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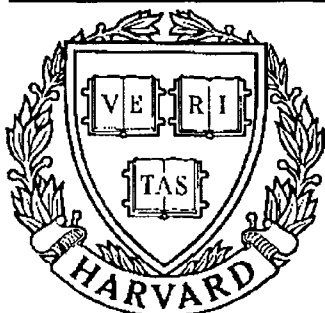
Asymptotic Behavior in Nonlinear Stochastic Filtering

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ASYMPTOTIC BEHAVIOR IN NONLINEAR STOCHASTIC FILTERING

By

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ABSTRACT

A lower and upper bound approach on the optimal mean square error is used to study the asymptotic behavior of one dimensional nonlinear filters. Two aspects are treated: (1) The long time behavior $(t \rightarrow \infty)$. (2) The asymptotic behavior as a small parameter $\epsilon \rightarrow 0$. Lower and upper bounds that satisfy Riccati equations are derived and it is shown that for nonlinear systems with linear limiting systems, the Kalman filter designed for the limiting systems is asymptotically optimal in a reasonnable sense. In the case of nonlinear systems with low measurement noise level, three asymptotically optimal filters are provided one of which is linear. In chapter 4, the stationary behavior of the Benes filter is investigated.

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1 INTRODUCTION; OVERVIEW

The filtering problem for diffusion processes involves the estimation of an unobserved stochastic process $\{x_t\}$ given observations of a related process $\{y_t\}$. The classic formulation involves the computation, for each t, of conditional statistics such as the conditional mean and variance of $\{x_t\}$ given observations in additive white noise.

A typical model is

$$
\begin{aligned}
dx_t &= g(t,x_t)\ dt + \sigma(t)\ dw_t \\
dy_t &= h(t,x_t)\ dt + \rho(t)\ dv_t \\
x(0) &= x_0 \quad ; \quad 0 \le t \le T
\end{aligned}
\tag{1}
$$

where g, h, α and ρ are smooth functions of their arguments, $\{v_t\}$, $\{w_t\}$ are independent Wiener processes, x_0 a random variable independent of $\{v_t\}$, $\{w_t\}$.

Given this model one is interested in computing least squares estimates of functions of the signal x_t given $\sigma\{\,y_s\,,\,0 \le s \le t\,\}$, the σ-algebra generated by the observations, i.e. quantities of the form $E\,[\,\phi(x_t)\ |\ \sigma\{y_s\,,\,0 \le s \le t\,\}\,]$. In many applications this computation must be done recursively. This involves the conditional probability density $p^y(t,x)$ which satisfies a nonlinear stochastic partial differential equation, the Kushner-Stratonovich equation [1].

The filtering problem was completely solved in the context of finite dimensional linear Gaussian systems by Kalman and Bucy [2], [3] in 1960-61, and the resulting Kalman filter (KF) has been widely applied. Apart from a few special cases [4], [5] the nonlinear case is far more complicated; the evolution of the conditional statistics is, in general, an infinite dimensional system.

By factoring $p^y(t,x) = u(t,x)\,/\int u(t,z)dz$, a characterization of p^y may be obtained via the unnormalized conditional probability density $u(t,x)$ which solves the

Duncan-Mortenson-Zakai (DMZ) equation [6], a linear parabolic stochastic PDE driven by the observation process. A further transformation, $u(t,x) = v(t,x) \exp < h(t,x), y_t >$, eliminates the stochastic differentials in this equation resulting in the robust equation (RDMZ) [6] which may be analyzed by classical methods for parabolic PDE's. This is one of the key factors making the DMZ equation a useful basis for the study of nonlinear filtering of diffusion processes.

Although progress has been made using these tools, optimal algorithms are not generally available. Suboptimal filters are thus of interest. The performance of suboptimal designs, however derived, may be based on lower and upper bounds on the minimum mean square error (MMSE) $P(t)$. This approach is used in this thesis to investigate the asymptotic behavior of a class of nonlinear filtering problems.

Two aspects are treated in detail:

(1)   the long time behavior, that is, the asymptotic behavior of the filter as $t \to \infty$.

(2)   the asymptotic behavior as $\epsilon \to 0$, with ϵ a small parameter in the model.

To illustrate the ideas, consider the one-dimensional version of the model above where g and h have continuous bounded derivatives, say

$$\underline{\alpha}(t) \leq g_x(t,x) \leq \overline{\alpha}(t) \tag{2-a}$$

$$\underline{\beta}(t) \leq h_x(t,x) \leq \overline{\beta}(t) \tag{2-b}$$

and let

$$
\begin{aligned}
p(t) &:= E\left[\, x_t - E(x_t \mid \mathcal{Y}_0^{\,t})\,\right]^2 \\
p^*(t) &:= E\left(\, x_t - x_t^* \,\right)^2
\end{aligned}
\tag{3}
$$

where $\mathcal{Y}_0^{\,t} = \sigma\{\, y_s, 0 \leq s \leq t \,\}$ and x_t^* is given by :

$$dx_t^* = g(t,x_t^*)dt + \frac{\beta(t)}{\rho^2(t)} u(t)[dy_t - h(t,x_t^*)dt] \quad ; \quad x^*(0) = 0$$

$$\dot{u}(t) = \sigma^2(t) + 2\overline{\alpha}(t) u(t) - \frac{\beta^2(t)}{\rho^2(t)} u^2(t) \quad ; \quad u(0) = \sigma_0^2 \qquad \text{(BOF)}$$

$$(x_0 \sim N(0,\sigma_0^2) \quad assumed)$$

Clearly the BOF (bound optimal filter) is readily implementable, with precomputable gain. It coincides with the Kalman filter if g and h are linear. In Chapter 2 it is shown by applying results from [9], [10] that the BOF is a " best bound " filter in the sense that the associated upper bound $u(t)$ of $p^*(t)$ is the tightest over a class of nonlinear Kalman-like filters and that $p(t)$ is bounded as follow:

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t)$$

where $l(t)$ satisfies another Riccati equation.

In Chapter 3 these bounds are used to address the long time behavior of asymptotically time invariant systems. In the particular case where

$$g(t,x) = a\ x + \lambda(t)\ f(t,x) \quad \text{------------>}_{t\to\infty} \quad a\ x$$

and

$$h(t,x) = c\ x + \nu(t)\ k(t,x) \quad \text{------------>}_{t\to\infty} \quad c\ x$$

it is shown that the BOF is asymptotically optimal, i.e., $\lim\limits_{t\to\infty} (p^*(t) - p(t)) = 0$

and that as far as the long time performance is concerned, the nonlinearities f and k can be ignored in the original model. In other words the " KF " and even the " SSKF " (steady state) designed for the underlying linear system are asymptotically optimal.

In chapter 3, we consider the stationary behavior of the Benes filter [4]. Benes proved that if

$$dx_t = f(x_t)\ dt + dw_t \quad , \quad x_0 = x$$
$$dy_t = x_t\ dt + dv_t$$

with

$$f_x(x) + f^2(x) = ax^2 + bx + c \quad ; \quad a \geq 0$$

then the filtering problem is completely specified by two sufficient statistics, in that the solution to the DMZ equation is given by:

$$u(t,x) = \exp\left\{ \int_0^x f(z)dz - \frac{1}{2} \frac{(x - \mu_t)^2}{\sigma(t)} \right\}$$

where

$$\dot{\sigma}(t) = 1 - k^2 \sigma^2(t) \quad ; \quad \sigma(0) = 0$$

$$d\mu_t = -k^2 \sigma(t) \mu_t \, dt - \frac{1}{2} b \, \sigma(t) \, dt + \sigma(t) \, dy_t$$

$$\mu_0 = x \quad ; \quad k = (1 + a)^{1/2}$$

We then note that $\lim_{t \to \infty} \sigma(t) = \frac{1}{k} := \bar{\sigma}$ and define

$$\bar{u}(t,x) = \exp\left\{ \int_0^x f(z)dz - \frac{1}{2} \frac{(x - \bar{\mu}_t)^2}{\bar{\sigma}} \right\}$$

where

$$d\bar{\mu}_t = -k^2 \bar{\sigma} \bar{\mu}_t \, dt - \frac{1}{2} b \, \bar{\sigma} \, dt + \bar{\sigma} \, dy_t \quad ; \quad \bar{\mu}_0 = x$$

as a natural limiting function of the unnormalized density $u(t,x)$ and we prove that

$\lim_{t \to \infty} E (\mu_t - \bar{\mu}_t)^2 = 0$. (Note that this is not a complete characterization of the asymptotic behavior of the Benes filtering problem).

The bound approach combined with perturbation methods is used in Chapter 5 to treat two types of perturbed systems; namely, weakly nonlinear systems [12], i.e., those for which

$$g(t,x) = a(t) x + \epsilon f(t,x) \quad ; \quad h(t,x) = c(t) x$$

where ϵ is a small positive parameter; and systems with low measurement noise level [13], [14], [15] that is of the form

$$dx_t = g(t,x_t) \, dt + \sigma(t) \, dw_t$$

$$dy_t = c(t) x_t \, dt + \epsilon \, dv_t$$

For the former type it is shown that the BOF is asymptotically optimal in the sense that the corresponding MMSE $p^*(t,\epsilon)$ is identical to $p(t,\epsilon)$ up to $O(\epsilon)$; and that the " KF " designed for the underlying linear system (i.e., ignoring ϵf) is asymptotically optimal as $\epsilon \to 0$. In the case of low measurement noise level, three asymptotically optimal algorithms are provided, one of which has the very simple form

$$dx_t^L = \frac{\sigma(t)}{\epsilon} \{ dy_t - c(t) x_t^L \, dt \} \quad ; \quad x_0^L = 0$$

which does not involve the drift g .

2   LOWER AND UPPER BOUNDS ON THE A PRIORI MMSE

Since the explicit solution of nonlinear filtering problems is impossible in general, one is naturally interested in suboptimal solutions, the performance of which may be avaluated using upper and lower bounds on the (unknown) MMSE.

In fact, the structural complexity which arises is also present at the level of performance testing in the sense that simple and tractable bounds are not generally available for suboptimal estimators unless one puts further restrictions on the type of nonlinearities considered.

Consider the n dimensional *Itô* stochastic differential equation

$$
\begin{aligned}
dx_t &= g\,(\,t\,,x_t\,)\ dt\ +\ B\,(\,t\,)\ dw_t \quad ,\quad t \geq 0 \\
dy_t &= h\,(\,t\,,x_t\,)\ dt\ +\ D\,(\,t\,)\ dv_t \\
x_0 &\sim p_0(x) \quad ,\quad E\ x_0\ =\ 0 \quad ,\quad E\ x_0 x_0^T\ =\ \sigma_0^2
\end{aligned}
\tag{1}
$$

where $\{w_t\}$ and $\{v_t\}$ are independent standard wiener processes, $x_0 = x(0)$ is a random variable (generally taken to be Gaussian) independent of $\{w_t\}$ and $\{v_t\}$; g and h are such (1) has a unique solution. Given this model one is interested in finding bounds on the MMSE :

$$
P(t) = E\ [\ (\ x_t\ -E\,(x_t\ \mid\ \mathcal{Y}_0^t\,)\,)\ (\ x_t\ -E\,(x_t\ \mid\ \mathcal{Y}_0^t\,)\,)^T\ ]
\tag{2}
$$

where $\mathcal{Y}_0^t\ =\ \sigma\,\{\ y_s\ ,0 \leq s\ \leq t\ \}$ is the σ - algebra generated by the observations up to time t, i.e., find matrices $L\,(t)$, $U(t)$ such that :

$$
0\ \leq\ L(t)\ \leq\ P(t)\ \leq\ U(t)
\tag{3}
$$

where the matrix inequality $X \geq Y$ should be understood as $(X-Y)$ is positive semi-definite.

In this chapter, existing results are applied to one dimensional systems for which the nonlinearities have bounded derivatives to obtain lower and upper bounds involving ordinary differential equations of the Riccati type. The upper bound is derived in section 2-2 by considering a class of nonlinear, Kalman-like suboptimal filters.

To each such filter is associated an upper bound on the corresponding MSE and the BOF (bound optimal filter) is defined as the one with the tightest upper bound. The latter is used in (3).

2-1 Lower bound :

We consider (1) with the same assumptions stated there. In addition it is assumed that :

(1)   g and h are differentiable with continuous, bounded partial derivatives

(2) $B(t)$ and $D(t)$ are non singular

(3) $p_0(x)$ satisfies the assumptions of Theorem (4) in [9] (e.g., (3) holds if x_0 is Gaussian)

Associated with (1) is the following linear system :

$$
\begin{aligned}
dz_t &= A(t)\, z_t\; dt\; + B(t)\, dw_t \\
dy_t &= C(t)\, z_t\; dt\; + D(t)\, dv_t \\
z_0 &\sim N(0, \sigma_0^2)
\end{aligned}
\tag{4}
$$

where :

$$
A(t) = E\; g_x(t,x_t)
\tag{5}
$$

$$
C^T(t)[D(t)D^T(t)]^{-1}C(t) =
$$

$$
E\left[(g_x(t,x_t) - A(t))^T (B_t^T B_t)^{-1}(g_x(t,x_t)-A(t)) + h_x^T(t,x_t)(D_t^T D_t)^{-1}h_x(t,x_t) \right]
\tag{6}
$$

Let

$$P(t) = E\ [\ (x_t - E(x_t \mid \overset{t}{\mathcal{y}_0}))\ (x_t - E(x_t \mid \overset{t}{\mathcal{y}_0}))^T\ ] \tag{7}$$

and

$$L(t) = E\ [\ (z_t - E(z_t \mid \overset{t}{\widehat{\mathcal{y}}_0}))\ (z_t - E(z_t \mid \overset{t}{\widehat{\mathcal{y}}_0}))^T\ ] \tag{8}$$

then the following result, due to B. Z. Bobrovsky and M. Zakai [9], holds

Theorem 2-1-1 : The MMSE in the original problem (1) is lower bounded by the MMSE corresponding to the associated linear one (4), i.e.,

$$0 \leq L(t) \leq P(t)$$

* * *

Note : Recall that (e.g., [16])

$$\dot{L}(t) = B(t)B^T(t) + A(t)L(t) + L(t)A^T(t) - L(t)C^T(t)(D(t)D^T(t))^{-1}C(t)L(t)$$
$$L(0) = \sigma_0^2 \qquad (\ \in \mathrm{R}^{n x n}\ ) \tag{10}$$

furthermore $\quad \hat{z}_t := E\ (\ z_t \mid \overset{t}{\widehat{\mathcal{y}}_0}\ )\quad$ is given by :

$$d\hat{z}_t = A(t)\,\hat{z}_t\,dt + G(t)\,[\ d y_t - C(t)\,\hat{z}_t\,dt\ ]\quad,\quad \hat{z}_0 = 0$$
$$G(t) = L(t)\,C(t)\,(\ D(t)\,D^T(t)\ )^{-1} \tag{11}$$

which is the Kalman filter (KF).

The one dimensional case :

Since the coefficients $A(t)$ and $C(t)$ are in general unknown, the lower bound $L(t)$ cannot be computed exactly for a general n dimensional system. The following additional assumptions make it possible to derive a simple, tractable lower bound in the one dimensional case :

$$H_1: \qquad |\ g_x(t,x) - \alpha(t)\ | \leq \Delta\alpha(t)$$

$$H_2 : \qquad |\ h_x(t,x)\ -\ \beta(t)\ |\ \leq\ \Delta\beta(t)\ ,\quad \underline{\beta}(t)\ =\ \beta(t) - \Delta\beta(t)\ \geq\ 0$$

We will denote this by :

$$g\ \in\ <\ [\ \alpha(t)\ ,\ \Delta\alpha(t)\ ]$$

$$h\ \in\ <\ [\ \beta(t)\ ,\ \Delta\beta(t)\ ]$$

Note : the symbol Δ has no particular mathematical meaning here, it is used only to exhibit the fact that $\Delta\alpha$ is a slope departure function.

