ABSTRACT

Title of dissertation: MATROIDS AND GEOMETRIC INVARIANT THEORY OF TORUS ACTIONS ON FLAG SPACES

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Let λ and μ be weights of $G = \mathrm{SL}(n, \mathbb{C})$ such that λ is dominant. Let V_λ be the irreducible representation of G with highest weight λ, and let $V_\lambda[\mu]$ denote the μ-th weight space within V_λ. That is, $V_\lambda[\mu]$ is an isotypic component of the representation V_λ pulled back to the maximal torus $T \subset G$ of diagonal matrices. The vector space

$$R_{\lambda,\mu} = \bigoplus_{N=0}^{\infty} V_{N\lambda}[N\mu]$$

has a natural structure as a graded ring, which is graded by N. We study the structure of the rings $R_{\lambda,\mu}$. The motivation is that $R_{\lambda,\mu}$ is the projective coordinate ring of a Geometric Invariant Theory quotient of the flag space G/B by the natural left action of T, where B is the Borel subgroup of upper triangular matrices. We have

$$T \backslash\backslash (G/B) = \mathrm{Proj}(R_{\lambda,\mu}).$$

MATROIDS AND GEOMETRIC INVARIANT THEORY OF TORUS ACTIONS ON FLAG SPACES

by

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I owe a lot to Ravi Vakil who generously shared his ideas regarding projective invariants of points on the Riemann sphere. It was his idea that the relations among the projective invariants might be only quadrics, given by very simple relations among regular graphs. The last chapter of this thesis contains a proof that this is true when the points are all weighted by even integers.

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Chapter 1

Introduction

The geometry (both symplectic and algebraic) of the quotients $T\backslash\backslash F$ have been extensively studied in recent years; Allen Knutson called them "weight varieties"[1] in his thesis [K]. The dependence of the geometry of the quotient on the choice of linearization was studied by Yi Hu in [Hu] and in a more general setting by Igor Dolgachev and Yi Hu in [DoHu]. The cohomology of nonsingular weight varieties for $G = \mathrm{SL}(n, \mathbb{C})$ was computed by Rebecca Goldin [G]. Special cases of weight varieties have been studied since the nineteenth century; for example a G.I.T. quotient $(\mathbb{CP}^{k-1})^n//\mathrm{PGL}(k, \mathbb{C})$ is isomorphic to a G.I.T. quotient $T\backslash\backslash \mathrm{Gr}_k(\mathbb{C}^n)$ by the Gel'fand MacPherson correspondence (here $\mathrm{Gr}_k(\mathbb{C}^n)$ denotes the Grassmannian). The projective invariants of n-tuples of points on projective space are still not understood today; we do not even know a minimal set of generators for the ring of projective invariants (see page 8 of [Ha]).

The thesis contains three main theorems about the rings $R_{\lambda,\mu}$. The first result is the discovery of an explicit finite (but not minimal) set of generators for $R_{\lambda,\mu}$. In [LG] a degeneration of partial flag spaces to toric varieties was given. In [FH] Phillip Foth and Yi Hu observed that these degenerations may be restricted to T invariants

[1]The term "weight variety" actually refers to more general quotients; they are G.I.T. quotients of G/P by a maximal torus T in G, where G is a reductive connected complex Lie group and P is a parabolic subgroup of G containing T.

and consequently they give flat degenerations of weight varieties to toric varieties. These toric varieties are defined by polytopes known as Gel'fand Tsetlin polytopes. A recent theorem of Jesús De Loera and Tyrell McAllister (see [dLMc]) concerns the vertices of Gel'fand Tsetlin polytopes $GT(\lambda, \mu)$. They find an upper bound on the denominators of these vertices. Consequently one gets a finite set of generators for the associated semigroup of integral points in the cone on the Gel'fand Tsetlin polytopes. According to [FH] there is a filtration F of $R_{\lambda,\mu}$ for which the associated graded ring $\mathrm{gr}_F(R_{\lambda,\mu})$ is the semigroup algebra of these integral points. The lifts of the generators of $\mathrm{gr}_F(R_{\lambda,\mu})$ to $R_{\lambda,\mu}$ must generate $R_{\lambda,\mu}$. These lifted generators are far from minimal. In fact the degree bound on the generators is

$$M_n = \frac{n^2 - 3n + 4}{2}\Big((n-1)^{n(n+1)/2-n-1}\Big),$$

a terribly large number in n, though it applies for all pairs (λ, μ) of length n. It remains an open problem to find a minimal set of generators.

The second theorem is the following. Suppose $N > 0$ is the minimal integer such that $R^{(N)}_{\lambda,\mu} = V_{N\lambda}[N\mu]$ is nonzero. Let $v_1, \ldots, v_m$ be a basis for $V_{N\lambda}[N\mu]$. Then for each semistable point $p \in (G/B)$ there exists some v_i such that $v_i(p)$ is nonzero. In particular, the map $T \backslash\backslash (G/B) \to \mathbb{CP}^{m-1}$ given by $p \mapsto [v_1(p), \ldots, v_m(p)]$ is well-defined. This is remarkable since in general the v_i's do not generate the ring. This result applies a theorem of Gel'fand, Goresky, MacPherson, and Serganova concerning matroids and momentum images of closures of T-orbits in Grassmannian spaces $\mathrm{Gr}_k(\mathbb{C}^n)$. Each such momentum image is a polytope P for which a matroid $M(P)$ is defined in which the bases of $M(P)$ correspond to the vertices of P. Their main theorem is that the edges of P are parallel to roots of $G = \mathrm{SL}(n, \mathbb{C})$.

The third theorem addresses the special case where λ is a multiple of the second fundamental weight ϖ_2 connected to the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$. This special case is of particular interest since the quotient $T \backslash\backslash (G/B)$ is the space of n tuples of points on the projective line modulo automorphisms of the line. In this case, the lowest degree T invariants generate $R_{\lambda,\mu}$. This was proved in 1894 by Alfred Bray Kempe [Ke]. However the relations in these generators were not discovered until very recently in [HMSV1]. They discover that the relations are generated by relations of degree four and less. If the weight μ is such that all components are even integers, then the relations are only quadratic. Here it is shown that these quadratic relations have a very simple form. This result is motivated by [HMSV2] where it is proven that relations of this sort cut out the projective variety (though perhaps not the ideal) for *any* weight μ, except for the case $n = 6$ and $\mu = (1, 1, 1, 1, 1, 1)$.

Chapter 2

The Geometric Invariant Theory Quotients $T\backslash\backslash(G/B)$

We begin by defining Geometric Invariant Theory (G.I.T.) quotients of projective varieties by reductive affine algebraic groups.

2.1 G.I.T. quotients

We refer the reader to [Do] for additional details. Suppose that G is a reductive algebraic group, V is a quasi-projective variety, and $\eta : G \times V \to V$ is a regular action of G. Let $\pi : \mathcal{L} \to V$ be an ample line bundle of V. A G–linearization of $\mathcal{L}$ is a regular action $\widetilde{\eta} : G \times \mathcal{L} \to \mathcal{L}$ which is linear on fibers and makes the following diagram commute:

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\widetilde{\eta}} & \mathcal{L} \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times V & \xrightarrow{\eta} & V \end{array}$$

Given such a linearization, we automatically get linearizations on all tensor powers $\mathcal{L}^{\otimes N}$ of $\mathcal{L}$. Thus G has an action on sections s of $\mathcal{L}^{\otimes N}$ given by $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x) = \widetilde{\eta}(g, s(\eta(g^{-1}, x)))$. Let $\Gamma(V, \mathcal{L}^{\otimes N})^G$ denote the G–invariant sections of $\mathcal{L}^{\otimes N}$. The G.I.T. quotient $V//_{\widetilde{\eta}}\, G$ is defined as

$$V//_{\widetilde{\eta}}\, G = \mathrm{Proj}\Big(\bigoplus_{N=0}^{\infty} \Gamma(V, \mathcal{L}^{\otimes N})^G \Big).$$

If the linearization is understood, sometimes we denote $V//_{\widetilde{\eta}}G$ by $V//G$.

If s is a section let $V_s = \{x \in V \mid s(x) \neq 0\}$. The set $V^{ss}_{\widetilde{\eta}}$ of semistable points of V is defined as

$$V_{\tilde{\eta}}^{ss} = \bigcup_{N \geq 0} \bigcup \{V_s \mid s \in \Gamma(V, \mathcal{L}^{\otimes N})^G \text{ and } V_s \text{ is affine}\}.$$

(Note that when V is affine or projective the distinguished open sets V_s are automatically affine.) If x is a semistable point let $cl(G \cdot x)$ be the (Zariski) closure of the orbit $G \cdot x$ in $V_{\tilde{\eta}}^{ss}$. As a topological space $V/\!/_{\tilde{\eta}}\, G$ is the quotient space of $V_{\tilde{\eta}}^{ss}$ where points x, y are identified iff $cl(G \cdot x)$ and $cl(G \cdot y)$ intersect nontrivially.

2.1.1 The Gel'fand-MacPherson correspondence

The Gel'fand–MacPherson correspondence says that a G.I.T. quotient of Grassmannian space $\mathrm{Gr}_k(\mathbb{C}^n)$ by the torus $(\mathbb{C}^*)^n$ is isomorphic to a G.I.T. quotient of the product space $(\mathbb{CP}^{k-1})^n$ by the diagonal action of $\mathrm{PGL}(k, \mathbb{C})$.

Let $\mathcal{L}$ be the trivial line bundle $\mathbb{C}^{n \times k} \times \mathbb{C} \to \mathbb{C}^{n \times k}$. Given any group G acting on $\mathbb{C}^{n \times k}$, a character $\chi : G \to \mathbb{C}^*$ defines a linearization of $\mathcal{L}$ by $g \cdot (A, z) = (g \cdot A, \chi(g)z)$.

The group $\mathrm{GL}(k, \mathbb{C})$ acts on the right of $\mathbb{C}^{n \times k}$ by matrix multiplication. The group T of diagonal matrices in $\mathrm{GL}(n)$ acts on the left of $\mathbb{C}^{n \times k}$. Let $\det^a : \mathrm{GL}(k, \mathbb{C}) \to \mathbb{C}^*$ be $\det^a(g) = (\det(g))^a$ and let $\chi_{\mathbf{r}} : T \to \mathbb{C}^*$ be

$$\chi_{\mathbf{r}}(\mathrm{diag}(z_1, \ldots, z_n)) = \prod_{i=1}^{n} z_i^{r_i}$$

where $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Z}_+^n$. The one–dimensional subgroup $K = \{(zI_n, z^{-1}I_k) : z \in \mathbb{C}^*\}$ of $T \times \mathrm{GL}(k, \mathbb{C})$ acts trivially on $\mathbb{C}^{n \times k}$. Let G be the quotient of $T \times \mathrm{GL}(k, \mathbb{C})$ by K. The character $\chi_{\mathbf{r}} \times \det^a$ descends to G iff $|\mathbf{r}| = \sum_i r_i = ka$, and we assume that is the case so that we have a G–linearization of the trivial line bundle.

Let $\mathcal{L}_{k,n}$ be the ample generator of the Picard group of $\mathrm{Gr}_k(\mathbb{C}^n)$; we may realize the total space of $\mathcal{L}_{k,n}$ by equivalence classes $V^{n\times k} \times \mathbb{C}/\sim$ where $V^{n\times k}$ is the open subset of $\mathbb{C}^{n\times k}$ of matrices with independent columns and $(A, z) \sim (Ag, \det(g)z)$ for $g \in \mathrm{GL}(k, \mathbb{C})$. Denote the equivalence class $(A, z)_\sim$ by $[A, z]$. The character $\chi_{\mathbf{r}}$ defines a T–linearization of $\mathcal{L}_{k,n}^a = \mathcal{L}_{k,n}^{\otimes a}$ by

$$t \cdot [A, z] = [tA, \chi_{\mathbf{r}}(t)z].$$

Let $\mathcal{H}$ be the ample generator of the Picard group of $\mathbb{CP}^{k-1}$, and let $\mathcal{H}^{\mathbf{r}}$ be the ample line bundle over the product $(\mathbb{CP}^{k-1})^n$ given by

$$\mathcal{H}^{\mathbf{r}} = \mathcal{H}^{\otimes r_1} \boxtimes \cdots \boxtimes \mathcal{H}^{\otimes r_n}.$$

We may identify the total space of $\mathcal{H}$ with $(\mathbb{C}^k \setminus \{0\}) \times \mathbb{C}/\sim$ where $(v, z) \sim (v\lambda, \lambda z)$ for $\lambda \in \mathrm{GL}(1)$; let $[v, z]$ denote the equivalence class. There is a unique linearization of $\mathcal{H}^{\mathbf{r}}$ for the (right) diagonal action of $\mathrm{PGL}(k, \mathbb{C})$ on $(\mathbb{CP}^{k-1})^n$.

By the First Fundamental Theorem of Invariant Theory [Do, Theorem 2.1], the homogeneous coordinate ring of the Grassmannian is generated by Plücker coordinates, hence, for any $N \geq 0$ we have

$$\Gamma(\mathrm{Gr}_k(\mathbb{C}^n), (\mathcal{L}_{k,n}^a)^{\otimes N}) \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes akN})^{\mathrm{GL}(k,\mathbb{C})}$$

and consequently

$$\Gamma(\mathrm{Gr}_k(\mathbb{C}^n), (\mathcal{L}_{k,n}^a)^{\otimes N})^T \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes akN})^G.$$

On the other hand the sections of the outer tensor product of hyperplane section bundles over n copies of $\mathbb{CP}^{k-1}$ are products of *homogeneous* polynomials in the homogeneous coordinates, that is,

$$\Gamma((\mathbb{CP}^{k-1})^n, (\mathcal{H}^{\mathbf{r}})^{\otimes N}) \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes |\mathbf{r}|N})^T$$

and consequently

$$\Gamma((\mathbb{CP}^{k-1})^n, (\mathcal{H}^{\mathbf{r}})^{\otimes N})^{\mathrm{PGL}(k,\mathbb{C})} \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes |\mathbf{r}|N})^G.$$

Hence we have an isomorphism of the G.I.T. quotients

$$\mathrm{Gr}_k(\mathbb{C}^n)//_{\chi_{\mathbf{r}}}T \cong \mathbb{C}^{n\times k}//_{\chi_{\mathbf{r}}\times \det^a} G \cong (\mathbb{CP}^{k-1})^n//_{\mathbf{r}}\mathrm{PGL}(k,\mathbb{C}).$$

2.2 The G.I.T. quotients $T\backslash\backslash(G/B)$

A weight variety of $G = \mathrm{SL}(n,\mathbb{C})$ is a G.I.T. quotient of the flag manifold $F = G/B$ by the (left) action of the Cartan subgroup T. The construction of such a quotient involves the choice of a T-linearized line bundle L of $F = G/B$. Since T acts by left multiplication, it is more natural to denote the G.I.T. quotient by $T\backslash\backslash(G/B)$ (or $T\backslash\backslash F$) rather than by $(G/B)//T$. We will keep this notation throughout the remainder of this thesis.

If L has any nonzero sections, then it's isomorphism class is determined by a choice of dominant weight λ. The T-linearization of L will also depend on a choice of a weight μ, but μ need not be dominant.

2.2.1 Elementary representation theory of $\mathrm{SL}(n,\mathbb{C})$

Since $\mathrm{SL}(n,\mathbb{C})$ is simply connected, the set of $\mathrm{SL}(n,\mathbb{C})$ weights are the differentials evaluated at the identity element of characters $\chi : T \to \mathbb{C}^*$, which are holomorphic homomorphisms (that is, the character lattice coincides with the weight lattice). The differential $d\chi$ (evaluated at the identity element of T) of χ lies within the dual Lie algebra $\mathfrak{t}^*$ of T. On the other hand, if $\varpi \in \mathfrak{t}^*$ is a weight, we shall denote e^ϖ as the unique character $e^\varpi : T \to \mathbb{C}^*$ such that $d(e^\varpi) = \varpi$.

A character e^λ applied to $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$ must be equal to $\prod_{i=1}^n t_i^{a_i}$ for some fixed integers a_i. Since $\prod_{i=1}^n t_i = 1$ for all $t \in T$, we have that the n-tuple of exponents $(a_1, \ldots, a_n)$ and $(a_1 + a, a_2 + a, \ldots, a_n + a)$ determine the same character. We may thus view the abelian group of characters of T as $\mathbb{Z}^n/\Delta$ where Δ is the diagonal. On the other hand, the weight $\lambda \in \mathfrak{t}^*$ takes a complex vector $(z_1, \ldots, z_n) \in \mathfrak{t}$ (where $z_1 + z_2 + \cdots + z_n = 0$) to $\sum_{i=1}^n a_i z_i$. Again, adding a constant to each a_i results in the same function, and so again we have that the additive group of weights is isomorphic to $\mathbb{Z}^n/\Delta$. We shall henceforth identify characters and weights as n-tuple of integers modulo the diagonal Δ.

The Borel-Weil construction

Let λ be any weight of $G = SL(n, \mathbb{C})$. The character e^λ defined on T extends uniquely to a character $\chi_\lambda : B \to \mathbb{C}^*$. We will abuse notation and denote χ_λ by e^λ as well. The weight λ determines a holomorphic line bundle L_λ of G/B. The total space of L_λ is the set of equivalence classes of pairs (g, z) for $g \in G$ and $z \in \mathbb{C}$, where $(g, z) \sim (gb, e^\lambda(b)z)$ for all $b \in B$. The map π from the total space to G/B is given by $\pi : (g, z) \mapsto gB$. Each global section of L_λ is given by $s_f(gB) = (g, f(g))$ where $f : G \to \mathbb{C}$ is a holomorphic function such that $f(gb) = e^\lambda(b)f(g)$ for all $b \in B$ and

$g \in G$.

