ABSTRACT

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In the present work, a framework is proposed for studying autonomous agents which interact locally yet effect a globally coherent behavior. This problem of locally induced organization is ubiquitous in decentralized multi-robot environments and various micro- and macroscopic biological contexts (e.g., cellular chemotaxis, avian flocking). In analogy with the local equations of motion which arise in various elastic rod and vorticity theories, we pursue this question in a continuum setting where agents are uniquely associated with material points of a virtual filament. The governing dynamics for this filament are chosen so that an established set of control objectives is achieved. The appropriate configuration space of continua is shown to be an infinite dimensional Hilbert Lie group admitting a separable topology. A class of filament models is studied in a Lagrangian formalism on this manifold, leading to a natural curvature feedback law.

The Virtual Filament Model

by

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Chapter 1

A Continuum Approach to Formation Control

The problem of formation control is one of prescribing the relative and collective motion of a group of autonomous agents. In this context, control laws are often proposed for a collection of isolated agents. The evolution of a large swarm is then described by a limiting equation derived from the underlying finite agent model. However, the most elegant theories, while not restricting interaction of agents by arbitrary locality constraints, often require full interaction of the assembly. This inherent lack of locality leads to a computationally intractable solution in many robotic applications, and a significant conceptual break with natural swarming phenomenon. In contrast, we propose a continuum formulation in which individual agents are identified with material points of a filament. Here we consider a filament to be a one dimensional continuum analogous to a physical string. A control mechanism is then naturally established in an infinite dimensional setting by prescribing the filament evolution. The trajectory of each material point of the filament is recovered by spatial discretization.

The theory of vorticity in fluid mechanics offers a compelling inspiration for this continuum approach. Given a flow field v, the associated vorticity is given by $\omega = \nabla \times v$. Inverting this curl relationship by the the Law of Biot-Savart, the flow can be regarded as induced by the vorticity. With this perspective, consider the induced flow from a singular vorticity distribution along a filament γ in $\mathbb{R}^3$. Under the assumption that the flow of the filament at any point is induced only by its local geometry, DaRios [6] and Betchov [2] independently showed that the evolution of the filament is governed by the equation

$$\gamma_t = \gamma_s \times \gamma_{ss}, \tag{1.1}$$

where the subscripts t and s denote temporal and spatial differentiation, respectively. The assumption that the flow is locally induced is classically known as the Localized Induction Approximation (LIA) and the corresponding equation (1.1) is referred to as the DaRios-Betchov filament equation [16].

One of the remarkable properties of the LIA model is that (1.1) leads to a nonstretching filament evolution. While this is a general property of planar vortex filaments, dynamic length variation is an important flow characteristic of higher dimensional vortex filaments. In fact vortex stretching is thought to be an important mechanism underlying various modes of turbulence [15]. In the interest of the present work, however, the inextensibility of the filament is an interesting quality since it suggests that material points of this idealized vorticity distribution will persist indefinitely as constitutive elements of the continuum. This is essentially a stable formation.

The filament equation (1.1) admits an elegant characterization in terms of a corresponding curvature evolution. Let κ_1 and κ_2 denote the principal curvatures [3] of γ. In has been shown by Hasimoto [1] that $\kappa = \kappa_1 + i\kappa_2$ is governed by the

[1]The original derivation offered in [10] involves the curvature and torsion of Frenét Serret

cubic nonlinear Schrödinger equation given by

$$\frac{1}{i}\kappa_t = \kappa_{ss} + |\kappa|^2 \frac{\kappa}{2}. \tag{1.2}$$

This equation is featured prominently in an extensive literature related to soliton theory and the Nonlinear Schrödinger Hierarchy [13]. Immediately we conclude that the vortex filament dynamics induced from LIA are both Hamiltonian and completely integrable. Furthermore, the evolution of a vortex filament under LIA is entirely characterized by a corresponding evolution of its intrinsic curvature. Conversely, this curvature evolution induces the filament flow given by (1.1). The interest of this work is to study whether there exist other curvature flows which induce similarly interesting behavior.

This model study of vorticity suggests the importance of two filament flow characteristics: locality and cohesiveness. In order to effectively guide the collective motion of a formation, we will develop these ideas in the context of a general framework for studying filament evolution. Our approach is to conceive of a virtual filament as an abstract object to which a particular governing mechanics is assigned. The pseudo-physical properties of the filament are chosen so as to achieve desired control objectives. In particular we derive equations of motion for this infinite dimensional virtual system by appealing to Lagrange D'Alembert Mechanics.

The construction of the filament Lagrangian is motivated by an insightful model offered by avian flocking. Birds in flight often assume a filament-like,

frame theory. Here we have stated a slightly more general result derived using natural frames and principle curvatures. This avoids the intrinsic singularity present in the former framing convection.

V configuration. The emergence of this pattern is driven largely by its aerodynamic efficiency as well as various local interactions among the flock members (e.g, a preferred separation distance). Another significant feature of avian flight is exhibited by migratory bird flocks which are able to distinguish between north and south directions primarily by sensing the Earth's magnetic field. The filament model proposed in this work draws directly from these local and global elements of flocking, penalizing both geodesic stretching of the filament and misalignment of the material point trajectories with an imposed symmetry breaking vector field. The latter field is referred to as the orientation field. In many instances these local and global objectives will constitute competing interests. It is the responsibility of the controller to manage this tradeoff. Our approach in this work is to seek a natural policy governing this tradeoff which is informed by a continuum mechanical perspective.

Implicitly our choice of the filament Lagrangian is motivated by extremizing a cost functional consisting of terms that ought to be minimized. It is important to note that, while this variational argument is insightful, our derivation of the governing filament equations is subtly different due to the manner in which external forcing and constraints are introduced into the problem. Many applications of formation control – such as motion planning for unmanned arial vehicles and orbiting satellites – require minimal variation in the speed of individual formation members. Hence the flow of each material point in the proposed filament model is constrained to observe a nonholonomic constant speed constraint. The non-integrability of the induced constraint distribution leads to a subtle distinction between minimizing a Lagrangian action functional and deriving the real dynamics of mechanical motion.

In general, nonholonomic mechanics is not a variational theory, requiring instead the application of the Lagrange-d'Alambert principle of virtual work. We have chosen to study mechanics rather than optimal control because this perspective offers a broader framework in which to develop an effective theory of virtual filaments. In the penultimate chapter of this thesis, for example, we argue that the introduction of external forcing is an essential dissipation mechanism required for stable filament evolution.

The outline of this thesis may be summarized as follows. In Chapter 2 we briefly introduce the essential ideas and notation from differential geometry that will be employed throughout throughout this work. We then develop the concept of an oriented filament as a curve on a matrix Lie group. The algebraic and topological structure of the configuration space of filaments is then established as an infinite dimensional Hilbert Lie group. Chapter 3 begins with an intrinsic statement of the Lagrange D'Alembert principle of virtual work. Exploiting the algebraic structure of the filament configuration space, we pull back the classic Euler-Lagrange equations of motion to the trivialization of its tangent bundle. Finally, a class of Lagrangians is considered in which each member admits a local description in terms of a Lagrangian density. For this class of models, a representation of the Euler-Lagrange equations is derived which appeals only to finite dimensional calculus.

In Chapter 4 we argue for a particular Lagrangian model of a planar filament which incorporates both local and global aspects of control. The governing equations for the filament are derived using the Lagrangian apparatus established in the preceding chapter. A natural choice of smooth curvature feedback is proposed which is motivated by the governing equations. We then introduce the concept of an oriented orbit for an integral curve of a vector field. Oriented integral curve orbits of the orientation field are shown to be invariant under the flow induced by the proposed feedback. We conclude this chapter by demonstrating in simulation that this induced filament evolution asymptotically aligns itself with a variety of nontrivial orientation fields. In Chapter 5 we offer concluding remarks and possible avenues for future work.

Chapter 2

The Geometry and Calculus of Continua

As outlined in the previous chapter, we are interested in characterizing collective particle dynamics by studying the evolution of a related filament. In order to appeal to a classical Lagrangian formalism we must establish the algebraic and geometric structure of an appropriate space of filaments. We begin with a brief review of smooth differential geometry and proceed to introduce the concept of a collection modeled on a space of oriented filaments. We conclude by showing that such a space forms an infinite dimensional Hilbert Lie group under a natural topology.

2.1 Differential Geometry

The following discussion of differential geometry is offered primarily to establish notation. The reader is referred to either [1],[7], or [8] for a more comprehensive treatment. Let U and V be n-dimensional vector spaces over $\mathbb{R}$ and let $\Lambda^p(V)$ denote the p^{th} exterior algebra of V. Let $\wedge : \Lambda^p(V) \times \Lambda^q(V) \to \Lambda^{p+q}(V)$ denote the exterior product on V. The set of tensors of type (r, s) on V is given the notation

$$\mathcal{T}_r^s(V) = \overbrace{V \otimes \cdots \otimes V}^{r} \otimes \overbrace{V^* \otimes \cdots \otimes V^*}^{s}.$$

Then $\Lambda^p(V^*) \simeq \Lambda^p(V)^*$ is the set of smooth p-linear alternating maps $\mathcal{T}_p(V) \to \mathbb{R}$. The symbol $\cdot$ is reserved for the action of a covariant tensor on a contravariant tensor. In particular, this notation represents the natural pairing of vectors and covectors.

For notational clarity we adopt the follow summation convection: Greek characters are summed over repeated indices beginning at unity while Roman characters are summed analogously beginning at zero. Furthermore we adopt the Einstein convention, subscripting and superscripting covariant and contravariant coordinates respectively. In contrast, the subscripting of a map by its argument denotes partial differentiation.

Given an inner product $< \cdot, \cdot >$ on V, the morphisms $\flat : V \to V^*$ and $\sharp : V^* \to V$ are defined for each $u, v \in V$ and $\omega \in V^*$ as

$$u^\flat \cdot v \;\; = \;\; < u, v >, \qquad \omega \cdot v = < \omega^\sharp, v > .$$

The existence of the objects $u^\flat$ and $\omega^\sharp$ is guaranteed for any separable Hilbert space by the Riesz Representation theorem (recall that any finite dimensional inner product space is a separable Hilbert space). The dual of a linear map $A : U \to V$ is a map $A^* : V^* \to U^*$ defined for each $u \in U$ as

$$\omega \cdot Au = A^*\omega \cdot u. \tag{2.1}$$

The p^{th} exterior power of linear map A, denoted as $\Lambda^p A : \Lambda^p(U) \to \Lambda^p(V)$ is defined by

$$\Lambda^p A(v_1 \wedge \cdots \wedge v_p) = Av_1 \wedge \cdots \wedge Av_p. \tag{2.2}$$

One of the most natural and important concepts in differential geometry is a smooth manifold. A smooth manifold is a topological space which is locally diffeomorphic to a linear space on which it is said to be modelled. The dimension of a manifold is inherited from the underlying modelling space.

A natural notion of calculus is established on a smooth manifold M through the classic language of differential forms. Let $C^\infty(M, p)$ denote the set of smooth functions defined on a neighborhood of $p \in M$. Let $S^\infty(E \to M)$ denote the set of smooth sections of a vector bundle $E \to M$. The space of sections then is a module over the ring of smooth functions. A section of the bundle $\Lambda^p(T^*M) \to M$ is referred to as a differential form of degree p. The set of all such differential p-forms is denoted by $\Omega^p(M) = S^\infty(\Lambda^p(T^*M) \to M)$. Note that the zeroth exterior power of a real vector space is $\mathbb{R}$. Hence a 0-form is a section of the real line bundle over M. This implies that a 0-form is simply a function on M; i.e. $\Omega^0(M) = C^\infty(M)$.

The derivative of a differential form is defined inductively as follows. A section X of the tangent bundle is interpreted as a derivation on $C^\infty(M, p)$ by defining the action

$$X_p f = \frac{d}{d\epsilon} f(\gamma(\epsilon))\bigg|_{\epsilon=0}, \tag{2.3}$$

where $f \in C^\infty(M, p)$ and $\epsilon \mapsto \gamma(\epsilon)$ is curve passing through $p \in M$ whose tangent vector is X_p. The exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is defined as the unique operator satisfying

$$df \cdot X = Xf$$

for $f \in C^\infty(M)$, and

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{p-k}(M)$. Hence, the exterior derivative is a derivation of unity degree over the graded algebra $\oplus_{r=0}^{p} \Omega^p(M)$. By this construction, it immediately follows that exterior derivative of a covectorfield is expressed as

$$d\omega \cdot u \wedge v \quad = \quad u(\omega \cdot v) - v(\omega \cdot u) - \omega \cdot [u, v]. \tag{2.4}$$

The exterior derivative of higher degree forms admit similar identities. Given a Riemannian structure on M, the gradient of a function is defined as the sharpening of the exterior derivative; i.e., for each $f \in C^\infty(M, p)$, the gradient is expressed as $\nabla_p f = (df)_p^\#$.

Let M and N be smooth manifolds. If $\varphi : M \to N$ is a diffeomorphism, then it induces an invertible linear transformation between the domain and target tangent bundles. This induced map $D\varphi_p : T_pM \to T_{\varphi(p)}N$ is the differential of φ at $p \in M$ defined for each function $f \in C^\infty(M, p)$ as

$$(D\varphi X)_p f = X_p(f \circ \varphi). \tag{2.5}$$

The pushforward of $X \in S^\infty(\mathcal{T}_k(TM) \to M)$ by φ is a section $\varphi_* X \in S^\infty(\mathcal{T}_k(TN), \pi, M)$ defined as

$$\varphi_* X_p = D\varphi_{\varphi^{-1}(p)} X_{\varphi^{-1}(p)}. \tag{2.6}$$

Analogously, the differential map induces a linear transformation of covectors on N to covectors on M through the adjoint of its exterior power:

$$(\Lambda^k D\varphi)_p^* : \Lambda^k(T^*_{\varphi(p)}N) \to \Lambda^k(T^*_p M). \tag{2.7}$$

This leads to a important natural isomorphism betweens forms on N and M known as the pullback. Specifically, the pullback of the k-form $\omega \in \Omega^k(N)$ by the diffeomorphism φ is a k-form $\varphi^*\omega \in \Omega^k(M)$ defined for each $p \in M$ by

$$\varphi^*\omega_p = (\Lambda^k D\varphi)_p^* \, \omega_{\varphi(p)}. \tag{2.8}$$

Explicitly we may write the pullback of ω by φ as

$$\varphi^*\omega_p \cdot V_1(p) \wedge \ldots \wedge V_k(p) = \omega_{\varphi(p)} \cdot D\varphi_p V_1(p) \wedge \ldots \wedge D\varphi_p V_k(p), \tag{2.9}$$

where $p \in M$ and $V_1, \ldots, V_k$ are smooth sections of the tangent bundle. Note that for the case of functions, the pullback is simply a change of coordinates. As a generalized mechanism of coordinate change for higher order forms, the pullback is an essential construction in differential calculus. The pullback and pushforward are compactly related by the identity

$$\varphi_* X(\omega) = \varphi^*\omega \cdot X, \tag{2.10}$$

where X is a vector field on M and ω is a 1-form. Furthermore, the pushforward of a diffeomorphism is the pullback of its inverse.

Many of the familiar vector field operations in $\mathbb{R}^n$, such as the curl and divergence, can be generalized to a manifold setting through the Hodge star operator. This operator *: $\Omega^p(V) \rightarrow \Omega^{(n-p)}(V)$ is the unique isomorphism satisfying

$$\ll *\alpha, \sigma \gg \mu = *\alpha \wedge \sigma, \tag{2.11}$$

where $\ll \cdot, \cdot \gg$ denotes a nondegenerate inner product on $\Omega^{(n-p)}(V)$, μ is a volume form on V, and σ is an $(n-p)$-form. If F is vector field on M, then the curl of F is intrinsically expressed on M as

$$curl(F) = (*dF^\flat)^\#. \tag{2.12}$$

Similarly, the divergence of F may be written

$$div(F) = \big(d(*F^\flat)\big)^\#. \tag{2.13}$$

A manifold with a smooth group structure is given the special status of a a Lie group. Given a Lie group G and smooth manifold Q, the left action Φ : $G \times Q \to Q$ of G on Q is the smooth map given by

$$(g, q) \longmapsto \Phi(g, q) = \Phi_g(q) = g \cdot q. \tag{2.14}$$

The right action is defined analogously. The action of a group on itself is referred to as translation. The tangent lift action $\Phi^T : G \times TQ \to TQ$ is the induced action on the tangent bundle of Q:

$$(g, v_q) \longmapsto \Phi_g^T(v_q) = T_q\Phi_g(v_q), \tag{2.15}$$

where, for classical reasons, T denotes the tangent map or linearization operator. Similarly the cotangent lift action is the mapping $\Phi^{T^*} : G \times T^*Q \to T^*Q$ given by

$$(g, \omega_q) \longmapsto \Phi_g^{T^*}(\omega_q) = T_q^*\Phi_{g^{-1}}\omega_q, \tag{2.16}$$

where $T_q^*\Phi_{g^{-1}} = (T_q\Phi_{g^{-1}})^*$. If Φ denotes the left action of G on itself, then with respect to a left invariant Riemannian structure on G,

$$\Phi_{g^{-1}}^{T^*}(\omega_g) = \left(T_g\Phi_{g^{-1}}\omega_g^\#\right)^\flat. \tag{2.17}$$

Given the vector space V, a Lie bracket $[\cdot,\cdot] : V \times V \to V$ is a bilinear, antisymmetric operator satisfying the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \tag{2.18}$$

for each $X, Y, Z \in V$. A Lie algebra is a vector space which is closed under a Lie bracket. The bracket of vector fields on Q is a Lie bracket $[[\cdot,\cdot]] : S^\infty(TQ) \times S^\infty(TQ) \to S^\infty(TQ)$ defined for $f \in C^\infty(M)$ as

$$[[X,Y]]f = X(Y(f)) - Y(X(f)), \tag{2.19}$$

for $X, Y \in S^\infty(TM)$. A left invariant vector field X on G is a section of the tangent bundle that is invariant is under the tangent lift action; i.e., for each $g \in G$ and $f \in C^\infty(Q)$

$$(T_q\Phi_g)X_q f = X_q f. \tag{2.20}$$

The Lie bracket of vectors $\xi, \eta \in T_eG$ is defined as

$$[\xi,\eta] = [[X_\xi, X_\eta]], \tag{2.21}$$

where X_ξ and X_η are left invariant extensions of ξ and η, respectively, in a neighborhood of the identity. The tangent space of G at the identity, under the Lie bracket (2.21), is referred to as the Lie algebra of G. The Lie algebra is isomorphic to the set of left invariant vector fields on G. An operator which is featured prominently in the present work is the adjoint map $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ defined for each $\xi \in \mathfrak{g}$ as

$$ad_\xi \eta = [\xi,\eta]. \tag{2.22}$$

Let $\chi_0, \chi_1, \ldots, \chi_{n-1}$ be a basis for the Lie algebra $\mathfrak{g}$ of an n-dimensional Lie group and let $\chi_*^0, \chi_*^1, \ldots, \chi_*^{n-1}$ be the basis for $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$. An explicit expansion for the coadjoint action on $\omega \in \mathfrak{g}^*$ in terms of the dual basis and structure coefficients for the Lie algebra is given for each $\xi, \eta \in \mathfrak{g}$ as

$$\begin{aligned}
ad_\xi^* \omega \cdot \eta &= \omega \cdot ad_\xi \eta \\
&= \omega \cdot [\xi, \eta] \\
&= \omega \cdot [\xi^i \chi_i, \eta^j \chi_j] \\
&= \omega \cdot \xi^i \eta^j [\chi_i, \chi_j] \\
&= \big(\omega \cdot \xi^i [\chi_i, \chi_j]\big) \eta^j \\
&= \big(\omega \cdot \xi^i [\chi_i, \chi_j]\big) \chi_*^j \cdot \eta \\
&= \big(\omega \cdot \xi^i \Upsilon_{ij}^k \chi_k\big) \chi_*^j \cdot \eta \\
&= \Big(\omega_k \, \xi^i \, \Upsilon_{ij}^k\Big) \chi_*^j \cdot \eta,
\end{aligned} \tag{2.23}$$

where the structure constants Υ_{ij}^k are defined as

$$[\chi_i, \chi_j] = \Upsilon_{ij}^k \chi_k. \tag{2.24}$$

Since $\eta \in \mathfrak{g}$ is arbitrary,

$$ad_\xi^* \omega = \Big(\omega_k \, \xi^i \, \Upsilon_{i,j}^k\Big) \chi_*^j. \tag{2.25}$$

Having introduced the necessary concepts of differential calculus, we now develop a continuum description of a collection of agents.

