ABSTRACT

Title of dissertation: ON THE GROMOV-WITTEN THEORY OF $\mathbb{P}^1$-BUNDLES OVER RULED SURFACES

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Let C be a smooth, connected, complex, projective curve of genus g and let D_1, D_2 be divisors of degree k_1, k_2 respectively. Let S be the decomposable ruled surface given by the total space of the following $\mathbb{P}^1$-bundle over C:

$$p_C : \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D_1)) \to C.$$

Let C_0 be the locus of $(1 : 0)$ in $S \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D_1))$. Then let X be the threefold given by the total space of the following $\mathbb{P}^1$-bundle over S:

$$p_S : \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-E)) \to S$$

where $E = aC_0 + p_C^{-1}(D_2)$. This determines an $\mathcal{H}_a$-bundle over C where $\mathcal{H}_a$ is a Hirzebruch surface.

In this thesis we determine the equivariant Gromov-Witten partition function for all "section classes" of the form $s + m_1 f_1 + m_2 f_2$ where s is a section of the map $X \to C$ and f_1, f_2 are fiber classes, in the case that $a = 0, -1$. A class is Calabi-Yau

if $K_X \cdot \beta = 0$. For $a = 0$, the partition function of Calabi-Yau section classes is given by

$$Z(g|k_1,k_2) = \begin{cases} 4^g \phi^{2g-2} v_1^{\frac{g-1+k_1}{2}} v_2^{\frac{g-1+k_2}{2}} & (g-1) \equiv k_1 \equiv k_2 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

where v_1, v_2 count the number of fibers and $\phi = 2\sin\frac{u}{2}$. In the case that $a = -1$ the partition functions of Calabi-Yau section classes satisfy the following relations

$$Z(g|k_1,k_2) = Z(g|k_1-2,k_2+1)$$

$$Z(g|k_1,k_2) = v_1^2 Z(g|k_1-3,k_2) + v_1^2 v_2 Z(g|k_1-4,k_2)$$

$$Z(g|k_1,k_2) = -\phi^2 v_1^2 v_2^{-2} Z(g-1|k_1,k_2)$$

$$+6\phi^4 v_1^2 v_2^{-1} Z(g-2|k_1,k_2) + (256\phi^8 v_1^2 v_2 + 27\phi^8 v_1^4 v_2^{-2}) Z(g-4|k_1,k_2)$$

which allow us to compute all the Calabi-Yau section class invariants from the following base cases:

	$g=0$	1	2	3
$k_1 = 0$	0	4	$-\phi^2 v_1^2 v_2^{-2}$	$12\phi^4 v_1^2 v_2^{-1} + \phi^4 v_1^4 v_2^{-4}$
1	ϕ^{-2}	0	$\phi^2 v_1^2 v_2^{-1}$	$16\phi^4 v_1^2 - \phi^4 v_1^4 v_2^{-3}$
2	0	0	$8\phi^2 v_1^2$	$64\phi^4 v_1^2 v_2 + \phi^4 v_1^4 v_2^{-2}$
3	0	$3v_1^2$	$16\phi^2 v_1^2 v_2$	$-\phi^4 v_1^4 v_2^{-1}$

As a corollary, we establish the Gromov-Witten/Donaldson-Thomas/Stable Pairs correspondence for the Calabi-Yau section class partition functions for these families of non-toric threefolds.

ON THE GROMOV-WITTEN THEORY OF $\mathbb{P}^1$-BUNDLES OVER RULED SURFACES

by

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Dedication

To Amy and baby Henry, and a life of learning and growing together

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Chapter 0: Introduction

0.1 Gromov-Witten Theory

Enumerative geometry is a branch of algebraic geometry which studies how to count the number of solutions to various geometric questions. We are particularly interested in *curve counting*. Some excellent introductory resources on curve counting are [12, 28]. A good curve counting theory needs two important properties. The first is we would like a compact moduli space with an open component corresponding to embeddings of non-singular curves. Then non-singular curves may degenerate to more general objects in flat families. The second issue is related to transversality. In the ideal situation, our moduli space would cut be out of an ambient space by equations defining subschemes which meet transversely. In practice, moduli spaces often have irreducible components of different dimensions.

Remarkably, the first theory to adequately address these issues arose from the interaction between mathematics and theoretical physics (string theory). A string traces out a two-dimensional manifold in space-time. In closed A-model topological string theory, the space-time is a six dimensional manifold and worldsheets are parameterized by pseudo-holomorphic curves [11]. The path integrals in this theory are integrals over the moduli space of pseudo-holomorphic curves and are called Gromov-Witten invariants. These are named for Gromov's work on pseudo-holomorphic curves in symplectic geometry and Witten's work on topological strings.

Kontsevich introduced the moduli space of stable maps in 1992. This provides a mathematically rigorous framework for these integrals. Let the target space X be a non-singular, complex, projective variety. The moduli space $\overline{\mathcal{M}}_g(X, \beta)$ parameterizes maps from nodal curves C to X with homology class $\beta \in H_2(X, \mathbb{Z})$. These maps must be stable, which means they have finite automorphism groups.

This built on previous work on moduli spaces of curves. Deligne-Mumford stacks were introduced by Pierre Deligne and David Mumford in 1969. These have an atlas given by a surjective étale map from a scheme. Roughly, these Deligne-Mumford stacks parameterize objects which have finite automorphism groups. Stable curves are Riemann surfaces with at worst nodal singularities and finite automorphism groups. They introduced Deligne-Mumford stacks $\overline{\mathcal{M}}_g$ which parameterize stable curves (see [1] for a good account). In the case that X is a point, Kontsevich's space reduces to $\overline{\mathcal{M}}_g$.

The moduli space of stable maps $\overline{\mathcal{M}}_g(X, \beta)$ is a compact Deligne-Mumford stack. While it does not have a well-defined dimension, there is an expected or virtual dimension

$$\text{vdim } \overline{\mathcal{M}}_g(X, \beta) = -K_X \cdot \beta + (\text{dim} X - 3)(1 - g).$$

For a threefold the virtual dimension has no genus dependence. For a Calabi-Yau threefold (that is K_X is trivial), or more generally for a Calabi-Yau class on a threefold (that is $K_X \cdot \beta = 0$) the virtual dimension is zero. There is virtual fundamental class $[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(X,\mathbb{Z})$ (see [2] for the construction). When X is a threefold and β is a Calabi-Yau class the Gromov-Witten partition function may be defined by

$$Z_\beta^{\mathrm{GW}}(u) = \sum_{g\in\mathbb{Z}} u^{2g-2} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1.$$

0.2 Calculations in Gromov-Witten Theory

A general quintic threefold has finitely many lines, each of which has normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. What is the contribution of each line to the Gromov-Witten partition function? Faber and Pandharipande [7] computed this contribution using the technique of (virtual) localization. The $\mathbb{C}^*$-action on the line $\mathbb{P}^1$ induces an action on the moduli space of stable maps to $\mathbb{P}^1$. Localization reduces the calculation of Gromov-Witten invariants to contributions from the fixed points of the moduli space. For degree one maps the contribution to the partition function is ϕ^{-2} where $\phi := 2\sin(u/2)$.

More generally, Pandharipande [23] calculated the contribution of a nonsingular embedded curve of genus g, representing an infinitesimally isolated solution to incidence conditions to be $\phi^{2g-2-K_X\cdot\beta}$.

Bryan and Pandharipande [6] considered isolated genus g curves with normal bundle $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ (with no incidence conditions). In general the total space of the normal bundle is not a Calabi-Yau threefold, so we don't have interesting numerical invariants without imposing incidence conditions. Therefore they define invariants by equivariant pushforward to the point. These invariants take values in

$\mathbb{Q}(t_1, t_2)$. For degree one maps the result is $(t_1 t_2)^{g-1} t_1^{-k_1} t_2^{-k_2} \phi^{k_1+k_2}$.

If one can degenerate a smooth target X to the union of two threefolds X', X'' glued along a smooth divisor, Li's degeneration formula [14] relates the Gromov-Witten invariants of the threefold X to the (relative) Gromov-Witten invariants of X', X''. In [6], this degeneration formula leads to a 1+1 dimensional topological quantum field theory (TQFT) formalism for the Gromov-Witten theory of local curves. Gholampour [9] used degeneration and TQFT techniques to study Gromov-Witten invariants of $\mathbb{P}^2$-bundles over a curve C. If g is the genus of C and β is a Calabi-Yau section class then the partition function is $3^g \phi^{2g-2}$.

This raises the question of what other toric surface bundles over curves are amenable to these methods. For $\mathbb{P}^1 \times \mathbb{P}^1$-bundles we obtain an explicit formula for the equivariant partition functions of section classes. For certain Calabi-Yau section classes, the partition function is $4^g \phi^{2g-2}$, and for others the partition function vanishes. Studying the case of $\mathcal{H}_1$-bundles (where $\mathcal{H}_1$ is a Hirzebruch surface), we find the enumerative geometry of section classes for toric surface bundles over a curve to be richer than anticipated from the first two examples. We fully determine the equivariant partition functions for all section classes. In the non-equivariant limit, we find recursions relating the Calabi-Yau section class partition functions for bundles with different parameters.

0.3 GW/DT/PT Correspondence

Two other curve counting theories are Donaldson-Thomas theory [17] and Stable Pairs (Pandharipande-Thomas) [29] theory. The Hilbert scheme $\mathrm{Hilb}_n(X, \beta)$ parameterizes one dimensional subschemes $C \subset X$ such that $\chi(\mathcal{O}_C) = n$ and $[C] = \beta \in H_2(X, \mathbb{Z})$. Let $I_n(X, \beta)$ be the moduli space of ideal sheaves $\mathcal{J}$ such that $\mathrm{ch}(\mathcal{J}) = (1, 0, -\beta, -n)$. The map $C \mapsto \mathcal{J}_C$ taking a subscheme to its ideal sheaf determines an isomorphism of schemes $\mathrm{Hilb}_n(X, \beta) \to I_n(X, \beta)$. However, the deformation theory of ideal sheaves leads to a virtual fundamental class. The support C may have zero-dimensional components. These may be embedded points or free points. This is a technical disadvantage of the theory.