Let the scalar system be:

$$
\begin{aligned}
dx_t &=\ g\,(t,x_t)\ dt\ +\ \sigma(t)\ dw_t \\
dy_t &=\ h\,(t,x_t)\ dt\ +\ \rho(t)\ dv_t \\
E\ x_0 &=\ 0\ ,\qquad E\ x_0^2\ =\ \sigma_0^2
\end{aligned}
\tag{14}
$$

with similar assumptions as in (1).

Proposition 2-1-2 :

Assume H_1 , H_2 hold and let $p\,(t) := E\ (\ x_t\ -\ E\,(x_t\ |\ \mathcal{Y}_0^t\,)\,)^{\,2}$; then $p\,(t)$ is lower bounded by $l\,(t)$, i.e., $0\ \leq\ l\,(t)\ \leq\ \hat{p}\,(t)$ where $l\,(t)$ satisfies the following Riccati equation :

$$
\begin{aligned}
\dot{l}\,(t) &=\ \sigma^2(t)\ +\ 2\underline{\alpha}(t)\,l\,(t)\ -\ \frac{1}{\rho^2(t)}\ [\ \overline{\beta}^2(t)\ +\ 4\frac{\rho^2(t)}{\sigma^2(t)}\ (\Delta\alpha(t))^2\ ]\ l^2(t) \\
l\,(0) &=\ \sigma_0^2
\end{aligned}
\tag{15}
$$

with the notation : $\quad \overline{\alpha} = \alpha + \Delta\alpha\quad ,\quad \underline{\alpha} = \alpha - \Delta\alpha$

$$* \ * \ *$$

Proof :

From Theorem 2-1-1 we have $\quad L\,(t)\ \leq\ p\,(t)\quad$ where

$$\dot{L}(t) = \sigma^2(t) + 2a(t)L(t) - \frac{c^2(t)}{\rho^2(t)} L^2(t)$$

$$L(0) = \sigma_0^2$$

$$a = E \ g_x(t, x_t)$$

$$c^2(t) = E \ h_x^2(t, x_t) + \frac{\rho^2(t)}{\sigma^2(t)} \ var(g_x(t, x_t))$$

Clearly H_1 implies : $\quad \underline{\alpha}(t) \leq g_x \leq \overline{\alpha}(t) \quad$ a.s., and hence , $\underline{\alpha}(t) \leq a(t) \leq \overline{\alpha}(t)$.

Thus

$$| \ g_x(t, x_t) - a(t) \ | \ \leq \ 2 \ \Delta\alpha(t) \qquad a.s$$

and

$$var \ g_x(t, x_t) \ \leq \ 4 \ (\Delta\alpha(t))^2$$

Similarly H_2 implies : $\quad 0 \leq \underline{\beta}(t) \leq h_x(t, x_t) \leq \overline{\beta}(t)$

hence $\quad E \ h_x^2(t, x_t) \leq \overline{\beta}^2(t) \quad$.

Therefore :

$$c^2(t) \ \leq \ \overline{\beta}^2(t) + 4 \ \frac{\rho^2(t)}{\sigma^2(t)} \ (\Delta\alpha(t))^2$$

Since by Theorem 3 in the appendix $L(t) > 0$, the right hand side of $\dot{L}(t)$ is greater than

$$\sigma^2(t) + 2\underline{\alpha}(t)L(t) - \frac{1}{\rho^2(t)} \ [\ \overline{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)} \ (\Delta\alpha(t))^2 \ ] \ L^2$$

By the comparison theorem we obtain : $\quad l(t) \leq L(t)$

* * *

2.2 Upper bound and bound optimal filter (BOF)

In [10], A. S. Gilman and I. B. Rhodes derived upper bounds on the minimum mean square error (MMSE) for systems of the form

$$
\begin{aligned}
dx_t &= g(t, x_t)\, dt + B(t)\, dw_t \\
dy_t &= h(t, x_t)\, dt + D(t)\, dv_t \\
E\, x_0 &= 0 \quad, \quad E\, x_0 x_0^T = \sigma_0^2
\end{aligned}
\tag{16}
$$

where g and h are incrementally conic nonlinearities i.e. satisfy the following hypotheses:

There exist matrices $A(t)$, $C(t)$ and positive functions a and c such that

$$
|\ |\ g(t, x+\delta) - g(t, x) - A(t)\delta\ |\ | \ \leq\ a(t)\ |\delta|
\tag{17}
$$

$$
|\ |\ h(t, x+\lambda) - h(t, x) - C(t)\lambda\ |\ | \ \leq\ c(t)\ |\lambda|
\tag{18}
$$

for each t, x, δ and λ.

An outline of this work and its applications to the one dimensional case, for which better bounds can be achieved, is given in the next two sections.

2-2-1 Upper bounds for incrementally conic nonlinearities : [10]

Let us consider (16) with the assumptions (17) and (18) which will be respectively denoted by :

$$
g \in IC\ [\ A(t),\ a(t)\ ]
$$

$$
h \in IC\ [\ C(t),\ c(t)\ ]
$$

The upper bound for such systems was derived in [10] by considering the following class of suboptimal filters which is suggested, in part, by the Kalman filter :

$$
\begin{aligned}
dx_t^K &= g(t, x_t^K)\, dt + K(t)\, [\ dy_t - h(t, x_t^K)\, dt\ ] \\
x_0^K &= 0
\end{aligned}
\tag{19}
$$

where $K(t)$ is a non random, piecewise continuous bounded matrix.

Thus to each gain $K(t)$ is associated a suboptimal filter given by (19), which we denote by $\{x\}_K$. The main results are as follows :

(1) For each $K(.)$ there exist an upper bound matrix U^K such that :

$$P_K(t) := E\left[(x_t - x_t^K)(x_t - x_t^K)^T\right] \leq U^K(t)$$

and U^K satisfies the following linear matrix equation (where the time dependence is omitted) :

$$\begin{aligned}
\dot{U}^K &= BB^T + KDD^T K + (A - KC)U^K + U^K(A - KC)^T \\
&\quad + (a + c)U^K + a\ tr(U^K)I + c\ tr(U^K)KK^T \\
U_K(0) &= \sigma_0^2
\end{aligned} \tag{20}$$

(2) The suboptimal filter $\{x\}_{K^*}$ defined by :

$$K^* = U^* C^T W^{-1} \tag{21}$$

$$\dot{U}^* = V + \tilde{A}U^* + U^*\tilde{A}^T - U^*\tilde{C}^T W^{-1}\tilde{C}U^* \quad ; \quad U^*(0) = \sigma_0^2 \tag{22}$$

$$\tilde{A} = A + \frac{1}{2}(a + c)I \tag{23}$$

$$V = B B^T + a\ tr(U^*) I \tag{24}$$

$$W = D D^T + c\ tr(U^*) I \tag{25}$$

has the smallest upper bound i.e. :

$$P_{K^*}(t) \leq U^{K^*}(t) = U^*(t) \leq U^K(t)$$

for every K

Remarks:

(1) This says that $U^{K^*} = U^*$ is the smallest element of the family of upper bounds defined by (20).

(2) The resulting filter $\{x\}_{K^*}$ given by (19), (21)-(25) has the smallest upper bound and can thus be referred to as a bound optimal filter (BOF) or the BOF relative to the family of upper bounds (20). This point will become clearer in the next section.

(3) We obviously have:

$$P(t) := E\left[(x_t - E(x_t \mid \overset{t}{y}_0))(x_t - E(x_t \mid \overset{t}{y}_0)^T\right] \leq P_{K^*}(t) \leq U^*(t) \tag{26}$$

(4) Application to scalar systems :

Let

$$\begin{aligned}
dx_t &= g(t,x_t)\, dt + \sigma(t)\, dw_t \\
dy_t &= h(t,x_t)\, dt + \rho(t)\, dw_t \\
E\, x_0 &= 0 \quad , \quad E\, x_0^2 = \sigma_0^2
\end{aligned} \tag{27}$$

where $\quad g \in IC\,[\,\alpha(t)\,,\,\Delta\alpha(t)\,]\quad$ and $\quad h \in IC\,[\,\beta(t)\,,\,\Delta\beta(t)\,]$.

Then one easily gets by applying the above results that the smallest element u^* of the family of upper bounds

$$\begin{aligned}
\dot{u}_k &= \sigma^2 + \rho^2 k^2 + 2\,(\,\overline{\alpha} - k\,\underline{\beta}\,)u^k + \Delta\beta\,(\,k-1\,)^2\,u^k \\
u_k(0) &= \sigma_0^2 \quad ; \quad \overline{\alpha} = \alpha + \Delta\alpha \quad ; \quad \underline{\beta} = \beta - \Delta\beta
\end{aligned} \tag{28}$$

is given by

$$\begin{aligned}
\dot{u}^* &= \sigma^2 + 2\,[\,\overline{\alpha} + \Delta\beta\,]\,u^* - \frac{\underline{\beta}^2}{\rho^2 + u^*\,\Delta\beta}\,(u^*)^2 \\
u^*(0) &= \sigma_0^2
\end{aligned} \tag{29}$$

The BOF relative to (28) being

$$\begin{aligned}
dx_t^* &= g(t,x_t^*)\, dt + \frac{\beta(t)\,u^*(t)}{\rho^2(t) + u^*(t)\,\Delta\beta(t)}\,[\,dy_t - h(t,x_t^*)\, dt\,] \\
x_0^* &= 0
\end{aligned} \tag{30}$$

and we have:

$$p(t) := E\,(\,x_t - E(x_t \mid \overset{t}{y}_0)\,)^2 \leq p_{k^*}(t) := E\,(\,x_t - x_t^*\,)^2 \leq u^*(t).$$

It is however possible to derive a simpler and tighter upper bound, hence a better BOF, in the scalar case as is shown in the next section.

2-2-2 Upper bounds for one dimensional systems :

Let x_t and y_t be given by (27) and assume that

$$H_1 \quad : \quad g_x(t,x) \quad \text{is continuous and} \quad g_x(t,x) \leq \overline{\alpha}(t) \tag{31}$$

$$H_2 \quad : \quad h_x(t,x) \quad \text{is continuous and} \quad h_x(t,x) \geq \underline{\beta}(t) \geq 0 \tag{32}$$

Notice that H_1, H_2 hold if f and g are incrementally conic.

Proposition 2-2-1 :

The MMSE $p(t)$ is upper bounded by $u(t)$ where $u(t)$ satisfies the Riccati equation :

$$\dot{u}(t) = \sigma^2(t) + 2\,\overline{\alpha}(t)\,u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}\,u^2(t)$$
$$u(0) = \sigma_0^2 \tag{33}$$

$$* \quad * \quad *$$

Note: This says that the MMSE in the nonlinear filtering problem (27) is upper bounded by the MMSE in the following linear one :

$$\begin{aligned}
dz_t &= \overline{\alpha}(t)\,z_t\,dt + \sigma(t)\,dw_t \\
dy_t &= \underline{\beta}(t)\,z_t\,dt + \rho(t)\,dv_t
\end{aligned} \tag{34}$$

Proof :

The conditional mean $\hat{x}_t := E\left(x_t \mid \mathcal{Y}_0^t\right)$ and the conditional MMSE

$$\hat{p}_t := E\left[\,(x_t - \hat{x}_t)^2 \mid \mathcal{Y}_0^t\,\right] \tag{35}$$

are given by : [1]

$$d\hat{x}_t = \hat{g}(t,x_t)\,dt + \frac{\hat{e}_t}{\rho^2(t)}\,d\overline{w}_t \quad ; \quad \hat{x}_0 = 0 \tag{35}$$

$$d\hat{p}_t = \left[\,\sigma^2(t) + 2(\,(x_t\,g_t)\,\hat{} - \hat{x}_t\,\hat{g}_t\,) - \frac{1}{\rho^2(t)}\,(\hat{e}_t)^2\,\right]\,dt + \frac{T_t}{\rho^2(t)}\,d\overline{w}_t$$
$$\hat{p}_0 = \sigma_0^2 \tag{36}$$

where $\hat{}$ denotes conditional expectation and

$$g_t \;=\; g(t,x_t) \quad ; \quad h_t \;=\; h(t,x_t) \tag{37}$$

$$\hat{e}_t \;=\; (x_t\,h_t)\,^\wedge - \hat{x}_t\,\hat{h}_t$$

$$T_t \;=\; (x_t{}^2 h_t)\,^\wedge - x_t{}^{\wedge 2}\hat{h}_t \;-\; 2\hat{x}_t\,(x_t\,h_t)\,^\wedge + 2(\hat{x}_t)^2\hat{h}_t \tag{38}$$

and $\overline{dw_t} := dy_t - \hat{h}(t,x_t)dt$ is the innovation process which is a wiener process on $y_0^{\,t}$.

Since the expectation of $It\hat{o}$ integrals is zero and $E\,\hat{p}_t = E\,(x_t - \hat{x}_t)^2 = p(t)$, we get by taking the expectation on both sides of (36) that :

$$\dot{p}(t) \;=\; \sigma^2(t) + 2E\big((x_t\,g_t)\,^\wedge - \hat{x}_t\,\hat{g}_t\big) - \frac{E(\,\hat{e}_t\,)^2}{\rho^2(t)}$$

$$p(0) \;=\; \sigma_0^2$$

The smoothing property of conditional expectations [17] implies

$$E\big((x_t\,g_t)\,^\wedge - \hat{x}_t\,\hat{g}_t\big) \;=\; E\big(\,x_t - \hat{x}_t\,\big)\big(\,g_t - g(t,\hat{x}_t)\,\big) \tag{39}$$

$$=\; E\,\tilde{x}_t\,\big(\,g_t - g(t,\hat{x}_t)\,\big)$$

Therefore

$$\dot{p}(t) \;=\; \sigma^2(t) + 2\,E\,\tilde{x}_t\,\big(\,g_t - g(t,\hat{x}_t)\,\big) - \frac{E(e_t)^2}{\rho^2(t)}$$

$$p(0) \;=\; \sigma_0^2 \tag{40}$$

Jensen's inequality [17] implies that :

$$E\,(\,\hat{e}_t\,)^2 \;\geq\; (\,E\,\hat{e}_t\,)^2$$

$$E\,\hat{e}_t \;=\; E\big((x_t\,h_t)\,^\wedge - \hat{x}_t\,\hat{h}_t\big) \;=\; E\,\tilde{x}_t\,\big(\,h_t - h(t,\hat{x}_t)\,\big)$$

now

$$h(t,x_t) - h(t,\hat{x}_t) \;=\; \tilde{x}_t\,\int_0^1 h_x[\,t\,,\,\hat{x}_t + s\tilde{x}_t\,]ds \;:=\; \tilde{x}_t\,\psi_h$$

hence

$$E\,\hat{e}_t \;=\; E\,\tilde{x}_t{}^2\,\psi_h$$

H_2 implies that

$$\psi_h \;\geq\; \underline{\beta}(t) \quad \text{a.s.}$$

$$E\,\hat{e}_t \;\geq\; \underline{\beta}(t)\,E\,\tilde{x}_t{}^2 \;=\; \underline{\beta}(t)\,p(t)$$

$$E\,(\hat{e}_t)^2 \;\geq\; (\,E\hat{e}_t\,)^2 \;\geq\; \underline{\beta}^2(t)\,p^2(t) \tag{41}$$

Similarly H_1 implies that

$$E \; \tilde{x}_t \; (\; g_t \; - g \, (t \, , \hat{x}_t) \;) \;\; = \;\; E \; \psi_g \; \tilde{x}_t^{\,2} \;\; \leq \;\; \overline{\alpha}(t) \, E \; \tilde{x}_t^{\,2} \;\; = \;\; \overline{\alpha}(t) \, p \, (t) \qquad (42)$$

Combining (40)-(42) and the comparison theorem (See the appendix) yields:

$p \, (t) \; \leq \; u \, (t)$.