2.2 The Oriented Virtual Filament

An oriented particle is an abstract object consisting of a pair (R, γ), where R is an element of the Special Orthogonal group and γ is an element of Euclidean space. The space of oriented particles in $\mathbb{R}^n$ is identified with the n-dimensional Special Euclidean group, represented as a set of matrices given by

$$SE_n = \left\{ \begin{bmatrix} R & \gamma \\ \emptyset & 1 \end{bmatrix} : R \in SO_n, \gamma \in \mathbb{R}^n \right\},$$

where SO_n denotes the n-dimensional Special Orthogonal group, consisting of all matrices with unity determinant, and $\emptyset$ denotes a row vector containing n zeros. It is elementary to show that SE_n forms a Lie group under matrix multiplication and inversion.

In the continuum setting, an oriented filament is modelled as a 1-dimensional continuum of oriented particles. Formally a configuration of an oriented filament in $\mathbb{R}^n$ is a map from a 1-dimensional compact manifold with boundary Ω into SE_n. Hence a filament is a curve on SE_n, with each material point uniquely identified with an element of SE_n. Clearly such a system evolves on an infinite dimensional space. In order to effectively characterize its evolution, we develop a manifold structure for the configuration space of filaments.

Let us denote by $P^Q = C^\infty(Q, P)$ the set of smooth maps between manifolds Q and P. Let Ω be a compact manifold and G a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. Define multiplication on G^Ω pointwise as

$$(\psi_1 \cdot \psi_2)(\omega) = \psi_1(\omega) \cdot \psi_2(\omega), \tag{2.26}$$

for each ψ_1,$\psi_2 \in G^\Omega$ and $\omega \in \Omega$. Similarly, define inversion for each $\psi \in G^\Omega$ as

$$\psi^{-1}(\omega) = (\psi(\omega))^{-1}, \tag{2.27}$$

for each $\omega \in \Omega$. If e is the identity map on G, let $id \in G^\Omega$ denote the constant map $\Omega \xmapsto{id} e$. The natural geometry and algebraic structure of G^Ω is established in the following theorem.

Theorem 1. *The set of maps G^Ω, with group operations (2.26) and (2.27), is a separable Hilbert Lie Group under the uniform convergence topology.*

Proof. Let $\mathfrak{g}$ be equipped with an inner product $< \cdot, \cdot >_e$ and define an inner product on $\mathfrak{g}^\Omega$ as

$$< \xi, \eta >_{id} = \int_\omega < \xi(\omega), \eta(\omega) >_e d\omega,$$

where $\xi, \eta \in \mathfrak{g}^\Omega$. We begin by showing that $\mathfrak{g}^\Omega$, equipped with a associated non-degenerate inner product $< \cdot, \cdot >_{id}$, is a separable Hilbert space. Let $\| \cdot \|_e$ and $\| \cdot \|_{id}$ denote the standard induced norms on $\mathfrak{g}$ and $\mathfrak{g}^\Omega$, respectively. Recall that every finite dimensional inner product space is complete with respect to its induced norm. Hence $\mathfrak{g}$ is complete. Let $\{\eta_k\}$ be a cauchy sequence in $\mathfrak{g}^\Omega$. By the completeness of $\mathfrak{g}$, there exists $\eta \in \mathfrak{g}^\Omega$ such that $\{\eta_k\}$ converges pointwise to η. It is necessary to show that this convergence is uniform. Let $\mathbb{N}$ denote the positive integers. By pointwise convergence, for each $\omega \in \Omega$ and $\epsilon > 0$ there exist maps $\mu : \Omega \to \mathbb{N}$ and $\rho : \Omega \to \mathbb{N}$ such that for $\rho(\omega) > \mu(\omega)$

$$\|\eta_{\rho(\omega)}(\omega) - \eta(\omega)\|_e < \epsilon. \tag{2.28}$$

By the compactness of Ω, the map μ attains a maximum, denoted by M, such that for $m > M$

$$\|\eta_m(\omega) - \eta(\omega)\|_e < \epsilon, \tag{2.29}$$

for each $\omega \in \Omega$. Since Ω is compact, $\int_\Omega d\omega$ is finite and positive. Therefore, for $\epsilon > 0$ there exists $M > 0$ such that for $m > M$

$$\|\eta_m(\omega) - \eta(\omega)\|_e < \frac{\epsilon}{\sqrt{\int_\Omega d\omega}}, \tag{2.30}$$

for each $\omega \in \Omega$. Hence, given $\epsilon > 0$, there exists $M > 0$ such that for $m > M$

$$\begin{aligned} \|\eta_m - \eta\|_{id} &= \sqrt{\int_\Omega < \eta_m(\omega) - \eta(\omega), \eta_m(\omega) - \eta(\omega) >_e d\omega} \\ &= \sqrt{\int_\Omega \|\eta_m(\omega) - \eta(\omega)\|_e^2 d\omega} \\ &< \sqrt{\int_\Omega \frac{\epsilon^2}{\int_\Omega d\omega} d\omega} \\ &= \epsilon. \end{aligned} \tag{2.31}$$

Hence $\{\eta_k\}$ converges uniformly to η. Since $\{\eta_k\}$ is an arbitrary cauchy sequence, $\mathfrak{g}^\Omega$ is complete. Therefore $(g^\Omega, < \cdot, \cdot >_{id})$ is a Hilbert space.

To establish separability, we observe that for each $\eta \in \mathfrak{g}^\Omega$ there exists $\eta^0, \ldots, \eta^{n-1} \in C^\infty(\Omega)$ such that for each $\omega \in \Omega$,

$$\eta(\omega) = \eta^k(\omega)\chi_k, \tag{2.32}$$

where $\chi_0, \ldots, \chi_{n-1}$ is a basis for $\mathfrak{g}$. Let $L^2(\Omega)$ be the set of Lebesgue square integrable function on Ω. Since Ω is compact, $C^\infty(\Omega) \subseteq L^2(\Omega)$. Hence (2.32) implies that $\mathfrak{g}^\Omega$ is a submodule of the Lie algebra $\mathfrak{g}$ over the ring of L^2 functions on Ω.

Since $L^2(\Omega)$ is separable, any finite module over $L^2(\Omega)$ is separable. Hence $\mathfrak{g}^\Omega$ is a separable Hilbert space.

We proceed by showing that G^Ω is a smooth Lie group modelled on the separable Hilbert space $\mathfrak{g}^\Omega$. Recall that the uniform topology on G^Ω admits the subbase consisting of sets of form

$$\mathfrak{B}(U, V) = \{\rho \in G^\Omega \mid \rho(V) \subseteq U\}, \tag{2.33}$$

where $U \subseteq G$ and $V \subseteq \Omega$ are open and compact, respectively. Let $U_e \subseteq G$ be an open neighborhood of the identity in G that is diffeomorphic, by the exponential map, to V, an open neighborhood of the origin in $\mathfrak{g}$. Define a neighborhood of $\psi \in G^\Omega$ as

$$\Phi_\psi(U_e^\Omega) = \{\psi \cdot \phi \mid \phi \in U_e^\Omega\}. \tag{2.34}$$

Define the left translation of U_e by ψ as

$$U_\psi = \bigcup_{\omega \in \Omega} \{\psi(\omega) \cdot u \mid u \in U_e\}. \tag{2.35}$$

By continuity of translation on G, the image of U_e under left translation by $\psi(\omega)$ for a fixed $\omega \in \Omega$ is open. Thus U_ψ in (2.35) is an infinite union of open sets; hence it is open. We can now write

$$\Phi_\psi(U_e^\Omega) = \{\rho \in G^\Omega \mid \rho(\Omega) \subseteq U_\psi\}. \tag{2.36}$$

Since U_ψ is open and Ω is compact, $\Phi_\psi(U_e^\Omega)$ is contained in the subbasis of the uniform topology; hence it is open. Since the map $\Omega \mapsto e$ is in U_e^Ω, clearly $\psi \in \Phi_\psi(U_e^\Omega)$. Hence $\Phi_\psi(U_e^\Omega)$ is an open neighborhood of ψ.

Let exp denote the exponential map on G and define the map $\Theta_\psi : V^\Omega \to \Phi_\psi(U_e^\Omega)$ as

$$\Theta_\psi(v)(\omega) = \Phi_{\psi(\omega)}\Big(exp(v(\omega))\Big), \tag{2.37}$$

for each for each $v \in V^\Omega$ and $\omega \in \Omega$. Since every pointwise convergent sequence in G^Ω is uniformly convergent in the uniform topology, the the pointwise smoothness of an operator on G^Ω with respect to the topology on G guarantees is smoothness on G^Ω. Since Θ_ψ is clearly pointwise smooth on G, it is smooth in the uniform topology on G^Ω. Since $\psi \in G^\Omega$ is arbitrary the following holds: For each $\psi \in G^\Omega$ there exist an open neighborhood of ψ that is diffeomorphic by Θ_ψ to an open to an open neighborhood of the origin in $\mathfrak{g}^\Omega$. Therefore G^Ω is a smooth manifold.

It remains to be shown that the multiplication (2.26) and inversion (2.27) operations are smooth. Since the multiplication (2.26) and inversion (2.27) operations inherit pointwise smoothness from the smooth group structure of G, they are smooth maps on G^Ω. Therefore G^Ω is a smooth Lie Group modelled on the separable Hilbert space $\mathfrak{g}^\Omega$. $\square$

A tangent vector v to G^Ω at the point φ is a map from Ω into the tangent bundle of G^Ω. More precisely, if π denotes the canonical projection of TG onto G, then $v(\omega) \in \pi^{-1}(\varphi(\omega))$ for each $\omega \in \Omega$. The set of all such tangent maps constitutes the tangent space of G^Ω at the point φ. A Riemannian metric is defined on G^Ω for each $\varphi \in G^\Omega$ and $u, v \in T_\varphi G^\Omega$ by

$$< u, v >_\varphi = \int_\Omega < u(\omega), v(\omega) >_{\varphi(\omega)} d\omega, \tag{2.38}$$

where $< \cdot, \cdot >_g$ is a left invariant Riemannian metric on G.

Given a Riemannian metric on G, the induced supremum topology on G^Ω coincides with the uniform convergence topology. Furthermore one can easily show that locally G^Ω is compact and admits unique geodesics. In fact the exponential map for G^Ω is defined as the pointwise exponential map on G. In the case of a matrix Lie group such as SE_n, the exponential map for SE_n^Ω is obtained by pointwise matrix exponentiation.

Chapter 3

A Class of Virtual Filament Models

Having established the Lie group structure of the space of oriented filaments, we now propose a class of dynamic filament models through a Lagrangian formalism. As outlined earlier, the virtual filament dynamics will be governed by the equations of motion for a constrained Lagrangian system. In an effort to derive these governing equations, we first discuss filament kinematics and then introduce a broad class of filament models defined in terms of a Lagrangian density. We then establish an intrinsic characterization of Lagrangian mechanics for this class of models by appealing to the Lagrange D'Alembert Principle of Virtual Work. The governing equations of motion are then derived as the central result of this chapter. We conclude by showing that the governing equations for an unconstrained, unforced mechanical system describe the evolution of extremal maps for a natural cost functional. This foundational idea will establish a basis for the model construction methodology introduced in the following chapter.

3.1 Oriented Filament Kinematics

In order to discuss the dynamics of a filament we must first establish a convention for describing its kinematics. Let $\mathcal{I}$ be a time interval of $\mathbb{R}$ and let Ω be a compact 1-dimensional manifold with boundary. Then $\Pi = C^\infty(\mathcal{I}, SE_n^\Omega)$ represents the space of filament trajectories. The flow of an oriented filament $\varphi \in \Pi$ is uniquely determined by a corresponding evolution $\xi : \mathcal{I} \to \mathfrak{se}_n^\Omega$ on the Lie algebra of G^Ω; i.e.,

$$\varphi_t = T_{id}\Phi_\varphi \xi, \tag{3.1}$$

where the subscript t denotes partial differentiation and Φ denotes the left action of SE_n^Ω on itself. At each point $t \in \mathcal{I}$, $\varphi(t)$ is a map $\Omega \to SE_n$. Hence φ may be interpreted as an SE_n-valued field over $\mathcal{I} \times \Omega$. Therefore we naturally define a partial spatial derivative of φ as

$$\varphi_\omega(t, \omega) = (\varphi(t))_\omega(\omega), \tag{3.2}$$

for each $t \in \mathcal{I}$ and $\omega \in \Omega$. Under this notation, we write

$$\varphi_\omega = T_{id}\Phi_\varphi \eta, \tag{3.3}$$

for some $\eta : \mathcal{I} \to \mathfrak{se}_n^\Omega$. Equations (3.1) and (3.3) constitute the filament kinematics. This temporal and spatial evolution on the group induces a kinematic flow on the Lie algebra described by the partial differential equation in the following proposition.

Proposition 1. *Given a smooth map $\varphi \in C^\infty(\mathcal{I}, SE_n^\Omega)$, let $\xi = T_\varphi \Phi_{\varphi^{-1}} \varphi_t$ and $\eta = T_\varphi \Phi_{\varphi^{-1}} \varphi_\omega$. Then*

$$\eta_t = \xi_\omega - [\xi, \eta]. \tag{3.4}$$

Proof. Given that $T_{id}\Phi_\varphi = \varphi$ on G^Ω, where G is a matrix Lie group,

$$\begin{aligned}
\eta_t = \frac{\partial}{\partial t}(\varphi^{-1}\varphi_\omega) &= -\varphi^{-1}\varphi_t\varphi^{-1}\varphi_\omega + \varphi^{-1}(\varphi_t)_\omega \\
&= -\varphi^{-1}\varphi_t\,(\varphi^{-1}\varphi_\omega) + \varphi^{-1}\frac{\partial}{\partial\omega}\Big(\varphi\big(\varphi^{-1}\varphi_t\big)\Big) \\
&= -\varphi^{-1}\varphi_t\,(\varphi^{-1}\varphi_\omega) \\
&\quad +\varphi^{-1}\left(\varphi_\omega\,\varphi^{-1}\varphi_t + \varphi\frac{\partial}{\partial\omega}(\varphi^{-1}\varphi_t)\right) \\
&= [\varphi^{-1}\varphi_\omega, \varphi^{-1}\varphi_t] + \big(\varphi^{-1}\varphi_t\big)_\omega \\
&= [\eta, \xi] + \xi_\omega
\end{aligned}$$

This establishes the desired result. □

This elementary kinematic relation is referred to as the compatibility condition. A similar result for curves on a general Lie group is established in [4]. The compatibility condition is used extensively in our Lagrangian formulation of filament dynamics.

We now proceed to develop a general theory of Lagrangian mechanics for systems modeled on G^Ω, where G is a general finite dimensional Lie group and Ω is a compact manifold with boundary. This general theory is then applied to the study of virtual filaments where $G = SE_n$ and Ω is a compact subset of $\mathbb{R}$.

3.2 The Lagrangian Density Formulation

Consider a filament model that is both unconstrained and G^Ω-invariant. Then without external forcing, its evolution is governed by the classical Euler-Poincare equations on the reduced quotient bundle $TG^\Omega/G^\Omega \simeq \mathfrak{g}^\Omega$. More generally, we are interested in anisotropic models which do not admit G^Ω-symmetry. However, we would like to preserve the structure of the Euler Poincare equations. Consequently, the approach taken here is to construct a Lagrangian model on the trivialization of the tangent bundle. Therefore, the class of models proposed in this section will be introduced through a Lagrangian defined on $G^\Omega \times \mathfrak{g}^\Omega$, and related bundles.

Let $\Sigma_m = G \times \overbrace{\mathfrak{g} \times \mathfrak{g} \cdots \times \mathfrak{g}}^{m+1}$ and $T^mG = \overbrace{TG \oplus TG \oplus \cdots \oplus TG}^{m+1}$. Define the bundle map $\phi_m : \Sigma_m \to T^mG$ as

$$\phi_m(g, \xi, \vec{\eta}) = (g, T_{id}\Phi_g \xi, T_{id}\Phi_g \vec{\eta}), \tag{3.5}$$

where $(g, \xi, \vec{\eta}) = (g, \xi, \eta^1, \eta^2, \ldots, \eta^m) \in \Sigma_m$. Then the bundle $\Sigma_m \to G$ is the pullback bundle of $T^mG \to G$ under the isomorphism ϕ_m. Let $\Psi = \Sigma_0^\Omega \simeq G^\Omega \times \mathfrak{g}^\Omega$ and define $\phi : \Psi \to TG^\Omega$ as

$$\phi(\psi)(\omega) = \phi_0(\psi(\omega)), \tag{3.6}$$

for each $\psi \in \Psi$ and $\omega \in \Omega$. By construction of ϕ,

$$\Psi = \phi^* TG^\Omega. \tag{3.7}$$

Hence Ψ is the pullback bundle of TG^Ω over G^Ω. Furthermore, the pair (Ψ, ϕ) is referred to as the trivialization of the tangent bundle TG^Ω.

Consider a class of Lagrangians on Ψ of the form

$$\mathbb{L}[\varphi, \xi] = \int_\Omega \mathcal{L}\Big(\varphi(\omega), \xi(\omega), T_\varphi \Phi_{\varphi^{-1}} \varphi_\omega(\omega), \xi_\omega(\omega) - [\xi(\omega), T_\varphi \Phi_{\varphi^{-1}} \varphi_\omega(\omega)]\Big)\, d\omega, \tag{3.8}$$

where $(\varphi, \xi, \eta, \zeta) \mapsto \mathcal{L}(\varphi, \xi, \eta, \zeta)$ is a smooth lagrangian density on Σ_2. Here the the Lagrangian density depends on the location of the filament, the temporal temporal velocity ξ, the spatial gradient η, and the time rate of change of the spatial gradient. The final term is properly interpreted by the compatibility condition (3.4) as the time rate of change of η.

It is important to note that (3.8) represents a broad class of Lagrangian models. Consider a system with a Lagrangian L defined on the tangent bundle TG^{Ω} which admits a local description in terms of Lagrangian density analogous to (3.8). Then there exists a Lagrangian on Ψ of the form (3.8) which is the pullback of L under the bundle map ϕ.

3.3 Lagrange D'Alembert Mechanics

We now outline an intrinsic theory of Lagrangian mechanics for smooth manifolds as originally developed by Vershik and Gershkovich [17]. The language adopted here is motivated by the subsequent work of Wang [18] and Yang [19]. Given a smooth manifold M, a distribution of a vector bundle $E \xrightarrow{\pi} M$ on $M' \subseteq M$ is a smooth assignment of a subspace of $\pi^{-1}(p)$ to each $p \in M'$. Hence a distribution of TM on M′ is a subbundle of the tangent bundle. A codistribution of $E^* \to M$ on $M' \subseteq M$ is similarly a subbundle which annihilates a corresponding distribution on M'.

Let Q be a smooth manifold and let $\sigma^1, \ldots, \sigma^p \in C^\infty(TQ)$ be smooth, mutually independent functions on the tangent bundle. Recall that functions $\sigma^i : TQ \to \mathbb{R}$ and $\sigma^j : TQ \to \mathbb{R}$ are independent on $U \subseteq TQ$ if $\forall\, c \in \mathbb{R}$ there exists $p \in U$ such that $\sigma^i(p) \neq \sigma^j(p)$. The functions σ^k induce a natural codimension p

foliation of TQ. Furthermore, the p-dimensional subbundle of TQ given as

$$\mathcal{C}_\alpha = \{v \in TQ \mid \sigma^k(v) = \alpha \; \forall \; k = 1, \ldots, p\}, \tag{3.9}$$

is a leaf of this foliation for each α. Suppose that a physical system of interest is constrained to the α-leaf of this induced foliation. (Note that the constraints can always be chosen such that the system evolves on the zero leaf.) In this context, the functions $\sigma^1, \ldots, \sigma^p$ are interpreted as constraints, giving rise to the constraint codistribution defined as

$$\Xi = span\{d\sigma^k \mid k = 1, \ldots, p\}. \tag{3.10}$$

An element of the constraint codistribution is called a constraint reaction force.