Proposition 2.2.1 *(see [BL]) The Picard group of holomorphic line bundles of G/B (up to isomorphism) is isomorphic to the weight lattice $P(R)$ of G, via the correspondence*

$$P(R) \to Pic(G/B), \quad \lambda \mapsto L_\lambda.$$

The vector space $H^0(G/B, L_\lambda) = \Gamma(G/B, L_\lambda)$ of global sections is nonzero iff λ is dominant (this means that if $\lambda = (a_1, \ldots, a_n) \in \mathbb{Z}^n/\Delta$ then $a_i \geq a_{i+1}$ for all i, $1 \leq i \leq n-1$.) Additionally L_λ is very ample iff λ is strictly dominant, i.e. $\lambda = (a_1, \ldots, a_n)$ satisfies $a_i > a_{i+1}$ for each i, $1 \leq i \leq n-1$.

There is a natural action of G on the total space of L_λ, given by $g \cdot (g', z) = (gg', z)$. This defines an action on sections by

$$(g \cdot s)(g'P) = g \cdot s(g^{-1}g'P) = g \cdot (g^{-1}g', f(g^{-1}g')) = (g'P, f(g^{-1}g')).$$

If λ is dominant then the vector space V_λ of global sections is an irreducible representation of G; the action of $g \in G$ on s_f is $(g \cdot s_f)(g'P) = g \cdot s_f(g^{-1}g'P)$.

Proposition 2.2.2 *(see [FuHa]) There is a one-to-one correspondence between irreducible representations of G and line bundles L_λ where λ is dominant. The correspondence is given by $L_\lambda \mapsto \Gamma(G/B, L_\lambda)$.*

Choosing a T-linearization of L_λ

There is a canonical T-linearization of L_λ, given by restricting the action of G on L_λ to T. We shall call this the "democratic" linearization. A weight μ may be used to twist the democratic linearization;

$$t \cdot (g, z) = (tg, \mu(t)z).$$

We shall call this the "μ–linearization". Indeed the set of all T-linearizations are given by the characters μ of T.

The μ-twisted action of T on a section s_f is given by the formula,

$$(t \cdot s_f)(gB) = (gB, e^{\mu}(t)f(t^{-1}g)).$$

Hence s_f is T-invariant iff $\mu(t)f(t^{-1}g) = f(g)$ for all $t \in T$. Equivalently, we have that s_f is T-invariant iff for all $t \in T$ and $g \in G$,

$$f(tg) = e^{\mu}(t)f(g).$$

The action on a section s_f of $L_\lambda^{\otimes N}$ is given by $(t \cdot s_f)(gB) = (gB, e^{N\mu}(t)f(g))$, and so the T-invariant sections s_f of the N-th tensor power of L_λ are those which satisfy

$$f(t \cdot g) = e^{N\mu}(t)f(g).$$

The G.I.T. quotient $T \backslash\backslash (G/B)$ associated to the pair (λ, μ) is the projective variety,

$$T \backslash\backslash (G/B) = \mathrm{Proj}\left(\bigoplus_{N=0}^{\infty} \Gamma(G/B, L_\lambda^{\otimes N})^T\right),$$

where T acts on L_λ via the μ-linearization.

Recall that the set of semistable points $F^{ss} \subset F$ is defined by $p \in F^{ss}$ iff there exists some positive integer N and a T-invariant global section s of $L_\lambda^{\otimes N}$ such that $s(p) \neq 0$. (Normally there is the additional requirement that $X_s = \{p \in F \mid s(p) \neq 0\}$ is affine but this is automatic since F is a projective variety.) If we take the μ-linearization of L_λ, then we shall say that a semistable point is μ-semistable.

2.3 Partial flags

If λ is dominant but not strictly dominant, then the line bundle L_λ is not ample. However, there is a quotient space of G/B consisting of partial flags, which has a line bundle L'_λ which is very ample, and L_λ is the pullback of L'_λ.

The fundamental weights ϖ_i for $1 \leq i \leq n-1$ are a basis for the weight lattice of $\mathrm{SL}(n, \mathbb{C})$, where $\varpi_i = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ has i ones followed by $n-i$ zeroes. If the dominant weight λ is normalized so that the last component of λ is zero, then $\lambda = \sum_i a_i \varpi_i$ where each $a_i \geq 0$. The partial flag space corresponding to λ is then the space of sequences of vector spaces $V_{i_1}, \ldots, V_{i_k} \in \mathbb{C}^n$ where $i_1, \ldots, i_k$ are the indices for which $a_i > 0$, the dimension of V_{i_t} is i_t for $1 \leq t \leq k$, and $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_k}$. Let P_λ be the largest subgroup of $\mathrm{SL}(n, \mathbb{C})$ containing B such that the character e^λ extends to P_λ. Then the partial flag space above is G/P_λ. The construction of L'_λ is similar to that of L_λ. The total space is given by equivalence classes of pairs (g, z), where $g \in G$ and $z \in \mathbb{C}$, such that $(g, z) \sim (gp, e^\lambda(p)z)$ for all $p \in P_\lambda$. The quotient map to G/P_λ is then $\pi'(g, z) = gP_\lambda$. Let $\rho : G/B \to G/P_\lambda$ be the canonical quotient map. In terms of flags, ρ sends a full flag $V_1 \subset V_2 \cdots \subset V_{n-1}$ to $V_{i_1} \subset \cdots \subset V_{i_k}$. Now, the pullback $\rho^*(L'_\lambda)$ is L_λ, and the sections of L_λ are all pulled back from sections of L'_λ. Additionally, the line bundle L'_λ is very ample.

As before, L'_λ has a canonical G-action and a democratic T-linearization of L'_λ gotten from restricting the action of G to T. As before, we may twist the democratic linearization by a character e^μ of T. The μ-linearization of L'_λ is compatible with the μ-linearization of L_λ via the pullback map ρ^*. There is an isomorphism of graded rings,

$$R_{\lambda,\mu} \cong \bigoplus_{N=0}^{\infty} \Gamma(G/P_\lambda, L'_{N\lambda})^T.$$

Therefore, we have an isomorphism of G.I.T. quotients,

$$\mathrm{Proj}(R_{\lambda,\mu}) \cong \mathrm{Proj}\Big(\bigoplus_{N=0}^{\infty} \Gamma(G/P_\lambda, L'_{N\lambda})^T\Big).$$

Chapter 3

A finite set of generators

3.1 Lifting generators and relations from the associated graded ring of a filtered ring

For background on filtrations and gradings we refer to [Bou1]. Our filtrations will be increasing and indexed by the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Lemma 3.1.1 *Suppose that M is a filtered module over a filtered ring R and that their filtrations are compatible in the sense that*

$$\mu(F_i(R) \otimes F_j(M)) \subset F_{i+j}(M)$$

where μ is the module structure. Suppose that $x_1, x_2, \ldots, x_n$ are elements of M such that their images $\overline{x}_i, 1 \leq i \leq n$, under the leading term map generate $\mathrm{gr}(M)$ *as a* $\mathrm{gr}(R)$ *module. Then the x_i, for $1 \leq i \leq n$ generate M.*

Remark 3.1.2 An analogous argument shows that if the images in $\mathrm{gr}(R)$ *of a finite set of elements $w_1, w_2, \ldots, w_n$ of R generate* $\mathrm{gr}(R)$ *then $w_1, w_2, \ldots, w_n$ generate R.*

Our goal in this section is to prove the statement for *relations* that is the analogue of the statement in the remark for generators.

3.1.1. *Definition.* Let M be a filtered module and $x \in M$. We define the filtration level (or order) $v(x) \in \mathbb{N}$ of x to be the smallest n such that $x \in F_n(M)$.

Assume that R is graded as a vector space and that we have chosen homogeneous generators $f_1, f_2, \ldots, f_n$ for R such that the images $\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n$ of these generators in $\mathrm{gr}(R)$ generate $\mathrm{gr}(R)$. We assume that the degree of f_i is e_i, for $1 \leq i \leq n$. We obtain two exact sequences,

$$I \xrightarrow{\iota} \mathbb{C}[x_1, x_2, \ldots, x_n] \xrightarrow{\pi} R$$

and

$$J \longrightarrow \mathrm{gr}(\mathbb{C}[x_1, x_2, \ldots, x_n]) \xrightarrow{\mathrm{gr}(\pi)} \mathrm{gr}(R).$$

Here π sends x_i to f_i, for $1 \leq i \leq n$. In the above the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is a weighted polynomial ring, the variable x_i has weight e_i. We define a filtration on R by defining the filtration level of r to be the minimum of the degrees of the polynomials in $\pi^{-1}(r)$. The reader will verify that this filtration coincides with the quotient filtration of the standard filtration on $\mathbb{C}[x_1, x_2, \ldots, x_n]$. We remind the reader that the quotient filtration is characterized by the fact that the induced map on each filtration level is a surjection, see [Bou1], pg. 164.

We note that $\mathrm{gr}(\mathbb{C}[x_1, x_2, \ldots, x_n])$ is the polynomial ring $\mathbb{C}[\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]$. We leave to the reader the task of proving (by induction on the filtration level):

Lemma 3.1.3 *Suppose R is a filtered $\mathbb{C}$–algebra which is graded as a vector space and $f_1, \ldots, f_n$ have the property that their images $\overline{f}_1, \ldots, \overline{f}_n$ generate $\mathrm{gr}(R)$. Then the given filtration on R coincides with the quotient filtration associated to the surjection $\pi : \mathbb{C}[x_1, x_2, \ldots, x_n] \to R$ given by $\pi(x_i) = f_i$.*

Since we give I the filtration induced as a submodule of the polynomial ring both I and R have the filtrations needed to apply Proposition 2 of [Bou1], pg. 169

to deduce that we have an exact sequence

$$\mathrm{gr}(I) \xrightarrow{\mathrm{gr}(\iota)} \mathrm{gr}(\mathbb{C}[x_1, x_2, \ldots, x_n]) \xrightarrow{\mathrm{gr}(\pi)} \mathrm{gr}(R).$$

and consequently $\mathrm{gr}(\iota) : \mathrm{gr}(I) \to J$ is an isomorphism.

We are now ready to state and prove the result we want on lifting relations from $\mathrm{gr}(R)$ to R. We emphasize that we are assuming that the generators for R map to generators for $\mathrm{gr}(R)$ under the leading term map.

Proposition 3.1.4 *Suppose* $p_1, p_2, \ldots, p_k \in \mathrm{gr}(\mathbb{C}[x_1, x_2, \ldots, x_n])$ *generate the ideal of relations in the given generators for* $\mathrm{gr}(R)$. *Then*

1. *There exist lifts* $\tilde{p}_i, 1 \leq i \leq k$, *to* $\mathbb{C}[x_1, x_2, \ldots, x_n]$ *such that for all* i *the polynomial* $\tilde{p}_i$ *is a relation for* R.

2. *For any choice of such lifts* $\tilde{p}_i$, $1 \leq i \leq k$, *the lifts generate the ideal of relations of* R.

Proof. Since we have shown that $J \cong \mathrm{gr}(I)$ the first statement in the proposition is obvious (since the leading term map is onto by definition of $\mathrm{gr}(I)$). However, the lift of a homogeneous element will usually not be homogeneous (the ideal I may not contain any nonzero homogeneous elements). The second statement follows from Lemma 3.1.1 - the images of the lifts generate the ideal $\mathrm{gr}(I)$ so the lifts generate I. $\square$

3.2 Gel'fand Tsetlin polytopes and a toric degeneration of $R_{\lambda,\mu}$

We shall assume that λ is dominant, and that $\lambda - \mu$ is in the root lattice of $\mathrm{SL}(n, \mathbb{C})$, and that μ lies in the convex hull of the Weyl orbit of λ. We can now normalize the coordinates of λ and μ so that the last coordinate of λ is zero, and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$. Now it is also true that each $\mu_i \geq 0$. If these conditions do not hold then the degree one part of $R_{\lambda,\mu}$ is zero.

The Gel'fand Tsetlin polytope $GT(\lambda, \mu)$ is given by the set of triangular arrays $[x_{i,j}]$, $1 \leq i \leq n$, $1 \leq j \leq i$, such that $\lambda = (x_{n,1}, \ldots, x_{n,n})$, the row sums $s_i = \sum_{j=1}^{i} x_{i,j} = \mu_1 + \cdots + \mu_i$ for each i, $1 \leq i \leq n$, together with the inequalities,

1. $x_{i,j} \geq 0$ for each i, j (nonnegativity).

2. $x_{i+1,j} \geq x_{i,j} \geq x_{i+1,j+1}$ for each i, j (interlacing).

Illustration of the triangular array:

$$
\begin{array}{ccccccccccc}
x_{n,1} & & x_{n,2} & & x_{n,3} & & \cdots & & & & x_{n,n} \\
& x_{n-1,1} & & x_{n-1,2} & & \cdots & & & & x_{n-1,n-1} & \\
& & \ddots & & & \vdots & & & & & \\
& & & x_{3,1} & & x_{3,2} & & x_{3,3} & & & \\
& & & & x_{2,1} & & x_{2,2} & & & & \\
& & & & & x_{1,1} & & & & &
\end{array}
$$

The integral points in $GT(\lambda, \mu)$ are in one-to-one correspondence with a basis of $R^{(1)}_{\lambda,\mu}$, the degree one elements of $R_{\lambda,\mu}$. In general, the dimension of $R^{(N)}_{\lambda,\mu}$ is equal to the number of integral points in $GT(N\lambda, N\mu)$ for each N. Let $S_{\lambda,\mu}$ be the graded semigroup $\bigcup_{N=0}^{\infty} GT(N\lambda, N\mu)$, where the grading is by N and the semigroup operation is addition. We can think of the elements of $S_{\lambda,\mu}$ as symbolically representing a special basis of $R_{\lambda,\mu}$ as a complex vector space. In fact the usual basis chosen is that of semistandard tableaux of shape $N\lambda$ filled by $N\mu$ for the degree N part of the ring, but we will not use this explicitly.

Theorem 3.2.1 *(Foth and Hu [FH]) There is a one parameter flat degeneration of $R_{\lambda,\mu}$ with special fiber a toric ring $R'_{\lambda,\mu}$ which is isomorphic to the graded semigroup algebra of $S_{\lambda,\mu}$:*

$$R'_{\lambda,\mu} \cong \mathbb{C}[S_{\lambda,\mu}].$$

In fact there is an $\mathbb{N}$-filtration of $R_{\lambda,\mu}$ such that $R'_{\lambda,\mu}$ is the associated graded algebra.

Now, as we saw in the previous section, a presentation for the semigroup $S_{\lambda,\mu}$ can lifted to a presentation for the ring $R_{\lambda,\mu}$. In particular there is a canonical choice of lifts given the basis for $R_{\lambda,\mu}$ corresponding to lattice points in the Gel'fand Tsetlin cone. In particular, if $S_{\lambda,\mu}$ is generated by elements of degree $\leq m$, then $R_{\lambda,\mu}$ is also generated by elements of degree $\leq m$.

We now illustrate a couple of examples where we apply the above theorem to study the structure of the ring $R_{\lambda,\mu}$. Let $n = 4$, and $\lambda = (3, 1, 0, 0)$ and $\mu = (1, 1, 1, 1)$. The G.I.T. quotient is the democratic quotient of the flag space $F_{1,2}(\mathbb{C}^4)$ by T. Using the computer program cdd+[1] we find that the polytope $GT(\lambda, \mu)$ has

[1]See http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html

three vertices v_1, v_2, v_3:

$$v_1 = \begin{array}{ccccccc} 3 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array} \qquad v_2 = \begin{array}{ccccccc} 3 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{array} \qquad v_3 = \begin{array}{ccccccc} 3 & & 1 & & 0 & & 0 \\ & 3 & & 0 & & 0 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{array}$$

All of these vertices are integral, and their convex hull $GT(\lambda, \mu)$ is a unimodular triangle with no interior integral points. Hence the semigroup $S_{\lambda,\mu}$ is generated by these three elements. They have no relations since they form a simplex. Let $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$ be lifts in $R_{\lambda,\mu}$ of degree one. Now these three elements will generate $R_{\lambda,\mu}$. There are no relations in these generators $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$ since their leading terms v_1, v_2, v_3 have no relations in them. Therefore, the ring $R_{\lambda,\mu}$ is the polynomial ring $\mathbb{C}[\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3]$, and the G.I.T. quotient is $\mathbb{CP}^2$.

Another example is given by $n = 6$, $\lambda = (2, 2, 2, 0, 0, 0)$ and $\mu = (1, 1, 1, 1, 1, 1)$. The G.I.T. quotient is the space of six points on $\mathbb{CP}^2$ modulo $\mathrm{PGL}(3, \mathbb{C})$ via the Gel'fand MacPherson correspondence. It is known as the Igusa's quartic $\mathcal{I}_4$. It has an interesting self-duality by association of point sets; see [DO] or [HM] for a generalization of association of points sets to weight varieties. The Gel'fand Tsetlin polytope $GT(\lambda, \mu)$ consists of numbers a, b, c, d, e, f, g in the pattern,

$$\begin{array}{ccccccccccc}
2 & & 2 & & 2 & & 0 & & 0 & & 0 \\
& 2 & & 2 & & 1 & & 0 & & 0 & \\
& & 2 & & f & & g & & 0 & & \\
& & & c & & d & & e & & & \\
& & & & a & & b & & & & \\
& & & & & 1 & & & & &
\end{array}$$

Using cdd+ we find there are seven vertices $x_1, x_2, x_3, x_4, x_5, y_1, y_2$, given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 & 0 \\ 3/2 & 1/2 & 3/2 & 3/2 & 0 & 3/2 & 1/2 \\ 3/2 & 1/2 & 2 & 1/2 & 1/2 & 3/2 & 1/2 \end{pmatrix}.$$

In particular two of the vertices are not integral. This shows that the semigroup $S_{\lambda,\mu}$ is not generated in degree one. In fact a minimal presentation of the ring $R_{\lambda,\mu}$ was given in [DO]. There are lifts of the above vertices, say $X_1, X_2, X_3, X_4, X_5, Y_1, Y_2$, such that the X_i's and the difference $Y_1 - Y_2$ generate $R_{\lambda,\mu}$. Since the x_i's form a 4-simplex we can already see that there can be no relations in the X_i's. In [DO] it is shown that there is a single relation, which expresses $(Y_1 - Y_2)^2$ as quartic polynomial in the X_i's.

