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Controllability of Multiparameter Singularly Perturbed Systems

by

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Abstract. The controllability of a general linear time-invariant multiparameter singularly perturbed system is studied with no *a priori* assumptions on the relative magnitudes of the small parameters. It is shown how Kokotovic and Haddad's result on persistence of controllability under singular perturbations in the single parameter case extends to this more general setting. The separation of the system eigenvalues into 'slow' and 'fast' groups is proved here for the first time and employed in the analysis. It is found that one does not expect controllability for all sufficiently small values of the parameters, but conditions are given for this property to hold for almost all sufficiently small values of the small parameters. Moreover, one can describe the set in parameter space for which the system is not controllable.

Keywords: Singular perturbation. Controllability. Linear systems. Asymptotic Analysis. Multiparameter singular perturbation.

INTRODUCTION

The characterization of controllability of linear time-invariant singularly perturbed systems involving one singular perturbation parameter has been studied by Kokotovic and Haddad (1975) and extended by Chow (1977). These systems often take the form

$$\dot{x} = A_{11}x + A_{12}y + B_1u \tag{1a}$$

$$\epsilon\dot{y} = A_{21}x + A_{22}y + B_2u. \tag{1b}$$

Here $x \in R^n$, $y \in R^m$, $u \in R^{n_1}$ is the control, the A_{ij}, B_i are matrices of conformable dimensions, and the dot denotes differentiation with respect to time t. The basic result (Kokotovic and Haddad, 1975) asserts that Eq. (1) is controllable for all sufficiently small positive ϵ if (i) the associated reduced system obtained by formally setting $\epsilon = 0$ in (1) is controllable, and (ii) the boundary-layer system of (1), obtained by making a change in time-scale $\tau = t/\epsilon$ in (1) and then setting $\epsilon = 0$, $x = 0$, is controllable. This can be summarized as follows.

Lemma 1. Assume A_{22} is nonsingular and let the pairs (A_{22}, B_2), $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ be controllable, then there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the system (1) is controllable.

Recently there has been considerable interest in the analysis of singularly perturbed systems involving several independent small parameters $\epsilon_1, \ldots, \epsilon_M$. These so-called *multiparameter singular perturbation problems* facilitate a more realistic approach to the modeling of unknown parasitics. Denoting by ϵ the vector $(\epsilon_1, \ldots, \epsilon_M)$ of small parameters, these systems may be displayed compactly as follows (Khalil and Kokotovic, 1979; Khalil, 1979; Abed, 1985; Abed, 1986a)

$$\dot{x} = A_{11}x + A_{12}y + B_1u \tag{2a}$$

$$E(\epsilon)\dot{y} = A_{21}x + A_{22}y + B_2u \tag{2b}$$

where $E(\epsilon) :=$ block diag $(\epsilon_1 I_{m_1}, \ldots, \epsilon_M I_{m_M})$ with each parameter ϵ_i positive. Here, I_k denotes the $k \times k$ identity matrix. Results on the asymptotic stability of Eq. (2) (with $u \equiv 0$) have been obtained by Khalil and Kokotovic (1979) for the (two time-scale) case in which the mutual ratios ϵ_i/ϵ_j are assumed bounded away from zero and infinity by fixed constants. Abed (1986a, 1986b) has recently obtained stability results for Eq. (2) which apply regardless of the mutual ratios ϵ_i/ϵ_j.

The purpose of this paper is to obtain results analogous to Lemma 1 for the controllability of the multiparameter singularly perturbed system (2). As a complementary result, interesting in its own right, a formal proof of the separation of fast and slow modes of (2) is given for the first time. The development of the paper is as follows. The next section contains relevant background material. In the third section a transformation of variables is employed to separate fast and slow dynamics, and a matrix eigenvalue inequality is used to prove that the modes of (2) separate into distinct fast and slow groups for small $|\epsilon|$. In the fourth section, the controllability of (2) is investigated, with special attention to the dependence of the controllability of (2) on the small parameters ϵ_i. The final section contains a simple example illustrating some of the ideas.