Let π be the canonical projection of the tangent bundle $TQ \to Q$ and define $\tau : T^*TQ \to T^*TQ$ as the bundle isomorphism

$$\tau = (T\pi)^*\rho^*,$$

where $\rho : T_qQ \to T_{(q,v)}TQ$ is the canonical isomorphism between the tangent and vertical tangent space. Let $L : TQ \to \mathbb{R}$ be a smooth Lagrangian and define $\theta^L \in \Omega^1(Q)$ as

$$\theta^L = \tau \circ dL.$$

Define Θ^L by exterior differentiation as

$$\Theta^L = -d\theta^L.$$

The Lagrangian force F^L is defined as

$$F^L(X) = \Theta^L(X, \cdot) - dH^L, \tag{3.11}$$

where $H^L : TQ \to \mathbb{R}$ is given by

$$H^L = \rho^* dL - L.$$

Readers familiar with hamiltonian mechanics will note that when the lagrangian force (3.22) vanishes, the present construction of the hamiltonian function H^L coincides with its classical definition. Informally then, one would expect the Lagrangian force to vanish along the trajectories of physical motion. In fact, for an unconstrained and unforced system, this is essentially the Lagrange D'Alembert Principle of Virtual Work. A more general and rigorous expression of this idea, which incorporates constraints and external forcing, is presented below (see [17], [18] and [19]).

Principle 1 (Lagrange D'Alembert Principle of Virtual Work)**.** *Let $\mathcal{S}$ be a system with a lagrangian L subject to an external force F^E and a set of constraints $\sigma^1, \ldots, \sigma^p$ with a corresponding constraint distribution Ξ. Then there exists $F^{\mathcal{C}} \in \tau(\Xi)$ such that the trajectories of motion for $\mathcal{S}$ are integral curves of the special vector field X satisfying*

$$F^L(X) + F^E + F^{\mathcal{C}} = 0 \tag{3.12}$$

and

$$\Xi(X) = 0. \tag{3.13}$$

We proceed by expressing the Lagrangian force in coordinates in order to derive a more explicit representation of the force balancing equation (3.12). Since we are ultimately interesting is studying an infinite dimensional system, the calculus pursued in the subsequent derivation must reflect this generality. Given that the manifold of oriented filaments is separable, one could express the derivation of forms by the usual construction of basis. However, we adopt a more elegant approach by appealing to the abstract notion of exterior differentiation introduced in Chapter 1. Let d be the unique exterior derivation on Q. Then the 2-form Θ^L can be expressed in coordinates at $(\varphi, v) \in TQ$ as

$$\begin{aligned}
\Theta^L_{(\varphi,v)} \cdot U \wedge V &= -d\theta^L \cdot U \wedge V \\
&= -\Big(d\big(\theta^L \cdot V\big) \cdot U - d\big(\theta^L \cdot U\big) \cdot V - \theta^L \cdot [U,V]\Big) \\
&= -\big(DD_vL \cdot v_1\big) \cdot (u_1, u_2) + \big(DD_vL \cdot u_1\big) \cdot (v_1, v_2) \\
&\quad -\theta^L \cdot (dV \cdot U) + \theta^L \cdot (dU \cdot V) - \theta^L \cdot [U,V] \\
&= -\big(DD_vL \cdot v_1\big) \cdot (u_1, u_2) + \big(DD_vL \cdot u_1\big) \cdot (v_1, v_2) \\
&= -\big(D_\varphi D_vL \cdot u_1\big) \cdot v_1 - \big(D_vD_vL \cdot u_2\big) \cdot v_1 + \big(D_\varphi D_vL \cdot v_1\big) \cdot u_1 \\
&\quad +\big(D_vD_vL \cdot v_2\big) \cdot u_1.
\end{aligned} \tag{3.14}$$

Similarly the exterior derivative of the Hamiltonian can be expanded as

$$\begin{aligned}
dH^L_{(\varphi,v)} \cdot V &= d\big(\rho^* dL(\varphi, v) - L(\varphi, v)\big) \cdot V \\
&= d\big(D_vL \cdot v\big) \cdot V - dL \cdot V \\
&= (D_\varphi D_VL \cdot v_1) \cdot v + (D_\varphi D_vL \cdot v_2) \cdot v + D_vL \cdot v_2 - D_\varphi L \cdot v_1 - D_vL \cdot v_2 \\
&= (D_\varphi D_VL \cdot v_1) \cdot v + (D_vD_vL \cdot v_2) \cdot v - D_\varphi L \cdot v_1.
\end{aligned} \tag{3.15}$$

Therefore, given a principal vector field $X^p(\varphi, v) = (v, w)$, the Lagrangian force may be expressed in coordinates as

$$
\begin{aligned}
F^L_{(\varphi,v)}(X^p)\cdot(u,z) &= \Theta^L(X^p)\cdot(w,z) - dH^L\cdot(w,z)\\
&= -\big(D_\varphi D_v L\cdot v\big)\cdot u - \big(D_v D_v L\cdot w\big)\cdot u + \big(D_\varphi D_v L\cdot u\big)\cdot v\\
&\quad +\big(D_v D_v L\cdot z\big)\cdot v - (D_\varphi D_V L\cdot u)\cdot v - (D_v D_v L\cdot z)\cdot v + D_\varphi L\cdot u\\
&= -\big(D_\varphi D_v L\cdot v\big)\cdot u - \big(D_v D_v L\cdot w\big)\cdot u + D_\varphi L\cdot u\\
&= \Bigg(D_\varphi L - D_\varphi D_v L\cdot v - D_v D_v L\cdot w\Bigg)\cdot u \qquad (3.16)
\end{aligned}
$$

The integral curve (φ, v) of a vector field $X(\varphi, v) = (v, w)$ on the tangent bundle is given in coordinates as $(\varphi, v)_t = (v, w)$. Equipped with this identification, the Lagrangian force may be expressed as

$$
\begin{aligned}
F^L_{(\varphi,v)}(X^p)\cdot(u,z) &= \Bigg(D_\varphi L - D_\varphi D_v L\cdot v - D_v D_v L\cdot w\Bigg)\cdot u\\
&= D_\varphi L_{(\varphi,v)}\cdot u - \Bigg(\frac{d}{dt}D_v L(\varphi,v) - D_v D_v L(\varphi,v)\cdot v_t\Bigg)\cdot u\\
&\quad -(D_v D_v L(\varphi,v)\cdot v_t)\cdot u.\\
&= \Bigg(D_\varphi L_{(\varphi,v)} - \frac{d}{dt}D_v L(\varphi,v)\Bigg)\cdot u. \qquad (3.17)
\end{aligned}
$$

Given an unconstrained Lagrangian system with no external forcing, the Lagrange D'Alembert Principle states that the Lagrangian force vanishes on the special vector field whose integral curves are the trajectories of motion. Hence, equation (3.17) implies that motion of such a system is governed by the classical Euler-Lagrange equation:

$$D_\varphi L - \frac{d}{dt}D_v L = 0. \qquad (3.18)$$

The elegance of this equation arises from its subtle abstractness and generality: The underlying calculus and smooth manifold structure which support this equation are entirely unspecified.

3.4 The Euler-Lagrange Equations

There are two primary ideas that are addressed in this section. Letting $Q = TG^\Omega$, an expression for the Lagrangian force is derived on the trivialization of TG^Ω. This representation of the Lagrangian will be expressed abstractly, ostensibly demanding an infinite dimensional calculus to evaluate. Hence the second idea pursued in this section involves deriving a finite dimensional representation of the Lagrangian force for models in the class of interest (3.22). These formal concepts will be made explicit in the subsequent discussion.

We proceed by expressing the Lagrangian force in terms of the pullback Lagrangian $\mathbb{L}$ define by

$$\mathbb{L} = \phi^* L, \tag{3.19}$$

where ϕ is the bundle isomorphism between Ψ and TG^Ω introduced in Section 3.2, and L is a smooth Lagrangian define on TG^Ω. Observe that $D_\varphi L$ can be expressed in terms of $\mathbb{L}$ as

$$\begin{aligned}
D_\varphi L_{(\varphi,v)} \cdot u &= D_\varphi \phi_* \mathbb{L}_{(\varphi,v)} \cdot u \\
&= D_\varphi \mathbb{L}(\phi^{-1}(\varphi, v)) \cdot u \\
&= D_\varphi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot u \\
&= D_\varphi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot \left(D_\varphi \phi^{-1}_{(\varphi,v)}(u) \right) \\
&\quad + D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v)) \cdot \left(D_v \phi^{-1}_{(\varphi,v)}(u) \right) \\
&= D_\varphi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot u \\
&\quad + D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot (-T_\varphi \Phi_{\varphi^{-1}} u \, T_\varphi \Phi_{\varphi^{-1}} v).
\end{aligned} \tag{3.20}$$

Similarly, the time derivative of the fiber derivative of L can be written as

$$\begin{aligned}
\frac{d}{dt} D_v L(\varphi, v) \cdot u &= \frac{d}{dt}\big(D_v \phi_* \mathbb{L}_{(\varphi,v)} \big) \cdot u \\
&= \frac{d}{dt}\big(D_v \mathbb{L}(\phi^{-1}(\varphi, v)) \big) \cdot u \\
&= \frac{d}{dt}\big(D_v \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \big) \cdot u \\
&= \frac{d}{dt} \left(\frac{d}{d\epsilon} \mathbb{L}\big(\varphi, T_\varphi \Phi_{\varphi^{-1}}(v + \epsilon u)\big) \bigg|_{\epsilon=0} \right) \\
&= \frac{d}{dt} \left(D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot T_\varphi \Phi_{\varphi^{-1}} u \right) \\
&= \frac{d}{dt} (D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v)) \cdot T_\varphi \Phi_{\varphi^{-1}} u \\
&\quad + D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot \frac{d}{dt} (T_\varphi \Phi_{\varphi^{-1}} u) \\
&= \frac{d}{dt} (D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v)) \cdot T_\varphi \Phi_{\varphi^{-1}} u \\
&\quad - D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot T_\varphi \Phi_{\varphi^{-1}} \varphi_t T_\varphi \Phi_{\varphi^{-1}} u) \\
&= \frac{d}{dt} (D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v)) \cdot T_\varphi \Phi_{\varphi^{-1}} u \\
&\quad - D_\xi \mathbb{L}(\varphi, T_\varphi \Phi_{\varphi^{-1}} v) \cdot T_\varphi \Phi_{\varphi^{-1}} v \, T_\varphi \Phi_{\varphi^{-1}} u.
\end{aligned} \tag{3.21}$$

Substituting (3.20)-(3.21) into (3.17) yields

$$
\begin{aligned}
F^L_{(\varphi,v)}(X^p)\cdot(u,z) &= \Big(D_\varphi L_{(\varphi,v)} - \frac{d}{dt}D_v L(\varphi,v)\Big)\cdot u \\
&= D_\varphi \mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\cdot u - D_\xi\mathbb{L}(\varphi, T_\varphi \mathbb{L}_{T_\varphi\Phi_{\varphi^{-1}}v})\cdot(T_\varphi\Phi_{\varphi^{-1}}u\; T_\varphi\Phi_{\varphi^{-1}}v) \\
&\quad -\frac{d}{dt}(D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v))\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad +D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\cdot T_\varphi\Phi_{\varphi^{-1}}v\; T_\varphi\Phi_{\varphi^{-1}}u \\
&= D_\varphi \mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\cdot\Big(T_{id}\Phi_\varphi T_\varphi\Phi_{\varphi^{-1}}\Big)u \\
&\quad -\frac{d}{dt}(D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v))\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad -D_\xi\mathbb{L}(\varphi, T_\varphi\mathbb{L}_{\varphi^{-1}}v)\cdot(T_\varphi\Phi_{\varphi^{-1}}u\; T_\varphi\Phi_{\varphi^{-1}}v) \\
&\quad +D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\cdot T_\varphi\Phi_{\varphi^{-1}}v\; T_\varphi\Phi_{\varphi^{-1}}u \\
&= T^*_{id}\Phi_\varphi\Big(D_\varphi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\Big)\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad -\frac{d}{dt}(D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v))\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad +D_\xi\mathbb{L}(\varphi, T_\varphi\mathbb{L}_{\varphi^{-1}}v)\cdot[T_\varphi\Phi_{\varphi^{-1}}v, T_\varphi\Phi_{\varphi^{-1}}u] \\
&= \Phi^{T^*}_{\varphi^{-1}}\Big(D_\varphi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v)\Big)\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad -\frac{d}{dt}(D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v))\cdot T_\varphi\Phi_{\varphi^{-1}}u \\
&\quad +ad^*_{T_\varphi\Phi_{\varphi^{-1}}v}D_\xi\mathbb{L}(\varphi, T_\varphi\Phi_{\varphi^{-1}}v),
\end{aligned}
$$

where $\Phi^{T^*}_{\varphi^{-1}}$ denotes the cotangent lift action induced by φ^{-1}. Let $\Delta^{\mathbb{L}}_{(\varphi,T_\varphi\Phi_{\varphi^{-1}}v)}(X^p) = F^L_{(\varphi,v)}(X^p)$ be defined as the lagrangian force on Ψ. Then $\Delta^{\mathbb{L}}$ is the pullback of F^L under the bundle isomorphism ϕ and may be written compactly as

$$\Delta^{\mathbb{L}}_{(\varphi,\xi)}(X^p) = \Phi^{T^*}_{\varphi^{-1}}D_\varphi\mathbb{L}_{(\varphi,\xi)} - \frac{d}{dt}D_\xi\mathbb{L}_{(\varphi,\xi)} + ad^*_\xi D_\xi\mathbb{L}_{(\varphi,\xi)}, \tag{3.22}$$

Let X be the special vector field on Ψ satisfying the pullback by ϕ of both the constraint distribution equation (3.13) and equation (3.12). Then integral curves on X, evolving on Ψ, will project onto the base manifold G^Ω as the real trajectories of motion for the constrained Lagrangian system S defined in Principle 1. More explicitly, the virtual filament will evolve along integral curves of the special vector field satisfying

$$\Delta^{\mathbb{L}}(X) + \Delta^E + \Delta^{\mathcal{C}} = 0, \tag{3.23}$$

and

$$\Xi(X) = 0, \tag{3.24}$$

where Ξ is now defined as a constraint distribution of Ψ on G^Ω, Δ^E is a section of the bundle $\Psi^* \to G^\Omega$ representing of the external force, and $\Delta^{\mathcal{C}}$ is similarly a constraint reaction force.

The Lagrangian force in (3.22) involves derivatives defined on the infinite dimensional manifold G^Ω. Recall, however, that we are primarily interested in a class of Lagrangians (3.8) defined locally in terms of a density. Using the abstract notion of derivation introduced in Chapter 1, one can exploit this form of the Lagrangian, offering a representation of the Lagrangian force in terms of finite dimensional derivatives. Let μ be a differential p-form on G^Ω. Then $\mu(\omega)$ is a finite dimensional form on G for each $\omega \in \Omega$. Therefore μ is effectively a map from Ω into p-forms on G. This identification suggests that for each $v \in T^p G^\Omega$,

$$\mu \cdot v = \int_\Omega \mu(\omega) \cdot v(\omega) d\omega. \tag{3.25}$$

Exploiting this natural connection between Ω-parameterized finite dimensional forms and differential forms on G^Ω yields the following result.

Theorem 2. *Let $\mathbb{L} : \Psi(G,\Omega) \to \mathbb{R}$ be a smooth lagrangian given by*

$$\mathbb{L}[\varphi,\xi] = \int_\Omega \mathcal{L}\Big(\varphi(\omega), \xi(\omega), T_\varphi\Phi_{\varphi^{-1}}\varphi_\omega(\omega), \xi_\omega(\omega) - [\xi(\omega), T_\varphi\Phi_{\varphi^{-1}}\varphi_\omega(\omega)]\Big)\, d\omega,$$

where $(\varphi,\xi,\eta,\zeta) \mapsto \mathcal{L}(\varphi,\xi,\eta,\zeta)$ is a smooth lagrangian density on Σ_2. Then the lagrangian force on Ψ is given by

$$\begin{aligned}\Delta^{\mathbb{L}}_{(\varphi,\eta)}(X^p) &= \Phi^{T^*}_{\varphi^{-1}} D_\varphi\mathcal{L}_{(\varphi,\xi)} - \frac{\partial}{\partial t}D_\xi\mathcal{L} - \frac{\partial}{\partial\omega}\Big(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\Big)\\ &\quad + ad^*_\xi D_\xi\mathcal{L} + ad^*_\eta\Big(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\Big) + \mathfrak{B}(\varphi,\xi), \end{aligned} \tag{3.26}$$

where $\eta = T_\varphi\Phi_{\varphi^{-1}}\varphi_\omega$, and $\mathfrak{B}$ depends only on the boundary of Ω.

Proof. We begin by computing the constitutive elements of Δ^L as given in (3.22). For notational clarity all time arguments will be suppressed. Let $\epsilon \mapsto \varphi(\epsilon)$ be a curve on G^Ω passing through φ whose tangent vector at φ is u. Similarly, let $\xi(\epsilon) = \xi + \epsilon\,(T_\varphi\Phi_{\varphi^{-1}}u)$. Then

$$\begin{aligned} D_\varphi\mathbb{L}\cdot u &= \frac{\partial}{\partial\epsilon}L(\varphi(\epsilon),\xi)\Big|_{\epsilon=0}\\ &= \int_\Omega D_\varphi\mathcal{L}(\omega)\cdot u(\omega) + D_\eta\mathcal{L}(\omega)\cdot\left(\frac{\partial}{\partial\epsilon}T_{\varphi(\epsilon)}\Phi_{\varphi(\epsilon)^{-1}}\varphi_\omega(\epsilon,\omega)\right)\Big|_{\epsilon=0} d\omega\\ &\quad - \int_\Omega D_\zeta\mathcal{L}(\omega)\cdot\frac{\partial}{\partial\epsilon}\Big[\xi(\omega), T_{\varphi(\epsilon)}\Phi_{\varphi(\epsilon)^{-1}}\varphi_\omega(\epsilon,\omega)\Big]_{\epsilon=0} d\omega. \end{aligned} \tag{3.27}$$

By an argument analogous to that offered in Proposition 1,

$$\frac{\partial}{\partial\epsilon}T_{\varphi(\epsilon)}\Phi_{\varphi(\epsilon)^{-1}}\varphi_\omega(\epsilon)\Big|_{\epsilon=0} = [\eta, T_\varphi\Phi_{\varphi^{-1}}u] + \big(T_\varphi\Phi_{\varphi^{-1}}u\big)_\omega. \tag{3.28}$$

Substituting (3.28) into (3.27) yields

$$
\begin{aligned}
D_\varphi \mathbb{L} \cdot u &= D_\varphi \mathcal{L} \cdot u + \int_\Omega D_\eta \mathcal{L}(\omega) \cdot \Big([\eta(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)] + \big(T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big)_\omega \Big) d\omega \\
&\quad - \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \Big[\xi(\omega), [\eta, T_\varphi \Phi_{\varphi^{-1}} u(\omega)] + \big(T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big)_\omega \Big] d\omega \\
&= D_\varphi \mathcal{L} \cdot u + ad^*_\eta D_\eta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) + D_\eta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega} \\
&\quad - \int_\Omega \frac{\partial}{\partial \omega}(D_\eta \mathcal{L})(\omega) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big) d\omega - \int_\Omega ad^*_\eta ad^*_\xi D_\zeta \mathcal{L}(\omega) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big) d\omega \\
&\quad - \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \big[\xi(\omega), \big(T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big)_\omega\big] d\omega \\
&= D_\varphi \mathcal{L} \cdot u + ad^*_\eta D_\eta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial \omega}(D_\eta \mathcal{L}) \cdot (T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad - ad^*_\eta ad^*_\xi D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) - \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \frac{\partial}{\partial \omega}\big[\xi(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big)] d\omega \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \big[\xi_\omega(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big] d\omega + D_\eta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega} \\
&= D_\varphi \mathcal{L} \cdot u + ad^*_\eta D_\eta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial \omega}(D_\eta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad - ad^*_\eta ad^*_\xi D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) + \int_\Omega \frac{\partial}{\partial \omega}(D_\zeta \mathcal{L})(\omega) \cdot \big[\xi(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big] d\omega \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \big[\xi_\omega(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big] d\omega + D_\eta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega} \\
&\quad - (D_\zeta \mathcal{L}) \cdot \big[\xi, T_\varphi \Phi_{\varphi^{-1}} u\big]\big|_{\partial\Omega} \qquad (3.29) \\
&= D_\varphi \mathcal{L} \cdot u + ad^*_\eta D_\eta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial \omega}(D_\eta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad - ad^*_\eta ad^*_\xi D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) + ad^*_\xi \left(\frac{\partial}{\partial \omega} D_\zeta \mathcal{L}\right) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \big[\xi_\omega(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big] d\omega + D_\eta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega} \\
&\quad - ad^*_\xi D_\zeta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u)\big|_{\partial\Omega} \qquad (3.30)
\end{aligned}
$$