Stable pairs are two-term complexes $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ where $\mathcal{F}$ is pure with one-dimensional support C, s has a zero dimensional cokernel Q, $\chi(\mathcal{F}) = n$ and $[C] = \beta$. The curve C has no embedded points, but the support of Q consists of points on C. Thus this theory avoids the free points which occur in Donaldson-Thomas theory. There is a moduli scheme $P_n(X, \beta)$ parameterizing stable pairs. The deformation theory of complexes in the derived category leads to a virtual fundamental class.

If X is a threefold, and β is a Calabi-Yau class ($K_X \cdot \beta = 0$) then conjecturally [18, 29] the three partition functions[1]

$$Z_\beta^{\mathrm{GW}}(u) = \sum_{g \in \mathbb{Z}} u^{2g-2} \int_{[\overline{\mathcal{M}}_g^\bullet(X,\beta)]^{\mathrm{vir}}} 1,$$

$$Z_\beta^{\mathrm{DT}}(q) = \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1,$$

[1] here we allow disconnected domain curves for stable maps

$$Z_\beta^{\mathrm{PT}}(q) = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1,$$

are related by the change of variables $q = -e^{iu}$,

$$Z_\beta^{\mathrm{GW}}(u) = \frac{Z_\beta^{\mathrm{DT}}(q)}{Z_0^{\mathrm{DT}}(q)} = Z_\beta^{\mathrm{PT}}(q).$$

Maulik, Oblomkov, Okounkov, and Pandharipande [19] proved the GW/DT correspondence for toric threefolds. Pandharipande and Pixton [27] proved the GW/PT correspondence for toric threefolds. In [26] Pandharipande and Pixton proved the GW/PT correspondence for a number of non-toric varieties including Calabi-Yau complete intersections in products of projective spaces.

We prove the correspondence for the section class partition functions for our non-toric threefolds, providing new examples of the correspondence.

Chapter 1: Geometry of $\mathbb{P}^1$-Bundles Over Ruled Surfaces

In this chapter we will introduce a family of threefolds whose Gromov-Witten invariants will be our main object of study. Let C be a smooth, connected, complex, projective curve of genus g and let D_1, D_2 be divisors of degree k_1, k_2 respectively. Let S be the decomposable ruled surface given by the total space of the following $\mathbb{P}^1$-bundle over C:

$$p_C : \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D_1)) \to C.$$

We let $s = c_1(\mathcal{O}_S(1)) \in H^2(S, \mathbb{Z})$ and let f_1 denote the cohomology class of a fiber. Let C_0 be the locus of $(1:0)$ in $S \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D_1))$ and let C_1 be the locus of $(0:1)$ in $S \cong \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}_C)$. The cohomology classes of C_0 and C_1 are s and $s + k_1 f_1$, respectively.

Then let X be the threefold given by the total space of the following $\mathbb{P}^1$-bundle over S:

$$p_S : \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-E)) \to S$$

where $E = aC_0 + p_C^{-1}(D_2)$. Following [6], [9] we refer to (k_1, k_2) as the level. The inverse image by p_S of a fiber class in S is isomorphic to a Hirzebruch surface $\mathcal{H}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$. Thus $p_C \circ p_S : X \to C$ is an $\mathcal{H}_a$-bundle over C. Without loss of generality, we may assume $a \leq 0$.

Let $F \in H^2(X, \mathbb{Z})$ denote the cohomology class of a fiber $(p_C \circ p_S)^{-1}(c)$ for $c \in C$. Identify S with the locus of $(1 : 0)$ in $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-E))$ and let S' denote the locus of $(0 : 1)$ in $X \cong \mathbb{P}(\mathcal{O}_S(E) \oplus \mathcal{O}_S)$. Let $C_2 = p_S^{-1}(C_0) \cap S'$ and $C_3 = p_S^{-1}(C_1) \cap S'$. We denote the cohomology classes of the surfaces $S, S', p_S^{-1}(C_0)$, and $p_S^{-1}(C_1)$ by H_1, H_1', H_2, and H_2' respectively. Then $\{H_1, H_2, F\}$ is a set of generators for $H^2(X, \mathbb{Z})$ and

$$H_1' = H_1 + aH_2 + k_2 F,$$

$$H_2' = H_2 + k_1 F.$$

The canonical class of X is given by

$$K_X = -2H_1 - (2 + a)H_2 - (k_1 + k_2 + 2 - 2g)F.$$

The classes s and f_1 satisfy

$$s = H_1 \cdot H_2, \qquad f_1 = H_1 \cdot F,$$

and we define

$$f_2 := H_2 \cdot F.$$

Then $\{s, f_1, f_2\}$ is a set of generators for $H^4(X, \mathbb{Z})$. A class $\beta \in H^4(X, \mathbb{Z})$ is called a section class if $F \cdot \beta = 1$, and it is called a Calabi-Yau class if $K_X \cdot \beta = 0$. Any section class can be expressed as $s + m_1 f_1 + m_2 f_2$.

Remark 1.0.1. *A section class is not necessarily represented by a geometric section of $p_C \circ p_S : X \to C$. For instance, we may have the union of a section with fiber curves.*

We have the following relations in the cohomology ring:

$$H_1 \cdot s = ak_1 - k_2, \quad H_1 \cdot f_1 = -a, \quad H_1 \cdot f_2 = 1,$$

$$H_2 \cdot s = -k_1, \quad H_2 \cdot f_1 = 1, \quad H_2 \cdot f_2 = 0,$$

$$F \cdot s = 1, \quad F \cdot f_1 = 0, \quad F \cdot f_2 = 0.$$

We see that C_0, C_1, C_2, C_3 represent the cohomology classes

$$s, \quad s + k_1 f_1, \quad s + (k_2 - ak_1) f_1, \quad s + k_1 f_1 + k_2 f_2,$$

respectively.

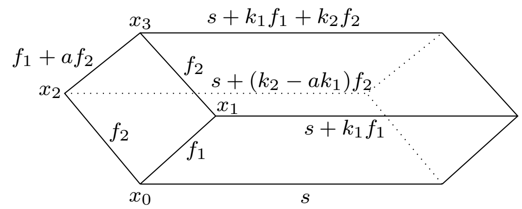


Figure 1.1: Curve and point classes

1.1 Torus Action

The complex torus $\mathbb{T} = (\mathbb{C}^*)^2$ acts on X. The element $(z_0, z_1) \in \mathbb{T}$ acts as follows: The first coordinate acts on $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D_1))$ by scaling the line bundle $\mathcal{O}_C(-D_1)$. Since the divisor E is invariant under this $\mathbb{C}^*$-action, this extends canonically to a compatible action on $X = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-E))$. The second coordinate acts by scaling the line bundle $\mathcal{O}_C(-E)$. The curves C_0, C_1, C_2, C_3 are invariant under the torus action. For any fiber F we let x_i denote the point given by the intersection of C_i and F. In the case that $g = 0$, the normal bundles of the torus invariant sections C_0, C_1, C_2, C_3 are given by

$$\mathcal{N}_{C_0/X} \cong \mathcal{O}(-k_1) \oplus \mathcal{O}(ak_1 - k_2), \qquad \mathcal{N}_{C_1/X} \cong \mathcal{O}(k_1) \oplus \mathcal{O}(-k_2), \tag{1.1}$$

$$\mathcal{N}_{C_2/X} \cong \mathcal{O}(-k_1) \oplus \mathcal{O}(k_2 - ak_1), \qquad \mathcal{N}_{C_3/X} \cong \mathcal{O}(k_1) \oplus \mathcal{O}(k_2).$$

Let t_1, t_2 be generators of the equivariant Chow group of a point,

$$A^*_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Q}[t_1, t_2].$$

The tangent weights $w_1(x_i), w_2(x_i)$ at the fixed points x_i corresponding to the directions given by the normal bundles above are given by

$$(t_1, t_2 - at_1), \qquad (-t_1, t_2), \tag{1.2}$$

$$(t_1, -t_2 + at_1), \qquad (-t_1, -t_2),$$

respectively. We need to fix a basis $\mathcal{B}_a$ (we may write $\mathcal{B}$ when no confusion can arise) for the equivariant Chow group of a fiber F which is a copy of the Hirzebruch surface $\mathcal{H}_a$. The equivariant Chow ring is generated by t_1, t_2 and the classes of the torus fixed divisors D'_0, D'_1, D'_2, D'_3 with relations

$$D'_0 D'_2 = D'_1 D'_3 = 0,$$

$$D'_3 = D'_1 + t_1,$$

$$D'_2 = D'_0 + t_2 + aD'_3.$$

The point classes are

$$x_0 := D'_0 D'_3, x_1 := D'_0 D'_1, x_2 := D'_2 D'_3, x_3 := D'_1 D'_2,$$

and we have

$$x_i^2 = T(x_i)x_i$$

$$x_i x_j = 0 \text{ for } i \neq j$$

where

$$T(x_0) := t_1(t_2 - at_1), \qquad T(x_1) := -t_1 t_2, \tag{1.3}$$

$$T(x_2) := -t_1(t_2 - at_1), \qquad T(x_3) := t_1 t_2.$$

We let $\mathcal{B}_a = \{x_0, x_1, x_2, x_3\}$ be the fixed point basis for $A^*_{\mathbb{T}}(\mathcal{H}_a)$.

1.2 Deformation Theory

In this section we discuss deformations of ruled surfaces and vector bundles on ruled surfaces. The deformations in this section are non-equivariant so they don't preserve the equivariant Gromov-Witten partition functions, but they do preserve the Calabi-Yau section class partition functions. We first address the question of which $\mathbb{P}^1$-bundles over ruled surfaces are deformation equivalent to our construction. Since any rank two vector bundle on a curve is an extension of line bundles, we may deform to the split case and assume our surface is of the form $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ where $\mathcal{L}$ is a line bundle on C. Furthermore, line bundles on C with the same degree are deformation equivalent (see [8] 19.3). However, we will see that not all rank two vector bundles over S can be deformed to split bundles.

Remark 1.2.1. *Any (Zariski locally trivial) $\mathbb{P}^1$-bundle over a ruled surface S is given by $\mathbb{P}(\mathcal{V})$ where $\mathcal{V}$ is a rank two vector bundle over S. Two vector bundles $\mathcal{V}, \mathcal{V}'$*

determine the same $\mathbb{P}^1$-bundle if and only if $\mathcal{V}' \cong \mathcal{V} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ (see [10] exercise II.7.10).

