The Partially Observed Stochastic Minimum Principle

by

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1. INTRODUCTION.

Various proofs have been given of the minimum principle satisfied by an optimal control in a partially observed stochastic control problem. See, for example, the papers by Bensoussan [1], Elliott [5], Haussmann [7], and the recent paper [9] by Haussmann in which the adjoint process is identified. The simple case of a partially observed Markov chain is discussed in the University of Maryland lecture notes [6] of the second author.

We show in this article how a minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. The results of Bismuit [2], [3] and Kunita [10], on stochastic flows enable us to compute in an easy and explicit way the change in the the cost due to a 'strong variation' of the an optimal control. The only technical difficulty is the justification of the differentiation. As we wished to exhibit the simplification obtained by using the ideas of stochastic flows the result is not proved under the weakest possible hypotheses. Finally, in Section 6, we show how Bensoussan's minimum principle follows from our result if the drift coefficient is differentiable in the control variable.

2. DYNAMICS.

Suppose the state of the system is described by a stochastic differential equation

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t,$$

$$\xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leq t \leq T. \tag{2.1}$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. We shall make the following assumptions:

A_1: x_0 is given; if x_0 is a random variable and P_0 its distribution the situation when $\int |x|^q P_0(dx) < \infty$ for some $q > n + 1$ can be treated, as in [9], by including an extra integration with respect to P_0.

A_2: $f : [0, T] \times R^d \times U \rightarrow R^d$ is Borel measurable, continuous in u for each (t, x), continuously differentiable in x and for some constant K

$$(1 + |x|)^{-1} |f(t, x, u)| + |f_x(t, x, u)| \leq K_1.$$

A_3: $g : [0, T] \times R^d \rightarrow R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some constant K_2

$$|g(t, x)| + |g_x(t, x)| \leq K_2.$$

The observation process is given by

$$dy_t = h(\xi_t)dt + dv_t$$

$$y_t \in R^m, \quad y_0 = 0, \quad 0 \leq t \leq T. \tag{2.2}$$

In the above equations $w = (w^1, \ldots, w^n)$ and $v = (v^1, \ldots, v^d)$ are independent Brownian motions. We also assume

A_4: $h : R^d \rightarrow R^m$ is Borel measurable, continuously differentiable in x, and for some constant K_3

$$|h(t, x)| + |h_x(t, x)| \leq K_3.$$

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REMARKS 2.1. These hypotheses can be weakened. For example, in A_4, h can be allowed linear growth in x. Because g is bounded a delicate argument then implies the exponential Z of (2.3) is in some L^p space, $1 < p < \infty$. (See, for example, Theorem 2.2 of [8]). However, when h is bounded Z is in all the L^p spaces, (see Lemma 2.3). Also, if we require f to have linear growth in u then the set of control values U can be unbounded as in [9]. Our objective, however, is not the greatest generality but to demonstrate the simplicity of the techniques of stochastic flows.

Let $\hat{P}$ denote Wiener measure on the $C([0,T], R^n)$ and μ denote Wiener measure on $C([0,T], R^m)$. Consider the space $\Omega = C([0,T], R^n) \times C([0,T], R^m)$ with coordinate functions (x_t, y_t) and define Wiener measure P on Ω by

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. Write $Y = \{Y_t\}$ for the right continuous complete filtration on $C([0,T], R^m)$ generated by $Y_t^0 = \sigma\{y_s : s \leq t\}$. The set of admissible control functions $\underline{U}$ will be the Y-predictable functions on $[0,T] \times C([0,T], R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$ write $\xi_{s,t}^u(x)$ for the strong solution of (2.1) corresponding to control u, and with $\xi_{s,s}^u(x) = x$. Write

$$Z_{s,t}^u(x) = \exp\left(\int_s^t h(\xi_{s,r}^u(x))' dy_r - \frac{1}{2} \int_s^t h(\xi_{s,r}^u(x))^2 dt \right) \tag{2.3}$$

and define a new probability measure P^u on Ω by $\frac{dP^u}{dP} = Z_{0,T}^u(x_0)$. Then under P^u $(\xi_{0,t}^u(x_0), y_t)$ is a solution of (2.1) and (2.2), that is $\xi_{0,t}^u(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion v such that y_t satisfies (2.2). A version of Z defined for every trajectory y of the observation process is obtained by integrating by parts the stochastic integral in (2.3).

