslash\{0\}$. Furthermore, the mappings q_j summing up to 1, we get

$$|u_k(x)| \leq \max\left(|u_k^+(x)|, |u_k^-(x)|\right), \quad x\in\mathbb{R}^2,$$

and for each $k>0$, continuity of u_k at the origin follows from that of u_k^- and u_k^+.

Invariance of the sets W_j :

We now show that there exists $k_0>0$ such that for each n in $\mathbb{Z}$, the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ (resp. $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$) are invariant with respect to the vector-field $[f_-, u_{k_0}]^t$ (resp. $[f_+, u_{k_0}]^t$).

Recall that $Q_{\beta_{2n+1}}$ and $T_{\beta_{2n}}$ denote the sectors of W_{2n+1} and W_{2n} [or equivalently $W^-_{\beta_{2n+1}}$ and $W^+_{\beta_{2n}}$] defined by (6.1) and (6.2) respectively.

We note that for each m in $\mathbb{Z}$, the boundary ∂W_m is included in $W_{m-1}\backslash\overline{W}_{m+1}$ and does not intersect with any other set $W_{j-1}\backslash\overline{W}_{j+1}$. Because the support of the mapping q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$ for each j in $\mathbb{Z}$, and the mappings q_j sum up to 1, we obtain

$$u_k(x) = u_k^-(x), \quad x\in\partial W_{2n+1}, \quad n\in\mathbb{Z}, \tag{6.6}$$

and

$$u_k(x) = u_k^+(x), \quad x\in\partial W_{2n}, \quad n\in\mathbb{Z}. \tag{6.7}$$

Next, because $u_k^-(x)$ and $u_k^+(x)$ are both equal to $-kx_2$ for x in the set $\Sigma_{-\frac{b_-}{2a_-}}\cup\Sigma_{-\frac{b_+}{2a_+}}$, we get

$$u_k(x) = -kx_2, \quad x \in \Sigma_{-\frac{b_-}{2a_-}}\cup\Sigma_{-\frac{b_+}{2a_+}}. \tag{6.8}$$

By definition of u_k we also have

$$u_k(x) > 0, \quad x \in \Sigma_{-\frac{b_-}{a_-}}\cup\widehat{\Sigma}_{-\frac{b_+}{a_+}}, \tag{6.9}$$

and

$$u_k(x) < 0, \quad x \in \widehat{\Sigma}_{-\frac{b_-}{a_-}}\cup\Sigma_{-\frac{b_+}{a_+}}. \tag{6.10}$$

Let k_- and k_+ be obtained through Lemmas 6.1 and 6.2 [with μ and δ as defined here] and set $k_0\triangleq\max(k_-,k_+)$. We now fix n in $\mathbb{Z}$ and show that the set $\overline{W}_{2n+1}$ is invariant with respect to the vector-field $[f_-, u_{k_0}]^t$. This will be proved if for each x in the boundary ∂W_{2n+1}, the vector $[f_-(x), u_{k_0}(x)]^t$ points inside the set W_{2n+1}.

Because the intersection of more than two sets in $\{R_1, \ldots, R_4, \widehat{R}_1, \ldots, \widehat{R}_4\}$ is empty, for each x in ∂W_{2n+1}, the vector $[f_-(x), u_{k_0}^-(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemma 6.1, and therefore points inside W_{2n+1}, or is a convex combination of two of them. In the latter case, $[f_-(x), u_{k_0}^-(x)]^t$ points inside W_{2n+1} either because we have a convex combination, or because we have $f_-(x) < 0$ (resp. $f_-(x) > 0$) on the segments $[A_2, A_3]$ (resp. $[\widehat{A}_2, \widehat{A}_3]$) of ∂W_{2n+1}. By (6.6), we have $u_{k_0} = u_{k_0}^-$ on ∂W_{2n+1} and it follows that the vector $[f_-(x), u_{k_0}(x)]^t$ points inside W_{2n+1} for each x in ∂W_{2n+1}.

Therefore, the set $\overline{W}_{2n+1}$ is invariant with respect to the vector-field $[f_-, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

Similarly, (6.8), (6.9) and Assertion **(i)** of Lemma 6.1 yield the invariance of the set $\overline{Q}_{\beta_{2n+1}}$ with respect to the vector-field $[f_-, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

On the other hand, (6.7), (6.8), (6.10) and Lemma 6.2 imply that the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$ are invariant with respect to the vector-field $[f_+, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

Asymptotic stability:

We now show that the feedback law u_{k_0} globally asymptotically stabilizes the system S_-. Let $\widetilde{S}_-$ denote the system obtained once u_{k_0} is fed back into S_-. Fix n in $\mathbb{Z}$ and let x_0 be in $\overline{W}_{2n+1}$. In view of (6.9), we have $u_{k_0}(x) \neq 0$ for all x in $\mathbb{R}^2 \setminus \{0\}$ with $f_-(x) = 0$, so that the origin is the unique equilibrium point of $\widetilde{S}_-$ in $\overline{W}_{2n+1}$. Thus, by the invariance with respect to $\widetilde{S}_-$ of the compact set $\overline{W}_{2n+1}$, and the Poincaré-Bendixson Theorem [38, p. 151], the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_{2n+1}$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$. If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [38, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of $\overline{Q}_{\beta_{2n+1}}$ with respect to $\widetilde{S}_-$, and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\widetilde{S}_-$ starting in $\overline{W}_{2n+1}$ remains in $\overline{W}_{2n+1}$ and converges to the origin [38, Corollary 1.1 p. 146].

Because this last result holds for each n in $\mathbb{Z}$, and the family $\{W_{2n+1}\}_{n \in \mathbb{Z}}$ is a base at the origin that covers $\mathbb{R}^2$, the feedback law u_k globally asymptotically stabilizes the system S_-.

Similarly, from the invariance of the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$ with respect to the vector-field $[f_+, u_{k_0}]^t$ for each n in $\mathbb{Z}$, and the Poincaré-Bendixson Theorem [38] combined with (6.10), it follows that u_{k_0} globally asymptotically stabilizes the system S_+, which completes the proof of the theorem. ■

6.3 Concluding remarks

The idea of constructing a feedback law u_{k_0} together with a base at the origin $\{W_{2n+1}\}_{n\in\mathbb{Z}}$ and a "sector" $T_{\beta_{2n+1}}$ in each neighborhood W_{2n+1}, such that both $\overline{W}_{2n+1}$ and $\overline{T}_{\beta_{2n+1}}$ are invariant with respect to the closed-loop S_- (with $u = u_{k_0}$), originated from Dayawansa, Martin and Knowles [21]. In [21], this technique is applied to the construction of a feedback law that asymptotically stabilizes a single system in the plane. Here, given two systems S_- and S_+, we produce a controller that achieves these "invariance requirements" simultaneously for both systems S_- and S_+, so that it simultaneously asymptotically stabilizes S_- and S_+. We design such a controller by using the interpolation method of Chapter 2.

The proof of Theorem 6.1, can actually be adapted in order to prove the simultaneous asymptotic stabilizability by means of time-invariant feedback, of any finite (resp. countably infinite) family $\{S_i,\ i = 1,\ldots,I\}$ (resp. $\{S_i,\ i = 1,2,\ldots\}$) of systems

$$S_i : \begin{cases} \dot{x}_1 &= f_i(x_1,x_2) \\ \dot{x}_2 &= u \end{cases},$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}$ is real analytic on a neighborhood of the origin, with $\frac{\partial f_i(x_1,x_2)}{\partial x_1}|_{(0,0)} \neq 0$ and $\frac{\partial f_i(x_1,x_2)}{\partial x_2}|_{(0,0)} \neq 0$ for each $i = 1,\ldots,I$ (resp. $i = 1,2,\ldots$). However, because this result is a particular case of a more general result established in the next chapter, we will not prove it.

6.4 Technical lemmas

We now present two technical lemmas that were used in the proof of Theorems 6.1.

Lemma 6.1 *Assume that a_-, a_+ and b_+ are positive, and that b_- is negative. Let μ and δ be some positive reals with $\mu < -\frac{b_-}{a_-}$ and $\frac{2b_+}{a_+} < \delta$. Then, there exists $k_- > 0$ such that the following holds:*

i) *For each $k \geq k_-$, the vector $[f_-(x), -kx_2]^t$ points into the region below $\Sigma_{-\frac{b_-}{2a_-}}$ for each x in $\Sigma_{-\frac{b_-}{2a_-}}$, and into the region above $\widehat{\Sigma}_{-\frac{b_-}{2a_-}}$ for each x in $\widehat{\Sigma}_{-\frac{b_-}{2a_-}}$.*

ii) *For each $\beta > 0$, let D_β denote the set $D_\beta \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 + \beta,\ x_1 > 0\}$. Then, for each $\beta > 0$, the vector $[f_-(x), \frac{1}{\mu}(2a_-x_1 + b_-x_2)]^t$ points towards the left of D_β for each x in D_β below $\Sigma_{\frac{-b_-}{a_-}}$, and towards the right of $\widehat{D}_\beta$ for each x in $\widehat{D}_\beta$ above $\widehat{\Sigma}_{\frac{-b_-}{a_-}}$.*

iii) *For each $\tau > 0$, let L_τ denote the set $L_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 - \tau,\ x_1 > 0\}$. Then, for each $\tau > 0$, the vector $[f_-(x), -\frac{1}{\delta}(\frac{a_-}{2}x_1 + b_-x_2)]^t$ points into the region above L_τ for each x in L_τ, and into the region below $\widehat{L}_\tau$ for each x in $\widehat{L}_\tau$.*

Proof: We only prove the first part of the assertions of the lemma as the arguments carry over to the second part of the assertions.

(i) Let x be in $\Sigma_{\frac{-b_-}{2a_-}}$. We have $f_-(x) = \frac{b_-}{2}x_2$, so that $-\frac{f_-(x)}{kx_2} = -\frac{b_-}{2k}$. Because $-\frac{f_-(x)}{kx_2}$ is less than $-\frac{b_-}{2a_-}$ for k large enough, the claim follows.

(ii) Let $\beta > 0$ and let x be in D_β below $\Sigma_{\frac{-b_-}{a_-}}$. As we have $f_-(x) > 0$ and $a_-x_1 > 0$, we immediately obtain $\frac{f_-(x)}{\frac{2a_-x_1 + b_-x_2}{\mu}} < \mu$, for all $\beta > 0$. Hence the claim.

(iii) Let $\tau > 0$ and let x be in D_τ. Because we have $\frac{a_-}{2\delta}x_1 > 0$, we easily get $-\frac{-\frac{1}{\delta}(\frac{a_-}{2}x_1 + b_-x_2)}{f_-(x)} < \frac{1}{\delta}$, for all $\tau > 0$. Hence the result. ■

The proof of the following lemma is similar to that of Lemma 6.1 and is therefore omitted.

Lemma 6.2 *Assume that a_-, a_+ and b_+ are positive, and that b_- is negative. Let μ and δ be some positive reals with $\mu < \frac{b_+}{a_+}$ and $-\frac{2b_-}{a_-} < \delta$. Then, there exists $k_+ > 0$ such that the following holds.*

i) *For each $k \geq k_+$, the vector $[f_+(x), -kx_2]^t$ points into the region above $\Sigma_{-\frac{b_+}{2a_+}}$ for each x in $\Sigma_{-\frac{b_+}{2a_+}}$, and into the region below $\widehat{\Sigma}_{-\frac{b_+}{2a_+}}$ for each x in $\widehat{\Sigma}_{-\frac{b_+}{2a_+}}$.*

ii) *For each* $\beta > 0$, *let* D_β *denote the set* $D_\beta \triangleq \{x \in \mathbb{R}^2 : x_1 = -\mu x_2 + \beta,\ x_1 > 0\}$. *Then, for each* $\beta > 0$, *the vector* $[f_+(x), -\frac{1}{\mu}(2a_+x_1 + b_+x_2)]^t$ *points towards the left of* D_β *for each* x *in* D_β *above* $\Sigma_{\frac{-b_+}{a_+}}$, *and towards the right of* $\widehat{D}_\beta$ *for each* x *in* $\widehat{D}_\beta$ *below* $\widehat{\Sigma}_{\frac{-b_+}{a_+}}$.

iii) *For each* $\tau > 0$, *let* L_τ *denote the set* $L_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \delta x_2 - \tau,\ x_1 > 0\}$. *Then, for each* $\tau > 0$, *the vector* $[f_+(x), \frac{1}{\delta}(\frac{a_+}{2}x_1 + b_+x_2)]^t$ *points into the region below* L_τ *for each* x *in* L_τ, *and into the region above* $\widehat{L}_\tau$ *for each* x *in* $\widehat{L}_\tau$.

Chapter 7

Robust Asymptotic Stabilization in the Plane

Throughout this chapter we consider a class of parameterized families of nonlinear systems in the plane and we discuss their robust asymptotic stabilization around a parameter value at which the corresponding families of linearized systems are not controllable. Because these families do not admit C^1 robust asymptotic stabilizers, we investigate the existence of **merely continuous** robust asymptotic stabilizers. In particular, we introduce a new approach to the robust asymptotic stabilization, where a robust asymptotic stabilizer of a parameterized family of systems is considered as a feedback law that simultaneously robustly asymptotically stabilizes two distinct sub-families of the original family. More precisely, we construct in Section 7.3 a robust asymptotic stabilizer for a parameterized family of systems as follows: We design the robust asymptotic stabilizers of two of its sub-families by extending the ideas introduced in the previous chapter. We then piece together these two robust asymptotic stabilizers using the interpolation method of Chapter 2.

The chapter is organized as follows: We first present the problem under consideration, and then discuss the robust asymptotic stabilization case by case in Section 7.2, 7.3, and 7.4. In Section 7.5, we show that in some cases, the families under consideration, are robustly asymptotically stabilizable by means of continuous and almost C^∞ feedback, while they are not by means of Lipschitz continuous feedback. Finally, we give in Section 7.6, simple expressions for the robust asymptotic stabilizers constructed in Section 7.2.

7.1 Problem definition

In this chapter, we consider a parameterized family of systems in the plane

$$S(\gamma): \begin{cases} \dot{x}_1 &= f_\gamma(x_1, x_2) \\ \dot{x}_2 &= u \end{cases},$$

where γ is a real parameter, $f_\gamma : \mathbb{R}^2 \to \mathbb{R}$ is real analytic on a neighborhood of the origin and u is a scalar control. We assume that γ lies in some interval $[-\zeta_0, \zeta_1]$ where ζ_0 and ζ_1 are positive and that the mappings $a, b : [-\zeta_0, \zeta_1] \to \mathbb{R}$ defined by

$$a(\gamma) = \frac{\partial f_\gamma(x_1, x_2)}{\partial x_1}|_{(0,0)} \qquad \text{and} \qquad b(\gamma) = \frac{\partial f_\gamma(x_1, x_2)}{\partial x_2}|_{(0,0)}, \tag{7.1}$$

are C^∞ on $[-\zeta_0, \zeta_1]$. We let Γ denote the set $[-\zeta_0, 0) \cup (0, \zeta_1]$ and we assume that $b(0) = 0$, $a(\gamma) \neq 0$ and $b(\gamma) \neq 0$ for all γ in Γ.

We investigate here the existence of a time-invariant feedback law u that is continuous at the origin and that robustly asymptotically stabilizes the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$ according to Definition 1.5. In fact we study the robust asymptotic stabilization of the family around a parameter value at which the corresponding family of linearized systems is not controllable.

Our primary motivation for studying this robust stabilization problem is to try to understand the robust stabilizability of a family of systems around a parameter value at which the family has a singularity. In this particular case the singularity is the loss of controllability of the family of linearized systems corresponding to $S(\gamma)$. More precisely, we wish to answer the following question: Under the assumption that the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by means of continuous feedback, does it exist a continuous feedback law which robustly asymptotically stabilizes the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$ and which stabilizes (not necessarily asymptotically) the system $S(0)$?

It is natural to believe that the answer to this question is positive and that the robust asymptotic stabilizability of the family of systems $S(\gamma)$ is in a certain sense a property which is continuous with respect to the parameter γ.

To answer this question, it would be necessary to also study the robust asymptotic stabilization of the family of systems $\{S(\gamma),\ \gamma \in [-\zeta_0, \zeta_1]\}$. Although there exist some cases where we can solve this last robust stabilization problem, we have not been able to complete this investigation, and we leave this issue for further research.

To our knowledge the only published work addressing the robust stabilization of a family of systems around a parameter value at which the family "loses" its controllability is that of Colonius and Kliemann [16]. They consider parameterized families of scalar control systems whose corresponding families of uncontrolled systems undergo bifurcation at some parameter value. Using the notion of control sets [15], they derive necessary and sufficient conditions for the robust asymptotic stabilizability of such families by means of piecewise constant controls and around the bifurcation parameter.

One of the originalities of our work lies in the fact that we show that dynamic feedback laws may achieve robust stabilization for families of systems that do not admit a continuous static robust stabilizer. Indeed, consider the family $\{\Sigma(\gamma),\, \gamma \in \Gamma\}$ of scalar systems

$$\Sigma(\gamma):\ \dot{x} = f_\gamma(x,u),$$

where the state x and the input u lie in $\mathbb{R}$. For each γ in Γ, the system $S(\gamma)$ is the system resulting from the application of dynamic feedback to $\Sigma(\gamma)$. If the mapping f_γ is such that $a(\gamma) > 0$ on Γ, $b(\gamma) < 0$ whenever γ is negative and $b(\gamma) > 0$ whenever γ is positive, then it is easily seen that any two systems $\Sigma(\gamma_-)$ and $\Sigma(\gamma_+)$ with γ_- and γ_+ respectively negative and positive, are not simultaneously asymptotically stabilizable by means of continuous feedback. Thus, there exists no continuous static feedback law that robustly asymptotically stabilizes the family $\{\Sigma(\gamma),\, \gamma \in \Gamma\}$. However, as we shall see, there exists a continuous static feedback law that robustly asymptotically stabilizes $\{S(\gamma),\, \gamma \in \Gamma\}$, i.e., there exists a continuous dynamic feedback law that robustly asymptotically stabilizes the family $\{\Sigma(\gamma),\, \gamma \in \Gamma\}$. This robust asymptotic stabilizer is constructed through a new approach where a robust asymptotic stabilizer is considered as a feedback law that simultaneously robustly asymptotically stabilizes two sub-families of the original family. More precisely, we first construct the robust asymptotic stabilizers of two particular sub-families using the ideas introduced in the previous chapter. We then piece together the two robust asymptotic stabilizers using the interpolation method of Chapter 2, in order to obtain a robust asymptotic stabilizer for the original family.

We emphasize that the robust asymptotic stabilization [in this sense] of parameterized families of **nonlinear** systems by means of merely continuous feedback has not been addressed yet in the literature.

We complete this section with a few comments on the system $S(\gamma)$: As usual for a given x in $\mathbb{R}^2$, we let x_1 and x_2 denote its coordinates. We note that for each γ in Γ, Lemma A.6 combined with the enforced assumptions imply that there exist some neighborhoods of the origin U_γ and I_γ in $\mathbb{R}^2$ and $\mathbb{R}$ respectively, and some real analytic mappings $h_\gamma : U_\gamma \to \mathbb{R}$ and $\phi_\gamma : I_\gamma \to \mathbb{R}$ such that

$$f_\gamma(x_1, x_2) \;=\; h_\gamma(x_1, x_2)\,(x_1 - \phi_\gamma(x_2)), \quad (x_1, x_2) \in U_\gamma, \tag{7.2}$$

with $h_\gamma(0) = a(\gamma)$, $\phi_\gamma(0) = 0$ and $\phi'_\gamma(0) = -\dfrac{b(\gamma)}{a(\gamma)}$. Therefore, for each γ in Γ we have

$$h_\gamma(x) = a(\gamma) + \widehat{h}_\gamma(x) \qquad \text{with} \qquad \widehat{h}_\gamma(x) \to 0 \;\; \textit{as} \;\; x \to 0, \tag{7.3}$$

and

$$\phi_\gamma(x) \;=\; -\frac{b(\gamma)}{a(\gamma)}x_2 + \widehat{\phi}_\gamma(x_2) \qquad \text{with} \qquad \frac{\widehat{\phi}_\gamma(x_2)}{x_2} \to 0 \;\; \textit{as} \;\; x_2 \to 0. \tag{7.4}$$

Next, for each γ in Γ, because $a(\cdot)$ and $b(\cdot)$ do not vanish on Γ, we obtain from (7.2), (7.3) and (7.4), the following lemma.

Lemma 7.1 *For each γ in Γ, there exists a neighborhood of the origin $U_\gamma = I_\gamma \times J_\gamma$ of the origin, such that the following equality holds*

$$\{x \in U_\gamma : f_\gamma(x) = 0\} \;=\; \{(\phi_\gamma(x_2), x_2) : \; x_2 \in J_\gamma\},$$

where $\phi_\gamma(\cdot)$ is strictly monotone on J_γ. Moreover, $h_\gamma(\cdot)$ and $\phi_\gamma(\cdot)$ are analytic on U_γ and J_γ respectively, and satisfy (7.2), (7.3) and (7.4).

Finally, a few words about the notation and terminology used in this chapter.

For any subset I of $\mathbb{R}$, we denote respectively by I^- and I^+, the sets $I^- \triangleq \{\rho \in I : \; \rho < 0\}$ and $I^+ \triangleq \{\rho \in I : \; \rho > 0\}$. For a subset Y of $\mathbb{R}^2$, we let $\widehat{Y}$ denote its symmetric with respect to the origin and Y^s its symmetric with respect to the x_1-axis, i.e,

$$\widehat{Y} \;\triangleq\; \{-y : \; y \in Y\} \qquad \text{and} \qquad Y^s \;\triangleq\; \{(y_1, -y_2) \in \mathbb{R}^2 : \; (y_1, y_2) \in Y\}$$

Finally, for each positive reals α and β, and for each γ in Γ, we define

$$\begin{aligned}
\Omega_\alpha \;&\triangleq\; \{x \in \mathbb{R}^2 : \; x_1 = (x_2)^{1+\alpha}, \; x_2 > 0\} \\
\Delta_\alpha \;&\triangleq\; \{x \in \mathbb{R}^2 : \; x_1 = \frac{(x_2)^{1+\alpha}}{2}, \; x_2 > 0\} \\
\Psi_\beta \;&\triangleq\; \{x \in \mathbb{R}^2 : \; x_2 = x_1 \ln(\frac{x_1}{\beta}), \; x_1 > \beta\} \\
\Pi_\gamma \;&\triangleq\; \{x \in \mathbb{R}^2 : \; f_\gamma(x) = 0\} \\
\Pi_\gamma^+ \;&\triangleq\; \{x \in \mathbb{R}^2 : \; f_\gamma(x) = 0, \; x_2 > 0\} \\
\Pi_\gamma^- \;&\triangleq\; \{x \in \mathbb{R}^2 : \; f_\gamma(x) = 0, \; x_2 < 0\}
\end{aligned}$$

In order to discuss the robust asymptotic stabilization of the family $\{S(\gamma),\, \gamma \in \Gamma\}$, we introduce the family $\{S_L(\gamma),\, \gamma \in \Gamma\}$ of linearized systems

$$S_L(\gamma): \quad \left\{ \begin{array}{rcl} \dot{x}_1 & = & a(\gamma)x_1 + b(\gamma)x_2 \\ \dot{x}_2 & = & u \end{array} \right. .$$

We distinguish several cases based on the sign of $a(\cdot)$ and $b(\cdot)$. Recall that $a(\cdot)$ and $b(\cdot)$ take nonzero values on Γ so that, by continuity, both have a constant sign on Γ^- and Γ^+.

In each section, without further reference, we omit the cases $a(\cdot) < 0$ on Γ, as in that case any feedback law $u(x) = -kx_2$, where k is a positive real, robustly asymptotically stabilizes the family $\{S_L(\gamma),\, \gamma \in \Gamma\}$ and the family $\{S(\gamma),\, \gamma \in \Gamma\}$.

7.2 Robust stabilization when the sign of $b(\cdot)$ is constant on Γ

In this section, we assume that $b(\cdot)$ is either negative on Γ or positive on Γ. Moreover, we assume that $a(\cdot)$ is either positive on the entire set Γ, or negative on Γ^- and positive on Γ^+. We restrict our discussion to these two cases as the remaining case $a(\cdot)$ positive on Γ^- and negative on Γ^+ is obtained from the latter by replacing Γ^- by Γ^+ and vice versa.

Under these assumptions, because the mappings $a(\cdot)$ and $b(\cdot)$ are C^∞ on $\Gamma \cup \{0\}$ and do not vanish on Γ, it is easily checked by using elementary linear algebra, that the family $\{S_L(\gamma),\, \gamma \in \Gamma\}$ and the family $\{S(\gamma),\, \gamma \in \Gamma\}$ are robustly asymptotically stabilizable by C^∞ (linear) feedback if and only if $\frac{b(\gamma)}{a(\gamma)}$ does not converge to 0 as γ goes to 0. If $\frac{b(\gamma)}{a(\gamma)}$ converges to 0 as γ goes to 0, a linear feedback law with an "infinite gain" would be necessary in order to robustly asymptotically stabilize the family $\{S(\gamma),\, \gamma \in \Gamma\}$. As we shall see below, it turns out that the family $\{S(\gamma),\, \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by means of continuous, almost C^∞ feedback. Indeed, we will establish the following theorem:

Theorem 7.1 *Assume that either $a(\cdot)$ is positive on the entire set Γ, or $a(\cdot)$ is negative on Γ^- and positive on Γ^+. Furthermore, assume that $b(\cdot)$ is either negative on Γ, or positive on Γ. Assume that $\frac{b(\gamma)}{a(\gamma)}$ converges to 0 as γ tends to 0. Then, there exists a feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ which is continuous and almost*

C^∞ on a neighborhood of the origin, and which robustly asymptotically stabilizes the family of systems $\{S(\gamma),\, \gamma \in \Gamma\}$.