* * *

Proposition 2-2-2 :

Let x_t , y_t be as in (27) and assume that H_1, H_2 hold. Let

$$\begin{aligned} dx_t^k \;\; &= \;\; g \, (t \, , x_t^k) \; dt \; + \; k \, (t) \, [\; dy_t \; - \; h \, (t \, , x_t^k) dt \;] \\ x_0^k \;\; &= \;\; 0 \end{aligned} \qquad (43)$$

$k \, (t)$ continuous, nonnegative.

Then

   (i)   the MSE $\;\; p^k \, (t) \; := \; E \; (\; x_t \; - \; x_t^k \,)^2 \;$ is upper bounded by $\; u^k \, (t) \;$ where

$$\begin{aligned} \dot{u}^k \, (t) \;\; &= \;\; \sigma^2(t) \; + \; \rho^2(t) \, k^2(t) \; + \; 2 \; [\; \overline{\alpha}(t) \; - \; k \, (t) \underline{\beta}(t) \;] \; u^k \, (t) \\ u^k \, (0) \;\; &= \;\; \sigma_0^2 \end{aligned} \qquad (44)$$

   (ii)   the upper bound $\; u^{k\,*}(t) \;$ corresponding to the particular choice

$k^*(t) \;\; = \;\; \dfrac{\beta(t)}{\rho^2(t)} \; u^{k\,*}(t) \;$ is the minimizer over the class of upper bounds (44).

Furthermore $u^{k\,*}(t) \;\; = \;\; u \, (t) \;$ where $\; u \, (t) \;$ is as in (33)

* * *

Proof:

(i)   Let $\; \tilde{x}_t \;\; = \;\; x_t \; - \; x_t^k$. Then (27) and (43) imply that

$$d\tilde{x}_t \;\; = \;\; [\; \tilde{g}_t \; - \; k \, (t) \tilde{h}_t \,] dt \; + \; \sigma(t) dw_t \; - \; k \, (t) \rho^2(t) dv_t \qquad (45)$$

where

$$\tilde{g}_t = g(t,x_t) - g(t,x_t^k)$$
$$\tilde{h}_t = h(t,x_t) - h(t,x_t^k)$$

Applying *Itô's* chain rule [1] gives

$$d\tilde{x}_t^2 = [\sigma^2(t) + \rho^2(t)k^2(t)] dt + 2\tilde{x}_t d\tilde{x}_t$$

taking the expectation on both sides yields :

$$\frac{d}{dt}E \ \tilde{x}_t^2 = \dot{p}^k(t) = \sigma^2(t) + \rho^2(t)k^2(t) + 2E\tilde{x}_t[\tilde{g}_t - k(t)\tilde{h}_t]$$
$$\dot{p}^k(t) = \sigma^2(t) + \rho^2(t)k^2(t) + 2E\tilde{x}_t\tilde{g}_t - 2k(t)E\tilde{x}_t\tilde{h}_t$$

(46)

using :

$$\tilde{g}_t = g(t,x_t) - g(t,x_t^k) = \tilde{x}_t \int_0^1 g_x(t, x_t^k + s\tilde{x}_t)ds := \tilde{x}_t \psi_g$$

we can write:

$$E \ \tilde{x}_t \tilde{g}_t = E \ \psi_g \tilde{x}_t^2$$

similarly

$$E \ \tilde{x}_t \tilde{h}_t = E \ \psi_h \tilde{x}_t^2$$

H_1 and H_2 imply that

$$E \ \tilde{x}_t \tilde{g}_t \leq \overline{\alpha}(t)E \ \tilde{x}_t^2 = \overline{\alpha}(t)p^k(t) \qquad (47)$$

$$E \ \tilde{x}_t \tilde{h}_t \geq \underline{\beta}(t)E \ \tilde{x}_t^2 = \underline{\beta}(t)p^k(t) \qquad (48)$$

by combining (46)-(48) and the comparison theorem in the appendix we readily get the result in (i).

(ii) To show that $u^{k^*}(t) = u(t) \leq u^k(t)$ for every continuous k we consider the following optimal control problem: Find k which minimizes $u^k(T)$ subject to (44). It is well known that the optimal control $k^*(t)$ solves [18]

$$\left\{ \frac{\partial H}{\partial k} \right\}_{k^*} = 0$$

where the Hamiltonian H is given by

$$H = \lambda\,[\,\sigma^2(t) + \rho^2(t)k^2(t) + 2(\overline{\alpha}(t) - k(t)\underline{\beta}(t))u^k(t)\,]$$

with

$$\left\{\frac{\partial H}{\partial k}\right\}_{k^*} = \lambda\,[\,2\rho^2(t)k^*(t) - 2\underline{\beta}(t)u^k(t)\,] = 0$$

This implies that $\qquad k^*(t) = \dfrac{\underline{\beta}(t)}{\rho^2(t)}\,u^k(t)$

Plugging this into (44) yields (33). Moreover

$$\frac{\partial^2 H}{\partial k^2} = 2\rho^2(t)\lambda(t)$$

where λ solves

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial u^k} = -2(\overline{\alpha}(t) - k(t)\underline{\beta}(t))\lambda(t) \quad , \quad \lambda(T) = 1$$

and is :

$$\lambda(t) = \exp 2\!\int_t^T (\overline{\alpha}(s) - k(s)\underline{\beta}(s))ds \ \geq\ 0$$

therefore $\qquad \dfrac{\partial^2 H}{\partial k^2} > 0 \qquad$ i.e. $\quad k^*\quad$ achieves the minimum.

* * *

The consequences of this proposition are as follows :

the suboptimal filter

$$dx_t^* = g(t,x_t^*)dt + \frac{\beta^2(t)}{\rho^2(t)}\,u(t)[dy_t - h(t,x_t^*)dt] \quad ; \quad x_0^* = 0 \qquad (49)$$

where

$$\dot{u}(t) = \sigma^2(t) + 2\overline{\alpha}(t)u(t) - \frac{\beta^2(t)}{\rho^2(t)}\,u^2(t) \quad ; \quad u(0) = \sigma_0^2 \qquad (50)$$

has the best upper bound (relatively to (44)) , i.e., $u(t) \leq u^k(t)$ for every continuous nonnegative function $k(t)$. Furthermore

$$p(t) := E \ (\ x_t - E(x_t \mid \mathcal{Y}_0^t)\ )^2 \ \leq \ p^*(t) := E \ (\ x_t - x_t^*)^2 \ \leq \ u(t) \qquad (51)$$

We have also seen (remark (2) and (4) in previous section) that applying the results obtained in [10] to the 1 dimensional case yielded that the upper bound $u^*(t)$ is the smallest element of the family of upper bounds (28).

The next proposition states that $u(t)$ is tighter than $u^*(t)$ which in turn means that the BOF (49)-(50) is better (and simpler) than the one given by (30).

For the remainder of this thesis we will use the term BOF (omitting the family with respect to which it is bound optimal) to refer to (49)-(50).

Remark: (51) is another proof of proposition 2-2-1.

Proposition 2-2-3 :

$u(t) \ \leq \ u^*(t)$ where $u(t)$ and $u^*(t)$ satisfy (50) and (29) respectively.

* * *

Proof :

Let us recall that $u^*(t)$ is the smallest element of the family

$$\dot{u}^k \ = \sigma^2 + \rho^2 k^2 + 2(\overline{\alpha} - k\,\underline{\beta})u^k + \Delta\beta(k-1)^2 u^k \quad ; \quad u^k(0) = \sigma_0^2 \qquad (28)$$

over all $k(t)$ and is achieved for $k^* = \dfrac{\beta u^*}{\rho^2 + u^*\Delta\beta}$. But H_2 implies that $\beta > 0$, i.e., $k^* > 0$ for every $t \geq 0$, hence $u^*(t)$ is also the minimizer of (28) over all nonnegative k's.

Next we notice that if $g \in IC\,[\alpha(t)\,,\,\Delta\alpha(t)]$ and $h \in IC\,[\beta(t)\,,\,\Delta\beta(t)]$ $\quad(\underline{\beta} \geq 0)$ then g and h satisfy H_1 and H_2 with $\overline{\alpha}(t) = \alpha(t) + \Delta\alpha(t)$ and $\underline{\beta}(t) = \beta(t) - \Delta\beta(t)$ respectively.

Now, we clearly have $u^k(t) > 0$, and by virtue of the comparison theorem: $u^k(t) \geq v^k(t)$ where

$$\dot{v}^k = \sigma^2 + \rho^2 k^2 + 2(\overline{\alpha} - k\,\underline{\beta})v^k \quad ; \quad v^k(0) = \sigma_0^2.$$

But $u(t)$ is precisely the smallest element of this family of functions (by last proposition). Therefore $u(t)$ is necessarily smaller than $u^*(t)$.

$$* \quad * \quad *$$

In the next section we combine results from the previous two sections in a single statement ready to be used in subsequent chapters.

2-3 Summary :

We consider systems modeled by one dimensional *Itô* SDE's of the form :

$$
\begin{aligned}
dx_t &= g(t, x_t)\, dt + \sigma(t)\, dw_t \\
dy_t &= h(t, x_t)\, dt + \rho(t)\, dv_t \\
E\ x_0 &= 0 \quad , \quad E\ x_0^2 = \sigma_0^2
\end{aligned}
\tag{51}
$$

where g and h satisfy

$$A_1 \quad : \quad |\, g_x(t, x) - \alpha(t)\,| \ < \ \Delta\alpha(t) \quad denoted\ by \quad g \in \ <[\alpha(t), \Delta\alpha(t)] \tag{52}$$

$$A_2 \quad : \quad |\, h_x(t, x) - \beta(t)\,| \ < \ \Delta\beta(t) \quad denoted\ by \quad h \in \ <[\beta(t), \Delta\beta(t)] \tag{53}$$

Note : $f \in IC\ [\alpha, \Delta\alpha]$ implies $f \in < [\alpha, \Delta\alpha]$. Let:

$$\overline{\alpha}(t) := \alpha(t) + \Delta\alpha(t) \quad ; \quad \underline{\alpha}(t) := \alpha(t) - \Delta\alpha(t) \tag{54}$$

$$\overline{\beta}(t) := \beta(t) + \Delta\beta(t) \quad ; \quad \underline{\beta}(t) = \beta(t) - \Delta\beta(t) \tag{55}$$

$$p(t) := E\ (\ x_t - E(x_t \mid y_0^t)\ )^2 \tag{56}$$

$$p^*(t) := E\ (\ x_t - x_t^*\ )^2 \tag{57}$$

where x_t^* is the BOF and is given by

$$dx_t^* = g(t, x_t^*)\, dt + \frac{\underline{\beta}(t)}{\rho^2(t)}\, u(t)\, [\ dy_t - h(t, x_t^*)dt\ ] \quad ; \quad x_0^* = 0 \tag{58}$$

$$\dot{u}(t) = \sigma^2(t) + 2\overline{\alpha}(t)u(t) - \frac{\beta^2(t)}{\rho^2(t)} u^2(t) \qquad ; \qquad u(0) = \sigma_0^2. \tag{59}$$

Then by combining the results from the previous two sections we readily get the following :

$$0 \le l(t) \le p(t) \le p^*(t) \le u(t) \tag{60}$$

where

$$\dot{l}(t) = \sigma^2(t) + 2\underline{\alpha}(t)l(t) - \frac{1}{\rho^2(t)} [\,\overline{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)} (\Delta\alpha(t))^2\,]\, l^2(t)$$
$$l(0) = \sigma_0^2 \tag{61}$$

and $u(t)$ satisfies (59).

3   ASYMPTOTICALLY TIME INVARIANT SYSTEMS

In this chapter we discuss systems that are asymptotically time invariant, i.e.,

$$
\begin{aligned}
dx_t &= g(t,x_t)\, dt + \sigma\, dw_t \\
dy_t &= h(t,x_t)\, dt + \rho dv_t
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
g(t,x) &= g(x) + \lambda(t)\, f(t,x) \\
h(t,x) &= h(x) + \nu(t)\, h(t,x)
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
g &\in <[\,a\ ,\Delta a\,] \quad ; \quad f \in <[\,\mu(t)\,,\Delta\mu(t)\,] \\
h &\in <[\,c\ ,\Delta c\,] \quad ; \quad k \in <[\,\varsigma(t)\,,\Delta\varsigma(t)\,]
\end{aligned}
\tag{3}
$$

and

$$
\lim_{t\to\infty} [\lambda(t)\,,\nu(t)] = [0,0]
\tag{4}
$$

In the particular case where $g(x)$ and $h(x)$ are linear (the limiting system is linear), one is interested in knowing whether the Kalman filter (KF) for the limiting linear system, driven by y_t in (1), is asymptotically optimal as t becomes large.

More specifically, let

$$
\begin{aligned}
dx_t &= a\, x_t\, dt + \lambda(t)\, f(t,x_t)\, dt + \sigma\, dw_t \\
dy_t &= c\, x_t\, dt + \nu(t)\, k(t,x_t)\, dt + \rho\, dv_t \\
E\, x_0 &= 0 \quad , \quad E\, x_0^2 = \sigma_0^2 > 0
\end{aligned}
\tag{5}
$$

Then the " KF " designed for the limiting system is

$$
dx_t^k = a\, x_t^k\, dt + \frac{c}{\rho^2}\, r(t)\,[\,dy_t - c\, x_t^k\, dt\,] \quad ; \quad x^k(0) = 0
\tag{6}
$$

$$
\dot{r}(t) = \sigma^2 + 2\, a\, r - \frac{c^2}{\rho^2}\, r^2 \quad ; \quad r(0) = \sigma_0^2
\tag{7}
$$

and the questions of interest are :

- under what conditions is x_t^k (or the BOF x_t^*) asymptotically optimal as
$t \to \infty$, i.e.

$$\lim_{t \to \infty} (\, p^{\,k}(t) - p\,(t)\,) \;=\; 0 \qquad (\, \lim_{t \to \infty} (\, p^{\,*}(t) - p\,(t)\,) \;=\; 0) \qquad ?$$

where

$$p^{\,k}(t) \;=\; E\,(\,x_t - x_t^{\,k}\,)^2 \tag{8}$$

$$p^{\,*}(t) \;=\; E\,(\,x_t - x_t^{\,*}\,)^2 \tag{9}$$

$$p\,(t) \;=\; E\,(\,x_t - E(x_t \mid \overset{t}{y_0})\,)^2 \tag{10}$$

- would the same result hold for the steady state KF (SSKF), obtained by setting

$$r\,(t) \;=\; r\,(\infty) \quad \text{in (6)} \quad ?$$

The bounds on the MMSE derived in the previous chapter are used to answer these questions in the linear limiting case. However, the bounds on the nonlinearities partial derivatives do not contain " enough information " to treat similar questions in the general case where $g\,(x)$ and $h\,(x)$ are nonlinear.