This brings us to the question of how large denominators can be in Gel'fand Tsetlin polytopes. The following theorem gives an upper bound.

Theorem 3.2.2 *(De Loera and McAllister [dLMc]) The denominators of vertices of the polytope $GT(\lambda, \mu)$ are bounded above by*

$$D_n = (n-1)^{C(n+1,2)-n-1},$$

where $C(n,k)$ is the number of k-sized subsets of $\{1, 2, \ldots, n\}$, i.e. "n choose k". (By denominator of a rational vector v we mean the least integer k such that kv is integral.)

Now as a quick consequence of the above two theorems we get a finite set of generators for $R_{\lambda,\mu}$.

Theorem 3.2.3 *The ring $R_{\lambda,\mu}$ is generated by elements of degree less than or equal to $(C(n-1,2)+1)\,D_n$.*

Proof. Suppose that P is a rational polytope. Let M be a natural number such that for each vertex $v \in P$ there exists a natural number $k_v \leq M$ such that $k_v v$ is an integral vector. Suppose that the dimension of P is d. We claim the integral points in the convex hull of the origin with the dilation $(d+1)M\,P$ generate the semigroup of integral points in the cone CP of P, where $CP = \{tp \mid t \in \mathbb{R}, t \geq 0, p \in P\}$. First triangulate P into simplices $P_1, \ldots, P_\ell$ such that each vertex of each simplex P_i is a vertex of P (see [Z]). Now fix $q \in CP$ such that q is integral. Then $q \in CP_i$ for some i, since the P_i's cover P. Let Π_i be the parallelopiped generated by the edges

$[0, k_v v]$ for v a vertex of P_i. That is,

$$\Pi_i = \Big\{ \sum_{v \in \mathrm{vert}(P_i)} t_v k_v v \mid 0 \le t_v \le 1 \Big\}.$$

There are natural numbers a_v for v a vertex of P_i such that q lies in the translate,

$$\Pi_i + \sum_{v \in \mathrm{vert}(P_i)} a_v k_v v.$$

Now, $q - \sum_v a_v k_v v$ is an integral vector $\bar{q}$ in Π_i. The points $k_v v$ and $\bar{q}$ are all in the convex hull of the origin with the $(d+1)M$-th dilation of P, since each $k_v \le M$ and there are only $d+1$ vertices of P_i.

We claim the dimension of $GT(\lambda, \mu)$ is at most $C(n-1, 2)$. Indeed the dimension of G/B is $C(n, 2)$, and the dimension of T is $n-1$. Hence the dimension of the G.I.T. quotient is at most $C(n,2) - (n-1) = C(n-1, 2)$. Since there is flat degeneration of $R_{\lambda,\mu}$ to the semigroup algebra associated to the cone on $GT(\lambda, \mu)$, we have,

$$\dim_{\mathbb{R}} GT(\lambda, \mu) = \dim_{\mathbb{C}} \mathrm{Proj}(R_{\lambda,\mu}) \le C(n-1, 2).$$

Now from the DeLoera McAllister bound on denominators of vertices, we get that the semigroup $S_{\lambda,\mu}$ is generated by elements of degree at most

$$(C(n-1,2)+1)\Big((n-1)^{C(n+1,2)-n-1}\Big).$$

We may lift these generators to get generators of $R_{\lambda,\mu}$. $\square$

Unfortunately this is a terribly large bound on degree. Most likely the ring is generated by a much smaller set of generators. The advantage to the theorem is that the bound is not dependent on the pair of weights but only on n.

Chapter 4

Matroids and semistability of flags

In this chapter[1] we apply a theorem of Gel'fand, Goresky, MacPherson, and Serganova about matroids and matroid polytopes to study semistability of flags relative to a given T-linearization of L_λ. In this chapter, the space F is meant to be the space of partial flags associated to the dominant weight λ. The main theorem of this chapter is the following: if there exists a nonzero T-invariant global section of L_λ, then for each semistable flag $x \in F$, there exists a T-invariant global section s of L_λ such that $s(x) \neq 0$. Hence the global T-invariant sections of L_λ determine a well-defined map from $T \backslash\backslash F$ to projective space, provided there is at least one such which is nonzero.

A related result in this note is that the closure of any T-orbit in F is projectively normal for any projective embedding of F. The proof of this fact uses essentially the same argument given for the semistability theorem above.

We take one step towards a solution to the generators problem (for $G = \mathrm{SL}(n, \mathbb{C})$) with Theorem 4.1.1, which implies that the lowest degree T–invariants in the graded ring of F are sufficient to give a well-defined map from $T \backslash\backslash F$ to projective space. Consequently these global sections determine an ample line bundle M of $T \backslash\backslash F$. We are left with the problem of determining which tensor power of M

[1] The contents in this chapter are intended for publication; the preprint [H] is available on the archive.

is very ample.

The proof of Theorem 4.1.1 involves a simple combinatorial argument involving Minkowski sums of weight polytopes of flags. These weight polytopes are also known as flag matroid polytopes, see [BGW]. Two facts are essential to the argument:

- Any subset of $\mathrm{SL}(n, \mathbb{C})$ roots which are linearly independent may be extended to a basis of the root lattice.

- Each edge of a matroid polytope is parallel to a root of $\mathrm{SL}(n, \mathbb{C})$ (due to Gel'fand, Goresky, MacPherson, Serganova).

Remark 4.0.4 The first fact is specific to $\mathrm{SL}(n, \mathbb{C})$*. The root systems of other classical complex simple Lie algebras do not have this remarkable saturation property. However, the second result is a special case of the Gel'fand–Serganova theorem which is one of the central theorems in the new subject of Coxeter matroids, see [BGW]. It should be noted that Theorem 4.1.1 easily follows from a theorem of Neil White [W] in the case that F is a Grassmannian.*

Additionally, the tools we develop in proving Theorem 4.1.1 also allow us to show that the closure of a T-orbit $cl(T \cdot x)$ for any $x \in F$ (for any projective embedding of F) is a projectively normal toric variety. Again Neil White [W] showed this holds when F is a Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. Additionally, R. Dabrowski [Dab] proved that projective normality holds for closures of certain *generic* T-orbits in other homogeneous spaces G/P (he covered the case that G is any semi-simple complex Lie group).

4.1 The semistability theorem

The proof of the following theorem will be given in section §4.3. This theorem allows us to explicitly construct an ample line bundle of $T \backslash\backslash F$, and to cover $T \backslash\backslash F$ by explicit affine varieties.

Theorem 4.1.1 *(Semistability Theorem) Suppose that λ is a dominant weight and μ is any weight, such that $\lambda - \mu$ lies in the root lattice of* $\mathrm{SL}(n, \mathbb{C})$*. Then if $p \in F$ is μ–semistable there is a global T-invariant section s of L_λ such that $s(p) \neq 0$.*

Remark 4.1.2 If $\lambda - \mu$ is not in the root lattice, then $\Gamma(F, L_\lambda)^T$ is zero. In fact, $\Gamma(F, L_\lambda)^T$ is nonzero if and only if $\lambda - \mu$ is in the root lattice and μ lies within the convex hull of the Weyl orbit of λ. If μ does not lie in the convex hull of the Weyl orbit of λ, then $\Gamma(F, L_\lambda^{\otimes N})^T$ is zero for all $N > 0$; in this case there are no semistable points in F, and the quotient $T \backslash\backslash F$ is empty.

The following is taken from [Do], chapter 8. Let $s_1, \ldots, s_m$ be a basis of the T-invariant sections of L_λ for the μ-linearization. By theorem 4.1.1, the semistable points F^{ss} are covered by the affine open subsets X_{s_i}, where $X_{s_i} = \{x \in F \mid s_i(x) \neq 0\}$. Let Y_i be the affine quotient $T \backslash\backslash X_{s_i}$; the affine coordinate ring of Y_i is $\mathcal{O}(X_{s_i})^T$. The Y_i's may be glued together via the transition functions s_i/s_j to form the G.I.T. quotient $T \backslash\backslash F$, and simultaneously an ample line bundle M of $T \backslash\backslash F$, such that $\pi^*(M) = \iota^*(L_\lambda)$, where $\iota : F^{ss} \to F$ is the inclusion map.

As stated in the introduction, it remains an open problem to compute the minimal integer N such that $M^{\otimes N}$ is very ample.

4.2 Matroid polytopes and weight polytopes

A matroid is a pair $M = (E, \mathcal{B})$ where E is a finite set called the ground set of M, and $\mathcal{B}$ is a nonempty collection of subsets of E called bases that satisfy the exchange condition, which is that for any two bases $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \setminus B_2$ then there is an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ is a basis. Necessarily it follows that all bases $B \in \mathcal{B}$ have the same cardinality, which is called the rank of M. Matroids are a generalization of finite configurations of vectors, where the only data known about the set of vectors is which subsets are maximal independent subsets. The collection of maximal independent subsets satisfies the exchange condition. Similarly, a linear subspace Λ of dimension k of $\mathbb{C}^n$ determines a matroid $M(\Lambda)$, given by the vector configuration $\{\pi_\Lambda(e_1), \ldots, \pi_\Lambda(e_n)\}$ where π_Λ is orthogonal projection onto Λ (for the standard Hermitian form) and the e_i's are the standard basis vectors of $\mathbb{C}^n$.

4.2.1 Matroid polytopes

Suppose that $M = (E, \mathcal{B})$ is a matroid, and $E = \{1, 2, \ldots, n\}$. For each $B \in \mathcal{B}$ let $v^B \in \mathbb{R}^n/\Delta$ (Δ is the diagonal in $\mathbb{R}^n$) be given by $v_i^B = 0$ if $i \notin B$ and $v_i^B = 1$ if $i \in B$. Let P_M be the convex hull of $\{v^B \mid B \in \mathcal{B}\}$. We call P_M a matroid polytope. Each v^B is a vertex of P_M and so M may be recovered from P_M.

Theorem 4.2.1 *(Gel'fand Goresky MacPherson Serganova [GGMS]) Two vertices v^{B_1}, v^{B_2} of P_M form an edge of P_M iff $v^{B_1} - v^{B_2} = e_i - e_j$ for some $i \neq j$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. In other words, edges of P_M are* *parallel to roots of* $\mathrm{SL}(n,\mathbb{C})$. *(In fact, the bases* B_1 *and* B_2 *differ by a single element iff* v^{B_1} *and* v^{B_2} *form an edge of* P_M*.)*

Conversely, if P *is a polytope where all vertices are* $0/1$ *vectors (each component is either* 0 *or* 1*), and each edge of* P *is parallel to an* $\mathrm{SL}(n,\mathbb{C})$ *root, then there is a matroid* M *such that* $P = P_M$.

Remark 4.2.2 A natural way that matroid polytopes arise is by restricting the momentum mapping $\rho : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathfrak{t}_0^*$ *for the action of the maximal compact subtorus* T_0 *in* T *on the Grassmannian to the closure of an orbit* $T\cdot\Lambda$*, see [GGMS] or [BGW]. The polytope* $P_{M(\Lambda)}$ *is the image of* ρ *restricted to the closure of* $T\cdot\Lambda$.

4.2.2 Weight polytopes

Suppose that V is a finite dimensional complex representation of a torus T. Then V is a direct sum of weight spaces,

$$V = \bigoplus_{\mu} V[\mu],$$

where $V[\mu] = \{v \in V \mid t\cdot v = e^{\mu}(t)v \text{ for all } t \in T\}$. Note that a section $s \in V_\lambda = \Gamma(F, L_\lambda)$ is T-invariant for the μ-linearization of L_λ if and only if $s \in V_\lambda[\mu]$.

Given a dominant weight λ let P_λ denote the associated parabolic subgroup. For each $g \in G$, let

$$wt_\lambda(g) = \{\mu \mid (\exists s \in V_\lambda[\mu])(s(gP_\lambda) \neq 0)\}.$$

Let the *weight polytope* $\overline{wt}_\lambda(g)$ be the convex hull of $wt_\lambda(g)$ (the convex hull is taken in $\mathfrak{t}_0^*$, where $\mathfrak{t}_0$ is the Lie algebra of the maximal compact torus $T_0 \subset T$).

Lemma 4.2.3 *For any two dominant weights λ_1 and λ_2, we have*

$$wt_{\lambda_1}(g) + wt_{\lambda_2}(g) = wt_{\lambda_1+\lambda_2}(g),$$

where the summation denotes the Minkowski sum, $A + B = \{a + b \mid a \in A, b \in B\}$.

Proof. Suppose that $\mu_1 \in wt_{\lambda_1}(g)$ and $\mu_2 \in wt_{\lambda_2}(g)$. Let $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$ such that $s_1(gP_{\lambda_1}) \neq 0$ and $s_2(gP_{\lambda_2}) \neq 0$. Recall there are functions $f_1, f_2 : G \to \mathbb{C}$ such that $s_1 = s_{f_1}$ and $s_2 = s_{f_2}$. We have that $f_1(g) \neq 0$ and $f_2(g) \neq 0$. Hence, $f_1(g)f_2(g) \neq 0$. The section $s_{f_1 f_2}$ lies in $V_{\lambda_1+\lambda_2}[\mu_1 + \mu_2]$, and is nonzero at $gP_{\lambda_1+\lambda_2}$.

Now suppose that $\mu \in wt_{\lambda_1+\lambda_2}(g)$. We may identify the irreducible representation V_λ as the space of global sections of $\pi^*(L_\lambda)$ of G/B where B is the Borel subgroup of G and $\pi : G/B \to G/P_\lambda$. This is justified since the pullback $\pi^* : \Gamma(G/P_\lambda, L_\lambda) \to \Gamma(G/B, \pi^*(L_\lambda))$ is an isomorphism of vector spaces. We shall also abuse notation and identify L_λ with the pullback $\pi^* L_\lambda$.

The tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ is the vector space of sections of the outer tensor product $L_{\lambda_1} \boxtimes L_{\lambda_2}$ of $G/B \times G/B$, where B is the Borel subgroup. The irreducible representation $V_{\lambda_1+\lambda_2}$ is a direct summand of $V_{\lambda_1} \otimes V_{\lambda_2}$, and the projection $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_1+\lambda_2}$ is realized by pulling back $L_{\lambda_1} \boxtimes L_{\lambda_2}$ to the diagonal $\Delta \subset G/B \times G/B$. We have assumed there is a section $s \in V_{\lambda_1+\lambda_2}[\mu]$ such that $s(gB) \neq 0$. Clearly $(V_{\lambda_1} \otimes V_{\lambda_2})[\mu]$ surjects onto $V_{\lambda_1+\lambda_2}[\mu]$. Furthermore,

$$(V_{\lambda_1} \otimes V_{\lambda_2})[\mu] = \sum_{\mu_1+\mu_2=\mu} V_{\lambda_1}[\mu_1] \otimes V_{\lambda_2}[\mu_2].$$

Hence there must exist weights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$ and some component $s' = s_1 s_2$ of s such that $s_1(gB)s_2(gB) \neq 0$ and $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$. $\square$

Corollary 4.2.4 *Suppose that* $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$ *is dominant, i.e. each* a_i *is non-negative and* ϖ_i *denotes the* i*-th fundamental weight connected to the Grassmannian* $\mathrm{Gr}_i(\mathbb{C}^n)$*. Then for any* $g \in G$*,*

$$wt_\lambda(g) = \sum_{i=1}^{n-1} a_i \cdot wt_{\varpi_i}(g),$$

where the sum indicates Minkowski sum and $a_i \cdot wt_{\varpi_i}(g)$ *denotes the* a_i*-fold Minkowski sum of* $wt_{\varpi_i}(g)$*.*

The weight polytope $\overline{wt}_\lambda(g)$ is a *flag matroid polytope* within the more general setting of Coxeter matroid polytopes, see [BGW]. However, we will only need to consider standard matroid polytopes, as they are the building blocks for flag matroid polytopes.

Proposition 4.2.5 *Suppose that* ϖ_k *is the* k*-th fundamental weight. Then* $\overline{wt}_{\varpi_k}(g)$ *is a matroid polytope for any* $g \in G$*.*

Proof. A basis for the sections of L_{ϖ_k} is given by bracket functions $[i_1, i_2, \ldots, i_k]$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The section $s = [i_1, i_2, \ldots, i_k]$ is equal to s_f, where $f : G \to \mathbb{C}$ assigns the determinant of the k by k submatrix given by columns $1, 2, \ldots, k$ and rows $i_1, i_2, \ldots, i_k$ of $g \in G$. The bracket $[i_1, i_2, \ldots, i_k]$ belongs to the weight space $V_{\varpi_k}[\mu]$ where $e^\mu = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n/\Delta$ is given by $a_i = 1$ if $i = i_t$ for some t, $1 \leq t \leq k$, otherwise $a_i = 0$. Now suppose that $gP_{\varpi_k} \in G/P_{\varpi_k} = \mathrm{Gr}_k(\mathbb{C}^n)$. The linear subspace defined by gP_{ϖ_k} is the span of the first k columns of g. We have that $\mu \in \overline{wt}_{\varpi_k}(g)$ iff μ is a $0/1$ vector (mod Δ) with k ones (occuring at $I = (i_1, i_2, \ldots, i_k)$) and $n - k$ zeros such that the I-th minor is nonzero.