Remark 1.2.2. *Given an extension $0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{L}' \to 0$ where $\mathcal{L}, \mathcal{L}'$ are line bundles we can deform $\mathcal{V}$ to a split bundle by deforming the extension class $v \in \operatorname{Ext}^1(\mathcal{L}', \mathcal{L})$ to 0 inside the vector space. However, there are rank two vector bundles on ruled surfaces which cannot be deformed to split bundles. As an example let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{V}$ be a rank two vector bundle. The Chern classes $c_1(\mathcal{V}) \in H^2(S, \mathbb{Z}), c_2(\mathcal{V}) \in H^4(S, \mathbb{Z})$ are preserved in flat families. If $\mathcal{V}$ is the direct sum of two line bundles $\mathcal{L}, \mathcal{L}'$ with $c_1(\mathcal{L}) = as + bf_1$ and $c_2(\mathcal{L}') = cs + df_1$ then $c_1(\mathcal{V}) = (a + c)s + (b + d)f_1$ and $c_2(\mathcal{V}) = ad + bc$. Suppose $\mathcal{V}$ is a direct sum of two line bundles and $c_1(\mathcal{V}) = xs + yf_1$ with x, y even and $c_2(\mathcal{V}) = z$ with z odd. Then $c_2(\mathcal{V}) = ad + bc = a(y - b) + b(x - a) = ay + bx - 2ab$ which is even, giving a contradiction. So any rank two vector bundle with such Chern classes cannot be deformed to a split bundle. In fact by Theorem 4 in [3] such vector bundles exist: for (any ruled surface S) given $\gamma_1, \lambda \in H^2(S, \mathbb{Z})$ and $\gamma_2 \in \mathbb{Z}$ such that $2(\lambda \cdot f_1) > \gamma_1 \cdot f_1$ and $\gamma_2 + \lambda(\lambda - \gamma_1) \geq 0$ there exists a vector bundle $\mathcal{V}$ with $c_1(\mathcal{V}) = \gamma_1, c_2(\mathcal{V}) = \gamma_2$.*

1.2.1 Kodaira Deformation

In [13] Kodaira gives an explicit deformation from the Hirzebruch surface $\mathcal{H}_r$ to $\mathcal{H}_{r+2k}$. Note that this is a non-equivariant deformation, and in fact Hirzebruch surfaces are torically rigid, but this deformation will have consequences for the partition functions of Calabi-Yau section classes. See [4] for a good discussion on Kodaira's deformation and applications to enumerative geometry. We take the following viewpoint on this deformation: There is a one-to-one correspondence between sections $\mathbb{P}^1 \to \mathcal{H}_r$ in class $s + (r+d)f$ and surjections $\mathcal{O} \oplus \mathcal{O}(-r) \to \mathcal{O}(d)$ (see [10] Proposition V.2.9). We deform an extension corresponding to a section in class $s+(r+d)f$ in $\mathcal{H}_r$

$$0 \to \mathcal{O}(-r-d) \to \mathcal{O} \oplus \mathcal{O}(-r) \to \mathcal{O}(d) \to 0$$

to a split extension by deforming $v \in \mathrm{Ext}^1(\mathcal{O}(d), \mathcal{O}(-e-d))$ to zero inside the vector space. We twist the resulting extension by $\mathcal{O}(-d)$ to get

$$0 \to \mathcal{O}(-r-2d) \to \mathcal{O} \oplus \mathcal{O}(-r-2d) \to \mathcal{O} \to 0$$

which corresponds to a section in class $s + (r+2d)f$ in $\mathcal{H}_{r+2d}$. We need to prove a generalization to ruled surfaces over higher genus curves which we will refer to as (generalized) Kodaira deformation.

Lemma 1.2.1 (Generalized Kodaira Deformation)**.** *Let C be a non-singular projective curve of genus g. Let E, E' be effective divisors of degree r, r'. If $r \equiv r' \mod 2$ then $S = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-E))$ is deformation equivalent to $S' = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-E'))$. The curve of class s in S is deformed to a curve of class $s + \frac{r'-r}{2} f$ in S'.*

Proof. A surjection $\mathcal{O}_C \oplus \mathcal{O}_C(-E) \to \mathcal{O}_C(D)$ where D is an effective divisor of degree d corresponds to a section of class $s + (r+d)f$ in S (see [10] Proposition V.2.9). If $d + r > 2g$ then $\mathcal{O}_C(E+D)$ is very ample (see [10] Corollary IV.3.2), and we may find a section $\mathcal{O}_C \to \mathcal{O}_C(E+D)$ cutting out a divisor G of degree $d+r$ avoiding D. This determines a map from $\mathcal{O}_C(-E) \to \mathcal{O}_C(D)$ which is surjective on

$C - G$. The canonical section $\mathcal{O}_C \to \mathcal{O}_C(D)$ is surjective on $C - D$. Thus we can find an extension

$$0 \to \mathcal{G} \to \mathcal{O}_C \oplus \mathcal{O}_C(-E) \to \mathcal{O}_C(D) \to 0$$

where $\mathcal{G}$ is a line bundle. We deform to a split extension which corresponds to a section in class $s + (r + 2d)f$ in $S'' := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-D) \otimes \mathcal{G})$ (see [10] Proposition V.2.9). Since line bundles on smooth complex projective curves are determined up to deformation by their degree (see [8] 19.3), up to deformation, S'' depends only on g and the degree of $\mathcal{O}_C(-D) \otimes \mathcal{G}$ which is $-r - 2d$. Thus if $r \equiv r' \mod 2$, and taking d such that $d + \min(r, r') > 2g$, we see we can deform S and S' to a common surface S''. $\square$

Chapter 2: Relative Gromov-Witten Invariants

The moduli space of stable maps $\overline{\mathcal{M}}^{\bullet}_h(X,\beta)$ is a compact Deligne-Mumford stack parametrizing maps $q : C' \to X$ such that C' is a nodal curve of genus h, $f_*[C] = \beta$ and $\mathrm{Aut}(f)$ is finite. The superscript $\bullet$ indicates that we allow disconnected domain curves. Let vdim $\overline{\mathcal{M}}^{\bullet}_h(X,\beta)$ denote the virtual dimension. Then

$$\mathrm{vdim}\, \overline{\mathcal{M}}^{\bullet}_h(X,\beta) = -K_X \cdot \beta$$

and in particular, for a section class $s + m_1 f_1 + m_2 f_2$ we have

$$\mathrm{vdim}\, \overline{\mathcal{M}}^{\bullet}_h(X,\beta) = (a-1)k_1 - k_2 + 2 - 2g + (2-a)m_1 + 2m_2. \tag{2.1}$$

Note that the virtual dimension does not depend on the genus h of the domain curve. This is a property of Gromov-Witten invariants of threefolds. The virtual fundamental class $[\overline{\mathcal{M}}^{\bullet}_h(X,\beta)] \in A^{\mathbb{T}}_d\left(\overline{\mathcal{M}}^{\bullet}_h(X,\beta)\right)$ is in the dth equivariant Chow group for $d = -K_X \cdot \beta$. The partition function of degree β equivariant Gromov-Witten invariants is given by

$$Z_\beta(g|k_1,k_2) = \sum_{h\in\mathbb{Z}} u^{2h-2-K_X\cdot\beta} \int_{[\overline{\mathcal{M}}^{\bullet}_h(X,\beta)]^{\mathrm{vir}}} 1$$

where the integral is defined by equivariant push-forward to a point. For h sufficiently negative, the moduli space $\overline{\mathcal{M}}^{\bullet}_h(X,\beta)$ is empty, so $Z_\beta(g|k_1,k_2)$ is a Laurent series in u whose coefficients are homogenous polynomials in t_1, t_2 of degree $K_X \cdot \beta$.

Remark 2.0.1. *Equivariant Gromov-Witten partitions functions are invariant under equivariant deformations. The space X is determined up to equivariant deformation by a, the genus g, and the level (k_1, k_2).*

We define the partition function for section class Gromov-Witten invariants in $\mathbb{Q}[t_1, t_2]((u, v_1, v_2))$ by

$$Z(g|k_1, k_2) = \sum_{m_1, m_2} Z_{s+m_1 f_1 + m_2 f_2}(g|k_1, k_2) v_1^{m_1} v_2^{m_2}.$$

The variables v_1, v_2 keep track of the number of f_1 and f_2 fibers. We denote the non-equivariant limit (given by setting $t_1 = t_2 = 0$) by $Z(g|k_1, k_2)|_{t_1 = t_2 = 0}$. This corresponds to the invariants of Calabi-Yau section classes.

Remark 2.0.2. *Since X is compact, the equivariant Gromov-Witten invariants have non-negative degree in t_1, t_2.*

Remark 2.0.3. *Equivariant Gromov-Witten invariants may be non-vanishing even when the virtual dimension is negative. For example, if $a = 0$ so that X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over C, by Theorem 3.1.1 we have*

$$Z_s(0|1, 2) = -t_2 \phi^{-3}$$

By (2.1) the virtual dimension is -1.

2.1 Degeneration

We now discuss degenerated targets and the moduli space of stable relative maps, following [9, 16] and their applications to our family of varieties. Let D be a smooth divisor in X. Let $Y[0] = X$ and $D[0] = D$. The spaces $Y[n]$ are constructed by iterating deformation to the normal cone: $Y[n]$ is the blowup of $Y[n-1] \times \mathbb{A}^1$ along $D[n-1] \times 0$ and $D[n]$ is the proper transform of $D[n-1] \times \mathbb{A}^1$. We refer to the central fiber $X[n] := Y[n]_0$ over $0 \in \mathbb{A}^n$ as the n-step degeneration of X. In the case that the divisor is a fiber F, $X[n]$ is a chain of varieties whose irreducible components are X and n copies of the space $C \times \mathcal{H}_a$. Let $p_1, \ldots, p_r$ be distinct points on C, and let F_i be the fiber over p_i. Let $\vec{L} = (l_1 \ldots, l_r) \in (\mathbb{Z}_{>0})^r$ and let $X[\vec{L}]$ be the $\vec{L}$-step degeneration of X along each F_i. Given a subvariety $V \subset X$ we let $V[\vec{L}]$ denote the $\vec{L}$-step degeneration of V along each $F_i \cap V$.