LEMMA 2.3. Under hypothesis A_4, for $t \leq T$,

$$E[(Z_{0,t}^u(x_0))^p] < \infty \quad \text{for all } u \in \underline{U} \text{ and all } p, \quad 1 \leq p < \infty.$$

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PROOF.

$$Z_{0,t}^u(x_0) = 1 + \int_0^t Z_{0,r}^u(x_0) h(\xi_{0,r}^u(x_0))' dy_r.$$

Therefore, for any p there is a constant C_p such that

$$E\left[(Z_{0,t}^u(x_0))^p\right] \le C_p\left[1 + E\left(\int_0^t (Z_{0,r}^u(x_0))^2 h(\xi_{0,r}^u(x_0))^2 dr\right)^{p/2}\right].$$

The result follows by Gronwall's inequality.

COST 2.4. We shall suppose the cost is purely terminal and given by some bounded, differentiable function

$$c\left(\xi_{0,T}^u(x_0)\right)$$

which has bounded derivatives. Then the expected cost if control $u \in \underline{U}$ is used is

$$J(u) = E_u\left[c\left(\xi_{0,T}^u(x_0)\right)\right].$$

In terms of P, under which y_t is always a Brownian motion, this is

$$J(e) = E\left[Z_{0,T}^u(x_0)\, c\left(\xi_{0,T}^u(x_0)\right)\right]. \tag{2.4}$$

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3. STOCHASTIC FLOWS.

For $u \in \underline{U}$ write

$$\xi_{s,t}^u(x) = x + \int_s^t f(r, \xi_{s,r}^u(x), u_r)dr + \int_s^t g(r, \xi_{s,r}^u(x))dw_r \qquad (3.1)$$

for the solution of (2.1) over the time interval $[s, t]$ with initial condition $\xi_{s,s}^u(x) = x$. In the sequel we wish to discuss the behaviour of (3.1) for each trajectory y of the observation process. We have already noted there is a version of Z defined for every y. The results of Bismut [2] and Kunita [10] extend easily and show the map

$$\xi_{s,t}^u : R^d \to R^d$$

is, almost surely, for each $y \in C([0,T], R^m)$ a diffeomorphism. Bismut [2] initially gives proofs when the coefficients f and g are bounded, but points out that a stopping time argument extends the results to when, for example, the coefficients have linear growth.

Write $\|\xi^u(x_0)\|_t = \sup_{0 \le s \le t} |\xi_{0,s}^u(x_0)|$. Then, as in Lemma 2.1 of [8], for any p, $1 \le p < \infty$ using Gronwall's and Jensen's inequalities

$$\|\xi^u(x_0)\|_T^p \le C\left(1 + |x_0|^p + \left|\int_0^t g(r, \xi_{0,r}^u(x_0))dw_r\right|^p\right)$$

almost surely, for some constant C.

Therefore, using Burkholder's inequality and hypothesis A_3, $\|\xi^u(x_0)\|_T$ is in L^p for all p, $1 \le p < \infty$.

Suppose $u^* \in U$ is an optimal control so $J(u^*) \le J(u)$ for any other $u \in U$. Write $\xi_{s,t}^*(\cdot)$ for $\xi_{s,t}^{u^*}(\cdot)$. The Jacobian $\dfrac{\partial \xi_{s,t}^*}{\partial x}(x)$ is the matrix solution C_t of the equation for $s \le t$,

$$dC_t = f_x(t, \xi_{s,t}^*(x), u^*)C_t dt + \sum_{i=1}^n g_x^{(i)}(t, \xi_{s,t}^*(x))C_t dw_t^i \qquad (3.2)$$

with $C_s = I$.