BACKGROUND AND PRELIMINARIES

As this study is motivated by the considerations of various authors, especially Kokotovic and Haddad (1975) and Chow (1977), on the preservation of controllability of single-parameter singularly perturbed systems, it seems appropriate to recall the basic concepts and results for the single-parameter case. Lemma 1 above first appeared in (Kokotovic and Haddad, 1975) and provides a useful sufficient condition for the controllability of (1) for all sufficiently small values of the singular perturbation parameter ϵ. The results of Chow

(1977) nicely complement Lemma 1. For instance, it is natural to ask whether or not the converse of Lemma 1 holds. Chow (1977) showed by example that the converse does *not* hold. That is, it is possible that (1) is controllable for all sufficiently small values of the parameter ϵ and yet the associated fast and slow subsystems are not (both) controllable. This observation led Chow (1977) to introduce the following definitions.

Definition 1. System (1) is said to be *weakly controllable* if it is controllable for all sufficiently small and positive values of the parameter ϵ, but loses its controllability in the limit $\epsilon \to 0$.

Definition 2. System (1) is said to be *strongly controllable* if it is controllable for all sufficiently small values of the parameter ϵ, as well as in the limit $\epsilon \to 0$.

In studying the implications of weak and strong controllability, Chow (1977) noted that the feedback gain needed to place the weakly controllable eigenvalues of a singularly perturbed system (1) approaches infinity as $\epsilon \to 0$ with order $1/\epsilon$ or higher. Hence weak controllability is not desirable in practice. As a necessary and sufficient condition for strong controllability, Chow (1977) gave the following result.

Lemma 2. Assume A_{22} is nonsingular. Then (1) is strongly controllable near $\epsilon = 0$ if and only if the pairs (A_{22}, B_2), $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ are controllable.

The next assumption will hold throughout the remainder of the paper.

Assumption 1. The matrix A_{22} is nonsingular.

DECOUPLING OF FAST AND SLOW DYNAMICS

The system equations (2) can be rewritten as

$$\dot{x} = A_{11}x + A_{12}y + B_1u \tag{3a}$$

$$\dot{y} = E^{-1}(\epsilon)A_{21}x + E^{-1}(\epsilon)A_{22}y + E^{-1}(\epsilon)B_2u. \tag{3b}$$

The eigenvalues of this system can be grouped into two distinct sets, one set corresponding to the 'slow subsystem' and the other to the 'fast subsystem.' Separation of these modes enables one to obtain two decoupled subsystems. In order to accomplish this decomposition it is useful to block-diagonalize the Jacobian matrix

$$J(\epsilon) = \begin{pmatrix} A_{11} & A_{12} \\ E^{-1}(\epsilon)A_{21} & E^{-1}(\epsilon)A_{22} \end{pmatrix}. \tag{4}$$

Consider the similarity transformation (Chang, 1972)

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} I & -M(\epsilon) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -L(\epsilon) & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$=: T(\epsilon)\begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

where M and L are continuous matrix functions of ϵ specified next. The matrix L is calculated so that

$$\begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} J(\epsilon) \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} + A_{12}L & A_{12} \\ E^{-1}g(L, \epsilon) & -LA_{12} + E^{-1}A_{22} \end{pmatrix} \tag{6}$$

is block upper triangular, where the notation $E = E(\epsilon)$ has been introduced, and $g(L, \epsilon)$ is defined as

$$g(L, \epsilon) = A_{21} + A_{22}L - ELA_{11} - ELA_{12}L. \tag{7}$$

That is, one requires L to satisfy

$$g(L, \epsilon) = 0. \tag{8}$$

Note that $g(-A_{22}^{-1}A_{21}, 0) = 0$ and that the Fréchet derivative

$$\frac{\partial g}{\partial L} = A_{22}$$

is nonsingular by Assumption 1. The Implicit Function Theorem therefore implies the existence of a continuous matrix function $L = L(\epsilon)$ with $L(0) = -A_{22}^{-1}A_{21}$ and $g(L(\epsilon), \epsilon) = 0$ in a neighborhood of this solution and of $\epsilon = 0$.