Similarly the derivative of the Lagrangian $\mathbb{L}$ with respect to its second factor is

$$
\begin{aligned}
D_\xi \mathbb{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) &= \int_\Omega D_\xi \mathcal{L}(\omega) \cdot T_\varphi \Phi_{\varphi^{-1}} u(\omega)\, d\omega \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \frac{\partial}{\partial \epsilon} \xi_\omega(\epsilon, \omega)\bigg|_{\epsilon=0} d\omega \\
&\quad - \int_\Omega \frac{\partial}{\partial \epsilon} [\xi(\epsilon, \omega), T_\varphi \Phi_{\varphi^{-1}} \varphi_\omega(\omega)]\bigg|_{\epsilon=0} d\omega \\
&= D_\xi \mathcal{L} \cdot T_\varphi \Phi_{\varphi^{-1}} u + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot (T_\varphi \Phi_{\varphi^{-1}} u)_\omega\, d\omega \\
&\quad - \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot [T_\varphi \Phi_{\varphi^{-1}} u, T_\varphi \Phi_{\varphi^{-1}} \varphi_\omega(\omega)]\, d\omega \\
&= D_\xi \mathcal{L} \cdot T_\varphi \Phi_{\varphi^{-1}} u - \frac{\partial}{\partial \omega} D_\zeta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad + ad^*_\eta D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) + D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega} . \qquad (3.31)
\end{aligned}
$$

Computing the time derivative of $D_\xi \mathbb{L}$ yields

$$\begin{aligned}
\frac{d}{dt} D_\xi \mathbb{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) &= \frac{\partial}{\partial t} D_\xi \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial t\, \partial \omega} D_\zeta \mathcal{L} \cdot T_\varphi \Phi_{\varphi^{-1}} u \\
&\quad + \frac{\partial}{\partial t} ad^*_\eta D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) + \frac{\partial}{\partial t} (D_\zeta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\Big|_{\partial\Omega} \\
&= \frac{\partial}{\partial t} D_\xi \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial t\, \partial \omega} D_\zeta \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad + ad^*_\eta \frac{\partial}{\partial t} D_\zeta \mathcal{L}(\omega) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot [\eta_t(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)] d\omega \\
&\quad + \frac{\partial}{\partial t} (D_\zeta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\Big|_{\partial\Omega} \\
&= \frac{\partial}{\partial t} (D_\xi \mathcal{L}) \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial t\, \partial \omega} (D_\zeta \mathcal{L} \cdot)(T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad + ad^*_\eta \left(\frac{\partial}{\partial t} D_\zeta \mathcal{L} \right) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \Big[\xi_\omega(\omega) - [\xi(\omega), \eta(\omega)], T_\varphi \Phi_{\varphi^{-1}} u\Big] d\omega \\
&\quad + \frac{\partial}{\partial t} (D_\zeta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\Big|_{\partial\Omega} \\
&= \frac{\partial}{\partial t} (D_\xi \mathcal{L}) \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - \frac{\partial}{\partial t\, \partial \omega} (D_\zeta \mathcal{L} \cdot)(T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad + ad^*_\eta \left(\frac{\partial}{\partial t} D_\zeta \mathcal{L} \right) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad + \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \big[\xi_\omega(\omega), T_\varphi \Phi_{\varphi^{-1}} u(\omega)\big] d\omega \\
&\quad - \int_\Omega D_\zeta \mathcal{L}(\omega) \cdot \Big[[\xi(\omega), \eta(\omega)], T_\varphi \Phi_{\varphi^{-1}} u(\omega)\Big] d\omega \\
&\quad + \frac{\partial}{\partial t} (D_\zeta \mathcal{L}) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\Big|_{\partial\Omega}.
\end{aligned} \tag{3.32}$$

Finally we write the translation of $D_\xi \mathbb{L}$ under the coadjoint action as

$$\begin{aligned}
ad^*_\xi D_\xi \mathbb{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) &= ad^*_\xi D_\xi \mathcal{L} \cdot (T_\varphi \Phi_{\varphi^{-1}} u) - ad^*_\xi \left(\frac{\partial}{\partial \omega} D_\zeta \mathcal{L} \right) \cdot (T_\varphi \Phi_{\varphi^{-1}} u) \\
&\quad + ad^*_\xi \big(ad^*_\eta D_\zeta \mathcal{L} \big) \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big) \\
&\quad + ad^*_\xi D_\zeta \mathcal{L} \cdot \big(T_\varphi \Phi_{\varphi^{-1}} u\big)\big|_{\partial\Omega}.
\end{aligned} \tag{3.33}$$

Note that by the Jacobi identity on $\mathfrak{g}^{\Omega}$,

$$\big[\xi, [\eta, T_{\varphi}\Phi_{\varphi^{-1}}u]\big] + \big[T_{\varphi}\Phi_{\varphi^{-1}}u, [\xi, \eta]\big] + \big[\eta, [T_{\varphi}\Phi_{\varphi^{-1}}u, \xi]\big] = 0 \qquad (3.34)$$

Hence substituting (3.29)-(3.33) into (3.22) yields

$$\begin{aligned}
\Delta^{\mathbb{L}}_{(\varphi,\eta)}(X^p)\cdot(u,z) \;=\;& D_{\varphi}\mathcal{L}\cdot u + ad^*_{\eta}D_{\eta}\mathcal{L}\cdot(T_{\varphi}\Phi_{\varphi^{-1}}u) - \frac{\partial}{\partial\omega}(D_{\eta}\mathcal{L})\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\\
&-ad^*_{\eta}ad^*_{\xi}D_{\zeta}\mathcal{L}\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big) + ad^*_{\xi}\left(\frac{\partial}{\partial\omega}D_{\zeta}\mathcal{L}\right)\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\\
&+\int_{\partial\Omega} D_{\zeta}\mathcal{L}(\omega)\cdot\big[\xi_{\omega}(\omega), T_{\varphi}\Phi_{\varphi^{-1}}u(\omega)\big]d\omega + D_{\eta}\mathcal{L}\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\big|_{\partial\Omega}\\
&- ad^*_{\xi}(D_{\zeta}\mathcal{L})\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\big|_{\partial\Omega} - \Bigg\{\frac{\partial}{\partial t}(D_{\xi}\mathcal{L})\cdot T_{\varphi}\Phi_{\varphi^{-1}}u\\
&-\frac{\partial}{\partial t\,\partial\omega}(D_{\zeta}\mathcal{L})\cdot(T_{\varphi}\Phi_{\varphi^{-1}}u) + ad^*_{\eta}\left(\frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\right)\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\\
&+\int_{\partial\Omega} D_{\zeta}\mathcal{L}(\omega)\cdot\big[\xi_{\omega}(\omega), T_{\varphi}\Phi_{\varphi^{-1}}u(\omega)\big]d\omega\\
&-\int_{\partial\Omega} D_{\zeta}\mathcal{L}(\omega)\cdot\Big[[\xi(\omega),\eta(\omega)], T_{\varphi}\Phi_{\varphi^{-1}}u(\omega)\Big]d\omega\\
&+\frac{\partial}{\partial t}(D_{\zeta}\mathcal{L})\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\Big|_{\partial\Omega}\Bigg\} + ad^*_{\xi}D_{\xi}\mathcal{L}\cdot T_{\varphi}\Phi_{\varphi^{-1}}u\\
&-ad^*_{\xi}\left(\frac{\partial}{\partial\omega}D_{\zeta}\mathcal{L}\right)\cdot(T_{\varphi}\Phi_{\varphi^{-1}}u) + ad^*_{\xi}\big(ad^*_{\eta}D_{\zeta}\mathcal{L}\big)\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\\
&+ ad^*_{\xi}D_{\zeta}\mathcal{L}\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\big|_{\partial\Omega}\\
=\;& D_{\varphi}\mathcal{L}\cdot u - \frac{\partial}{\partial t}(D_{\xi}\mathcal{L})\cdot T_{\varphi}\Phi_{\varphi^{-1}}u + ad^*_{\xi}D_{\xi}\mathcal{L}\cdot(T_{\varphi}\Phi_{\varphi^{-1}}u)\\
&-\frac{\partial}{\partial\omega}(D_{\eta}\mathcal{L})\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big) + ad^*_{\eta}D_{\eta}\mathcal{L}\cdot(T_{\varphi}\Phi_{\varphi^{-1}})u\\
&+\frac{\partial}{\partial t\,\partial\omega}(D_{\zeta}\mathcal{L})\cdot(T_{\varphi}\Phi_{\varphi^{-1}}u) - ad^*_{\eta}\left(\frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\right)\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\\
&+ D_{\eta}\mathcal{L}\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\big|_{\partial\Omega} - \frac{\partial}{\partial t}(D_{\zeta}\mathcal{L})\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big)\Big|_{\partial\Omega}\\
&-ad^*_{\eta}ad^*_{\xi}D_{\zeta}\mathcal{L}\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big) + D_{\zeta}\mathcal{L}\cdot\Big[[\xi,\eta], T_{\varphi}\Phi_{\varphi^{-1}}u\Big]\\
&+ad^*_{\xi}\big(ad^*_{\eta}D_{\zeta}\mathcal{L}\big)\cdot\big(T_{\varphi}\Phi_{\varphi^{-1}}u\big).
\end{aligned}$$

$$
\begin{aligned}
&= D_\varphi\mathcal{L}\cdot u - \frac{\partial}{\partial t}\big(D_\xi\mathcal{L}\big)\cdot(T_\varphi\Phi_{\varphi^{-1}}u) - \frac{\partial}{\partial\omega}\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot(T_\varphi\Phi_{\varphi^{-1}}u)\\
&\quad + ad^*_\xi D_\xi\mathcal{L}\cdot(T_\varphi\Phi_{\varphi^{-1}}u) + ad^*_\eta\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot(T_\varphi\Phi_{\varphi^{-1}}u)\\
&\quad + \bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot\big(T_\varphi\Phi_{\varphi^{-1}}u\big)\bigg|_{\partial\Omega}\\
&\quad - D_\zeta\mathcal{L}\cdot\bigg(\big[\xi,[\eta,T_\varphi\Phi_{\varphi^{-1}}u]\big] + \big[T_\varphi\Phi_{\varphi^{-1}}u,[\xi,\eta]\big] + \big[\eta,[T_\varphi\Phi_{\varphi^{-1}}u,\xi]\big]\bigg)\\
&= \Phi^{T^*}_{\varphi^{-1}}D_\varphi\mathcal{L}\cdot(T_\varphi\Phi_{\varphi^{-1}}u) - \frac{\partial}{\partial t}D_\xi\mathcal{L}\cdot(T_\varphi\Phi_{\varphi^{-1}}u) - \frac{\partial}{\partial\omega}\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot(T_\varphi\Phi_{\varphi^{-1}}u)\\
&\quad + ad^*_\xi D_\xi\mathcal{L}\cdot(T_\varphi\Phi_{\varphi^{-1}}u) + ad^*_\eta\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot(T_\varphi\Phi_{\varphi^{-1}}u)\\
&\quad + \bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\cdot\big(T_\varphi\Phi_{\varphi^{-1}}u\big)\bigg|_{\partial\Omega}.
\end{aligned}
\tag{3.35}
$$

Therefore

$$
\begin{aligned}
\Delta^{\mathbb{L}}_{(\varphi,\eta)}(X^p) &= \Phi^{T^*}_{\varphi^{-1}}D_\varphi\mathcal{L} - \frac{\partial}{\partial t}D_\xi\mathcal{L} - \frac{\partial}{\partial\omega}\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\\
&\quad + ad^*_\xi D_\xi\mathcal{L} + ad^*_\eta\bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg) + \mathfrak{B}(\varphi,\xi),
\end{aligned}
$$

where

$$
\mathfrak{B}(\varphi,\xi) = \bigg(D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}\bigg)\bigg|_{\partial\Omega}. \tag{3.36}
$$

This yields the desired result. □

For a manifold Ω without boundary, equation (3.36) shows that $\mathfrak{B}$ in (3.26) vanishes. Hence for closed curves, effectively modelled on $\Omega = S^1$, there is no boundary term.

One of the fascinating aspects of this representation of the Lagrangian force is the emergence of the term

$$
D_\eta\mathcal{L} - \frac{\partial}{\partial t}D_\zeta\mathcal{L}. \tag{3.37}
$$

Since ζ is properly interpreted as η_t by the compatibility condition (3.4), this term is simply the Euler-Lagrange operator. When the evolution of η coincides with the extremal trajectory of some cost functional defined on $\mathfrak{g}^\Omega$, this term is identically zero.

3.5 A Comparative Analysis of Variational Calculus

In this section we consider the governing equations which characterized extremal maps of a cost functional $J : \Pi \to \mathbb{R}$ defined as

$$J(\varphi) \quad = \quad \int_{\mathcal{I}} \mathbb{L}\big(\varphi(t), T_\varphi \Phi_{\varphi^{-1}} \varphi_t(t)\big)\, dt, \tag{3.38}$$

where $\mathcal{I} \subseteq \mathbb{R}$ and $\mathbb{L}$ is a lagrangian defined on Ψ. The following theorem shows that extremal maps of J satisfy the equations of mechanical motion for an unconstrained, unforced system with Lagrangian L.

Theorem 3. *Given a lagrangian density $\mathcal{L}$ related to $\mathbb{L}$ by (3.8), extremal maps of (3.38) satisfy*

$$\begin{aligned} &\Phi^{T^*}_{\varphi^{-1}} D_\varphi \mathcal{L}_{(\varphi,\xi)} - \frac{\partial}{\partial t} D_\xi \mathcal{L} - \frac{\partial}{\partial \omega}\left(D_\eta \mathcal{L} - \frac{\partial}{\partial t} D_\zeta \mathcal{L}\right) \\ &+ ad^*_\xi D_\xi \mathcal{L} + ad^*_\eta \left(D_\eta \mathcal{L} - \frac{\partial}{\partial t} D_\zeta \mathcal{L}\right) + \mathfrak{B}(\varphi, \xi) = 0. \end{aligned} \tag{3.39}$$

Proof. Let $\delta\varphi = \left.\dfrac{d}{d\epsilon}\varphi^\epsilon\right|_{\epsilon=0}$ where φ^ϵ is a smooth 1-parameter variation agreeing with φ on ∂I. Since $\eta = \varphi^{-1}\varphi_\omega$,

$$\begin{aligned} \delta\eta &= \delta(\varphi^{-1})\varphi_\omega + \varphi^{-1}\delta(\varphi_\omega) \\ &= -\varphi^{-1}\delta\varphi\varphi^{-1}\varphi_\omega + \varphi^{-1}\delta(\varphi_\omega) \\ &= -\varphi^{-1}\delta\varphi\eta + \varphi^{-1}\delta(\varphi_\omega). \end{aligned} \tag{3.40}$$

Similarly, we observe that

$$
\begin{aligned}
(\varphi^{-1}\delta\varphi)_\omega &= (\varphi^{-1})_\omega\delta\varphi + \varphi^{-1}\delta\varphi_\omega \\
&= -\varphi^{-1}\varphi_\omega\varphi^{-1}\delta\varphi + \varphi^{-1}\delta\varphi_\omega \\
&= -\eta\varphi^{-1}\delta\varphi + \varphi^{-1}\delta\varphi_\omega. \qquad (3.41)
\end{aligned}
$$

Rearranging the above equation yields

$$
\varphi^{-1}\delta\varphi_\omega = (\varphi^{-1}\delta\varphi)_\omega + \eta\varphi^{-1}\delta\varphi. \qquad (3.42)
$$

Substituting (3.42) into (3.40) yields

$$
\begin{aligned}
\delta\eta &= -\varphi^{-1}\delta\varphi\eta + (\varphi^{-1}\delta\varphi)_\omega + \eta\varphi^{-1}\delta\varphi \\
&= (\varphi^{-1}\delta\varphi)_\omega + \eta\varphi^{-1}\delta\varphi - \varphi^{-1}\delta\varphi\eta \\
&= (\varphi^{-1}\delta\varphi)_\omega + [\eta, \varphi^{-1}\delta\varphi].
\end{aligned}
$$

Under the identification $\zeta = \eta_t$, observe that $\delta\zeta = (\delta\eta)_t$. Therefore, since $\delta\varphi$

vanished on $\partial\mathcal{I}$, the first variation of J is

$$
\begin{aligned}
\frac{\delta J}{\delta\varphi}\cdot\delta\varphi &= \frac{d}{d\epsilon}\int_{\mathcal{I}}\int_{\Omega}\mathcal{L}(\varphi^{\epsilon},\xi^{\epsilon},\eta^{\epsilon},\zeta^{\epsilon})\,d\omega\,dt\bigg|_{\epsilon=0} \\
&= \int_{\mathcal{I}}\int_{\Omega} D_{\varphi}\mathcal{L}\cdot\delta\varphi + D_{\xi}\mathcal{L}\cdot\delta\xi + D_{\eta}\mathcal{L}\cdot\delta\eta + D_{\zeta}\mathcal{L}\cdot(\delta\eta)_t\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{D_{\varphi}\mathcal{L}\cdot\delta\varphi + D_{\xi}\mathcal{L}\cdot\Big((\varphi^{-1}\delta\varphi)_t + [\xi,\varphi^{-1}\delta\varphi]\Big) \\
&\quad + D_{\eta}\mathcal{L}\cdot\big((\varphi^{-1}\delta\varphi)_{\omega} + [\eta,\varphi^{-1}\delta\varphi]\big) \\
&\quad + D_{\zeta}\mathcal{L}\cdot\big((\varphi^{-1}\delta\varphi)_{\omega} + [\eta,\varphi^{-1}\delta\varphi]\big)_t\bigg\}\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{D_{\varphi}\mathcal{L}\cdot\delta\varphi - \frac{\partial}{\partial t}D_{\xi}\mathcal{L}\cdot\varphi^{-1}\delta\varphi + D_{\xi}\mathcal{L}\cdot\varphi^{-1}\delta\varphi|_{\partial\mathcal{I}} + D_{\xi}\mathcal{L}\cdot ad_{\xi}\varphi^{-1}\delta\varphi \\
&\quad - \frac{\partial}{\partial\omega}D_{\eta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi + D_{\eta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi|_{\partial\Omega} + D_{\eta}\mathcal{L}\cdot ad_{\eta}\varphi^{-1}\delta\varphi \\
&\quad - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\cdot\big((\varphi^{-1}\delta\varphi)_{\omega} + ad_{\eta}\varphi^{-1}\delta\varphi\big) + D_{\zeta}\mathcal{L}\cdot\big((\varphi^{-1}\delta\varphi)_{\omega} + [\eta,\varphi^{-1}\delta\varphi]\big)|_{\partial\mathcal{I}}\bigg\}\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{\Phi^{T^*}_{\varphi^{-1}}D_{\varphi}\mathcal{L}\cdot\varphi^{-1}\delta\varphi - \frac{\partial}{\partial t}D_{\xi}\mathcal{L}\cdot\varphi^{-1}\delta\varphi + ad^*_{\xi}D_{\xi}\mathcal{L}\cdot\varphi^{-1}\delta\varphi \\
&\quad - \frac{\partial}{\partial\omega}D_{\eta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi + D_{\eta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi|_{\partial\Omega} + ad^*_{\eta}D_{\eta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi \\
&\quad \frac{\partial}{\partial\omega}\frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi|_{\partial\Omega} - ad^*_{\eta}\frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\cdot\varphi^{-1}\delta\varphi\bigg\}\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{\Phi^{T^*}_{\varphi^{-1}}D_{\varphi}\mathcal{L} - \frac{\partial}{\partial t}D_{\xi}\mathcal{L} + ad^*_{\xi}D_{\xi}\mathcal{L} - \frac{\partial}{\partial\omega}D_{\eta}\mathcal{L} + D_{\eta}\mathcal{L}|_{\partial\Omega} + ad^*_{\eta}D_{\eta}\mathcal{L} \\
&\quad \frac{\partial}{\partial\omega}\frac{\partial}{\partial t}D_{\zeta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}|_{\partial\Omega} - ad^*_{\eta}\frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\bigg\}\cdot\varphi^{-1}\delta\varphi\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{\Phi^{T^*}_{\varphi^{-1}}D_{\varphi}\mathcal{L} - \frac{\partial}{\partial t}D_{\xi}\mathcal{L} - \frac{\partial}{\partial\omega}\Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big) + ad^*_{\xi}D_{\xi}\mathcal{L} \\
&\quad + ad^*_{\eta}\Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big) + \Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big)\bigg|_{\partial\Omega}\bigg\}\cdot\varphi^{-1}\delta\varphi\,d\omega\,dt \\
&= \int_{\mathcal{I}}\int_{\Omega}\bigg\{\Phi^{T^*}_{\varphi^{-1}}D_{\varphi}\mathcal{L} - \frac{\partial}{\partial t}D_{\xi}\mathcal{L} - \frac{\partial}{\partial\omega}\Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big) + ad^*_{\xi}D_{\xi}\mathcal{L} \\
&\quad + ad^*_{\eta}\Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big) + \mathfrak{B}(\varphi,\xi)\bigg\}\cdot\varphi^{-1}\delta\varphi\,d\omega\,dt,
\end{aligned}
$$

where $\mathfrak{B} = \Big(D_{\eta}\mathcal{L} - \frac{\partial}{\partial t}D_{\zeta}\mathcal{L}\Big)\Big|_{\partial\Omega}$. Setting the first variation of J equal to zero and applying the Fundamental Lemma of Variational Calculus yields the desired result. $\square$

Comparing Theorems 2 and 3 suggests that that governing equation for an unconstrained system without external forcing (3.8) coincides with the evolution of extremal trajectories of the related cost functional (3.38). This observation is important since, in subsequent work, we will construct a particular Lagrangian for a virtual filament by appealing to a purely variational argument. The essential reasonableness of this model will arise from this fact that, in the absence of external forcing and constraints, a mechanical system with Lagrangian L evolves as an extremal map of J.