The moduli space of stable relative maps $\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta)$ parameterizes stable maps $q : C' \to X[\vec{L}]$ representing the class β for some $\vec{L}$. Here C' is a possibly disconnected, nodal, genus h curve. The image $q(C')$ must meet the transforms of $F_1, \ldots, F_r$ transversely, and must meet each of the singular divisors of $X[\vec{L}]$ at a single node joining two irreducible components of $q(C')$. This is a simplified requirement which suffices for the case of section classes. If β is a section class, $\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta)$ is a Deligne-Mumford stack of virtual dimension $-K_X \cdot \beta$.

For each $1 \leq i \leq r$ we have a $\mathbb{T}$-equivariant evaluation map

$$\mathrm{ev}_i : \overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta) \to F_i \cong \mathcal{H}_a$$

and the partition functions of relative, equivariant Gromov-Witten invariants of degree β are given by

$$Z_\beta(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r} = \sum_{h \in \mathbb{Z}} u^{2h-2-K_X \cdot \beta} \int_{[\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1) \cup \ldots \cup \mathrm{ev}_r^*(\alpha_r)$$

where $\alpha_1, \ldots, \alpha_r \in A^*_{\mathbb{T}}(\mathcal{H}_a)$.

The partition functions of section class, relative, Gromov-Witten invariants are given by

$$Z(g|k_1,k_2)_{\alpha_1\ldots\alpha_r} = \sum_{m_1,m_2} Z_{s+m_1f_1+m_2f_2}(g|k_1,k_2)_{\alpha_1\ldots\alpha_r} v_1^{m_1} v_2^{m_2}.$$

As a polynomial in t_1, t_2 the partition function has degree

$$\sum_{i=1}^{r} \deg(\alpha_i) - ((a-1)k_1 - k_2 + 2 - 2g + (2-a)m_1 + 2m_2) \tag{2.2}$$

The relative partition functions with raised indices are defined by

$$Z_\beta(g|k_1,k_2)^{\gamma_1\ldots\gamma_t}_{\alpha_1\ldots\alpha_s} := \left(\prod_{i=1}^{t} T(\gamma_i)^{-1}\right) Z_\beta(g|k_1,k_2)_{\alpha_1\ldots\alpha_s\gamma_1\ldots\gamma_t} \tag{2.3}$$

where $T(\gamma_i)$ is given by (1.3). The functions $Z(g|k_1,k_2)^{\gamma_1\ldots\gamma_r}_{\alpha_1\ldots\alpha_r}$ are defined similarly.

The following gluing lemma is the primary workhorse of this thesis,

Lemma 2.1.1 (Gluing). *For any elements $\alpha_1,\ldots,\alpha_s$ and $\beta_1,\ldots,\beta_t$ in $\mathcal{B}$ and integers $g'+g''=g$, $k_1'+k_1''=k_1$, $k_2'+k_2''=k_2$ we have*

$$Z(g|k_1,k_2)_{\alpha_1\ldots\alpha_s\gamma_1\ldots\gamma_t} = \sum_{\lambda\in\mathcal{B}} Z(g'|k_1',k_2')_{\alpha_1\ldots\alpha_s\lambda} Z(g''|k_1'',k_2'')^{\lambda}_{\gamma_1\ldots\gamma_t}$$

and

$$Z(g|k_1,k_2)_{\alpha_1\ldots\alpha_s} = \sum_{\lambda\in\mathcal{B}} Z(g-1|k_1,k_2)^{\lambda}_{\alpha_1\ldots\alpha_s\lambda}.$$

Proof. Consider the case $s=t=0$; the other cases are similar. Let C' and C'' be irreducible curves of genera g' and g'' respectively. Let C be a genus g curve given by gluing C' and C'' at the points $p'\in C'$ and $p''\in C''$. Let X' be an $\mathcal{H}_a$-bundle over C' with level (k_1',k_2'). Let X'' be an $\mathcal{H}_a$-bundle over C'' with level (k_1'',k_2''). Then we construct an $\mathcal{H}_a$-bundle X over C by gluing the fibers F' and F'' over p' and

p''. Let $W \to \mathbb{A}^1$ be a one parameter deformation of X, equivariant with respect to the action of the two dimension torus $\mathbb{T}$, and such that $W_0 = X$ and the fibers W_t for $t \neq 0 \in \mathbb{A}^1$ are level (k_1, k_2) $\mathcal{H}_a$-bundles over a smooth curve of genus g. Following [9, 14], let $\mathfrak{W}$ be the stack of expanded degenerations of W, and let $\mathfrak{W}_0$ be its central fiber. Let $\overline{\mathcal{M}}^{\bullet}_h(\mathfrak{W}, \beta)$ be the stack of non-degenerate, pre-deformable, genus h maps to $\mathfrak{W}$ representing the section class β. Let $\mathrm{ev}', \mathrm{ev}''$ be the evaluation maps

$$\mathrm{ev}' : \overline{\mathcal{M}}^{\bullet}_{h'}(X'/F', \beta') \to F', \quad \mathrm{ev}'' : \overline{\mathcal{M}}^{\bullet}_{h''}(X''/F'', \beta'') \to F''$$

and let F be the fiber of W_0 given by gluing F' and F''. Let $\eta = (h', h'', \beta', \beta'')$ where $h' + h'' = h, \beta' + \beta'' = \beta$. In [14], Li constructs a map

$$\Phi_\eta : \overline{\mathcal{M}}^{\bullet}_{h'}(X'/F', \beta') \times_F \overline{\mathcal{M}}^{\bullet}_{h''}(X''/F', \beta'') \to \overline{\mathcal{M}}^{\bullet}_h(\mathfrak{W}_0, \beta)$$

and proves the following formula for the virtual fundamental class:

$$\left[\overline{\mathcal{M}}^{\bullet}_h(\mathfrak{W}_0, \beta)\right]^{\mathrm{vir}} = \sum_\eta (\Phi_\eta)_* \Delta^! \left(\left[\overline{\mathcal{M}}^{\bullet}_{h'}(X'/F', \beta')\right]^{\mathrm{vir}} \times \left[\overline{\mathcal{M}}^{\bullet}_{h''}(X''/F'', \beta'')\right]^{\mathrm{vir}} \right)$$

where $\Delta : F \to F \times F$ is the diagonal map. The action of the torus $\mathbb{T}$ extends to an action on the moduli spaces $\overline{\mathcal{M}}^{\bullet}_{h'}(X'/F', \beta'), \overline{\mathcal{M}}^{\bullet}_{h''}(X''/F'', \beta'')$ and $\overline{\mathcal{M}}^{\bullet}_h(\mathfrak{W}, \beta)$, and Li's formula holds for equivariant cycles.

The dual basis to the fixed point basis $\mathcal{B}$ is given by $\{x_0^\vee, x_1^\vee, x_2^\vee, x_3^\vee\}$ where

$$x_i^\vee = \frac{x_i}{T(x_i)},$$

so the equivariant class of the diagonal is given by

$$\Delta_*[F] = \sum_{i=0}^{3} x_i \times x_i^\vee = \sum_{i=0}^{3} x_i \times \frac{x_i}{T(x_i)}$$

Thus we have,

$$\left[\overline{\mathcal{M}}^\bullet_h(\mathfrak{W}_0,\beta)\right]^{\mathrm{vir}} = \sum_\eta (\Phi_\eta)_* \left(\sum_{i=0}^{3} (\mathrm{ev}')^*(x_i) \cap \left[\overline{\mathcal{M}}^\bullet_{h'}(X'/F',\beta')\right]^{\mathrm{vir}}\right.$$

$$\left. \times \frac{(\mathrm{ev}'')^*(x_i)}{T(x_i)} \cap \left[\overline{\mathcal{M}}^\bullet_{h''}(X''/F'',\beta'')\right]^{\mathrm{vir}}\right)$$

from which the lemma follows. The case of irreducible degenerations of X follows similarly. $\square$

2.2 Localization

The local Gromov-Witten theory of curves was computed in [6]. We will only need the case that the base curve is $\mathbb{P}^1$ and the maps have degree one. Let N be the non-compact space given by the total space of the rank two vector bundle

$$\mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \to \mathbb{P}^1$$

and let F_i denote the fibers of distinct points $p_1, \ldots, p_r \in \mathbb{P}^1$. Given a collection of divisors $\vec{F} = (F_1, \ldots, F_r)$ and a subvariety V, let $\vec{F} \cap V$ denote $(F_1 \cap V, \ldots, F_r \cap V)$. Then the local invariants may be written as equivariant integrals

$$Z^{\mathrm{loc}}(n_1,n_2)_{p_1\ldots p_r}(t_1,t_2)$$

$$= \sum_{h\in\mathbb{Z}} u^{2h-2-K_N\cdot[\mathbb{P}^1]} \int_{\overline{\mathcal{M}}^\bullet_h(\mathbb{P}^1/\vec{F}\cap\mathbb{P}^1,1)} e(-R^\bullet\pi_* f^*(\mathcal{O}(n_1)\oplus\mathcal{O}(n_2)))$$

where

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}^\bullet_h(\mathbb{P}^1/\vec{F}\cap\mathbb{P}^1,1) & & \end{array}$$

is the universal diagram and e is the equivariant Euler class. Note that relative stable maps may map to an $\vec{L}$-step degeneration $\mathbb{P}^1[\vec{L}]$, and the map f includes a contraction. The shift in the exponent is given by

$$-[\mathbb{P}^1]\cdot K_N = 2 + n_1 + n_2. \tag{2.4}$$

Then we recall the results from [6]:

$$Z^{\mathrm{loc}}(n_1,n_2)_{p_1\dots p_r}(t_1,t_2) = \phi^{n_1+n_2}\frac{1}{t_1t_2}t_1^{-n_1}t_2^{-n_2} \tag{2.5}$$

where

$$\phi := 2\mathrm{sin}\frac{u}{2}.$$

The action of $\mathbb{T}$ on X induces an action on the moduli space $\overline{\mathcal{M}}^\bullet_h(X/\vec{F},\beta)$ and we denote the fixed locus by $\overline{\mathcal{M}}^\bullet_h(X/\vec{F},\beta)^{\mathbb{T}}$. In general the moduli space $\overline{\mathcal{M}}^\bullet_h(X/\vec{F},\beta)^{\mathbb{T}}$ is rather complicated, but in special circumstances we can reduce the calculation of invariants to the invariants of local curves:

Lemma 2.2.1 (Localization). *Let $g=0$ and suppose that the only $\mathbb{T}$-fixed relative stable maps $q: C' \to X[\vec{L}]$ representing class β which meet the transforms of each of the divisors $F_1,\dots,F_r$ at the point x_i have image $C_i[\vec{L}]$ and that $\mathcal{N}_{C_i|X}\cong \mathcal{O}(n_1)\oplus\mathcal{O}(n_2)$. Then*

$$Z_\beta(0|k_1,k_2)_{x_i\dots x_i} = Z^{loc}(n_1,n_2)_{p_1\dots p_r}(w_1(x_i),w_2(x_i))T(x_i)^r$$

where $w_1(x_i), w_2(x_i)$ are the tangent weights given by (1.2) and $T(x_i)$ is given by (1.3).