Likewise, M is chosen such that

$$\begin{pmatrix} I & -M \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} + A_{12}L & A_{12} \\ 0 & -LA_{12} + E^{-1}A_{22} \end{pmatrix} \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} + A_{12}L & h(M, \epsilon)E^{-1}A_{22} \\ 0 & -LA_{12} + E^{-1}A_{22} \end{pmatrix} \tag{9}$$

is block diagonal, where $h(M, \epsilon)$ is defined as

$$h(M, \epsilon) := A_{11}MA_{22}^{-1}E + A_{12}LMA_{22}^{-1}E$$

$$+ A_{12}A_{22}^{-1}E + MLA_{12}A_{22}^{-1}E \quad M. \tag{10}$$

Therefore M should satisfy

$$h(M, \epsilon) = 0. \tag{11}$$

Now $h(0,0) = 0$ and the Fréchet derivative

$$\frac{\partial h}{\partial M} = -I$$

is nonsingular. Hence there exists a continuous matrix function $M(\epsilon)$ such that $M(0) = 0$ and $h(M(\epsilon), \epsilon) = 0$ in some neighborhood of $\epsilon = 0$.

From these considerations it follows that $T(\epsilon)$ of Eq. (5) satisfies

$$T(\epsilon)J(\epsilon)T^{-1}(\epsilon)$$

$$= \begin{pmatrix} A_{11} + A_{12}L & 0 \\ 0 & -LA_{12} + E^{-1}A_{22} \end{pmatrix} \tag{12}$$

with

$$T(0) = \begin{pmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{pmatrix}. \tag{13}$$

The change of coordinates (5) therefore yields the system

$$\dot{\xi} = (A_{11} + A_{12}L)\xi + B_1^*(\epsilon)u \tag{14a}$$

$$E(\epsilon)\dot{\eta} = (A_{22} - ELA_{12})\eta + B_2^*(\epsilon)u \tag{14b}$$

where $B_1^*(\epsilon)$, $B_2^*(\epsilon)$ are defined as

$$B_1^*(\epsilon) = B_1 + MLB_1 - A_{11}MA_{22}^{-1}B_2 \tag{15a}$$

$$B_2^*(\epsilon) = ELB_1 + B_2. \tag{15b}$$

This follows from (11) and (13). Note that $B_1^*(0) = B_1 - A_{12}A_{22}^{-1}B_2$ and $B_2^*(0) = B_2$.

In (14), the first equation corresponds to the slow subsystem and the second to the fast subsystem. Note that the eigenvalues of the slow subsystem remain bounded as $\epsilon_i \to 0$, $i = 1,...,M$, indeed approach those of $A_{11} - A_{12}A_{22}^{-1}A_{21}$. The remainder of this section is devoted to proving that the eigenvalues of the fast subsystem approach ∞ in magnitude as $\|\epsilon\| \to 0$. This, then, justifies the terminology "fast subsystem" and "slow subsystem." Note that the result holds as the small parameters ϵ_i approach 0 without regard to their relative magnitudes. The exact result proved next is summarized in the following theorem.

Theorem 1. The magnitude of each of the eigenvalues of $E^{-1}(\epsilon)(A_{22} - E(\epsilon)L(\epsilon)A_{12})$ approaches ∞ as $\|\epsilon\| \to 0$, while the eigenvalues of $A_{11} + A_{12}L(\epsilon)$ remain bounded as $\|\epsilon\| \to 0$.

Proof: From elementary linear algebra, it follows that the magnitude of each eigenvalue of a matrix X is bounded below by $\|X^{-1}\|^{-1}$. This holds for any induced matrix norm. Applying this to the matrix $E^{-1}(A_{22} - ELA_{12})$ gives, using the Schwarz inequality:

$$\lambda_m(E^{-1}(A_{22} - ELA_{12})) \geq \|(A_{22} - ELA_{12})^{-1}E\|^{-1} \quad (16)$$

$$\geq \|(A_{22} - ELA_{12})^{-1}\|^{-1} \|E(\epsilon)\|^{-1}, \quad (17)$$

where $\lambda_m(X)$ denotes the eigenvalue of a square matrix X of least magnitude. Choosing a sufficiently small neighborhood of $\epsilon = 0$, say $|\epsilon_i| < \epsilon_0$, $i = 1,...,M$, such that

$$\|(A_{22} - E(\epsilon)L(\epsilon)A_{12})^{-1}\|^{-1} > \frac{1}{2}\|A_{22}^{-1}\|^{-1}$$

for all $|\epsilon_i| < \epsilon_0$. It follows that in that neighborhood of $\epsilon = 0$,

$$\lambda_m(E^{-1}(\epsilon)(A_{22} - E(\epsilon)L(\epsilon)A_{12})) > \frac{K}{2}\|A_{22}^{-1}\|^{-1}\frac{1}{\epsilon_0} \quad (18)$$

for some K which can be chosen independent of ϵ_0. The theorem statement follows.