Chapter 4

A Virtual Filament Model with Nonholonomic Constraints

We now consider a particular Lagrangian model of a filament evolving as a curve on SE_2. In the following section we begin by arguing for a reasonable set of local and global control objectives. In an effort to achieve these objectives, we judiciously establish a Lagrangian density for a virtual filament. This Lagrangian construction is pursued without regard to either constraints or external forcing. As noted earlier, the equations of motion for an unforced Lagrangian system without constraints are simply the classic Euler-Lagrange equations. Yet these constitute the first order necessary conditions for extremal maps of a naturally induced cost functional (see discussion in Section 3.5). Hence it is reasonable to construct a Lagrangian density which consists additively of elements which ought to be extremized. In this sense, the model construction is motivated by a variational principle. However, since the complete model will involve nonholonomic constraints, it is important to understand that the model equations for the virtual filament, derived subsequently, describe the evolution of a mechanical system and not extremal maps of a cost functional.

Having established a reasonable Lagrangian, the appropriate external forcing and constraints will be introduced. We then apply the Lagrange D'Alembert Principle of Virtual Work, deriving the governing partial differential equations of mechanical motion for this virtual filament. Since the proposed Lagrangian is degenerate, the governing equations lack uniqueness. However, an appropriate choice of the Lagrange multipliers, which enter through the constraints, leads to a natural set of well-posed equations. Hence we interpret these multipliers as control parameters for our virtual system and study the induced flow of the filament under a particular choice of these parameters. In particular we will show that integral curve orbits of the orientation field are invariant under this flow. In the final section of this chapter, we simulate a filament aligning with various orientation fields of interest.

4.1 The Lagrangian Density

There are two primary control objectives for the evolution of the virtual filament which will be formalized in the context of the subsequent discussion. Globally it is necessary to align the orientation of the filament with a fixed planar vector field called the orientation field. Naturally, we must first make sense of what is meant by filament orientation. The second objective is that geodesic stretching of the filament be marginalized.

We begin by addressing the mechanism for global control of the filament and then proceed to local considerations. Here particular care must be taken to develop the appropriate language for describing the alignment of a virtual filament with an orientation field. Let $\mathbb{E}_2$ denote the Euclidean plane equipped with the standard Euclidean inner product $< \cdot, \cdot >$ and induced norm $\| \cdot \|$. Let γ

denote the canonical Euclidean projection of the vector bundle

$$\begin{array}{ccc} SO_2^\Omega & \hookrightarrow & SE_2^\Omega \\ & & \gamma \downarrow \\ & & \mathbb{E}_2^\Omega. \end{array}$$

Then the linearization $D\gamma$ is the canonical projection of TSE_2^Ω onto $\mathbb{E}_2^\Omega$. Similarly, let $\mathcal{R}$ denote the canonical rotational projection for the bundle $SE_2^\Omega \xrightarrow{\mathcal{R}} SO_2^\Omega$. Then $\mathcal{R}(G^\Omega)$ acts naturally as a Lie group on $\mathbb{E}_2$. Let $\varphi \in C^\infty(\mathcal{I}, SE_2^\Omega)$ represent a typical trajectory of a filament. The orientation or flow of the filament is defined as the planar vector field $D\gamma_\varphi(\varphi_t(t,\cdot))$ along the curve $\gamma(\varphi(t,\cdot))$. Observe that

$$\begin{aligned} D\gamma_\varphi\big((\varphi_t(t,\cdot)\big) &= \gamma_t(\varphi) \\ &= \mathcal{R}(\varphi)\mathcal{E}(\xi), \end{aligned} \tag{4.1}$$

where $\mathcal{E}$ denotes the canonical coordinate map for the projection of $\mathfrak{se}_2$ onto the subalgebra spanned by the basis elements χ_1 and χ_2. Consequently $\mathcal{R}(\varphi)$ is often referred to as the orientation of the filament φ.

We adopt the standard notational convection for vectors in the Euclidean plane, representing elements of $\mathbb{E}_2$ as column vectors and dual elements as row vectors. Let $D : \mathbb{E}_2 \to \mathbb{E}_2$ be a smooth vector field representing the desired orientation of the filament. Let $\mathcal{F}_1 : G^\Omega \to \mathbb{R}$ and $\mathcal{F}_2 : G^\Omega \to \mathbb{R}$ be smooth maps such that $\varphi \to \big(\mathcal{F}_1(\varphi)\ \ \mathcal{F}_2(\varphi)\big)$, a section of the dual $\mathbb{E}_2$ bundle over G^Ω, annihilates the vector field $\varphi \longmapsto \mathcal{R}^{-1}(\varphi)D(\gamma(\varphi))$; in particular we choose

$$\begin{pmatrix} \mathcal{F}_1(\varphi) \\ \\ \mathcal{F}_2(\varphi) \end{pmatrix} = \mathcal{R}^{-1}(\varphi)\frac{D(\varphi)^\perp}{\|D(\varphi)\|}, \tag{4.2}$$

where $D^\perp$ is the vector orthogonal to D obtained by counterclockwise rotation in the plane by $\frac{\pi}{2}$ radians. Note that $\mathcal{F}_\alpha$ is well defined for any nonvanishing orientation field D. Let χ_0, χ_1, χ_2 be a basis for $\mathfrak{se}_2^\Omega$ with nonzero structure coefficient signature

$$\Upsilon^2_{01} = 1 \qquad \Upsilon^2_{10} = -1 \qquad \Upsilon^1_{02} = -1 \qquad \Upsilon^1_{20} = 1. \tag{4.3}$$

Let $\chi^0_*, \chi^1_*, \chi^2_*$ denote the dual basis. Define the map $\mathcal{F} : SE_2^\Omega \to (\mathfrak{se}_2^*)^\Omega$ as

$$\mathcal{F}(\varphi) = \mathcal{F}_\alpha(\varphi)\chi^\alpha_*. \tag{4.4}$$

By construction, $\mathcal{F}$ preserves a unit norm. Furthermore, if $\gamma = \gamma(\varphi)$, then by the kinematics (3.1),

$$\begin{aligned}
\mathcal{F}\cdot\xi &= \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\cdot(\xi^1\ \xi^2)^T \\
&= \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\cdot\mathcal{R}^{-1}\gamma_t \\
&= \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\cdot\mathcal{R}^{-1}\left(< D,\gamma_t > \frac{D}{\|D\|} + < D^\perp,\gamma_t > \frac{D^\perp}{\|D\|}\right) \\
&= < D,\gamma_t > \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\cdot\mathcal{R}^{-1}\frac{D}{\|D\|} + < D^\perp,\gamma_t > \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\cdot\mathcal{R}^{-1}\frac{D^\perp}{\|D\|} \\
&= < D^\perp,\gamma_t > \frac{\left(\mathcal{R}(\varphi)^{-1}D^\perp\right)^\flat}{\|D\|}\cdot\mathcal{R}^{-1}\frac{D^\perp}{\|D\|} \\
&= < D^\perp,\gamma_t > \frac{\|\mathcal{R}(\varphi)^{-1}D^\perp\|^2}{\|D\|^2} \\
&= < D^\perp,\gamma_t >
\end{aligned} \tag{4.5}$$

Therefore $|\mathcal{F}\cdot\xi|$ is a measure of the misalignment of the current orientation γ_t and the desired orientation D. In particular, when γ_t and D are aligned, $|\mathcal{F}\cdot\xi| = 0$. Similarly, $|\mathcal{F}\cdot\xi|$ is maximized when γ_t and $D^\perp$ are aligned (maximum misalignment with D). Clearly then the square of $\mu = \mathcal{F}\cdot\xi$ ought to be minimized to encourage alignment of γ_t with D; hence it will be incorporated additively in the lagrangian density.

In terms of local considerations, the primary concern is to avoid the collapse or excessive expansion of the formation. In a continuum setting, this sort of behavior is discouraged infinitesimally by penalizing flows which lead to either contraction or elongation of the filament. We proceed by characterizing the non-stretching flow of a filament by introducing a geodesic distance measure. Let $\omega_0, \omega_1 \in \Omega$ and define the length of the filament from ω_0 to ω_1 as

$$\delta(t) = \int_{\omega_0}^{\omega_1} \|\gamma_\omega(t,\omega)\| \, d\omega. \tag{4.6}$$

Then

$$\frac{d\delta(t)}{dt} = \int_{\omega_0}^{\omega_1} \frac{< \gamma_{t\omega}(t,\omega), \gamma_\omega(t,\omega) >}{\|\gamma_\omega(t,\omega)\|^2} \, d\omega. \tag{4.7}$$

Clearly if $< \gamma_{t'\omega}(t,\omega), \gamma_\omega(t',\omega) >= 0$ on (ω_0, ω_1), then there is no change in length of the filament at time $t = t'$; i.e., $\dfrac{d\delta(t')}{dt} = 0$. The converse is similarly true in the limit as $\omega_0 \to \omega_1$. Hence the flow is non-stretching at $t = t'$ if and only if $< \gamma_{t\omega}(t',\omega), \gamma_\omega(t',\omega) >= 0$. Note that $\frac{\partial}{\partial t}\mathcal{R}(\varphi) = \mathcal{R}(\varphi)Q(\varphi)$ for some antisymmetric Q. Hence

$$\begin{aligned}
< \gamma_{t\omega}, \gamma_\omega > &= < \frac{\partial}{\partial t}\Big(\mathcal{R}(\varphi)(\eta^1\ \eta^2)^T\Big), \mathcal{R}(\varphi)(\eta^1\ \eta^2)^T > \\
&= < \frac{\partial}{\partial t}\big(\mathcal{R}(\varphi)\big)(\eta^1\ \eta^2)^T + \mathcal{R}(\varphi)(\eta^1\ \eta^2)^T_t, \mathcal{R}(\varphi)(\eta^1\ \eta^2)^T > \\
&= < \mathcal{R}(\varphi)Q(\varphi)(\eta^1\ \eta^2)^T + \mathcal{R}(\varphi)(\eta^1\ \eta^2)^T_t, \mathcal{R}(\varphi)(\eta^1\ \eta^2)^T > \\
&= < Q(\varphi)(\eta^1\ \eta^2)^T + (\eta^1\ \eta^2)^T_t, (\eta^1\ \eta^2)^T > \\
&= \eta^\alpha \eta^\alpha_t.
\end{aligned}$$

We define the stretch rate at time t and material point ω as

$$\tau(t,\omega) = \frac{1}{2}\frac{\nu_t(t,\omega)}{\nu(t,\omega)}. \tag{4.8}$$

where $\nu = \|\gamma_\omega(t,\omega)\|^2$. Therefore

$$\tau = \frac{\eta^\alpha \eta_t^\alpha}{\nu}. \tag{4.9}$$

Penalizing τ is equivalent to penalizing the change in geodesic distance between material points of the filament. Hence, incorporating the symmetry breaking term μ and the stretching penalty τ, we consider the Lagrangian density given by

$$\mathcal{L}(\varphi,\xi,\eta,\zeta) = \frac{\mathcal{A}}{2}\,\mu^2(\varphi,\xi) + \frac{\mathcal{B}}{2}\,\tau^2(\eta,\zeta), \tag{4.10}$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants. Here, ζ is interpreted as the partial time derivative of η by definition of the Lagrangian density (3.8) and compatibility condition (3.4). Hence τ is expressed as

$$\tau(\eta,\zeta) = \frac{\mathcal{E}^*(\eta)\cdot\zeta}{\nu}, \tag{4.11}$$

where $\mathcal{E}^*(\eta)$ denotes the dual of $\mathcal{E}(\eta)$. This filament model admits an SO_2^Ω symmetry group. This fact is partially obscured by the absorption of vector field D in the dual vector field $\mathcal{F}$. However, recall that D depends only on the Euclidean component of G^Ω and, by construction, τ depends only on the flow of the projection $\gamma(\varphi)$.

4.2 The Lagrangian Force

We now compute $\Delta^{\mathcal{L}}$, the Lagrangian force on Ψ, given by equations (3.26). The partial derivative at $\sigma = (g, \xi, \eta, \zeta) \in \Sigma_2$ in the direction $u \in T_\varphi SE_2$ of the Lagrangian density with respect to its first factor is given by

$$\begin{aligned} D_\varphi \mathcal{L}_\sigma \cdot u &= \frac{\mathcal{A}}{2} D_\varphi(\mu^2(\varphi, \xi)) \cdot u \\ &= \mathcal{A}\mu \, D_\varphi(\mathcal{F}_\alpha \chi_*^\alpha \cdot \xi) \cdot u \\ &= \mathcal{A}\mu \, (d\mathcal{F}_\alpha \cdot u)(\chi_*^\alpha \cdot \xi) \\ &= \mathcal{A}\mu \, (d\mathcal{F}_\alpha \xi^\alpha) \cdot u. \end{aligned} \tag{4.12}$$

Hence,

$$D_\varphi \mathcal{L}_\sigma = \mathcal{A}\mu \, d\mathcal{F}_\alpha \xi^\alpha. \tag{4.13}$$

Note that

$$F_\alpha(\varphi) = \frac{\left(R^{-1}(\varphi) D^\perp\right)^\alpha}{\|D\|}, \tag{4.14}$$

where components are taken with respect to $\{e_i\}$, the standard basis for $\mathbb{E}_2$. Here we have employed the established convention that a covector superscripted by α denotes the α component of that covector. Let $\bar{D} = \dfrac{D}{\|D\|}$ denote the normalization of D. Then

$$\begin{aligned} \frac{\partial \bar{D}^\perp}{\partial \varphi} \cdot \varphi u &= \frac{\partial}{\partial \varphi} \bar{D}^\perp(\gamma(\varphi)) \cdot \varphi u \\ &= \left(\frac{\partial}{\partial \gamma} \bar{D}^\perp(\gamma(\varphi)) \cdot \frac{\partial \gamma}{\partial \varphi} \right) \cdot \varphi u \\ &= \left(\frac{\partial}{\partial \gamma^\alpha} \bar{D}^\perp(\gamma(\varphi)) e_\alpha \cdot \frac{\partial \gamma^\alpha e_\alpha}{\partial \varphi} \right) \cdot \varphi u \\ &= \frac{\partial}{\partial \gamma^\alpha} \bar{D}^\perp(\gamma(\varphi)) \, d\gamma^\alpha \cdot \varphi u. \end{aligned} \tag{4.15}$$

Therefore, exploiting the form of $\mathcal{F}$,

$$\begin{aligned} d\mathcal{F}_\alpha \cdot \varphi u &= -\left(R^{-1}(\varphi)\left(\frac{\partial R}{\partial \varphi}\cdot \varphi u\right) R^{-1}(\varphi)\frac{D^\perp}{\|D\|}\right)^\alpha + \left(\mathcal{R}^{-1}\frac{\partial \bar{D}^\perp}{\partial \gamma}\, d\gamma \cdot \varphi\, u\right)^\alpha \\ &= -\Big((-\mathcal{F}_2 \quad \mathcal{F}_1)^T\Big)^\alpha \Phi_\varphi^{T^*}\chi_*^0 \cdot \varphi\, u + \left(\mathcal{R}^{-1}\frac{\partial \bar{D}^\perp}{\partial \gamma^\beta}\right)^\alpha d\gamma^\beta \cdot \varphi\, u. \end{aligned}$$

More explicitly, we write

$$d\mathcal{F}_1(\varphi) = \mathcal{F}_2\ \Phi_\varphi^{T^*}\chi_*^0 + \frac{\partial \mathcal{F}_1}{\partial \gamma^\alpha}\, d\gamma^\alpha, \tag{4.16}$$

and

$$d\mathcal{F}_2(\varphi) = -\mathcal{F}_1\ \Phi_\varphi^{T^*}\chi_*^0 + \frac{\partial \mathcal{F}_2}{\partial \gamma^\alpha}\, d\gamma^\alpha. \tag{4.17}$$

Equations (4.12), (4.16) and (4.17) imply that the cotangent lift of $D_\varphi\mathcal{L}$ induced by φ^{-1} is given by

$$\begin{aligned} \Phi_{\varphi^{-1}}^{T^*} D_\varphi \mathcal{L}_\sigma &= \mathcal{A}\mu\Phi_{\varphi^{-1}}^{T^*}\Big(\mathcal{F}_2\ \Phi_\varphi^{T^*}\ \chi_*^0 + \frac{\partial \mathcal{F}_1}{\partial \gamma^\alpha}\, d\gamma^\alpha\Big)\xi^1 \\ &\quad + \mathcal{A}\mu\Phi_\varphi^{T^*}\Big(-\mathcal{F}_1\ \Phi_\varphi^{T^*}\ \chi_*^0 + \frac{\partial \mathcal{F}_2}{\partial \gamma^\alpha}\, d\gamma^\alpha\Big)\xi^2 \\ &= \mathcal{A}\mu(\mathcal{F}_2\xi^1 - F_1\xi^2)\ \chi_*^0 + \mathcal{A}\mu\left(\frac{\partial \mathcal{F}_1}{\partial \gamma^1}\xi^1 + \frac{\partial \mathcal{F}_2}{\partial \gamma^1}\xi^2\right)\ \Phi_{\varphi^{-1}}^{T^*}\, d\gamma^1 \\ &\quad + \mathcal{A}\mu\left(\frac{\partial \mathcal{F}_1}{\partial \gamma^2}\xi^1 + \frac{\partial \mathcal{F}_2}{\partial \gamma^2}\xi^2\right)\ \Phi_{\varphi^{-1}}^{T^*}\, d\gamma^2 \\ &= \mathcal{A}\mu(\mathcal{F}_2\xi^1 - F_1\xi^2)\ \chi_*^0 + \mathcal{A}\mu\,\frac{\partial \mu}{\partial \gamma^\alpha}\ \Phi_{\varphi^{-1}}^{T^*}\, d\gamma^\alpha. \end{aligned} \tag{4.18}$$