Proof. In this case the moduli space $\overline{\mathcal{M}}^\bullet_h(X/\vec{F},\beta)^{\mathbb{T}}$ contains a connected component $\overline{\mathcal{M}}^\bullet_h(C_i/\vec{F}\cap C_i,1)$ parameterizing maps $q: C'\to C_i[\vec{L}]$. We will denote the inclusion of this component by $j : \overline{\mathcal{M}}^{\bullet}_h(C_i/\vec{F}\cap C_i, 1) \to \overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta)$ and we denote the virtual normal bundle of the component by $\mathrm{Norm}^{\mathrm{vir}}$. Since this is the unique component whose virtual fundamental class has non-trivial intersection with $\mathrm{ev}_1^*(x_i) \cup \ldots \cup \mathrm{ev}_r^*(x_i)$ we have

$$\int_{[\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F},\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(x_i) \cup \ldots \cup \mathrm{ev}_r^*(x_i)$$
$$= \int_{[\overline{\mathcal{M}}^{\bullet}_h(C_i/\vec{F}\cap C_i,1)]^{\mathrm{vir}}} \frac{(\mathrm{ev}_1 \circ j)^*(x_i) \cup \ldots \cup (\mathrm{ev}_r \circ j)^*(x_i)}{e(\mathrm{Norm}^{\mathrm{vir}})}.$$

As in [9] Lemma 3.2, the equivariant Euler class of the virtual normal bundle of the component $\overline{\mathcal{M}}^{\bullet}_h(C_i/\vec{F} \cap C_i, 1)$ of the fixed locus is given by

$$e(\mathrm{Norm}^{\mathrm{vir}}) = e(R^{\bullet}\pi_* f^*\mathcal{N}_{C_i/X}) \cong e(R^{\bullet}\pi_* f^*(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)))$$

and we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\bullet}_h(C_i/\vec{F} \cap C_i, 1) & \to & F_p \cap C_i \\ j \downarrow & & \downarrow \\ \overline{\mathcal{M}}^{\bullet}_h(X/\vec{F}, \beta) & \xrightarrow{\mathrm{ev}_p} & F_p \end{array}$$

from which we can see that

$$(\mathrm{ev}_p \circ j)^*(x_k) = \begin{cases} T(x_i) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}.$$

In conclusion we have

$$Z_\beta(0|k_1, k_2)_{x_i \ldots, x_i} = \sum_{h\in\mathbb{Z}} u^{2h-2-K_X\cdot\beta} \int_{[\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F},\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(x_i) \cup \ldots \cup \mathrm{ev}_r^*(x_i)$$
$$= \sum_{h\in\mathbb{Z}} u^{2h-2-K_N\cdot[\mathbb{P}^1]} \int_{[\overline{\mathcal{M}}^{\bullet}_h(C_i/\vec{F}\cap C_i,1)]^{\mathrm{vir}}} T(x_i)^r e(-R^{\bullet}\pi_* f^*(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)))$$
$$= Z^{\mathrm{loc}}(n_1, n_2)_{p_1 \ldots p_r}(w_1(x_i), w_2(x_i)) T(x_i)^r.$$

where $N = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2))$. The agreement of shifted exponents $-K_N \cdot [\mathbb{P}^1] = -K_X \cdot \beta$ is a consequence of equivariant localization, and can easily be checked directly. $\square$

Chapter 3: Gromov-Witten Theory of $\mathbb{P}^1 \times \mathbb{P}^1$-Bundles

3.1 Summary of Results

In this chapter we restrict to the case that $a = 0$, so that X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a curve C. Throughout we use the notation

$$\phi := 2\sin\frac{u}{2}.$$

Then the following theorem is the main result of this chapter:

Theorem 3.1.1. *Let $a = 0$, so that X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a genus g curve C. Then $Z(g|k_1, k_2) = tr\left(G^{g-1}U_1^{k_1}U_2^{k_2}\right)$ where*[1]

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left(\phi^{-1}\begin{bmatrix} t_1 & 0 \\ 0 & 0 \end{bmatrix} + \phi \begin{bmatrix} 1/t_1 & 1/t_1 \\ -1/t_1 & -1/t_1 \end{bmatrix} v_1\right)$$

$$U_2 = \left(\phi^{-1}\begin{bmatrix} t_2 & 0 \\ 0 & 0 \end{bmatrix} + \phi \begin{bmatrix} 1/t_2 & 1/t_2 \\ -1/t_2 & -1/t_2 \end{bmatrix} v_2\right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[1] Here the Kronecker product is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

$$G = \begin{bmatrix} t_2 + 2\phi^2 v_2/t_2 & 2\phi^2 v_2/t_2 \\ -2\phi^2 v_2/t_2 & -t_2 - 2\phi^2 v_2/t_2 \end{bmatrix} \otimes \begin{bmatrix} t_1 + 2\phi^2 v_1/t_1 & 2\phi^2 v_1/t_1 \\ -2\phi^2 v_1/t_1 & -t_1 - 2\phi^2 v_1/t_1 \end{bmatrix}.$$

Remark 3.1.1. *We see from the formula above, that if β is a section class, $Z_\beta(g|k_1, k_2)$ is of the form $p(t_1, t_2)\phi^{2g-2-K_X\cdot\beta}$ where $p(t_1, t_2)$ is a homogeneous polynomial of degree $K_X \cdot \beta$ in t_1, t_2.*

Recall $Z(g|k_1, k_2)|_{t_1=t_2=0}$ denotes the non-equivariant limit which is obtained by setting $t_1 = t_2 = 0$. Its terms correspond to Calabi-Yau section classes. The Calabi-Yau section class partition functions satisfy the following recursions:

Corollary 3.1.1.

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = \begin{cases} 4^g \phi^{2g-2} v_1^{\frac{g-1+k_1}{2}} v_2^{\frac{g-1+k_2}{2}} & (g-1) \equiv k_1 \equiv k_2 \mod 2 \\ 0 & \textit{otherwise} \end{cases}$$

Remark 3.1.2. *The (generalized) Kodaira deformation (Lemma 1.2.1) implies symmetry for the Calabi-Yau section class partition functions in Corollary 3.1.1.*

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = v_1 Z(g|k_1 - 2, k_2)|_{t_1=t_2=0},$$

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = v_2 Z(g|k_1, k_2 - 2)|_{t_1=t_2=0}.$$

Remark 3.1.3. *Corollary 3.1.1 invites a comparison with Theorem 1.7 in [9] which states that if X is any $\mathbb{P}^2$-bundle over a curve C of genus g and β is a Calabi-Yau section class then*

$$Z_\beta(g) = 3^g \phi^{2g-2}.$$

Corollary 3.1.2. *The GW/DT/PT correspondence 5.3.1 holds for the Calabi-Yau section class partition functions when $a = 0$. This will be proved in Chapter 5.*

3.2 Calculations

We will see that the full theory is determined by the following basic partition functions:

$$Z(0|0,0)_\alpha, \quad Z(0|0,0)_{\alpha_1\alpha_2},$$

$$Z(0|1,0)_{\alpha_1\alpha_2}, \quad Z(0|-1,0)_{\alpha_1\alpha_2}, \quad Z(0|0,1)_{\alpha_1\alpha_2}, \quad Z(0|0,-1)_{\alpha_1\alpha_2},$$

$$Z(0|0,0)_{\alpha_1\alpha_2\alpha_3}.$$

Lemma 3.2.1. *The basic partition functions depend on only the following cohomology classes:*

$Z(0\|0,0)_\alpha$	s
$Z(0\|0,0)_{\alpha_1\alpha_2}$	$s,\ s+f_1,\ s+f_2$
$Z(0\|1,0)_{\alpha_1\alpha_2}$	$s,\ s+f_1,\ s+f_2$
$Z(0\|-1,0)_{\alpha_1\alpha_2}$	$s-f_1,\ s,\ s-f_1+f_2$
$Z(0\|0,1)_{\alpha_1\alpha_2}$	$s,\ s+f_1,\ s+f_2$
$Z(0\|0,-1)_{\alpha_1\alpha_2}$	$s-f_2,\ s,\ s+f_1-f_2$
$Z(0\|0,0)_{\alpha_1\alpha_2\alpha_3}$	$s,\ s+f_1,\ s+f_2,\ s+2f_1,\ s+2f_2,\ s+f_1+f_2$

Proof. In the cases above, X is a toric threefold. By (2.2) for $\beta = s+m_1f_1+m_2f_2$, the degree of $Z_\beta(0|k_1,k_2)_{\alpha_1,\ldots,\alpha_j}$ as polynomial in t_1,t_2 is given by

$$N = \sum_{i=1}^{j} \deg(\alpha_i) - \mathrm{vdim}\ \overline{\mathcal{M}}_h^\bullet(X/\vec{F},\beta) = 2j - (2m_1+2m_2-k_1-k_2+2).$$

Since X is compact, the degree is nonnegative. This provides an upper bound on m_1, m_2. To obtain a lower bound, recall that the toric cone theorem implies the Mori cones are generated by the torus-invariant curves. For levels $(0,0),(1,0),(0,1)$ the cones are generated by $\langle s, f_1, f_2\rangle$. For levels $(-1,0),(0,-1)$ the cones are generated by $\langle s-f_1, f_1, f_2\rangle$ and $\langle s-f_2, f_1, f_2\rangle$ respectively. $\square$