CONTROLLABILITY

Controllability of the overall system (14) follows from the controllability of the slow and fast subsystems (14a) and (14b), respectively, as long as the eigenvalues of these subsystems never coincide. This follows by an application of the Popov-Belevitch-Hautus (PBH) rank test (see (Kailath, 1980)) for controllability of linear time-invariant systems to (14). (Recall that this test simply says that the pair (A, B) is controllable iff the matrix $(\lambda I - A, B)$ is full rank for all eigenvalues λ of A.) The next result guarantees that the eigenvalues of the slow and fast subsystem do not coincide, for all sufficiently small $|\epsilon|$. It follows directly from Theorem 1.

Corollary 1. None of the eigenvalues of $(A_{11} + A_{12}L)$ and $E^{-1}(A_{22} - ELA_{12})$ coincide, for all sufficiently small $|\epsilon|$.

It is now possible to conclude controllability of the overall system from that of the (decoupled) slow and fast subsystems (14a), (14b), respectively. The slow subsystem (14a) is controllable for ϵ in some neighborhood of 0 if $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1)$ is a controllable pair. This follows easily from continuity of Eq. (14a) in ϵ and the fact that controllability is a robust property of linear time-invariant systems. The fast subsystem (14b) is controllable if the pair

$$(E^{-1}(\epsilon)A_{22} - L(\epsilon)A_{12}, E^{-1}(\epsilon)B_2^*(\epsilon)) \quad (19)$$

is controllable. To guarantee that this will be the case for all sufficiently small $|\epsilon|$, one seeks conditions such that the pair

$$(E^{-1}(\epsilon)A_{22}, E^{-1}(\epsilon)B_2). \quad (20)$$

is controllable for all sufficiently small $|\epsilon|$, *and moreover that this remains true under slight perturbations of the matrices A_{22} and B_2.* This is because the pair (19) arises from (20) by an $O(|\epsilon|)$ perturbation of the matrices A_{22}, B_2:

$$(E^{-1}(A_{22} - ELA_{12}), E^{-1}(B_2 + ELB_1)). \quad (21)$$

At this stage it is useful to introduce a definition characterizing the type of controllability required for the pair in Eq. (20) as it relates to A_{22} and B_2. The definitions below are motivated by the notion of "strong D-stability" introduced by Abed (1986a).

Definition 3. The pair (A, B) of matrices is said to be *D-controllable* if for any diagonal matrix D with strictly positive diagonal elements, (DA, DB) is a controllable pair. Moreover, if the diagonal matrix D is restricted to be of the form

$$D = \text{block diag}\,(d_1 I_{m_1}, d_2 I_{m_2}, \ldots, d_M I_{m_M})$$

with $d_i > 0$, $i = 1,...,M$, then (A, B) is said to be *block D-controllable (with respect to the multiindex $(m_1, \ldots, m_M)$).*

Definition 4. The pair (A, B) of matrices is *strongly D-controllable* if it is D-controllable and if it remains so under arbitrary small perturbations of the matrices A, B. The analogous notion of *strong block D-controllability* is similarly defined.

Note the difference in the meanings of Chow's (1977) "strong controllability" (cf. Definition 2 above) and the foregoing notion of "strong D-controllability." The former concerns the influence on controllability of changes in the singular perturbation parameters, while the latter deals exclusively with perturbations in the *plant model.*

The following result follows from the preceding discussion.