In order to avoid a lengthy proof we introduce an intermediate lemma, which is the key in proving Theorem 7.1.

7.2.1 Stabilizability when $a(\cdot) > 0$ on Γ^+, $a(\cdot) < 0$ on Γ^- and $b(\cdot) < 0$ on Γ

In the following lemma we establish the robust asymptotic stabilizability of the family $\{S(\gamma),\, \gamma \in \Gamma\}$ in the case $a(\cdot) > 0$ on Γ^+, $a(\cdot) < 0$ on Γ^- and $b(\cdot) < 0$ on Γ.

The main lines of the proof are as follows: We first construct a feedback law u_{k_0} based on some partition of unity. Next, we introduce a base at the origin $\{W_\beta\}_{\beta>0}$ which is independent of the parameter γ. We show that for each parameter value γ in the set Γ, there exists a positive real β_γ such that for each β in $(0, \beta_\gamma]$ the set $\overline{W_\beta}$ is invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$. This enables us to conclude stability of the corresponding closed-loop system. Furthermore, by proving that the only positive limit set in $\overline{W_\beta}$ is the origin, we deduce that u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$.

Lemma 7.2 *Theorem 7.1 holds if in addition to the assumptions of the theorem we have $a(\cdot) > 0$ on Γ^+, $a(\cdot) < 0$ on Γ^- and $b(\cdot) < 0$ on Γ.*

Proof:

Construction of the stabilizing feedback law :

Recall that $a(\cdot)$ and $b(\cdot)$ are C^∞ on Γ and that $a(\cdot)$ does not vanish on Γ. Thus, because $\frac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$, there exists $\theta > 0$ such that $|\frac{b(\gamma)}{a(\gamma)}| < \theta$ for all γ in Γ. Therefore, for each γ in Γ^+, the half-line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)}x_2,\, x_2 > 0\}$ (resp. $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)}x_2,\, x_2 < 0\}$) is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 > 0\}$ (resp. below the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\, x_2 < 0\}$).

Let α be a constant in $(0, 1)$, and consider Fig. 7.1: For each $\beta > 0$, let W_β denote the neighborhood of the origin bounded by the closed curve in bold.

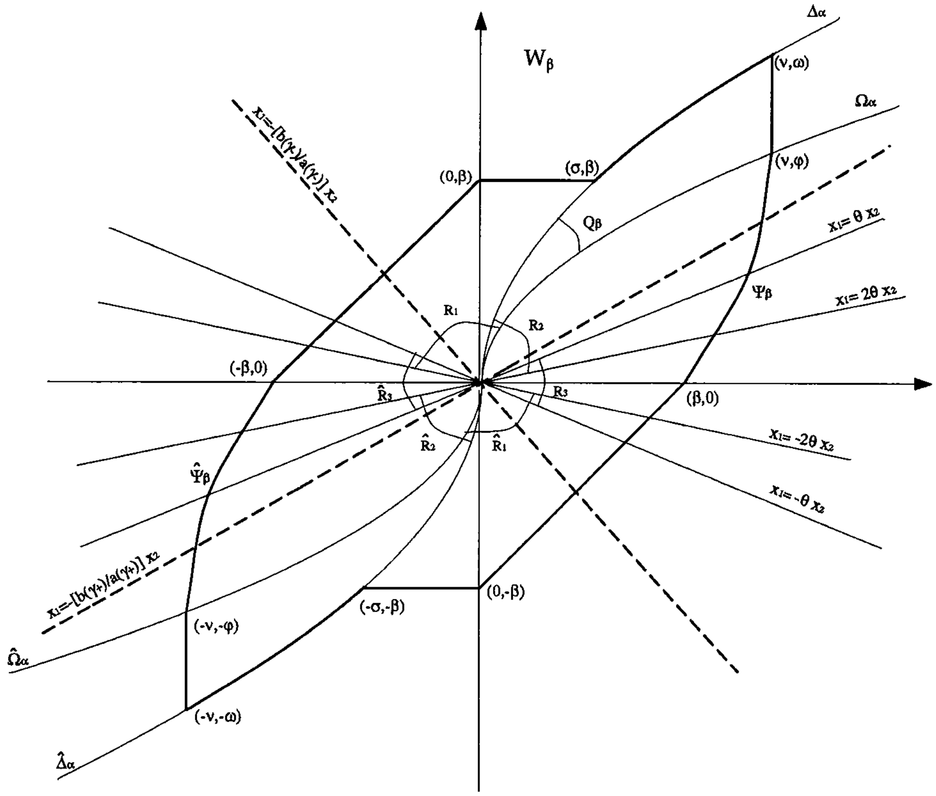

Figure 7.1: Neighborhood W_β

Because the curves Ψ_β and Ω_α intersect for each $\beta > 0$ (Lemma C.1), the neighborhood W_β is well-defined for each $\beta > 0$.

Besides, as Ω_α is tangent to the x_2-axis at the origin, Ω_α is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \ x_2 > 0\}$ for x_2 small enough. Moreover, by Lemma C.1, the unique point $[h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$ at which the sets $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \ x_2 > 0\}$ and Ψ_β intersect is such that $h(\beta) \to 0 \ as \ \beta \to 0$. Thus, there exists $\bar{\beta} > 0$ such that for each β in $(0, \bar{\beta}]$, the point $[h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$ is below Ω_α. Furthermore, as $h(\beta) \to 0 \ as \ \beta \to 0$, it is easily seen from the definition of W_β, that $\{W_\beta\}_{\beta \in (0,\bar{\beta}]}$ is a base at the origin.

In view of the comments made above, we can now define the following open subsets of $W_{\bar{\beta}} \backslash \{0\}$:

$R_1 \triangleq$ region in $W_{\bar{\beta}}$ between the curves $\{x \in \mathbb{R}^2 : \ x_1 = -2\theta x_2, \ x_2 > 0\}$ and Ω_α,

$R_2 \triangleq$ region in $W_{\bar{\beta}}$ between the curves Δ_α

$$
\begin{aligned}
&\quad\text{and } \{x \in \mathbb{R}^2 : \ x_1 = 2\theta x_2, \ x_2 > 0\}, \\
R_3 &\triangleq \text{region in } W_{\bar\beta} \text{ between the half-lines } \{x \in \mathbb{R}^2 : \ x_1 = \theta x_2, \ x_2 > 0\} \\
&\quad\text{and } \{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 < 0\}, \\
Q_\beta &\triangleq \text{region delimited by } \Omega_\alpha, \ \Delta_\alpha \text{ and the segment } [(\nu,\varphi),(\nu,\omega)].
\end{aligned}
$$

We note that $\{R_1, R_2, R_3, \widehat{R}_1, \widehat{R}_2, \widehat{R}_3\}$ [where $\widehat{R}_i$ is the symmetric of R_i with respect to the origin for each $i = 1, 2, 3$] is an open cover of $W_{\bar\beta}\backslash\{0\}$. Thus, by Theorem 1.1, there exists a partition of unity $\{p_1, p_2, p_3, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3\}$ subordinate to this cover and such that for the support of p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) for each $i = 1, 2, 3$.

For each $k > 0$, we now define the feedback law $u_k : W_{\bar\beta} \to \mathbb{R}$ by setting

$$
u_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ k\ [-(x_2)^{1-\alpha}p_1(x) + (x_1+x_2)p_2(x) + x_1p_3(x) & \\ \quad +(-x_2)^{1-\alpha}\widehat{p}_1(x) + (x_1+x_2)\widehat{p}_2(x) + x_1\widehat{p}_3(x)\,] & \text{otherwise}. \end{cases} \tag{7.5}
$$

We note that the regions R_1 and $\widehat{R}_1$ do not contain any point of the form $(x_1, 0)$, and that the support of p_1 and $\widehat{p}_1$ are included in R_1 and $\widehat{R}_1$, respectively. Therefore, it follows from the smoothness of the mapping p_i and $\widehat{p}_i$ on $W_{\bar\beta}\backslash\{0\}$ for each $i = 1, 2, 3$, that u_k is C^∞ on $W_{\bar\beta}\backslash\{0\}$, for each $k > 0$. Furthermore, the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k that

$$
|u_k(x)| \ \le \ k \ \max\left(|x_2|^{1-\alpha}, |x_1+x_2|, |x_1|\right), \quad x \in W_{\bar\beta},
$$

and continuity of u_k at the origin follows for each $k > 0$.

Invariance of the sets $\overline{W}_\beta$ and $\overline{Q}_\beta$:

The following claim is the key argument to establish robust asymptotic stability.

Claim 1: *There exists $k_0 > 0$, and for each γ in Γ there exists β_γ in $(0, \bar\beta]$ such that the sets $\overline{W}_\beta$ and $\overline{Q}_\beta$ are invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$ for each β in $(0, \beta_\gamma]$.*

We note that the invariance of $\overline{W}_\beta$ will be proved if for each x in the boundary ∂W_β, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside the set W_β.

By applying Lemmas C.3 and C.4 (with θ, $\bar{\beta}$, α as given here, and $I = \Gamma$, $\mu = 1$, $\eta = 1 - \alpha$, $\eta' = 1 - \alpha$), we obtain two positive reals k_1 and k_2. We set $k_0 \triangleq \max(k_1, k_2)$. Thus, for each γ in Γ there exists a neighborhood of the origin U_γ such that the assertions of both lemmas hold for each $k \geq k_0$. For each γ in Γ, let $\bar{\beta}_\gamma$ in $(0, \bar{\beta}]$ be such that for each β in $(0, \bar{\beta}_\gamma]$, we have $W_\beta \subset U_\gamma$.

For each γ in Γ^-, Lemma 7.1 combined with Assertion (7.4) yield the existence of $\beta_\gamma \leq \bar{\beta}_\gamma$ such that Π_γ^+ (resp. Π_γ^-) is in the sector $\{x \in W_{\beta_\gamma} : x_1 < 0, x_2 > 0\}$ (resp. $\{x \in W_{\beta_\gamma} : x_1 > 0, x_2 < 0\}$) with $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) for each x above (resp. below) $\Pi_\gamma^+ \cup \Pi_\gamma^-$.

Similarly, for each γ be in Γ^+, there exists $\beta_\gamma \leq \bar{\beta}_\gamma$ such that Π_γ^+ (resp. Π_γ^-) is in the sector $\{x \in W_{\beta_\gamma} : x_1 > 0, x_2 > 0\}$ (resp. $\{x \in W_{\beta_\gamma} : x_1 < 0, x_2 < 0\}$) with $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) for each x above (resp. below) $\Pi_\gamma^+ \cup \Pi_\gamma^-$. Further, because the curves Ω_α and $\widehat{\Omega}_\alpha$ are tangent to the x_2-axis it follows from the fact that $\Pi_\gamma^+ \cup \Pi_\gamma^-$ is tangent to the line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)} x_2\}$ that β_γ can be chosen small enough so that for each β in $(0, \beta_\gamma]$ the curve Ω_α (resp. $\widehat{\Omega}_\alpha$) is above Π_γ^+ (resp. below Π_γ^-).

Next, we fix γ in Γ and β in $(0, \beta_\gamma]$. From the definition of β_γ, it is easily checked that for each x in the segment $[(\nu, \varphi), (\nu, \omega)]$ (resp. $[(-\nu, -\varphi), (-\nu, -\omega)]$) of ∂W_β, we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$), so that $[f_\gamma(x), u_{k_0}(x)]^t$ points into W_β.

Further, recall that for each $i = 1, 2, 3$, the support of the mappings p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) and note that the intersection of more than two sets of the family $\{R_1, R_2, R_3, \widehat{R}_1, \widehat{R}_2, \widehat{R}_3\}$ is empty. Thus, for each x in ∂W_β, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemmas C.3 and C.4, and therefore points inside W_β, or is a convex combination of two of them. In the latter case, we obtain that $[f_\gamma(x), u_{k_0}(x)]^t$ points inside W_β either from the fact that we have a convex combination or from the fact that we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) on the segments $[(\nu, \varphi), (\nu, \omega)]$ (resp. $[(-\nu, -\varphi), (-\nu, -\omega)]$) of ∂W_β.

Finally, for each x in $\Omega_\alpha \cap W_\beta$, because $u_{k_0}(x)$ is positive and $f_\gamma(x)$ is negative [follows from the definition of β_γ], the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points into Q_β. This, combined with the assertion of Lemma C.3 **(ii)** and the fact that $\beta \leq \beta_\gamma$, implies that $\overline{Q}_\beta$ is an invariant set with respect to the vector-field $[f_\gamma, u_{k_0}]^t$. The proof of Claim 1 is completed upon noting that the previous results hold for each γ in Γ and each β in $(0, \beta_\gamma]$.

Robust asymptotic stability :

We now prove that the feedback law u_{k_0}, where k_0 is as given in Claim 1, robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$. Fix γ in Γ and let $\widetilde{S}(\gamma)$ denote the closed-loop system obtained once u_{k_0} is fed back into $S(\gamma)$. Let β_γ be as defined in Claim 1, let β be in $(0, \beta_\gamma]$, and let x_0 be in $\overline{W}_\beta$. In view of the definition of β_γ, we have $u_{k_0}(x) \neq 0$ for all x in $\overline{W}_\beta \backslash \{0\}$ such that $f_\gamma(x) = 0$. Thus, the origin is the unique equilibrium point of $\widetilde{S}(\gamma)$ in $\overline{W}_\beta$ and from the invariance with respect to $\widetilde{S}(\gamma)$ of the compact set $\overline{W}_\beta$ (Claim 1) combined with the Poincaré-Bendixson Theorem [38], it follows that the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_\beta$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$.

If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [38, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of the set $\overline{Q}_\beta$ and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\widetilde{S}(\gamma)$ starting in $\overline{W}_\beta$ converges to the origin [38, Corollary 1.1 p. 146].

As $\{W_\beta\}_{0<\beta\leq\beta_\gamma}$ is a base at the origin, we easily obtain that the feedback law u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$ for each γ in Γ. In short, u_{k_0} robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$, which completes the proof. ∎

Using this lemma we now prove Theorem 7.1.

7.2.2 Proof of Theorem 7.1

We are now able to prove Theorem 7.1.

Proof of Theorem 7.1 :

We distinguish three cases.

a) $a(\cdot) > 0$ **on** Γ^+, $a(\cdot) < 0$ **on** Γ^- **and** $b(\cdot) < 0$ **on** Γ **:**

The results follows from Lemma 7.2.

b) $a(\cdot) > 0$ **on** Γ^+, $a(\cdot) < 0$ **on** Γ^- **and** $b(\cdot) > 0$ **on** Γ **:**

We consider the family $\{\bar{S}(\gamma),\ \gamma \in \Gamma\}$ of systems

$$\bar{S}(\gamma): \quad \begin{cases} \dot{x}_1 &= f_\gamma(x_1, -x_2) \\ \dot{x}_2 &= u \end{cases}.$$

Thus, we have $\dfrac{\partial f_\gamma(x_1,-x_2)}{\partial x_1}|_{(0,0)} = a(\gamma)$ and $\dfrac{\partial f_\gamma(x_1,-x_2)}{\partial x_2}|_{(0,0)} = -b(\gamma)$. Because $-b(\cdot)$ is negative on Γ, by **(a)**, there exists a feedback laws u_{k_0} that robustly asymptotically stabilize $\{\bar{S}(\gamma),\ \gamma \in \Gamma\}$. In other words, the system

$$\begin{cases} \dot{x}_1 &= f_\gamma(x_1, -x_2) \\ \dot{x}_2 &= u_{k_0}(x_1, x_2) \end{cases} \tag{7.6}$$

is asymptotically stable for each γ in Γ. By the change of variable $(x_1, x_2) \mapsto (x_1, -x_2)$, the system (7.6) is transformed into the asymptotically stable system

$$\begin{cases} \dot{x}_1 &= f_\gamma(x_1, x_2) \\ \dot{x}_2 &= -u_{k_0}(x_1, -x_2) \end{cases}$$

and we conclude that the feedback law v_{k_0} given by $v_{k_0}(x_1, x_2) = -u_{k_0}(x_1, -x_2)$, robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

c) $a(\cdot) > 0$ on Γ, and $b(\cdot)$ is either positive or negative on Γ :

In that case, the result follows easily from the arguments given in **(a)** and **(b)** by replacing Γ^+ by Γ and Γ^- by $\emptyset$.

■

It is easily seen from the proof of Theorem 7.1 that each one of the feedback law of the collection $\{u_k,\ k \in [k_0, \infty)\}$ robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

7.3 Robust stabilization when $a(\cdot)$ is positive and the sign of $b(\cdot)$ changes

In this section, we only consider the case $b(\cdot)$ negative on Γ^- and positive on Γ^+. The symmetric case $b(\cdot)$ positive on Γ^- and negative on Γ^+ is obtained from the former by replacing Γ^+ by Γ^- and vice versa. It is easily checked that given any two systems $S(\gamma_-)$ and $S(\gamma_+)$, with $\gamma_- < 0$ and $\gamma_+ > 0$, there exist neither a C^1 feedback law that simultaneously asymptotically stabilizes these two systems, nor a continuous static feedback law that simultaneously stabilizes $\dot{x} = f_{\gamma_-}(x,u)$ and $\dot{x} = f_{\gamma_+}(x,u)$. Thus, the family $\{\dot{x} = f_\gamma(x,u),\ \gamma \in \Gamma\}$ (resp. $\{S(\gamma),\ \gamma \in \Gamma\}$) is not robustly stabilizable by means of continuous (resp. C^1) feedback. However, as we shall see below, the family $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by means of **continuous** feedback.

This result is proved by designing a feedback law that simultaneously robustly asymptotically stabilizes the families $\{S(\gamma),\ \gamma \in \Gamma^-\}$ and $\{S(\gamma),\ \gamma \in \Gamma^+\}$. More precisely, we define two mappings $u^+_{k_0}$ and $u^-_{k_0}$ that robustly asymptotically stabilize the family $\{S(\gamma),\ \gamma \in \Gamma^+\}$ and $\{S(\gamma),\ \gamma \in \Gamma^-\}$ respectively, by using a first partition of unity similar to that introduced in the proof of Theorem 7.1. Then, in order to obtain a feedback law that robustly asymptotically stabilizes the entire family $\{S(\gamma),\ \gamma \in \Gamma\}$, we "piece" together $u^+_{k_0}$ and $u^-_{k_0}$ by using the interpolation method of Chapter 2. Robust asymptotic stability is shown through an argument similar to that used in the proof of Theorem 7.1.

Theorem 7.2 *Assume that $a(\cdot)$ is positive on Γ, and that $b(\cdot)$ is respectively negative on Γ^- and positive on Γ^+. Then, there exists a feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ which is continuous and almost C^∞ on a neighborhood of the origin, and which robustly asymptotically stabilizes the family of systems $\{S(\gamma),\ \gamma \in \Gamma\}$.*

Proof: For the sake of clarity we divide the proof of the theorem into two cases.

a) **The case** $\dfrac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$:

Construction of u^-_k and u^+_k :

In that case, the assumptions on $a(\cdot)$ and $b(\cdot)$ yield the existence of $\theta > 0$ such that $|\dfrac{b(\gamma)}{a(\gamma)}| < \theta$ for all γ in Γ. Let α be a constant in $(0,1)$ and consider Fig. 7.2 and Fig. 7.3: For each $\beta > 0$, we let W^-_β and W^+_β be the open subsets of $\mathbb{R}^2$ bounded by the closed curves in bold, in Fig. 7.2 and Fig. 7.3 respectively. The neighborhood W^+_β is obtained by rotating W^-_β around the x_1-axis by 180 degrees. In Fig. 7.2, the segments $[\widehat{A}_6, A_1]$ and $[A_2, A_3]$ are respectively horizontal and vertical, while the segments $[A_6, A_5]$ and $[A_4, A_5]$ have respective slopes $\dfrac{dx_1}{dx_2} = -\delta$ and $\dfrac{dx_1}{dx_2} = \mu$ where μ and δ are fixed positive reals such that $\delta > 2\theta$. Combining this last inequality with the fact that the curves Ψ_β and Ω_α intersect for each $\beta > 0$ (Lemma C.1), we obtain that the neighborhoods W^-_β and W^+_β are well-defined for each $\beta > 0$.

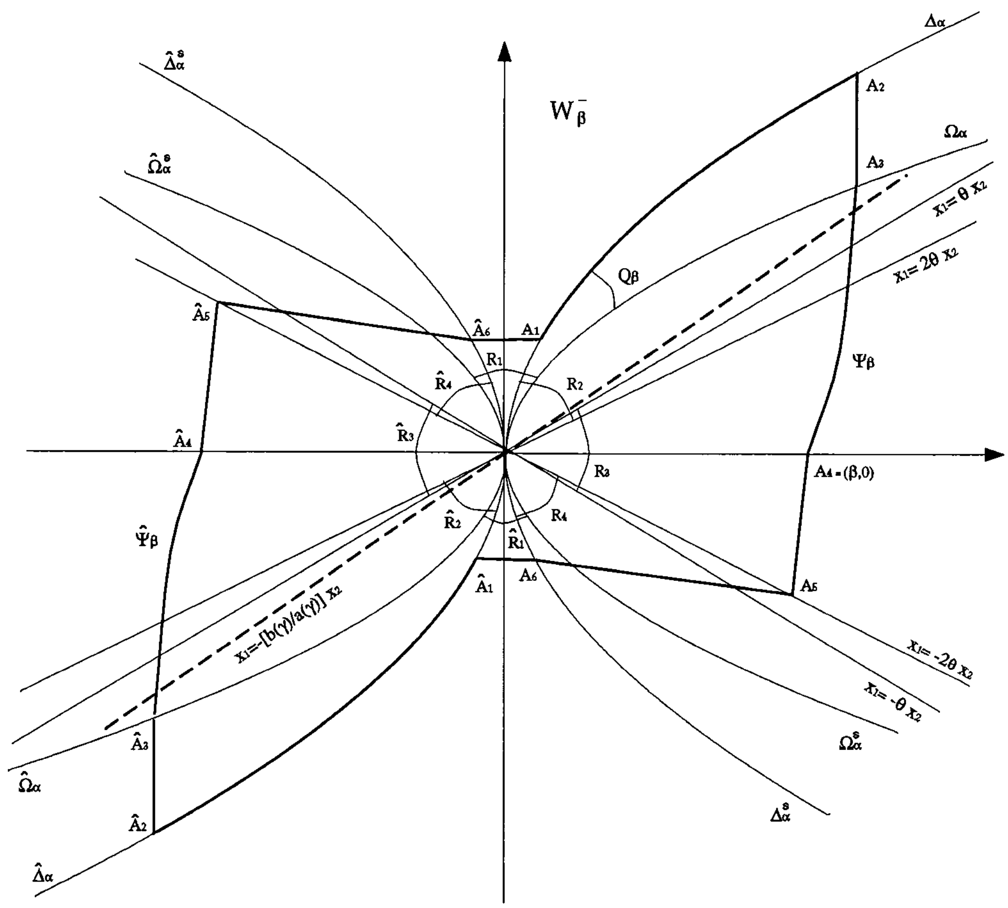

Figure 7.2: Neighborhood W_β^- in the case $\dfrac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$

Besides, because the curve Ω_α is tangent to the x_2-axis at the origin, it is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\ x_2 > 0\}$ for x_2 small enough. Furthermore, Lemma C.1 yields the existence of $\bar{\beta} > 0$ such that for each β in $(0, \bar{\beta}]$, the intersection of $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\ x_2 > 0\}$ with Ψ_β is below Ω_α. Finally, it is easily checked that $\bar{\beta}$ can be chosen such that for each β in $(0, \bar{\beta}]$, both A_5 and the intersection of $[A_6, A_5]$ with $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2,\ x_2 < 0\}$ are above Ω_α^s. We now define the set W by

$$W \triangleq W_{\bar{\beta}}^- \cup W_{\bar{\beta}}^+.$$

In view of the comments made above and the symmetry of the neighborhoods W_β^- and W_β^+, we can define the following open subsets of $W\backslash\{0\}$:

$R_1 \triangleq$ region in W between the curves $\widehat{\Omega}_\alpha^s$ and Ω_α,
$R_2 \triangleq$ region in W between the curves Δ_α and $\{x :\ x_1 = 2\theta x_2,\ x_2 > 0\}$,
$R_3 \triangleq$ region in W between the half-lines $\{x :\ x_1 = \theta x_2,\ x_2 > 0\}$ and

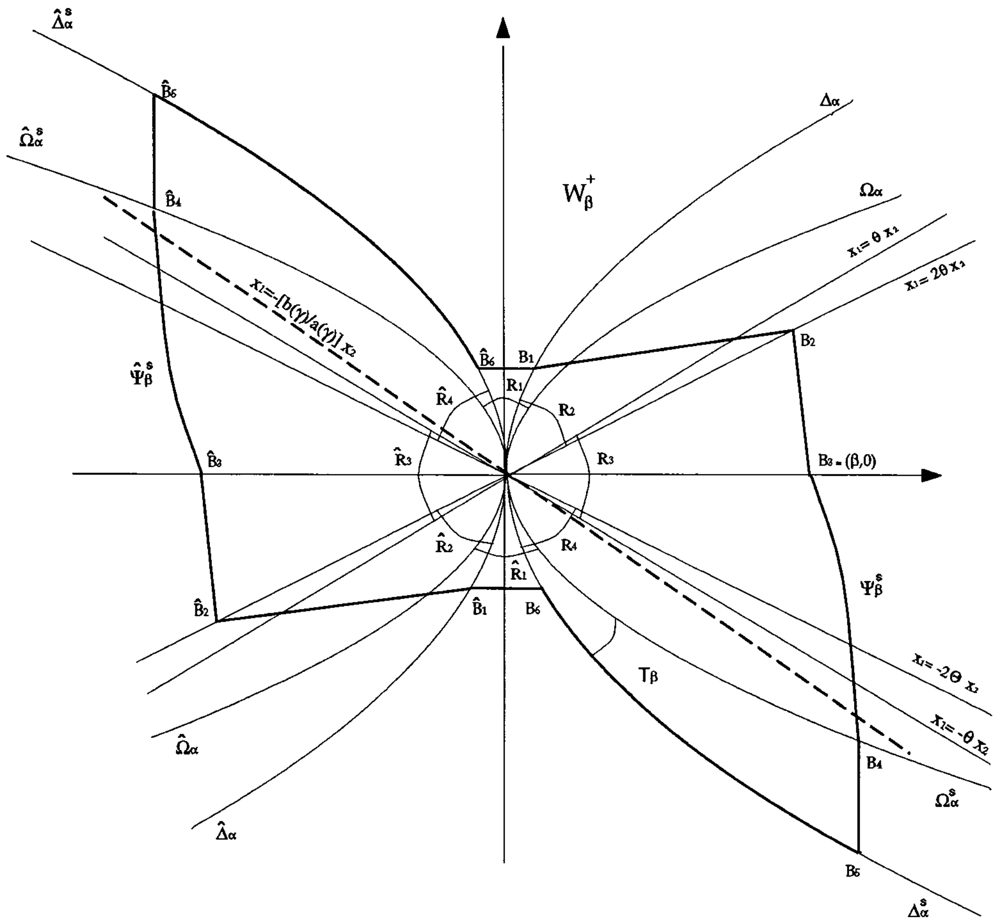

Figure 7.3: Neighborhood W^+_β in the case $\dfrac{b(\gamma)}{a(\gamma)} \to 0 \;\; as \;\; \gamma \to 0$

$\{x : \; x_1 = -\theta x_2, \; x_2 < 0\}$,

$R_4 \triangleq$ region in W between the half-line $\{x : \; x_1 = -2\theta x_2, \; x_2 < 0\}$ and the curve Δ^s_α,

$Q_\beta \triangleq$ region delimited by Δ_α, Ω_α and the segment $[A_2, A_3]$,

$T_\beta \triangleq$ region delimited by Ω^s_α, Δ^s_α and the segment $[B_4, B_5]$,

Because $\{R_1, .., R_4, \widehat{R}_1, .., \widehat{R}_4\}$ is an open cover of $W \backslash \{0\}$, by Theorem 1.1 there exists a partition of unity $\{p_1, .., p_4, \widehat{p}_1, .., \widehat{p}_4\}$ subordinate to this cover such that the support of p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) for each $i = 1, \ldots, 4$.