Here I is the $n \times n$ identity matrix and $g^{(i)}$ is the ith column of g. From hypotheses A_2 and A_3, f_x and g_x are bounded. Writing $\|C\|_T = \sup_{0 \le s \le t} |C_s|$ an application of Gronwall's

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Jensen's and Burkholder's inequalities again implies $\|C\|_T$ is in L^p for all p, $1 \le p < \infty$. Consider the related matrix valued stochastic differential equation

$$D_t = I - \int_s^t D_r f_x(r, \, \xi^*_{s,r}(x), \, u^*_r)' dr$$

$$- \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \, \xi^*_{s,r}(x))' dw_r^i$$

$$+ \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \, \xi^*_{s,r}(x))')^2 dr. \tag{3.3}$$

Then it can be checked that $D_t C_t = I$ for $t \ge s$, so that D_t is the inverse of the Jacobian, that is $D_t = \left(\frac{\partial \xi^*_{s,t}(x)}{\partial x} \right)^{-1}$. Again, because f_x and g_x are bounded we have that $\|D\|_t$ is in every L^p, $1 \le p < \infty$.

For a d-dimensional semimartingale z_t Bismut [2] shows one can consider the flow $\xi^*_{s,t}(z_t)$ and gives the semimartingale representation of this process. In fact if $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw_r^i$ is the d-dimensional semimartingale, Bismut's formula states that

$$\xi^*_{s,t}(z_t) = z_s + \int_s^t \left(f(r, \, \xi^*_{s,r}(z_r), \, u^*_r) \right.$$

$$+ \sum_{i=1}^n g_x^{(i)}(r, \, \xi^*_{s,r}(z_r), \, u^*_r) \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \xi^*_{s,r}(z_r)}{\partial x^2}(H_i, H_i) \bigg) dr$$

$$+ \int_s^t \frac{\partial \xi^*_{s,r}(z_r)}{\partial x} dA_r + \sum_{i=1}^n \int_s^t \left(g^{(i)}(r, \, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \right) dw_r^i. \tag{3.4}$$

DEFINITION 3.1. *We shall consider perturbations of the optimal control u^* of the following kind: For $s \in [0, T)$, $h > 0$ such that $0 \le s < s + h \le T$, for any other admissible control $u \in \underline{U}$ and $A \in Y_s$ define a strong variation of u^* by*

$$u(t, w) = \begin{cases} u^*(t, w) & \text{if } (t, w) \notin [s, s+h] \times A \\ \tilde{u}(t, w) & \text{if } (t, w) \in [s, s+h] \times A. \end{cases}$$

Applying (3.4) as in Theorem 5.1 of [4] we have the following result.

THEOREM 3.2. *For the perturbation u of the optimal control u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right)^{-1} \left(f(r, \, \xi^*_{s,r}(z_r), \, u_r) - f(r, \, \xi^*_{s,r}(z_r), \, u^*_r) \right) dr. \tag{3.5}$$

Then the process $\xi_{s,t}^*(z_t)$ is indistinguishable from $\xi_{s,t}^u(x)$.

PROOF. Note the equation defining z_t involves only an integral in time; there is no martingale term, so to apply (3.4) we have $H_i = 0$ for all i. Therefore, from (3.4)

$$
\begin{aligned}
\xi_{s,t}^*(z_t) = x &+ \int_s^t f(r, \, \xi_{s,r}^*(z_r), \, u_r^*) dr \\
&+ \int_s^t \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x}\right) \left(\frac{\partial \xi_{s,r}^*(z_r)}{\partial x}\right)^{-1} \left(f(r, \, \xi_{s,r}^*(z_r), \, u_r) - f(r, \, \xi_{s,r}^*(z_r), \, u_r^*)\right) dr \\
&+ \int_s^t g(r, \, \xi_{s,r}^*(z_r)) dw_r.
\end{aligned}
$$

However, the solution of (3.2) is unique so

$$
\xi_{s,t}^*(z_t) = \xi_{s,t}^u(x).
$$

REMARKS 3.3. Note that the perturbation $u(t)$ equals $u^*(t)$ if $t > s+h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$
\xi_{s,r}^*(z_r) = \xi_{s,t}^*(z_{s+h}) = \xi_{s+h,t}^*(\xi_{s,s+h}^u(x)).
$$