Let $S=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-k_1))$ and let X_{k_1,k_2} be the threefold given by the total space

$$X_{k_1,k_2} := \mathrm{Tot}\left(\mathbb{P}(\mathcal{O}_S\oplus\mathcal{O}_S(-k_2 f_1))\right) \tag{3.1}$$

The following observation about symmetries among the spaces X_{k_1,k_2} will simplify the calculations:

Remark 3.2.1. *The space $X_{1,0}$ is isomorphic to $X_{-1,0}$ because $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1))$ is equivalent to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))$ which we may see may tensoring with $\mathcal{O}_{\mathbb{P}^1}(1)$ and exchanging the order of the line bundles. The transformation exchanges the classes and weights as follows:*

$$s\leftrightarrow s-f_1, x_0\leftrightarrow x_1, x_2\leftrightarrow x_3, t_1\leftrightarrow -t_1.$$

The space $X_{0,1}$ is isomorphic to $X_{0,-1}$ because $\mathbb{P}(\mathcal{O}_S\oplus\mathcal{O}_S(-f_1))$ is equivalent to $\mathbb{P}(\mathcal{O}_S\oplus\mathcal{O}_S(f_1))$. The transformation exchanges the classes and weights as follows:

$$s\leftrightarrow s-f_2, x_0\leftrightarrow x_2, x_1\leftrightarrow x_3, t_2\leftrightarrow -t_2.$$

The space $X_{1,0}$ is isomorphic to $X_{0,1}$ because $\mathbb{P}(\mathcal{O}_{\mathcal{H}_1}\oplus\mathcal{O}_{\mathcal{H}_1})\cong\mathbb{P}^1\times\mathcal{H}_1$ which is also a $\mathbb{P}^1$-bundle over $\mathbb{P}^1\times\mathbb{P}^1$. The transformation exchanges the classes and weights as follows:

$$f_1\leftrightarrow f_2, x_1\leftrightarrow x_2, t_1\leftrightarrow t_2.$$

3.2.1 Localization Calculations

In this section we use equivariant localization to calculate certain partition functions corresponding to the lowest degree effective cohomology classes.

Lemma 3.2.2. *The partition functions for the degree s, level $(0,0)$ cap, tube, and pants are given by*

$$Z_s(0|0,0)_{x_a} = 1$$

$$Z_s(0|0,0)_{x_a x_b} = \begin{cases} T(x_a) & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$Z_s(0|0,0)_{x_a x_b x_c} = \begin{cases} T(x_a)^2 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

where $a, b, c \in \{0, 1, 2, 3\}$.

Proof. Any $\mathbb{T}$-fixed stable relative map representing the class s must have image $C_i[\vec{L}]$ for some C_i (see Figure 1). Each C_i has normal bundle $\mathcal{O} \oplus \mathcal{O}$ (see (1.1)). Applying Lemma 2.2.1 we get

$$Z_s(0|0,0)_{x_i \ldots x_i} = Z^{\mathrm{loc}}(0,0)_{p_1 \ldots p_r}(w_1(x_i), w_2(x_i))T(x_i)^r = T(x_i)^{r-1}$$

and the other invariants vanish.

□

Lemma 3.2.3. *Partition functions for the tubes of degree s, level $(1,0)$ and $(0,1)$, degree $s - f_1$, level $(-1,0)$, and degree $s - f_2$, level $(0,-1)$ are given by*

$$\left[Z_s(0|1,0)^{x_a}_{x_b}\right] = \phi^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} t_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left[Z_{s-f_1}(0|-1,0)^{x_a}_{x_b}\right] = \phi^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & -t_1 \end{bmatrix},$$

$$\left[Z_s(0|0,1)^{x_a}_{x_b}\right] = \phi^{-1}\begin{bmatrix} t_2 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\left[Z_{s-f_2}(0|0,-1)^{x_a}_{x_b}\right] = \phi^{-1}\begin{bmatrix} 0 & 0 \\ 0 & -t_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. First let the level be $(1,0)$. Any $\mathbb{T}$-fixed stable relative map representing the class s must have image $C_0[\vec{L}]$ or $C_2[\vec{L}]$ (see Figure 1). The curves C_0 and C_2 each have normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}$ (see (1.1)). Applying Lemma 2.2.1 and (1.2) we get

$$Z_s(0|1,0)^{x_0}_{x_0} = T(x_0)Z^{\mathrm{loc}}(-1,0)_{0,\infty}(t_1,t_2) = T(x_0)\phi^{-1}\frac{1}{t_2} = t_1\phi^{-1}$$

$$Z_s(0|1,0)^{x_2}_{x_2} = T(x_2)Z^{\mathrm{loc}}(-1,0)_{0,\infty}(t_1,-t_2) = T(x_2)\phi^{-1}\frac{1}{-t_2} = t_1\phi^{-1}$$

and $Z_s(0|1,0)^{x_a}_{x_b} = 0$ otherwise. The results for the other levels follow by symmetry (Remark 3.2.1).

$\square$

3.2.2 Degeneration Calculations

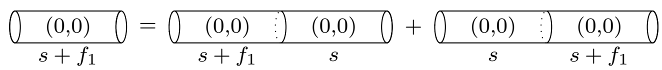
In this section we use the gluing formula to solve for more invariants in terms of the invariants calculated in the previous section.

Lemma 3.2.4. *The partition functions for the degree $s+f_1$ and $s+f_2$, level $(0,0)$ tubes vanish*

$$Z_{s+f_1}(0|0,0)^{x_a}_{x_b} = 0$$

$$Z_{s+f_2}(0|0,0)^{x_a}_{x_b} = 0$$

Proof. By attaching two level $(0,0)$ tubes and applying the gluing formula and Lemma 3.2.2



we deduce the relation

$$Z_{s+f_1}(0|0,0)^{x_a}_{x_b} = \sum_{c=0}^{3} \left(Z_{s+f_1}(0|0,0)^{x_a}_{x_c} \delta^c_b + \delta^a_c Z_{s+f_1}(0|0,0)^{x_c}_{x_b} \right)$$

$$= Z_{s+f_1}(0|0,0)^{x_a}_{x_b} + Z_{s+f_1}(0|0,0)^{x_a}_{x_b}.$$

We conclude $Z_{s+f_1}(0|0,0)^{x_a}_{x_b} = 0$. By a similar argument $Z_{s+f_2}(0|0,0)^{x_a}_{x_b} = 0$. □

Lemma 3.2.5. *Partition functions for the tubes of degree s, level $(-1,0)$ and $(0,-1)$, degree $s+f_1$, level $(1,0)$, and degree $s+f_2$, level $(0,1)$ are given by*

$$\left[Z_{s+f_1}(0|1,0)^{x_a}_{x_b}\right] = \left[Z_s(0|-1,0)^{x_a}_{x_b}\right] = \phi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1/t_1 & 1/t_1 \\ -1/t_1 & -1/t_1 \end{bmatrix}$$

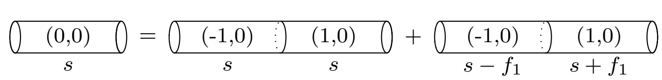
$$\left[Z_{s+f_2}(0|0,1)^{x_a}_{x_b}\right] = \left[Z_s(0|0,-1)^{x_a}_{x_b}\right] = \phi \begin{bmatrix} 1/t_2 & 1/t_2 \\ -1/t_2 & -1/t_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proof. First let the level be $(1,0)$. We notice that the following invariants vanish by geometric constraints:

$$Z_{s+f_1}(0|1,0)^{x_2}_{x_0} = Z_{s+f_1}(0|1,0)^{x_3}_{x_0} = Z_{s+f_1}(0|1,0)^{x_2}_{x_1} = Z_{s+f_1}(0|1,0)^{x_3}_{x_1} = 0.$$

To see this observe that $X_{1,0} \cong \mathcal{H}_1 \times \mathbb{P}^1$ (3.1), so has a projection $X_{1,0} \to \mathbb{P}^1 \times \mathbb{P}^1$. This projection sends the class $s + f_1$ to the class s. The points $x_0, x_1 \in F$ map to $0 \in \mathbb{P}^1$ and the points $x_2, x_3 \in F$ map to $\infty \in \mathbb{P}^1$. By naturality of deformation to the normal cone, for each $\vec{L}$-step degeneration the projection extends to a projection $\pi : X_{1,0}[\vec{L}] \to (\mathbb{P}^1 \times \mathbb{P}^1)[\vec{L}]$. First suppose the intersection of the image $q(C')$ of the $\mathbb{T}$-fixed stable relative map with each component of $X_{1,0}[\vec{L}]$ is irreducible. Then its image in $(\mathbb{P}^1 \times \mathbb{P}^1)[\vec{L}]$ is a curve with class s which cannot connect the point 0 on the transform of $\pi(F_1)$ to the point ∞ on the transform of $\pi(F_2)$ (see Section 2.1 for an explanation of $q(C')$, F_i etc.). Thus $q(C')$ cannot meet the transform of F_1 at x_0 or x_1 and meet the transform of F_2 at x_2 or x_3. Now suppose the intersection of the image of the stable map with some component of $X_{1,0}[\vec{L}]$ is reducible. Then the (possibly disconnected) image is the union of $C_0[\vec{L}]$ or $C_2[\vec{L}]$ and a fiber in class f_2 (see Figure 1). Since q is admissible, this fiber is disjoint from the transforms of F_1 and F_2. Thus the image $q(C')$ intersects the transforms of F_1 and F_2 in the same point x_i where $x_i \in \{x_0, x_2\}$. We conclude the above invariants vanish.