For each $k > 0$, we now define the mappings $u_k^-, u_k^+ : W \to \mathbb{R}$, by setting

$$u_k^-(x) = \begin{cases} 0 & \text{if } x = 0 \\ -k(x_2)^{1-\alpha} p_1(x) + k(x_1 + x_2) p_2(x) + k x_1\, p_3(x) - (x_1)^2 p_4(x) \\ +k(-x_2)^{1-\alpha} \widehat{p}_1(x) + k(x_1 + x_2) \widehat{p}_2(x) + k x_1\, \widehat{p}_3(x) + (x_1)^2 \widehat{p}_4(x) \\ & \text{otherwise}, \end{cases}$$

and

$$u_k^+(x) = \begin{cases} 0 & \text{if } x = 0 \\ -k(x_2)^{1-\alpha} p_1(x) + (x_1)^2 p_2(x) - k x_1\, p_3(x) + k(-x_1 + x_2) p_4(x) \\ +k(-x_2)^{1-\alpha} \widehat{p}_1(x) - (x_1)^2 \widehat{p}_2(x) - k x_1\, \widehat{p}_3(x) + k(-x_1 + x_2) \widehat{p}_4(x) \\ & \text{otherwise}. \end{cases}$$

The argument given in the proof of Theorem 7.1 to show the smoothness of u_k transposes easily here, and for each $k > 0$, both mappings u_k^- and u_k^+ are C^∞ on $W \backslash \{0\}$ and continuous at the origin.

Using u_k^- and u_k^+, we now construct the desired stabilizing feedback law u_k.

Construction of u_k :

It is not hard to see from Lemma C.2 that both families $\{W_\beta^-\}_{\beta \in (0, \bar{\beta}]}$ and $\{W_\beta^+\}_{\beta \in (0, \bar{\beta}]}$ are bases at the origin with

$$W_\beta^- \subset W_{\beta'}^- \quad \text{and} \quad W_\beta^+ \subset W_{\beta'}^+ \quad \text{whenever} \quad \beta < \beta'. \tag{7.7}$$

Thus, there exists a sequence of positive reals $\{\beta_j\}_{j=0}^\infty$ included in $(0, \bar{\beta}]$ such that

$$\overline{W}_{j+1} \subset W_j, \quad j = 0, 1, 2, \ldots. \tag{7.8}$$

and

$$\beta_j \to 0 \ \ as \ \ j \to \infty \tag{7.9}$$

where we have set

$$\begin{aligned} W_{2n} &\triangleq W_{\beta_{2n}}^+, \quad n = 0, 1, 2, \ldots, \\ W_{2n+1} &\triangleq W_{\beta_{2n+1}}^-, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Combining the inclusions (7.8) with the fact that $\{W_j\}_{j=0}^\infty$ is a base at the origin [which follows from (7.9)], it is not hard to check that $\{W_{j-1} \backslash \overline{W}_{j+1}\}_{j=1}^\infty$ is an open cover of $W_0 \backslash \{0\}$. Let $\{q_j\}_{j=1}^\infty$ be a partition of unity subordinate to the cover $\{W_{j-1} \backslash \overline{W}_{j+1}\}_{j=1}^\infty$ such that the support of q_j is included in $W_{j-1} \backslash \overline{W}_{j+1}$, for each $j = 1, 2, \ldots$ [follows from Theorem 1.1].

For each $k > 0$, we now define the feedback law $u_k : W_0 \to \mathbb{R}$ by setting

$$u_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ u_k^+(x) \sum_{n=1}^{\infty} q_{2n}(x) + u_k^-(x) \sum_{n=0}^{\infty} q_{2n+1}(x) & \text{otherwise}. \end{cases}$$

Next, we fix $k > 0$ and we show that u_k is C^∞ on $W_0 \backslash \{0\}$ and continuous at the origin. Let x be in $W_0 \backslash \{0\}$. It is easily checked that there exists a neighborhood U_x of x such that U_x intersects with at most three sets of the collection $\{W_{j-1} \backslash \overline{W}_{j+1}\}_{j=1}^{\infty}$. Because the support of each mapping q_j is included in $W_{j-1} \backslash \overline{W}_{j+1}$, the infinite sums in the expressions of u_k reduce to the sum of at most three fixed terms on U_x. Therefore, the smoothness of u_k on $W_0 \backslash \{0\}$ follows from that of the mappings u_k^-, u_k^+ and q_j, $j = 1, 2, \ldots$.

Furthermore the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k that

$$|u_k(x)| \;\leq\; \max(\,|u_k^+(x)|, |u_k^-(x)|\,), \quad x \in W_0 \backslash \{0\},$$

and for each $k > 0$, continuity of u_k at the origin follows from that of u_k^- and u_k^+.

The key argument for proving robust asymptotic stabilizability lies in the following claim.

Invariance of the sets W_j :

Claim 1: *There exists $k_0 > 0$, and for each γ in Γ^- (resp. in Γ^+) there exists an integer n_γ such that the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ (resp. $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$) are invariant with respect to the vector field $[f_\gamma, u_{k_0}]^t$ for each $n = n_\gamma, n_\gamma + 1, \ldots$.*

The invariance of $\overline{W}_j$ will be proved if for each x in the boundary ∂W_j, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside the set W_j.

By definition of the partition of unity $\{q_j\}_{j=1}^{\infty}$, we have

$$q_m(x) \;=\; 1 \quad \text{and} \quad q_j(x) = 0, \quad j \neq m, \tag{7.10}$$

for each x in some set $W_{m-1} \backslash \overline{W}_{m+1}$ which does not belong to any other set of the family $\{W_{j-1} \backslash \overline{W}_{j+1}\}_{j=1}^{\infty}$. Therefore, because for each $m = 1, 2, \ldots$, the boundary ∂W_m of the neighborhood W_m is included in $W_{m-1} \backslash \overline{W}_{m+1}$ and does not intersect with any other set $W_{j-1} \backslash \overline{W}_{j+1}, j \neq m$, the definition of u_k yields

$$u_k(x) = u_k^+(x), \quad x \in \partial W_{2n}, \quad n = 1, 2, \ldots, \tag{7.11}$$

and

$$u_k(x) = u_k^-(x), \quad x \in \partial W_{2n+1}, \quad n = 0, 1, \ldots . \tag{7.12}$$

Recall that for each $j = 1, 2, \ldots$ we have $\overline{W}_j \subset W_0$. Because $u_k^-(x)$ and $u_k^+(x)$ are both equal to $-k(x_2)^{1-\alpha}$ if x is in $\Delta_\alpha \cap W_0$ and to $k(-x_2)^{1-\alpha}$ if x is in $\Delta_\alpha^s \cap W_0$, we get

$$u_k(x) = -k(x_2)^{1-\alpha}, \quad x \in \Delta_\alpha \cap \overline{W}_j, \quad j = 1, 2, \ldots \tag{7.13}$$

and

$$u_k(x) = k(-x_2)^{1-\alpha}, \quad x \in \Delta_\alpha^s \cap \overline{W}_j, \quad j = 1, 2, \ldots . \tag{7.14}$$

Furthermore, by definition of u_k we have

$$u_k(x) > 0, \quad x \in \Omega_\alpha \cap \overline{W}_j, \quad j = 1, 2, \ldots \tag{7.15}$$

and

$$u_k(x) < 0, \quad x \in \Omega_\alpha^s \cap \overline{W}_j, \quad j = 1, 2, \ldots . \tag{7.16}$$

From Lemmas C.6 applied with $I = \Gamma^-$, and θ, δ as given here (resp. Lemma C.7 applied with $I = \Gamma^+$, and θ, δ as given here), we obtain for each γ in Γ^- (resp. in Γ^+), a neighborhood V_γ such that the assertions of Lemma C.6 (resp. Lemma C.7) hold.

We now apply Lemmas C.3 and C.4 (with θ, μ, $\bar{\beta}$ and α as given here, and $I = \Gamma^-$, $\eta = 1 - \alpha$, $\eta' = 1 - \alpha$) and Lemma C.5 (with θ, μ, α as given here, and $I = \Gamma^+$, $\eta = 1 - \alpha$): we obtain positive reals k_1, k_2 and k_3, and for each γ in Γ, we get a neighborhood of the origin U_γ such that the assumptions of the lemmas hold for each $k \geq \max(k_1, k_2, k_3)$ and each x in U_γ. Without loss of generality, for each γ in Γ, we take $U_\gamma \subset V_\gamma$. Moreover, for each γ in Γ^- (resp. Γ^+) we let $\bar{\beta}_\gamma$ in $(0, \bar{\beta}]$ be such that $W_{\bar{\beta}_\gamma}^- \subset U_\gamma$ (resp. $W_{\bar{\beta}}^+ \subset U_\gamma$). We also set $k_0 \triangleq \max(k_1, k_2, k_3)$, so that the assertions of the three lemmas hold for each $k \geq k_0$.

Next, we fix γ in Γ^- and we define β_γ as follows: In view of lemma 7.1 and Assertion (7.4), there exists $\beta_\gamma \leq \bar{\beta}_\gamma$ such that $\Pi_\gamma^+ \cap W_{\beta_\gamma}^-$ (resp. $\Pi_\gamma^- \cap W_{\beta_\gamma}^-$) is in the sector $\{x \in W_{\beta_\gamma}^- : x_1 > 0, x_2 > 0\}$ (resp. $\{x \in W_{\beta_\gamma}^- : x_1 < 0, x_2 < 0\}$) with $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) for each x above (resp. below) the curve Π_γ.

Further, because the curves Ω_α and $\widehat{\Omega}_\alpha$ are tangent to the x_2-axis, it follows from the the fact that Π_γ is tangent to the line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma)}{a(\gamma)} x_2\}$ at the origin that β_γ can be chosen small enough so that for each β in $(0, \beta_\gamma]$ the curve Ω_α (resp. $\widehat{\Omega}_\alpha$) is above Π_γ^+ (resp. below Π_γ^-).

Similarly, for each γ in Γ^+, there exists β_γ in $(0, \bar{\beta}_\gamma]$ such that the set $\Pi_\gamma^+ \cap W_{\beta_\gamma}^+$ (resp. $\Pi_\gamma^- \cap W_{\beta_\gamma}^+$) is in the sector $\{x \in W_{\beta_\gamma}^+ : x_1 < 0, x_2 > 0\}$ (resp. $\{x \in W_{\beta_\gamma}^+ : x_1 > 0, x_2 < 0\}$) with $f_\gamma(x) > 0$ (resp. $f_\gamma(x) < 0$) for each x in $W_{\beta_\gamma}^+$ above (resp. below) the curve Π_γ.Moreover, β_γ can be chosen such that we also have the curve $\widehat{\Omega}_\alpha^s \cap W_{\beta_\gamma}^+$ (resp. $\Omega_\alpha^s \cap W_{\beta_\gamma}^+$) above the curve Π_γ^+ (resp. below Π_γ^-).

Because $\beta_j \to 0$ *as* $j \to \infty$, the family $\{W_j\}_{j=1}^\infty$ is a base at the origin, and for each γ in Γ, there exists an integer n_γ such that

$$\beta_j \;\leq\; \beta_\gamma, \quad j = 2n_\gamma, 2n_\gamma + 1, \ldots \tag{7.17}$$

and

$$W_j \;\subset\; U_\gamma, \quad j = 2n_\gamma, 2n_\gamma + 1, \ldots \;. \tag{7.18}$$

It follows from (7.17) and the inclusions (7.7) that for each $n = n_\gamma, n_\gamma + 1, \ldots$, the neighborhood $W_{\beta_{2n+1}}^-$ (resp. $W_{\beta_{2n}}^+$) is included in $W_{\beta_\gamma}^-$ (resp. $W_{\beta_\gamma}^+$). In other words, for each $n = n_\gamma, n_\gamma + 1, \ldots$, the neighborhood W_{2n+1} (resp. W_{2n}) is included in $W_{\beta_\gamma}^-$ (resp. $W_{\beta_\gamma}^+$). Thus, the definition of β_γ implies that for each x on the vertical segments of ∂W_{2n+1} (resp. ∂W_{2n}), we have $f_\gamma(x) < 0$ for x in $[A_2, A_3]$ (resp. $[B_4, B_5]$) while $f_\gamma(x) > 0$ for x in $[\widehat{A}_2, \widehat{A}_3]$ (resp. $[\widehat{B}_4, \widehat{B}_5]$).

Fix γ in Γ^- and n in $\{n_\gamma, n_\gamma + 1, \ldots\}$. Recall that for each $i = 1, 2, 3, 4$, the support of the mappings p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$). Further, note that the intersection of more than two sets of the family $\{R_1, \ldots, R_4, \widehat{R}_1, \ldots, \widehat{R}_4\}$ is empty. Thus, for each x in ∂W_{2n+1}, the vector $[f_\gamma(x), u_{k_0}^-(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemmas C.3, C.4 and C.6, and therefore points inside W_{2n+1}, or is a convex combination of two of them. In the latter case, $[f_\gamma(x), u_{k_0}^-(x)]^t$ points inside W_{2n+1} either because we have a convex combination, or because we have $f_\gamma(x) < 0$ (resp. $f_\gamma(x) > 0$) on the vertical segments $[A_2, A_3]$ (resp. $[\widehat{A}_2, \widehat{A}_3]$) of ∂W_{2n+1}.

By (7.12), $u_{k_0} = u_{k_0}^-$ on ∂W_{2n+1} and it follows that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside W_{2n+1}, for each x in ∂W_{2n+1}.

Because we have $u_{k_0}(x) > 0$ [by (7.15)] and $f_\gamma(x) < 0$ for each x in $\Omega_\alpha \cap W_{2n+1}$, the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$. Further, (7.13) combined with the assertions of Lemma C.3 **(ii)** imply that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$ for each x in $\Delta_\alpha \cap W_{2n+1}$. Thus, as $\beta_{2n+1} \leq \beta_\gamma$, it follows from the definition of β_γ that the vector $[f_\gamma(x), u_{k_0}(x)]^t$ points inside $Q_{\beta_{2n+1}}$ for each x in $\partial Q_{\beta_{2n+1}}$.

Therefore, for each n in $\{n_\gamma, n_\gamma+1, \ldots\}$, the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ are invariant with respect to the vector-field $[f_\gamma, u_{k_0}]^t$.

Similarly, (7.11), (7.14), (7.16), (7.17) and (7.18), together with the assertions of Lemmas C.5 and C.7 yield the invariance with respect to the vector-field $[f_\gamma, u_{k_0}]^t$ of the sets $\overline{W}_{2n}$ and $\overline{Q}_{\beta_{2n}}$, for each γ in Γ^+ and for each n in $\{n_\gamma, n_\gamma+1, \ldots\}$. The proof of Claim 1 is now complete.

Robust asymptotic stability :

Let k_0 be as defined in Claim 1. Fix γ in Γ^-. Let n_γ be as given in Claim 1 and let $n = n_\gamma, n_\gamma+1, \ldots$. In view of (7.17) and the definition of β_γ, we have $u_{k_0}(x) \neq 0$ for all x in $\overline{W}_{2n+1}\backslash\{0\}$ with $f_\gamma(x) = 0$, so that the origin is the unique equilibrium point in $\overline{W}_{2n+1}$ of the system $\tilde{S}(\gamma)$ obtained once u_{k_0} is fed back into $S(\gamma)$. Thus, by the invariance of the compact set $\overline{W}_{2n+1}$ with respect to $\tilde{S}(\gamma)$ (Claim 1), and the Poincaré-Bendixson Theorem [38], the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_{2n+1}$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$.

If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [38, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of the set $\overline{Q}_{\beta_{2n+1}}$ and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\tilde{S}(\gamma)$ starting in $\overline{W}_{2n+1}$ remains in $\overline{W}_{2n+1}$ and converges to the origin [38, Corollary 1.1 p. 146].

As $\{W_{2n+1}\}_{n=n_\gamma}^{\infty}$ is a base at the origin, we obtain that the feedback law u_{k_0} locally asymptotically stabilizes the system $S(\gamma)$ for each γ in Γ^-.

Similarly, by using the invariance of the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$, we get that u_{k_0} locally asymptotically stabilizes $S(\gamma)$ for each γ in Γ^+.

We have therefore proved that if $\dfrac{b(\gamma)}{a(\gamma)} \to 0$ *as* $\gamma \to 0$ then $\{S(\gamma),\, \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by means of continuous feedback. To complete the proof of the theorem we now consider the remaining case.

b) The case $\dfrac{b(\gamma)}{a(\gamma)}$ does not converge to 0 as γ tends to 0

In this case, because $a(\cdot)$ and $b(\cdot)$ are C^∞ and do not vanish on Γ, we have $|\frac{b(\gamma)}{a(\gamma)}| \to +\infty$ *as* $\gamma \to 0$. Therefore, there exists $\theta > 0$ such that $|\frac{b(\gamma)}{a(\gamma)}| > \theta$, $\gamma \in \Gamma$. The result is now obtained through the same arguments as those in the proof of **(a)** with θ as defined above. ■

We note that the robust asymptotic stabilizer that we have found is not unique. In fact, each one of the feedback laws of the collection $\{u_k,\ k \in [k_0, \infty)\}$ robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

To complete our study, we now investigate the case when the sign of both $a(\cdot)$ and $b(\cdot)$ change as γ takes the value 0.

7.4 Robust stabilization when the signs of $a(\cdot)$ and $b(\cdot)$ change

The next theorem yields a necessary condition for robust asymptotic stabilizability of the family $\{S(\gamma),\ \gamma \in \Gamma\}$, in case $a(\cdot)$ is negative on Γ^- and positive on Γ^+. The companion case $a(\cdot)$ positive on Γ^- and negative on Γ^+ is easily deduced from the following results by replacing Γ^- by Γ^+ and vice versa.

Theorem 7.3 *Assume that $a(\cdot)$ is negative on Γ^- and positive on Γ^+, and that $b(\cdot)$ is negative (resp. positive) on Γ^- and positive (resp. negative) on Γ^+. If the family $\{S(\gamma),\ \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by a feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ which is continuous on a neighborhood of the origin, then we have*

$$\frac{b(\gamma_-)}{a(\gamma_-)} \le \frac{b(\gamma_+)}{a(\gamma_+)} \qquad (\textit{resp.} \quad \frac{b(\gamma_+)}{a(\gamma_+)} \le \frac{b(\gamma_-)}{a(\gamma_-)}) \,, \tag{7.19}$$

for all γ_- in Γ^- and all γ_+ in Γ^+.

Proof: We only consider the case $b(\cdot)$ negative on Γ^- and positive on Γ^+, as the arguments presented below carry over to the case $b(\cdot)$ positive on Γ^- and negative on Γ^+.

We prove the theorem by a contradiction argument. Suppose that there exists a continuous feedback law v that robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$ and that (7.19) does not hold. Then, there exist γ_- and γ_+ in Γ^- and Γ^+ respectively such that

$$\frac{b(\gamma_-)}{a(\gamma_-)} > \frac{b(\gamma_+)}{a(\gamma_+)},$$

and it follows that the half-line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma_-)}{a(\gamma_-)} x_2, x_2 > 0\}$ is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -\frac{b(\gamma_+)}{a(\gamma_+)} x_2, x_2 > 0\}$. This together with Lemma 7.1 and Assertion (7.4) yield the existence of a ball $B_\varepsilon(0)$ of radius ε such that the following holds.

i) $\Pi^+_{\gamma_-} \cap B_\varepsilon(0)$ and $\Pi^+_{\gamma_+} \cap B_\varepsilon(0)$ are in the sector $\{x \in \mathbb{R}^2 : x_1 < 0, x_2 > 0\}$,

ii) $\Pi^+_{\gamma_-} \cap B_\varepsilon(0)$ is below $\Pi^+_{\gamma_+} \cap B_\varepsilon(0)$,

iii) For each x in $\{x \in B_\varepsilon(0) : x_1 < 0\}$ we have $f_{\gamma_-}(x) > 0$ whenever x is below $\Pi^+_{\gamma_-}$ and $f_{\gamma_-}(x) < 0$ whenever x is above $\Pi^+_{\gamma_-}$,

iv) For each x in $\{x \in B_\varepsilon(0) : x_1 < 0\}$ we have $f_{\gamma_+}(x) > 0$ whenever x is above $\Pi^+_{\gamma_+}$ and $f_{\gamma_-}(x) < 0$ whenever x is below $\Pi^+_{\gamma_-}$.

Because v locally asymptotically stabilizes $S(\gamma_-)$ and $S(\gamma_+)$, it follows from **(iii)** and **(iv)** that ε can be chosen small enough so that

$$v(x) < 0, \quad x \in \Pi^+_{\gamma_-} \cap B_\varepsilon(0) \qquad \text{and} \qquad v(x) > 0, \quad x \in \Pi^+_{\gamma_+} \cap B_\varepsilon(0).$$

Let S be the region of $B_\varepsilon(0)$ below $\Pi^+_{\gamma_-}$ that is between $\Pi^+_{\gamma_-}$ and $\{(0, x_2) \in \mathbb{R}^2 : x_2 < 0\}$. The negativeness of f_{γ_+} on S [follows from **(iv)**] combined with the stability of the system associated with the vector-field $[f_{\gamma_+}, v]^t$ imply that each trajectory $x(\cdot, x_0)$ of this system starting in S leaves S. Thus, because f_{γ_+} and v are negative on $\Pi^+_{\gamma_-} \cap B_\varepsilon(0)$, and f_{γ_+} is negative on $\{(0, x_2) \in B_\varepsilon(0) : x_2 < 0\}$, it follows that $x(\cdot, x_0)$ cannot leave S, neither through $\Pi^+_{\gamma_-}$, nor through the x_2-axis. We conclude that $x(\cdot, x_0)$ leaves S through the boundary of $B_\varepsilon(0)$. In short $x(\cdot, x_0)$ leaves $B_\varepsilon(0)$ whenever x_0 lies in S, a contradiction with the fact that v stabilizes $S(\gamma_+)$. Hence the theorem. ■

In the situation where the necessary condition (7.19) is satisfied, we do not have any sufficient condition for the robust asymptotic stabilizability of the family $\{S(\gamma), \gamma \in \Gamma\}$. However, for the family of linearized systems $\{S_L(\gamma), \gamma \in \Gamma\}$ obtained from $\{S(\gamma), \gamma \in \Gamma\}$, the previous theorem can be refined and we obtain the following **necessary and sufficient** condition for the robust asymptotic stabilizability of the family $\{S_L(\gamma), \gamma \in \Gamma\}$.