4. AUGMENTED FLOWS.

Consider the augmented flow which includes as an extra coordinate the stochastic exponential $Z_{s,t}^*$ with a 'variable' initial condition $z \in R$ for $Z_{s,s}^*(\cdot)$. That is, consider the $(d+1)$ dimensional system given by:

$$\xi_{s,t}^*(x) = x + \int_s^t f(r, \xi_{s,r}^*(x), u_r^*)dr + \int_s^t g(r, \xi_{s,r}^*(x))dw_r$$

$$Z_{s,t}^*(x,z) = z + \int_s^t Z_{s,r}^*(x,z)h(\xi_{s,r}^*(x))'dy_r.$$

Therefore,

$$Z_{s,t}^*(x,z) = z Z_{s,r}^*(x)$$

$$= z \exp\left(\int_s^t h(\xi_{s,r}^*(x))'dy_r - \frac{1}{2}\int_s^t h(\xi_{s,r}^*(x))^2 dr \right)$$

and we see there is a version of the enlarged system defined for each trajectory y by integrating by parts the stochastic integral. The augmented map $(x,z) \rightarrow (\xi_{s,t}^*(x), Z_{s,t}^*(x,z))$ is then almost surely a diffeomorphism of R^{d+1}. Note that $\frac{\partial \xi_{s,t}^*(x)}{\partial z} = 0$, $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial g}{\partial z} = 0$. The Jacobian of this augmented map is, therefore, represented by the matrix

$$\widetilde{C}_t = \begin{pmatrix} \frac{\partial \xi_{s,t}^*(x)}{\partial x} & 0 \\ \frac{\partial Z_{s,t}^*(x,z)}{\partial x} & \frac{\partial Z_{s,t}^*(x,z)}{\partial z} \end{pmatrix},$$

and for $1 \leq i \leq d$ from equation (3.3)

$$\frac{\partial Z_{s,t}^*(x,z)}{\partial x_i} = \sum_{j=1}^m \int_s^t \left(Z_{s,r}^*(x,z)\frac{\partial h^j(\xi_{s,r}^*(x))}{\partial \xi_k} \cdot \frac{\partial \xi_{k,s,r}^*(x)}{\partial x_i} \right.$$

$$\left. + h^j(\xi_{s,r}^*(x))\frac{dZ_{s,r}^*(x,z)}{\partial x_i} \right) dy_r^j. \tag{4.1}$$

(Here the double index k is summed from 1 to n).

We shall be interested in the solution of this differential system (4.1) only in the situation when $z = 1$ so we shall write $Z_{s,t}^*(x)$ for $Z_{s,t}^*(x,1)$. The following result is motivated by formally differentiating the exponential formula for $Z_{s,t}^*(x)$.

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LEMMA 4.1.

$$\frac{\partial Z^{*}_{s,t}(x)}{\partial x} = Z^{*}_{s,t}(x)\Big(\int_{s}^{t} h_x\big(\xi^{*}_{s,r}(x)\big)\cdot \frac{\partial \xi^{*}_{s,r}(x)}{\partial x}\cdot dv_r\Big)$$

where $v = (v^1,\ldots,v^n)$ is the Brownian motion in the observation process.

PROOF. From (4.1) we see $\frac{\partial Z^{*}_{s,t}(x)}{\partial x}$ is the solution of the stochastic differential equation

$$\frac{\partial Z^{*}_{s,t}(x)}{\partial x} = \int_{s}^{t}\Big(\frac{\partial Z^{*}_{s,r}(x)}{\partial x}h'\big(\xi^{*}_{s,r}(x)\big) + Z^{*}_{s,r}(x)h_x\big(\xi^{*}_{s,r}(x)\big)\frac{\partial \xi^{*}_{s,r}(x)}{\partial x}\Big)dy_r. \qquad (4.2)$$

Write

$$L_{s,t}(x) = Z^{*}_{s,t}(x)\Big(\int_{s}^{t} h_x\cdot \frac{\partial \xi^{*}_{s,r}}{\partial x}\cdot dv_r\Big)$$

where

$$dy_r = h\big(\xi^{*}_{s,t}(x)\big)dt + dv_t.$$

Because

$$Z^{*}_{s,t}(x) = 1 + \int_{s}^{t} Z^{*}_{s,r}(x)h'\big(\xi^{*}_{s,r}(x)\big)dy_r$$