Consequently, we will only consider the class of nonlinear filtering problems (5) with the assumptions :

$H_1 \quad : \quad f \in < [\,\mu(t)\,,\,\Delta\mu(t)\,]\quad ; \quad k \in < [\,s(t)\,,\,\Delta s(t)\,]$

$H_2 \quad : \quad \lambda(t) \quad \text{and} \quad \nu(t) \quad \text{are continuous, vanishing functions}$

$\qquad \qquad \text{on } [0,\infty[\text{ and nonnegative for simplicity}$

$H_3 \quad : \quad \mu(t)\,,\ \Delta\mu(t)\,,\ s(t) \text{ and } \Delta s(t) \text{ are bounded continuous}$

$\qquad \qquad \text{functions on } [0,\infty[$

$H_4 \quad : \quad c + \nu(t)\,s(t) \;\geq\; \delta_0 \;>\; 0 \qquad ; \qquad c \quad \text{nonzero} \ .$

In the next two sections we show that :

$$\lim_{t \to \infty} (\, p^{\,*}(t) - p\,(t)\,) \;=\; 0 \quad \text{and} \quad \lim_{t \to \infty} (\, p^{\,k}(t) - p\,(t)\,) \;=\; 0; \tag{11}$$

this is done by bounding $p\,(t)$ as

$$0 \;\leq\; l(t) \;\leq\; p\,(t) \;\leq\; p^{\,*}(t) \;\leq\; u\,(t) \tag{12}$$

$$0 \leq l(t) \leq p(t) \leq p^k(t) \leq q(t) \tag{13}$$

and showing that

$$\lim_{t \to \infty} (u(t) - l(t)) = 0 \quad \text{and} \quad \lim_{t \to \infty} (q(t) - l(t)) = 0 \tag{14}$$

The result is then generalized to the case

$$g(t,x) = a \; x + \sum_{i=1}^{n} \lambda_i(t) \; f_i(t,x) \tag{15}$$

$$h(t,x) = c \; x + \sum_{i=1}^{m} \nu_i(t) \; k_i(t,x) \tag{16}$$

which in turn can be applied to treat cases where a and c are time varying functions.

3-1   Asymptotic optimality of the BOF:

We recall that

(1)   the BOF corresponding to

$$\begin{aligned} dx_t &= g(t,x_t) \; dt + \sigma \; dw_t \\ dy_t &= h(t,x) \; dt + \rho \; dv_t \\ E \; x_0 &= 0 \quad , \quad E \; x_0^2 = \sigma_0^2 \end{aligned} \tag{17}$$

where

$$\begin{aligned} g &\in < [\; \alpha(t) \; , \; \Delta\alpha(t) \;] \\ h &\in < [\; \beta(t) \; , \; \Delta\beta(t) \;] \quad ; \quad \underline{\beta}(t) > 0 \quad ; \quad t \geq 0 \end{aligned}$$

is given by

$$dx_t^* = g(t,x_t^*) \; dt + \frac{\beta(t)}{\rho^2} \; u(t) \; [\; dy_t - h(t,x_t^*)dt \;] \quad ; \quad x^*(0) = 0 \tag{18}$$

$$\dot{u}(t) = \sigma^2 + 2 \; \overline{\alpha}(t) \; u(t) - \frac{\beta^2(t)}{\rho^2} \; u^2(t) \quad ; \quad u(0) = \sigma_0^2 \tag{19}$$

(2)   the MSE's

$$p(t) := E \; (x_t - E(x_t \mid y_0^t))^2 \quad \text{and} \quad p^*(t) := E \; (x_t - x_t^*)^2$$

satisfy

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{20}$$

where:

$$\dot{l}(t) = \sigma^2 + 2 \underline{\alpha}(t) l(t) - \frac{1}{\rho^2} [\overline{\beta}^2(t) + 4\frac{\rho^2}{\sigma^2} (\Delta\alpha(t))^2] l^2$$

$$l(0) = \sigma_0^2 \tag{21}$$

In the case of (5), we note that H_1 implies:

$$g(t,x) = ax + \lambda(t)f(t,x) \in \ <[a + \lambda(t)\mu(t) , \lambda(t)\Delta\mu(t)] \tag{22}$$

$$h(t,x) = cx + \nu(t)k(t,x) \in \ <[c + \nu(t)\varsigma(t) , \nu(t)\Delta\varsigma(t)] \tag{23}$$

Thus the results (18)-(21) apply with

$$\overline{\alpha} = a + \lambda(t)\mu(t) + \lambda(t)\Delta\mu(t) = a + \lambda(t)\overline{\mu}(t) \tag{24}$$

$$\underline{\alpha} = a + \lambda(t)\underline{\mu}(t) \tag{25}$$

$$\overline{\beta} = c + \nu(t)\overline{\varsigma}(t) \tag{26}$$

$$\underline{\beta} = c + \nu(t)\underline{\varsigma}(t) \tag{27}$$

The asymptotic optimality of the BOF is a consequence of the following lemma:

Lemma 3-1 :

Let θ_1, θ_2, γ_1 and γ_2 be continuous functions on R^+ such that

- $\lim_{t \to \infty} \theta_i(t) = a$

- $\lim_{t \to \infty} \gamma_i^2(t) = c^2$; $t \geq 0$; $i = 1,2$

and consider the Riccati equations :

$$\dot{v}_1 = \sigma^2 + 2\theta_1 v_1 - \frac{\gamma_1^2}{\rho^2} v_1^2 \quad ; \quad v_1(0) = \sigma_0^2 \tag{28}$$

$$\dot{v}_2 = \sigma^2 + 2\theta_2 v_2 - \frac{\gamma_2^2}{\rho^2} v_2^2 \quad ; \quad v_2(0) = \sigma_0^2 \tag{29}$$

If $v_1(t) \geq v_2(t)$ and if one of the assumptions given below holds then:

$$\lim_{t \to \infty} v_1(t) = \lim_{t \to \infty} v_2(t)$$

$A_1 \quad : \quad a \; < \; 0$

$A_2 \quad : \quad v_2(t) \; \geq \; r(t), \quad t \geq 0 \quad \text{and} \quad \gamma_1^2 \; \geq \; \delta^2 \; > \; 0 \quad \text{for } some \; \delta$

Recall that :

$$\dot{r}(t) \; = \; \sigma^2 + 2ar \; - \; \frac{c^2}{\rho^2} \, r^2 \quad , \quad r(0) \; = \; \sigma_0^2 \tag{30}$$

<p align="center">* * *</p>

Proof:

Let $w(t) \; = \; v_1(t) - v_2(t) \; \geq \; 0$. Then a straightforward computation yields

$$\dot{w} \; = \; 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 \; + \; 2(\theta_1 - \frac{\gamma_1^2}{\rho^2} \, v_2)w \; - \; \frac{\gamma_1^2}{\rho^2} \, w^2$$
$$w(0) \; = \; 0 \tag{31}$$

which we rewrite as

$$\dot{w}(t) \; = \; i(t) + 2j(t)w \; - \; \frac{\gamma_1^2}{\rho^2}w^2 \quad , \quad w(0) \; = \; 0 \tag{32}$$

where

$$i(t) \; = \; 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 \tag{33}$$

$$j(t) \; = \; \theta_1 - \frac{\gamma_1^2}{\rho^2} \, v_2 \tag{34}$$

(32) clearly implies:

$$\dot{w} \; \leq \; i(t) + 2j(t)w. \tag{35}$$

Depending on the assumption used ($A_1 \; or \; A_2$) we will bound $w(t)$ differently using the comparison theorem.

(1) Assumption A_1 :

Since $i(t)$ and $w(t)$ are nonnegative, $w(t)$ can be bounded as

$$\dot{w} \leq i(t) + 2\theta_1 w \qquad (36)$$

thus $0 \leq w(t) \leq z(t)$ where

$$\dot{z}(t) = i(t) + 2\theta_1 z \quad ; \quad z(0) = 0 \qquad (37)$$

Similarly $v_1(t) \leq V_1(t)$ where

$$\dot{V}_1 = \sigma^2 + 2\theta_1 V_1 \quad , \quad V_1(0) = \sigma_0^2 \qquad (38)$$

If $a < 0$ then $\displaystyle\lim_{t \to \infty} \theta_1 = a < 0$ and Perron's theorem (See the appendix)

can be applied to (37) and (38). We get

$$V_1(\infty) = -\frac{\sigma^2}{2a} \qquad (39)$$

Since $v_2(t) \leq v_1(t) \leq V_1(t)$ for every $t \geq 0$, (39) implies

$$\lim_{t \to \infty} i(t) = 0$$

Re-applied to (37) Perron's theorem yields

$$\lim_{t \to \infty} z(t) = 0 \quad that \quad is \quad \lim_{t \to \infty} w(t) = 0$$

(2) Assumption A_2 :

Since $v_2(t) \geq r(t), \quad j(t) \leq \theta_1 - \dfrac{\gamma_1^2}{\rho^2} r(t),$ (35) then implies that

$w(t) \leq z(t)$, where:

$$\dot{z} = i(t) + 2 \left(\theta_1 - \frac{\gamma_1^2}{\rho^2} r(t) \right) z(t) \quad ; \quad z(0) = 0 \qquad (40)$$

$$\lim_{t \to \infty} \left(\theta_1 - \frac{\gamma_1^2}{\rho^2} r(t) \right) = a - \frac{c^2}{\rho^2} r(\infty) ; \quad \text{but } r(\infty) \text{ is the positive root of}$$

$$\sigma^2 + 2ax - \frac{c^2}{\rho^2} x^2 = 0 \qquad (41)$$

i.e.

$$r(\infty) = \frac{\rho^2}{c^2} \left[a + \left(a^2 + \frac{\sigma^2}{\rho^2} c^2 \right)^{1/2} \right]$$

and $a - \dfrac{c^2}{\rho^2} r(\infty) = -\left(a^2 + \dfrac{\sigma^2}{\rho^2} c^2 \right)^{1/2}$ Thus $\displaystyle\lim_{t \to \infty} z(t) = 0$ provided

$$\lim_{t \to \infty} i(t) = 0.$$

For this to happen it suffices that $v_2(t)$ be bounded ($v_1(t)$ be bounded).

Using the assumptions and the comparison theorem we immediately get:

$$v_1(t) \leq V_1(t) \qquad \text{where}$$

$$\dot{V}_1 = \sigma^2 + 2\theta_M V_1 - \frac{\delta^2}{\rho^2} V_1^2 \quad ; \quad V_1(0) = \sigma_0^2 \tag{42}$$

and θ_M is a nonzero upper bound of $\theta_1(t)$. $V_1(t)$ is clearly bounded.

We conclude that $\lim_{t \to \infty} z(t) = 0$, i.e., $\lim_{t \to \infty} w(t) = 0$

* * *

Note:

We can conclude in particular that:

(1) $v_1(t) \geq v_2(t) \geq r(t)$; $t \geq 0$ implies $v_1(\infty) = v_2(\infty) = r(\infty)$

(2) $v_1(t) \geq r(t) \geq v_2(t)$ and $a < 0$ imply $v_1(\infty) = v_2(\infty) = r(\infty)$

(3) $r(t) \geq v_1(t) \geq v_2(t)$ and $a < 0$ imply the same

Proposition 3-2 :

If $H_1 - H_4$ and H_5 or H_6 hold, where

$H_5 \quad : \quad a < 0$

$H_6 \quad : \quad l(t) \geq r(t) \qquad t \geq 0$

then the BOF given by (18), (19) and (24)-(27) is asymptotically optimal as $t \to \infty$.

* * *

Remark: It follows directly from lemma 3-1 that if H_5 is replaced by

$$H'_5 \quad : \quad a < 0 \quad \text{and either} \quad u(t) \geq r(t) \quad \text{or} \quad u(t) \leq r(t)$$

$$\text{then} \qquad p\,(\infty) = p^*(\infty) = r\,(\infty) = \frac{\rho^2}{c^2}\,[\ a\ +\ (a^2 + \frac{\sigma^2}{\rho^2}\ c^2)^{1/2}\,]$$

Proof:

We have that : $\qquad 0\ \leq\ l\,(t)\ \leq\ p\,(t)\ \leq\ p^*(t)\ \leq\ u\,(t)$

where $l\,(t)$ and $u\,(t)$ are given by (19), (21) and (24)-(27). Lemma 3-1 can then be applied to $u\,(t)$ and $l\,(t)$ by taking :

$$\theta_1(t)\ =\ \overline{\alpha}(t)\ =\ a\ +\ \lambda(t)\overline{\mu}(t) \tag{43}$$

$$\theta_2(t)\ =\ \underline{\alpha}(t)\ =\ a\ +\ \lambda(t)\underline{\mu}(t) \tag{44}$$

$$\gamma_1^2(t)\ =\ \underline{\beta^2}(t)\ =\ [\ c\ +\ \nu(t)\,\underline{s}(t)\,]^2 \tag{45}$$

$$\gamma_2^2(t)\ =\ \overline{\beta^2}(t) + 4\,\frac{\rho^2}{\sigma^2}\,(\ \Delta\alpha(t)\,)^2$$

$$\gamma_2^2\ =\ [\ c\ +\ \nu(t)\,\overline{s}(t)\,]^2 + 4\,\frac{\rho^2}{\sigma^2}\,\lambda^2(t)\,(\Delta\mu(t))^2 \tag{46}$$

It is readily checked that all hypotheses in lemma 3-1 are satisfied and the result follows.

*   *   *

Next we generalize proposition 3-2 to the nonlinearities

$$g\,(t,x)\ =\ a\ x\ +\ \sum_{i\,=\,1}^{n}\ \lambda_i(t)f_i(t,x)$$

$$h\,(t,x)\ =\ c\ x\ +\ \sum_{i\,=\,1}^{m}\ \nu_i(t)k_i(t,x) \tag{47}$$

with the assumptions H_1, H_2 and H_3 holding for each i, namely :

H_1 : $\qquad f_i\ \in\ <\ [\ \mu_i(t)\,,\Delta\mu_i(t)\,]\qquad i\ =\ 1,...,n$

$\qquad\quad k_i\ \in\ <\ [\ s_i(t)\,,\Delta s_i(t)\,]\qquad i\ =\ 1,....,m$

H_2 : $\qquad \lambda_i(t)$ and $\nu_j(t)$ are continuous vanishing functions and nonnegative for $i\ =\ 1,..,n\ ;\quad j\ =\ 1,...,m$

H_3 : μ_i , $\Delta\mu_i$, ς_j , $\Delta\varsigma_j$ are   bounded   continuous   functions   for   each

$i = 1,....,n$; $j = l,...,m$.

Using a vector notation, e.g., $\Delta\mu = (\Delta\mu_1 , \cdots , \Delta\mu_n)^T$, and $< , >_n$ to denote the inner product in $\mathbb{R}^n$, the problem above can be reformulated in the more condensed form:

$$g(t,x) = ax + <\lambda(t), f(t,x)>_n$$
$$h(t,x) = cx + <\nu(t), k(t,x)>_m \tag{48}$$

and we clearly have

$$g \in <[a + <\lambda, \mu>_n ; <\lambda, \Delta\mu>_n]$$
$$h \in <[c + <\nu, \varsigma>_m ; <\nu, \Delta\varsigma>_m] \tag{49}$$

Thus if we make the additional hypothesis

H_4 : $\underline{\beta} = c + <\nu, \underline{\beta}>_m \geq \delta_0 > 0$

then the same results hold. More precisely the BOF is given by

$$dx_t^* = ax_t^* dt + <\lambda(t), f(t,x_t^*)> dt + \frac{\beta(t)}{\rho^2} u^2(t)[dy_t - cx_t^* dt - <\nu(t), k(t,x_t^*)> dt] \tag{50}$$

$$x^*(0) = 0$$

with the corresponding MSE $p^*(t)$ and the MMSE $p(t)$ satisafying

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{51}$$

where $l(t)$ and $u(t)$ are given by (19) and (21). Here:

$$\overline{\alpha}(t) = a + <\lambda(t), \overline{\mu}(t)> \tag{52}$$

$$\underline{\alpha}(t) = a + <\lambda(t), \underline{\mu}(t)> \tag{53}$$

$$\Delta\alpha = <\lambda(t), \Delta\mu> \tag{54}$$

$$\overline{\beta}(t) = c + <\nu(t), \overline{\varsigma}(t)> \tag{55}$$

$$\underline{\beta}(t) = c + <\nu(t), \underline{\varsigma}(t)> \tag{56}$$

The corollary below is a direct application of lemma 3-1.