Next we attach the level $(-1, 0)$ tube to the $(1, 0)$ tube to get a $(0, 0)$ tube as represented in the following diagram:



Applying the gluing lemma and Lemma 3.2.3, we deduce the relations

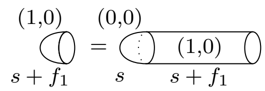
$$1 = Z_s(0|0,0)_{x_1}^{x_1} = Z_{s-f_1}(0|-1,0)_{x_1}^{x_1} Z_{s+f_1}(0|1,0)_{x_1}^{x_1}$$

$$1 = Z_s(0|0,0)_{x_3}^{x_3} = Z_{s-f_1}(0|-1,0)_{x_3}^{x_3} Z_{s+f_1}(0|1,0)_{x_3}^{x_3}$$

which imply

$$Z_{s+f_1}(0|1,0)_{x_1}^{x_1} = -\frac{\phi}{t_1}, \qquad Z_{s+f_1}(0|1,0)_{x_3}^{x_3} = -\frac{\phi}{t_1}.$$

Now we attach the $(0,0)$ cap to the $(1,0)$ tube to get a $(1,0)$ cap



The degree $s+f_1$ level (1,0) cap vanishes by dimension constraints, and $Z_s(0|0,0)_{x_a} = 1$ by Lemma 3.2.2 so we get the relations

$$0 = \sum_{c=0}^{3} Z_{s+f_1}(0|1,0)_{x_a}^{x_c}$$

which imply

$$Z_{s+f_1}(0|1,0)_{x_3}^{x_2} = \frac{\phi}{t_1}, \qquad Z_{s+f_1}(0|1,0)_{x_2}^{x_2} = \frac{\phi}{t_1}$$

$$Z_{s+f_1}(0|1,0)_{x_1}^{x_0} = \frac{\phi}{t_1}, \qquad Z_{s+f_1}(0|1,0)_{x_0}^{x_0} = \frac{\phi}{t_1}.$$

The results for the other levels follow by symmetry (Remark 3.2.1).

□

Lemma 3.2.6. *The partition functions for the tubes of degree $s+f_2$ level $(1,0)$, degree $s+f_1$ level $(0,1)$, degree $s-f_1+f_2$ level $(-1,0)$, and degree $s-f_2+f_1$ level $(0,-1)$ vanish*

$$Z_{s+f_2}(0|1,0)_{x_b}^{x_a} = Z_{s-f_1+f_2}(0|-1,0)_{x_b}^{x_a} = 0$$

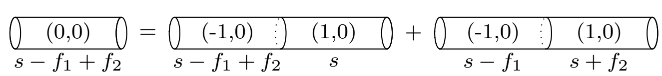
$$Z_{s+f_1}(0|0,1)_{x_b}^{x_a} = Z_{s-f_2+f_1}(0|0,-1)_{x_b}^{x_a} = 0$$

Proof. First let the level be $(1,0)$. The following invariants vanish by geometric constraints,

$$Z_{s+f_1}(0|1,0)_{x_0}^{x_1} = Z_{s+f_1}(0|1,0)_{x_0}^{x_3} = Z_{s+f_1}(0|1,0)_{x_1}^{x_2} = Z_{s+f_1}(0|1,0)_{x_2}^{x_3} = 0.$$

To see this consider the projection $X_{1,0} \to \mathcal{H}_1$ (3.1). This sends the class $s + f_2$ to the section class s. By naturality of deformation to the normal cone, for each $\vec{L}$-step degeneration this extends to a projection $\pi : X_{1,0}[\vec{L}] \to \mathcal{H}_1[\vec{L}]$. The fiber curves in $\mathcal{H}_1$ are isomorphic to $\mathbb{P}^1$. The points $x_0, x_2 \in F$ map to $0 \in \mathbb{P}^1$ and the points x_1, x_3 map to $\infty \in \mathbb{P}^1$. First suppose the intersection of the image $q(C')$ of the $\mathbb{T}$-fixed stable relative map with each component of $X_{1,0}[\vec{L}]$ is irreducible. Then its image in $\mathcal{H}_1[\vec{L}]$ is a curve with class s which cannot connect the point 0 on the transform of $\pi(F_1)$ to the point ∞ on the transform of $\pi(F_2)$ (see Section 2.1 for an explanation of $q(C')$, F_i etc.). Thus $q(C')$ cannot meet the transform of F_1 at x_0 or x_2 and meet the transform of F_2 at x_1 or x_3. Now suppose the intersection of the image of the stable map with some component of $X_{1,0}[\vec{L}]$ is reducible. Then the (possibly disconnected) image is the union of $C_0[\vec{L}]$ or $C_2[\vec{L}]$ and a fiber in class f_1 (see Figure 1). Since q is admissible, this fiber is disjoint from the transforms of F_1 and F_2. Thus the image $q(C')$ intersects the transforms of F_1 and F_2 in the same point x_i where $x_i \in \{x_0, x_2\}$. We conclude the above invariants vanish.

Now we attach the level $(-1, 0)$ tube to the $(1, 0)$ tube to get a $(0, 0)$ tube as represented in the following diagram:



We apply the gluing lemma and Lemma 3.2.3, and use that the degree $s - f_1 + f_2$ level (0,0) tube vanishes (the class is not effective), to get the relations

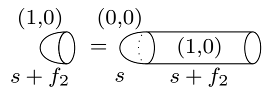
$$0 = Z_{s-f_1+f_2}(0|0,0)^{x_1}_{x_1} = Z_{s-f_1}(0|-1,0)^{x_1}_{x_1} Z_{s+f_2}(0|1,0)^{x_1}_{x_1} = -\phi^{-1} t_1 Z_{s+f_2}(0|1,0)^{x_1}_{x_1}$$

$$0 = Z_{s-f_1+f_2}(0|0,0)^{x_3}_{x_3} = Z_{s-f_1}(0|-1,0)^{x_3}_{x_3} Z_{s+f_2}(0|1,0)^{x_3}_{x_3} = -\phi^{-1} t_1 Z_{s+f_2}(0|1,0)^{x_3}_{x_3}$$

which imply

$$Z_{s+f_2}(0|1,0)_{x_1}^{x_1} = 0, \qquad Z_{s+f_2}(0|1,0)_{x_3}^{x_3} = 0.$$

We attach the $(0,0)$ cap to the $(1,0)$ tube to get a $(1,0)$ cap



and since the degree $s+f_2$ level $(1,0)$ cap vanishes by dimension constraints we get the relations

$$0 = \sum_{c=0}^{3} Z_{s+f_2}(0|1,0)_{x_a}^{x_c}$$

which imply

$$Z_{s+f_2}(0|1,0)_{x_3}^{x_2} = 0, \qquad Z_{s+f_2}(0|1,0)_{x_2}^{x_2} = 0$$

$$Z_{s+f_2}(0|1,0)_{x_1}^{x_0} = 0, \qquad Z_{s+f_2}(0|1,0)_{x_0}^{x_0} = 0.$$

The results for the other levels follow by symmetry (Remark 3.2.1).

$\square$

Lemma 3.2.7. *The partition functions for the level (0,0) pants in degrees $s+f_1$ and $s+f_2$ are given by*

$$Z_{s+f_1}(0|0,0)_{x_a x_b x_c} = \begin{cases} \phi^2 t_2^2 & a,b,c \in \{0,1\} \text{ or } a,b,c \in \{2,3\} \\ 0 & \text{otherwise} \end{cases}$$

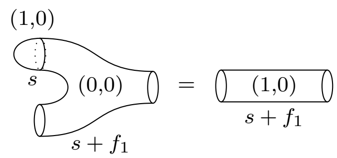
$$Z_{s+f_2}(0|0,0)_{x_a x_b x_c} = \begin{cases} \phi^2 t_1^2 & a,b,c \in \{0,2\} \text{ or } a,b,c \in \{1,3\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. First consider the case of degree $s+f_1$. By geometric constraints (similar to the proofs of Lemmas 3.2.5, 3.2.6) the invariants $Z_{s+f_1}(0|0,0)_{x_a x_b x_c}$ vanish unless

$a, b, c \in \{0, 1\}$ or $a, b, c \in \{2, 3\}$. The partition function for the $(1, 0)$ cap is

$$Z_s(0|1,0)_{x_a} = \begin{cases} \phi^{-1}t_1 & a \in \{0,2\} \\ 0 & \text{otherwise} \end{cases}.$$

This can easily be calculated by gluing the $(0, 0)$ cap to the $(1, 0)$ tube. Now we glue the $(1, 0)$ cap to the $(0, 0)$ pants



to get the relations

$$Z_{s+f_1}(0|0,0)_{x_0x_0x_0}Z_s(0|1,0)_{x_0}T(x_0)^{-1} = Z_{s+f_1}(0|1,0)_{x_0x_0}$$

$$Z_{s+f_1}(0|0,0)_{x_0x_1x_0}Z_s(0|1,0)_{x_0}T(x_0)^{-1} = Z_{s+f_1}(0|1,0)_{x_0x_1}$$

$$Z_{s+f_1}(0|0,0)_{x_2x_2x_2}Z_s(0|1,0)_{x_2}T(x_2)^{-1} = Z_{s+f_1}(0|1,0)_{x_2x_2}$$

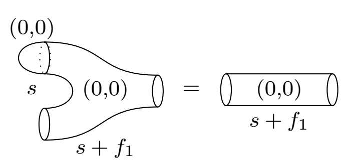
$$Z_{s+f_1}(0|0,0)_{x_2x_3x_2}Z_s(0|1,0)_{x_2}T(x_2)^{-1} = Z_{s+f_1}(0|1,0)_{x_2x_3}$$

and applying Lemma 3.2.5 we get

$$Z_{s+f_1}(0|0,0)_{x_0x_0x_0} = \phi^2t_2^2, \qquad Z_{s+f_1}(0|0,0)_{x_0x_1x_0} = \phi^2t_2^2$$

$$Z_{s+f_1}(0|0,0)_{x_2x_2x_2} = \phi^2t_2^2, \qquad Z_{s+f_1}(0|0,0)_{x_2x_3x_2} = \phi^2t_2^2$$

We glue the $(0, 0)$ cap to the $(0, 0)$ pants



to get the relations

$$\sum_{b=0}^{3} Z_{s+f_1}(0|0,0)_{x_a x_a x_b} T(x_b)^{-1} = 0$$

which imply

$$Z_{s+f_1}(0|0,0)_{x_1x_1x_1} = Z_{s+f_1}(0|0,0)_{x_1x_1x_0} = \phi^2 t_2^2,$$

$$Z_{s+f_1}(0|0,0)_{x_3x_3x_3} = Z_{s+f_1}(0|0,0)_{x_3x_3x_2} = \phi^2 t_2^2.$$

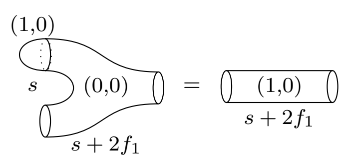
The case of degree $s+f_2$ follows by a similar argument. $\square$