the product rule gives

$$
\begin{aligned}
L_{s,t}(x) &= \int_{s}^{t} Z^{*}_{s,r}(x)h_x\cdot \frac{\partial \xi^{*}_{s,r}}{\partial x}dv_r \\
&\quad + \int_{s}^{t}\Big(\int_{s}^{r} h_x\cdot \frac{\partial \xi^{*}_{s,\sigma}}{\partial x}\cdot dv_\sigma\Big)Z^{*}_{s,r}(x)h'\big(\xi^{*}_{s,r}(x)\big)dy_r \\
&\quad + \int_{s}^{t} Z^{*}_{s,r}(x)h'\big(\xi^{*}_{s,r}(x)\big)\cdot h_x\cdot \frac{\partial \xi^{*}_{s,r}}{\partial x}dr \\
&= \int_{s}^{t} L_{s,r}(x)h'\big(\xi^{*}_{s,r}(x)\big)dy_r + \int_{x}^{t} Z^{*}_{s,r}(x)h_x\cdot \frac{\partial \xi*_{s,r}}{\partial x}\cdot dy_r.
\end{aligned}
$$

Therefore, $L_{s,t}(x)$ is also a solution of (4.2) so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^{*}_{s,t}(x)}{\partial x}.$$

REMARKS 4.2. As noted at the beginning of this section we can consider the augmented flow

$$(x,z) \to \big(\xi^{*}_{s,t}(x),\; Z^{*}_{s,t}(x,z)\big) \quad \text{for } x\in R^d,\; z\in R,$$

and we are only interested in the situation when $z = 1$, so we write $Z^{*}_{s,t}(x)$.

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LEMMA 4.3. $Z_{s,t}^*(z_t) = Z_{s,t}^u(x)$ where z_t is the semimartingale defined in (3.6).

PROOF. $Z_{s,t}^u(x)$ is the process uniquely defined by

$$Z_{s,t}^u(x) = 1 + \int_s^t Z_{s,r}^u(x) h'(\xi_{s,r}^u(x)) dy_r. \qquad (4.2)$$

Consider an augmented $(d+1)$ dimensional version of (3.6) defining a semimartingale $\bar{z}_t = (z_t, 1)$, so the additional component is always identically 1. Then applying (3.5) to the new component of the augmented process we have

$$\begin{aligned} Z_{s,r}^*(z_r) &= 1 + \int_s^t Z_{s,r}^*(z_r) h'(\xi_{s,r}^*(z_r)) dy_r \\ &= 1 + \int_s^t Z_{s,r}^*(z_r) h'(\xi_{s,r}^u(x)) dy_r \end{aligned}$$

by Theorem 3.2. However, (4.2) has a unique solution so $Z_{s,t}^*(z_t) = Z_{s,t}^u(x)$.

REMARKS 4.4. Note that for $t > s + h$

$$Z_{s,t}^*(z_t) = Z_{s,t}^*(z_{s+h}).$$

5. THE MINIMUM PRINCIPLE.

Control u will be the perturbation of the optimal control u^* as in Definition 3.1. We shall write $x = \xi^*_{0,s}(x_0)$. Then the minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0)c(\xi^*_{0,T}(x_0))]$$
$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(x)c(\xi^*_{s,T}(x))].$$

The cost corresponding to the perturbed control u is

$$J(u) = E[Z^*_{0,s}(x_0)Z^u_{s,T}(x)c(\xi^u_{s,T}(x))]$$
$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))]$$

by Theorem 3.2 and Lemma 4.3. Now $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable with continuous and uniformly integrable derivatives. Therefore

$$J(u) - J(u^*) = E[Z^*_{0,s}(x_0)(Z^*_{s,T}(z_{s+h})c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x)c(\xi^*_{s,T}(x)))]$$
$$= E\Big[\int_s^{s+h} \Gamma(s,z_r)(f(r,\,\xi^*_{s,r}(z_r),\,u_r) - f(r,\,\xi^*_{s,r}(x),\,u^*_r))dr\Big]$$

where

$$\Gamma(s,z_r) = Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\Big\{c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}(z_r)}{\partial x} +$$
$$c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\frac{\partial \xi^*_{s,\sigma}}{\partial x}(z_r)dv_\sigma\Big)\Big\}\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}.$$

Note that this expression gives an explicit formula for the change in the cost resulting from a variation in the optimal control. The only remaining problem is to justify differentiating the right hand side.

From Lemma 2.3, Z is in every L^p space, $1 \le p < \infty$ and from the remarks at the beginning of Section 3, $C_T = \frac{\partial \xi^*_{s,T}}{\partial x}$ and $D_T = \Big(\frac{\partial \xi^*_{s,T}}{\partial x}\Big)^{-1}$ are in every L^p space, $1 \le p < \infty$. Consequently, Γ is in every L^p space, $1 \le p < \infty$.