Theorem 7.4 *Assume that $a(\cdot)$ is negative on Γ^- and positive on Γ^+, and that $b(\cdot)$ is negative (resp. positive) on Γ^- and positive (resp. negative) on Γ^+. Then there exists a feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ which is continuous on a neighborhood of* *the origin and which robustly asymptotically stabilizes the family* $\{S_L(\gamma),\ \gamma \in \Gamma\}$, *if and only if*

$$\frac{b(\gamma_-)}{a(\gamma_-)} < \frac{b(\gamma_+)}{a(\gamma_+)} \qquad \textit{(resp.}\quad \frac{b(\gamma_+)}{a(\gamma_+)} < \frac{b(\gamma_-)}{a(\gamma_-)}\textit{)}\,, \tag{7.20}$$

for all γ_- *in* Γ^- *and all* γ_+ *in* Γ^+.

Proof: We only consider the case $b(\cdot)$ negative on Γ^- and positive on Γ^+, as the arguments presented below carry over to the case $b(\cdot)$ positive on Γ^- and negative on Γ^+.

We first show that under (7.20), the family $\{S_L(\gamma),\ \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by C^∞ feedback. Under the assumptions made on $a(\cdot)$ and $b(\cdot)$, $\frac{b(\gamma)}{a(\gamma)}$ converges either to 0, to $+\infty$, or to a positive real τ. Because $\frac{b(\cdot)}{a(\cdot)}$ is positive on Γ, it is easily checked that $\frac{b(\gamma)}{a(\gamma)}$ converges to some positive real τ whenever (7.20) holds.

Note that under (7.20), there does not exist any γ_- in Γ^- and γ_+ in Γ^+ such that $\frac{b(\gamma_-)}{a(\gamma_-)} = \frac{b(\gamma_+)}{a(\gamma_+)} = \tau$. Let $k < -\sup\{a(\gamma) : \gamma \in \Gamma\}$ and define the feedback law $u : \mathbb{R}^2 \to \mathbb{R}$ as follows: If $\frac{b(\gamma)}{a(\gamma)} = \tau$ for some γ in Γ^-, set $u(x) = k[\frac{1}{\tau}x_1 + x_2] + x_1^3$. If $\frac{b(\gamma)}{a(\gamma)} = \tau$ for some γ in Γ^+, set $u(x) = k[\frac{1}{\tau}x_1 + x_2] - x_1^3$. Finally, if $\frac{b(\gamma)}{a(\gamma)} \neq \tau$ for all γ in Γ, set $u(x) = k[\frac{1}{\tau}x_1 + x_2]$. By adapting Example 3.8 in [51, p. 118] to our setup, it is not hard to check that u robustly asymptotically stabilizes the family $\{S_L(\gamma),\ \gamma \in \Gamma\}$.

Now, if $\{S_L(\gamma),\ \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by some continuous feedback law v, then by Theorem 7.3 we have

$$\frac{b(\gamma_-)}{a(\gamma_-)} \leq \frac{b(\gamma_+)}{a(\gamma_+)}, \quad \gamma_- \in \Gamma^-, \quad \gamma_+ \in \Gamma^+.$$

Assume that there exists γ_- and γ_+ in Γ^- and Γ^+ respectively such that

$$\frac{b(\gamma_-)}{a(\gamma_-)} = \frac{b(\gamma_+)}{a(\gamma_+)},$$

and define the set Φ by setting

$$\Phi \triangleq \{x \in \mathbb{R}^2 : \; x_1 = -\frac{b(\gamma_+)}{a(\gamma_+)} x_2, \; x_2 > 0\}$$

Because the feedback law v stabilizes $S_L(\gamma_-)$ and $S_L(\gamma_+)$, the origin is an isolated equilibrium of the corresponding closed-loop systems and there exists a neighborhood of the origin U such that $v(x) \neq 0, \; x \in \Phi \cap U$. Because, $a(\gamma_-)x_1 + b(\gamma_-)x_2 < 0$ for all x in the region above Φ and $a(\gamma_+)x_1 + b(\gamma_+)x_2 < 0$ for all x in the region below Φ, the neighborhood U can be chosen small enough so that

$$v(x) < 0 \qquad \text{and} \qquad v(x) > 0, \quad x \in \Phi \cap U,$$

which is impossible. Therefore, if $\{S_L(\gamma), \, \gamma \in \Gamma\}$ is robustly asymptotically stabilizable by continuous feedback, then we have

$$\frac{b(\gamma_-)}{a(\gamma_-)} < \frac{b(\gamma_+)}{a(\gamma_+)}, \quad \gamma_- \in \Gamma^-, \quad \gamma_+ \in \Gamma^+,$$

and the proof of the theorem is complete. ■

The investigation of the robust stabilization of the family $\{S(\gamma), \; \gamma \in \Gamma\}$ is now complete. In case the signs of $a(\cdot)$ and $b(\cdot)$ both change, we provide a necessary condition the existence of a robust asymptotic stabilizer for the family $\{S(\gamma), \, \gamma \in \Gamma\}$, while we give a necessary and sufficient condition for the asymptotic stabilizability of the family $\{S_L(\gamma), \, \gamma \in \Gamma\}$. In all the other cases, whenever there exists no C^1 feedback law that robustly asymptotically stabilizes $\{S(\gamma), \, \gamma \in \Gamma\}$, we construct a merely continuous robust asymptotic stabilizer.

The robust stabilization of the particular family $\{S_L(\gamma), \, \gamma \in \Gamma\}$ has been investigated in Ho-Mock-Qai and Dayawansa [42]. The methods used for this simpler problem are basically the same as those presented here. However, because less technicalities are involved these methods may seem clearer.

7.5 Obstruction to Lipschitz robust stabilization

It is of theoretical interest to establish a classification between classes of feedback laws with different degree of regularity, according to what they can achieve for stabilization purpose. For example, by proving that every fully controllable system can be asymptotically stabilized by piecewise analytic feedback, Sussmann [74], showed that the class of piecewise analytic feedback laws is "superior" to the class of continuous feedback laws. Here, one may wonder whether Lipschitz continuous feedback can achieve more than continuous and almost C^∞ feedback, for robust stabilization problems .

The following result partially answers this question: We prove that some family of systems satisfying the assumptions of Theorem 7.1, cannot be robustly asymptotically stabilized by a feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ which is Lipschitz continuous at the origin, while Theorem 7.1 implies that they can be robustly asymptotically stabilized by means of continuous and almost C^∞ feedback.

Recall that a mapping $v : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous at the origin, if there exists a neighborhood U of the origin and $L > 0$ such that

$$|v(x) - v(y)| \leq L\|x - y\|, \quad (x, y) \in U^2.$$

Although, this result may seem to be of limited importance, the construction provided in the proof is by itself interesting and somewhat surprising.

Theorem 7.5 *Assume that $a(\cdot)$ is positive on Γ and $b(\cdot)$ is either negative or positive on Γ. If*

$$\frac{a(\gamma)^2}{b(\gamma)} \to +\infty \;\; as \;\; \gamma \to 0, \tag{7.21}$$

then, there does not exist any feedback law $v : \mathbb{R}^2 \to \mathbb{R}$ that is Lipschitz continuous at the origin and that robustly asymptotically stabilizes the family $\{S(\gamma),\, \gamma \in \Gamma\}$.

We prove the theorem only in the case $b(\cdot)$ negative on Γ, as the proof of the companion case $b(\cdot)$ positive on Γ is similar. Under this assumption Lemma 7.1 yield the existence of a neighborhood of the origin $U_\gamma \triangleq I_\gamma \times J_\gamma$ for each γ in Γ, such that

$$\Pi_\gamma \triangleq \{x \in U_\gamma : f_\gamma(x) = 0\} = \{(\phi_\gamma(x_2), x_2) : x_2 \in J_\gamma\}, \tag{7.22}$$

with $\phi_\gamma(\cdot)$ strictly increasing on J_γ, and (7.2), (7.3) and (7.4) fulfilled. Moreover, for each x_2 in J_γ^+ (resp. J_γ^-) we have $\phi_\gamma(x_2) > 0$ (resp. $\phi_\gamma(x_2) < 0$). Finally, f_γ is negative (resp. positive) above (resp. below) $\{(\phi_\gamma(x_2), x_2) :\; x_2 \in J_\gamma\}$.

Consider Fig. 7.4. For each γ in Γ, we define :

$P_\gamma \triangleq$ Open region in U_γ between the curves $\{(2\phi_\gamma(x_2), x_2),\; x_2 \in J_\gamma^+\}$ and $\{(-2\phi_\gamma(x_2), x_2),\; x_2 \in J_\gamma^+\}$,

$P_\gamma^- \triangleq$ Open region in U_γ between the curves $\{(-2\phi_\gamma(x_2), x_2),\; x_2 \in J_\gamma^-\}$ and $\{(2\phi_\gamma(x_2), x_2),\; x_2 \in J_\gamma^-\}$,

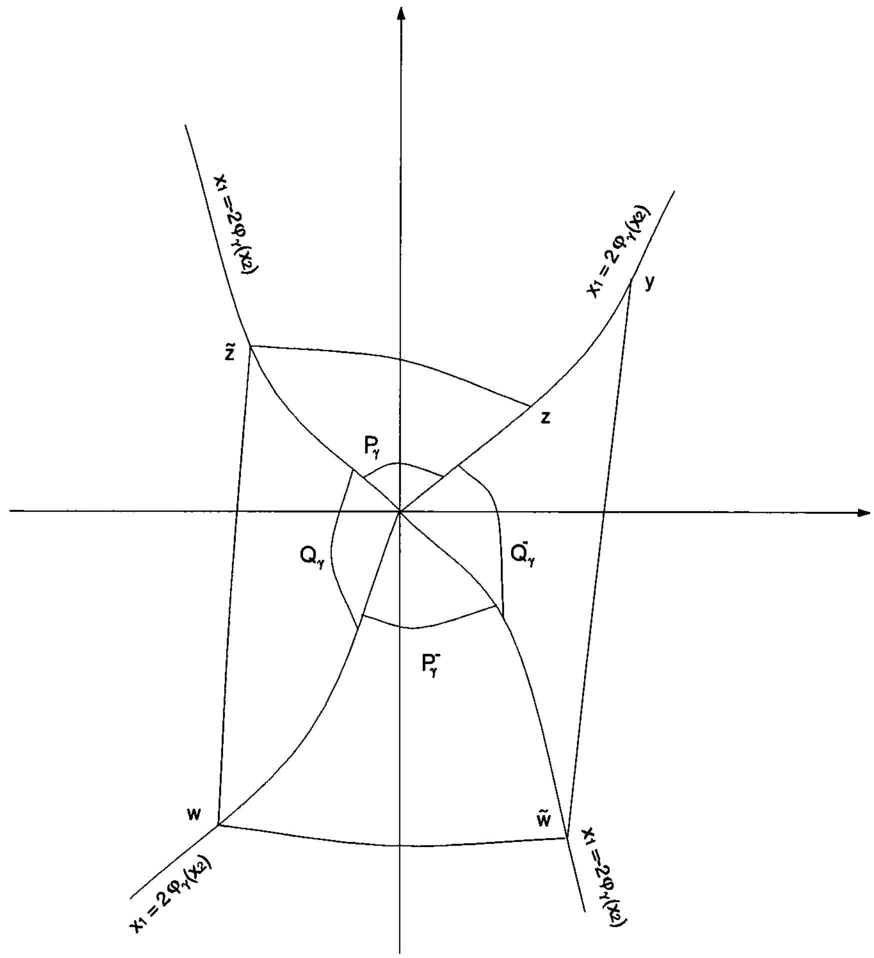

Figure 7.4: Trajectory $(z, \tilde{z}, w, \tilde{w}, y)$

$Q_\gamma \triangleq$ Open region in U_γ between the curves $\{(-2\phi_\gamma(x_2), x_2),\ x_2 \in J_\gamma^+\}$ and $\{(2\phi_\gamma(x_2), x_2),\ x_2 \in J_\gamma^-\}$,

$Q_\gamma^- \triangleq$ Open region in U_γ between the curves $\{(-2\phi_\gamma(x_2), x_2),\ x_2 \in J_\gamma^-\}$ and $\{(2\phi_\gamma(x_2), x_2),\ x_2 \in J_\gamma^+\}$.

To prove the theorem, we introduce a key intermediate lemma.

Lemma 7.3 *Suppose that the assumptions of Theorem 7.5 hold and that* $b(\cdot) < 0$ *on* Γ. *Furthermore, assume that there exists a feedback law* $v : \mathbb{R}^2 \to \mathbb{R}$ *which is Lipschitz continuous at the origin and which robustly asymptotically stabilizes the family* $\{S(\gamma),\ \gamma \in \Gamma\}$. *Then, there exists an interval* $I = (0, \zeta)$ *included in* Γ *and for each* γ *in* I, *there exists some neighborhoods of the origin* $\tilde{I}_\gamma$ *and* $\tilde{J}_\gamma$ *in* $\mathbb{R}$, *included in* I_γ *and* J_γ *respectively [where* I_γ *and* J_γ *are as given in (7.22)] such that:*

i) *If the vector* $[f_\gamma(z), u(z)]^t$ *points into* P_γ *for some* γ *in* I *and some* z *in* $\{(2\phi_\gamma(x_2), x_2) : x_2 \in \tilde{J}_\gamma^+\}$, *then* $v(y) \geq 0$ *for all* y *in* $P_\gamma \bigcap \{x \in \mathbb{R}^2 : x_2 = z_2\} \cap \left(\tilde{I}_\gamma \times \tilde{J}_\gamma\right)$.

ii) *If the vector* $[f_\gamma(w), u(w)]^t$ *points into* P_γ^- *for some* γ *in* I *and some* w *in* $\{(2\phi_\gamma(x_2), x_2) : x_2 \in \tilde{J}_\gamma^-\}$, *then* $v(y) \leq 0$ *for all* y *in* $P_\gamma^- \bigcap \{x \in \mathbb{R}^2 : x_2 = w_2\} \cap \left(\tilde{I}_\gamma \times \tilde{J}_\gamma\right)$.

Proof: We only prove **(i)** as the proof of **(ii)** is similar.

We assume that the claim does not hold. Then, for each $n = 1, 2, \ldots$, there exists γ_n in $(0, \frac{1}{n})$ such that either one of **(i)** or **(ii)** does not hold. We assume that **(i)** is violated infinitely many times as $n = 1, 2, \ldots$ and we fix n in $\{1, 2, \ldots\}$. Then, for each $m = 1, 2, \ldots$ there exists $z_{n,m} = (z_1^{n,m}, z_2^{n,m})$ in $(0, \frac{1}{m}) \times (0, \frac{1}{m})$ and $y_{n,m}$ in $P_{\gamma_n} \cap \{x \in \mathbb{R}^2 : x_2 = z_2^{n,m}\}$ such that

- $z_1^{n,m} = 2\phi_{\gamma_n}(z_2^{n,m})$ and $[f_{\gamma_n}(z_{n,m}), v(z_{n,m})]^t$ points into P_{γ_n} [note that this implies that $v(z_{n,m}) > 0$ for m large enough],
- $v(y_{n,m}) < 0$.

Because $(0, \dfrac{1}{m}) \times (0, \dfrac{1}{m})$ is included in $I_{\gamma_n} \times J_{\gamma_n}$ for m greater than some integer m_n, it follows from the fact that $[f_{\gamma_n}(z_{n,m}), v(z_{n,m})]^t$ points into P_{γ_n} that

$$\frac{f_{\gamma_n}(z_{n,m})}{v(z_{n,m})} \leq 2\phi'_{\gamma_n}(z_2^{n,m}).$$

with

$$v(z_{n,m}) > 0 \qquad \text{and} \qquad \phi'_{\gamma_n}(z_{n,m}) > 0, \quad m = m_n, m_n + 1, \ldots, \tag{7.23}$$

where ϕ'_{γ_n} denotes the derivative of ϕ_{γ_n}. Hence,

$$v(z_{n,m}) \geq \frac{f_{\gamma_n}(z_{n,m})}{2\phi'_{\gamma_n}(z_2^{n,m})}, \quad m = m_n, m_n + 1, \ldots. \tag{7.24}$$

Furthermore, in view of (7.23) and the inequality $v(y_{n,m}) < 0$, we get

$$\frac{|v(z_{n,m}) - v(y_{n,m})|}{\|z_{n,m} - y_{n,m}\|} \geq \frac{v(z_{n,m})}{2z_1^{n,m}}, \quad m = m_n, m_n + 1, \ldots.$$

Thus, the equalities $z_1^{n,m} = 2\phi_{\gamma_n}(z_2^{n,m})$, $f_{\gamma_n}(z_{n,m}) = h_{\gamma_n}(z_{n,m})(z_1^{n,m} - \phi_{\gamma_n}(z_2^{n,m}))$ combined with (7.24), yield

$$\frac{|v(z_{n,m}) - v(y_{n,m})|}{\|z_{n,m} - y_{n,m}\|} \geq \frac{h_{\gamma_n}(z_{n,m})}{8\phi'_{\gamma_n}(z_2^{n,m})}, \quad m = m_n, m_n + 1, \ldots \tag{7.25}$$

Recall that n is fixed in $\{1, 2, \ldots\}$ and that $h_{\gamma_n}(0) = a(\gamma_n)$ with $\phi'_{\gamma_n}(0) = -\frac{b(\gamma_n)}{a(\gamma_n)}$. Because $z_{n,m}$ converges to the origin as m tends to ∞, the analyticity of the mappings h_{γ_n} and ϕ'_{γ_n} combined with (7.25) yield

$$\frac{|v(z_{n,m}) - v(y_{n,m})|}{\|z_{n,m} - y_{n,m}\|} \quad \to \quad \frac{a^2(\gamma_n)}{-8b(\gamma_n)} \quad \text{as} \quad m \to \infty,$$

so that there exists an integer $m'_n \geq m_n$ satisfying

$$\frac{|v(z_{n,m}) - v(y_{n,m})|}{\|z_{n,m} - y_{n,m}\|} \geq \frac{a^2(\gamma_n)}{-8b(\gamma_n)}, \quad m = m'_n, m'_n + 1, \ldots \tag{7.26}$$

For each $n = 1, 2, \ldots$, we pick $m \geq m'_n$ such that $\|z_{n,m}\| < \frac{1}{n}$, and we set $\widehat{z}_n \triangleq z_{n,m}$ with $\widehat{y}_n \triangleq y_{n,m}$. Because, $\widehat{z}_n$ and $\widehat{y}_n$ converge to the origin, and γ_n converges to 0 as n tends to ∞, it follows from (7.21) and (7.26) that

$$\frac{|v(\widehat{z}_n) - v(\widehat{y}_n)|}{\|\widehat{z}_n - \widehat{y}_n\|} \to \infty \;\; as \;\; n \to \infty, \tag{7.27}$$

a contradiction with the fact that v is Lipschitz continuous at the origin.

If **(i)** is violated only finitely many times, then **(ii)** does not hold infinitely many times, and by a similar argument one can exhibit some sequences $\{\widehat{z}_n\}_{n=1}^{\infty}$ and $\{\widehat{y}_n\}_{n=1}^{\infty}$, converging to the origin [where $\widehat{y}_n$ lies in $\{x \in \mathbb{R}^2 : \; x_2 = \widehat{z}_2^n\}$ for each $n = 1, 2, \ldots$], such that (7.27) holds. This contradicts also the Lipschitz continuity of v at the origin and the lemma follows. ■

By using this lemma, we now prove the theorem.

Proof of Theorem 7.5 :

We only study the case $b(\cdot)$ negative on Γ, since the proof in the case $b(\cdot)$ positive on Γ is similar.

Assume that there exists a feedback law v that is Lipschitz continuous at the origin and that robustly asymptotically stabilizes the family $\{S(\gamma), \, \gamma \in \Gamma\}$. Then, there exists an interval I included in Γ and some neighborhoods of the the origin $\widetilde{I}_\gamma$ and $\widetilde{J}_\gamma$ included in I_γ and J_γ respectively [where I_γ and J_γ are as given in (7.22)], such that Lemma 7.3 **(i)** and **(ii)** hold.

Fix γ in I and let $\varepsilon > 0$ be such that $B_\varepsilon(0)$ is included both in $\widetilde{I}_\gamma \times \widetilde{J}_\gamma$ and in the domain of attraction of the closed-loop system $\widetilde{S}(\gamma)$ obtained once v is fed-back into $S(\gamma)$. Let $\delta > 0$ be such that every trajectory that starts in $B_\delta(0)$ remains in $B_\varepsilon(0)$ forever. Let $z = (z_1, 0)$ be in $B_\delta(0)$ with $z_1 > 0$. Because

f_γ is positive in the region $\left(P_\gamma^- \cup Q_\gamma^-\right) \cap B_\varepsilon(0)$, it is easily checked that the trajectory $x(\cdot, z)$ of $\widetilde{S}(\gamma)$ leaves $P_\gamma^- \cup Q_\gamma^-$ through the curve $\{(2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$. Therefore, we may take z is in $B_\varepsilon(0) \cap \{(2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$ such that the vector $[f_\gamma(z), v(z)]^t$ points into P_γ.

Because f_γ is negative on $P_\gamma \cap B_\varepsilon(0)$ and $v(y) \geq 0$ for each y in $\{y \in B_\varepsilon(0) :\ y_2 = z_2\}$ [follows from Lemma 7.3 **(i)**] the trajectory $x(\cdot, z)$ of $\widetilde{S}(\gamma)$ leaves P_γ through the curve $\{(-2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$. Further, because f_γ is negative on the set $Q_\gamma \cap B_\varepsilon(0)$ and v is non-negative on $\{y \in B_\varepsilon(0) : y_2 = z_2\}$ it follows from the fact that the mapping $-2\phi_\gamma(\cdot)$ is decreasing that $x(\cdot, z)$ enters the region P_γ^- through $\{(2\phi_\gamma(x_2), x_2),\ x_2 < 0\}$. Next, because $u(y) \leq 0$ for each y in $\{x \in B_\varepsilon(0) : x_2 = w_2\}$ [follows from Lemma 7.3 **(ii)**] and f_γ is positive in P_γ^-, the trajectory $x(\cdot, z)$ leaves P_γ^- through the curve $\{(-\phi_\gamma(x_2), x_2),\ x_2 < 0\}$. Finally, because the mapping f_γ is positive on Q_γ^- and $u(y) < 0$ for each y in $\{x \in \mathbb{R}^2 :\ x_2 = w_2\}$, we obtain from the fact that the mapping $-2\phi_\gamma(\cdot)$ is decreasing that the trajectory $x(\cdot, z)$ leaves Q_γ^- to enter P_γ through $\{(2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$.

Therefore, the trajectory $x(\cdot, z)$ spirals toward the origin and the origin is a focus.

Consider Fig. 7.4. Let $\widetilde{z}$, w, $\widetilde{w}$ be the points where the trajectory $x(\cdot, z)$ of $\widetilde{S}(\gamma)$ crosses for the first time the curves $\{(-2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$, $\{(2\phi_\gamma(x_2), x_2),\ x_2 < 0\}$ and $\{(-2\phi_\gamma(x_2), x_2),\ x_2 < 0\}$ respectively, and let y be the point at which $x(\cdot, z)$ returns to $\{(2\phi_\gamma(x_2), x_2),\ x_2 > 0\}$ for the first time.

From Lemma 7.3 and the fact that $\phi_\gamma(\cdot)$ is strictly increasing, we obtain

$$\widetilde{z}_2 > z_2 \qquad \text{and} \qquad -\widetilde{z}_1 > z_1. \tag{7.28}$$

Further, because f_γ is negative on Q_γ, we get

$$-w_1 > -\widetilde{z}_1. \tag{7.29}$$

Next, Lemma 7.3 combined with the fact that ϕ_γ is strictly increasing yield

$$-\widetilde{w}_2 > w_2 \qquad \text{with} \qquad \widetilde{w}_1 > w_1, \tag{7.30}$$

and f_γ being negative on Q_γ^- we get

$$y_1 > \widetilde{w}_1. \tag{7.31}$$

We now obtain from (7.28)-(7.31) that

$$y_1 > \widetilde{w}_1 > -w_1 > -\widetilde{z}_1 > z_1.$$

This, together with the fact that the mapping ϕ_γ is strictly increasing yield

$$y_2 > z_2, \tag{7.32}$$

and it is easily checked that $x(\cdot, z)$ does not spiral towards the origin. A contradiction with the stability of $\widetilde{S}(\gamma)$, which completes the proof.

■

Finally, following the ideas of Section 2.3, we again show that the use of partitions of unity for the design of controllers does not necessarily yield feedback laws that cannot be computed and implemented.

7.6 A simple expression for a robust stabilizer

Using the ideas of Section 2.3 of Chapter 2, we now construct an explicit feedback law that robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$ and that does not involve a partition of unity.