Let $M(g)$ be the matroid with ground set $\{1, 2, \ldots, n\}$ of the vector configuration $r_1, r_2, \ldots, r_n \in \mathbb{C}^k$ where r_i is the i-th row of g restricted to the first k columns, i.e. $r_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,k})$. It is clear that the matroid polytope of $M(g)$ is the weight polytope $\overline{wt}_{\varpi_k}(g)$. $\square$

4.3 Saturation properties of weight polytopes

We shall prove the following lemma by a combinatorial argument. The main theorem 4.1.1 easily follows from this lemma. Neil White proved in [W] the exact same statement for $\lambda = \varpi_k$, using a theorem of Edmonds in matroid theory.

Lemma 4.3.1 *Suppose $g \in G$ and λ is a dominant weight. Suppose μ is a weight such that $\lambda - \mu$ is in the root lattice. Then $N\mu \in wt_{N\lambda}(g)$ implies $\mu \in wt_\lambda(g)$ for all $N > 0$.*

Remark 4.3.2 If G is any complex semisimple group, and λ is a dominant weight, and $\lambda - \mu$ is in the root lattice of G, then $V_{N\lambda}[N\mu] \neq 0$ implies $V_\lambda[\mu] \neq 0$. However, the lemma is a much stronger statement than this (and it does not hold for groups other than $\mathrm{SL}(n, \mathbb{C})$) because g is fixed (i.e., the point $gP_\lambda \in G/P_\lambda$ is fixed).

Let R be the set of $\mathrm{SL}(n, \mathbb{C})$ roots. Let $Q(R)$ (resp. $P(R)$) denote the root lattice (resp. weight lattice). Convex hulls of subsets of the weight lattice, denoted by an overline, should take place in $\mathfrak{t}_0^*$, which is isomorphic to $P(R) \otimes \mathbb{R} = \overline{P(R)} = \mathbb{R}^n/\Delta$. The map $\epsilon : P(R) \to \mathbb{Z}/n\mathbb{Z}$ given by $\epsilon(a_1, \ldots, a_n) = \sum_i a_i \mod n$ is a homomorphism of abelian groups, and $Q(R) = \ker(\epsilon)$.

Definition 4.3.3 *A finite subset A of $Q(R)$ is called root–saturated if*

- *the convex hull $\overline{A}$ is such that each edge e_i is parallel to a root γ_i in R, (i.e. $\overline{A}$ is a flag matroid polytope, see [BGW].)*
- *$A = \overline{A} \cap Q(R)$.*

We will eventually prove that $wt_\lambda(g) - \lambda$ (the set $wt_\lambda(g)$ translated by $-\lambda$) is root-saturated for any dominant weight λ.

Lemma 4.3.4 *Suppose that $\alpha_1, \ldots, \alpha_{n-1} \in R$ are independent over $\mathbb{Q}$. Then they are a basis for the root lattice $Q(R)$.*

Proof. The proof goes by induction on n. If $n = 2$ there are only two roots $\alpha, -\alpha$ and they generate the same lattice. Now suppose that $n > 2$. Let $\mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}]$ be the $\mathbb{Z}$–span of $\alpha_1, \ldots, \alpha_{n-1}$. Without loss of generality we may assume that each α_i is a positive root since negating α_i does not change the span over $\mathbb{Z}$. Let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard simple roots of $\mathrm{SL}(n, \mathbb{C})$. That is, $\sigma_i = e_i - e_{i+1}$. Note that any positive root $e_i - e_j = \sum_{t=i}^{j-1} \sigma_t$ is a sum of consecutive simple roots. Conversely any consecutive sum of simple roots is a positive root. We may choose some $w \in W$ (where W is the Weyl group) such that $w(\alpha_{n-1}) = \sigma_{n-1}$. In particular if $\alpha_{n-1} = e_i - e_j$ let w be the product of two cycles $(n-1 \quad i)(n \quad j)$. Since elements of W induce isomorphisms of the lattice $Q(R)$, we have that $w(\alpha_1), \ldots, w(\alpha_{n-1})$ is a basis of $Q(R)$ if and only if $\alpha_1, \ldots, \alpha_{n-1}$ is a basis of $Q(R)$. Reassign $\alpha_i := w(\alpha_i)$. For each $i \leq n-2$, if $\alpha_i = e_s - e_n = \sigma_s + \cdots + \sigma_{n-1}$ replace α_i with $\alpha_i - \sigma_{n-1} = \sigma_s + \cdots + \sigma_{n-2} = e_s - e_{n-1}$. Now the roots $\alpha_1, \ldots, \alpha_{n-2}$ may be identified with roots of $\mathrm{SL}(n-1,\mathbb{C})$. By the induction hypothesis $\mathbb{Z}[\alpha_1,\ldots,\alpha_{n-2}] = \mathbb{Z}[\sigma_1,\ldots,\sigma_{n-2}]$. Since $\alpha_{n-1} = \sigma_{n-1}$ we have that $\mathbb{Z}[\alpha_1,\ldots,\alpha_{n-1}] = Q(R)$.

$\square$

Lemma 4.3.5 *Suppose that A and B are root-saturated, and $\overline{A}\cap\overline{B}$ is nonempty. Then $A\cap B$ is nonempty.*

Proof. The proof is by induction on the dimension of $\overline{A}$. If $\dim\overline{A} = 0$ then $A = \{a\}$ for some $a \in Q(R)$. Then $\overline{A}\cap\overline{B} = A\cap B = \{a\}$. Now suppose that $\dim\overline{A} \geq 1$.

We have two cases, the first case is that the intersection $\overline{A}\cap\overline{B}$ contains a boundary point of $\overline{A}$. Then there is some facet F of $\overline{A}$ such that $F\cap\overline{B}$ is nonempty. We claim $F\cap A$ is root–saturated. The vertices of F are within $F\cap A$, so $\overline{F\cap A} \supset F$. On the other hand $F \subset F\cap A$ so $F \subset \overline{F\cap A}$; therefore $F = \overline{F\cap A}$. The edges of F are also edges of $\overline{A}$ hence they are parallel to roots. Furthermore, for any $x \in F\cap A$, we have $\overline{F\cap A}\cap Q(R) \subset A$ since A is root–saturated, and it follows that $\overline{F\cap A}\cap Q(R) = F\cap A$ since $F\cap A \subset A \subset Q(R)$. Since $\dim F < \dim\overline{A}$ we may apply the induction hypothesis to get that $F\cap A\cap B$ is nonempty and hence $A\cap B$ is nonempty.

On the other hand suppose that $\overline{A}\cap\overline{B}$ contains no boundary point of $\overline{A}$. Let $L_A(R)$ be the sub–lattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{A}$. Let $a_0 \in A$ be a vertex of $\overline{A}$. Note that the affine space $H_A = a_0 + \overline{L_A(R)}$ is the smallest affine space containing $\overline{A}$. We claim $H_A\cap\overline{B} = \overline{A}\cap\overline{B}$. Suppose that $z \in H_A\cap\overline{B}$. Let $a \in \overline{A}\cap\overline{B}$. Since H_A has the same dimension as $\overline{A}$, there are linear inequalities $\eta_i(x) \leq f_i$ where the interior of $\overline{A}$ consists of points $x \in H_A$

where the inequalities are strict; that is, $\eta_i(x) < f_i$ for all i if and only if x is an interior point of $\overline{A}$. The boundary points of $\overline{A}$ are those points $x \in \overline{A}$ such that $\eta_i(x) = f_i$ for some i. Let $c(t) = (1-t)a + tz$ for $0 \leq t \leq 1$. Suppose that $z \notin \overline{A}$. Then there is some i such that $\eta_i(z) > f_i$. However a is an interior point of $\overline{A}$ and so $\eta_i(a) < f_i$. Hence there is some t_0 such that $\eta(c(t_0)) = f_i$ in which case $c(t_0)$ is a boundary point of $\overline{A}$. But $c(t) \in \overline{B}$ for each t by convexity of $\overline{B}$. This contradicts that $\overline{A} \cap \overline{B}$ is disjoint from the boundary of A. Hence $H_A \cap \overline{B} = \overline{A} \cap \overline{B}$. Therefore $(H_A \cap Q(R)) \cap B = A \cap B$ since $\overline{A} \cap Q(R) = A$ and $\overline{B} \cap Q(R) = B$.

We now show by induction on $\dim \overline{B}$, that for any B which is root-saturated, that $H_A \cap \overline{B}$ is nonempty implies $(H_A \cap Q(R)) \cap B$ is nonempty. Suppose that $\dim \overline{B} = 0$. Then $B = \{b\}$ for some $b \in Q(R)$, and so $b \in (H_A \cap Q(R)) \cap B$. Now suppose that $\dim \overline{B} \geq 1$. We have two cases.

First suppose that $H_A \cap \overline{B}$ intersects the boundary of $\overline{B}$ nontrivially. Then there is a face F of $\overline{B}$ such that $H_A \cap F$ is nonempty. Since $F \cap B$ is root–saturated, $\overline{F \cap B} = F$, $H_A \cap F$ is nonempty, and $\dim F < \dim B$, we may apply the induction hypothesis and we're finished.

Now suppose that $H_A \cap \overline{B}$ is disjoint from the boundary of $\overline{B}$. Let $L_B(R)$ be the sub–lattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{B}$. Let $b_0 \in B$ be a vertex of $\overline{B}$. The affine space $H_B = b_0 + \overline{L_B(R)}$ is the smallest affine space containing $\overline{B}$. As above, we have that $H_A \cap H_B = H_A \cap \overline{B}$ and so $(H_A \cap Q(R)) \cap (H_B \cap Q(R)) = A \cap B$. Since H_A does not intersect the boundary of $\overline{B}$, we have that $H_A \cap H_B$ is a single point z_0, since if the dimension of the intersection $H_A \cap H_B = H_A \cap \overline{B}$ is greater than zero then $H_A \cap \overline{B}$ is unbounded.

But $\overline{B}$ is compact since B is finite and this cannot happen. We now show that $z_0 \in Q(R)$. We have that $z_0 = a_0 + v_A = b_0 + v_B$ where $a_0 \in A$, $b_0 \in B$, $v_A \in \overline{L_A(R)}$, $v_B \in \overline{L_B(R)}$. Let $\{\alpha_1, \ldots, \alpha_p\} \subset R$ be a basis of $L_A(R)$ and let $\{\beta_1, \ldots, \beta_q\} \subset R$ be a basis of $L_B(R)$. Since the intersection of H_A and H_B is a point, we have that $\overline{L_A(R)} \cap \overline{L_B(R)} = \{0\}$. Hence the set $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ is linearly independent in $\overline{Q(R)}$. Choose $\{\gamma_1, \ldots, \gamma_r\} \subset R$ so that $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_r\}$ is a basis for $\overline{Q(R)}$. By the Lemma above this is also a basis for the lattice $Q(R)$. Now $v_A = \sum_i c_i \alpha_i$ and $v_B = \sum_j d_j \beta_j$ are unique expressions for v_A, v_B. But also the difference $a_0 - b_0 = v_B - v_A = (\sum_j d_j \beta_j) - (\sum_i c_i \alpha_i)$ lies within the lattice $Q(R)$, and so the coefficients c_i and d_j must be integers. Hence, z_0 is a lattice point and we've finished the proof of the Lemma. $\square$

Theorem 4.3.6 *Suppose that A and B are root-saturated. Then the Minkowski sum $A + B = \{a + b \mid a \in A, b \in B\}$ is root-saturated.*

Proof. We show that the Minkowski sum $A + B$ is root-saturated if A and B are each root-saturated. Clearly $A+B$ is finite, and the elements are within $Q(R)$ since $Q(R)$ is closed under addition. First we show that the edges of $\overline{A+B}$ are parallel to roots. Clearly $\overline{A} + \overline{B} = \overline{A+B}$. The Minkowski sum of two polytopes P, Q has edges of the following types:

- (vertex of P) + (edge of Q).
- (edge of P) + (vertex of Q).
- (edge of P) + (edge of Q), providing these edges are parallel.

We leave the proof to the reader (the proof is easily obtained by observing that the fan of $P+Q$ is the meet of the fan of P with the fan of Q). In all three cases, the resulting edge is parallel to an edge of either P or Q or both, and hence it is parallel to some root in R.

Next we must show that $A+B = (\overline{A+B}) \cap Q(R)$. Suppose that $z \in (\overline{A+B}) \cap Q(R)$. Hence there exists $x \in \overline{A}$ and $y \in \overline{B}$ such that $x + y = z$. Hence $x \in (z + \overline{-B}) \cap \overline{A}$, where $-B = \{-b : b \in B\}$. Clearly $z + (-B)$ is root-saturated. Hence, we may apply the Lemma above to get a lattice point x_0 in the intersection. Since A is saturated, we have that $x_0 \in A$. Now we have that $z = x_0 + y_0$ where $y_0 \in \overline{B}$. But since $z, x_0 \in Q(R)$ we have that $y_0 = z - x_0 \in Q(R)$, and so $y_0 \in B$ since B is root-saturated, and we're finished. $\square$

Lemma 4.3.7 *If ϖ_k is a fundamental weight and $g \in G$ then the translation $wt_{\varpi_k}(g) - \varpi_k$ is root-saturated.*

Proof. Note that all elements of $wt_{\varpi_k}(g)$ are $0/1$ vectors (mod Δ) having k ones and $n-k$ zeros. Translating by $-\varpi_k$ results in vectors whose first k components may be either 0 or -1 and last $n-k$ components are 0 or $+1$, and the sum of all components is zero. Hence the first k components define a vertex of the negated unit k-cube $[0,1]^k$, and the last $n-k$ components are vertices of the $(n-k)$-cube. Therefore, there can be no additional lattice points in the convex hull. We already showed that the convex hull of $wt_{\varpi_k}(g)$ is a matroid polytope, so the edges are parallel to roots. This property is preserved by translations. $\square$

Corollary 4.3.8 *For any dominant weight λ and $g \in G$, the set $wt_\lambda(g) - \lambda$ is root-saturated.*

Proof. We have that $\lambda = \sum_{k=1}^{n-1} a_k \varpi_k$, where the a_k's are non-negative integers. Also, $wt_\lambda(g) = \sum_{k=1}^{n-1} a_k \cdot wt_{\varpi_k}(g)$ (Minkowski sum). Hence,

$$wt_\lambda(g) - \lambda = \sum_{k=1}^{n-1} a_k \cdot (wt_{\varpi_k}(g) - \varpi_k).$$

Since the root-saturated property is preserved under Minkowski sums, we have that $wt_\lambda(g) - \lambda$ is root-saturated. □

Proof of lemma 4.3.1. Proof. Suppose that $N\mu \in wt_{N\lambda}(g)$. Then $N(\mu - \lambda) \in wt_{N\lambda}(g) - N\lambda$. The convex hull of $wt_{N\lambda}(g) - N\lambda$ scaled by $1/N$ is equal to the convex hull of $wt_\lambda(g) - \lambda$ since $N \cdot wt_\lambda(g) = wt_{N\lambda}(g)$. Therefore $\mu - \lambda$ is in the convex hull of $wt_\lambda(g) - \lambda$. But since $\mu - \lambda \in Q(R)$ and $wt_\lambda(g) - \lambda$ is root-saturated, we have that $\mu - \lambda \in wt_\lambda(g) - \lambda$, so $\mu \in wt_\lambda(g)$. □

Proof of main theorem 4.1.1. Proof. Suppose that gP_λ is semistable relative to the μ-linearization of the line bundle L_λ. This means there is some $N > 0$ and a section $s \in \Gamma(G/P_\lambda, L_\lambda^{\otimes N})^T$ such that $s(gP_\lambda) \neq 0$. This means that $N\mu \in wt_{N\lambda}(g)$. By Lemma 4.3.1 we have that $\mu \in wt_\lambda(g)$. So there must exist a section $s' \in \Gamma(G/P_\lambda, L_\lambda)^T$ such that $s'(gP_\lambda) \neq 0$. □

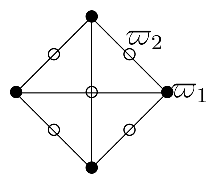
4.3.1 Failure of semistability theorem for $G = \mathrm{SO}(5, \mathbb{C})$

Let $B(z, w)$ be the bilinear form on $\mathbb{C}^5$ given by

$$B(z, w) = z_1w_5 + z_2w_4 + z_3w_3 + z_4w_2 + z_5w_1 = 2z_1w_5 + 2z_2w_4 + z_3w_3.$$

Now $\mathrm{SO}(5,\mathbb{C}) \subset \mathrm{SL}(5,\mathbb{C})$ is the subgroup preserving B. The maximal torus T may be taken to the diagonal matrices in $\mathrm{SO}(5,\mathbb{C})$. Elements of T have the form $diag(t_1, t_2, 1, 1/t_2, 1/t_1)$ for $t_1, t_2 \in \mathbb{C}^*$. Let ϖ_1 denote the first fundamental weight of $\mathrm{SO}(5,\mathbb{C})$. We have $e^{\varpi_1}(t_1, t_2, 1, 1/t_1, 1/t_2) = t_1$, but the second fundamental weight does not lift to a character of $\mathrm{SO}(5,\mathbb{C})$ - one needs to go the universal cover to find such a character. Let $P_{\varpi_1} \subset \mathrm{SO}(5,\mathbb{C})$ be the associated parabolic subgroup. The quotient space $\mathrm{SO}(5,\mathbb{C})/P_{\varpi_1}$ may be identified with the space of isotropic lines in $\mathbb{C}^5$.