In order to simplify the notation, define Γ as

$$\Gamma_\alpha^\beta(\varphi) = \left(\mathcal{R}^{-1}(\varphi)\frac{\partial \mathcal{F}_\alpha(\varphi)}{\partial \gamma}\right)^\beta. \tag{4.19}$$

Then

$$\begin{aligned}
\frac{\partial \mu}{\partial \gamma^\alpha}\, \Phi^{T^*}_{\varphi^{-1}} d\gamma^\alpha &= \left(\frac{\partial \mu}{\partial \gamma^\alpha} \varphi^{-1} \chi_\alpha \right)^\flat \\
&= \left(\left(\mathcal{R}^{-1} \frac{\partial \mu}{\partial \gamma} \right)^\alpha \chi_\alpha \right)^\flat \\
&= \left(\mathcal{R}^{-1} \frac{\partial \mathcal{F}_\beta}{\partial \gamma} \xi^\beta \right)^\alpha \chi^\alpha_* \\
&= \Gamma^\alpha_\beta \xi^\beta \chi^\alpha_*. \qquad (4.20)
\end{aligned}$$

In this notation,

$$\Phi^{T^*}_{\varphi^{-1}} D_\varphi \mathcal{L}_\sigma = \mathcal{A}\mu(\mathcal{F}_2 \xi^1 - F_1 \xi^2)\, \chi^0_* + \mathcal{A}\mu\, \Gamma^\alpha_\beta \xi^\beta \chi^\alpha_*. \qquad (4.21)$$

We now compute the the partial derivative of $\mathcal{L}$ with respect to its second factor. This is given by

$$D_\xi \mathcal{L}_\sigma = \mathcal{A}\mu\mathcal{F}. \qquad (4.22)$$

Therefore by equation (2.25) and the Lie algebra structure constants (4.3),

$$ad^*_\xi \omega = (\omega_1 \xi^2 - \omega_2 \xi^1)\chi^0_* + (\omega_2 \xi^0)\chi^1_* - (\omega^1 \xi^0)\chi^2_*, \qquad (4.23)$$

for each $\omega = \omega_k \chi^k_* \in \mathfrak{se}^*(2)$ and $\xi = \xi^k \chi_k \in \mathfrak{se}(2)$. Therefore

$$\begin{aligned}
ad^*_\xi D_\xi \mathcal{L}_\sigma &= \mathcal{A}\mu\Big((\mathcal{F}_1 \xi^2 - \mathcal{F}_2 \xi^1)\chi^0_* + \mathcal{F}_2 \xi^0 \chi_* 1 - \mathcal{F}_1 \xi^0 \chi^2_* \Big) \\
&= \mathcal{A}\mu\Big((\mathcal{F}_1 \xi^2 - \mathcal{F}_2 \xi^1)\chi^0_* - \xi^0 \mathcal{F}^\perp \Big), \qquad (4.24)
\end{aligned}$$

where $\mathcal{F}^{\perp} = -F^2\chi_*^1 + F^1\chi_*^2$. Then the time derivative of $D_\xi\mathcal{L}$ is given by

$$
\begin{aligned}
\frac{\partial}{\partial t}D_\xi\mathcal{L} &= \mathcal{A}\mu_t\mathcal{F} + \mathcal{A}\mu\mathcal{F}_t \\
&= \mathcal{A}\mu_t\mathcal{F} + \mathcal{A}\mu\left(\frac{\partial\mathcal{F}}{\partial\varphi}\cdot\varphi_t\right) \\
&= \mathcal{A}\mu_t\mathcal{F} + \mathcal{A}\mu\left(\chi_*^\alpha\otimes d\mathcal{F}_\alpha\cdot\varphi_t\right) \\
&= \mathcal{A}\mu_t\mathcal{F} + \mathcal{A}\mu\mathcal{F}_2\xi^0\chi_*^1 + \left(\left(\frac{\partial\mathcal{F}_1}{\partial\gamma^\alpha}\,d\gamma^\alpha\right)\cdot\varphi_t\right)\chi_*^1 \\
&\quad -\mathcal{A}\mu\mathcal{F}_1\xi^0\chi_*^2 + \left(\left(\frac{\partial\mathcal{F}_2}{\partial\gamma^\alpha}\,d\gamma^\alpha\right)\cdot\varphi_t\right)\chi_*^2. \\
&= \mathcal{A}\mu_t\mathcal{F} - \mathcal{A}\mu\,\xi^0\mathcal{F}^{\perp} + \mathcal{A}\mu\,\left(\frac{\partial\mathcal{F}_1}{\partial\gamma}\cdot\gamma_t\right)\chi_*^1 \\
&\quad +\mathcal{A}\mu\left(\frac{\partial\mathcal{F}_2}{\partial\gamma}\cdot\gamma_t\right)\chi_*^2 \\
&= \mathcal{A}\mu_t\mathcal{F} - \mathcal{A}\mu\,\xi^0\mathcal{F}^{\perp} + \mathcal{A}\mu\,\left(\frac{\partial\mathcal{F}_\alpha}{\partial\gamma}\cdot\gamma_t\right)\chi_*^\alpha. \qquad (4.25)
\end{aligned}
$$

Note that in term of Γ we can write

$$
\begin{aligned}
\left(\frac{\partial\mathcal{F}_\alpha}{\partial\gamma}\cdot\gamma_t\right)\chi_*^\alpha &= \left(\mathcal{R}^{-1}\frac{\partial\mathcal{F}_\alpha}{\partial\gamma}\right)\cdot\mathcal{E}(\xi)\chi_*^\alpha \\
&= \left(\mathcal{R}^{-1}\frac{\partial\mathcal{F}_\alpha}{\partial\gamma}\right)^\beta\xi^\beta\chi_*^\alpha \\
&= \Gamma_\alpha^\beta\xi^\beta\chi_*^\alpha. \qquad (4.26)
\end{aligned}
$$

Therefore

$$
\frac{\partial}{\partial t}D_\xi\mathcal{L} = \mathcal{A}\mu_t\mathcal{F} - \mathcal{A}\mu\,\xi^0\mathcal{F}^{\perp} + \mathcal{A}\mu\,\Gamma_\alpha^\beta\xi^\beta\chi_*^\alpha. \qquad (4.27)
$$

Similarly, observe that the partial derivative of the Lagrangian density with respect to η is given as

$$
\begin{aligned}
D_\eta\mathcal{L}_\sigma &= \mathcal{B}\tau\left(\frac{\zeta^\alpha}{\nu}\chi_*^\alpha - \frac{\eta^\alpha\zeta^\alpha}{\nu^2}\Big(2\eta^\alpha\chi_*^\alpha\Big)\right) \\
&= \mathcal{B}\tau\left(\frac{\zeta^\alpha}{\nu} - 2\,\tau\frac{\eta^\alpha}{\nu}\right)\chi_*^\alpha \\
&= \mathcal{B}\frac{\tau}{\nu}\big(\mathcal{E}^*(\zeta) - 2\,\tau\mathcal{E}^*(\eta)\big). \qquad (4.28)
\end{aligned}
$$

Similarly,

$$D_\zeta \mathcal{L}_\sigma = \mathcal{B}\tau \frac{\eta^\alpha}{\nu} \chi_*^\alpha = \mathcal{B}\frac{\tau}{\nu}\, \mathcal{E}^*(\eta). \tag{4.29}$$

The time derivative of $D_\zeta \mathcal{L}$ is given by

$$\begin{aligned}
\frac{\partial}{\partial t} D_\zeta \mathcal{L}_\sigma &= \mathcal{B}\tau_t \frac{\eta^\alpha}{\nu}\chi_*^\alpha + \mathcal{B}\tau\Big(\frac{\eta_t^\alpha}{\nu} - \frac{\eta^\alpha}{\nu^2}\nu_t\Big)\chi_*^\alpha \\
&= \mathcal{B}\tau_t \frac{\eta^\alpha}{\nu}\chi_*^\alpha + \mathcal{B}\tau\Big(\frac{\eta_t^\alpha}{\nu} - 2\,\tau\frac{\eta^\alpha}{\nu}\Big)\chi_*^\alpha \\
&= \mathcal{B}\frac{\tau_t}{\nu}\, \mathcal{E}^*(\eta) + \mathcal{B}\frac{\tau}{\nu}\Big(\mathcal{E}^*(\zeta) - 2\,\tau\mathcal{E}^*(\eta)\Big).
\end{aligned} \tag{4.30}$$

Let $\psi = D_\eta \mathcal{L}_\sigma - \frac{\partial}{\partial t} D_\zeta \mathcal{L}_\sigma$. Then because the Lagrangian density is symmetric with respect to η and ζ, equations (4.28) and (4.30) yield

$$\begin{aligned}
\psi &= \mathcal{B}\frac{\tau}{\nu}\big(\mathcal{E}^*(\zeta) - 2\,\tau\mathcal{E}^*(\eta)\big) - \mathcal{B}\frac{\tau_t}{\nu}\, \mathcal{E}^*(\eta)\chi_*^\alpha - \mathcal{B}\frac{\tau}{\nu}\Big(\mathcal{E}^*(\zeta) - 2\,\tau\mathcal{E}^*(\eta)\Big) \\
&= -\mathcal{B}\frac{\tau_t}{\nu}\, \mathcal{E}^*(\eta).
\end{aligned} \tag{4.31}$$

Therefore, the coajoint action applied to ψ is given as

$$\begin{aligned}
ad_\eta^*\psi &= -\mathcal{B}\frac{\tau_t}{\nu} ad_\eta^* \mathcal{E}^*(\eta) \\
&= -\mathcal{B}\frac{\tau_t}{\nu}\eta^0\big(\eta^2\chi_*^1 - \eta^1\chi_*^2\big).
\end{aligned} \tag{4.32}$$

Also the spacial derivative of that quantity ψ is

$$\psi_\omega = -\mathcal{B}\frac{\partial}{\partial \omega}\left(\frac{\tau_t}{\nu}\right)\, \mathcal{E}^*(\eta) - \mathcal{B}\frac{\tau_t}{\nu}\, \mathcal{E}^*(\eta_\omega). \tag{4.33}$$

Appealing to Theorem 2 and equations (4.18), (4.24) and (4.27), we write the

Lagrangian force on Ψ is as

$$
\begin{aligned}
\Delta^{\mathcal{L}}_{(\varphi,\xi)}(X^p) &= \Phi^{T^*}_{\varphi^{-1}} D_\varphi \mathcal{L} - \frac{\partial}{\partial t} D_\xi \mathcal{L} - \frac{\partial}{\partial \omega}\Big(D_\eta \mathcal{L} - \frac{\partial}{\partial t} D_\zeta \mathcal{L}\Big) \\
&\quad + ad^*_\xi D_\xi \mathcal{L} + ad^*_\eta \Big(D_\eta \mathcal{L} - \frac{\partial}{\partial t} D_\zeta \mathcal{L}\Big) \\
&= \mathcal{A}\mu(\mathcal{F}_2 \xi^1 - F_1 \xi^2)\, \chi^0_* + \mathcal{A}\mu \Gamma^\alpha_\beta \xi^\beta \chi^\alpha_* \\
&\quad -\Big(\mathcal{A}\mu_t \mathcal{F} - \mathcal{A}\mu \xi^0 \mathcal{F}^\perp\Big) - \mathcal{A}\mu\, \Gamma^\beta_\alpha \xi^\beta \chi^\alpha_* \\
&\quad -\psi_\omega + \mathcal{A}\mu\Big((\mathcal{F}_1 \xi^2 - \mathcal{F}_2 \xi^1)\chi^0_* - \xi^0 \mathcal{F}^\perp\Big) + ad^*_\eta \psi \\
&= ad^*_\eta \psi - \psi_\omega + \mathcal{A}\mu\, \Gamma^\alpha_\beta \xi^\beta \chi^\alpha_* - \mathcal{A}\mu\, \Gamma^\beta_\alpha \xi^\beta \chi^\alpha_* - \mathcal{A}\mu_t \mathcal{F} \\
&= ad^*_\eta \psi - \psi_\omega - \mathcal{A}\mu_t \mathcal{F} + \mathcal{A}\mu \sum_{\alpha \neq \beta} \xi^\beta \Big(\Gamma^\alpha_\beta - \Gamma^\beta_\alpha\Big) \chi^\alpha_*.
\end{aligned}
$$

The difference $\Gamma^2_1 - \Gamma^1_2$ can be expressed purely in terms of D. Under the identification $\mathcal{R} = [X, X^{\perp}]$,

$$\begin{aligned}
\Gamma_2^1 - \Gamma_1^2 &= < \left(\mathcal{R}^{-1}\frac{\partial \mathcal{F}_2}{\partial \gamma}\right), e_1 > - < \left(\mathcal{R}^{-1}\frac{\partial \mathcal{F}_1}{\partial \gamma}\right), e_2 > \\
&= < \left(\mathcal{R}^{-1}\frac{\partial \mathcal{F}_2}{\partial \gamma}\right), e_1 > - < \left(\mathcal{R}^{-1}\frac{\partial \mathcal{F}_1}{\partial \gamma}\right), e_2 > \\
&= < \left(\mathcal{R}^{-1}\frac{\partial}{\partial \gamma} < X^{\perp}, \bar{D}^{\perp} >\right), e_1 > - < \left(\mathcal{R}^{-1}\frac{\partial}{\partial \gamma} < X, \bar{D}^{\perp} >\right), e_2 > \\
&= X_k < X^{\perp}, \frac{\partial \bar{D}^{\perp}}{\partial \gamma^k} > - X_k^{\perp} < X, \frac{\partial \bar{D}^{\perp}}{\partial \gamma^k} > \\
&= X_k X_i^{\perp} \frac{\partial \bar{D}_i^{\perp}}{\partial \gamma^k} - X_k^{\perp} X_i \frac{\partial \bar{D}_i^{\perp}}{\partial \gamma^k} \\
&= (X_k X_i^{\perp} - X_k^{\perp} X_i)\frac{\partial \bar{D}_i^{\perp}}{\partial \gamma^k} \\
&= \sum_{i \neq k} (X_k X_i^{\perp} - X_k^{\perp} X_i)\frac{\partial \bar{D}_i^{\perp}}{\partial \gamma^k} \\
&= (X_1 X_2^{\perp} - X_1^{\perp} X_2)\frac{\partial \bar{D}_2^{\perp}}{\partial \gamma^1} + (X_2 X_1^{\perp} - X_2^{\perp} X_1)\frac{\partial \bar{D}_1^{\perp}}{\partial \gamma^2} \\
&= \frac{\partial \bar{D}_2^{\perp}}{\partial \gamma^1} - \frac{\partial \bar{D}_1^{\perp}}{\partial \gamma^2} \\
&= \frac{\partial}{\partial \gamma^1}\frac{D_1}{\|D\|} + \frac{\partial}{\partial \gamma^2}\frac{D_2}{\|D\|} \\
&= \nabla \cdot \frac{D}{\|D\|}, \qquad (4.34)
\end{aligned}$$

where $\nabla\cdot$ denote the divergence operator on $\mathbb{E}_2$. Therefore the Lagrangian force on Ψ can be expressed as

$$\Delta^{\mathcal{L}}_{(\varphi,\xi)}(X^p) = ad^*_\eta \psi - \psi_\omega - \mathcal{A}\mu_t \mathcal{F} + \mathcal{A}\mu \sum_{\alpha \neq \beta} (-1)^\beta \xi^\beta \left(\nabla \cdot \frac{D}{\|D\|}\right) \chi_*^\alpha. \qquad (4.35)$$

Having established the Lagrangian force corresponding to the density (4.10), we proceed in the following section to complete the model by developing the appropriate constraints and externally applied force.

4.3 External Forcing and Constraints

As noted earlier, the coordination of multiple unmanned arial vehicles is a primary application of interest for this work. In the interest of conserving fuel and maintaining a fixed elevation, it is often desired that the speed of each vehicle remain constant. As a result, we consider here a nonholomically constrained filament with fixed speed; specifically we require $\|\gamma_t(\varphi)\| = v$, for a fixed constant $v > 0$. Since $\gamma_t(\varphi) = \mathcal{R}(\varphi)\mathcal{E}(\xi)$, the constant speed constraint is equivalent to requiring $\|\mathcal{E}(\xi)\| = v$.

Suppose that orientation $\mathcal{R}(\varphi)$ is interpreted as a pair of framing vectors for the curve by the identification $\mathcal{R}(\varphi) = [X(\varphi), X^{\perp}(\varphi)]$. Then we define an adapted flow as one for which $\gamma_t(\varphi) = X(\varphi)$. Since we have no particular concern for the nature of this framing, we chosen an adapted flow to simplify the calculations. Coupled with the constant speed constraint, we chose the two constraints $\sigma^1 : \Psi \to \mathbb{R}$ and $\sigma^2 : T\Psi \to \mathbb{R}$ given as

$$\begin{aligned} \sigma^1(\varphi, \xi) &= \chi^1_* \cdot \xi - v, \\ \sigma^2(\varphi, \xi) &= \chi^2_* \cdot \xi. \end{aligned} \tag{4.36}$$

Under these constraints $\mu = \mathcal{F}_1 v$. Similarly, to simplify notation, let $\rho = \mathcal{F}_2$. In coordinates, $d\sigma^1(\varphi, \xi) = (0, \chi^1_*)$ and $d\sigma^2(\varphi, \xi) = (0, \chi^2_*)$. Therefore, by (3.10), the constraint distribution on Ψ is given by

$$\Xi(\varphi, v) = \{\lambda_1(0, \chi^1_*) + \lambda_2(0, \chi^2_*) \mid \lambda^1, \lambda^2 \in \mathbb{R}\}.$$

Therefore all constraint reaction forces will lie in Ξ.

We now seek to establish a reasonable external forcing on the filament. One of the difficulties in studying equations which arise purely from a variational principle is the lack of dissipation. The latter is an essential mechanism for convergence of the filament to an established orientation field. We consider the external force represent by the covector

$$\Delta^E = -\mathcal{A}\mu\mathfrak{S}(\rho)\mathcal{F}, \tag{4.37}$$

where $\mathfrak{S}$ denotes a sigmoidal function which inherits the sign of its argument and satisfies

$$\lim_{|\rho|\to 1} |\mathfrak{S}(\rho)| \quad = \quad \mathfrak{s}(\rho), \tag{4.38}$$

where $\mathfrak{s}$ denotes the sign of ρ.

Incidentally, $\Delta^E = -\mathcal{A}\mathfrak{S}(\rho)D_\xi\mathcal{L}$ for the Lagrangian density (4.10). Since μ is a measure of misalignment, the Δ^E is zero when the filament achieves alignment with D. Also, since the vector $(\mathcal{F}_1, \mathcal{F}_2)^T$ is a representation of $\bar{D}^\perp$ in the frame $\mathcal{R}(\varphi) = [X(\varphi), X^\perp(\varphi)]$, the covector Δ^E reflects a force proportional to the misalignment that is directed perpendicular to D. The choice of orientation along this perpendicular direction is governed by the sign of $\rho = \mathcal{F}_2$. Recall that the proposed lagrangian density does not distinguish between flows which align with D and those which align with its negation. Hence ρ appears in the external force as the mechanism for resolving this ambiguity. Observe that ρ is positive when the angle between the direction of motion and the vector D is acute and negative otherwise. This leads to rotation towards alignment with D and away from $-D$. Consequently the external force Δ^E may conceptually be interpreted as a proportional controller guiding the orientation of the filament towards D.

4.4 The Virtual Filament Equations

Drawing from the forgoing work, we now write down the mechanical equations of motion for our virtual filament as given by Principle 1. Given the Lagrangain force (3.17), the governing dynamics for a Lagrangian system with density (4.10), constraints (4.36), and external forcing (4.37), are given by the pair of equations

$$ad^*_\eta \psi - \psi_\omega - \mu_t \mathcal{F} + \mu \sum_{\alpha \neq \beta} (-1)^\beta \xi^\beta \left(\nabla \cdot \frac{D}{\|D\|} \right) \chi^\alpha_* - \mu \mathfrak{S}(\rho) \mathcal{F} + \lambda \;=\; 0,$$

and

$$\mathcal{C}(\varphi, \xi) \;=\; 0,$$

where λ represents the constraint reaction force. Here we have chosen $\mathcal{A} = \mathcal{B} = 1$ for notional clarity. Enforcing the constraints yields

$$ad^*_\eta \psi - \psi_\omega - \mu_t \mathcal{F} - \mu\, v \left(\nabla \cdot \frac{D}{\|D\|} \right) \chi^2_* - \mu \mathfrak{S}(\rho) \mathcal{F} + \lambda \;=\; 0. \tag{4.39}$$

These are the constrained virtual filament equations.