In the sequel, we assume that the sign of $b(\cdot)$ is constant on Γ. In this case, because there exists a linear feedback law that robustly the family $\{S(\gamma),\ \gamma \in \Gamma\}$ if $\dfrac{b(\gamma)}{a(\gamma)}$ does not converge to 0 as γ goes to 0, we also assume that $\dfrac{b(\gamma)}{a(\gamma)}$ converges to 0 as γ goes to 0.

We start by assuming that $a(\cdot) > 0$ on Γ^+, $a(\cdot) < 0$ on Γ^- and $b(\cdot) < 0$ on Γ, and we show that we can circumvent the computation of the partition of unity $\{p_1, p_2, p_3, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3\}$ that appears in the expression of the robust stabilizer constructed in Section 7.2. We exhibit a family $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ of mappings that are explicit and by replacing the mappings $\{p_1, p_2, p_3, \widehat{p}_1, \widehat{p}_2, \widehat{p}_3\}$ by these new mappings in the expression of the robust stabilizer (7.5), we obtain a feedback law that robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

First, we let α be in $(0,1)$. We define θ, $\bar{\beta}$, as well as the sets W_β, Q_β, R_1, R_2 and R_3 exactly as we did in the proof of Lemma 7.2. We then introduce the following subsets of $W_{\bar{\beta}}$:

$S_1 \triangleq$ closed region in $W_{\bar{\beta}}$ between $\{x \in \mathbb{R}^2 :\ x_1 = -2\theta x_2,\ x_2 > 0\}$ and $\{x \in \mathbb{R}^2 :\ x_1 = -\theta x_2,\ x_2 > 0\}$, minus the origin,

$S_2 \triangleq$ closed region in $W_{\bar{\beta}}$ between $\{x \in \mathbb{R}^2 :\ x_1 = -2\theta x_2,\ x_2 > 0\}$ and Δ_α, minus the origin,

$S_3 \triangleq$ closed region in $W_{\bar{\beta}}$ between Δ_α and Ω_α, minus the origin,

$S_4 \triangleq$ closed region in $W_{\bar{\beta}}$ between Ω_α and $\{x \in \mathbb{R}^2 :\ x_1 = \theta x_2,\ x_2 > 0\}$,

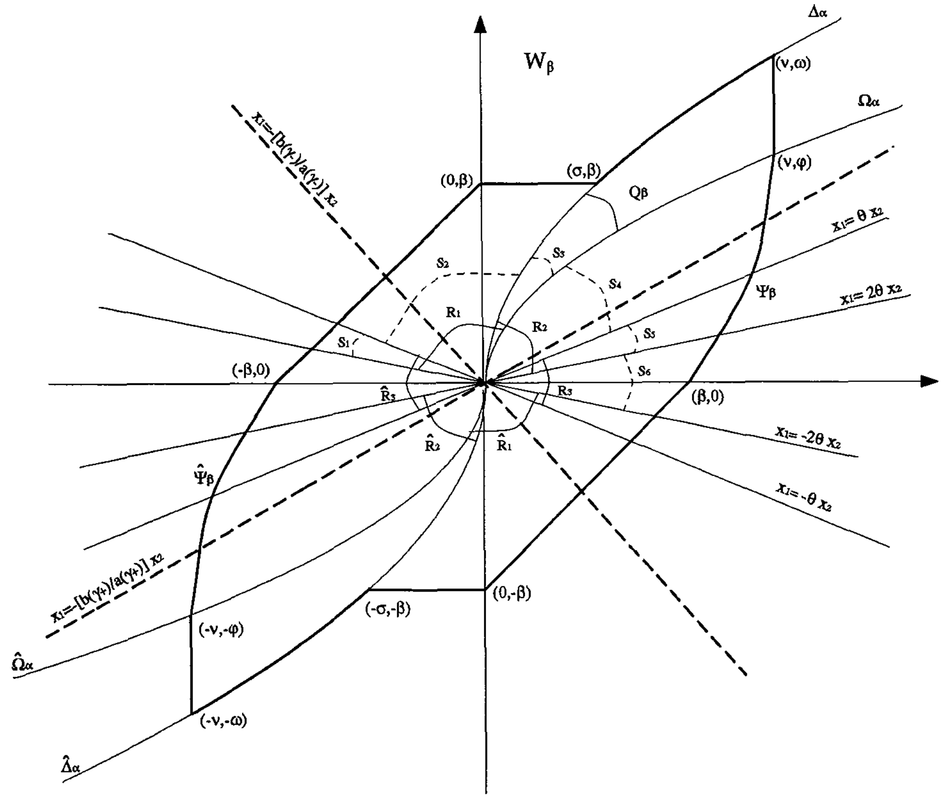

Figure 7.5: Neighborhood W_β for explicit robust asymptotic stabilizer

minus the origin,

$S_5 \triangleq$ closed region in $W_{\bar{\beta}}$ between $\{x \in \mathbb{R}^2 : \ x_1 = \theta x_2, \ x_2 > 0\}$ and $\{x \in \mathbb{R}^2 : \ x_1 = 2\theta x_2, \ x_2 > 0\}$, minus the origin,

$S_6 \triangleq$ closed region in $W_{\bar{\beta}}$ between $\{x \in \mathbb{R}^2 : \ x_1 = \theta x_2, \ x_2 > 0\}$ and $\{x \in \mathbb{R}^2 : \ x_1 = -2\theta x_2, \ x_2 < 0\}$, minus the origin,

as shown in Fig. 7.5. The following lemma will be the key for constructing the desired family of explicit mappings.

Lemma 7.4 *Assume that $a(\cdot) > 0$ on Γ^+, $a(\cdot) < 0$ on Γ^- and $b(\cdot) < 0$ on Γ. Let $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ be a collection of mappings defined from $W_{\bar{\beta}} \backslash \{0\}$ into $[0,1]$, that satisfy the following assertions :*

i) *The mappings of the collection are C^∞ on $W_{\bar{\beta}} \backslash \{0\}$,*

ii) *There exists $m > 0$ such that*

$$\sum_{i=1}^{3} (q_i + \widehat{q}_i)(x) \ \geq \ m, \quad x \in W_{\bar{\beta}} \backslash \{0\}.$$

iii) *$q_1 \equiv 1$ on S_2 (resp. $\widehat{q}_1 \equiv 1$ on $\widehat{S}_2$), while all the other mappings of the collection $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ vanish on this set,*

iv) *$q_2 \equiv 1$ on S_4 (resp. $\widehat{q}_2 \equiv 1$ on $\widehat{S}_4$), while all the other mappings of the collection $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ vanish on this set,*

v) *$q_3 \equiv 1$ on S_6 (resp. $\widehat{q}_3 \equiv 1$ on $\widehat{S}_6$), while all the other mappings of the collection $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ vanish on this set.*

Further, for each $k > 0$, let the mapping $\bar{u}_k : W_{\bar{\beta}} \to \mathbb{R}$ be given by

$$\bar{u}_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ k\,[\,-(x_2)^{1-\alpha}q_1(x) + (x_1+x_2)q_2(x) + x_1 q_3(x) & \\ \quad +(-x_2)^{1-\alpha}\widehat{q}_1(x) + (x_1+x_2)\widehat{q}_2(x) + x_1\widehat{q}_3(x)\,] & \text{otherwise.} \end{cases} \tag{7.33}$$

Then, for each $k > 0$, the mapping $\bar{u}_k$ is continuous and almost C^∞ on $W_{\bar{\beta}}$. Moreover, there exists $k_0' > 0$ such that the feedback law $\bar{u}_{k_0'}$ robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

Proof: By using the fact that the mappings of the collection $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ sum up to a real in $[0, 6]$, together with an argument similar to that used in the proof of Lemma 7.2, it is easily checked that $\bar{u}_k$ is continuous on $W_{\bar{\beta}}$ and C^∞ on $W_{\bar{\beta}}\backslash\{0\}$, for each $k > 0$.

We now define k_0 and β_γ for each γ in Γ, exactly as in the proof of Lemma 7.2, and we set

$$k_0' \triangleq \max\left(\frac{k_0}{m}, k_0\right)$$

By using the assertions of Lemmas C.3 and C.4 together with an argument similar to that used in the proof of Lemma 7.2, it is not hard to see that for each γ in Γ, the sets $\overline{W}_\beta$ and $\overline{Q}_\beta$ are invariant with respect to the vector-field $[f_\gamma, \overline{u}_{k_0'}]^t$ for each $\beta < \beta_\gamma$. It follows from the Poincaré-Bendixson Theorem that $\overline{u}_{k_0'}$ robustly stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$. ■

We now seek a collection of mappings $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ that satisfy the assertions of this last lemma. For each $i = 1, 2, 3$, we let the mappings q_i, $\widehat{q}_i$: $\mathbb{R}^2 \backslash\{0\} \to [0, 1]$ be given by

$$q_1(x) = \begin{cases} \exp\left[\frac{(x_1+\theta x_2)^2}{(x_1+\theta x_2)^2-(\theta x_2)^2}\right], & x \in S_1 \\ 1, & x \in S_2 \\ \exp\left[\frac{\left(x_1-\frac{(x_2)^{1+\alpha}}{2}\right)^2}{\left(x_1-\frac{(x_2)^{1+\alpha}}{2}\right)^2-\left(\frac{(x_2)^{1+\alpha}}{2}\right)^2}\right], & x \in S_3 \\ 0, & \text{otherwise} \end{cases}$$

$$
q_2(x) = \begin{cases} \exp\left[\dfrac{\left(x_1-(x_2)^{1+\alpha}\right)^2}{(x_1-(x_2)^{1+\alpha})^2-\left(\frac{(x_2)^{1+\alpha}}{2}\right)^2}\right], & x \in S_3 \\ 1, & x \in S_4 \\ \exp\left[\frac{(x_1-\theta x_2)^2}{(x_1-\theta x_2)^2-(\theta x_2)^2}\right], & x \in S_5 \\ 0, & otherwise \end{cases}
$$

$$
q_3(x) = \begin{cases} \exp\left[\frac{(x_1-2\theta x_2)^2}{(x_1-2\theta x_2)^2-(\theta x_2)^2}\right], & x \in S_5 \\ 1, & x \in S_6 \\ \exp\left[\frac{(x_1+2\theta x_2)^2}{(x_1+2\theta x_2)^2-(\theta x_2)^2}\right], & x \in \widehat{S}_1 \\ 0, & otherwise \end{cases}
$$

$$
\widehat{q}_1(x) = \begin{cases} \exp\left[\frac{(x_1+\theta x_2)^2}{(x_1+\theta x_2)^2-(\theta x_2)^2}\right], & x \in \widehat{S}_1 \\ 1, & x \in \widehat{S}_2 \\ \exp\left[\dfrac{\left(x_1+\frac{(x_2)^{1+\alpha}}{2}\right)^2}{\left(x_1+\frac{(-x_2)^{1+\alpha}}{2}\right)^2-\left(\frac{(-x_2)^{1+\alpha}}{2}\right)^2}\right], & x \in \widehat{S}_3 \\ 0, & otherwise \end{cases}
$$

$$
\widehat{q}_2(x) = \begin{cases} \exp\left[\dfrac{\left(x_1+(-x_2)^{1+\alpha}\right)^2}{(x_1+(-x_2)^{1+\alpha})^2-\left(\frac{(-x_2)^{1+\alpha}}{2}\right)^2}\right], & x \in \widehat{S}_3 \\ 1, & x \in \widehat{S}_4 \\ \exp\left[\frac{(x_1-\theta x_2)^2}{(x_1-\theta x_2)^2-(\theta x_2)^2}\right], & x \in \widehat{S}_5 \\ 0, & otherwise \end{cases}
$$

$$
\widehat{q}_3(x) = \begin{cases} \exp\left[\frac{(x_1-2\theta x_2)^2}{(x_1-2\theta x_2)^2-(\theta x_2)^2}\right], & x \in \widehat{S}_5 \\ 1, & x \in \widehat{S}_6 \\ \exp\left[\frac{(x_1+2\theta x_2)^2}{(x_1+2\theta x_2)^2-(\theta x_2)^2}\right], & x \in S_1 \\ 0, & otherwise \end{cases}
$$

Lemma 7.5 *The mappings of the collection* $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ *(as defined above) satisfy the assumptions of Lemma 7.4*

Proof: From the definition of the sets S_i and $\widehat{S}_i$ for each $i = 1, \ldots, 6$, it is not hard to see that we have

$$\begin{aligned}
x \in S_1 &\Leftrightarrow -2\theta x_2 \leq x_1 \leq -\theta x_2 \quad \text{and} \quad x_2 > 0, \\
x \in S_2 &\Leftrightarrow -\theta x_2 \leq x_1 \leq \frac{(x_2)^{1+\alpha}}{2} \quad \text{and} \quad x_2 > 0, \\
x \in S_3 &\Leftrightarrow \frac{(x_2)^{1+\alpha}}{2} \leq x_1 \leq (x_2)^{1+\alpha} \quad \text{and} \quad x_2 > 0, \\
x \in S_4 &\Leftrightarrow (x_2)^{1+\alpha} \leq x_1 \leq \theta x_2 \quad \text{and} \quad x_2 > 0, \\
x \in S_5 &\Leftrightarrow \theta x_2 \leq x_1 \leq 2\theta x_2 \quad \text{and} \quad x_2 > 0, \\
x \in S_6 &\Leftrightarrow -\frac{x_1}{2\theta} \leq x_2 \leq \frac{x_1}{2\theta} \quad \text{and} \quad x_1 > 0.
\end{aligned} \tag{7.34}$$

Similar assertions can be obtained for $\widehat{S}_1, \ldots, \widehat{S}_5$ and $\widehat{S}_6$. In view of the definition of the mappings q_i and $\widehat{q}_i$ for each $i = 1, 2, 3$ and Lemma B.7, (7.34) combined with the fact that the sets S_3 and $\widehat{S}_3$ do not contain any point of the form $(x_1, 0)$, imply that the mappings q_i and $\widehat{q}_i$ are C^∞ on $W_{\bar{\beta}} \backslash \{0\}$ for each $i = 1, 2, 3$.

Next, on the sets S_2, S_4, S_6, $\widehat{S}_2$, $\widehat{S}_4$, and $\widehat{S}_6$ the sum $\sum_{i=1}^{3} p_i + \widehat{p}_i$ is identically equal to 1. Further, it is not hard to check from the expression of this sum that for each $j = 1, 3, 5$, there exist two mappings g_j, $h_j : S_j \to \mathbb{R}$ such that g_j does not vanish on S_j, and for each x in S_j we have

$$\sum_{i=1}^{3} (q_i + \widehat{q}_i)(x) = \exp\left[\frac{h_j(x)^2}{h_j(x)^2 - g_j(x)^2}\right] + \exp\left[\frac{(h_j(x) - g_j(x))^2}{(h_j(x) - g_j(x))^2 - g_j(x)^2}\right],$$

with $h_j(x)$ in the interval $[0, g_j(x)]$ (resp. $[g_j(x), 0]$) if $g_j(x)$ is positive (resp. negative). Similarly, for each $j = 1, 3, 5$ there exist mappings $\widehat{g}_j$, $\widehat{h}_j : \widehat{S}_j \to \mathbb{R}$ such that $\widehat{g}_j$ does not vanish on $\widehat{S}_j$ and for each x in $\widehat{S}_j$ we have

$$\sum_{i=1}^{3} (p_j + \widehat{p}_i)(x) = \exp\left[\frac{\widehat{h}_j(x)^2}{\widehat{h}_j(x)^2 - \widehat{g}_j(x)^2}\right] + \exp\left[\frac{\left(\widehat{h}_j(x) - \widehat{g}_j(x)\right)^2}{\left(\widehat{h}_j(x) - \widehat{g}_j(x)\right)^2 - \widehat{g}_j(x)^2}\right],$$

with $\widehat{h}_j(x)$ in the interval $[0, \widehat{g}_j(x)]$ (resp. $[\widehat{g}_j(x), 0]$) if $\widehat{g}_j(x)$ is positive (resp. negative). It follows from Lemma B.8 that the sum

$$\sum_{i=1}^{3} (p_i + \widehat{p}_i)(x) \geq e^{-\frac{1}{3}}, \quad x \in W_{\bar{\beta}} \backslash \{0\}.$$

Therefore, the collection of mappings $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ satisfies the assumptions **(i)** and **(ii)** of Lemma 7.4, and it is easily seen from the definition of the mappings q_i and $\widehat{q}_i$ that the remaining assumptions of the lemma are also fulfilled. ■

Because, the collection $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ (as defined here) satisfies the assumptions of Lemma 7.4, there exists $k_0' > 0$ such that the feedback law $\bar{u}_{k_0'}$ given by the formula (7.33) robustly asymptotically stabilizes the family $\{S(\gamma),\ \gamma \in \Gamma\}$.

Finally, in the case $a(\cdot)$ negative on Γ^-, $a(\cdot)$ negative on Γ^- and $b(\cdot)$ is positive on Γ, as well as in the case $a(\cdot)$ positive on Γ with $b(\cdot)$ either positive or negative on Γ, it follows from the proof of Theorem 7.1, that we can find an explicit robust stabilizer for the family $\{S(\gamma),\ \gamma \in \Gamma\}$, by constructing a collection of mappings $\{q_1, q_2, q_3, \widehat{q}_1, \widehat{q}_2, \widehat{q}_3\}$ similar to that introduced above. Because, these mappings can be easily obtained following the steps of the construction presented here, we will not construct them here.

Chapter 8

Conclusions and Future Research

In this dissertation, we have constructed continuous time-invariant or time-varying controllers that achieve simultaneous stabilization, simultaneous asymptotic stabilization or robust asymptotic stabilization. The results that we obtained can be classified in three categories: The first category encompasses the results of Chapter 2 and 3, where time-invariant simultaneous stabilizers are constructed. The second one consists of the results of Chapter 4 and 5, where time-varying controllers are designed for simultaneous asymptotic stabilization purpose. Finally, the third one comprises the results of Chapter 6 and 7 on the simultaneous asymptotic stabilization and the robust asymptotic stabilization of systems in the plane.

We come back to the results of Chapter 2 and 3: It is shown there that every countable family of control systems with continuous dynamics, is simultaneously stabilizable by means of continuous time-invariant feedback, if each system of the family is asymptotically stabilizable by means of continuous feedback.

If, in addition, each one of the systems of the family is globally asymptotically stabilizable, we established the existence of a continuous feedback law that not only simultaneously stabilizes the family, but also yields boundedness of the trajectories of the closed-loop systems starting at any initial state in $\mathbb{R}^n$.

For any countable family of stabilizable systems, we find two feedback laws that solve the simultaneous stabilization problem. The first one depends on a partition of unity while the second one is simpler and does not involve any partition of unity.

Future work could consist in modifying the construction introduced in Chapter 2, in order to derive time-invariant simultaneous **asymptotic** stabilizers for more general families of systems than those considered in Chapter 6. This could be done by following the ideas introduced in Chapter 6.

It would also be interesting to try to extend the results of Chapter 2 and 3 to uncountable families of systems.

In our quest for feedback laws achieving simultaneous asymptotic stabilization for countable families of systems, we introduced time-varying feedback laws, and enriched our previous approach to the design of time-invariant simultaneous stabilizers. The results that followed concern both the simultaneous asymptotic stabilization of families of LTI systems and those of nonlinear systems.

Given any finite family of LTI systems that are individually asymptotically stabilizable by means of LTI feedback, we established the existence of a continuous time-varying feedback law that simultaneously globally exponentially stabilizes the family. We then provided sufficient conditions for the existence of a continuous time-varying feedback law that simultaneously globally asymptotically stabilizes a countably infinite family of LTI systems.

Furthermore, we obtained sufficient conditions for the existence of a continuous time-varying feedback law that simultaneously locally or globally asymptotically stabilizes a finite family of nonlinear systems. Using these sufficient conditions, we then established the simultaneous asymptotic stabilizability of the elements of a class of pairs of homogeneous systems.

In terms of further research on this second part, many directions can be investigated. The most natural one is of course to try to apply our results to more concrete examples and to practical applications. This is possible because we have provided complete design procedures as well as rather simple expressions for our controllers. From a more theoretical standpoint, it would be interesting to find whether or not the sufficient conditions for the simultaneous asymptotic stabilization of nonlinear systems are also necessary. We also believe that it is possible to modify the construction of the simultaneous local asymptotic stabilizer produced in the proof of Theorem 5.1, so that the domains of attraction of the closed-loop systems be larger.

Another important direction for future research consists in finding conditions under which a feedback law that simultaneously asymptotically stabilizes a given countable family of systems also asymptotically stabilizes a hybrid system whose dynamics switches between the systems of the family. This investigation could begin with testing the robustness to autonomous switchings of the simultaneous asymptotic stabilizers found in this dissertation.

Finally, by modifying the construction of Chapter 2 and by using the Poincaré-Bendixson theory, we managed to construct time-invariant simultaneous **asymptotic** stabilizers for some pairs of systems in the plane [in Chapter 6], together with time-invariant robust asymptotic stabilizers for some families of systems in the plane [in Chapter 7].

More precisely, in chapter 7, we considered a class of parameterized families of nonlinear systems in the plane and discussed their robust asymptotic stabilization around a parameter value at which the corresponding families of linearized systems are not controllable. In most of the cases where these families do not admit a C^1 robust asymptotic stabilizer, we succeeded in constructing a continuous time-invariant robust asymptotic stabilizer. In particular, we introduced a new approach to robust stabilization, where a robust asymptotic stabilizer is considered as a feedback law that simultaneously robustly asymptotically stabilizes a finite number of sub-families of the original parameterized family.

We believe that the idea of viewing the robust asymptotic stabilization as a "simultaneous design" is worth investigating further. Indeed, given a parameterized family of systems, it may be easier to construct robust asymptotic stabilizers for particular sub-families of this family, and then to find a way to produce a robust asymptotic stabilizer for the entire family using these "partial" robust asymptotic stabilizers.

We also feel necessary to continue our investigation on the robust stabilization of parameterized families of systems around parameter values at which the families have some singularities.

To summarize, we have obtained the first results related to a difficult and promising problem, namely that of simultaneous asymptotic stabilization and robust asymptotic stabilization of **uncertain nonlinear** systems. Along our way, we have also derived new results on the simultaneous stabilization of countable families of nonlinear systems as well as on the simultaneous asymptotic stabilization of countable families of LTI systems. Merely continuous time-invariant and time-varying feedback laws have proved to be powerful tools for addressing such problems, and we believe that their use for the control of uncertain systems will yield a great deal of interesting results in the future.

Appendix A

General Facts

In this appendix, we recall more or less easy facts that are used to prove Lemma A.6. This lemma is repeatedly used in Chapter 7.

A.1 Facts on power series

Throughout we let p be a positive integer, $\mathbb{N}$ be the set $\{0, 1, 2, \ldots\}$ and $\mathbf{C}$ be the set of complex numbers. For each z in $\mathbf{C}$ and each $\nu = (n_1, \ldots, n_p)$ in $\mathbb{N}^p$, we set $z^\nu \triangleq z_1^{n_1} z_2^{n_2} \ldots, z_p^{n_p}$. Further, we let K be either equal to $\mathbf{C}$ or $\mathbb{R}$ and we let $r = (r_1, \ldots, r_p)$ be in $(0, \infty)^p$. Finally for each z in $\mathbf{C}$, we let $\overline{z}$ denote its complex conjugate and $|z|$ its modulus.

The following lemma is a direct consequence of the Remark 9.1.6 in [24, pp. 205].