Let x be the (isotropic) line through $(1, \sqrt{-1}, 0, \sqrt{-1}, 1)$. Let $g_x \in \mathrm{SO}(5,\mathbb{C})$ be such that $g_x P_{\varpi_1} = x$. The set $wt_{\varpi_1}(g_x)$ is equal to $\{\varpi_1, 2\varpi_2 - \varpi_1, -2\varpi_2 + \varpi_1, -\varpi_1\}$. Depiction:

$wt_{\varpi_1}(g_x) =$ 

This set is missing the origin, although $V_{\varpi_1}[0] \neq 0$ and $\varpi_1 \in Q(\mathrm{SO}(5,\mathbb{C}))$, so $wt_{\varpi_1}(g_x) - \varpi_1$ is not root-saturated. Note the origin does belong to $wt_{2\varpi_1}(g_x) = wt_{\varpi_1}(g_x) + wt_{\varpi_1}(g_x)$. Therefore x is semistable for the democratic linearization of L_{ϖ_1}. It follows that for the democratic linearization of L_{ϖ_1}, one requires a T-invariant section of $L_{\varpi_1}^{\otimes 2}$ to pick out the semistable point x.

4.4 Projective normality of T-orbits in the space of partial flags

Let H be the group of diagonal matrices in $\mathrm{GL}(n,\mathbb{C})$. Hence $T \subset H$ is the set of diagonal matrices with determinant one. Let $\chi_1, \ldots, \chi_m$ be m characters of H. That is, each $\chi_i : H \to \mathbb{C}^*$ is an algebraic homomorphism of groups. Each χ_i is given by a point $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{Z}^n$, where

$$\chi_i(h_1, \ldots, h_n) = \prod_{j=1}^{n} h_j^{a_{i,j}}.$$

These characters determine an action of H on $\mathbb{A}^m$ by

$$h \cdot (z_1, z_2, \ldots, z_m) = (\chi_1(h)z_1, \chi_2(h)z_2, \ldots, \chi_m(h)z_m).$$

Now take any point $z \in \mathbb{A}^m$, and let $X(z)$ be the Zariski closure of the H-orbit of z. That is, $X(z) = cl(H \cdot z)$. Certainly $X(z)$ contains a dense torus and there is a natural action of this torus on $X(z)$; so $X(z)$ is a (possibly non-normal) toric variety.

But when is $X(z)$ a *normal* toric variety, i.e. when is the coordinate ring of $X(z)$ integrally closed in its field of fractions? Some notation: if A is a finite subset of $\mathbb{Z}^d$ then let $\mathbb{Z}(A)$ be the sub-lattice generated by A, let $\mathbb{N}(A)$ be the semigroup of all non-negative integral combinations of elements of A, and let $\mathbb{Q}_0^+(A)$ be the rational cone in $\mathbb{Q}^d$ given by all non-negative rational combinations of elements of A. According to Proposition 13.5 of [St] we have that the semigroup algebra $\mathbb{C}[\mathbb{N}(A)]$ is normal iff $\mathbb{N}(A) = \mathbb{Z}(A) \cap \mathbb{Q}_0^+(A)$.

The following proposition is likely well known but we give a proof for lack of reference.

Proposition 4.4.1 *Let* $\operatorname{supp}(z) = \{i \mid z_i \neq 0\}$. *Let* $A(z) = \{\chi_i \mid i \in \operatorname{supp}(z)\}$. *Then* $X(z)$ *is isomorphic to the affine toric variety defined by* $A(z) \subset \mathbb{Z}^n$. *That is,* $X(z)$ *is isomorphic to the affine variety* $V \subset \mathbb{C}^{\#A(n)}$ *of the semigroup algebra* $\mathbb{C}[\mathbb{N}(A(z))]$, *where* $\mathbb{N}(A(z))$ *is the semigroup in* $\mathbb{Z}^n$ *generated by* $A(z)$. *Hence* $X(z)$ *is normal if and only if* $\mathbb{Z}(A(z)) \cap \mathbb{Q}_0^+(A(z)) = \mathbb{N}(A(z))$.

Proof. Let $\bar{z} \in \mathbb{C}^m$ be given by $\bar{z}_i = 1$ if $i \in \mathrm{supp}(z)$ and $\bar{z}_i = 0$ otherwise. Let $s_i = 1/z_i$ if $z_i \neq 0$ and $s_i = 1$ if $z_i = 0$. Then the matrix $diag(s_1, \ldots, s_m)$ defines an algebraic automorphism of $\mathbb{A}^m$ which takes $X(z)$ to $X(\bar{z})$, so $X(\bar{z})$ is isomorphic to $X(z)$. Hence we may assume that all components of z are either 0 or 1. Additionally, $X(z)$ lives entirely within the components i where z_i is nonzero. Hence, we may project $X(z)$ onto the linear subspace given by the components in $\mathrm{supp}(z)$, which defines an isomorphism of $X(z)$ onto its image. Thus, we may assume that each component of z is equal to one. If $\chi_i = \chi_j$ for some i, j, we may also project away one of these. Hence we have reduced to the case that the χ_i's are distinct, and z is the vector of all ones. The coordinate ring of $X(z)$ is now easily seen to be the semigroup algebra $\mathbb{C}[\mathbb{N}(A(z))]$. $\square$

A dominant weight λ of $\mathrm{SL}(n, \mathbb{C})$ may be lifted to a dominant weight $\widetilde{\lambda}$ of $\mathrm{GL}(n, \mathbb{C})$ by normalizing λ so that the last component is zero. That is, the image of $\widetilde{\lambda} \in \mathbb{Z}^n$ in $\mathbb{Z}^n/\Delta$ is λ, and $\widetilde{\lambda}_n = 0$. Let

$$|\widetilde{\lambda}| = \sum_{i=1}^{n} \widetilde{\lambda}_i.$$

Now V_λ is also an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$, where $zI_n \in \mathrm{GL}(n, \mathbb{C})$ acts by scaling each vector $s \in V_\lambda$ by $z^{|\widetilde{\lambda}|}$, and so if $\widetilde{g} = zg$ where $z \in \mathbb{C}^*$ and $g \in \mathrm{SL}(n, \mathbb{C})$ then the action of $\widetilde{g}$ is defined by $\widetilde{g} \cdot s = z^{|\widetilde{\lambda}|}(g \cdot s)$. A basis for the representation V_λ is given by semi-standard tableaux τ of shape $\widetilde{\lambda}$ (with total number of slots equal to $|\widetilde{\lambda}|$), filled with indices from 1 to n. A section $s_\tau \in V_\lambda[\mu]$ iff the number of times the index i appears in τ is equal to μ_i. Here we are treating μ as a weight of $\mathrm{GL}(n, \mathbb{C})$. Note that if $V_\lambda[\mu] \neq 0$ then $|\mu| = \sum_{i=1}^{n} \mu_i = |\widetilde{\lambda}|$ since $|\mu|$ must equal the total number of slots in τ, where $s_\tau \in V_\lambda[\mu]$.

Recall that $H = \mathbb{C}^*(T)$ is the maximal torus in $\mathrm{GL}(n,\mathbb{C})$ consisting of diagonal matrices. For each $g \in \mathrm{GL}(n,\mathbb{C})$ let

$$wt_{\widetilde{\lambda}}(g) = \{\mu \mid (\exists s \in V_\lambda[\mu])(s(gP_{\widetilde{\lambda}}) \neq 0)\},$$

where $P_{\widetilde{\lambda}} \subset \mathrm{GL}(n,\mathbb{C})$ is the parabolic subgroup $\mathbb{C}^*(P_\lambda)$ associated to $\widetilde{\lambda}$. Each $\mu \in wt_{\widetilde{\lambda}}(g) \subset \mathbb{Z}^n$ satisfies $|\mu| = |\widetilde{\lambda}|$.

Note that the root lattice of $\mathrm{SL}(n,\mathbb{C})$ may be identified with integral vectors $v \in \mathbb{Z}^n$ whose components sum to zero. Hence, for any $g \in \mathrm{SL}(n,\mathbb{C})$ we have an identification of $wt_\lambda(g) - \lambda$ with $wt_{\widetilde{\lambda}}(g) - \widetilde{\lambda}$. In particular, $wt_{\widetilde{\lambda}}(g) - \widetilde{\lambda}$ is root-saturated.

Let $N_{\widetilde{\lambda}}$ be the sub-lattice of $\mathbb{Z}^n$ given by

$$N_{\widetilde{\lambda}} = \{v \in \mathbb{Z}^n \mid \ |v| = \sum_{i=1}^{n} v_i \equiv 0 \mod |\widetilde{\lambda}|\}.$$

Lemma 4.4.2 *For any $g \in \mathrm{SL}(n,\mathbb{C})$,*

$$\mathbb{Q}_0^+(wt_{\widetilde{\lambda}}(g)) \cap N_{\widetilde{\lambda}} = \mathbb{N}(wt_{\widetilde{\lambda}}(g)).$$

Proof. Suppose that $v \in \mathbb{Q}_0^+(wt_\lambda(g)) \cap N_{\widetilde{\lambda}}$. Then $|v| = d|\widetilde{\lambda}|$ for some $d \in \mathbb{N}$. Hence v belongs to the convex hull of the d-th dilate of $wt_{\widetilde{\lambda}}(g)$, so v is in the convex hull of $wt_{d\widetilde{\lambda}}(g)$, since $wt_{d\widetilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\widetilde{\lambda}}(g)$. But $wt_{d\widetilde{\lambda}}(g) - d\widetilde{\lambda}$ is root-saturated, and since $v - d\widetilde{\lambda} \in Q(R)$ we have that $v - d\widetilde{\lambda} \in wt_{d\widetilde{\lambda}}(g) - d\widetilde{\lambda}$. Equivalently, $v \in wt_{d\widetilde{\lambda}}(g)$. Since $wt_{d\widetilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\widetilde{\lambda}}(g)$, we have that $v \in \mathbb{N}(wt_{\widetilde{\lambda}}(g))$. $\square$

Corollary 4.4.3 *The semigroup algebra $\mathbb{C}[\mathbb{N}(wt_{\widetilde{\lambda}}(g))]$ is normal.*

Now suppose that λ is dominant and P_λ is the associated parabolic subgroup. Choose a basis $(s_1, s_2, \ldots, s_N)$ of $V_\lambda = \Gamma(\mathrm{SL}(n, \mathbb{C})/P_\lambda, L_\lambda)$ such that each basis vector is a generalized eigenvector for the democratic T-action. (Recall the democratic action is the restriction of the natural action of $\mathrm{SL}(n, \mathbb{C})$ on V_λ to T.) Let $\iota_\lambda : \mathrm{SL}(n, \mathbb{C})/P_\lambda \to \mathbb{P}(V_\lambda)$ be the projective embedding determined by this choice of basis. Note that one typically embeds G/P_λ into $\mathbb{P}(V_\lambda^*)$ as there is no need for a choice of basis, but it is more convenient for us to embed into $\mathbb{P}(V_\lambda)$.

The following theorem has been proven by R. Dabrowski for certain *generic* T-orbits in G/P for G an arbitrary semisimple complex Lie group, see [Dab]. Herein lies the first proof for *arbitrary* T-orbits in the case $G = \mathrm{SL}(n, \mathbb{C})$.

Theorem 4.4.4 *The Zariski closure of any T-orbit in* $\mathrm{SL}(n, \mathbb{C})/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ *is a projectively normal toric variety.*

Proof. Let $x \in \mathrm{SL}(n, \mathbb{C})/P_\lambda \subset \mathbb{P}(V_\lambda)$. Let $cl(T \cdot x)$ denote the Zariski closure of the orbit $T \cdot x$. Let $\mathrm{Aff}(cl(T \cdot x)) \subset V_\lambda$ denote the associated affine cone; it is easy to see that $\mathrm{Aff}(cl(T \cdot x)) = cl(H \cdot v_x)$ where v_x is any nonzero vector on the line x, since the scalar matrices in H fill out all nonzero multiples of points in $T \cdot v_x$.

Let $g \in \mathrm{SL}(n, \mathbb{C})$ be such that $gP_\lambda = x$. Now $wt_{\widetilde{\lambda}}(g) = A(v_x)$. Hence by Proposition 4.4.1, the affine toric variety $\mathrm{Aff}(cl(T \cdot x))$ is normal if and only if the semigroup algebra $\mathbb{C}[\mathbb{N}(wt_{\widetilde{\lambda}}(g))]$ is normal, which we have already shown. This means that the projective toric variety $cl(T \cdot x)$ is projectively normal. $\square$

Chapter 5

Evenly weighted points on the Riemann sphere

Here we investigate the special case where λ is a multiple of the second fundamental weight ϖ_2. Recall that by the Gel'fand MacPerhson correspondence, we are truly studying $\mathrm{SL}(2, \mathbb{C})$ invariants of n-tuples of points on the projective line $\mathbb{CP}^1$. Indeed in this case the ring $R_{\lambda,\mu}$ will be generated in degree one, by a theorem of Kempe [Ke] from 1894. The next step is to find the relations in the Kempe generators. In [HMSV1] it is found that the relations are generated in degree at most four, for general weighting μ and any number of points. However, if μ has all even components then it was found that the relations were only quadrics. In this thesis we will investigate the case that each μ_i is even in more detail. We find the quadric relations are given by very natural binomials inherited from simple relations among graphs. The idea is motivated by the main theorem of [HMSV2] where it is found that very simple binomial quadrics (together with some linear relations) cut out the projective variety, though perhaps not the ideal. It was left as on open problem if the ideal is generated by these simple relations. In this thesis we partially answer this question by restricting to the case that μ consists of all even integers, and by broadening the set of quadric binomials.

The condition that μ has even components vastly simplifies the study of the ring. Indeed in [HMSV1] the problem is studied for general μ by examining the degenerated toric ring of [FH]. The toric ring $R'_{\lambda,\mu}$ is not generated in degree one unless each μ_i is even.

We will slightly change notation in this section. Here we denote μ by $\mathbf{w}$ (the notation is taken to mean "weighting") and if $\mathbf{w} = (w_1, \ldots, w_n)$ we shall assume that each w_i is a positive even number. Now we set $\lambda = \sum_i w_i/2$ so that $\lambda_1 + \cdots + \lambda_n = w_1 + \cdots + w_n$. Since everything is determined by $\mathbf{w}$, we denote the moduli space by $M_{\mathbf{w}}$. We will denote the graded ring $R_{\lambda,\mathbf{w}}$ simply by $R_{\mathbf{w}}$.

5.1 A toric degeneration of $M_{\mathbf{w}}$

In this chapter we will need to know precisely how the degeneration of [FH] works. Firstly, we will see how generators of $R_{\mathbf{w}}$ may be interpreted as directed multigraphs on vertex set $\{1, 2, \ldots, n\}$ with valency $\mathbf{w}$.

Let $[X_i, Y_i]$ be the i-th point on $\mathbb{CP}^1$. We choose the embedding $\iota_{\mathbf{w}} : (\mathbb{CP}^1)^n \to \mathbb{CP}^{N-1}$, where $N = \prod_i (w_i + 1)$, given by sending the n-tuple $([X_1, Y_1], \ldots, [X_n, Y_n])$ to all monomials $\prod_i X_i^{a_i} Y_i^{b_i}$ where $a_i + b_i = w_i$. Let $\widetilde{R}_{\mathbf{w}}$ denote the subring of $\mathbb{C}[X_1, Y_1, \ldots, X_n, Y_n]$ generated by these monomials. Indeed, the coordinate ring of $(\mathbb{CP}^1)^n$ for the $\iota_{\mathbf{w}}$ embedding may be identified with $\widetilde{R}_{\mathbf{w}}$. There is a unique action of $\mathrm{SL}(2, \mathbb{C})$ on $\widetilde{R}_{\mathbf{w}}$ compatible with the standard diagonal action on $(\mathbb{CP}^1)^n$ via linear fractional transformations. Indeed, for $g \in \mathrm{SL}(2, \mathbb{C})$, let $[X'_i, Y'_i] = [X_i Y_i] g^{-1}$ (right matrix multiplication). Now the action of $g \in \mathrm{SL}(2, \mathbb{C})$ on the monomial $\prod_i X_i^{a_i} Y_i^{b_i}$ is given by

$$g \cdot \prod_i X_i^{a_i} Y_i^{b_i} = \prod_i (X'_i)^{a_i} (Y'_i)^{b_i}.$$

Now we claim,

$$R_{\mathbf{w}} = (\widetilde{R}_{\mathbf{w}})^{\mathrm{SL}(2,\mathbb{C})}.$$

This is quite easy to see. Let $\mathcal{H}$ denote the ample generator of the Picard group of $\mathbb{CP}^1$. Thus the global sections of $\mathcal{H}$ are linear combinations of the homogeneous coordinates X and Y. It is clear that

$$\widetilde{R}_{\mathbf{w}} = \bigoplus_{N=0}^{\infty} \Gamma((\mathbb{CP}^1)^n, (\mathcal{H}^{\otimes w_1} \boxtimes \cdots \boxtimes \mathcal{H}^{\otimes w_n})^{\otimes N}).$$

There is a unique linearization of $\mathrm{SL}(2,\mathbb{C})$ on this line bundle, and the action of $\mathrm{SL}(2,\mathbb{C})$ on the global sections is the same as we illustrated above. By the Gel'fand MacPherson correspondence this is the same as the ring of T invariants of the $\mathbf{w}$-linearization of the line bundle L_λ of G/B where $\lambda = (1/2)(w_1 + \cdots + w_n)\varpi_2$.

The determinants of two by two minors of the matrix,

$$\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_n & Y_n \end{pmatrix},$$

generate all the invariants under the action of $\mathrm{SL}(2,\mathbb{C})$ acting on the right, by the first fundamental theorem of invariant theory. We shall denote the determinant function $X_iY_j - X_jY_i$ by a directed edge $[i,j]$ drawn as a directed graph with vertex set $\{1,\ldots,n\}$. Now a monomial $\mathcal{M} = \prod_t [i_t, j_t]$ in these determinants may be depicted as the directed multigraph having edges $[i_t, j_t]$ for each t. Given a directed multigraph Γ with vertex set $\{1,2,\ldots,n\}$ let the valency of Γ be $(e_1, e_2, \ldots, e_n)$

where e_i is the number of edges (counted with multiplicity) containing the vertex i. It is easy to see that Γ lies in the image of $\iota_{\mathbf{w}}$ iff the valency of Γ is a multiple of $\mathbf{w}$.