Since the Lagrangian density proposed in (4.10) is degenerate, there is broad flexibility in choosing a constraint reaction force, λ, such that the virtual filament equations (4.39) are consistent. The covector λ emerges as a control by which one can manage the fundamental tradeoff between filament stretching and alignment with the vector field D.

To elucidate this tradeoff in governing filament equations (4.39), we allow the form $\Delta^{\mathcal{L}} + \Delta^{E} + \Delta^{\mathcal{C}}$ to act on the flow rate ξ. Equivalently, let the left hand sides of equation (4.39) act on ξ, yielding

$$\begin{aligned}
&\left(ad^*_\eta \psi - \psi_\omega - \mu_t \mathcal{F} - \mu\, v \left(\nabla \cdot \frac{D}{\|D\|}\right) \chi^1_* - \mu \mathcal{F} + \lambda\right) \cdot \xi \\
&= ad^*_\eta \psi \cdot \xi - \psi_\omega \cdot \xi - \mu_t \mu - \mu^2 + \lambda \cdot \xi \\
&= -\frac{\tau_t}{\nu} \eta^0 \left(\eta^2 \chi^1_* - \eta^1 \chi^2_*\right) \cdot \xi - \psi_\omega \cdot \xi - \mu_t \mu - \mu^2 + \lambda \cdot \xi \\
&= -\tau_t \tau v + \frac{\partial}{\partial \omega} \left(\frac{\tau_t}{\nu}\, \mathcal{E}^*(\eta)\right) \cdot \xi - \mu_t \mu - \mu^2 + \lambda \cdot \xi \\
&= -\frac{1}{2}(\tau)^2_t v - \frac{1}{2}(\mu)^2_t - v \frac{\partial}{\partial \omega} \left(\frac{\tau_t}{\nu} \eta^1\right) - \mu^2 + \lambda_1 v. \qquad (4.40)
\end{aligned}$$

Here we have used the fact that under the constraint $\mathcal{C}(\varphi, \xi) = 0$, the compatibility condition (3.4) yields

$$\begin{aligned}
\tau = \frac{1}{2}\frac{\nu_t}{\nu} &= \frac{1}{\nu}(\eta^1 \eta^1_t + \eta^2 \eta^2_t) \\
&= \frac{1}{\nu}\Big(\eta^1(\xi^0 \eta^2) + \eta^2(-\xi^0 \eta^1 + \eta^0)\Big) \\
&= \frac{\eta^0 \eta^2}{\nu}. \qquad (4.41)
\end{aligned}$$

Therefore, ignoring the higher order term $v \frac{\partial}{\partial \omega} \left(\frac{\tau_t}{\nu} \eta_1\right)$, we observe by (4.39) and (4.40) that

$$\frac{1}{2}(\tau)^2_t v \approx -\left(\frac{1}{2}(\mu)^2_t + \mu^2\right) + \lambda_1 v. \qquad (4.42)$$

Then clearly highlights an inverse relationship between changes in the respective magnitudes of the misalignment and stretching terms. Note that since v is fixed, λ_1, can be chosen such that the both τ and μ are decreasing in magnitude. This suggests that for certain vector fields, there may exist a nonstretching flow with a monotonically decreasing measure of misalignment. The choice of the constraint reaction force λ is an essential part of achieving this objective. Consequently, we now interpret the constraint reaction force as a control. In the subsequent work, we will make a particular choice of λ. This will generate a corresponding curvature feedback which we will examined in detail.

To further understand the virtual filament equations we proceed to establish a constraint reaction force for which the corresponding flow of the virtual filament is guaranteed to align with the vector field D. Our technique will be as follows: we will proposed a well defined constraint reaction force in the neighborhood of $\mu = 0$ and then relax this restriction by considering a related reaction force which is uniformly well defined.

Motivated by the $\mathcal{B} = 0$ dynamics, consider the constraint reaction force satisfying the equation

$$\lambda = \bar{\lambda} + \psi_\omega - ad^*_\eta \psi, \tag{4.43}$$

where $\bar{\lambda}$ is chosen to guarantee consistency of the corresponding filament dynamics given in (4.39) as

$$\mu_t \mathcal{F} = -\mu \mathfrak{S}(\rho)\mathcal{F} - \mu v \left(\nabla \cdot \frac{D}{\|D\|} \right) \chi_*^2 - \bar{\lambda}. \tag{4.44}$$

Observe that in the neighborhood of $\mu = 0$ the vectors ξ and $\mathcal{F}^\#$ span the Euclidean subalgebra of $\mathfrak{se}_2$. A set of equations is consistent in a neighborhood of μ if their projection onto these basis elements is consistent. Note that by construction, $\mathcal{F} \cdot \mathcal{F}^\sharp = 1$. Therefore, applying the covector equation (4.44) to ξ and $\mathcal{F}^\#$, respectively, generates the dual equations

$$\mu_t = -\mu\mathfrak{S}(\rho) - \mu v \left(\nabla \cdot \frac{D}{\|D\|}\right)\rho - \bar{\lambda}\cdot\mathcal{F}^{\#} \tag{4.45}$$

$$\mu_t\mu = -\mu^2\mathfrak{S}(\rho) - \bar{\lambda}_1 v. \tag{4.46}$$

A simple algebraic calculation reveals a natural choice of $\bar{\lambda}$ given in components as

$$\begin{aligned} \bar{\lambda}_1 &= \frac{\mu^2}{\rho}\left(\nabla \cdot \frac{D}{\|D\|}\right) \\ \bar{\lambda}_2 &= 0. \end{aligned} \tag{4.47}$$

Observe that $\bar{\lambda}$ is well defined since $\rho \neq 0$ in a neighborhood of $\mu = 0$. The governing equation (4.45) then becomes

$$\begin{aligned} \mu_t &= -\mu\mathfrak{S}(\rho) - \mu v \left(\nabla \cdot \frac{D}{\|D\|}\right)\rho - \mathcal{F}_1\frac{\mu^2}{\rho}\left(\nabla \cdot \frac{D}{\|D\|}\right) \\ &= -\mu\mathfrak{S}(\rho) - \left(\frac{\mu v\rho^2 + \mathcal{F}_1\mu^2}{\rho}\right)\left(\nabla \cdot \frac{D}{\|D\|}\right) \\ &= -\mu\mathfrak{S}(\rho) - \left(\frac{\mu v\left(1 - \left(\frac{\mu}{v}\right)^2\right) - \mathcal{F}_1\mu^2}{\rho}\right)\left(\nabla \cdot \frac{D}{\|D\|}\right) \\ &= -\mu\mathfrak{S}(\rho) - \left(\frac{\mu v}{\rho}\right)\left(\nabla \cdot \frac{D}{\|D\|}\right). \end{aligned} \tag{4.48}$$

Expanding the time derivative of μ by equation (4.16) yields

$$\begin{aligned} \mu_t &= v(\mathcal{F}_1)_t \\ &= v\left(\rho\,\Phi_\varphi^{T^*}\chi_*^0 + \frac{1}{v}\frac{\partial\mu}{\partial\gamma^\alpha}\,d\gamma^\alpha\right)\cdot\varphi\xi \\ &= v\left(\rho\xi^0 + \frac{1}{v}\frac{\partial\mu}{\partial\gamma^\alpha}X^\alpha\right) \\ &= v\rho\,\xi^0 + \frac{\partial\mu}{\partial\gamma^\alpha}X^\alpha, \end{aligned} \tag{4.49}$$

where $[X, X^{\perp}] = \mathcal{R}(\varphi)$. Substituting (4.49) into (4.48) and solving for $(\rho)^2\xi^0$ yields

$$(\rho)^2\xi^0 \quad = \quad -\frac{\rho}{v}\frac{\partial\mu}{\partial\gamma^\alpha}X^\alpha - \frac{\mu}{v}\mathfrak{S}(\rho)\rho - \mu\left(\nabla\cdot\frac{D}{\|D\|}\right).$$

Note that as $\mu \to 0$,

$$\rho = \mathfrak{s}(\rho)\sqrt{1-\left(\frac{\mu}{v}\right)^2} \to \mathfrak{s}(\rho). \tag{4.50}$$

Therefore we define κ as the positively scaled curvature $\rho^2\xi^0$ subject to the smooth relaxation $\rho \to \mathfrak{S}(\rho)$; i.e.,

$$\kappa(\varphi) \triangleq \left.\rho^2\xi^0\right|_{\rho\to\mathfrak{S}(\rho)} = -\frac{\mathfrak{S}(\rho)}{v}\frac{\partial\mu}{\partial\gamma^\alpha}X^\alpha - \frac{\mu}{v} - \mu\left(\nabla\cdot\frac{D}{\|D\|}\right). \tag{4.51}$$

Clearly κ is well defined everywhere and is completely specified in terms of the state φ. Note that $\lim_{\mu\to 0}\rho = \pm 1$. Therefore as a material point of the filament aligns with the orientation field, $\mu \to 0$ and κ is an approximation to the actual temporal curvature ξ^0. We now formally consider the virtual filament flow induced by the state dependent curvature feedback (4.51).

One of the most significant properties of the curvature feedback κ is that it is respected by integral curves of D. That is, once agreement has been established between the filament orientation and D, each material point of the filament persists along a corresponding integral curve. This invariance establishes the naturality of this feedback mechanism.

In order to more accurately characterize the fundamental invariance property of the feedback (4.51), we introduce the concept of an oriented integral curve orbit of a vector field. Let $\lambda : U \to \mathbb{E}_2$ be an integral curve of the orientation field

D. The oriented orbit of λ is the submanifold $\Lambda \subset SE_2$ given by

$$\Lambda(\lambda) = \left\{ \big(R(u), \lambda(u)\big) \in SE_2 \,\middle|\, R(u) = \big[\bar{D}(\lambda(u)), \bar{D}^{\perp}(\lambda(u))\big], u \in U \right\}.$$

Therefore the oriented integral curve orbit of a vector field represents the image of an integral curve and its normalized orientation. Naturally, the projection $\gamma(\Lambda)$ is an integral curve orbit. The following theorem establishes the invariance of oriented integral curve orbits of D under the curvature feedback (4.51).

Theorem 4. *Oriented integral curve orbits of the orientation field are invariant under the curvature feedback (4.51).*

Proof. Let Λ be an oriented integral curve orbit of the orientation field D and let φ be a temporal curve in SE_2 subject to the curvature feedback (4.51). Suppose that $\varphi(t) \in \Lambda$ for some t. It is enough to show that the projection of φ under γ is an integral curve of D. This is equivalent to establishing that $\gamma(\varphi)$ has the same curvature as $\gamma(\Lambda)$. Let λ be an integral curve of D with orbit $\gamma(\Lambda)$ described by the framing equations

$$\begin{aligned} \bar{D}_u &= c\bar{D}^{\perp} \\ \bar{D}^{\perp}_u &= -c\bar{D} \\ \lambda_u &= \bar{D}, \end{aligned} \tag{4.52}$$

where c is the intrinsic curvature of the manifold $\gamma(\Lambda)$ and u is the unit speed parameterization of λ. Note that such an integral curve can always be constructed from an arbitrary integral curve of D by reparameterization. Note that (4.52) holds for $\lambda = \gamma(\varphi)$ at time t. Since φ is aligned with D at t, $\mu(t) = 0$ and $\rho(t) = 1$. Therefore at time t, the curvature of φ is given as

$$
\begin{aligned}
\kappa(\varphi) &= -\frac{\mathfrak{S}(\rho)}{v}\frac{\partial \mu}{\partial \gamma^\alpha}X^\alpha - \frac{\mu}{v} - \mu\left(\nabla \cdot \frac{D}{\|D\|}\right) \\
&= -\frac{\mathfrak{s}(\rho)}{v}\frac{\partial \mu}{\partial \gamma} \cdot \bar{D} \\
&= -\frac{\partial}{\partial \gamma}\left(X \cdot \bar{D}^{\perp}\right) \cdot \bar{D} \\
&= -\left(\frac{\partial \bar{D}^{\perp}}{\partial \gamma} \cdot \bar{D}\right) \cdot X \\
&= -\left(\frac{\partial \bar{D}^{\perp}}{\partial \lambda} \cdot \lambda_u\right) \cdot \bar{D} \\
&= -\frac{\partial \bar{D}^{\perp}}{\partial u} \cdot \bar{D} \\
&= c. \qquad (4.53)
\end{aligned}
$$

Hence the curvature of $\gamma(\varphi)$ at t under the curvature feedback (4.51) is identical to the curvature of $\gamma(\Lambda)$. Therefore $\gamma(\varphi)$ is an integral curve of D, and φ lies in Λ for all time. This completes the proof. □

Consequently once agreement has been established between an oriented particle and the orientation field D, this particle will persists along an integral curve of D. Furthermore each material point of the filament will lie in an oriented integral curve orbit of D.

One may regard the particular constraint reaction force chosen in the above work as motivated solely by our interest in aligning the filament with the orientation field. Here we have established a starting point from which to study the virtual filament equations (4.39). In the limiting case in which the stretch penalty is effectively ignored, we achieve precisely the objective sought: alignment with the orientation field. Note that the proposed curvature feedback is independent of the filament stretch rate since all the relevant τ dependent terms have been absorbed in the constraint reaction force. In the next section we demonstrate the alignment of the virtual filament under the proposed curvature feedback to various orientation fields of interest.

4.5 The Orientation Field

4.5.1 A Elementary Orientation Field

We now consider the evolution of a virtual filament under the curvature feedback (4.51) for a variety of orientation fields. In the first instance we consider the simple vector field given by

$$D(\gamma) \;=\; -\gamma^2 e_1 + \gamma^1 e_2. \tag{4.54}$$

Recall that the dynamical system $\dot{\gamma} = D(\gamma)$ describes the evolution of an unforced harmonic oscillator. Since D is a divergence free field, the corresponding curvature feedback (4.51) is given as

$$\kappa = -\frac{\mathfrak{S}(\rho)}{v}\frac{\partial \mu}{\partial \gamma^\alpha} X^\alpha - \frac{\mu}{v}. \tag{4.55}$$

Observe that

$$
\begin{aligned}
\frac{\partial \mu}{\partial \gamma^{\alpha}} X^{\alpha} &= v \frac{\partial \mathcal{F}_1}{\partial \gamma} \cdot X \\
&= v \frac{\partial \mu}{\partial \gamma} \cdot X \\
&= v \left(\frac{\partial (\bar{D}^{\perp})^{\beta}}{\partial \gamma^{\alpha}} X^{\beta} \right) X^{\alpha} \\
&= \frac{v}{\|D\|} \left(X^1 X^1 + X^2 X^2 \right) - \frac{v \mathcal{F}_1}{\|D\|^2} \left(\gamma^1 X^1 - \gamma^2 X^2 \right) \\
&= \frac{v}{\|D\|} + \frac{v \mathcal{F}_1}{\|D\|^2} \mathcal{F}_1 \\
&= \frac{v}{\|D\|} + \frac{1}{v} \frac{\mu^2}{\|D\|^2}.
\end{aligned}
\tag{4.56}
$$

Substituting (4.56) into (4.55) yields

$$
\begin{aligned}
\kappa &= -\frac{\mathfrak{S}(\rho)}{v} \left(\frac{v}{\|D\|} + \frac{1}{v} \frac{\mu^2}{\|D\|^2} \right) - \frac{\mu}{v} \\
&= -\mathfrak{S}(\rho) \left(\frac{1}{\|D\|} + \frac{1}{v^2} \frac{\mu^2}{\|D\|^2} \right) - \frac{\mu}{v}.
\end{aligned}
\tag{4.57}
$$

The vector field D does not admit an isolated periodic orbit (limit cycle); however there exists a continuum of periodic orbits. In fact every nontrivial integral curve γ of D is a counter-clockwise circular trajectory of radius $\|D\| = \sqrt{(\gamma^1)^2 + (\gamma^2)^2}$, centered at the origin. Since the curvature of a circle is the reciprocal of its radius, integral curves of D have constant negative curvature $c = -\frac{1}{\|D\|}$. Consider the case when a material point of the filament aligns with D. In this case, $\mu = 0$ and $\rho = 1$. Hence, by (4.57) the curvature feedback is

$$
\kappa = -\mathfrak{S}(\rho) \left(\frac{1}{\|D\|} + \frac{1}{v^2} \frac{\mu^2}{\|D\|^2} \right) - \frac{\mu}{v} = -\frac{1}{\|D\|}. \tag{4.58}
$$

Therefore the curvature of this material point of the filament is identical to the curvature of the corresponding integral curve. Hence integral curves of (4.54) respect the curvature feedback (4.57).This is consistent with the general result established in Theorem 4.

In the following simulations we consider a unit speed flow with $v = 1$. Furthermore we choose $\mathfrak{S}(\rho) = \frac{\pi}{2} tan^{-1}(\rho)$ which satisfies property (4.38). Figure 4.1 depicts the evolution of the virtual filament induced by (4.58). The initial orientation of the filament is aligned with the positive γ^1 direction. Note that each material point of the filament aligns with an oriented integral curve of the circular orientation field D.

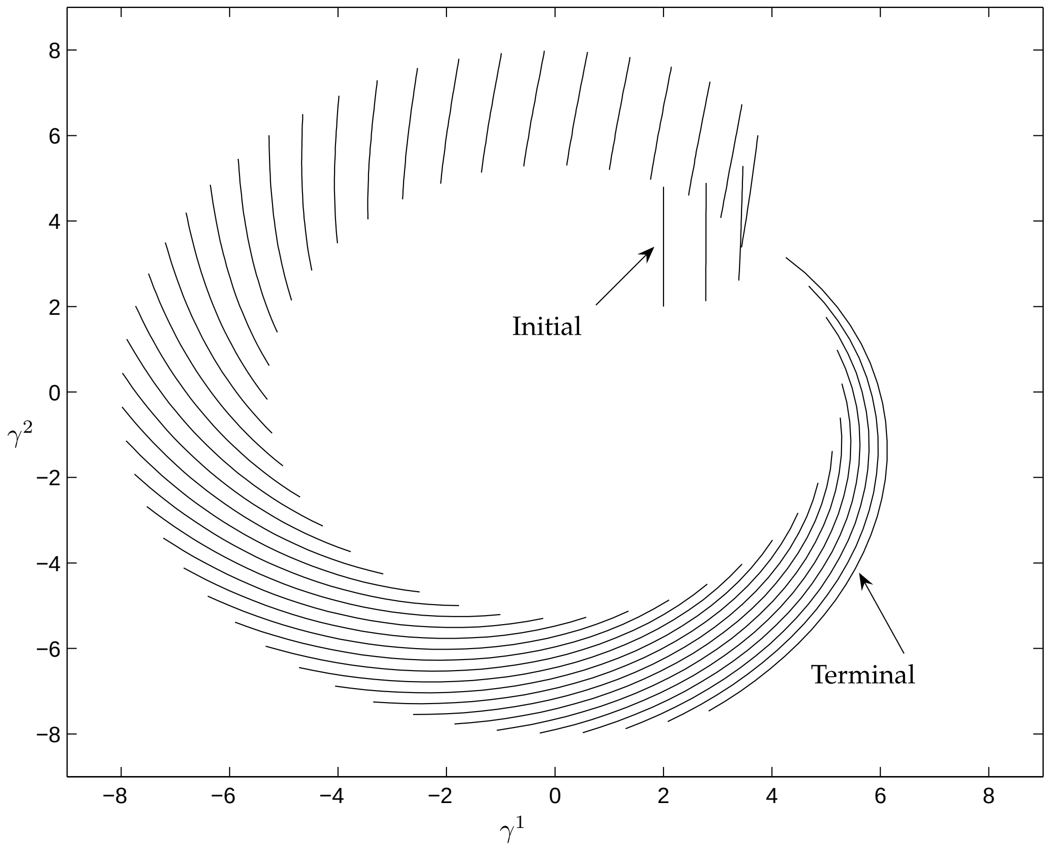

Figure 4.1: A Virtual Filament Aligning with the Orientation Field

To see this explicitly, consider a typical point on the filament which starts at the coordinate $(\gamma^1, \gamma^2) = (2, 2)$ and is oriented in the positive γ^1 direction. Figure

4.2 shows the evolution of this material point (the star and circle markers denote the initial and terminal points, respectively). The orientation field (4.59) is shown in the background.