Lemma A.1 *For each ν in $\mathbb{N}^p$, let a_ν and b_ν be in $\mathbf{C}$. Assume that for each z in some set $D \triangleq \{z \in K^p : |z_i| < r_i,\ i = 1, \ldots, p\}$, the series $\sum_{\nu \in \mathbf{N}^p} a_\nu z^\nu$ and $\sum_{\nu \in \mathbf{N}^p} b_\nu z^\nu$ converge absolutely and are equal. Then, we have $a_\nu = b_\nu$, for each ν in $\mathbb{N}^p$.*

Lemma A.2 *For each ν in $\mathbb{N}^p$, let a_ν be in $\mathbf{C}$. Assume that the series $\sum_{\nu \in \mathbf{N}^p} a_\nu z^\nu$ converges absolutely for each z in some set $D \triangleq \{z \in K^p : |z_i| < r_i,\ i = 1, \ldots, p\}$. Then, for each z in D, the series $\sum_{\nu \in \mathbf{N}^p} \overline{a}_\nu z^\nu$ converges absolutely and* *we have*

$$\sum_{\nu\in\mathbb{N}^p} \overline{a}_\nu \overline{z}^\nu = \overline{\sum_{\nu\in\mathbb{N}^p} a_\nu z^\nu}. \tag{A.1}$$

Proof: By assumption the series $\sum_{\nu\in\mathbb{N}^p} |a_\nu||z^\nu|$ converges for each z in P. Because, $|\overline{a}_\nu| = |a_\nu|$, the series $\sum_{\nu\in\mathbb{N}^p} |\overline{a}_\nu||z^\nu|$ converges for each z in D. Furthermore, for each z in D and each n in $\mathbb{N}$, we have

$$\begin{aligned} \Big| \sum_{\nu\in\mathbb{N}^p:|\nu|\le n} \overline{a}_\nu \overline{z}^\nu - \overline{\sum_{\nu\in\mathbb{N}^p} a_\nu z^\nu}\Big| &= \Big| \overline{\sum_{\nu\in\mathbb{N}^p:|\nu|\ge n+1} a_\nu z^\nu}\Big| \\ &= \Big| \sum_{\nu\in\mathbb{N}^p:|\nu|\ge n+1} a_\nu z^\nu\Big|, \end{aligned} \tag{A.2}$$

and assertion (A.1) follows from (A.2) upon letting n go to ∞. ■

Lemma A.3 *For each ν in $\mathbb{N}^p$, let c_ν be in $\mathbf{C}$. Assume that the complex series $\sum_{\nu\in\mathbb{N}^p} c_\nu x^\nu$ converges absolutely to a real number for each x in some set $P \triangleq \{x \in \mathbb{R}^p : |x_i| < r_i,\ i = 1,\ldots,p\}$. Then, for each ν in $\mathbb{N}^p$, the coefficient c_ν belongs to $\mathbb{R}$.*

Proof: The assumptions of the lemma imply that

$$\sum_{\nu\in\mathbb{N}^p} c_\nu x^\nu = \overline{\sum_{\nu\in\mathbb{N}^p} c_\nu x^\nu}, \quad x \in P. \tag{A.3}$$

Because, x is in $\mathbb{R}^p$, the equality (A.3) combined with Lemma A.2 yield

$$\sum_{\nu\in\mathbb{N}^p} c_\nu x^\nu = \sum_{\nu\in\mathbb{N}^p} \overline{c}_\nu x^\nu, \quad x \in P,$$

so that by Lemma A.1, we finally get $c_\nu = \overline{c}_\nu$, for each ν in $\mathbb{N}^p$, and the desired result follows. ■

A.2 Analytic mappings

For the sake of clarity, we produce below an usual definition of an analytic mapping.

Definition A.1 *[24, pp. 207] Let D be an open in K^p. A mapping $f : D \to K$ is analytic on D if for each z' in D there exists a neighborhood D' of z' and a sequence $\{c_\nu : \nu \in \mathbb{N}^p\}$ in K such that*

$$f(z) \;=\; \sum_{\nu \in \mathbf{N}^p} c_\nu (z - z')^\nu, \quad z \in D'. \tag{A.4}$$

If $K = \mathbb{R}$, the mapping f is said to be real analytic on D.

From Abel's lemma [24, pp. 203] we know that if a series $\sum_{\nu \in \mathbf{N}^p} c_\nu (z - z')^\nu$ converges at a point z, then it converges absolutely on a neighborhood of this point. This implies that the series introduced in (A.4) converges absolutely on D' and we easily obtain the following lemma.

Lemma A.4 *Let U be a neighborhood of the origin in $\mathbb{R}^p$ and let $f : U \to \mathbb{R}$ be a real analytic mapping on U given by $f(x) = \sum_{\nu \in \mathbf{N}^p} c_\nu x^\nu, x \in U$. Then, there exists a neighborhood D of the origin in $\mathbf{C}^p$ such that the mapping $F : D \to \mathbf{C}$ defined by*

$$F(z) \triangleq \sum_{\nu \in \mathbf{N}^p} c_\nu z^\nu, \quad z \in D,$$

is well-defined. Moreover, the series $\sum_{\nu \in \mathbf{N}^p} c_\nu z^\nu$ converges absolutely for each z in D.

Next, we give the definition of a Weierstrass polynomial.

Definition A.2 *[45, pp. 157]*

Let D be a neighborhood of the origin in $\mathbf{C} \times \mathbf{C}^{p-1}$. A mapping $q : D \to \mathbf{C}$ is a Weierstrass polynomial if

$$q(w, z) \;=\; w^d + e_1(z) w^{d-1} + \ldots + e_d(z), \quad (w, z) \in D,$$

where the mapping e_i is analytic on a neighborhood of the origin in $\mathbf{C}^{p-1}$, with $e_i(0) = 0$ for each $i = 1, \ldots, d$.

The next theorem is a direct consequence of the comments following Definition 6.1.5 [45, pp. 158] and Corollary 6.1.2 [45, pp. 157] of the Weierstrass Preparation Theorem.

Theorem A.1 *Let D be a neighborhood of the origin in $\mathbf{C} \times \mathbf{C}^{p-1}$ and let $F : D \to \mathbf{C}$ be analytic on D with $F(0,0) = 0$. Assume that the mapping $w \mapsto F(w,0)$ defined on a neighborhood of the origin in $\mathbf{C}$, does not vanish identically. Then, the following holds.*

i) *There exists a unique integer $d > 0$ such that the mapping $w \mapsto \dfrac{f(w,0)}{w^d}$ is analytic at the origin and takes a nonzero value at $w = 0$.*

ii) *There exists a neighborhood D' of the origin in $\mathbf{C} \times \mathbf{C}^{p-1}$ and two unique mappings $h,\ q : D' \to \mathbf{C}$ such that h is analytic at the origin with $h(0) \neq 0$, q is a Weierstrass polynomial and*

$$F(w,z) = h(w,z)\, q(w,z), \quad (w,z) \in D'. \tag{A.5}$$

By using this theorem, we now prove the following lemma.

Lemma A.5 *Let U be a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^{p-1}$, let $f : U \to \mathbb{R}$ be real analytic on U and assume that $f(w,0)$ does not vanish identically on U. Let D be a neighborhood of the origin in $\mathbf{C} \times \mathbf{C}^{p-1}$ and let the mapping $F : D \to \mathbf{C}$ be the extension of f to D as defined in Lemma A.4. Let h and g be the unique analytic mappings obtained through Theorem A.1, i.e. such that we have*

$$F(w,z) \;=\; h(w,z)q(w,z), \tag{A.6}$$

for each (w,z) in some neighborhood D'' of the origin in $\mathbf{C} \times \mathbf{C}^{p-1}$. Recall that $h(0) \neq 0$ and that q satisfies

$$q(w,z) \;=\; w^d + e_1(z)w^{d-1} + \ldots + e_d(z), \quad (z,w) \in D',$$

where for each $i = 1, \ldots, d$, the mapping e_i is analytic on a neighborhood of $z = 0$ in $\mathbf{C}^{p-1}$ with $e_i(0) = 0$.

Then, there exists a neighborhood U' of the origin in $\mathbb{R} \times \mathbb{R}^{p-1}$ such h and e_i are real analytic on U' for each $i = 1, \ldots, d$.

Proof: From (A.6) and the definition of F, it is easily seen that for each (x,y) in some neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^{p-1}$ we have

$$F(x,y) \;=\; f(x,y) \;=\; h(x,y)q(x,y) \;=\; \overline{h(x,y)q(x,y)} \;\in \mathbb{R}. \tag{A.7}$$

Because

$$q(x,y) \;=\; x^d + e_1(y)x^{d-1} + \ldots + e_d(y),$$

we get

$$\overline{q(x,y)} \;=\; x^d + \overline{e_1(y)}x^{d-1} + \ldots + \overline{e_d(y)}$$

and (A.7) yields

$$F(w,z) = \overline{h(w,z)}\left(w^d + \overline{e_1(z)}w^{d-1} + \ldots + \overline{e_d(z)}\right).$$

By uniqueness of the factorization (A.6), it follows that for each (x,y) in some neighborhood of the origin in $\mathbb{R}\times\mathbb{R}^{p-1}$, $h(x,y)$ and $e_i(x,y)$ are real for each $i = 1,\ldots,d$. Thus, Lemma A.3 implies that the mappings h and $e_i, i = 1,\ldots,d$ are real analytic on some neighborhood of the origin in $\mathbb{R}\times\mathbb{R}^{p-1}$. ■

We finally prove the lemma that we actually use in this dissertation.

Lemma A.6 *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be real analytic on U. Let $\alpha \neq 0$ and β be some reals such that $\frac{\partial f(x_1,x_2)}{\partial x_1}|_{(0,0)} = \alpha$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}|_{(0,0)} = \beta$. Then, there exist neighborhoods of the origin U and I in $\mathbb{R}^2$ and $\mathbb{R}$ respectively, and two real analytic mappings $h : U \to \mathbb{R}$ and $\phi : I \to \mathbb{R}$ such that*

$$f(x_1,x_2) = h(x_1,x_2)\,(x_1 - \phi(x_2)), \quad (x_1,x_2) \in U,$$

with $h(0) = \alpha$, $\phi(0) = 0$ and $\phi'(0) = -\frac{\beta}{\alpha}$.

Proof: Let D be a neighborhood of the origin in $\mathbf{C}^2$ and let $F : D \to \mathbf{C}$ be the extension of f obtained through Lemma A.4. Then, the mapping F is analytic on D and we have $F(x_1,x_2) = f(x_1,x_2)$, for each pair of reals (x_1,x_2) in D. As $\frac{\partial f(x_1,x_2)}{\partial x_1}|_{(0,0)} \neq 0$, it follows that $\frac{\partial F(z_1,z_2)}{\partial z_1}|_{(0,0)} \neq 0$, so that the mapping $z_1 \mapsto F(z_1,0)$ does not vanish identically on some neighborhood of $z_1 = 0$. Thus, by Theorem A.1, there exists a neighborhood D' of the origin in $\mathbf{C}^2$, a unique positive integer d and unique mappings h, $e_1,\ldots,$ $e_d : D' \to \mathbf{C}$, analytic on D', such that

$$F(z_1,z_2) = h(z_1,z_2)\left(z_1^d + e_1(z_2)z_1^{d-1} + \ldots + e_d(z_2)\right), \quad (z_1,z_2) \in D', \quad \text{(A.8)}$$

with $h(0) \neq 0$ and $e_i(0) = 0$ for each $i = 1,\ldots,d$. Now, from Lemma A.5 and the definition of F, there exists a neighborhood U of the origin in $\mathbb{R}^2$, such that the mapping h and $e_i, i = 1,\ldots,d$ are real analytic on U and

$$f(x_1,x_2) = h(x_1,x_2)\left(x_1^d + e_1(x_2)x_1^{d-1} + \ldots + e_d(x_2)\right), \quad (x_1,x_2) \in U.$$

Because $\frac{\partial f(x_1,x_2)}{\partial x_1}|_{(0,0)} = \alpha \neq 0$, we obtain that $d = 1$ and $h(0) = \alpha$. Moreover, $\frac{\partial f(x_1,x_2)}{\partial x_2}|_{(0,0)} = \beta$ yields $e_d'(0) = -\frac{\beta}{\alpha}$. The proof is completed upon setting $\phi = e_d$. ■

A.3 Miscellaneous

The following theorem can be found in [24, pp. 162].

Theorem A.2 (Mean Value Theorem) *Let x_0 be in K^p, let $\varepsilon > 0$ and let $f : B_\varepsilon(x_0) \to K$ be differentiable on $B_\varepsilon(x_0)$ [where $B_\varepsilon(x_0)$ is the open ball of radius ε centered at x_0]. Then, we have*

$$\|f(x) - f(x_0)\| \leq \|x - x_0\| \sup_{0 \leq \eta \leq 1} \|f'(x_0 + \eta(x - x_0))\|, \quad x \in B_\varepsilon(x_0).$$

Appendix B

Technical Lemmas for Simultaneous Stabilization

We present in this appendix several technical lemmas that are mainly used to prove the results on the simultaneous stabilization and asymptotic stabilization of countable family of systems.

Lemma B.1 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : D \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function), let W^β denote the set $W^\beta \triangleq \{x \in D : V(x) < \beta\}$. Then, the family $\{W^\beta\}_{\beta>0}$ is a base at the origin such that $W^\alpha \subset W^\beta$ whenever $\alpha \leq \beta$.*

Proof: By definition, $\{W^\beta\}_{\beta>0}$ is a neighborhood base at the origin if and only if for each $\varepsilon > 0$, there exists $\beta > 0$ such that $W^\beta \subset B_\varepsilon(0)$. Suppose now that the assertion of the Lemma does not hold. Then, there exists $\bar{\varepsilon} > 0$ and a sequence $\{x_n\}_{n=1}^\infty$ such that

$$x_n \in W^{\frac{1}{n}} \quad \text{and} \quad x_n \notin B_{\bar{\varepsilon}}(0), \quad n = 1, 2, \ldots . \tag{B.1}$$

By definition of $W^{\frac{1}{n}}$, the sequence $\{x_n\}_{n=1}^\infty$ is included in the set $\overline{D}$ and in the set $\overline{W}^1$. By assumption, either D is bounded and in this case $\overline{D}$ is bounded or $D = \mathbb{R}^n$ and in this case the set $\overline{W}^1$ is bounded because V is radially unbounded. It follows that in both situations $\{x_n\}_{n=1}^\infty$ is included in a compact set so that there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges to some point x_0 in $\overline{D}$. In view of (B.1), we have

$$V(x_{n_k}) < \frac{1}{n_k}, \quad k = 1, 2, \ldots,$$

and the continuity of V yields $V(x_0) = 0$. We conclude from the positive definiteness of V that x_0 is the origin, a contradiction with (B.1) and the fact that $x_{n_k} \to 0$ *as* $k \to \infty$.

Furthermore, it is plain that $W^\alpha \subset W^\beta$ for $\alpha \leq \beta$, which completes the proof. ■

It is easily seen that for the previous lemma to hold, it suffices that the mapping V be continuous and positive definite on D. We do not actually need the existence of a mapping $f : D \to \mathbb{R}^n$ such that

$$\nabla V(x)\, f(x) < 0, \quad x \in \overline{D}\backslash\{0\}.$$

Lemma B.2 *Let D be an open subset of $\mathbb{R}^n$ and let U and F be respectively open and closed subsets of $\mathbb{R}^n$ included in D. Let $f : [0, \infty) \times D \to \mathbb{R}^n$ be continuous and let (S) denote the system $\dot{x} = f(t, x)$. Throughout, let $x(\cdot, x_0, t_0) : [t_0, \infty) \to \mathbb{R}^n$ denote the trajectory of (S) starting from a given point x_0 in D at a given time $t_0 \geq 0$.*

i) *Let $t_0 \geq 0$, let x_0 be in D and let $\bar{t}$ be in $[t_0, \infty)$. If $x(\bar{t}, x_0, t_0)$ lies in U then there exists $\bar{h} > 0$ such that*

$$x(\bar{t} + h, x_0, t_0) \in U$$

for each h in $(-\bar{h}, \bar{h})$ if $\bar{t} > t_0$ or for each h in $[0, \bar{h})$ if $\bar{t} = t_0$.

ii) *Let $t_0 \geq 0$, let x_0 be in F and assume that the trajectory $x(\cdot, x_0, t_0)$ of (S), does not remain in F for ever. Then, there exists $\widehat{t} \geq t_0$ and $\bar{h} > 0$ such that*

$$x(\widehat{t}, x_0, t_0) \in \partial F \quad \text{and} \quad x(\widehat{t} + h, x_0) \notin F, \quad h \in (0, \bar{h}).$$

iii) *Let $t_0 \geq 0$, let x_0 be in U and assume that the trajectory $x(\cdot, x_0, t_0)$ of the system (S), does not remain in U for ever. Then, there exists $t_1 \geq t_0$ such that*

$$x(t_1, x_0, t_0) \in \partial U, \qquad \text{with} \qquad x(t, x_0, t_0) \in U, \quad t \in [t_0, t_1).$$

Proof: **(i)** Let $t_0 \geq 0$, let x_0 be in D and let $\bar{t}$ be in (t_0, ∞). Further, assume that $x(\bar{t}, x_0, t_0)$ lies in U. Because $x(\cdot, x_0, t_0)$ is continuous we obtain that $x^{-1}(U, x_0, t_0)$ is open in $[t_0, \infty)$ and it follows that $x^{-1}(U, x_0, t_0)$ is an union of open disjoint intervals in $[t_0, \infty)$ i.e.,

$$x^{-1}(U, x_0, t_0) = [t_0, b_0) \cup \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda).$$

Because $x(\bar{t}, x_0, t_0) \in U$, we either have $\bar{t} \in (t_0, b_0)$ or there exists $\bar{\lambda}$ in Λ such that $\bar{t} \in (a_{\bar{\lambda}}, b_{\bar{\lambda}})$. This yields the existence of $\bar{h} > 0$ such that $\bar{t} + h \in x^{-1}(U, x_0, t_0)$ for each h in $(-\bar{h}, \bar{h})$. In other words, $x(\bar{t} + h, x_0)$ lies in U for each h in $(-\bar{h}, \bar{h})$.

If $\bar{t} = t_0$, the argument above can be easily modified in order to prove the claim.

(ii) Let $t_0 \geq 0$ and let x_0 be in F. If we assume that the trajectory $x(\cdot, x_0, t_0)$ of (S) does not remain in F, then there exists $\bar{t} > t_0$ such that $x(\bar{t}, x_0, t_0) \in F^c$. Because the set $x^{-1}(F^c, x_0, t_0)$ is the union of open disjoint intervals in $[t_0, \infty)$ i.e.,

$$x^{-1}(F^c, x_0, t_0) \;=\; \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda), \tag{B.2}$$

there exists $\bar{\lambda}$ in Λ such that $\bar{t} \in (a_{\bar{\lambda}}, b_{\bar{\lambda}})$. From (B.2), $x(a_{\bar{\lambda}}, x_0, t_0)$ lies in F, and there exists $\bar{h} > 0$ such that

$$x(a_{\bar{\lambda}} + h, x_0, t_0) \;\in\; F^c, \quad h \in (0, \bar{h}).$$

By **(i)**, this implies that $x(a_{\bar{\lambda}}, x_0, t_0) \notin \operatorname{int}(F)$ and it follows that $x(a_{\bar{\lambda}}, x_0, t_0) \in \partial F$. Therefore **(ii)** holds with $\hat{t} = a_{\bar{\lambda}}$.

(iii) Let $t_0 \geq 0$ and let x_0 be in U. Then, $x^{-1}(U, x_0, t_0)$ is an union of open disjoint intervals in $[t_0, \infty)$ i.e.,

$$x^{-1}(U, x_0, t_0) \;=\; [t_0, b_0) \cup \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda),$$

and it follows that $x(b_0, x_0, t_0)$ lies in U^c with $x(b_0 - h, x_0, t_0)$ in U for $h > 0$ small enough. By **(i)**, this implies that $x(b_0, x_0, t_0)$ lies in $U^c \backslash \operatorname{int}(U^c)$ or equivalently in ∂U (follows from elementary topology [68, Proposition 2, pp. 172]), which completes the proof. ■

Lemma B.3 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function) for some arbitrary system $(S) : \dot{x} = g(x)$, where the mapping $g : \overline{D} \to \mathbb{R}^n$ is continuous on $\overline{D}$. Let β be in the interval $(0, \inf_{x \in \partial D} V(x))$ and define the set U by setting $U \triangleq D \cap V^{-1}([0, \beta))$. Then, the following holds:*

i) $\overline{U} = D \cap V^{-1}([0, \beta])$.

ii) *Let x_0 be in $\overline{U}$ and let the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If the trajectory $x(\cdot, x_0)$ of the system $(S') : \dot{x} = f(x)$ does not remain in $\overline{U}$ forever, then there exists $\hat{t} \geq 0$ and $\hat{h} > 0$ such that*

$$\begin{aligned} x(\hat{t}, x_0) &\in \partial U \quad \text{with} \quad V(x(\hat{t}, x_0)) = \beta, \\ x(\hat{t} + h, x_0) &\notin V^{-1}([0, \beta]), \quad h \in (0, \hat{h}), \\ x(\hat{t} + h, x_0) &\in D, \quad h \in (0, \hat{h}). \end{aligned}$$

Proof: (i) : From the definition of U and elementary topology we find

$$\overline{U} \subset \overline{D} \cap \overline{V^{-1}([0,\beta))}. \tag{B.3}$$

We now show that $\overline{U} \subset D$. Because, this inclusion trivially holds if $D = \mathbb{R}^n$, we assume that D is bounded and we prove the claim by contradiction. If the inclusion $\overline{U} \subset D$ does not hold, then because $\overline{U} \subset \overline{D}$, there exists y in $(\partial D) \cap \overline{U}$. Thus, we get

$$V(y) \geq \inf_{x \in \partial D} V(x) > \beta,$$

a contradiction with the fact that $V(y) \leq \beta$ [follows from (B.3)]. Therefore, we have $\overline{U} \subset D$ and it follows from [14, Proposition 5 pp. 24] combined with the definition of U that

$$\overline{U} = D \cap \overline{V^{-1}([0,\beta))}. \tag{B.4}$$

On the other hand, it is easily checked from the continuity of V that

$$D \cap \overline{V^{-1}([0,\beta))} \subset D \cap V^{-1}([0,\beta]). \tag{B.5}$$

We now show that these two sets are actually identical, by proving first that

$$D \cap V^{-1}(\beta) \subset D \cap \overline{V^{-1}([0,\beta))}.$$

If $D \cap V^{-1}(\beta) = \emptyset$, the inclusion trivially holds. Otherwise, let x_0 in $D \cap V^{-1}(\beta)$ and consider the trajectory $x(\cdot, x_0)$ of (S) starting from x_0 at time $t = 0$. Because V is a Lyapunov function for the system (S), we have

$$\frac{d}{dt} V(x(t,x_0)) < 0, \quad t \geq 0,$$

so that

$$V(x(t,x_0)) < V(x_0) = \beta, \quad t > 0. \tag{B.6}$$

By continuity of the mapping $x(\cdot, x_0) : [0,\infty) \to \mathbb{R}^n$, the sequence $\{x(\frac{1}{n}, x_0)\}_{n=1}^{\infty}$, converges to x_0 as n tends to ∞. Since x_0 lies in the open set D, there exists an integer N such that $x(\frac{1}{n}, x_0)$ belongs to D for each $n = N, N+1, \ldots$. This together with (B.6) yields

$$x(\frac{1}{n}, x_0) \in D \cap V^{-1}([0,\beta)), \quad n = N, N+1, \ldots,$$

and it follows that x_0 belongs to $\overline{D \cap V^{-1}([0,\beta))}$. In view of (B.4), this implies that x_0 lies in the set $D \cap \overline{V^{-1}([0,\beta))}$ and we get

$$D \cap V^{-1}(\beta) \subset D \cap \overline{V^{-1}([0,\beta))}. \tag{B.7}$$

Because we clearly have $D \cap V^{-1}([0,\beta)) \subset D \cap \overline{V^{-1}([0,\beta))}$, we conclude from (B.7) that

$$D \cap V^{-1}([0,\beta]) \subset D \cap \overline{V^{-1}([0,\beta))}.$$

The assertion **i)** follows from this last inclusion, (B.4) and (B.5).

(ii) : Let x_0 be in $\overline{U}$, and assume that the trajectory $x(\cdot, x_0)$ of (S') that starts from x_0 at time $t = 0$, does not remain in $\overline{U}$ forever. Then, by Lemma B.2, there exists $\widehat{t} \geq 0$ and $\bar{h} > 0$ such that

$$x(\widehat{t}, x_0) \in \partial U \tag{B.8}$$

and

$$x(\widehat{t} + h, x_0) \notin \overline{U}, \quad h \in (0, \bar{h}). \tag{B.9}$$

From **(i)** and the definition of U, we find

$$\partial U = D \cap \Big(V^{-1}([0,\beta]) \setminus V^{-1}([0,\beta))\Big), \tag{B.10}$$

so that (B.8) yields

$$V(x(\widehat{t}, x_0)) = \beta. \tag{B.11}$$

Moreover, in view of (B.8) and (B.10), $x(\widehat{t}, x_0)$ lies in the open set D. Thus, by Lemma B.2 **(ii)**, there exists $\widehat{h}$ in $(0, \bar{h})$ such that

$$x(\widehat{t} + h, x_0) \in D, \quad h \in (0, \widehat{h}). \tag{B.12}$$

Finally, upon noting that from (B.9) we have

$$x(\widehat{t} + h, x_0) \notin D \cap V^{-1}([0,\beta]), \quad h \in (0, \widehat{h}),$$

relation (B.12) yields

$$x(\widehat{t} + h, x_0) \notin V^{-1}([0,\beta]), \quad h \in (0, \widehat{h}). \tag{B.13}$$

The assertion **(ii)** now follows from (B.8), (B.11), (B.12), and (B.13). ■

We note that the previous proof does not yield the assertions **(i)** and **(ii)** of Lemma B.3 if the mapping V is only a positive definite mapping without being a Lyapunov function for some system.

Lemma B.4 *The following two assertions hold:*

i) *Let $\{U_i\}_{i=0}^{\infty}$ be a base at the origin such that*

$$U_i \supset \overline{U}_{i+1}, \quad i = 0, 1, \dots .$$

Then, the family of open sets $\{U_{i-1} \setminus \overline{U}_{i+1}\}_{i=1}^{\infty}$ is an open cover of $U_0 \setminus \{0\}$.

ii) *Let $\{U_i\}_{i \in \mathbb{Z}}$ be a base at the origin such that*

$$U_i \supset \overline{U}_{i+1}, \quad i \in \mathbb{Z}.$$

Assume that for each x in $\mathbb{R}^n \setminus \{0\}$ there exists an integer i such that x lies in U_i. Then, the family of open sets $\{U_{i-1} \setminus \overline{U}_{i+1}\}_{i \in \mathbb{Z}}$ is an open cover of $\mathbb{R}^n \setminus \{0\}$.