Corollary 3-3 :

If $H_1 - H_4$ and H_5 or H_6 stated below are satisfied then the BOF (50) is asymptotically optimal as $t \to \infty$.

$H_5 : \quad a < 0$

$H_6 : \quad l(t) \geq r(t) \quad ; \quad l(t) \quad$ given by (21), (52)--(56) and $\quad r(t) \quad$ by

$$\dot{r} = \sigma^2 + 2ar - \frac{c^2}{\rho^2} r^2 \quad ; \quad r(0) = \sigma_0^2 \tag{57}$$

<div align="center">* * *</div>

The corollary above can be used to treat the more general cases where a and c are time varying, i.e.,

$$g(t,x) = a(t)x + \sum_1^n \lambda_i(t)f_i(t,x)$$
$$h(t,x) = c(t)x + \sum_1^m \nu_i(t)k_i(t,x) \tag{58}$$

where

$$\lim_{t \to \infty} a(t) = a \quad \text{and} \quad \lim_{t \to \infty} c(t) = c$$

As an illustration, assume that $a(t)$ and $c(t)$ are continuous, $a(t) \geq a$ and $c(t) \geq c$, then rewrite (58) as :

$$g(t,x) = ax + (a(t) - a)x + <\lambda , f>_n$$
$$h(t,x) = cx + (c(t) - c)x + <\nu , k>_m \tag{59}$$

By letting:

$$\lambda_{n+1}(t) = a(t) - a$$
$$\nu_{m+1}(t) = c(t) - c$$
$$f_{n+1}(t,x) = x = k_{m+1}(t,x)$$

(59) becomes:

$$g(t,x) = ax + <\lambda , f>_{n+1}$$
$$h(t,x) = cx + <\nu , k>_{m+1} \tag{60}$$

Clearly, Corollary 3-3 can now be applied since λ_{n+1} and ν_{m+1} are continuous vanishing nonnegative functions, f_{n+1} , $k_{n+1} \in < [\,1\,,\delta\,]$ where $\delta \geq 0$ is an arbitrary parameter.

3-2 Asymptotic optimality of the KF :

Given the nonlinear filtering problem:

$$
\begin{aligned}
dx_t &= ax_t\,dt + \lambda(t)f\,(t,x_t)dt + \sigma dw_t \\
dy_t &= cx_t\,dt + \nu(t)k\,(t,x_t)dt + \rho dv_t \\
Ex_0 &= 0 \quad , \quad Ex_0^2 = \sigma_0^2 \quad , \quad c \quad nonzero
\end{aligned}
\tag{61}
$$

we mean by " KF " the following algorithm

$$
dx_t^k = ax_t^k dt + \frac{c}{\rho^2}\,r\,(t)\,[\,dy_t - cx_t^k dt\,] \quad ; \quad x^k(0) = 0
\tag{62}
$$

$$
\dot{r} = \sigma^2 + 2ar - \frac{c^2}{\rho^2}\,r^2 \quad , \quad r\,(0) = \sigma_0^2
\tag{63}
$$

It is clear that this corresponds to a regular Kalman filter designed for the underlying linear system obtained when one ignores the nonlinear terms in (61). It should be noted however that the " KF " (62), (63) is driven by nonlinear observations.

Nevertheless, we will continue to refer to (62), (63) as the " KF " and " SSKF " (steady state) when $r\,(t)$ is replaced by $r\,(\infty)$.

We make the assumptions $H_0 - H_4$ where $H_1 - H_4$ are as before and

H_0 : $f\,(t,0)$ and $k\,(t,0)$ are continuous, bounded on $\mathbb{R}^+$.

Proposition 3-4 :

If $H_0 - H_4$ hold and if $a < 0$ and either $l(t) \leq r(t)$ or $l(t) \geq r(t)$ for every $t \geq 0$ then both the " KF " and the " SSKF " are asymptotically optimal as $t \rightarrow \infty$. Moreover :

$$p\left(\infty\right) \;=\; p^{k}\left(\infty\right) \;=\; r\left(\infty\right) \;=\; \frac{\rho^2}{c^2}\left[\, a \,+\, \left(a^2 + \frac{\sigma^2}{\rho^2}\, c^2\right)^{1/2}\,\right]$$

$$*\quad *\quad *$$

Remark:

For example, a sufficient condition for $l\left(t\right) \;\leq\; r\left(t\right)$ is $\underline{\mu}(t) \;\leq\; 0$, $\overline{\varsigma}(t) \geq 0$

and $c \;>\; 0$.

Proof:

We first derive an upper bound on $p^{k}\left(t\right) \;:=\; E\left(x_t - x_t^{k}\right)^2$.

Following the same steps as in the proof of proposition 2-2-2 one gets:

$$\frac{dp^{k}}{dt} \;=\; \sigma^2 + \rho^2 G^2 + 2\,E\,\overline{x}_t\,\overline{g}_t \;-\; 2\,G\,E\,\overline{x}_t\,\overline{h}_t \quad,\quad p^{k}\left(0\right) \;=\; \sigma_0^2 \tag{64}$$

where

$$G\left(t\right) \;=\; \frac{c}{\rho^2}\,r\left(t\right) \qquad\left(\; or \quad \frac{c}{\rho^2}\,r\left(\infty\right)\;\right) \tag{65}$$

$$\overline{x}_t \;=\; x_t - x_t^{k} \tag{66}$$

$$\overline{g}_t \;=\; a\overline{x}_t + \lambda(t)f\left(t,x_t\right) \tag{67}$$

$$\overline{h}_t \;=\; c\overline{x}_t + \nu(t)k\left(t,x_t\right) \tag{68}$$

$$\dot{p}^{k} \;=\; \sigma^2 + \rho^2 G^2 + 2(a-cG)p^{k} + 2\lambda(t)E\,\overline{x}_t\,f\left(t,x_t\right) - 2\nu GE\,\overline{x}_t\,k\left(t,x\right) \tag{69}$$

Clearly

$$2E\,\overline{x}_t\,f\left(t,x_t\right) \;\leq\; E\,\overline{x}_t^{\,2} + E\,f^2(t,x_t) \;=\; p^{k}\left(t\right) + E\,f^2(t,x_t) \tag{70}$$

$$-2E\,\overline{x}_t\,k\left(t,x_t\right) \;\leq\; p^{k}\left(t\right) + E\,k^2(t,x_t) \tag{71}$$

By the comparison theorem : $p^{k}\left(t\right) \;\leq\; q\left(t\right)$; $q\left(0\right) \;=\; \sigma_0^2$ where

$$\dot{q}\left(t\right) \;=\; \sigma^2 + \rho^2 G^2 + 2(a-cG)q + \lambda(q + Ef^2) + \nu G(q + Ek^2)$$

$$=\; \sigma^2 + \rho^2 G^2 + \lambda Ef^2 + \nu GEk^2 + \left[\,2(a-cG) + \lambda + \nu G\,\right]q \tag{72}$$

which we rewrite:

$$\dot{q} \;=\; i(t) + j(t)q \qquad , \qquad q(0) \;=\; \sigma_0^2 \tag{73}$$

Now, $\displaystyle \lim_{t \to \infty} j(t) \;=\; 2\left(a - \frac{c^2}{\rho^2}\, r(\infty)\right) \;=\; -2\left(a^2 + \frac{\sigma^2}{\rho^2}\, c^2\right)^{1/2} \;<\; 0.$ Thus if

$$\lim_{t \to \infty} \lambda(t) E \; f^2(t,x_t) \;=\; \lim_{t \to \infty} \nu(t) E \; k^2(t,x_t) \;=\; 0 \tag{74}$$

then

$$\lim_{t \to \infty} i(t) \;=\; \sigma^2 + \frac{c^2}{\rho^2}\, r^2(\infty) \quad.$$

Applying Perron's theorem to (73) would give:

$$q(\infty) \;=\; -\,\frac{i(\infty)}{j(\infty)} \;=\; -\,\frac{\sigma^2 + \dfrac{c^2}{\rho^2}\, r^2(\infty)}{2\left(a - \dfrac{c^2}{\rho^2}\, r(\infty)\right)} \tag{75}$$

But $r(\infty)$ satisfies the algebraic Riccati equation:

$$\sigma^2 + 2a\; r(\infty) - \frac{c^2}{\rho^2}\, r^2(\infty) \;=\; 0$$

Hence rewrite $q(\infty)$ as :

$$q(\infty) \;=\; -\,\frac{\sigma^2 + 2a\; r(\infty) - \dfrac{c^2}{\rho^2}\, r^2(\infty) - 2\left(a - \dfrac{c^2}{\rho^2}\, r(\infty)\right) r(\infty)}{2\left(a - \dfrac{c^2}{\rho^2}\, r(\infty)\right)} \;=\; r(\infty) \tag{76}$$

If $a < 0$ and $(\; l(t) \leq r(t) \;$ or $\; l(t) \geq r(t)\;)$ we conclude by the note after lemma 3-1 that $l(\infty) = r(\infty) = q(\infty)$.

We now show that (74) holds if $a < 0$ and H_0 hold.

$$f \;\in\; <\,[\mu(t)\,,\,\Delta(t)]$$

implies

$$\underline{\mu}(t)x + f(t,0) \;\leq\; f(t,x) \;\leq\; \overline{\mu}(t)x + f(t,0) \tag{77}$$

were the time functions $\underline{\mu}(t)$, $\overline{\mu}(t)$ and $f(t,0)$ are all bounded continuous for $t \geq 0$.

(77) implies in turn that :

$$f^2(t,x) \;\leq\; A^2(t)x^2 + B^2(t) \tag{78}$$

for some continuous bounded functions A and B.

Therefore:

$$\lim_{t \to \infty} \lambda(t)Ef^{2}(t,x_t) = 0 \tag{79}$$

holds if

$$\lim_{t \to \infty} \lambda(t)Ex_t^{2} = 0 \tag{80}$$

Ex_t^{2} is given by [1] :

$$\frac{d}{dt} E \ x_t^{2} = 1 + 2\lambda(t)E \ x_t f\left(t,x_t\right) + 2aE \ x_t^{2} \tag{81}$$

$$2E \ x_t f\left(t,x_t\right) \le E \ x_t^{2} + E \ f^{2}(t,x_t) \tag{82}$$

Using (80) and (78) in (81), we conclude by the comparison that $E \ x_t^{2}$ is bounded by $V(t)$ where :

$$\dot{V} = 1 + \lambda(t)B^{2}(t) + \left(\ 2a \ + \lambda(t) + \lambda(t)A^{2}(t)\ \right)V(t)$$

Perron's theorem applies and :

$$V(\infty) = -\frac{1}{2a} \tag{83}$$

Therefore (79) holds.

Clearly the same thing is also true for $\nu(t) \ Ek^{2}(t,x_t)$.

$$*\quad*\quad*$$

We now consider:

$$\begin{aligned} g\left(t,x\right) &= ax \ + \ <\lambda(t)\ ,\ f\left(t,x\right)>_n \\ h\left(t,x\right) &= cx \ + \ <\nu(t)\ ,\ k\left(t,x\right)>_m \end{aligned} \tag{84}$$

where the notation and the assumptions are as in (47), (48). In addition to $H_2 - H_4$, we assume

$H_0 \qquad : \qquad f_i(t,0)$ and $k_j(t,0)$ are bounded continuous on R^+ for

$$i = 1,...,n \ ; \quad j = 1,...,m.$$

Corollary 3-5 :

If $H_0 - H_4$ hold and if $a < 0$ and either $l(t) \leq r(t)$ or $l(t) \geq r(t)$ for every $t \geq 0$ then both the " KF " and the " SSKF " are asymptotically optimal as $t \to \infty$. Moreover

$$p(\infty) = p^k(\infty) = r(\infty) = \frac{c^2}{c^2} [a + (a^2 \frac{\sigma^2}{\rho^2} c^2)^{1/2}]$$

* * *

Remarks :

(1)   We recall that, here, $l(t)$ is given by :

$$\dot{l} = \sigma^2 + 2\underline{\alpha}(t)l - \frac{1}{\rho^2} [\overline{\beta^2}(t) + 4\frac{\rho^2}{\sigma^2} (\Delta\alpha)^2] l^2 \quad ; \quad l(0) = \sigma_0^2 \tag{85}$$

$$\underline{\alpha} = a + <\lambda(t), \underline{\mu}(t)>_n \tag{86}$$

$$\Delta\alpha = <\lambda(t), \Delta\mu(t)>_n \tag{87}$$

$$\overline{\beta}(t) = c + <\nu(t), \overline{s}(t)>_m \tag{88}$$

(2)   A   sufficient   condition   for $l(t) \leq r(t)$, $t \geq 0$ is

$$<\lambda(t), \overline{\mu}(t)>_n \leq 0 ; \quad <\nu(t), \overline{s}(t)>_m \geq 0 \quad \text{and} \quad c > 0.$$

These conditions hold in particular if :

$$\overline{\mu}_i(t) \leq 0 , \quad \overline{s}_i(t) \geq 0 , \quad c > 0 \quad \text{for } each \ i \text{ and } t \geq 0. \tag{89}$$

(3)   Corollary 3-5 can be used to treat cases where :

$$g(t,x) = a(t)x + \sum_1^n \lambda_i(t)f_i(t,x) \quad ; \quad \lim_{t \to \infty} a(t) = a$$

$$h(t,x) = c(t)x + \sum_2^m \nu_i(t)k_i(t,x) \quad ; \quad \lim_{t \to \infty} c(t) = c \tag{90}$$

as was illustrated in the previous section.

Proof:

As in proof of proposition 3-4 :

$$\frac{d}{dt} p^k(t) = \sigma^2 + \rho^2 G^2 + 2(a - cG)p^k + 2E \, \overline{x}_t <\lambda , f>_n - 2GE \, \overline{x}_t <\nu , k>_m$$

$$2E \, \overline{x}_t <\lambda , f> = 2E<\lambda , \overline{x}_t f> \leq 2 | \, |\lambda| \, |E \, |\overline{x}_t| \, | \, |f| \, | \leq | \, |\lambda| \, |[E \, \overline{x}_t^2 + E \, | \, |f| \, |^2]$$

$$-2E \, \overline{x}_t <\nu , k> \leq | \, |\nu| \, |[E \, \overline{x}_t^2 + E \, | \, |k| \, |^2] = | \, |\nu| \, |[p^k(t) + E \, | \, |k| \, |^2]$$

As before, $p^k(t)$ is bounded by $q(t)$ given below :

$$\dot{q} = i(t) + j(t) \, q$$

where:

$$i(t) = \sigma^2 + \rho^2 G^2 + | \, |\lambda(t)| \, | \, |E \, | \, |f(t,x_t)| \, |^2 + G \, | \, |\nu(t)| \, | \, |E \, | \, |k(t,x_t)| \, |^2$$

$$j(t) = 2(a - cG) + | \, |\lambda(t)| \, | \, | + G \, | \, |\nu(t)| \, | \, |$$

$$\lim_{t \to \infty} j(t) = 2(a - \frac{c^2}{\rho^2} r(\infty)) = -2(a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2} < 0$$

and $$\lim_{t \to \infty} i(t) = 0$$ provided

$$\lim_{t \to \infty} | \, |\lambda(t)| \, | \, |E \, | \, |f| \, |^2 = \lim_{t \to \infty} | \, |\nu(t)| \, | \, |E \, | \, |k| \, |^2 = 0.$$

We need to show that : $\lim_{t \to \infty} | \, |\lambda(t)| \, | \, |E \, | \, |f(t,x_t)| \, |^2 = 0$

As in last proof, we have that:

$$f_i^2(t,x_t) \leq A_i^2(t)x_t^2 + B_i^2(t)$$

where A_i and B_i are bounded continuous.