Definition 5.1.1 *For each direct multigraph Γ with valency $N\mathbf{w}$ for some $N \in \mathbb{N}$, let X_Γ denote the associated element of $R_{\mathbf{w}}$.*

Proposition 5.1.2 *The ring $R_{\mathbf{w}}$ is generated by the X_Γ for Γ a directed multigraph on vertex set $\{1, 2, \ldots, n\}$ with valency a multiple of $\mathbf{w}$. Indeed the Nth graded piece $R_{\mathbf{w}}^{(N)}$ is spanned by the X_Γ for which Γ has valency $N\mathbf{w}$.*

Proof. Indeed the monomial X_Γ associated to the graph Γ is a T-invariant for the $\mathbf{w}$–linearization iff the valency of Γ is a multiple of $\mathbf{w}$. In general the subring of T-invariants is generated by T-invariant monomials, provided the generators are all generalized eigenvectors of T. $\square$

Theorem 5.1.3 *(Kempe [Ke], 1894) The ring $R_{\mathbf{w}}$ is generated by the X_Γ for Γ a directed multigraph with valency $\mathbf{w}$.*

Remark 5.1.4 Actually Kempe only handled the case that all w_i are equal to one. For a proof handling general weights see [HMSV1].

Let $\mathcal{G}_{\mathbf{w}}$ denote the set of directed multigraphs with valency $\mathbf{w}$. Now let $\widetilde{X}_\Gamma$ be formal variables for each Γ of valency $\mathbf{w}$. We now have a short exact sequence,

$$0 \to I \to \mathbb{C}[\widetilde{X}_\Gamma]_{\Gamma \in \mathcal{G}_{\mathbf{w}}} \to R_{\mathbf{w}} \to 0,$$

where $\widetilde{X}_\Gamma \mapsto X_\Gamma \in R_{\mathbf{w}}$. The remainder of this chapter will be an investigation of the kernel ideal I.

The set of directed graphs on the fixed vertex set $\{1, 2, \ldots, n\}$ form a monoidal semigroup via the operation of disjoint union of their edges. We will use use special notation $\Gamma_1 \cdot \Gamma_2$ for this operation:

Definition 5.1.5 *Suppose that Γ_1 and Γ_2 are directed multi-graphs on vertex set $\{1, 2, \ldots, n\}$. Let $\Gamma_1 \cdot \Gamma_2$ be the directed multi-graph on vertex set $\{1, 2, \ldots, n\}$, where the edge set of $\Gamma_1 \cdot \Gamma_2$ is the multi-set union of the edge sets of Γ_1 and Γ_2.*

Note that the graphs Γ whose valency is a multiple of $\mathbf{w}$ form a sub-semigroup under the above operation.

5.1.1 The Lakshmibai-Gonciulea inspired filtration of $R_{\mathbf{w}}$

We find it useful in defining the filtration to restrict to a linearly independent subset of the X_Γ for $\Gamma \in \mathcal{G}_{\mathbf{w}}$.

Definition 5.1.6 *(Kempe graphs) Let the vertices of a regular planar n-gon be denoted 1 through n, in clockwise order. Let $\mathcal{K}_{\mathbf{w}}^{(N)}$ denote the set of valency $N\mathbf{w}$ multigraphs ($N \geq 0$), with edges drawn as straight line segments joining the vertices of the n-gon above, such that:*

1. *No two edges cross. (If an edge has multiplicity $k > 1$, then it is to be drawn as a single edge labelled with multiplicity k.)*

2. *Each edge $[i, j]$ is oriented such that $i < j$.*

Let $\mathcal{K}_{\mathbf{w}} = \cup_{N \geq 0} \mathcal{K}_{\mathbf{w}}^{(N)}$. We shall call elements of $\mathcal{K}_{\mathbf{w}}$ Kempe graphs.

Since the orientation of each edge of a Kempe graph is determined, it is no longer necessary to think of Kempe graphs as directed graphs. From now on we will treat the Kempe graphs as non-directed multi-graphs, and we will use typical alphabetic characters (for example G) rather than Greek letters (such as Γ) to denote Kempe graphs.

The proof of the following theorem may be found in [HMSV2]. The proof is analogous to the proof that semistandard tableaux are linearly independent, for those familiar with the representation theory of $\mathrm{SL}(n, \mathbb{C})$.

Theorem 5.1.7 *The set* $\{X_G\}_{G \in \mathcal{K}_{\mathbf{w}}^{(N)}}$ *is a* $\mathbb{C}$*-basis for* $R_{\mathbf{w}}^{(N)}$.

Definition 5.1.8 *For each Kempe graph* G *let*

$$f(G) = \sum_{[i,j] \text{ an edge of } G} i + 2j.$$

Let

$$F_m(R_{\mathbf{w}}) = \langle X_G \rangle_{f(G) \leq m}.$$

The following theorem is central:

Theorem 5.1.9 *Suppose* G_1, G_2 *are Kempe graphs with valencies* $N\mathbf{w}$, $M\mathbf{w}$ *respectively. Let the integers* c_G *be the coefficients in the expansion of the product,*

$$X_{G_1} X_{G_2} = \sum_{G \in \mathcal{K}_{\mathbf{w}}^{(N+M)}} c_G X_G.$$

There exists an X_G *occurring on the right hand side with* $c_G = 1$*, such that* $f(G) = f(G_1) + f(G_2)$*; furthermore if* $G' \neq G$ *and* $c_{G'} \neq 0$ *then* $f(G') < f(G)$.

Proof. We will temporarily extend the domain of f to general multi-graphs Γ with properly oriented edges (but some edges may cross) by the same rule;

$$f(\Gamma) = \sum_{[i,j] \text{ an edge of } \Gamma} i + 2j.$$

Let Δ_1 be the graph with the two crossing edges, $[i,k]$, $[j,l]$, where $i < j < k < l$. Let Δ_2 be the graph with non-crossing edges $[i,l]$ and $[j,k]$, and let Δ_3 be the graph with non-crossing edges $[i,j]$ and $[k,l]$. The Plücker relations $X_{\Gamma'\cdot\Delta_1} = X_{\Gamma'\cdot\Delta_2} + X_{\Gamma'\cdot\Delta_3}$ applied two edges at a time are sufficient to enable one to re-express any Γ (with correctly oriented edges) as a sum of Kempe graphs. However, $f(\Delta_1) = f(\Delta_2) > f(\Delta_3)$ since $i+j+2(k+l) > i+k+2(j+l)$. Hence with each application of said Plücker relations $X_{\Gamma'\cdot\Delta_1} = X_{\Gamma'\cdot\Delta_2}+X_{\Gamma'\cdot\Delta_3}$, we have $f(\Gamma'\cdot\Delta_1) = f(\Gamma'\cdot\Delta_2) > f(\Gamma'\cdot\Delta_3)$. Finally once enough Plücker relations have been applied (starting from the initial $X_{G_1}X_{G_2}$) the final leftmost term X_G of the expansion will satisfy $f(G) = f(G_1) + f(G_2)$, and if $X_{G'}$ is any term other than the leftmost term X_G then $f(G') < f(G_1) + f(G_2)$. $\square$

Definition 5.1.10 *It will be useful to have a notation for the unique G above as a function of G_1, G_2. Let this G be denoted $G_1 * G_2$. It is obtained from $G_1 \cdot G_2$ by* un-crossing *crossing pairs of edges $[i,k]$, $[j,l]$, by replacing them with the edges $[i,l]$, $[j,k]$, until no crossing edges remain.*

Corollary 5.1.11 *The set of $F_m(R_{\mathbf{w}})$ form a filtration of $R_{\mathbf{w}}$, and for each Kempe graph $G \in \mathcal{K}_{\mathbf{w}}$,*

$$f(G) = \min\{m \mid X_G \in F_m(R_{\mathbf{w}})\}.$$

Definition 5.1.12 *Let* $\mathrm{gr}(R_{\mathbf{w}})$ *be the associated graded ring, with LG-graded components,* $F_m(R_{\mathbf{w}})/F_{m-1}(R_{\mathbf{w}})$*. Let the "standard" grading be given by*

$$\mathrm{gr}(R_{\mathbf{w}})^{(N)} = \langle Y_G \rangle_{G \in \mathcal{K}_{\mathbf{w}}^{(N)}},$$

where Y_G *is the image of* X_G *under the surjection*

$$F_{f(G)}(R_{\mathbf{w}}) \to F_{f(G)}(R_{\mathbf{w}})/F_{f(G)-1}(R_{\mathbf{w}}).$$

Hence $\mathrm{gr}(R_{\mathbf{w}})$ is a bi-graded ring.

Corollary 5.1.13 *(to Theorem 5.1.9) The set* $\{Y_G\}_{G \in \mathcal{K}_{\mathbf{w}}^{(N)}}$ *is a basis for* $\mathrm{gr}(R_{\mathbf{w}})^{(N)}$*.*

Corollary 5.1.14 *(to Theorem 5.1.9) If* G_1 *and* G_2 *are Kempe graphs then*

$$Y_{G_1} Y_{G_2} = Y_{G_1 * G_2}.$$

Corollary 5.1.15 *The set of all* Y_G *for ranging over Kempe graphs* $G \in \mathcal{K}_{\mathbf{w}}$ *form a graded semigroup. Furthermore the ring* $\mathrm{gr}(R_{\mathbf{w}})$ *is the graded semigroup algebra,*

$$\mathrm{gr}(R_{\mathbf{w}}) = \mathbb{C}[\{Y_G \mid G \in \mathcal{K}_{\mathbf{w}}\}].$$

Remark 5.1.16 We have called this filtration the LG-filtration since it motivated by the filtration of the ring of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ *given by Lakshmibai-Gonciulea [LG] which was designed to give a flat degeneration of* $\mathrm{Gr}_2(\mathbb{C}^n)$ *to a toric variety. (They also constructed flat degenerations for general flag varieties.)*

5.1.2 The polygonal semigroup algebra $\mathbb{C}[S_{\mathbf{w}}]$

We will now identify the semigroup $\{Y_G \mid G \in \mathcal{K}_{\mathbf{w}}\}$ explicitly as the set of lattice points in a rational cone.

Definition 5.1.17 *Let $C(\mathbf{w})$ in $\mathbb{R}^{n-2}$ be given by $(z, d_2, d_3, \ldots, d_{n-2}) \in C(\mathbf{w})$ iff $z \geq 0$, each $d_i \geq 0$, and*

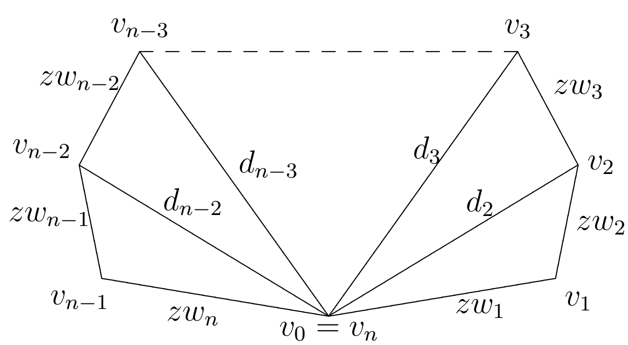
1. $d_i \leq d_{i+1} + zw_{i+1}$

2. $d_{i+1} \leq d_i + zw_{i+1}$

3. $zw_{i+1} \leq d_i + d_{i+1}$

for $1 \leq i \leq n-2$, where $d_1 := zw_1$ and $d_{n-1} := zw_n$. Let $D(N\mathbf{w})$ be intersection of the hyperplane $z = N$ with the cone $C(\mathbf{w})$.

These are the triangle inequalities that hold for an n-gon with vertices $v_0 = v_n, v_1, \ldots, v_{n-1}$ with side lengths $zw_i = |v_i - v_{i-1}|$ and diagonal lengths $d_i = |v_i - v_0|$.



Definition 5.1.18 *Let $\Lambda(\mathbf{w})$ be the lattice in $\mathbb{R}^{n-2}$ given by the conditions $(z, d_2, \ldots, d_{n-2}) \in \Lambda(\mathbf{w})$ iff z is an integer, and*

$$d_i \equiv z(w_1 + \cdots + w_i) \mod 2,$$

for each i, $2 \leq i \leq n-2$. (Note this is equivalent to the condition that each triple (d_i, zw_{i+1}, d_{i+1}) sums to an even integer.)

Definition 5.1.19 *Let $S_{\mathbf{w}}$ be the semigroup of lattice points in $C(\mathbf{w})$,*

$$S_{\mathbf{w}} = C(\mathbf{w}) \cap \Lambda(\mathbf{w}).$$

Let $S_{\mathbf{w}}^{(N)}$ be those elements $(z, d_2, \ldots, d_{n-2})$ in $S_{\mathbf{w}}$ such that $z = N$. Hence $S_{\mathbf{w}}^{(N)}$ is the set of lattice points within the bounded polytope $D(N\mathbf{w})$. This gives $S_{\mathbf{w}}$ the structure of a graded semigroup. Let $\mathbb{C}[S_{\mathbf{w}}]$ be the graded semigroup algebra over $\mathbb{C}$.

It should noted here that $S_{\mathbf{w}}$ is isomorphic to $S_{\lambda,\mathbf{w}}$, the lattice points in the cone on the Gel'fand Tsetlin polytope $GT(\lambda, \mathbf{w})$. The correspondence is given by the following map:

$$(N, d_2, \ldots, d_{n-2}) \mapsto \mathbf{x} \in GT(N\lambda, N\mathbf{w}),$$

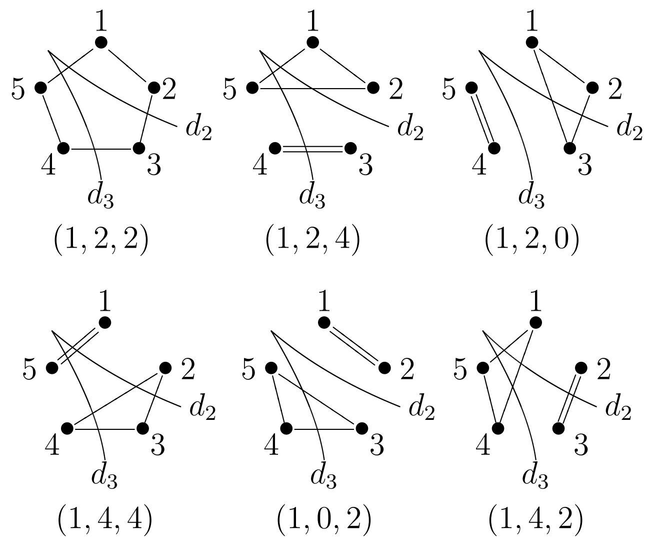
where $\mathbf{x}$ is the triangular array $x_{i,j}$, $1 \leq j \leq i \leq n$, given by $x_{1,1} = Nw_1$, $x_{n-1,2} = N(|\mathbf{w}|/2 - w_n)$, $x_{n,1} = x_{n,2} = x_{n-1,1} = N|\mathbf{w}|/2$, $x_{i,j} = 0$ for all $j \geq 3$, $x_{i,1} - x_{i,2} = d_i$ for all i, $2 \leq i \leq n-2$, and $x_{i,1} + x_{i,2} = N(w_1 + \cdots + w_i)$ for all i. For the proof this is a bijection see [HMSV1].

Definition 5.1.20 *For each Kempe graph $G \in \mathcal{K}_{\mathbf{w}}$, let*

$$\phi(Y_G) = (N, d_2, \ldots, d_{n-2}) \in \mathbb{Z}^{n-2},$$

where the degree of G is $N\mathbf{w}$, and d_i is the number of edges $[k, l]$ in G such that $k \leq i$ and $l \geq i+1$. Sometimes we abuse notation and write $\phi(G)$ instead of $\phi(Y_G)$.

An illustration for $\mathbf{w} = (2,2,2,2,2)$: There are six Kempe graphs of valency $(2,2,2,2,2)$, and the following illustrates their images under ϕ.



Lemma 5.1.21 *For each $N \geq 0$ the map ϕ is a bijection between $\{Y_G\}_{G \in \mathcal{K}_{\mathbf{w}}^{(N)}}$ and $S_{\mathbf{w}}^{(N)}$.*

Proof. First we will show that the image of ϕ is contained within the semigroup. Fix i, $1 \leq i \leq n-1$. Let S_i be the multi-set of edges $[k,l]$ such that $k \leq i$ and $l \geq i+1$, let S_{i+1} be the multi-set of edges $[k,l]$ such that $l \leq i+1$ and $l \geq i+2$, and let W_{i+1} be the multi-set of edges $[k,l]$ such that $k = i+1$ or $l = i+1$. We have $d_i = |S_i|$, $d_{i+1} = |S_{i+1}|$, and $Nw_{i+1} = |W_{i+1}|$. It is clear that if any edge $[k,l] \in S_i \cup S_{i+1} \cup W_{i+1}$, then it belongs to exactly two of these three sets. From this the triangle inequalities for the triple d_i, d_{i+1}, Nw_{i+1} follow easily, and it is also easy to see that $d_i + d_{i+1} + Nw_{i+1}$ must be an even integer, since each edge is counted twice in the sum.