Proof: We only prove **(i)** as the proof of **(ii)** is similar. We note that we have a sequence of nested neighborhoods

$$U_0 \supset U_1 \supset U_2 \supset \dots , \tag{B.14}$$

such that each neighborhood contains the closure of the neighborhood that follows.

Let x_0 be in $U_0 \backslash \{0\}$. Because the family $\{U_i\}_{i=0}^{\infty}$ is a base at the origin composed of nested neighborhoods, it is easily checked that there exists i in $\{1, 2, \dots\}$ such that

$$x_0 \in U_i \qquad \text{and} \qquad x_0 \notin U_{i+1}.$$

Besides, in view of (B.14), we have

$$U_i \backslash \overline{U}_{i+2} = U_i \cap \overline{U}_{i+2}^c \supset U_i \cap U_{i+1}^c.$$

Because x_0 lies in $U_i \cap \overline{U}_{i+1}^c$, we obtain that x_0 belongs to $U_i \backslash \overline{U}_{i+2}$ and it follows that the family $\{U_{i-1} \backslash \overline{U}_{i+1}\}_{i=1}^{\infty}$ is an open cover of $U_0 \backslash \{0\}$. ∎

Lemma B.5 *The mapping* $q : \mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty) \to [0, 1]$ *defined by*

$$q(x, a, \alpha, \beta) = \begin{cases} e^{\frac{(x-a-\alpha)^2}{(x-a-\alpha)^2-\alpha^2}} & \textit{if } x \in (a, a+\alpha) \\ e^{\frac{(x-a-\alpha)^2}{(x-a-\alpha)^2-\beta^2}} & \textit{if } x \in [a+\alpha, a+\alpha+\beta) \\ 0 \,, & \textit{otherwise} \end{cases} ,$$

is C^{∞} *on its domain of definition.*

Proof: Let the mappings $h, g : \mathbb{R} \times \mathbb{R} \times (0, \infty)$ be given by

$$h(y, a, \alpha) = \begin{cases} 0 \,, & \text{if} \quad y \leq a \\ e^{\frac{(y-a-\alpha)^2}{(y-a-\alpha)^2-\alpha^2}}, & \text{if} \quad y \in (a, a+\alpha) \\ 1 \,, & \text{if} \quad y \geq a+\alpha \end{cases} ,$$

and

$$g(y, b, \beta) = \begin{cases} 1 \,, & \text{if} \quad y \leq b \\ e^{\frac{(y-b)^2}{(y-b)^2-\beta^2}}, & \text{if} \quad y \in (b, b+\beta) \\ 0 \,, & \text{if} \quad y \geq b+\beta \end{cases} ,$$

respectively. Because we have

$$q(y, a, \alpha, \beta) = h(y, a, \alpha) + g(y, a+\alpha, \beta) - 1,$$

for each (y, a, α, β) in $\mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty)$, it suffices to show that h and g are C^∞ in order to prove that q is C^∞.

The mapping h is C^∞ :

We fix (x', a', α') in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ and we study the local behavior of the mapping h around (x', a', α'). For each positive reals δx, δa and $\delta\alpha$, with $\delta\alpha$ in $(0, \alpha)$, we set

$$\begin{aligned} I_{\delta x} &\triangleq (x' - \delta x, x' + \delta x), \\ I_{\delta a} &\triangleq (a' - \delta a, a' + \delta a), \\ I_{\delta\alpha} &\triangleq (\alpha' - \delta\alpha, \alpha' + \delta\alpha), \end{aligned}$$

and

$$U_{\delta x, \delta a, \delta\alpha} \triangleq (x' - \delta x, x' + \delta x) \times (a' - \delta a, a' + \delta a) \times (\alpha' - \delta\alpha, \alpha' + \delta\alpha).$$

We now distinguish several cases:

i) If $x' < a'$, then the positiveness of $a' - x'$ yields the existence of δx and δa such that

$$x' + \delta x \ < \ a' - \delta a,$$

and it follows that for all (x, a) in $I_{\delta x} \times I_{\delta a}$ we have $x < a$. By picking $\delta\alpha$ in $(0, \alpha')$, the previous comment yields

$$h(x, a, \alpha) \ = \ 0, \quad (x, a, \alpha) \in I_{\delta x} \times I_{\delta a} \times I_{\delta\alpha},$$

so that h is C^∞ on the neighborhood $U_{\delta x, \delta a, \delta\alpha}$ of (x', a', α').

ii) If $x' > a' + \alpha'$, by a similar argument, we find δx, δa and $\delta\alpha$ satisfying

$$x \ \geq \ a + \alpha, \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta\alpha}.$$

Thus, we get

$$h(x, a, \alpha) = 1, \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta\alpha},$$

and it is plain that h is C^∞ on $U_{\delta x, \delta a, \delta\alpha}$.

iii) Similarly, if x' is in $(a', a' + \alpha')$, there exists a neighborhood $U_{\delta x, \delta a, \delta\alpha}$ such that

$$x \ \in \ (a, a + \alpha), \quad (x, a, \alpha) \in U_{\delta x, \delta a, \delta\alpha},$$

and we obtain from the definition of h that it is C^∞ on $U_{\delta x, \delta a, \delta\alpha}$.

iv) If $x' = a'$, we pick $\delta\alpha$ in $(0, \alpha')$. Then, we select δx and δa satisfying

$$x' + \delta x < a' - \delta a \qquad \text{i.e.} \qquad \delta x + \delta a \ < \ \alpha' - \delta\alpha.$$

This implies that for each (x, a, α) in $U_{\delta x, \delta a, \delta \alpha}$, we have $x < a + \alpha$ and we get

$$h(x, a, \alpha) = \begin{cases} 0, & \text{if } x \leq a \\ e^{\frac{(x-a-\alpha)^2}{(x-a-\alpha)^2 - \alpha^2}}, & \text{if } x > a \end{cases}.$$

We now let the mapping $\bar{h} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be given by

$$\bar{h}(x, \alpha) = \begin{cases} 0, & \text{if } x \leq 0 \\ e^{\frac{(x-\alpha)^2}{(x-\alpha)^2 - \alpha^2}}, & \text{if } x > 0 \end{cases}.$$

It is readily seen from this definition that $\bar{h}$ is C^∞ on both sets $(-\infty, 0) \times (0, +\infty)$ and $(0, +\infty) \times (0, +\infty)$. Next, we fix $n = 1, 2, \ldots$ and $\bar{\alpha} > 0$. Because, $\frac{1}{x^m} e^{-\frac{1}{x}}$ converges to 0 as x tends to 0^+, for each m in $\mathbb{Z}$, we easily obtain that for each $\bar{\alpha} > 0$, each n-th order partial derivative of the mapping $(x, \alpha) \mapsto e^{\frac{(x-\alpha)^2}{(x-\alpha)^2 - \alpha^2}}$, converges to 0 as (x, α) tends to $(0^+, \bar{\alpha})$. Therefore, we can extend by continuity each n-th order partial derivatives of $\bar{h}$ at the point $(0, \bar{\alpha})$, and it follows from [24, Lemma (8.12.8) pp. 185], that $\bar{h}$ is C^∞ on $\mathbb{R} \times (0, \infty)$. Because $h(x, a, \alpha) = \bar{h}(x - a, \alpha)$, the previous result implies that h is C^∞ on $U_{\delta x, \delta a, \delta \alpha}$.

(v) If $x' = a' + \alpha'$, we pick $\delta\alpha$ in $(0, \alpha')$ and we select δa and δx such that

$$a' + \delta a \ < \ x' - \delta x \qquad \text{i.e.} \qquad \delta a + \delta x < \alpha'.$$

This implies that $a < x$ for each (x, a, α) in $U_{\delta x, \delta a, \delta \alpha}$ and it follows that

$$h(x, a, \alpha) = \begin{cases} e^{\frac{(x-a-\alpha)^2}{(x-a-\alpha)^2 - \alpha^2}}, & \text{if} \quad x \in (a, a + \alpha) \\ 1\,, & \text{if} \quad x \geq a + \alpha \end{cases}.$$

Let the mapping $\bar{h} : \mathbb{R}^2 \to [0, \infty)$ be given by

$$\bar{h}(y, z) = \begin{cases} e^{\frac{z^2}{z^2 - (y-z)^2}}, & \text{if} \quad z < 0 \\ 1\,, & \text{if} \quad z \geq 0 \end{cases}.$$

It is then easily seen that for each $n = 1, 2, \ldots$ and each $\bar{y}$ in $\mathbb{R}$, every n-th order partial derivative of $\bar{h}$ converges to 0 as (y, z) tends to $(\bar{y}, 0^-)$. We now extend by continuity each n-th order partial derivatives of $\bar{h}$ and we obtain that $\bar{h}$ is C^∞ on $\mathbb{R}^2$. Because $h(x, a, \alpha) = \bar{h}(x - a, x - a - \alpha)$ for each (x, a, α) in $U_{\delta x, \delta a, \delta \alpha}$, it follows that h is C^∞ on this neighborhood.

We conclude from **(i)**, **(ii)**, **(iii)**, **(iv)** and **(v)**, that h is C^∞ on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.

The mappings g and q are C^∞ :

It is not hard to check that we have

$$g(x, a, \alpha) \;=\; h(-x, -a-\alpha, \alpha), \quad (x, a, \alpha) \in \mathbb{R} \times \mathbb{R} \times (0, \infty),$$

which implies that g is C^∞ on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.

Therefore, the mapping q is C^∞ on $\mathbb{R} \times \mathbb{R} \times (0, \infty) \times (0, \infty)$. ■

The following result is a direct consequence of the previous lemma.

Lemma B.6 *Let the mappings $a, b, c : [0, \infty) \to \mathbb{R}$ be C^k on $[0, \infty)$ with*

$$a(t) \;<\; b(t) \;<\; c(t), \quad t \in \mathbb{R},$$

and let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^k mapping. Then, the mapping $q : [0, \infty) \times \mathbb{R}^n \to [0, 1]$ given by

$$q(t, x) = \begin{cases} e^{\frac{(V(x)-b(t))^2}{(V(x)-b(t))^2-(b(t)-a(t))^2}} & \text{if } V(x) \in (a(t), b(t)) \\ e^{\frac{(V(x)-b(t))^2}{(V(x)-b(t))^2-(c(t)-b(t))^2}} & \text{if } V(x) \in [b(t), c(t)) \\ 0\,, & \text{otherwise} \end{cases}, \tag{B.15}$$

for each (t, x) in $[0, \infty) \times \mathbb{R}^n$, is C^k on $[0, \infty) \times \mathbb{R}^n$.

Because the proof of the following lemma is similar to that of Lemma B.5, we omit it.

Lemma B.7 *The mapping $q : \mathbb{R} \times \mathbb{R} \times (0, \infty)^3 \to [0, 1]$ defined by*

$$q(x, a, \alpha, \beta, \gamma) \;=\; \begin{cases} e^{\frac{(x-a-\alpha)^2}{(x-a-\alpha)^2-\alpha^2}} & \text{if } x \in (a, a+\alpha) \\ 1 & \text{if } x \in [a+\alpha, a+\alpha+\beta) \\ e^{\frac{(x-a-\alpha-\beta)^2}{(x-a-\alpha-\beta)^2-\gamma^2}} & \text{if } x \in [a+\alpha+\beta, a+\alpha+\beta+\gamma) \\ 0\,, & \text{otherwise} \end{cases},$$

is C^∞.

Lemma B.8 *Let a be a nonzero real and let f be a mapping given by*

$$f(x) = e^{\frac{x^2}{x^2-a^2}} \;+\; e^{\frac{(x-a)^2}{(x-a)^2-a^2}},$$

for each x in $[0, a]$ (resp. $[a, 0]$) if a is positive (resp. negative). Then we have $f(x) \geq e^{-\frac{1}{3}}$ for each x in $[0, a]$ (resp. $[a, 0]$).

Proof: We only prove the case $a > 0$ as the proof is similar in the case $a < 0$.

We assume that a is positive and we note that $f(a-x) = f(x)$ for each x in $[0,a]$. Thus, the mapping f is symmetric with respect to the set $\{x = \frac{a}{2}\}$, and it suffices to prove that $f(x) \geq e^{-\frac{1}{3}}$ for each x in $[0, \frac{a}{2}]$. By studying the mapping $e^{\frac{x^2}{x^2-a^2}}$ as x lies in $[0, \frac{a}{2}]$, we find that this mapping takes values greater than $e^{-\frac{1}{3}}$, and because $e^{\frac{(x-a)^2}{(x-a)^2-a^2}}$ is non-negative the lemma follows. ■

Lemma B.9 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to [0,\infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function). Let I be a closed and bounded interval of $[0,\infty)$ and let $b : [0,\infty) \to [0, \inf_{x \in \partial D} V(x)\,)$ be a continuous mapping. Then, the set*

$$D \cap V^{-1}(\,b(I)\,) \;\triangleq\; D \cap \left(\bigcup_{\tau \in I} V^{-1}(\,b(\tau)\,) \right)$$

is compact.

Proof: Because I is a closed and bounded interval, the continuity of b yields

$$D \cap V^{-1}(\,b(I)\,) = D \cap V^{-1}(\,[\min_{\tau\in I} b(\tau), \max_{\tau\in I} b(\tau)]\,).$$

Thus, if $D = \mathbb{R}^n$, the continuity of V implies that $D \cap V^{-1}(\,b(I)\,)$ is closed. Moreover, because V is radially unbounded in this case, we obtain that this set is also bounded, thus compact.

We now assume that D is bounded. If $\min_{\tau\in I} b(\tau)$ is equal to 0, we get

$$D \cap V^{-1}(\,b(I)\,) = D \cap V^{-1}(\,[0, \max_{\tau\in I} b(\tau)]\,), \tag{B.16}$$

while we otherwise have

$$\begin{aligned} D \cap V^{-1}(\,b(I)\,) \;&= \\ &\left(D \cap V^{-1}(\,[0, \max_{\tau\in I} b(\tau)]\,)\right) \setminus \left(D \cap V^{-1}(\,[0, \min_{\tau\in I} b(\tau))\,)\right). \end{aligned} \tag{B.17}$$

Because the real $\max_{\tau\in I} b(\tau)$ belongs to the interval $(0, \inf_{x\in\partial D} V(x)\,)$, it follows from Lemma B.3 **(i)** (applied with V, D and $\max_{\tau\in I} b(\tau)$), that $D \cap V^{-1}(\,[0, \max_{\tau\in I} b(\tau)]\,)$ is compact. Therefore, in view of either (B.16) or (B.17), the set $D \cap V^{-1}(\,b(I)\,)$ is compact. ■

Lemma B.10 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$). Let K be a compact subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^n$. Further, let $V : \overline{D} \to \mathbb{R}$ be a Lyapunov function (resp. a radially unbounded Lyapunov function) and let β be in $(0, \inf_{x \in \partial D} V(x))$. Then the following holds.*

i) *If we have the inclusion $D \cap V^{-1}([0, \beta)) \supset K$, then there exists α in $(0, \beta)$ such that*

$$D \cap V^{-1}([0, \alpha)) \supset K.$$

ii) *If we have the inclusion $U \supset D \cap V^{-1}([0, \beta])$, then there exists γ in $(\beta, \inf_{x \in \partial D} V(x))$ such that*

$$U \supset D \cap V^{-1}([0, \gamma]).$$

Proof: (i) We prove the claim by a contradiction argument. For each $n = 1, 2, \ldots$, we set $\alpha_n \triangleq \beta - \frac{1}{n}$ and we assume that the lemma does not hold. Then, for each $n = 1, 2, \ldots$, there exists x_n in K such that

$$x_n \notin D \cap V^{-1}([0, \alpha_n)). \tag{B.18}$$

By compactness of K, the sequence $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ that converges to some point x_0 in K. It follows from the assumptions that

$$x_0 \in D \cap V^{-1}([0, \beta)). \tag{B.19}$$

Because D is open, there exists an integer I such that $x_{n_i} \in D$, $i = I, I+1, \ldots$ and the assertion (B.18) implies that

$$V(x_{n_i}) \geq \alpha_{n_i}, \quad i = I, I+1, \ldots.$$

By continuity of V, this yields $V(x_0) \geq \beta$, a contradiction with (B.19). The assertion **(i)** then follows by contradiction.

(ii) For each $n = 1, 2, \ldots$, we set $\gamma_n \triangleq \beta + \frac{1}{n}$ and we assume that the claim does not hold. Then, for each $n = 1, 2, \ldots$, there exists x_n satisfying

$$x_n \in D \cap V^{-1}([0, \gamma_n)), \tag{B.20}$$

with

$$x_n \notin U. \tag{B.21}$$

Because we have $D \cap V^{-1}([0, \beta]) \subset U$, the assertion (B.20) and (B.21) yield

$$x_n \in D \cap V^{-1}((\beta, \gamma_n)), \quad n = 1, 2, \ldots,$$

and it follows that

$$\beta \; < \; V(x_n) \; < \; \gamma_n, \quad n = 1, 2, \ldots . \tag{B.22}$$

Since $\beta \in (0, \inf_{x \in \partial D} V(x)\,)$, there exists an integer N such that $\gamma_N \in (0, \inf_{x \in \partial D} V(x)\,)$. Moreover, it is plain that $x_n \in D \cap V^{-1}([0, \gamma_N])$, for each $n = N, N+1, \ldots$. By Lemma B.9 (applied with D, V, $I = [0, \gamma_N]$, and b = identity), the set $D \cap V^{-1}([0, \gamma_N])$ is compact. Thus there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ that converges to a point x_0 in $D \cap V^{-1}([0, \gamma_N])$. By (B.22), we get

$$V(x_0) \; = \; \beta,$$

and it follows that x_0 lies in $D \cap V^{-1}([0, \beta])$. Thus, the assumption of the lemma implies that $x_0 \in U$. Now, the set U being open, there exists an integer I such that

$$x_{n_i} \; \in \; U, \quad i = I, I+1, \ldots ,$$

which contradicts (B.21) and yields **(ii)**. ■

Lemma B.11 *Let D_1 and D_2 be two bounded neighborhood of the origin (resp. $D_1 = D_2 = \mathbb{R}^n$), and let $V_1 : \overline{D_1} \to [0, \infty)$ and $V_2 : \overline{D_2} \to [0, \infty)$ be two Lyapunov functions (resp. radially unbounded Lyapunov functions). Further, let β_1 and β_2 be in $(0, \inf_{x \in \partial D_1} V_1(x)\,)$ and $(0, \inf_{x \in \partial D_2} V_2(x)\,)$ respectively, and assume that*

$$D_1 \cap V_1^{-1}(\,[0, \beta_1)\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \beta_2]\,). \tag{B.23}$$

Then, there exist α_1 in $(0, \beta_1)$ and γ_2 in $(\beta_2, \inf_{x \in \partial D_2} V_2(x)\,)$ such that

$$D_1 \cap V_1^{-1}(\,[0, \alpha_1)\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \gamma_2]\,) \tag{B.24}$$

Proof: By Lemma B.9 (applied with D_2, V_2, $I = [0, \beta_2]$ and b = identity), the set $D_2 \cap V_2^{-1}(\,[0, \beta_2]\,)$ is compact. Thus, Lemma B.10 **(i)** applied with (B.23), yields the existence of α_1 in $(0, \beta_1)$ such that

$$D_1 \cap V_1^{-1}(\,[0, \alpha_1)\,) \;\supset\; D_2 \cap V_2^{-1}(\,[0, \beta_2]\,). \tag{B.25}$$

Because $D_1 \cap V_1^{-1}(\,[0, \alpha_1)\,)$ is open, by using Lemma B.10 **(ii)** with (B.25), we obtain γ_2 in $(\beta_2, \inf_{x \in \partial D_2} V_2(x)\,)$ such that (B.24) holds, and the lemma is proved. ■

The following lemma follows from the previous one by continuity of b_1 and b_2.

Lemma B.12 *Let D_1 and D_2 be two bounded neighborhoods of the origin (resp. $D_1 = D_2 = \mathbb{R}^n$), and let $V_1 : \overline{D}_1 \to [0, \infty)$ and $V_2 : \overline{D}_2 \to [0, \infty)$ be two Lyapunov functions (resp. two radially unbounded Lyapunov functions). Further, let b_1 and b_2 be two continuous mappings from $[0, \infty)$ into $(0, \inf_{x \in \partial D_1} V_1(x))$ and $(0, \inf_{x \in \partial D_2} V_2(x))$ respectively. Finally, let $t \geq 0$ be such that*

$$D_1 \cap V_1^{-1}([0, b_1(t))) \supset D_2 \cap V_2^{-1}([0, b_2(t)]). \tag{B.26}$$

Then, if $t > 0$, there exists δ_t in $(0, t)$ such that

$$D_1 \cap V_1^{-1}([0, \min_{\tau \in [t-\delta_t, t+\delta_t]} b_1(\tau))) \supset D_2 \cap V_2^{-1}([0, \max_{\tau \in [t-\delta_t, t+\delta_t]} b_2(\tau)]), \tag{B.27}$$

and if $t = 0$ there exists $\delta_t > 0$ such that

$$D_1 \cap V_1^{-1}([0, \min_{\tau \in [0, \delta_t]} b_1(\tau))) \supset D_2 \cap V_2^{-1}([0, \max_{\tau \in [0, \delta_t]} b_2(\tau)]). \tag{B.28}$$

Proof: Fix $t > 0$ satisfying the inclusion (B.26). Because $b_1(t)$ and $b_2(t)$ lie in $(0, \inf_{x \in \partial D_1} V_1(x))$ and $(0, \inf_{x \in \partial D_2} V_2(x))$ respectively, Lemma B.11 [applied with D_1, D_2, V_1, V_2, $\beta_1 = b_1(t)$ and $\beta_2 = b_2(t)$] yields the existence of α_1 in $(0, b_1(t))$ and γ_2 in $(b_2(t), \inf_{x \in \partial D_2} V_2(x))$ such that

$$D_1 \cap V_1^{-1}([0, \alpha_1)) \supset D_2 \cap V_2^{-1}([0, \gamma_2]). \tag{B.29}$$

By continuity of b_1 and b_2 at t, there exists δ_t in $(0, t)$ such that

$$b_1(\tau) \geq \alpha_1 \qquad \text{with} \qquad \gamma_2 \geq b_2(\tau'), \quad \tau, \tau' \in [t - \delta_t, t + \delta_t],$$

and we get from the inclusion (B.29) that

$$D_1 \cap V_1^{-1}([0, b_1(\tau))) \supset D_2 \cap V_2^{-1}([0, b_2(\tau')]), \quad \tau, \tau' \in [t - \delta_t, t + \delta_t]. \tag{B.30}$$

The mappings b_1 and b_2 being continuous on the compact set $[t - \delta_t, t + \delta_t]$, they achieve their minimum and maximum on this interval, and the desired inclusion (B.27) follows easily from (B.30).

The arguments above easily transpose to accommodate the case $t = 0$ by replacing the intervals $[t - \delta_t, t + \delta_t]$ by $[0, \delta_t]$, which yields the inclusion (B.28) and completes the proof. ■

Lemma B.13 *Let r be in $(0,1)$, let $\theta > 0$ and let $\delta > 0$. Let $\bar{\beta}$ be defined by*

$$\bar{\beta} \triangleq \left[\theta \left(\frac{1-r}{2\delta}\right)^{\frac{1}{2}} e^{\frac{1}{2}}\right]^{\frac{1}{1-r}}.$$

Then, for each β in $(0,\bar{\beta})$ the mapping $h_\beta : [0,\infty) \to (0,\infty)$ given by

$$h_\beta(t) = \begin{cases} \beta, & t \leq \frac{\beta^{1-r}}{\theta} \\ \beta e^{-\delta(t-\frac{\beta^{1-r}}{\theta})^2}, & t > \frac{\beta^{1-r}}{\theta} \end{cases},$$

satisfies

$$\dot{h}_\beta(t) \geq -\theta\, h_\beta(t)^r, \quad t \geq 0. \tag{B.31}$$

Moreover, for each β and γ positive such that $\gamma > \beta$, we have $h_\gamma(t) > h_\beta(t)$ for each $t \geq 0$.

Proof: Let β be in $(0,\bar{\beta}]$. If $t \leq \dfrac{\beta^{1-r}}{\theta}$, the claim clearly holds, Thus, let $t > \dfrac{\beta^{1-r}}{\theta}$ and set $y = t - \frac{\beta^{1-r}}{\theta}$. It is easily checked that

$$f(y) = \frac{\dot{h}_\beta(t)}{h_\beta(t)^r} = -2\delta\beta^{1-r} y e^{-\delta(1-r)y^2}.$$

Now, by studying the mapping f as y lies in $[0,+\infty)$, we get

$$f(y) \geq -2\beta^{1-r}\left(\frac{\delta}{2(1-r)}\right)^{\frac{1}{2}} e^{-\frac{1}{2}}, \quad y \geq 0,$$

and the desired inequality (B.31) will hold if the right hand side of this last inequality is greater than the real $-\theta$, or equivalently if

$$\beta \leq \left[\theta \left(\frac{1-r}{2\delta}\right)^{\frac{1}{2}} e^{\frac{1}{2}}\right]^{\frac{1}{1-r}} = \bar{\beta},$$

which is satisfied by assumption.