Thus, it suffices to show that : $\lim_{t \to \infty} Ex_t^2 < \infty$

$$\frac{d}{dt} Ex_t^2 = 1 + 2Ex_t <\lambda , f> + 2aEx_t^2$$

$$\frac{d}{dt} Ex_t^2 \leq 1 + | \, |\lambda| \, | \, | [Ex_t^2 + E \, | \, |f| \, |^2] + 2aEx_t^2$$

But

$$E \, | \, |f| \, |^2 = \sum_1^n Ef_i^2 \leq \sum_1^n A_i^2(t)Ex_t^2 + \sum_1^n B_i^2(t)$$

Therefore : $Ex_t^2 \leq V(t)$ where:

$$\dot{V} \;=\; 1 + \;|\;|\,\lambda\,|\;|\; \sum_{1}^{n} B_i{}^2(t) + [\,2a\; + \;|\;|\,\lambda\,|\;|\; + \;|\;|\,\lambda\,|\;|\; \sum_{1}^{n} A_i{}^2(t)\,]\; V$$

$$\dot{V} \;:=\; I(t) + J(t)\, V$$

Clearly : $\lim_{t \to \infty} I(t) \;=\; 1$ and $\lim_{t \to \infty} J(t) \;=\; 2a \;<\; 0$

Therefore $\lim_{t \to \infty} V(t) \;=\; -\dfrac{1}{2a} \;<\; \infty$ and the proof is complete.

* * *

4   STATIONARY BEHAVIOR OF THE BENES FILTER

4-1 Problem statement :

V.E. Benes established the following [4]:

Given the one dimensional filtering problem

$$
\begin{aligned}
dx_t &= f\,(x_t)dt\,+\,dw_t \quad , \quad x\,(0)\;=\;x \\
dy_t &= x_t\,dt\,+\,dv_t
\end{aligned}
\tag{1}
$$

where f satisfies the Riccati equation:

$$
f_x(x)\,+\,f^2(x)\;=\;a\,x^2+\,b\;x\,+\,c \quad ; \quad a\;\geq\;0
\tag{2}
$$

then

$$
u\,(t,z)\;=\;\exp\left\{\;\int_0^z f\,(x)dx\;-\;\frac{1}{2}\;\frac{(\,z\,-\mu_t\,)^2}{\sigma(t)}\;\right\}
$$

determines an unnormalized conditional density in terms of the two statistics $\sigma(t)$ and μ_t given by

$$
\dot{\sigma}(t)\;=\;1-k^2\sigma^2(t) \quad ; \quad \sigma(0)\;=\;0 \qquad (\;k\;=\;(1+a\,)^{1/2}\;)
\tag{3}
$$

$$
\begin{aligned}
d\,\mu_t &=\;-\,k^2\sigma^2(t)\mu_t\,dt\;-\;\frac{1}{2}\;b\,\sigma(t)dt\;+\;\sigma(t)dy_t \\
\mu_0 &=\;x
\end{aligned}
\tag{4}
$$

As already mentioned, this is one of the few known (see [5]) finite dimensional, recursive non-linear filters.

In certain applications one is interested in the long time behavior of easily implement-able filtering algorithms, e.g., algorithms in which the sufficient statistics satisfy differential equations with constant coefficients.

In the context of this particular filter we first note that the diffusion process satisfying (1) and (2) has a stationary probability distribution only if f is affine [19] (see also next

remark).

Since $\lim_{t \to \infty} \sigma(t) = \dfrac{1}{k} := \bar{\sigma}$, we define

$$\bar{u}(t,z) = \exp\left\{ \int_0^z f(x)dx - \frac{1}{2} \frac{(z - \bar{\mu}_t)^2}{\bar{\sigma}} \right\} \tag{5}$$

where now

$$d\bar{\mu}_t = -k^2\bar{\sigma}\,\bar{\mu}_t\,dt - \frac{1}{2}\,b\,\bar{\sigma}dt + \bar{\sigma}dy_t$$

$$\bar{\mu}_0 = x \tag{6}$$

as a " natural limiting function " of $u(t,z)$.

The question of interest is then to know how $\bar{u}(t,z)$ relates to $u(t,z)$ as $t \to \infty$.

We shall show that μ_t and $\bar{\mu}_t$ are asymptotically indistinguishable in the quadratic mean sense, i.e.

$$\lim_{t \to \infty} E\left(\mu_t - \bar{\mu}_t \right)^2 = 0$$

4-2 A partial result :

We start by recalling that μ_t has the alternative form [4]:

$$\mu_t = \sigma(t)y_t + \frac{m_1 - r_{12} + r_{13}}{1 - kr_{11}} \tag{7}$$

where

$$\sigma(t) = \frac{r_{11}(t)}{1 - kr_{11}(t)} \tag{8}$$

$$\dot{r}_{11} = 1 - 2kr_{11} \quad ; \quad r_{11}(0) = 0 \tag{9}$$

$$\dot{r}_{12} = -kr_{12} + y_t \quad ; \quad r_{12}(0) = 0 \tag{10}$$

$$\dot{r}_{13} = -kr_{13} + r_{11}\left(ky_t - \frac{1}{2}b \right) \quad ; \quad r_{13}(0) = 0 \tag{11}$$

$$\dot{m}_1 = -km_1 \quad ; \quad m_1(0) = x \tag{12}$$

A similar form for $\bar{\mu}_t$ is readily obtainable.

Proposition 4-2-1 :

An alternative expression for $\bar{\mu}_t$ is

$$\bar{\mu}_t = \bar{\sigma}y_t + \frac{m_1 - r_{12} + \bar{r}_{13}}{1 - k\bar{r}_{11}} \tag{13}$$

where

$$\bar{r}_{11} = \frac{1}{2k} = \frac{\bar{\sigma}}{2} \tag{14}$$

$$\dot{\bar{r}}_{13} = -k\bar{r}_{13} + r_{11}\left(ky_t - \frac{1}{2} b \right) \quad ; \quad \bar{r}_{13}(0) = 0 \tag{15}$$

* * *

Proof :

Differentiate (13)

$$d\bar{\mu}_t = \bar{\sigma}dy_t + \frac{1}{1 - k\bar{r}_{11}}\left(\dot{m}_1 - \dot{r}_{12} + \dot{\bar{r}}_{13} \right)dt$$

$$\dot{m}_1 + \dot{r}_{12} + \dot{\bar{r}}_{13} = -km_1 - y_t + \frac{1}{2} b\bar{r}_{11}$$

$$= -k\left(m_1 - r_{12} + \bar{r}_{13} \right) - y_t + k\bar{r}_{11}y_t - \frac{1}{2} b\bar{r}_{11}.$$

Thus:

$$d\bar{\mu}_t = \bar{\sigma}dy_t + \frac{1}{1-k\bar{r}_{11}}\left[-k\left(m_1 - r_{12} + \bar{r}_{13} \right) - (1-k\bar{r}_{11})y_t - \frac{1}{2} b\bar{r}_{11}\right]dt$$

$$d\bar{\mu}_t = \bar{\sigma}dy_t - k\frac{m_1 - r_{12} + \bar{r}_{13}}{1-k\bar{r}_{11}} dt$$

$$= \bar{\sigma}dy_t - k\left(\bar{\mu}_t - \bar{\sigma}y_t \right)dt - y_t dt - \frac{1}{2} b\bar{\sigma}dt$$

$$= -k\bar{\mu}_t dt - \frac{1}{2} b\bar{\sigma}dt + \bar{\sigma}dy_t - (1 - k\bar{\sigma})y_t dt$$

but $\bar{\sigma} := \frac{1}{k}$ hence $(1 - k\bar{\sigma}) = 0$ and $k = k^2\bar{\sigma}$.

* * *

The following estimates are also needed.

Lemma 4-2-2 :

$E \ y_t{}^2$ and $E \ [\int_0^t | \ y_s \ | \ ds \ ]^2$ are at most $O (\ t \ e^{2\sqrt{a} \ t})$ and

$O (\ t^2 \ e^{2\sqrt{a} \ t})$ as $t \rightarrow \infty$, respectively.

* * *

Proof :

$$E \ y_t{}^2 \ = \ E \ [\ \int_0^t x_s \ ds \ + \ v_t \ ]^2$$

$$= \ E \ [\ \int_0^t x_s \ ds \ ]^2 + E \ v_t \ \int_0^t x_s \ ds \ + \ E \ v_t{}^2$$

$$= \ t \ + E \ [\ \int_0^t x_s \ ds \ ]^2$$

since $\{ v_t \}$ is independent of $\{ w_t \}$, x_0 and $E \ v_t{}^2 \ = \ t$.

$$E \ [\ \int_0^t x_s \ ds \ ]^2 \ = \ \int_0^t \int_0^t E \ x_s \ x_\tau \ ds d \tau \ \leq \ t \ \int_0^t E \ x_s{}^2 ds$$

Thus

$$E \ y_t{}^2 \ \leq \ t \ (\ 1 + \int_0^t E \ x_s{}^2 ds \ ) \tag{16}$$

Let's now solve explicitly for $E x_t{}^2$.

It is well known that if ϕ is twice differentiable then [1]:

$$\frac{d}{dt} E \ \phi(x_t) \ = \ E \ [\ \phi_x(x_t) f \ (x_t) + \frac{1}{2} \ \phi_{xx}(x_t) \ ] \tag{17}$$

hence:

$$\frac{d}{dt} E \ x_t{}^2 \ = \ 1 + 2 E \ x_t \ f \ (x_t) \tag{18}$$

Furthermore, note that (2) implies that f_x and f_{xx} exist and are continuous, so that (17) can be used again for $\phi(x) \ = \ xf \ (x)$.

$$\frac{d}{dt} E \ x_t f \ (x_t) \ = \ E \ [\ f_x(x_t) + f^2(x_t) + \frac{1}{2} \ x_t \ (\ f_{xx}(x_t) + 2f \ (x_t)f_x(x_t) \ ) \ ]$$

by differentiating (2) we get :

$$f_{xx} + 2f f_x \ = \ 2ax + b$$

$$\frac{d}{dt} E \ x_t f \ (x_t) \ = \ 2a \ E \ x_t^2 + \frac{3}{2} \ b \ E \ x_t + c \tag{19}$$

Similarly:

$$\frac{d}{dt} E \ x_t \ = \ E \ f \ (x_t)$$

and

$$\frac{d}{dt} E \ f \ (x_t) \ = \ \frac{1}{2} E \ [\ f_{xx}(x_t) + 2f_x(x_t)f \ (x_t) \ ] \tag{20}$$

$$= \ a \ E \ x_t + \frac{b}{2}$$

Therefore

$$\frac{d^2}{dt^2} E \ x_t \ = \ a \ E \ x_t + \frac{b}{2}$$

which clearly implies that $E \ x_t \ = \ O(\ e^{\sqrt{a} \ t} \ )$ as $t \ \rightarrow \ \infty$.

By differentiating (18) and using (19) we get

$$\frac{d^2}{dt^2} E \ x_t^2 \ = \ 4a \ E \ x_t^2 + 3b \ E \ x_t + 2c \tag{21}$$

clearly $E \ x_t^2 \ = \ O(\ e^{2\sqrt{a} \ t} \ )$ as $t \ \rightarrow \ \infty$, which in turn implies by virtue of (16) that $E \ y_t^2$ is at most $O(\ te^{2\sqrt{a} \ t} \ )$ as $t \ \rightarrow \ \infty$.

Now

$$E \ [\int_0^t \ | \ y_s \ | \ ds \ ]^2 \ \leq \ t \ \int_0^t Ey_s^2 ds$$

and the remaining assertion follows.

* * *

Remark :

It is well known that if f is linear (hence $a > 0$), the stationary probability density exists and is Gaussian. In the particular case where $a = 0$, it is clear that for a stationary probability density to exist, it is necessary that the left hand sides of (20) and (21) be zero, i.e. $b = c = 0$; this leaves the explosive nonlinearity $f(x) = \dfrac{1}{x}$ which is not allowed (notice that $f(x) = th(x)$ satisfies (2) with $a = b = 0$ and $c = 1$).

We now state the main result

Proposition 4-2-3 :

Let μ_t and $\bar{\mu}_t$ be given by (4) and (6) respectively, then

$$\lim_{t \to \infty} E \left(\mu_t - \bar{\mu}_t \right)^2 = 0$$

* * *

Proof :

Using the alternative expressions (7) and (13) for μ_t and $\bar{\mu}_t$ we compute:

$$\mu_t - \bar{\mu}_t = (\sigma(t) - \bar{\sigma})y_t + \frac{m_1 - r_{12} + r_{13}}{1 - kr_{11}} - \frac{m_1 - r_{12} + \bar{r}_{13}}{1 - k\bar{r}_{11}}$$

which can be rewritten as

$$\mu_t - \bar{\mu}_t = (\sigma(t) - \bar{\sigma})y_t + \frac{m_1 + r}{1 - kr_{11}} - \frac{m_1 + \bar{r}}{1 - k\bar{r}_{11}} \tag{22}$$

where $r = r_{13} - r_{12}$ and $r = \bar{r}_{13} - r_{12}$ satisfy

$$\dot{r} = -kr - \frac{1}{2} br_{11} - (1 - kr_{11})y_t \quad ; \quad r(0) = 0 \tag{23}$$

$$\dot{\bar{r}} = -k\bar{r} - \frac{1}{2} b\bar{r}_{11} - (1 - k\bar{r}_{11})y_t \quad ; \quad \bar{r}(0) = 0 \tag{24}$$

$$\mu_t - \mu_t = (\sigma(t) - \bar{\sigma})y_t + h(t) + R(t) \tag{25}$$

where

$$h(t) = \frac{km_1(r_{11} - \overline{r}_{11})}{(1 - kr_{11})(1 - k\overline{r}_{11})} \tag{26}$$

$$R(t) = \frac{r}{1 - kr_{11}} - \frac{\overline{r}}{1 - k\overline{r}_{11}} \tag{27}$$

Now

$$\sigma(t) - \overline{\sigma} = \frac{1}{k}(\, th\,(kt) - 1\,)$$

$$= -\frac{2}{k}\frac{e^{-2kt}}{1 + e^{-2kt}}$$

(9), (12) and (14) imply

$$m_1 = e^{-kt}\,x$$

$$h(t) = -\frac{2e^{-3kt}}{1 + e^{-2kt}}x$$

A straightforward computation involving the integration of (17) and (18) yields

$$R(t) = \frac{b}{2k^2}e^{-kt}\,th\,(kt) + \frac{e^{-3kt}}{1 + e^{-2kt}}\int_0^t e^{ks}\,y_s\,ds - \frac{e^{-kt}}{1 + e^{-2kt}}\int_0^t e^{-ks}\,y_s\,ds$$

hence

$$|R(t)| \leq 2e^{-kt}\,[\,\frac{|b|}{4k} + \int_0^t |y_s|\,ds\,] \qquad a.s$$

$$|\mu_t - \overline{\mu}_t| \leq \frac{2}{k}e^{-2kt}\,|y_t| + |h(t)| + |R(t)| \qquad a.s$$

i.e.

$$|\mu_t - \overline{\mu}_t| \leq \frac{2}{k}e^{-2kt}\,|y_t| + 2e^{-3kt}\,|x| + 2e^{-kt}\,[\,\frac{|b|}{rk} + \int_0^t |y_s|\,ds\,]$$

After squaring both sides of (28) and taking expectations, we find that $E(\mu_t - \overline{\mu}_t)^2$ is bounded by a quantity which leading terms (up to positive constants) are the following:

$$T_1 = e^{-2kt}\,E\int_0^t |y_s|\,ds$$

$$T_2 = e^{-2kt}\,E\,[\int_0^t |y_s|\,ds\,]^2$$

$$T_3 = e^{-5kt}\,E\,|xy_t| \leq +\frac{1}{2}e^{-5kt}\,Ex^2 + \frac{1}{2}e^{-5kt}\,Ey_t^2$$

$$T_4 = e^{-3kt} E \mid y_t \mid$$

$$T_5 = e^{-3kt} E \mid y_t \mid \int_0^t \mid y_s \mid ds \le \frac{1}{2} e^{-3kt} E y_t^2 + \frac{1}{2} e^{-3kt} E [\int_0^t \mid y_s \mid ds]^2$$

$$T_6 = e^{-4kt} E \mid x \mid \int_0^t \mid y_s \mid ds \le \frac{1}{2} e^{-4kt} E x^2 + \frac{1}{2} e^{-4kt} E [\int_0^t \mid y_s \mid ds]^2$$

So clearly all these terms are less than $e^{-2kt} O(t^2 e^{2\sqrt{a}\, t})$ as $t \to \infty$ by last proposition. Thus they all go to zero as $t \to \infty$ since $2k := 2(1+a)^{1/2}$ is always bigger than $2\sqrt{a}$.