Theorem 2. Assume A_{22} is nonsingular. If the pair $(A_{11} - A_{12}A_{22}^{-1}A_{21}, B_1 - A_{12}A_{22}^{-1}B_2)$ is controllable and the pair (A_{22}, B_2) is strongly block D-controllable (with respect to the multiindex $(m_1, m_2, \ldots, m_M)$), then there is an $\epsilon_0 > 0$ such that for $0 < |\epsilon| < \epsilon_0$ the system (2) is controllable.

It will become apparent toward the end of this section that typically the hypothesis of strong block D-controllability of (A_{22}, B_2) will not be satisfied. (Assuming, of course, that there are fewer inputs than state variables.) Indeed, even the lesser requirement of block D-controllability is rarely satisfied. Hence, Theorem 2 above is mainly of academic interest. Proposition 1 below gives a partial resolution of this issue.

To link Theorem 2 with Chow's (1977) result on strong controllability in the single parameter case, the following Conjecture is offered.

Conjecture 1. The assumptions of Theorem 2 are necessary and sufficient for the strong controllability of Eq. (2), where "strong controllability" is in the sense of Definition 2 (Chow, 1977) with ϵ replaced by $|\epsilon|$.

A similarity transformation can be used to show that the type of controllability for the pair in Eq. (20) referred to above (i.e., strong block D-controllability) is equivalent to controllability of the pair

$$(A_{22}E^{-1}(\epsilon), B_2) \quad (22)$$

for all small $|\epsilon|$ and robustness of this property to small perturbations in A_{22} and B_2.

The controllability matrix for this last pair (i.e., Eq. (22)) is

$$(B_2, A_{22}E^{-1}, \ldots, (A_{22}E^{-1})^{m-1}B_2). \quad (23)$$

Recall that A_{22} is $m \times m$ and B_2 is $m \times n_r$. Choose all possible combinations of m columns from the matrix in Eq. (23), and construct the corresponding finite set of $m \times m$ matrices. The determinants of these matrices are polynomials in the reciprocals of the parameters ϵ_i. Whether or not these polynomials are all zero at a point ϵ determines whether or not the pair (23) is controllable. Replace these polynomials in the reciprocals of the ϵ_i by polynomials in ϵ_i by multiplying each by the necessary powers of the ϵ_i. In this way, arrive at a finite set of polynomials

$$\psi_j(\epsilon), \; j = 1,...,J$$

such that (23) is not of full rank for some $\epsilon = \epsilon^*$ precisely when

$$\psi_j(\epsilon^*) = 0\,, \quad j = 1,\ldots,J.$$

The study of the zeros of a set of polynomials in several variables is a topic of algebraic geometry (Hartshorne, 1977). Without entering into this difficult topic it is none the less pertinent to note here that if any of the polynomials $\psi_j(\epsilon^*)$ generated from the controllability matrix (23) is nontrivial then the pair $(A_{22}E^{-1}(\epsilon), B_2)$ is controllable for almost all values of ϵ. Typically, if at some point $\epsilon^* = (\epsilon_1^*, \ldots, \epsilon_M^*)$ one has $\psi_j(\epsilon^*) = 0\,,\ j = 1,\ldots,J$ but

$$\frac{\partial \psi_j}{\partial \epsilon}\Big|_{\epsilon^*} = \Big(\frac{\partial \psi_j}{\partial \epsilon_1}\Big|_{\epsilon^*}, \ldots, \frac{\partial \psi_j}{\partial \epsilon_M}\Big|_{\epsilon^*}\Big) \neq 0$$

then $\psi_j^{-1}(0)$, the preimage of 0, is locally near $\epsilon = \epsilon^*$ a submanifold of dimension $m - 1$ (Guillemin and Pollack, 1974). One says that $\psi_j(\epsilon)$ is a submersion from R^M to R near ϵ^*. It will become apparent below that typically $\psi_j(\epsilon)$ is not a submersion at $\epsilon = 0$. The following claim can, however, be made.

Proposition 1. If not all the $\psi_j(\epsilon)$ are trivial then locally near $\epsilon = 0$, the pair $(A_{22}E^{-1}(\epsilon), B_2)$ is controllable except possibly on the union of finitely many submanifolds of dimension $M - 1$. Moreover, this conclusion will also hold for the pair $(E^{-1}A_{22} - LA_{12}, E^{-1}B_2^*)$.