Further, let β and γ be positive reals with $\gamma > \beta$. Then, we clearly have

$$h_\gamma(t) > h_\beta(t), \quad t \leq \frac{\gamma^{1-r}}{\theta}.$$

Moreover if $t > \dfrac{\gamma^{1-r}}{\theta}$ we get

$$\frac{h_\gamma(t)}{h_\beta(t)} = \left(\frac{\gamma}{\beta}\right) e^{\delta\left[2t - \frac{\gamma^{1-r}}{\theta} - \frac{\beta^{1-r}}{\theta}\right]\left[\frac{\gamma^{1-r}}{\theta} - \frac{\beta^{1-r}}{\theta}\right]} > 1,$$

and it follows that we have $h_\gamma(t) > h_\beta(t)$ for each $t \geq 0$. ■

Appendix C

Technical Lemmas for Robust Asymptotic Stabilization

Throughout this appendix, we let I be some interval in $\mathbb{R}$ and for each γ in I, we let the mapping $f_\gamma : \mathbb{R}^2 \to \mathbb{R}$ be real analytic on a neighborhood of the origin. Further, we let the mappings $a, b : I \to \mathbb{R}$ be given by

$$a(\gamma) = \frac{\partial f_\gamma(x_1, x_2)}{\partial x_1}|_{(0,0)} \qquad \text{and} \qquad b(\gamma) = \frac{\partial f_\gamma(x_1, x_2)}{\partial x_2}|_{(0,0)},$$

and we assume that $a(\cdot)$ and $b(\cdot)$ do not vanish on I.

For each γ in I, Lemma A.6 combined with the enforced assumptions imply that there exist some neighborhoods of the origin U_γ and I_γ in $\mathbb{R}^2$ and $\mathbb{R}$ respectively, and some real analytic mappings $h_\gamma : U_\gamma \to \mathbb{R}$ and $\phi_\gamma : I_\gamma \to \mathbb{R}$ such that

$$f_\gamma(x_1, x_2) \;=\; h_\gamma(x_1, x_2)\,(x_1 - \phi_\gamma(x_2)), \quad (x_1, x_2) \in U_\gamma, \tag{C.1}$$

with $h_\gamma(0) = a(\gamma)$, $\phi_\gamma(0) = 0$ and $\phi'_\gamma(0) = -\dfrac{b(\gamma)}{a(\gamma)}$. Therefore, for each γ in Γ we have

$$h_\gamma(x) = a(\gamma) + \widehat{h}_\gamma(x) \qquad \text{with} \qquad \widehat{h}_\gamma(x) \to 0 \;\; as \;\; x \to 0, \tag{C.2}$$

and

$$\phi_\gamma(x) \;=\; -\frac{b(\gamma)}{a(\gamma)} x_2 + \widehat{\phi}_\gamma(x_2) \qquad \text{with} \qquad \frac{\widehat{\phi}_\gamma(x_2)}{x_2} \to 0 \;\; as \;\; x_2 \to 0. \tag{C.3}$$

We now recall a few definitions that were introduced in Chapter 7.

For a subset Y of $\mathbb{R}^2$, we let $\widehat{Y}$ denote its symmetric with respect to the origin and Y^s its symmetric with respect to the x_1-axis, i.e,

$$\widehat{Y} \;\triangleq\; \{-y : \; y \in Y\} \qquad \text{and} \qquad Y^s \;\triangleq\; \{(y_1, -y_2) \in \mathbb{R}^2 : \; (y_1, y_2) \in Y\}$$

Finally, for each positive reals α and β, we define

$$\begin{aligned}\Delta_\alpha &\triangleq \{x \in \mathbb{R}^2 : \ x_1 = \frac{(x_2)^{1+\alpha}}{2}, \ x_2 > 0\} \\ \Psi_\beta &\triangleq \{x \in \mathbb{R}^2 : \ x_2 = x_1 \ln(\frac{x_1}{\beta}), \ x_1 > \beta\}\end{aligned}$$

We are now able to present here several technical lemmas that were used in the proofs of Theorems 7.1 and 7.2.

Lemma C.1 *Let $\eta > 0$ and $\theta \geq 1$. Then, for each $\beta > 0$, the intersection of the sets $\Omega \triangleq \{x \in \mathbb{R}^2 : x_1 = \eta\,(x_2)^\theta,\ x_2 > 0\}$ and Ψ_β contains a unique point $[h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$. Moreover, we have $h(\beta) \to 0$ as $\beta \to 0$.*

Proof: It is easily seen that the point $x = (x_1, x_2)$ belongs to the set $\Omega \cap \Psi_\beta$ if and only if $x_1 > \beta$ with

$$(\frac{1}{\eta})^{\frac{1}{\theta}} x_1^{\frac{1}{\theta}} = x_1 \ln(\frac{x_1}{\beta}). \tag{C.4}$$

Let $f : (\beta, \infty) \to [0, \infty)$ be given by

$$f(x_1) \ = \ (\frac{1}{\eta})^{-\frac{1}{\theta}} x_1^{\frac{\theta-1}{\theta}} \ln(\frac{x_1}{\beta}) - 1 \,.$$

As we have $f'(x_1) = (\frac{1}{\eta})^{\frac{-1}{\theta}} \left[\frac{\theta-1}{\theta}\ln(\frac{x_1}{\beta}) + 1\right] x_1^{\frac{-1}{\theta}}$, we conclude that f strictly increases from -1 to $+\infty$ as x goes from β to $+\infty$, and it follows that for each $\beta > 0$, the mapping f vanishes at a unique value $x_1 = h(\beta)$. Because (C.4) holds if and only if $f(x_1) = 0$, it follows that the point (x_1, x_2) lies in $\Omega \cap \Psi_\beta$ if and only if $(x_1, x_2) = [h(\beta), h(\beta)\ln(\frac{h(\beta)}{\beta})]$.

Next, we prove that the mapping $h : (0, \infty) \to (0, \infty)$ converges to 0 as β tends to 0. Set $k \triangleq (\frac{1}{\eta})^{\frac{1}{\theta}}$ and let $l : (0, \infty) \to (0, \infty)$ be defined by

$$l(x_1) \ = \ x_1 e^{-k(x_1)^{\frac{1-\theta}{\theta}}}.$$

We easily obtain

$$l'(x_1) \ = \ e^{-k(x_1)^{\frac{1-\theta}{\theta}}}[1 + k\frac{\theta - 1}{\theta}(x_1)^{\frac{1-\theta}{\theta}}]$$

so that l strictly increases from 0 to $+\infty$ as x_1 goes from 0 to $+\infty$. Therefore, the inverse mapping l^{-1} or equivalently h strictly increases from 0 to $+\infty$ as β goes from 0 to $+\infty$. Hence the lemma. ∎

Lemma C.2 *For each positive reals β and β', the curve Ψ_β is on the left of $\Psi_{\beta'}$ whenever $\beta < \beta'$.*

Proof: Let (x_1, x_2) be in Ψ_β and (x_1', x_2') be in $\Psi_{\beta'}$ with $x_2 = x_2'$. To prove the lemma, we show by contradiction that $x_1' < x_1$. Since $x_2 = x_2'$, we obtain from the definition of Ψ_β and $\Psi_{\beta'}$ that

$$\frac{x_1}{x_1'} = \frac{\ln(\frac{x_1'}{\beta'})}{\ln(\frac{x_1}{\beta})} \tag{C.5}$$

By assuming that $x_1' \geq x_1$ or equivalently that $\frac{x_1}{x_1'} \leq 1$, equality (C.5) yields $\ln(\frac{x_1'\beta}{x_1\beta'}) \leq 0$ and it follows that $\frac{x_1'}{x_1} \leq \frac{\beta'}{\beta} < 1$, a contradiction with the inequality $x_1' \geq x_1$. Therefore, we have $x_1' < x_1$ so that the lemma is proved. ■

The next two lemmas were needed in establishing Claim 1 in the proof of Theorem 7.1.

Lemma C.3 *Assume that $b(\cdot)$ is negative on I and $a(\cdot)$ is either positive on I or negative on I. Further, assume that $a(\cdot)$ and $b(\cdot)$ are bounded on I. Let θ, μ and $\bar{\beta}$ be positive reals, let α be in $(0,1)$, let η be in $(0, 1-\alpha]$ and let η' be in $(0,1]$. Then, there exists $k_1 > 0$ and for each γ in I there exists a neighborhood U_γ of the origin in $\mathbb{R}^2$ such that for each k in $[k_1, \infty)$ the following holds:*

i) a) *For each $\beta > 0$ and each x in $U_\gamma \cap \Psi_\beta$, the vector $[f_\gamma(x), k(x_1 + x_2)]^t$ points towards the left of Ψ_β,*

b) *For each $\beta > 0$ and each x in $U_\gamma \cap \Psi_\beta$, the vector $[f_\gamma(x), kx_1]^t$ points towards the left of Ψ_β if x is below the half-line $\{x \in \mathbb{R}^2 : \ x_1 = \theta x_2, \ x_2 > 0\}$.*

ii) *For each x in $U_\gamma \cap \Delta_\alpha$, the vector $[f_\gamma(x), -k(x_2)^\eta]^t$ points into the region below Δ_α if x is above the half-line $\{x \in \mathbb{R}^2 : \ x_1 = \theta x_2, \ x_2 > 0\}$.*

iii) *For each β in $(0, \bar{\beta}]$ and for each x in $U_\gamma \cap D_\beta$, where D_β denotes the segment $D_\beta \triangleq \{x \in \mathbb{R}^2 : \ \mu x_2 - x_1 = -\beta, \ x_1 \in [0, \beta]\}$, we have:*

a) *If x is above the line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 < 0\}$, then the vector $[f_\gamma(x), kx_1]^t$ points into the region above D_β.*

b) *If x is below the line $\{x \in \mathbb{R}^2 : \ x_1 = -2\theta x_2, \ x_2 < 0\}$, then the vector $[f_\gamma(x), k(-x_2)^{\eta'}]^t$ points into the region above D_β.*

Proof:

(i) Fix γ in I and $\beta > 0$. For each x in Ψ_β, the tangent to Ψ_β at x is given by $\frac{dx_1}{dx_2} = \frac{x_1}{x_1 + x_2}$. If x is in Ψ_β with $f_\gamma(x) \leq 0$, then **(a)** and **(b)** are immediate. Further, because for x in Ψ_β close enough of the origin, we have $f_\gamma(x) \leq 0$ whenever $a(\gamma) < 0$, throughout this paragraph we implicitly assume that $a(\gamma) > 0$ and that $f_\gamma(x) > 0$ for all the points x that we consider. From (C.2) and (C.3), we obtain the existence of a neighborhood V_γ such that we have

$$h_\gamma(x) > 0 \qquad \text{and} \qquad -\phi_\gamma(x) < 0, \quad x \in V_\gamma,$$

and it follows from (C.1) that

$$\frac{f_\gamma(x)}{k(x_1 + x_2)} \leq \frac{h_\gamma(x)}{k}\frac{x_1}{x_1 + x_2} \qquad \text{and} \qquad \frac{f_\gamma(x)}{kx_1} \leq \frac{h_\gamma(x)}{k}, \tag{C.6}$$

for each x in $V_\gamma \cap \Psi_\beta$. Now, from the uniform boundedness of $a(.)$ on I there exists $L > 0$ satisfying

$$a(\gamma) \leq L, \quad \gamma \in I,$$

and for each γ in I, the analyticity of h_γ yields the existence of a neighborhood of the origin $U_\gamma \subset V_\gamma$ such that

$$|h_\gamma(x)| < 2L, \quad x \in U_\gamma.$$

Thus, in view of (C.6), we get

$$\frac{f_\gamma(x)}{k(x_1 + x_2)} \leq \frac{2L}{k}\frac{x_1}{x_1 + x_2} \qquad \text{and} \qquad \frac{f_\gamma(x)}{kx_1} \leq \frac{2L}{k}, \quad x \in U_\gamma \cap \Psi_\beta. \tag{C.7}$$

The first inequality in (C.7) yields **(a)**. Furthermore, if x is below the line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2,\ x_2 > 0\}$, then the tangent to Ψ_β at x is greater than $\frac{\theta}{1+\theta}$ and claim **(b)** follows from the second inequality in (C.7).

(ii) Let x be in Δ_α and let γ be in I. If $f_\gamma(x) \geq 0$, then the claim clearly holds. On the other hand, if $f_\gamma(x) < 0$, by setting $x_1 = x_2^{1+\alpha}$ it is easily checked from (C.3) that

$$\begin{aligned} \frac{-f_\gamma(x)}{k(x_2)^\eta} &= \frac{-h_\gamma(x)(x_1 - \phi_\gamma(x))}{k(x_2)^\eta} \\ &= -\frac{h_\gamma(x)}{k}\left[\frac{b(\gamma)}{a(\gamma)} + x_2^\alpha - \frac{\widehat{\phi}_\gamma(x_2)}{x_2}\right] x_2^{1-\eta}, \end{aligned} \tag{C.8}$$

and that $-\frac{h_\gamma(x)}{k}\left[\frac{b(\gamma)}{a(\gamma)} + x_2^\alpha - \frac{\widehat{\phi}_\gamma(x_2)}{x_2}\right]$ converges to $\frac{-b(\gamma)}{k}$ as x tends to 0. Let L be a positive real such that $|b(\gamma)| < L, \gamma \in I$, then for each γ in I there exists a neighborhood of the origin V_γ such that we have

$$-\frac{h_\gamma(x)}{k}\left[\frac{b(\gamma)}{a(\gamma)}+x_2^\alpha-\frac{\widehat{\phi}_\gamma(x_2)}{x_2}\right] \leq \frac{2L}{k}, \quad x \in V_\gamma. \tag{C.9}$$

The claim now follows from (C.8) and (C.9), upon noticing that for $x_2 > 0$ close enough to the origin we have $x_2^{1-\eta} < x_2^\alpha$.

(iii) For each γ in I, because f_γ is analytic at the origin there exists $\varepsilon(\gamma) > 0$, such that we have

$$f_\gamma'(x) \;=\; a(\gamma)+b(\gamma)+g_\gamma(x) \quad \text{with} \quad g_\gamma(y) \to 0 \;\; as \;\; y \to 0,$$

for each x in $B_{\varepsilon(\gamma)}(0)$. The uniform boundedness of $a(\cdot)$ and $b(\cdot)$ on I implies that there exist a positive real L and for each γ in I there exists $\delta(\gamma) > 0$ such that $\delta(\gamma) < \varepsilon(\gamma)$ and

$$|f_\gamma'(x)| \;\leq\; L, \quad x \in B_{\delta(\gamma)}(0).$$

Thus, the Mean Value Theorem A.2 yields

$$\begin{aligned} |f_\gamma(x)| \;&\leq\; (|x_1|+|x_2|) \sup_{0\leq\eta\leq 1} |f_\gamma'(\eta x)| \\ &\leq\; L\,(|x_1|+|x_2|), \quad x \in B_{\delta(\gamma)}(0). \end{aligned} \tag{C.10}$$

Let $\beta > 0$ and let x be in D_β. Because **(a)** and **(b)** clearly hold if $f_\gamma(x) \leq 0$, we assume that $f_\gamma(x) > 0$. If x is above the line $\{x \in \mathbb{R}^2 : x_1 = -\theta x_2,\, x_2 < 0\}$, then $-x_2 < \frac{x_1}{\theta}$ and from (C.10) we get

$$\begin{aligned} \frac{f_\gamma(x)}{kx_1} \;&\leq\; \frac{L(x_1-x_2)}{kx_1} \\ &\leq\; \frac{L(\theta+1)}{k\theta}, \quad x \in B_{\delta(\gamma)}(0). \end{aligned} \tag{C.11}$$

On the other hand if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2,\, x_2 < 0\}$ we have $x_1 \in [0,\beta]$ and $\frac{\beta}{\mu+2\theta} \leq -x_2 \leq \frac{\beta}{\mu}$ and it follows from (C.10) that

$$\frac{f_\gamma(x)}{k(-x_2)^{\eta'}} \;\leq\; \frac{L}{k}(1+\frac{1}{\mu})(\mu+2\theta)^{\eta'}\beta^{1-\eta'}, \quad x \in B_{\delta(\gamma)}(0). \tag{C.12}$$

Claim **(a)** and **(b)**, follow easily from the fact that for k large enough the right hand sides of (C.11) and (C.12) are smaller than μ, for each γ in I. ∎

Lemma C.4 *Let I be some subset of $\mathbb{R}$. Assume that $b(\cdot)$ is negative on I and $a(\cdot)$ is either positive on I or negative on I. Suppose that $a(\cdot)$ and $b(\cdot)$ are* *bounded on I. Let θ, μ and $\bar{\beta}$ be positive reals, let α be in $(0,1)$, let η be in $(0, 1-\alpha]$ and let η' be in $(0,1]$. Then, there exists $k_2 > 0$ and for each γ in I there exists a neighborhood of the origin U_γ in $\mathbb{R}^2$ such that for each k in $[k_2, +\infty)$ the following holds:*

i) a) *For each $\beta > 0$ and each x in $U_\gamma \cap \widehat{\Psi}_\beta$, the vector $[f_\gamma(x), k(x_1 + x_2)]^t$ points towards the right of $\widehat{\Psi}_\beta$.*

b) *For each $\beta > 0$ and each x in $U_\gamma \cap \widehat{\Psi}_\beta$, the vector $[f_\gamma(x), kx_1]^t$ points towards the right of $\widehat{\Psi}_\beta$ if x is above the line $\{x \in \mathbb{R}^2 : \; x_1 = \theta x_2, \; x_2 < 0\}$.*

ii) *For each x in $U_\gamma \cap \widehat{\Delta}_\alpha$, the vector $[f_\gamma(x), k(-x_2)^\eta]^t$ points into the region above $\widehat{\Delta}_\alpha$ if x is below the half-line $\{x \in \mathbb{R}^2 : \; x_1 = \theta x_2, \, x_2 < 0\}$.*

iii) *For each β in $(0, \bar{\beta}]$ and each x in $U_\gamma \cap \widehat{D}_\beta$ where $\widehat{D}_\beta$ denotes the segment $\widehat{D}_\beta \triangleq \{x \in \mathbb{R}^2 : \; \mu x_2 - x_1 = \beta, \, x_1 \in [-\beta, 0]\,\}$, we have:*

a) *If x is below the line $\{x \in \mathbb{R}^2 : \; x_1 = -\theta x_2, \, x_2 > 0\}$, then the vector $[f_\gamma(x), kx_1]^t$ points into the region below $\widehat{D}_\beta$.*

b) *If x is above the line $\{x \in \mathbb{R}^2 : \; x_1 = -2\theta x_2, \, x_2 \geq 0\}$, then the vector $[f_\gamma(x), -k(-x_2)^{\eta'}]^t$ points into the region below $\widehat{D}_\beta$.*

When $b(\cdot)$ is positive on some subset I of $\mathbb{R}$, the assertions **(i)**, **(ii)** and **(iii) (a)** of Lemma C.3 and C.4 translate to the following lemma.

Lemma C.5 *Let I be some subset of $\mathbb{R}$. Assume that $b(\cdot)$ is positive on I and $a(\cdot)$ is either positive on I or negative on I. Suppose that $a(\cdot)$ and $b(\cdot)$ are bounded on I. Let θ and μ be positive reals, let α be in $(0,1)$ and let η be in $(0, 1-\alpha]$. Then, there exists $k_3 > 0$ and for each γ in I there exists a neighborhood of the origin U_γ in $\mathbb{R}^2$, such that for each k in $[k_3, +\infty)$ the following holds.*

i) a) *For each $\beta > 0$ and each x in $U_\gamma \cap \Psi^s_\beta$, the vector $[f_\gamma(x), k(-x_1 + x_2)]^t$ points towards the left of Ψ^s_β.*

b) *For each $\beta > 0$ and each x in $U_\gamma \cap \Psi^s_\beta$, the vector $[f_\gamma(x), -kx_1]^t$ points towards the left of Ψ^s_β if x is above the half-line $\{x \in \mathbb{R}^2 : \; x_1 = -\theta x_2, \; x_2 < 0\}$.*

ii) *For each x in $U_\gamma \cap \Delta^s_\alpha$, the vector $[f_\gamma(x), k(-x_2)^\eta]^t$ points into the region above Δ^s_α if x is below the half-line $\{x \in \mathbb{R}^2 : \; x_1 = -\theta x_2, \, x_2 < 0\}$.*

iii) *For each β in $(0, \bar{\beta}]$ and each x in D_β where D_β denotes the segment $D_\beta \triangleq \{x \in \mathbb{R}^2 : \; \mu x_2 + x_1 = \beta, \, x_1 \in [0, \beta]\,\}$, the vector $[f_\gamma(x), -kx_1]^t$ points into the region below D_β if x is below the half-line $\{x \in \mathbb{R}^2 : \; x_1 = \theta x_2, \, x_2 > 0\}$.*

iv) a) *For each $\beta > 0$ and each x in $U_\gamma \cap \widehat{\Psi}^s_\beta$, the vector $[f_\gamma(x), k(-x_1 + x_2)]^t$ points towards the right of $\widehat{\Psi}^s_\beta$.*

b) *For each $\beta > 0$ and each x in $U_\gamma \cap \widehat{\Psi}^s_\beta$, the vector $[f_\gamma(x), -kx_1]^t$ points towards the right of $\widehat{\Psi}^s_\beta$ if x is below the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 > 0\}$.*

v) *For each x in $U_\gamma \cap \widehat{\Delta}^s_\alpha$, the vector $[f_\gamma(x), -k(-x_2)^\eta]^t$ points into the region below $\widehat{\Delta}^s_\alpha$ if x is above the half-line $\{x \in \mathbb{R}^2 : \ x_1 = -\theta x_2, \ x_2 > 0\}$.*

vi) *For each β in $(0, \bar{\beta}]$ and each x in $U_\gamma \cap \widehat{D}_\beta$ where $\widehat{D}_\beta$ denotes the segment $\widehat{D}_\beta \triangleq \{x \in \mathbb{R}^2 : \ \mu x_2 + x_1 = -\beta, \ x_1 \in [-\beta, 0]\}$, the vector $[f_\gamma(x), -kx_1]^t$ points into the region above $\widehat{D}_\beta$ if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = \theta x_2, \ x_2 < 0\}$.*

Finally the last two lemmas are used in the proof of Theorem 7.2.

Lemma C.6 *Assume that $a(\cdot)$ is positive on some subset I of $\mathbb{R}$ and that $b(\cdot)$ is negative on I. Let θ and δ be fixed positive reals with $2\theta < \delta$. Then, for each γ in I, there exists a neighborhood V_γ of the origin such that for each $\tau > 0$ the following holds:*

i) *For each x in V_γ and in the half-line $D_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 - \tau, \ x_1 > 0\}$ the vector $[f_\gamma(x), -(x_1)^2]^t$ points into the region above D_τ, if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2, \ x_2 < 0\}$.*

ii) *For each x in V_γ and in the half-line $\widehat{D}_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 + \tau, \ x_1 < 0\}$ the vector $[f_\gamma(x), (x_1)^2]^t$ points into the region below $\widehat{D}_\tau$, if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = -2\theta x_2, \ x_2 > 0\}$.*

Proof: We only prove (**i**) as the proof of (**ii**) is similar. Fix γ in I. Since $a(\gamma) > 0$ and $b(\gamma) < 0$, it follows from (C.2) and (C.3) that there exists a neighborhood U_γ of the origin such that $f_\gamma(x) > 0$ with $h_\gamma(x) > 0$ and $-\phi_\gamma(x_2) > 0$ for all x in $U_\gamma \cap D_\tau$ and all $\tau > 0$. Therefore, we have

$$
\begin{aligned}
\frac{(x_1)^2}{f_\gamma(x)} &= \frac{x_1^2}{h_\gamma(x)(x_1 - R_\gamma(x_2))} \\
&\leq \frac{x_1}{h_\gamma(x)}, \quad x \in U_\gamma \cap D_\tau, \quad \tau > 0.
\end{aligned}
$$

This together with the fact that $h_\gamma(0) = a(\gamma) > 0$, imply that $\dfrac{(x_1)^2}{f_\gamma(x)}$ converges to 0 as x_1 tends to 0. Thus there exists a neighborhood of the origin $V_\gamma \subset U_\gamma$ such that if x is in $V_\gamma \cap D_\tau$ then we have $\dfrac{(x_1)^2}{f_\gamma(x)} \leq \delta$, for all $\tau > 0$, and the claim follows. ∎

Lemma C.7 *Assume that $a(\cdot)$ is positive on some subset I of $\mathbb{R}$ and that $b(\cdot)$ is positive on I. Let θ and δ be positive reals with $2\theta < \delta$. Then, for each γ in I, there exists a neighborhood U_γ of the origin such that for each $\tau > 0$ the following holds.*

i) *For each x in U_γ and in the half-line $D_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 - \tau,\ x_1 > 0\}$, the vector $[f_\gamma(x), (x_1)^2]^t$ points into the region below D_τ, if x is above the half-line $\{x \in \mathbb{R}^2 : x_1 = 2\theta\, x_2,\ x_2 > 0\}$.*

ii) *For each x in U_γ and in the half-line $\widehat{D}_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 + \tau,\ x_1 < 0\}$, the vector $[f_\gamma(x), -(x_1)^2]^t$ points into the region above $\widehat{D}_\tau$ if x is below the half-line $\{x \in \mathbb{R}^2 : x_1 = 2\theta\, x_2,\ x_2 < 0\}$.*

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