* * *

5   PERTURBED SYSTEMS

In this chapter we consider systems that contain a small parameter $\epsilon > 0$, namely weakly nonlinear systems and systems with low measurement noise level. Systems of the first type are modeled as :

$$
\begin{aligned}
dx_t &= a(t)x_t\, dt + \epsilon f(t,x_t)dt + \sigma(t)dw_t \\
dy_t &= c(t)x_t\, dt + \rho(t)dv_t
\end{aligned}
\tag{1}
$$

while those of the second type are :

$$
\begin{aligned}
dx_t &= g(t,x_t)dt + \sigma(t)dw_t \\
dy_t &= h(t,x_t)dt + \epsilon\, dv_t
\end{aligned}
\tag{2}
$$

It is well known that for filtering problems of this type there may be no finite set of equations which propagate the conditional mean.

We are interested in (one dimensional) suboptimal filters which are asymptotically optimal in the sense that the corresponding a priori mean square error (MSE) is identical, up to some power of ϵ, to the optimal one.

Weakly nonlinear systems have been studied in Brockett [12] where it was shown that in the general case, even to be optimal in the asymptotic sense, such filters must evolve in higher dimensional spaces than the state space. In section 5-1, it is shown that for a particular class of non linearities f (those with bounded derivatives), the " KF " and the BOF, which are both one dimensional filters with precomputable (non random) gains, are asymptotically optimal as $\epsilon \to 0$.

Next, the low measurement noise case, first studied in [7], [13], [14], [15] is treated in section 5-2 where the BOF and a constant gain version of it are shown to be asymptotically optimal; in addition, an even simpler asymptotically optimal filter is obtained. These same results have been obtained in [7], [15] by a different approach (e.g. an elaborate WKB procedure applied directly to the DMZ equation in [7]), while here, basic bounds on the a priori

MMSE and perturbation methods are used.

5-1 Weakly non linear systems :

Let x_t and y_t be given by

$$
\begin{aligned}
dx_t &= g(t,x_t)dt + \epsilon f(t,x_t) + \sigma(t)dw_t \\
dy_t &= h(t,x_t)dt + \rho(t)dv_t
\end{aligned}
\tag{3}
$$

where $\epsilon > 0$ is a small parameter and

$$
\begin{aligned}
&- \quad g \in <[a(t), \Delta a(t)] \quad ; \quad f \in <[\mu(t), \Delta\mu(t)] \\
&- \quad h \in <[c(t), \Delta c(t)] \quad with \quad \underline{c}(t) > 0 \quad ; \quad t \geq 0
\end{aligned}
$$

Upper and lower bounds on $p(t) := E(x_t - E(x_t \mid y_0^t))^2$, $p^*(t) := E(x_t - x_t^*)^2$ and $p^k(t) := E(x_t - x_t^k)^2$ (x_t^*, x_t^k being the BOF and KF estimators respectively) are used to establish that in the weakly nonlinear case, that is in the case g and h are linear, both filters are asymptotically optimal in the sense that p, p^* and p^k are the same up to first order in ϵ.

5-1-1 Asymptotic optimality of the BOF:

We recall that here the BOF x_t^* is given by:

$$
\begin{aligned}
dx_t^* &= g(t,x_t^*)dt + \epsilon f(t,x_t^*)dt + \frac{c(t)}{\rho^2(t)} u(t)[dy_t - h(t,x_t^*)dt] \\
x^*(0) &= 0 \\
\dot{u} &= \sigma^2(t) + 2(\overline{a}(t) + \epsilon\overline{\mu}(t)) u(t) - \frac{c^2(t)}{\rho^2(t)} u^2 \quad ; \quad u(0) = \sigma_0^2
\end{aligned}
\tag{4}
$$

Proposition 5-1 :

If $g(t,x_t) = a(t)x(t)$ and $h(t,x_t) = c(t)x(t)$; $c(t) > 0$, then, the BOF is asymptotically optimal as $\epsilon \to 0$, i.e.

$$p^*(t) = p(t) = r(t) + O(\epsilon) \tag{5}$$

where

$$\dot{r} = \sigma^2(t) + 2a(t)r(t) - \frac{c^2(t)}{\rho^2(t)} r^2 \quad ; \quad r(0) = r_0^2 \tag{6}$$

*   *   *

Remark :

If furthermore, the system is time invariant then

$$p^*(t) = p(t) = r(t) + 2\,\epsilon\,\mu \int_0^t \phi(t,s)r(s)ds + O(\epsilon, \Delta\alpha) \tag{7}$$

where

$$r(t) = \frac{\rho^2}{c^2}\left\{ a + \delta\frac{1-Ae^{-2\delta t}}{1+Ae^{-2\delta t}} \right\} \tag{8}$$

$$\delta = \sqrt{a^2 + \frac{\sigma^2}{\rho^2}c^2} \quad ; \quad A = \frac{\frac{c^2}{\rho^2}(a+\delta) - \sigma_0^2}{\sigma_0^2 - \frac{c^2}{\rho^2}(a-\delta)} \tag{9}$$

$$\phi(t,s) = e^{2a(t-s)}\exp\left\{ -2\frac{c^2}{\rho^2}\int_s^t r(\tau)d\tau \right\} \tag{10}$$

here $O(x,y)$ means order of each one of the arguments separately.

Proof :

From the summary in 2-3 we have

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{11}$$

where:

$$\dot{u} \;=\; \sigma^2(t) + 2(\,\bar{a}(t) + \epsilon\bar{\mu}(t)\,)u \;-\; \frac{c^{\,2}(t)}{\rho^2(t)}\,u^2$$

$$u(0) \;=\; \sigma_0^2 \tag{12}$$

$$\dot{l} \;=\; \sigma^2(t)+2(\underline{a}(t)+\epsilon\underline{\mu}(t))l-\frac{1}{\rho^2(t)}\,[\bar{c}^{\,2}(t)+4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta a(t)+\epsilon\Delta\mu(t))^2]l^2$$

$$l(0) = \sigma_0^2 \tag{13}$$

expanding $u(t)$ in the form :

$$u(t) \;\sim\; \sum_{i=0}^{\infty} u_i(t)\,\epsilon^i \tag{14}$$

gives :

$$u^2(t) \;\sim\; \sum_{k=0}^{\infty} c_k\,\epsilon^k$$

$$c_k \;=\; \sum_{j=0}^{n} u_j(t)u_{n-j}(t) \tag{15}$$

Plugging (14) and (15) in (12) and equating powers of ϵ yields:

$$\dot{u}_0 \;=\; \sigma^2(t) + 2\bar{a}(t)u_0 - \frac{c^{\,2}(t)}{\rho^2(t)}\,u_0^2 \qquad,\qquad u_0(0) \;=\; \sigma_0^2 \tag{16}$$

$$\dot{u}_1 \;=\; 2[\,\bar{a}(t) - \frac{c^{\,2}(t)}{\rho^2(t)}\,u_0(t)\,]u_1 + 2\bar{\mu}(t)u_0(t) \qquad,\qquad u_1(0) \;=\; 0 \tag{17}$$

Proceeding similarly for $l(t)$, one obtains:

$$\dot{l}_0 \;=\; \sigma^2(t) + 2\underline{a}(t)l_0 - \frac{1}{\rho^2}\,[\,\bar{c}^{\,2}(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}\,\Delta a^2(t)\,]\,l_0^2$$

$$l_0(0) \;=\; \sigma_0^2 \tag{18}$$

$$\dot{l}_1 = 2[\underline{a}(t)-\frac{1}{\rho^2(t)}\,(\bar{c}^{\,2}(t)+4\frac{\rho^2(t)}{\sigma^2(t)}\,\Delta a^2(t))l_0]l_1+2\underline{\mu}(t)l_0-8\frac{\Delta a\,\Delta\mu}{\sigma^2}\,l_0^2$$

$$l_1(0) \;=\; 0 \tag{19}$$

(here $\Delta a^2 := (\Delta a)^2$)

It is clear from (16) and (18) that $u_0(t)$ and $l_0(t)$ are different in the general case but coincide with $r(t)$ if $\Delta a = \Delta c = 0$ that is :

$$g(t,x) \;=\; a(t)x \qquad \text{and} \qquad h(t,x) \;=\; c(t)x$$

* * *

Now if the system is time invariant i.e. :

$$a(t) = a \quad ; \quad \mu(t) = \mu \quad ; \quad c(t) = c \quad ; \quad \sigma(t) = \sigma \quad \text{and} \quad \rho(t) = \rho$$

then one easily gets the results in the remark above by using the change of variable $r = \dfrac{\rho^2}{c^2} \dfrac{\dot{w}}{w}$ to solve (6) and the variation of constants formula in (17) and (19).

5-1-2 Asymptotic optimality of the KF :

The question considered here is whether one could, in the case of weakly nonlinear systems, ignore the nonlinear part in the drift, use the Kalman filter designed for the underlying linear system (driven by y_t) and be able to achieve asymptotic optimality as $\epsilon \to 0$. As already mentioned, this filtering scheme is what is referred to as the " KF " (See Chapter 3).

Proposition 5-2 :

If $\quad g(t,x) = a(t)x \quad , \quad h(t,x) = c(t)x \quad , \quad c(t) > 0 \quad ;$ then the KF given by :

$$dx_t^k = a(t)x_t^k dt + \frac{c(t)}{\rho^2(t)} r(t) [dy_t - c(t)x_t^k dt] \quad ; \quad x^k(0) = 0$$

where $r(t)$ is as in (6), is asymptotically optimal as $\epsilon \to 0$ in the sense that

$$p(t) = p^k(t) = r(t) + O(\epsilon) \qquad 0 \le t \le T$$

$$* \quad * \quad *$$

Proof :

Following the steps in Proposition 3-4 one gets :

$$l(t) \le p(t) \le p^k(t) \le q^\epsilon(t)$$

where:

$$\dot{l} = \sigma^2(t) + 2(a(t) + \epsilon \mu(t))l - \frac{1}{\rho^2(t)} [c^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)} \Delta \mu^2(t)\epsilon^2]l^2$$
$$l(0) = \sigma_0^2$$

and

$$\dot{q}^{\epsilon}(t) = i(t) + j(t)q^{\epsilon} \qquad ; \qquad q^{\epsilon}(0) = \sigma_0^2$$

$$i(t) = \sigma^2(t) + \frac{c^2(t)}{\rho^2(t)} r^2(t) + \epsilon E f^2(t,x_t)$$

$$j(t) = 2(a(t) - \frac{c(t)}{\rho^2(t)} r(t)) + \epsilon$$

Expanding $q^{\epsilon}(t)$ in the form :

$$q^{\epsilon}(t) \sim \sum_{i=0}^{\infty} q_i^{\epsilon}(t) \epsilon^i$$

and equating powers of ϵ yields :

$$\dot{q}_0^{\epsilon} = \sigma^2(t) + \frac{c^2(t)}{\rho^2(t)} r^2(t) + 2[a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)]q_0^{\epsilon} \qquad ; \qquad q_0^{\epsilon}(0) = \sigma_0^2$$

Let $w := q_0^{\epsilon}(t) - l_0(t)$. Then by the previous section: $w(t) = q_0^{\epsilon}(t) - r(t)$

$$\dot{w}(t) = \frac{c^2(t)}{\rho^2(t)} r^2(t) + 2[a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)]q_0^{\epsilon} - 2a(t)r(t) + \frac{c^2(t)}{\rho^2(t)} r^2(t)$$

This easily becomes :

$$\dot{w} = 2 [a(t) - \frac{c^2(t)}{\rho^2(t)} r(t)] w \qquad ; \qquad w(0) = 0$$

The solution clearly is $w(t) = 0$ which implies $q_0^{\epsilon} = r$.

* * *

5-2 Low measurement noise level :

Consider the system :

$$
\begin{aligned}
dx_t &= g(t,x_t)dt + \sigma(t)dw_t \\
dy_t &= h(t,x_t)dt + \epsilon\, dv_t
\end{aligned}
$$

(20)

where
$$g \in < [a(t) , \Delta a(t)]$$

$$h \in < [c(t) , \Delta c(t)] \qquad ; \qquad \underline{c}(t) \geq 0 \qquad ; \qquad t \geq 0$$

and $\epsilon > 0$ is a small parameter (this is the case in many practical situations [13], [15]).

The optimal a priori MSE is bounded from above and below ; perturbation methods for the bounds are used to show for three suboptimal filters that the upper bound approaches the lower one as ϵ becomes smaller.

The result is quoted for h linear but holds for nonlinearities h which tend asymptotically to be linear, i.e. Δc is small (see remark 2). This type of (almost linear) nonlinearities arise in practice and are usually modeled as being linear [10].

Proposition 5-3 :

If in (20) $\quad h(t,x) = c(t)x \quad$ and $\quad c(t) > 0 \quad$ then

$$p(t) = \frac{\sigma(t)}{c(t)} \epsilon + O(\epsilon) = E(x_t - x_t^F)^2 \tag{21}$$

where $\quad \lim\limits_{\epsilon \to 0} \dfrac{O(\epsilon)}{\epsilon} = 0 \quad$ and $\quad x_t^F$ denotes anyone of the three asymptotically optimal filters listed below.

(1)   The BOF :

$$dx_t^* = g(t,x_t^*)dt + \frac{c(t)}{\epsilon^2} u(t)\,[\,dy_t - c(t)x_t^*dt\,] \quad , \quad x^*(0) = 0 \tag{22}$$

$$\dot{u}(t) = \sigma^2(t) + 2\,\bar{a}(t)\,u(t) - \frac{c^2(t)}{\epsilon^2} u^2(t) \quad ; \quad u(0) = \sigma_0^2 \tag{23}$$

(2)   The constant gain BOF :

$$dx_t^c = g(t,x_t^c)dt + \frac{\sigma(t)}{\epsilon}\,[\,dy_t - cx_t^cdt\,] \quad ; \quad x_t^c(0) = 0 \tag{24}$$

(3)   The linear (first approximation) BOF :

$$dx_t^L = \frac{\sigma(t)}{\epsilon}\,[\,dy_t - c(t)\,x_t^L\,dt\,] \quad ; \quad x^L(0) = 0 \tag{25}$$

* * *

Proof :

(21) is proven for each case separately.