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Therefore

$$
J(u) - J(u^*) = \int_s^{s+h} E\Big[\big(\Gamma(s,z_r) - \Gamma(s,x)\big)\big(f(r,\ \xi_{s,r}^*(z_r),\ u_r) - f(r,\ \xi_{s,r}^*(z_r),\ u_r^*)\big)\Big]dr
$$

$$
+ \int_s^{s+h} E\Big[\big(\Gamma(s,x) - \Gamma(r,x)\big)\big(f(r,\ \xi_{s,r}^*(z_r),\ u_r) - f(r,\ \xi_{s,r}^*(z_r),\ u_r^*)\big)\Big]dr
$$

$$
+ \int_s^{s+h} E\Big[\Gamma(r,x)\big(f(r,\ \xi_{s,r}^*(z_r),\ u_r) - f(r,\ \xi_{s,r}^*(z_r),\ u_r^*)
$$

$$
- f(r,\ \xi_{s,r}^*(x),\ u_r) + f(r,\ \xi_{s,r}^*(x),\ u_r^*)\big)\Big]dr
$$

$$
+ \int_s^{s+h} E\Big[\Gamma(r,x)\big(f(r,\ \xi_{0,r}^*(x_0),\ u_r) - f(r,\xi_{0,r}^*(x_0),\ u_r^*)\big)\Big]dr
$$

$$
= I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.}
$$

Now,

$$
|I_1(h)| \le K_1 \int_s^{s+h} E\Big[|\Gamma(s,z_r) - \Gamma(s,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]dr
$$

$$
\le K_1 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,z_r) - \Gamma(s,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]
$$

$$
|I_2(h)| \le K_2 \int_s^{s+h} E\Big[|\Gamma(s,x) - \Gamma(r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]dr
$$

$$
\le K_2 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,z_r) - \Gamma(r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]
$$

$$
|I_3(h)| \le K_3 \int_s^{s+h} E\Big[|\Gamma(r,x)|\ \|x - z_r\|\Big]dr
$$

$$
\le K_3 h \sup_{s \le r \le s+h} E\Big[|\Gamma(r,x)|\ \|x - z.\|_{s+h}\Big].
$$

The differences $|\Gamma(s,z_r) - \Gamma(s,x)|$, $|\Gamma(s,x) - \Gamma(r,x)|$ and $\|x - z\|_{s+h}$ are all uniformly bounded in some L^p, $p \ge 1$, and

$$
\lim_{r \to s} |\Gamma(s,z_r) - \Gamma(s,x)| = 0 \quad \text{a.s.}
$$

$$
\lim_{r \to s} |\Gamma(s,x) - \Gamma(r,x)| = 0 \quad \text{a.s.}
$$

$$
\lim_{h \to 0} \|x - z.\|_{s+h} = 0.
$$

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Therefore,

$$\lim_{r \to s} \|\Gamma(s, z_r) - \Gamma(s, x)\|_p = 0$$

$$\lim_{r \to s} \|\Gamma(s, x) - \Gamma(r, x)\|_p = 0$$

and $\quad \lim_{h \to 0} \|(\|x - z\|_{s+h})\|_p = 0 \quad$ for some p.

Consequently, $\lim_{h \to 0} h^{-1} I_k(h) = 0$, for $k = 1, 2, 3$.

The only remaining problem concerns the differentiability of

$$I_4(h) = \int_s^{s+h} E\Big[\Gamma(r, x)(f(r, \xi_{0,r}^*(x_0), u_r) - f(r, \xi_{0,r}^*(x_0), u_r^*))\Big] dr.$$

The integrand is almost surely in $L^1([0, T])$ so $\lim_{h \to 0} h^{-1} I_4(h)$ exists for almost every $s \in [0, T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the control u. Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

1) Because the trajectories are, almost surely, continuous, Y_ρ is countably generated by sets $\{A_{i\rho}\}$, $i = 1, 2, \ldots$ for any rational number $\rho \in [0, T]$. Consequently Y_t is countably generated by the sets $\{A_{i\rho}\}$, $r \leq t$.