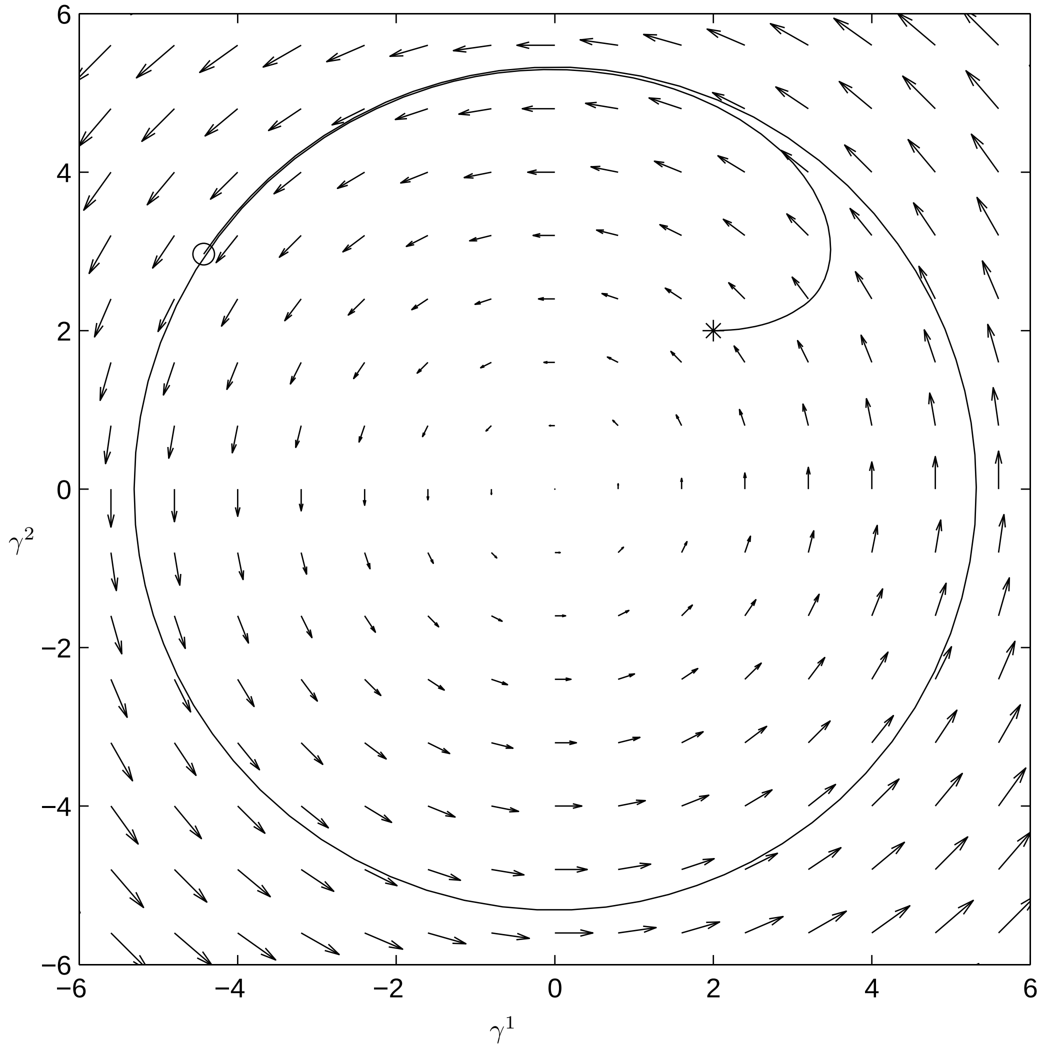

Figure 4.2: A Particle Aligning with the Orientation Field

Recall that for a unit speed flow, μ and ρ evolve on the unit circle. Hence the aligning particle flow observed in figure 4.2 suggests that $\mu \to 0$ and $\rho \to 1$ as $t \to \infty$. This is precisely the behavior seen in figure 4.3. Note that initially μ is increasing in norm. This reflects the fact that the particle is initially headed is a direction nearly opposite to that of the orientation field. During this phase the particle is turning around. This natural reversing phenomenon is due to our judicious choice of external forcing. As outline earlier, the proposed feedback curvature for a divergence free field leads to the monotonic convergence $\rho \rightarrow 1$ for each material point of the filament. This convergence, as depicted for a typical material particle in figure 4.3, is the underlying mechanism that aligns every material point of the filament with the orientation field.

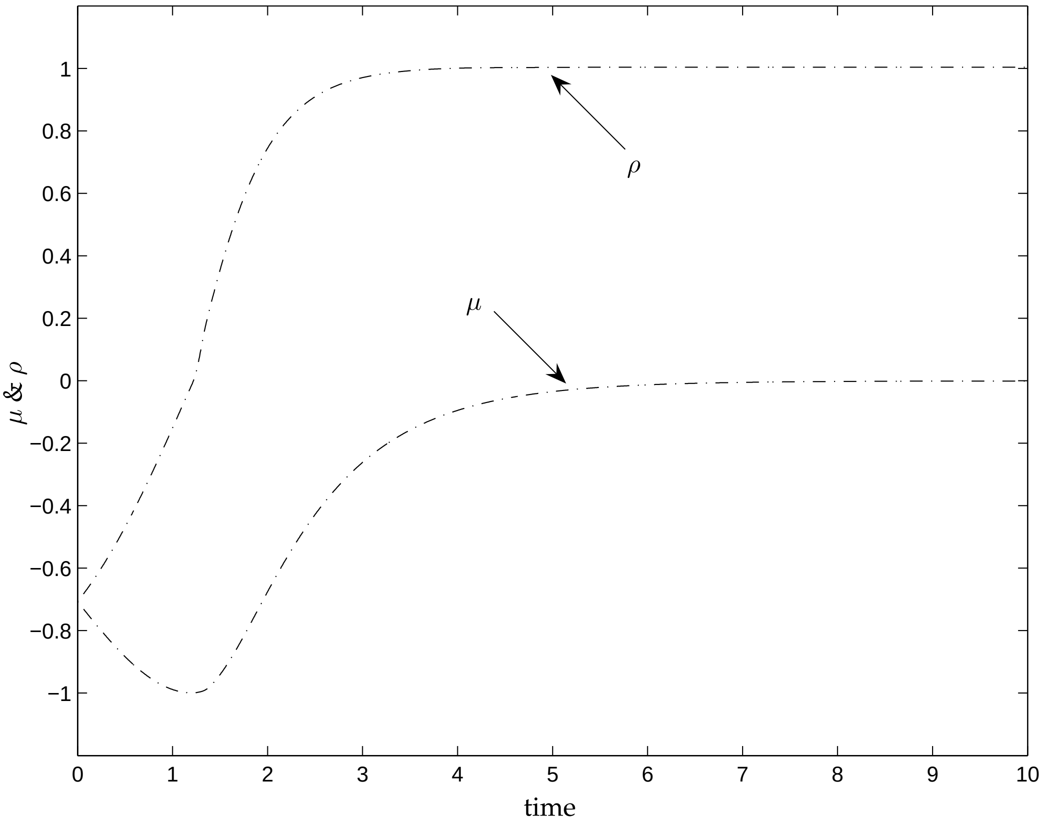

Figure 4.3: An Aligning Evolution of μ and ρ

While the flow observed in figure 4.1 has the desired alignment property, it clearly exhibits significant stretching. However, given that (4.51) captures only the alignment property of the proposed filament model (4.10), this is entirely expected.

4.5.2 A Non-trivial Orientation Field

We now consider a more complex orientation field by introducing a cubic nonlinearity into the field studied in the previous section. In particular we will examine the field

$$D(\gamma) \;\; = \;\; \Big(\gamma^1(\alpha - \|\gamma\|^2) - \gamma^2\Big)e_1 + \Big(\gamma^2(\alpha - \|\gamma\|^2) + \gamma^1\Big)e_2, \qquad (4.59)$$

for $\alpha > 0$. Integral curves of D undergo a subcritical Hopf bifurcation at $\alpha = 0$ (see [12]). For $\alpha > 0$ the circle of radius $\sqrt{\alpha}$, centered at the origin, forms a globally asymptotically stable limit cycle. We proceed by constructing the curvature feedback (4.51) for the vector field (4.59) with the expectation that a filament subject to this feedback will converge to the $\sqrt{\alpha}$ limit cycle. For $\alpha = v = 1$, an elementary calculation reveals that the divergence of D is given as

$$\nabla \cdot \frac{D}{\|D\|} = 2\alpha - 4(\gamma^1)^2 - 4(\gamma^2)^2. \qquad (4.60)$$

Also,

$$\begin{aligned}
\frac{\partial \mu}{\partial \gamma} \cdot X \;\; = \;\; & \frac{1}{\|D\|}\Big(2\gamma^1\gamma^2\Big((X^1)^2 - (X^2)^2\Big) + 2X^1X^2\Big((\gamma^2)^2 - (\gamma^1)^2\Big)\Big) \\
& + \frac{1}{\|D\|}\Big(\frac{\mu\rho}{v}\Big(\alpha - \|\gamma\|^2\Big) + \frac{\mu^2}{v^2} - 1\Big) \\
& + \frac{2\mu}{v\|D\|^2}\Big(\gamma^1\gamma^2\Big(X^1D^2 + X^2D^1\Big) + X^1D^1(\gamma)^2 - X^2D^2(\gamma^2)^2\Big).
\end{aligned}$$

Therefore the curvature is given by an elementary, though tedious, calculation as

$$
\begin{aligned}
\kappa \;=\; & -\frac{\mu}{v} - \frac{\mathfrak{S}(\rho)}{v\|D\|}\Big(2\gamma^1\gamma^2\Big((X^1)^2 - (X^2)^2\Big) + 2X^1X^2\Big((\gamma^2)^2 - (\gamma^1)^2\Big)\Big) \\
& - \frac{2\mathfrak{S}(\rho)\mu}{v^2\|D\|^2}\Big(\gamma^1\gamma^2\Big(X^1D^2 + X^2D^1\Big) + X^1D^1(\gamma)^2 - X^2D^2(\gamma^2)^2\Big) \\
& - \frac{\mathfrak{S}(\rho)}{v\|D\|}\Big(\frac{\mu\rho}{v}\Big(\alpha - \|\gamma\|^2\Big) + \frac{\mu^2}{v^2} - 1\Big) - \;2\mu\Big(\alpha - 2(\gamma^1)^2 - 2(\gamma^2)^2\Big).
\end{aligned}
$$

The evolution of a virtual filament under this curvature is depicted in figure 4.4. Initially the orientation of the filament is aligned in the northwest direction.

Immediately one observes that this flow exhibits the appropriate alignment and appears to stretch minimally. This stands in marked contrast to the elongating flow of the previous section. In the latter case, material points of the filament each aligned with an integral curve corresponding to a different periodic orbit. In the present case, however, each material point of the filament converges to an isolated periodic orbit; namely, the circular limit cycle of unity radius.

Another interesting characteristic of this flow is captured in figure 4.5. The filament is initially aligned in the positive γ^1 direction. While attempting to align with the orientation field, the filament collapses. While ostensibly undesirable, this behavior is natural since there is no provision in the proposed model to bound the spacial curvature of the filament. Other models which attempt to implement a spacial curvature penalty are currently being studied.

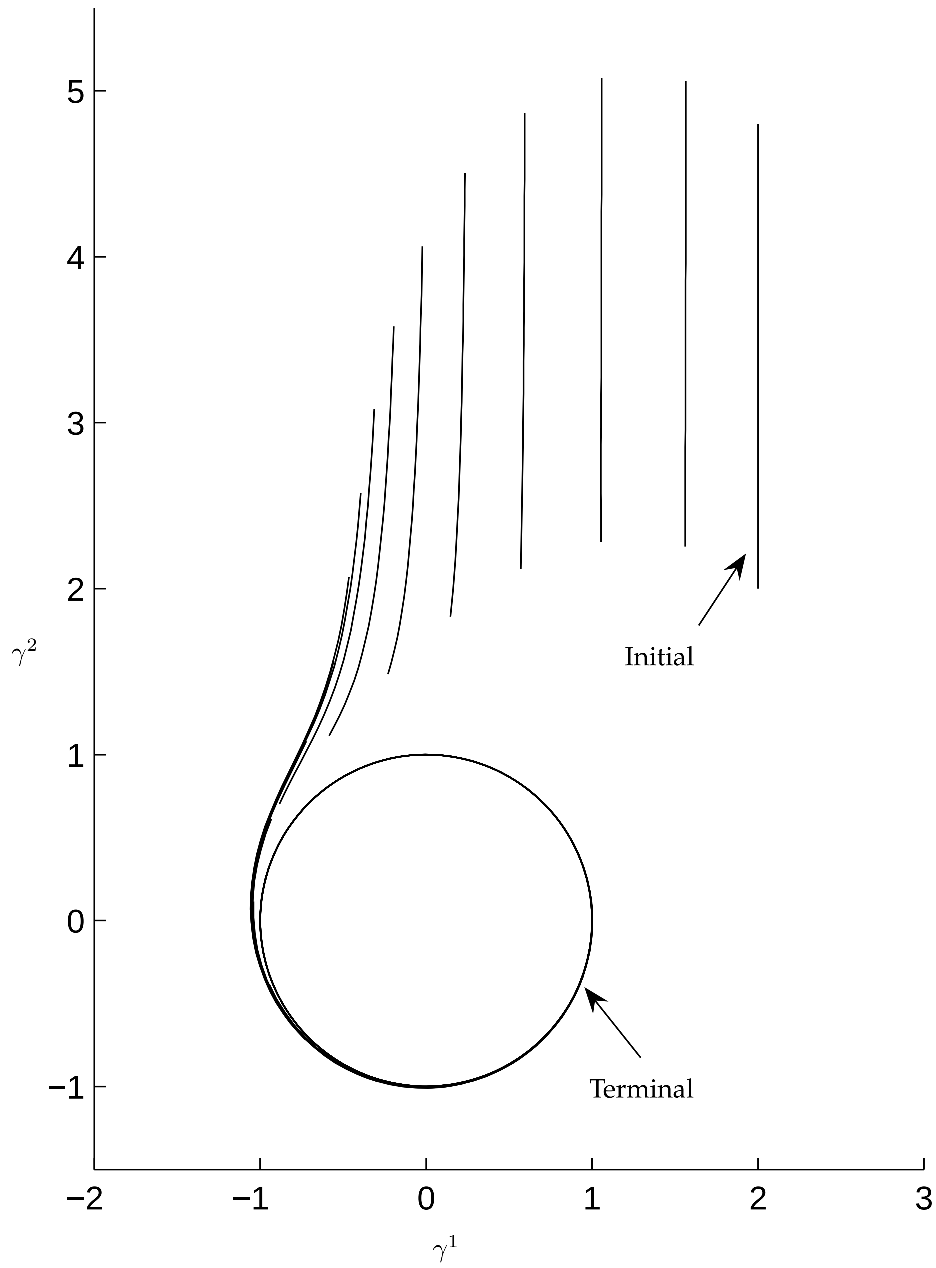

Figure 4.4: An Aligning Filament

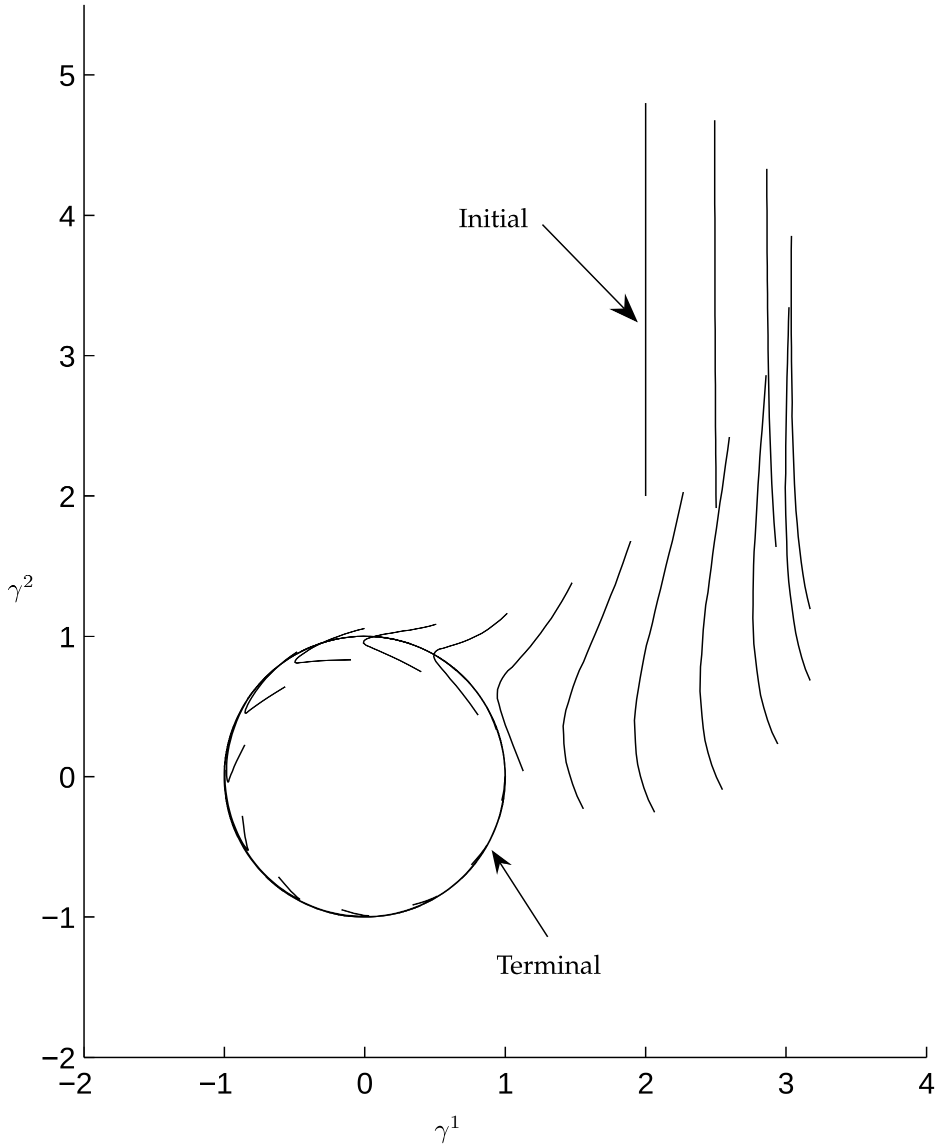

Figure 4.5: A Collapsing Filament

Chapter 5

Analysis and Future Directions

We began at the outset of this work with an interest in constructing a continuum theory of formations. In the proceeding chapters we have developed an infinite dimensional theory of Lagrangian mechanics for a broad class of filament models. The exploitation of intrinsic filament geometry achieved in this work leads naturally to higher dimensional models currently being explored. We have shown that a continuum perspective is a viable tool for studying formations. Furthermore, the concept of a virtual filament has served as a useful abstraction in characterizing the Lagrangian evolution of a formation.

The proposed virtual filament model has led to a prescription of the temporal curvature for each material point of the filament. The local nature of this feedback is a signature of the continuum approach. We have argued for the naturality of this control by noting that it leaves oriented integral curve orbits of the orientation field invariant. While this particular feedback is offered primarily as an argument for the viability of the proposed approach, this invariance property is an essential characteristic for any filament controller. In fact, this property may serve as a useful organizing principle for future models.

There are a number of notable extensions which emerge naturally from this work. One interesting idea is to study the form of the Lagrangian force on the trivialization of the filament tangent bundle in the context of higher dimensional continua. Another interesting extension currently being studied is a lagrangian model which considers only inextensible filaments. In this case, the constant speed and inextensibility conditions enter as holonomic and nonholonomic constraints, respectively. A reasonable Lagrangian density may then retain the current alignment term μ as well as introduce an additional penalty for the spacial curvature of the filament. These ideas constitute a basis for developing a more complete theory of the virtual filament.

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