Proof: First note that the last statement in the Proposition follows from the first, since the assumption on the polynomials ψ_j is robust to small perturbations in A_{22} and B_2. Now suppose $\psi_j(\epsilon)$ is nontrivial. Note that ψ_j is a homogeneous polynomial, i.e. $\psi_j(\epsilon) = 0$ implies $\psi_j(\lambda\epsilon) = 0$ for all $\lambda \in R$. Denote by q the degree of ψ_j. That is, $\psi_j(\epsilon)$ is a linear combination of terms of the form

$$\epsilon_1^{i_1}\epsilon_2^{i_2}\cdots\epsilon_M^{i_M}$$

where $i_1 + i_2 + \cdots + i_M = q$. It is easy to see that $\psi_j(0) = 0$, $\frac{\partial^r}{\partial^r \epsilon}\psi_j(\epsilon)\,|_{\epsilon=0} = 0$ for $r \le q - 1$ and $\partial^q \psi_j =$ constant (i.e. all partial derivatives of order q are constants). Because ψ_j is nontrivial one of these, say

$$\frac{\partial^q \psi_j}{\partial^{i_1}\epsilon_1 \partial^{i_2}\epsilon_2 \cdots \partial^{i_M}\epsilon_M}$$

is nonzero. This implies that any one of its primitives $\frac{\partial^{q-1}}{\partial^{q-1}\epsilon}\psi_j$ is a submersion at $\epsilon = 0$. That is, locally near $\epsilon = 0$, $(\frac{\partial^{q-1}}{\partial^{q-1}\epsilon}\psi_j)^{-1}(0)$ is a submanifold of dimension $M - 1$ which will be denoted M_1. Consider now one of the primitives of the primitive above, which for simplicity is denoted $\partial^{q-2}\psi_j$. Locally near $\epsilon = 0$ the set $(\partial^{q-2}\psi_j)^{-1}(0) - M_1$ is itself a submanifold of dimension $M - 1$. This follows from the fact that for any $\bar\epsilon \in (\partial^{q-2}\psi_j)^{-1}(0) - M_1$, $\partial^{q-2}\psi_j$ is a submersion at $\bar\epsilon$ since its derivative $\partial^{q-1}\psi_j$ is nonzero. Therefore locally near $\epsilon = \bar\epsilon$ $(\partial^{q-2}\psi_j)^{-1}(0) - M_1$ is a submanifold of dimension $M - 1$. As this is true for all ϵ near zero, $M_2 := (\partial^{q-2}\psi_j)^{-1}(0) - M_1$ is itself a submanifold of dimension $M - 1$ so that for ϵ near zero, $\epsilon \notin M_2$ implies $(\partial^{q-2}\psi_j)^{-1}(\epsilon) \neq 0$. Repeating this argument, the set $\partial^{q-3}\psi_j^{-1}(0) - (M_1 \cup M_2) =: M_3$ is a submanifold of dimension $M - 1$ and so on. Finally, one obtains a set $\psi_j^{-1}(0) - (M_1 \cup M_2 \cdots \cup M_{q-1})$ which is itself a submanifold M_q of dimension $M - 1$. Thus $\psi_j(\epsilon) \neq 0$ for ϵ not in the union $M_1 \cup M_2 \cdots \cup M_q$, proving the claim.

AN EXAMPLE

The following example of a system whose fast subsystem is two-dimensional may illuminate some of the foregoing remarks.

Consider a system with $m = 2$, $n_f = 1$. Denote the elements of A_{22} by a_{ij}, $i,j = 1,2$, and those of B_2 by b_1, b_2. Then the controllability matrix (23) is 2×2, and one can check that it is singular precisely when

$$-\epsilon_1(a_{12}b_2^2 + a_{22}b_1b_2) + \epsilon_2(a_{21}b_1^2 - a_{11}b_1b_2) = 0.$$

This is a homogeneous polynomial in two variables of degree one. Note that its zeros are the points of a line (a submanifold of dimension $m-1$) passing through the origin in the (ϵ_1, ϵ_2) plane. The pair (A_{22}, B_2) is strongly D-controllable if the line has negative slope. Otherwise controllability will be assured if this line is avoided, as in Proposition 1.

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