Next we show that ϕ is a bijection. We must show there is only one way to build a Kempe graph G from the data $(N, d_2, \ldots, d_{n-2})$. First we will show that some graph Γ exists (with properly oriented edges but possibly crossing edges) such that $\phi(\Gamma) = (N, d_2, \ldots, d_{n-1})$ (ϕ is extended to such graphs using the same rule, $\phi(\Gamma)$ the sum of $i+2j$ for $[i,j]$ an edge of Γ). The number N forces that the number of edges containing vertex i must be Nw_i. Let a_2 be the multiplicity of the edge $[1,2]$, let b_2 be the total number of edges $[2,j]$ with $j \geq 3$, and let c_2 be the total number of edges $[1,j]$ with $j \geq 3$. We have the three equations,

$$a_2 + c_2 = zw_1, \quad b_2 + c_2 = d_2, \quad a_2 + b_2 = zw_2.$$

Hence,

$$a_2 = (zw_1 + zw_2 - d_2)/2, \quad b_2 = (-zw_1 + zw_2 + d_2)/2, \quad c_2 = (zw_1 - zw_2 + d_2)/2.$$

Since the numerators in the above three equations are even, and since the triple (zw_1, zw_2, d_2) satisfies the triangle inequalities (making all numerators non-negative), these are valid non-negative integral values for a_2, b_2, and c_2, and they are uniquely determined by the data (N, d_2). In particular we know there must be a_2 multiples of the edge $[1,2]$. Now let a_i be the number of edges $[h,i]$, b_i is number of edges $[i,j]$, and c_i is the number of edges $[h,j]$ with $h < i < j$, where $i \leq n-1$. The number d_{i-1} tells us the number of edges $[h,j]$ with $h \leq i-1$ $j \geq i$, but the heads j of these edges have not yet been assigned. The number a_i tells us how many of these end at index i. Similarly as before, we get equations,

$$a_i + c_i = d_{i-1}, \quad b_i + c_i = d_i, \quad a_i + b_i = zw_i,$$

$$a_i = (d_{i-1} + zw_i - d_i)/2, \quad b_i = (-d_{i-1} + zw_i + d_i)/2, \quad c_i = (d_{i-1} - zw_i + d_i)/2.$$

Again each of a_i, b_i, c_i is a non-negative integer. Of the d_{i-1} edges $[h, j]$ with $h \leq i-1$, $j \geq i$, we must assign a_i of them so that $j = i$. But there is only one way to do this so that the resulting graph has no crossing edges: first we must assign those edges $[h, j]$ with $h = i - 1$ to terminate at i (i.e. $j := i$), then those with $h = i - 2$, etc. It is pictorially obvious this leads to no crossing edges, and any other choice of assignments would lead to an eventual crossing. $\square$

Lemma 5.1.22 *If G_1 and G_2 are Kempe graphs then*

$$\phi(Y_{G_1}Y_{G_2}) = \phi(Y_{G_1 * G_2}) = \phi(Y_{G_1}) + \phi(Y_{G_2}).$$

Hence ϕ is an isomorphism of semigroups, and induces an isomorphism (also denoted ϕ) on the semigroup algebras,

$$\phi : \mathrm{gr}(R_{\mathbf{w}}) \cong \mathbb{C}[S_{\mathbf{w}}].$$

Proof. This proof is similar to the proof of Theorem 5.1.9. We will temporarily extend the domain of ϕ to general multi-graphs Γ with properly oriented edges (but some edges may cross) by the same rule;

$$\phi(\Gamma) = (N, d_2, d_3, \ldots, d_{n-1}),$$

where N is the degree of X_Γ and d_i is the number of edges $[k, l]$ of Γ such that $k \leq i$ and $l \geq i + 1$. With this extension of the definition it is clear that $\phi(G_1 \cdot G_2) = \phi(G_1) + \phi(G_2)$.

Let Δ_1 be the graph with the two crossing edges, $[i, k]$, $[j, l]$, where $i < j < k < l$. Let Δ_2 be the graph with non-crossing edges $[i, l]$ and $[j, k]$, and let Δ_3 be the graph with non-crossing edges $[i,j]$ and $[k,l]$. We know the Plücker relations $X_{\Gamma'\cdot\Delta_1} = X_{\Gamma'\cdot\Delta_2} + X_{\Gamma'\cdot\Delta_3}$ applied two edges at a time are sufficient to enable one to re-express any X_Γ (with correctly oriented edges) as a sum of Kempe graphs. We have that $\phi(\Delta_1) = \phi(\Delta_2)$, hence with each application of said Plücker relations $X_{\Gamma'\cdot\Delta_1} = X_{\Gamma'\cdot\Delta_2} + X_{\Gamma'\cdot\Delta_3}$, we have $\phi(\Gamma'\cdot\Delta_1) = \phi(\Gamma'\cdot\Delta_2)$. Finally once enough Plücker relations have been applied (starting from the initial $X_{G_1}X_{G_2}$) the final leftmost term X_G of the expansion will satisfy $\phi(G) = \phi(G_1 \cdot G_2) = \phi(G_1) + \phi(G_2)$.

□

Corollary 5.1.23 *The rings* $\mathrm{gr}(R_{\mathbf{w}})$ *and* $\mathbb{C}[S_{\mathbf{w}}]$ *are isomorphic as graded rings, where the grading of* $\mathrm{gr}(R_{\mathbf{w}})$ *is by the* standard *grading, not the LG-grading.*

5.1.3 Filtrations give flat degenerations

It is well-known that if R is a filtered ring then there is a one-parameter flat degeneration with special fiber the associated graded ring of R. We sketch one way to do this, borrowed from [AB], using the Rees algebra. Let t be an indeterminant and let $\mathcal{R}$ be the Rees algebra

$$\mathcal{R} = \bigoplus_{m=0}^{\infty} F_m(R)t^m \subset R[z].$$

Theorem 5.1.24 *(see Alexeev–Brion [AB])*

- $\mathcal{R}$ *is flat over* $\mathbb{C}[t]$.
- $\mathcal{R} \otimes_{\mathbb{C}[t]} \mathbb{C}[t,t^{-1}] \cong R[t,t^{-1}]$.
- $\mathcal{R} \otimes_{\mathbb{C}[t]} (\mathbb{C}[t]/(t)) \cong \mathrm{gr}(R)$.

In [HMSV1] the following theorem was proven.

Theorem 5.1.25 *Suppose that each w_i is even. Then the semigroup $S_{\mathbf{w}}$ is generated by degree one elements, and the relations in these elements are only quadrics. Consequently, the relations among the Kempe generators of $R_{\mathbf{w}}$ are generated by quadric relations.*

We now look more closely at the appearance of these quadric relations.

5.2 Graphic relations generate the ideal of $R_{\mathbf{w}}$

We suppose that the weights w_i are all equal to 2:

$$\mathbf{w} = (2, 2, \ldots, 2) \in \mathbb{Z}^n.$$

Definition 5.2.1 *(Graphic binomials) Suppose that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d} \in S_{\mathbf{w}}^{(1)}$ such that $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$. Then we say that the quadric binomial relation $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$ is graphic if there exists graphs $\Gamma_{\mathbf{a}}$, $\Gamma_{\mathbf{b}}$, $\Gamma_{\mathbf{c}}$, $\Gamma_{\mathbf{d}}$ of degree $\mathbf{w}$ such that the following holds:*

1. *The leading terms of $X_{\Gamma_{\mathbf{a}}}$, $X_{\Gamma_{\mathbf{b}}}$, $X_{\Gamma_{\mathbf{c}}}$, and $X_{\Gamma_{\mathbf{d}}}$ are respectively $X_{\phi^{-1}(\mathbf{a})}$, $X_{\phi^{-1}(\mathbf{b})}$, $X_{\phi^{-1}(\mathbf{c})}$, and $X_{\phi^{-1}(\mathbf{d})}$.*
2. *$X_{\Gamma_{\mathbf{c}}} X_{\Gamma_{\mathbf{d}}} = X_{\Gamma_{\mathbf{a}}} X_{\Gamma_{\mathbf{b}}}$.*

Lemma 5.2.2 *Suppose that $\mathbf{a}, \mathbf{b} \in S_{\mathbf{w}}^{(1)}$, and $[\mathbf{a}, \mathbf{b}]$ are such that there exists some index i such that $|a_i - b_i| \geq 4$. Then there exists $\mathbf{c}$, $\mathbf{d}$ in $S_{\mathbf{w}}^{(1)}$ such that $\mathbf{c} + \mathbf{d} = \mathbf{a} + \mathbf{b}$, $\sum_{i=1}^{n-1} |c_i - d_i| < \sum_{i=1}^{n-1} |a_i - b_i|$, and the relation $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$ is graphic.*

Proof. Recall the rules for a point $\mathbf{a}$ to be in $S^{(1)}_{\mathbf{w}}$. Let $\partial a_i = a_i - a_{i-1}$, for $2 \leq i \leq n-1$. We have that each $|\partial a_i| \leq 2$ by the first two triangle inequalities. The lattice condition is that each a_i is even. Hence each $\partial a_i \in \{-2, 0, +2\}$. We have that $a_1 = a_{n-1} = 2$. Finally the third triangle inequality disallows any consecutive pair of components a_i, a_{i+1} to each be zero. These are necessary and sufficient conditions for $\mathbf{a}$ to be an element of $S^{(1)}_{\mathbf{w}}$.

We shall find vectors $\mathbf{w}'$, $\mathbf{w}''$, $\mathbf{a}'$, $\mathbf{a}''$, $\mathbf{b}'$, $\mathbf{b}''$ such that

- $\mathbf{w}' + \mathbf{w}'' = \mathbf{w}$, $\mathbf{a}' + \mathbf{a}'' = \mathbf{a}$, and $\mathbf{b}' + \mathbf{b}'' = \mathbf{b}$.
- $\mathbf{a}', \mathbf{b}' \in S^{(1)}_{\mathbf{w}'}$ and $\mathbf{a}'', \mathbf{b}'' \in S^{(1)}_{\mathbf{w}''}$.
- $\mathbf{c} = \mathbf{a}' + \mathbf{b}''$ and $\mathbf{d} = \mathbf{b}' + \mathbf{a}''$ will be the desired $\mathbf{c}, \mathbf{d}$ of the Lemma.

The construction will be in the form of an inductive algorithm.

Step 0: Let

$$i_1 = \min\{i \mid |a_i - b_i| \geq 4\}.$$

Note that $i_1 \geq 2$ since $a_1 = b_1 = 2$. If neither a_i nor b_i is zero for all $i < i_1$ then set $i_0 = 1$. Otherwise, let

$$i_0 = 1 + \max\{i < i_1 \mid a_i = 0 \text{ or } b_i = 0\}.$$

Note in this case that $\partial a_{i_0} \geq 0$ and $\partial b_{i_0} \geq 0$. Let $a''_i = b''_i = 0$ for $i < i_0$. Let $a''_i = b''_i = 1$ for $i_0 \leq i < i_1$. Assign $s := 1$.

Step 1: Let

$$i_{s+1} = \min\{i > i_s \mid |a_i - b_i| \leq 2\}.$$

If $a_{i_s} > b_{i_s}$ then let $a''_i = 2$ for $i_s \leq i < i_{s+1}$, and let $b''_i = 0$ for $i_s \leq i < i_{s+1}$. If $b_{i_s} > a_{i_s}$ then let $b''_i = 2$ for $i_s \leq i < i_{s+1}$, and let $a''_i = 0$ for $i_s \leq i < i_{s+1}$.

Step 2: If each of a_i and b_i is nonzero for all $i > i_{s+1}$ then let $i_{s+2} = n$. Otherwise, let

$$i_{s+2} = \min\{i > i_{s+1} \mid a_i = 0, \text{ or } b_i = 0, \text{ or } \partial a_i \leq 0 \text{ and } \partial b_i \leq 0\}.$$

Let $a''_i = b''_i = 1$ for $i_{s+1} \leq i < i_{s+2}$. If $i_{s+2} = n$ then we are finished constructing $\mathbf{a}''$ and $\mathbf{b}''$. If $a_{i_{s+2}}$ and $b_{i_{s+2}}$ are each nonzero then let $a''_i = b''_i = 0$ for all $i \geq i_{s+2}$, and we are again finished constructing $\mathbf{a}''$ and $\mathbf{b}''$. Otherwise $a_{i_{s+2}}$ or $b_{i_{s+2}}$ is zero, and we must continue. Note that not each of $a_{i_{s+2}}$ and $b_{i_{s+2}}$ can be zero since otherwise we would have $\partial a_{i_{s+2}}, \partial b_{i_{s+2}} \leq 0$. There are two cases:

Case $a_{i_{s+2}} > 0$ and $b_{i_{s+2}} = 0$. Claim $a_{i_{s+2}} \geq 4$: we have $\partial b_{i_{s+2}} = -2$, so $\partial a_{i_{s+2}} = +2$ is positive. We also have that $a_{i_{s+1}} \geq 2$ since $a_{i_{s+1}-1} \geq b_{i_{s+1}-1} + 4 \geq 4$. Also $b_{i_{s+1}} \geq 2$ since $\partial b_{i_{s+1}} > \partial a_{i_{s+1}}$. By definition of i_{s+2}, there is not an i such that $i_{s+1} < i < i_{s+3}$ where a_i or b_i is zero. In particular we have that $a_{i_{s+2}-1} \geq 2$ and $b_{i_{s+2}-1} \geq 2$. So our claim holds that $a_{i_{s+2}} \geq 4$. In particular, $a_{i_{s+2}} - b_{i_{s+2}} \geq 4$. Now re-assign $s := s + 2$, and go back to Step 1.

Case $b_{i_{s+2}} > 0$ and $a_{i_{s+2}} = 0$. As above, we have that $b_{i_{s+2}} \geq 4$. Re-assign $s := s + 2$ and go back to Step 1.

Example:

$$\mathbf{a} = (2,2,4,6,4,2,0,2,2,4,2)$$

$$\mathbf{b} = (2,0,2,2,0,2,4,2,4,2,2)$$

$$\mathbf{a}'' = (0,0,1,2,2,1,0,1,1,1,0)$$

$$\mathbf{b}'' = (0,0,1,0,0,1,2,1,1,1,0)$$

Let $\mathcal{I}$ be the set of indices $\{i_0, i_1, \ldots, i_k\}$ appearing in the construction above. Let $\mathbf{w}''$ be given by $w''_i = 1$ if $i \in \mathcal{I}$ and $w''_i = 0$ otherwise.

We claim that $\mathbf{a}'' \in S^{(1)}_{\mathbf{w}''}$. The proof that $\mathbf{b}'' \in S^{(1)}_{\mathbf{w}''}$ is identical by symmetry of definition. Note that each a''_i is a non-negative integer. Also, observe that $|\partial a''_i| = w''_i$ for each i. Hence the parity of components of $\mathbf{a}''$ change precisely at the indices in $\mathcal{I}$. Therefore $\mathbf{a}''$ is in the lattice relative to $\mathbf{w}''$. We have $a''_{i-1} \le a''_i + w''_i$ and $a''_i \le a''_{i-1} + w''_i$ since $|\partial a''_i| = w''_i$ for each i. Also, $w''_i \le a''_{i-1} + a''_i$ for the same reason.

Let $\mathbf{w}' = \mathbf{w} - \mathbf{w}''$, $\mathbf{a}' = \mathbf{a} - \mathbf{a}''$, and $\mathbf{b}' = \mathbf{b} - \mathbf{b}''$. We claim that $\mathbf{a}'$ is in $S^{(1)}_{\mathbf{w}'}$. The proof that $\mathbf{b}' \in S^{(1)}_{\mathbf{w}'}$ is identical. First we need to show the components of $\mathbf{a}'$ are non-negative. This is equivalent to showing that $a''_i \le a_i$ for each i. For each i we have that $0 \le a''_i \le 2$. We claim that $a''_i = 0$ whenever $a_i = 0$. Suppose that $a_i = 0$. Then, either $i_s < i < i_{s+1}$ for some $s \equiv 1 \mod 2$, or $i \in \mathcal{I}$, or $i < \min \mathcal{I}$, or $i > \max \mathcal{I}$. In the latter three cases a''_i is zero by construction. In the intervals $i_s < i < i_{s+1}$ for odd s, the greater of the two sequences $\mathbf{a}$ or $\mathbf{b}$ remains at least 4 greater than the lesser, and so in particular the greater sequence cannot have a zero component in this interval. Since $a_i = 0$, we have that $a_{i_s} + 4 \leq b_{i_s}$ and so $a''_i = 0$ by definition.

We claim that $\mathbf{a}'$ satisfies the parity conditions to be a lattice point relative to $\mathbf{w}'$. We have that $\partial a'_i = \partial a_i - \partial a''_i = \partial a_i \pm w''_i$. But the parity of w''_i is equal to the parity of w'_i since $w'_i + w''_i = 2$. Since ∂a_i is even we have $\partial a'_i \equiv w'_i \mod 2$.