2) Let G_t denote the set of measurable functions from (Ω, Y_t) to $U \subset R^k$. (If $u \in \underline{U}$ then $u(t, w) \in G_t$.) Using the L^1-norm, as in [5], there is a countable dense subset $H_\rho = \{u_{j\rho}\}$ of G_ρ, for rational $\rho \in [0, T]$. If $H_t = \bigcup_{\rho \leq t} H_\rho$ then H_t is a countable dense subset of G_t. If $u_{j\rho} \in H_\rho$ then, as a function constant in time, $u_{j\rho}$ can be considered as an admissible control over the time interval $[t, T]$ for $t \geq \rho$.

3) The countable family of perturbations is obtained by considering sets $A_{i\rho} \in Y_t$, functions $u_{j\rho} \in H_t$, where $\rho \leq t$, and defining as in 3.1

$$u_{j\rho}^*(s, w) = \begin{cases} u^*(s, w) & \text{if } (s, w) \notin [t, T] \times A_{i\rho} \\ u_{j\rho}(s, w) & \text{if } (s, w) \in [t, T] \times A_{i\rho}. \end{cases}$$

Then for each i, j, ρ

$$\lim_{h \to 0} h^{-1} \int_s^{s+h} E\Big[\Gamma(r, x)(f(r, \xi_{0,r}^*(x_0), u_{j\rho}^*) - f(r, \xi_{0,r}^*(x_0), u^*))\Big] dr \qquad (5.1)$$

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exists and equals

$$E\left[\Gamma(s,x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}\right]$$

for almost all $s \in [0,T]$.

Therefore, considering this perturbation we have

$$\lim_{h\to 0} h^{-1}\left(J(u^*_{j\rho}) - J(u^*)\right) = E\left[\Gamma(s,x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}\right]$$

$$\geq 0 \quad \text{for almost all } s \in [0,T].$$

Consequently there is a set $S \subset [0,T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (5.1) exists for all i,j,ρ, and gives

$$E\left[\Gamma(s,x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}\right] \geq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underline{U}$ we can deduce that if $s \notin S$

$$E\left[\Gamma(s,x)(f(s, \xi^*_{0,s}(x_0), u) - f(s, \xi^*_{0,s}(x_0), u^*))I_A\right] \geq 0 \quad \text{for any } A \in Y_s. \tag{5.2}$$

Write

$$p_s(x) = E^*\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}(x)}{\partial x} + c(\xi^*_{0,T}(x_0))\left(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\frac{\partial \xi^*_{s,\sigma}(x)}{\partial x}dv_\sigma\right) \mid Y_s \vee \{x\}\right]$$

where, as before, $x = \xi^*_{0,s}(x_0)$ and E^* denotes expectation under $P^* = P^{u^*}$. Then $p_s(x)$ is the co-state variable and we have in (5.2) proved the following 'conditional' minimum principle:

THEOREM 5.1. *If* $u^* \in \underline{U}$ *is an optimal control there is a set* $S \subset [0,T]$ *of zero Lebesgue measure such that if* $s \notin S$

$$E^*[p_s(x)f(s,x,u^*) \mid Y_s] \leq E^*[p_s(x)f(s,x,u) \mid Y_s] \quad \text{a.s.}$$

That is, the optimal control u^* *almost surely minimizes the conditional Hamiltonian and the adjoint variable is* $p_s(x)$.

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6. CONCLUSION.

Using the theory of stochastic flows the effect of a perturbation of an optimal control is explicitly calculated. The only difficulty was to justify its differentiation. The adjoint process is explicitly identified as $p_s(x)$.

THEOREM 6.1. *If f is differentiable in the control variable u, and if the random variable $x = \xi_{0,s}^*(x_0)$ has a conditional density $q_s(x)$ under the measure P^*, then the inequality of Theorem 5.1 implies*

$$\sum_{j=1}^{k}(u_j(s) - u_j^*(s)) \int_{R^d} \Gamma(s,x)\frac{\partial f}{\partial u_j}(s,x,u^*)q_s(x)dx \leq 0.$$

This is the result of Bensoussan's paper [1].

The method of this paper can be applied to completely observable systems by initially considering 'stochastic open loop' controls, systems with stochastic constraints and deterministic systems. The adjoint process can be explicitly identified. 'Almost minimum' principles for 'almost optimal' controls can be obtained. Some of these will be discussed in later work.

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