Case 1 :

From section 2-3 we get :

$$l(t) \leq p(t) \leq p^*(t) = E (x_t - x_t^*)^2 \leq u(t) \tag{26}$$

where:

$$\dot{u} = \sigma^2(t) + 2\bar{a}(t)u - \frac{c^2(t)}{\epsilon^2} u^2 \quad ; \quad u(0) = \sigma_0^2 \tag{27}$$

$$\dot{l} = \sigma^2(t) + 2\underline{a}(t)l - \frac{1}{\epsilon^2} [\bar{c}^2(t) + 4\frac{\epsilon^2}{\sigma^2(t)}(\Delta a)^2] l^2$$

$$l(0) = \sigma_0^2 \tag{28}$$

It can be easily seen by inspection of (27) and (28) that $u(t)$ and $l(t)$ are of different order in ϵ if Δc is nonzero. Let's show this explicitly.

Expanding $u(t)$ as

$$u(t) \sim \sum_{n=0}^{\infty} u_n(t) \epsilon^n \tag{29}$$

gives:

$$u^2(t) \sim \sum_{n=0}^{\infty} d_n \epsilon^n$$

$$d_n(t) = \sum_{j=0}^{n} u_j(t)u_{n-j}(t) \tag{30}$$

e.g.

$$d_0(t) = u_0^2(t)$$
$$d_1(t) = 2u_0(t)u_1(t)$$
$$d_2(t) = 2u_0 u_2 + u_1^2$$

Plugging (29) and (30) in (27) gives:

$$\sum_{n=0}^{\infty} \dot{u}_n \epsilon^n = \sigma^2(t) + 2\bar{a} \sum_{0}^{\infty} u_n \epsilon^n - \frac{\bar{c}^2}{\epsilon^2} \sum_{0}^{\infty} d_n \epsilon^n \tag{31}$$

Equating powers of ϵ, starting with ϵ^{-2}, yields:

$$d_0 = 0 \quad \text{i.e.} \quad u_0(t) = 0.$$

This in turn implies that $\quad d_1 = 0.$

Similarly $\quad \sigma^2 - \underline{c}^2 d_2 = 0.$ But $\quad d_2 = u_1^2 \quad$ hence

$$u_1(t) = \frac{\sigma(t)}{\underline{c}(t)} \tag{32}$$

i.e.

$$u(t) = \frac{\sigma(t)}{\underline{c}(t)} \epsilon + O(\epsilon^2) \quad \text{for } each \quad t > 0 \tag{33}$$

By a similar procedure we get $\quad l_0 = 0 \quad$ and $\quad l_1 = \frac{\sigma(t)}{\overline{c}(t)} \quad$ that is

$$l(t) = \frac{\sigma(t)}{\overline{c}(t)} \epsilon + O(\epsilon^2) \qquad t > 0 \tag{34}$$

Note: These approximations are obviously not valid in the immediate vicinity of $t = 0$ where $u(0) = l(0) = \sigma_0^2$. This (boundary layer) problem is neglected here.

We conclude from (33) and (34) that if $\Delta c = 0$, i.e., $h(t,x) = c(t)x$ then:

$$u(t) = l(t) = \frac{\sigma(t)}{c(t)} \epsilon + O(\epsilon^2) \qquad t > 0 \tag{35}$$

which establishes the asymptotic optimality of the BOF as $\epsilon \to 0$.

This suggests the following:

(i) since $u(t) = \epsilon u_1(t) + O(\epsilon^2)$, one can replace $u(t)$ in (22) by $\epsilon u_1 = \epsilon \frac{\sigma(t)}{c(t)}$ and hope to achieve asymptotic optimality as well. The new filter clearly would have the advantage that the gain $k(t) = \frac{\sigma(t)}{\epsilon}$, thus avoiding solving a Riccati equation and therefore resulting in faster computations.

(ii) If the answer to (i) is affirmative, the next question is whether the same thing would hold for the first approximation (when expanding x_t^c) filter:

$$dx_t{}^L \;=\; \frac{\sigma(t)}{\epsilon}\;[\; dy_t - c\,(t)\; x_t{}^L dt \;]$$

It turns out that both filters are asymptotically optimal as is shown in case 2 and 3.

Case 2 :

An upper bound on filters of the type 2 has already been given in (44) Proposition 2-2. In this case

$$E\;(\; x_t - x_t{}^c\;)^2 \;\leq\; u^k\,(t) \tag{36}$$

where $\;(\; k\,(t)\;=\;\dfrac{\sigma(t)}{\epsilon}\;)$:

$$
\begin{aligned}
\dot{u}^k &\;=\; 2\sigma^2(t) + 2\;[\;\overline{a}\,(t) - \frac{\sigma(t)c\,(t)}{\epsilon}\;]\;u^k \\
u^k\,(0) &\;=\; \sigma_0^2
\end{aligned}
\tag{37}
$$

By setting $\quad u^k\,(t,\epsilon)\;\sim\;\sum u_i{}^k(t)\epsilon^i\quad$ in (37), one easily obtains

$$u_0^k\,(t)\;=\;0 \qquad;\qquad u_1^k\,(t)\;=\;\frac{\sigma(t)}{c\,(t)}$$

hence:

$$p\,(t)\;=\;E\;(x_t - x_t{}^c)^2\;=\;\frac{\sigma(t)}{c\,(t)}\;\epsilon + O\,(\epsilon^2)\;,\qquad t\;>\;0$$

(Recall that : $\quad p\,(t)\;\geq\;l\,(t)\;=\;\dfrac{\sigma(t)}{c\,(t)}\;\epsilon + O\,(\epsilon^2)\;)$

Case 3 :

An upper bound on $\;p^L\,(t)\;:=\;E\;(\;x_t - x_t{}^L\;)^2\;$ is obtained by following the first steps in the proof of proposition 2-2. Instead of (46) we get by making

$$k\,(t)\;=\;\frac{\sigma(t)}{\epsilon}$$

$$\dot{p}^L\;=\;2\sigma^2(t) + 2\,E\;(\;x_t - x_t{}^L\;)g\,(t,x_t) - 2\frac{c\,(t)\sigma(t)}{\epsilon}\;p^L$$

Using the Schwartz inequality:

$$E\;ab\;\leq\;E^{\frac{1}{2}}\,a^2\;.\;E^{\frac{1}{2}}\,b^2$$

and the comparison theorem we get $\quad p^L(t) \leq u^L(t) \quad$ where

$$\dot{u}^L = 2\sigma^2(t) + 2\theta(t)\left(u^L\right)^{\frac{1}{2}} - 2\frac{c(t)\sigma(t)}{\epsilon} u^L \tag{38}$$

with $\qquad \theta(t) = E^{\frac{1}{2}} g^2(t, x_t)$.

Expanding $\qquad u^L \sim \sum_0^\infty u_i^L \epsilon^{\frac{i}{2}} \qquad$ in (38) and equating powers of ϵ gives

$$u_0^L = u_1^L = 0 \quad \text{and} \quad u_2^L = \frac{\sigma(t)}{c(t)}$$

hence

$$u^L(t) = \frac{\sigma(t)}{c(t)} \epsilon + O(\epsilon^{\frac{3}{2}}) \quad , \quad t > 0$$

Therefore

$$p(t) = p^L(t) = \frac{\sigma(t)}{c(t)} \epsilon + O(\epsilon) \qquad t > 0$$

<p style="text-align:center">* * *</p>

Remark (1) :

(i) If $\sigma(t) = \sigma$ and $c(t) = c$ then $\dot{u}_1(t) = \dot{l}_1(t) = 0$ and the next terms in the expansion of $u(t)$ and $l(t)$ are :

$$u_2(t) = \frac{1}{c^2} \overline{a}(t)$$

$$l_2(t) = \frac{1}{c^2} \underline{a}(t)$$

so that $\quad u(t) = l(t) + O(\epsilon^3) \quad$ if and only if $\quad \Delta a = 0$, i.e., both g and h are linear.

(ii) In [9], it was shown that for incrementally conic nonlinearities we have the following lower bound $L(t)$:

$$p(t) \geq L(t) = (1 - s(t)) r(t) \tag{39}$$

where $s(t)$ is the unique nonnegative root of

$$(1 - s (t)) \, e^{s(t)} = e^{-d(t)}$$

$$d (t) = \int_0^t [\frac{\Delta a (s)}{\sigma^2(s)} + \frac{\Delta c^2(s)}{\epsilon^2}] \, q (s)ds \tag{41}$$

$$\dot q = \sigma^2(t) + \frac{c^2(t)}{\epsilon^2} r^2(t) + 2[\, \overline a(t) - \frac{c^2(t)}{\epsilon^2} r \,] \, q$$

$$q (0) = \sigma_0^2 \tag{42}$$

$$\dot r = \sigma^2(t) + 2 a (t)r - \frac{c^2(t)}{\epsilon^2} r^2$$

$$r (0) = \sigma_0^2 \tag{43}$$

From (27) and (33) we readily get that

$$r (t) = \frac{\sigma(t)}{c (t)} \epsilon + O(\epsilon^2)$$

and it is therefore clear from (39) that if $s (t) = O(\epsilon)$ then $L (t) = \frac{\sigma(t)}{c (t)} \epsilon + O(\epsilon^2)$ the same as we have just obtained using the Zakai-Bobrovsky lower bound $l(t)$.

This is indeed the case: (42) implies $q (t) = O(\epsilon)$ and (41) that $d (t) = O(\epsilon)$ ($\Delta c = 0$).

Assuming $s (t) \sim \sum_0^\infty s_n \epsilon^n$ and letting ϵ go to zero in (40) gives that $1 - s_0 = e^{-s_0}$ necessarily. This has the unique solution $s_0 = 0$, hence $s (t) = O(\epsilon)$.

Remark (2) : almost linear observations.

The same results in previous proposition can be extended to the particular class of nonlinearities $h \in < [c , \Delta c]$ where Δc is also a small parameter. Indeed; the upper and lower bounds u and l on $p (t)$ and $p^*(t) := E (x_t - x_t^*)^2$ where x_t^* is the BOF in (1) with cx_t^* replaced by $h (x_t^*)$ are given by (33) and (34) ;

$$u (t) = \frac{\sigma(t)}{\underline c (t)} \epsilon + O(\epsilon^2)$$

$$= \frac{\sigma(t)}{c(t)} \epsilon \left(1 + \frac{\Delta c}{c} + O\left((\Delta c)^2\right) \right) + O\left(\epsilon^2\right)$$

$$= \frac{\sigma(t)}{c(t)} \epsilon + \frac{1}{c} \epsilon \Delta c + O\left(\epsilon^2\right) + \epsilon O\left((\Delta c)^2\right)$$

so if $\Delta c \to 0$ we certainly can consider that

$$u(t) = \frac{\sigma(t)}{c(t)} \epsilon + O\left(\epsilon^2\right)$$

Similarly

$$l(t) = \frac{\sigma(t)}{\overline{c}(t)} \epsilon + O\left(\epsilon^2\right)$$

$$= \frac{\sigma(t)}{c(t)} \epsilon \left(1 - \frac{\Delta c}{c} + O\left((\Delta c)^2\right) \right) + O\left(\epsilon^2\right)$$

$$= \frac{\sigma(t)}{c(t)} \epsilon + O\left(\epsilon^2\right)$$

It is not hard either to establish that for the analogous of the filters 2 and 3 (as in 2 and 3, but with cx replaced by $h(x)$) the upper bounds are

$$u^k(t) = \frac{\sigma(t)}{\underline{c}(t)} \epsilon + O\left(\epsilon^2\right)$$

and

$$u^L(t) = \frac{r(t)}{\underline{c}(t)} \epsilon + O\left(\epsilon^{\frac{3}{2}} \right)$$

which makes these filters asymptotically optimal too as Δc and ϵ become smaller with

$$p(t) = \frac{\sigma(t)}{c(t)} \epsilon + O(\epsilon).$$

6   CONCLUSION

We investigated the asymptotic behavior question of one dimensional nonlinear filtering problems involving drifts with bounded derivatives using an upper and lower bound approach to show that the a priori mean square error associated with some suboptimal filters approaches the optimal one asymptotically.

This approach proved that significant information relevant to this type of problems can be infered from the knowledge of the derivative bounds (i.e., of the cone in which the nonlinearities reside).

In particular, it is shown that in the case of asymptotically time invariant systems for which the limiting system is linear, the " KF " and " SSKF " (designed for the limiting linear system) are asymptotically optimal as $t \to \infty$ (chapter 3). In other words the nonlinearity can be ignored as long as the long time behavior is concerned.

The same " KF " designed for the underlying linear system in weakly nonlinear systems is shown to be asymptotically optimal as $\epsilon \to 0$, while a simple asymptotically optimal linear filter, not involving the drift, is provided for nonlinear models with low measurements noise level (chapter 5).

The overall performence of these filters, tough asymptotically optimal, will strongly depend on the derivatives bounds that is on the nonlinearities shape; the more variations there is, the poorest the overall performence.

An attempt to characterize the stationary behavior of the Benes filter has been made in chapter 4 where a partial result was obtained. The global question raised there remains unanswered.

APPENDIX

Theorem (1) :   comparison theorem [25]

Let $F(x,y)$ and $G(x,y)$ be continuous in the rectangle

$$D : \quad |\, x - x_0 \,| \; < \; a \quad , \quad |\, y - y_0 \,| \; < \; b$$

and suppose that $F(x,y) \; < \; G(x,y)$ everywhere in D. Let $y(x)$ and $z(x)$ be the solutions of

$$\dot{y} \; = \; F(x,y) \quad , \quad y(x_0) = \alpha$$
$$\dot{z} \; = \; G(x,y) \quad , \quad z(x_0) = \alpha$$

Let I be the largest subinterval of $(\, x_0 - a \; , \; x_0 + a \;)$ where both $y(x)$ and $z(x)$ are defined and continuous ; then for $x \in I$

$$z(x) \; < \; y(x) \quad , \quad x \, < \, x_0$$
$$z(x) \; > \; y(x) \quad , \quad x \, > \, x_0$$

Theorem (2) :   Perron [26]

If $F(t)$, $f_i(t)$, $t_0 \in [\, 0 \, , \infty\, [$, $i = 1,.....,n$, are real continuous functions of t having finite limits $\lim\limits_{t \to \infty} F(t) \; = \; b$, $\lim\limits_{t \to \infty} f_i \; = \; a_i$, if the roots

λ_i , $i = 1,....,n$ of the equation

$$\rho^n \, + \, a_1 \, \rho^{n-1} + \, ... + \, a_n \; = \; 0$$

are real, distinct, and different from 0, then the equation

$$\frac{d^n}{dt^n} \; y(t) + f_1(t) \, \frac{d^{n-1}}{dt^{n-1}} \; y(t) + \, ... + f_n(t) \, y(t) \; = \; F(t) \qquad (*)$$

has at least one solution $y(t)$ with

$$\lim\limits_{t \to \infty} y(t) \; = \; \frac{b}{a_n} \quad , \qquad \lim\limits_{t \to \infty} \frac{d^m}{dt^m} \; y(t) \; = \; 0 \, .$$

If $\lambda_i \; < \; 0$, $i = 1,........,n$, then all solutions of (*) have these properties.

Theorem (3) :

Let A , B and C be $n \times n$, $n \times m$ and $p \times n$ matrices respectively. If the triplet

(A,B,C) is minimal then the solution $P(t)$ of the matrix Riccati equation :

$$\dot{P}(t) = B \ B^T + A \ P(t) + P(t) \ A^T - P(t) \ C^T \ C \ P(t)$$
$$P(0) = P_0 \geq 0$$

exists and is a continuous, non-negative monotone $n \times n$ matrice. Furthermore

$\lim_{t \to \infty} P(t) = \overline{P}$ where $\overline{P}$ satisfies the algebraic matrix Riccati equation

$$B \ B^T + A \ \overline{P} + \overline{P} \ A^T - \overline{P} \ B^T \ B \ P = 0$$

Proof : see [27], [28], [29].

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