Also, we need $|\partial a'_i| \leq w'_i$ for all $i \geq 2$ (two of three triangle inequalities). We have that $\partial a'_i = \partial a_i - \partial a''_i$. If $\partial a''_i = 0$ then $i \notin \mathcal{I}$ and $w''_i = 0$. Hence $w'_i = w_i = 2$, and so $|\partial a'_i| = |\partial a_i| \leq 2 = w_i = w'_i$. Suppose that $\partial a''_i = \pm 1$. Then $i \in \mathcal{I}$ and so $w'_i = 1$. Suppose $i = i_0$; in this case we have that $\partial a''_i = +1$. We may exclude the case $i_0 = 1$ since $i \geq 2$. We have that $\partial a_{i_0} \geq 0$, as was pointed out in Step 0 of the construction. Hence $|\partial a'_{i_0}| = |\partial a_{i_0} - \partial a''_{i_0}| \leq 1 = w'_{i_0}$. Now suppose that $i = i_s$ and s is odd. First consider the case $s = 1$. If $\partial a''_{i_1} = +1$ then $a_{i_1} \geq b_{i_1} + 4$ and $a_{i_1 - 1} \leq b_{i_1 - 1} + 2$. Hence $\partial a_{i_1} \geq 0$ and so $|\partial a'_{i_1}| = |\partial a_{i_1} - \partial a''_{i_1}| \leq 1 = w'_{i_1}$. Similarly if $\partial a''_{i_1} = -1$ then $\partial a_{i_1} \leq 0$ and again $|\partial a'_{i_1}| = |\partial a_{i_1} - \partial a''_{i_1}| \leq 1$. Now suppose that $s \geq 3$ (and s is odd). There are three cases. Either $i_s = n$, or $\partial a_{i_s} \leq 0$ and $\partial b_{i_s} \leq 0$, or at least one of a_{i_s} or b_{i_s} is zero. We may exclude the case $i_s = n$ since $i \leq n - 1$. Suppose that $\partial a_{i_s} \leq 0$ and $\partial b_{i_s} \leq 0$. Then $\partial a''_{i_s} = -1$ by definition, and so $|\partial a'_{i_s}| = |\partial a_{i_s} - \partial a''_{i_s}| \leq 1$. If on the other hand at least one of a_{i_s} or b_{i_s} is zero, then the situation is identical to the case $s = 1$ as above. Suppose $i = i_{s+1}$ and s is odd. If $\partial a''_{i_{s+1}} = +1$ then $\mathbf{a}$ is less than $\mathbf{b}$ on the interval $[i_s, i_{s+1})$, and $\partial a_{i_{s+1}} \geq 0$. Thus we have $|\partial a'_{i_{s+1}}| = |\partial a_{i_{s+1}} - \partial a''_{i_{s+1}}| \leq 1$. Similarly if $\partial a''_{i_{s+1}} = -1$ then $\partial a_{i_{s+1}} \leq 0$, so again $|\partial a'_{i_{s+1}}| = |\partial a_{i_{s+1}} - \partial a''_{i_{s+1}}| \leq 1$.

We still must show the last inequality $w'_i \leq a'_{i-1} + a'_i$. First consider the case that $w'_i = 1 = w''_i$. Then $i \in \mathcal{I}$ so $\partial a''_i \neq 0$. Therefore $\partial a'_i \neq 0$ and so $a'_{i-1} + a'_i \geq 1 = w'_i$. Now suppose that $w'_i = 2$. Then $w''_i = 0$, so $i \notin \mathcal{I}$. Therefore $a''_{i-1} = a''_i$. The inequality $a'_{i-1} + a'_i \geq 2$ is equivalent to $a_{i-1} + a_i \geq a''_{i-1} + a''_i + 2$. Hence if $a''_{i-1} = a''_i = 0$ we are done. Suppose that $a''_{i-1} = a''_i = 1$. Then we must show that $a_{i-1} + a_i \geq 4$. We have that $i_{s-1} < i < i_s$ where s is odd. But $\mathbf{a}$ is nonzero on intervals $[i_{s-1}, i_s)$ where s is odd. Hence $a_{i-1} + a_i \geq 4$.

Let

$$\mathbf{c} = \mathbf{a}' + \mathbf{b}'', \quad \mathbf{d} = \mathbf{b}' + \mathbf{a}''.$$

We claim that $\sum_{i=1}^{n-1} |c_i - d_i| < \sum_{i=1}^{n-1} |a_i - b_i|$. Whenever $i_s \leq i < i_{s+1}$ where s is odd, then $|a_i - b_i| \geq 4$. Suppose without loss of generality that $a_i \geq b_i + 4$. Then $a''_i = 2$ and $b''_i = 0$. Hence, $c_i = a_i - 2$ and $d_i = b_i + 2$ thus $|c_i - d_i| = |a_i - b_i| - 4$. Now suppose that i is not in any interval $[i_s, i_{s+1})$ where s is odd. Then $a''_i = b''_i$. Hence $c_i = (a_i - a''_i) + b''_i = a_i$ and $d_i = (b_i - b''_i) + a''_i = b_i$, and so $|c_i - d_i| = |a_i - b_i|$. Since the intervals $[i_s, i_{s+1})$ exist (in particular $[i_1, i_2)$) the claim follows.

Let $G(\mathbf{a}') = \phi^{-1}(\mathbf{a}')$, $G(\mathbf{a}'') = \phi^{-1}(\mathbf{a}'')$, $G(\mathbf{b}') = \phi^{-1}(\mathbf{b}')$, and $G(\mathbf{b}'') = \phi^{-1}(\mathbf{b}'')$ be the associated Kempe graphs. Let

$$\Gamma_{\mathbf{a}} = G(\mathbf{a}') \cdot G(\mathbf{a}''),\ \Gamma_{\mathbf{b}} = G(\mathbf{b}') \cdot G(\mathbf{b}''),\ \Gamma_{\mathbf{c}} = G(\mathbf{a}') \cdot G(\mathbf{b}''),\ \Gamma_{\mathbf{d}} = G(\mathbf{b}') \cdot G(\mathbf{a}'').$$

Then, the polynomial

$$\widetilde{X}_{\Gamma_{\mathbf{a}}} \widetilde{X}_{\Gamma_{\mathbf{b}}} - \widetilde{X}_{\Gamma_{\mathbf{c}}} \widetilde{X}_{\Gamma_{\mathbf{d}}}$$

is a lift of the quadric relation $\widetilde{Y}_{\phi^{-1}(\mathbf{a})} \widetilde{Y}_{\phi^{-1}(\mathbf{b})} - \widetilde{Y}_{\phi^{-1}(\mathbf{c})} \widetilde{Y}_{\phi^{-1}(\mathbf{d})}$ for the toric ring $\mathrm{gr}(R_{\mathbf{w}})$.

$\square$

Lemma 5.2.3 *Suppose that* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$ *are in* $S_{\mathbf{w}}^{(1)}$ *and* $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$ *is a nontrivial relation, and that each of* $[\mathbf{a},\mathbf{b}]$ *and* $[\mathbf{c},\mathbf{d}]$ *satisfy* $|a_i-b_i|\le 2$ *and* $|c_i-d_i|\le 2$ *for each* i. *Then there exists* $\hat{\mathbf{a}}$ *and* $\hat{\mathbf{b}}$ *in* $S_{\mathbf{w}}^{(1)}$ *such that* $\mathbf{a}+\mathbf{b}=\hat{\mathbf{a}}+\hat{\mathbf{b}}$ *is graphic,* $[\hat{\mathbf{a}},\hat{\mathbf{b}}]$ *satisfies that* $|\hat{a}_i-\hat{b}_i|\le 2$ *for each* i, *and*

$$\sum_{i=1}^{n-1}|\hat{a}_i-c_i|<\sum_{i=1}^{n-1}|a_i-c_i|,$$
$$\sum_{i=1}^{n-1}|\hat{b}_i-d_i|<\sum_{i=1}^{n-1}|b_i-d_i|.$$

Proof. Let i_1 be the first i such that $a_i\ne c_i$. Note that i_1 is also the first i such that $b_i\ne d_i$ since $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$. Let $i_0=1$ if all a_i and b_i are nonzero for $i<i_1$. Otherwise, let

$$i_0=1+\max\{i<i_1\mid \min(a_i,b_i)=0\}.$$

Without loss of generality, suppose that $a_{i_1}>c_{i_1}$. Let i_2 be the first $i>i_1$ such that $a_i\le c_i$. Let $a''_i=b''_i=0$ for $i<i_0$. Let $a''_i=b''_i=1$ for $i_0\le i<i_1$. Let $a''_i=2$ and $b''_i=0$ for $i_1\le i<i_2$.

Case $a_{i_2}>0$. Let $i_3=n$ if all a_i and b_i are nonzero for $i>i_2$. Otherwise let i_3 be the first $i>i_2$ such that $\min(a_i,b_i)=0$. Let $a''_i=b''_i=1$ for $i_2\le i<i_3$, and let $a''_i=b''_i=0$ for $i\ge i_3$.

Let i'_2 be the first $i\ge i_2$ such that $\min(a_i,b_i)\ge 2$. Let $a''_i=0$ and $b''_i=2$ for $i-i_2$ even and $i_2\le i<i'_2$. Let $a''_i=2$ and $b''_i=0$ for $i-i_2$ odd and $i_2\le i<i'_2$. Let $i_3=n$ if each a_i and b_i is nonzero for $i\ge i'_2$. Otherwise let i_3 be the first $i>i'_2$ such that $\min(a_i,b_i)=0$. Let $a''_i=b''_i=1$ for $i'_2\le i<i_3$. Let $a''_i=b''_i=0$ for $i\ge i_3$.

Let $\mathcal{I}=\{i_0,i_1,i_2,\ldots,i'_2,i_3\}$ (the ellipsis means that $i\in\mathcal{I}$ if $i_2\le i\le i'_2$). Let $w''_i=0$ for all $i\notin\mathcal{I}$. Let $w''_i=1$ if $i\in\mathcal{I}$ and $i<i_2$ or $i\ge i'_2$. Let $w''_i=2$ if $i\in\mathcal{I}$ and

$i_2 \leq i < i_2'$. Let $\mathbf{w}'' = (w_1'', \ldots, w_n'')$, and let $\mathbf{w}' = \mathbf{w} - \mathbf{w}''$. Let $\mathbf{a}'' = (a_1'', \ldots, a_{n-1}'')$ and let $\mathbf{b}'' = (b_1'', \ldots, b_{n-1}'')$. Let $\mathbf{a}' = \mathbf{a} - \mathbf{a}''$ and let $\mathbf{b}' = \mathbf{b} - \mathbf{b}''$. We claim that $\mathbf{a}', \mathbf{b}' \in S_{\mathbf{w}'}^{(1)}$ and $\mathbf{a}'', \mathbf{b}'' \in S_{\mathbf{w}''}^{(1)}$.

First we check that $\mathbf{a}'', \mathbf{b}'' \in S_{\mathbf{w}''}^{(1)}$. Suppose that $i \notin \mathcal{I}$.

It is clear that it is enough to check the inequalities and parity conditions at the special indices $i \in \mathcal{I}$. The argument for $i \in \{i_0, i_1, i_2', i_3\}$ is similar to the argument given in the proof of Lemma 5.2.2. Now suppose that $i_2' > i_2$. Then, the components of $\mathbf{a}$ and $\mathbf{b}$ alternate between 0 and 2 respectively on the interval $[i_2, i_2')$. Similarly the components of $\mathbf{c}$ and $\mathbf{d}$ alternate between 0 and 2 on the interval $[i_2, i_2')$. By definition $\mathbf{a}'' = \mathbf{a}$, $\mathbf{b}'' = \mathbf{b}$, and $\mathbf{w}'' = \mathbf{w}$ on the this interval, so there can be no problems.

Now we must show that $\mathbf{a}', \mathbf{b}' \in S_{\mathbf{w}'}^{(1)}$. We will just check the special indices $i \in \mathcal{I}$. The argument for $i \in \{i_0, i_1, i_2', i_3\}$ is essentially the same as in the proof of Lemma 5.2.2.

Let

$$\hat{\mathbf{a}} = \mathbf{a}' + \mathbf{b}'', \quad \hat{\mathbf{b}} = \mathbf{b}' + \mathbf{a}''.$$

It is easy to check that $|\hat{a}_i - c_i| \leq |a_i - c_i|$ for all i, and $|\hat{a}_{i_1} - c_{i_1}| < |a_{i_1} - c_{i_1}|$. Similarly, $|\hat{b}_i - d_i| \leq |b_i - d_i|$ for all i, and $|\hat{b}_{i_1} - d_{i_1}| < |b_{i_1} - d_{i_1}|$.

Let $G(\mathbf{a}') = \phi^{-1}(\mathbf{a}')$, $G(\mathbf{a}'') = \phi^{-1}(\mathbf{a}'')$, $G(\mathbf{b}') = \phi^{-1}(\mathbf{b}')$, and $G(\mathbf{b}'') = \phi^{-1}(\mathbf{b}'')$ be the associated Kempe graphs. Let

$$\Gamma_{\mathbf{a}} = G(\mathbf{a}') \cdot G(\mathbf{a}''),\ \Gamma_{\mathbf{b}} = G(\mathbf{b}') \cdot G(\mathbf{b}''),\ \Gamma_{\hat{\mathbf{a}}} = G(\mathbf{a}') \cdot G(\mathbf{b}''),\ \Gamma_{\hat{\mathbf{b}}} = G(\mathbf{b}') \cdot G(\mathbf{a}'').$$

Then, the polynomial

$$\widetilde{X}_{\Gamma_{\mathbf{a}}}\widetilde{X}_{\Gamma_{\mathbf{b}}} - \widetilde{X}_{\Gamma_{\hat{\mathbf{a}}}}\widetilde{X}_{\Gamma_{\hat{\mathbf{b}}}}$$

is a lift of the quadric relation $\widetilde{Y}_{\phi^{-1}(\mathbf{a})}\widetilde{Y}_{\phi^{-1}(\mathbf{b})} - \widetilde{Y}_{\phi^{-1}(\hat{\mathbf{a}})}\widetilde{Y}_{\phi^{-1}(\hat{\mathbf{b}})}$ for the toric ring $\mathrm{gr}(R_{\mathbf{w}})$.

□

Theorem 5.2.4 *Suppose that each weight w_i is even. Then the ring $R_{\mathbf{w}}$ (using generators X_Γ where Γ is a directed graph of degree $\mathbf{w}$) is cut out by the following relations:*

- *The sign relations,*

$$X_{\Gamma'\cdot[i,j]} = -X_{\Gamma'\cdot[j,i]}.$$

- *The linear Plücker relations,*

$$X_{\Gamma'\cdot[i,k]\cdot[j,l]} = X_{\Gamma'\cdot[i,l]\cdot[j,k]} + X_{\Gamma'\cdot[i,j]\cdot[k,l]}.$$

- *The graphical quadrics,*

$$X_{\Gamma_1}X_{\Gamma_2} = X_{\Gamma_3}X_{\Gamma_4},$$

  whenever $\Gamma_1 \cdot \Gamma_2 = \Gamma_3 \cdot \Gamma_4$.

Proof. First we assume that each $w_i = 2$. The graphic quadric relations generate all the quadric relations in the toric ring. They each lift to relations of type $X_{\Gamma_1}X_{\Gamma_2} = X_{\Gamma_3}X_{\Gamma_4}$, so the linear combinations of these give all the quadratic relations in $R_{\mathbf{w}}$. Since the quadratic relations generate the ideal of $R_{\mathbf{w}}$, the claim follows.

If not all the $w_i = 2$ then let $\widetilde{\mathbf{w}} = (2, 2, \ldots, 2) \in \mathbb{Z}^m$ where $m = \sum w_i/2$. It is shown in [HMSV2] that $M_{\mathbf{w}}$ is a linear section of $M_{\widetilde{\mathbf{w}}}$, given as a subspace cut out by relations $X_\Gamma = 0$ whenever Γ is a graph which connects vertices in the same "clump". The first $w_1/2$ points form a clump, and the next $w_2/2$ points form another clump, etc. From the point of view of points on the projective line, it is described as the subspace corresponding to those point configurations where all the points in a given clump are equal to each other. Hence there is a surjection π from $R_{\widetilde{\mathbf{w}}}$ onto $R_{\mathbf{w}}$, such that the generator X_Γ is mapped to $X_{\overline{\Gamma}}$ where $\overline{\Gamma}$ is Γ collapsed so that all the vertices in the same clump are identified. If two vertices in the same clump are joined by an edge in Γ, then $\overline{\Gamma}$ will then contain a loop, and will therefore be zero by a sign relation. The map π was first introduced in [HMSV1] as the "side-splitting map".

The map π lifts (by the above description) to the polynomial rings in the $\widetilde{X}_\Gamma$ and the $\widetilde{X}_{\overline{\Gamma}}$. In [HMSV2] it is shown that $I_{\mathbf{w}} = \pi(I_{\widetilde{\mathbf{w}}})$, where $I_{\widetilde{\mathbf{w}}}$ and $I_{\mathbf{w}}$ are the kernels of the exact sequences:

$$0 \to I_{\widetilde{\mathbf{w}}} \to \mathbb{C}[\widetilde{X}_\Gamma]_\Gamma \to R_{\widetilde{\mathbf{w}}} \to 0,$$

$$0 \to I_{\mathbf{w}} \to \mathbb{C}[\widetilde{X}_{\overline{\Gamma}}]_{\overline{\Gamma}} \to R_{\mathbf{w}} \to 0.$$

Also, it is easy to see that π takes graphical quadrics to graphical quadrics, and the sign and Plücker relations are mapped into analogous sign and Plücker relations. $\square$

Embedding into a toric variety

Theorem 5.2.4 implies that if each w_i is even, then $M_{\mathbf{w}}$ embeds into a certain toric variety as a linear subspace. Let $\mathcal{G}_{\mathbf{w}}$ denote the set of directed multigraphs of degree a multiple of $\mathbf{w}$. The binary operation $\Gamma_1 \cdot \Gamma_2$ gives $\mathcal{G}_{\mathbf{w}}$ the structure of a graded semigroup. Let $\mathbb{C}[\mathcal{G}_{\mathbf{w}}]$ denote the associated semigroup algebra.

This semigroup is also the set of lattice points in a rational cone. Let the variables $x_{i,j}$ for $1 \leq i, j \leq n$ represent the multiplicity of the edge $[i, j]$. The condition that a graph has degree a multiple of $\mathbf{w}$ translates into a set of rational linear equalities in the $x_{i,j}$'s. Also we have the inequalities $x_{i,j} \geq 0$. The semigroup $\mathcal{G}_{\mathbf{w}}$ may be interpreted as the set of integral solutions to these equalities and inequalities.

Since each w_i is even, by the Petersen decomposition theorem, any graph of degree $N\mathbf{w}$ may be factored into subgraphs each of degree $\mathbf{w}$. Hence the degree one elements generate the semigroup, so $\mathbb{C}[\mathcal{G}_{\mathbf{w}}]$ is projectively normal.

By Theorem 5.2.4 the linear sign and Plücker relations cut out a subspace of the above toric variety which is isomorphic to $M_{\mathbf{w}}$.

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