Lemma 3.2.8. *The partition functions for the level (0,0) pants in degrees $s+2f_1$ and $s+2f_2$ vanish*

$$Z_{s+2f_1}(0|0,0)_{x_a x_b x_c} = 0$$

$$Z_{s+2f_2}(0|0,0)_{x_a x_b x_c} = 0$$

Proof. First consider the case of degree $s+2f_1$. By geometric constraints the invariants $Z_{s+2f_1}(0|0,0)_{x_a x_b x_c}$ vanish unless $a,b,c \in \{0,1\}$ or $a,b,c \in \{2,3\}$. Now we glue the $(1,0)$ cap to the $(0,0)$ pants



to get the relations

$$Z_{s+2f_1}(0|0,0)_{x_0x_0x_0} Z_s(0|1,0)_{x_0} T(x_0)^{-1} = 0$$

$$Z_{s+2f_1}(0|0,0)_{x_0x_1x_0} Z_s(0|1,0)_{x_0} T(x_0)^{-1} = 0$$

$$Z_{s+2f_1}(0|0,0)_{x_2x_2x_2} Z_s(0|1,0)_{x_2} T(x_2)^{-1} = 0$$

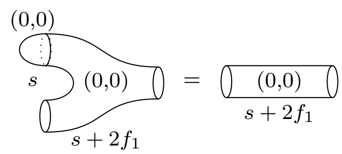
$$Z_{s+2f_1}(0|0,0)_{x_2x_3x_2} Z_s(0|1,0)_{x_2} T(x_2)^{-1} = 0$$

which imply

$$Z_{s+2f_1}(0|0,0)_{x_0x_0x_0} = 0, \qquad Z_{s+2f_1}(0|0,0)_{x_0x_1x_0} = 0$$

$$Z_{s+2f_1}(0|0,0)_{x_2x_2x_2} = 0, \qquad Z_{s+2f_1}(0|0,0)_{x_2x_3x_2} = 0$$

We glue the $(0,0)$ cap to the $(0,0)$ pants



to get the relations

$$\sum_{b=0}^{3} Z_{s+2f_1}(0|0,0)_{x_ax_ax_b} T(x_b)^{-1} = 0$$

which imply

$$Z_{s+2f_1}(0|0,0)_{x_1x_1x_1} = Z_{s+2f_1}(0|0,0)_{x_1x_1x_0} = 0$$

$$Z_{s+2f_1}(0|0,0)_{x_3x_3x_3} = Z_{s+2f_1}(0|0,0)_{x_3x_3x_2} = 0.$$

The case of degree $s + 2f_2$ follows by a similar argument.

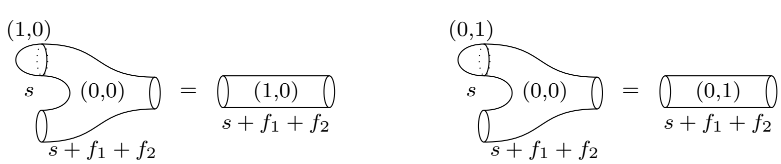
□

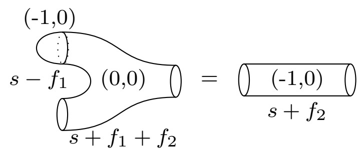
Lemma 3.2.9. *The partition functions for the level (0,0) pants in degree $s+f_1+f_2$ are independent of a, b, c*

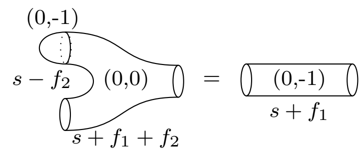
$$Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_c} = C$$

where $C \in \mathbb{Q}[t_1,t_2]((u))$ is independent of a, b, c.

Proof. Applying the gluing formula corresponding to the diagrams depicted below:







we deduce the following relations (respectively):

$$\phi^{-1}t_1\left(Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_0}T(x_0)^{-1}+Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_2}T(x_2)^{-1}\right)=0,$$

$$\phi^{-1}t_2\left(Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_0}T(x_0)^{-1}+Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_1}T(x_1)^{-1}\right)=0,$$

$$-\phi^{-1}t_1\left(Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_1}T(x_1)^{-1}+Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_3}T(x_3)^{-1}\right)=0,$$

$$-\phi^{-1}t_2\left(Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_2}T(x_2)^{-1}+Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_3}T(x_3)^{-1}\right)=0.$$

In fact this implies $Z_{s+f_1+f_2}(0|0,0)_{x_ax_bx_c}=C$ where C is independent of a,b,c. In the next section, by studying the level and genus raising operators, we will see that $C=0$. □

3.2.3 Raising Operators

The level creation operators are defined by

$$U_1 := \left[Z(0|1,0)^{x_a}_{x_b}\right], \qquad U_2 := \left[Z(0|0,1)^{x_a}_{x_b}\right].$$

We can compute these using Lemma 3.2.3 and Lemma 3.2.5,

$$U_1 = \left[Z_s(0|1,0)^{x_a}_{x_b}\right] + \left[Z_{s+f_1}(0|1,0)^{x_a}_{x_b}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left(\phi^{-1}\begin{bmatrix} t_1 & 0 \\ 0 & 0 \end{bmatrix} + \phi \begin{bmatrix} 1/t_1 & 1/t_1 \\ -1/t_1 & -1/t_1 \end{bmatrix} v_1\right)$$

$$U_2 = \left[Z_s(0|0,1)^{x_a}_{x_b}\right] + \left[Z_{s+f_2}(0|0,1)^{x_a}_{x_b}\right]$$

$$= \left(\phi^{-1} \begin{bmatrix} t_2 & 0 \\ 0 & 0 \end{bmatrix} + \phi \begin{bmatrix} 1/t_2 & 1/t_2 \\ -1/t_2 & -1/t_2 \end{bmatrix} v_2 \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly the level annihilation operators are defined by

$$L_1 := \left[Z(0|-1,0)^{x_a}_{x_b} \right], \qquad L_2 := \left[Z(0|0,-1)^{x_a}_{x_b} \right]$$

which we may again compute using Lemma 3.2.3 and Lemma 3.2.5,

$$L_1 = \left[Z_{s-f_1}(0|-1,0)^{x_a}_{x_b} \right] + \left[Z_s(0|-1,0)^{x_a}_{x_b} \right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left(\phi^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -t_1 \end{bmatrix} v_1^{-1} + \phi \begin{bmatrix} 1/t_1 & 1/t_1 \\ -1/t_1 & -1/t_1 \end{bmatrix} \right)$$

$$L_2 = \left[Z_{s-f_2}(0|0,-1)^{x_a}_{x_b} \right] + \left[Z_s(0|0,-1)^{x_a}_{x_b} \right]$$

$$= \left(\phi^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -t_2 \end{bmatrix} v_2^{-1} + \phi \begin{bmatrix} 1/t_2 & 1/t_2 \\ -1/t_2 & -1/t_2 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

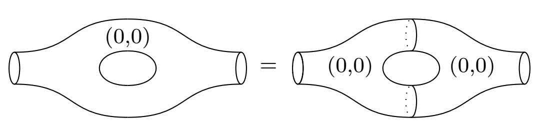
The level annihilation operators satisfy $L_1 = U_1^{-1}, L_2 = U_2^{-1}$.

The genus raising operator $G = \left[Z(1|0,0)^{x_a}_{x_b} \right]$ is given by

$$G = \left[Z_s(1|0,0)^{x_a}_{x_b} \right] + \left[Z_{s+f_1}(1|0,0)^{x_a}_{x_b} \right] v_1$$

$$+ \left[Z_{s+f_2}(1|0,0)^{x_a}_{x_b} \right] v_2 + \left[Z_{s+f_1+f_2}(1|0,0)^{x_a}_{x_b} \right] v_1 v_2.$$

We can calculate G by gluing two pairs of pants at two points and applying the gluing formula:



$$Z(1|0,0)^{x_a}_{x_b} = Z(0|0,0)_{x_b x_c x_d} Z(0|0,0)^{x_a x_c x_d}$$

from which we deduce that

$$G = \begin{bmatrix} t_2 + 2\phi^2 v_2/t_2 & 2\phi^2 v_2/t_2 \\ -2\phi^2 v_2/t_2 & -t_2 - 2\phi^2 v_2/t_2 \end{bmatrix} \otimes \begin{bmatrix} t_1 + 2\phi^2 v_1/t_1 & 2\phi^2 v_1/t_1 \\ -2\phi^2 v_1/t_1 & -t_1 - 2\phi^2 v_1/t_1 \end{bmatrix}$$

$$+C \begin{bmatrix} v_2/t_2 & v_2/t_2 \\ -v_2/t_2 & -v_2/t_2 \end{bmatrix} \otimes \begin{bmatrix} v_1/t_1 & v_1/t_1 \\ -v_1/t_1 & -v_1/t_1 \end{bmatrix}$$

where C is as in Lemma 3.2.9. The gluing lemma implies G commutes with the level creation operators. We calculate the commutator

$$[G, U_1] = C \begin{bmatrix} t_2 + 2\phi^2 v_2/t_2 & 2\phi^2 v_2/t_2 \\ -2\phi^2 v_2/t_2 & -t_2 - 2\phi^2 v_2/t_2 \end{bmatrix} \otimes \begin{bmatrix} 0 & v_1 \\ v_1 & 0 \end{bmatrix}$$

which implies that $C = 0$.

We now prove Theorem 3.1.1:

Proof. As in [6,9] the gluing formula implies the partition functions $Z(g|0,0)_{\alpha_1,\ldots,\alpha_r}$ give rise to a 1+1-dimensional topological quantum field theory (TQFT) taking values in $R = \mathbb{Q}(t_1, t_2)((u))$. The corresponding Frobenius algebra is

$$A = \bigoplus_{i=0}^{3} Re_{x_i}$$

for $x_i \in \mathcal{B}$ with multiplication given by

$$e_{x_i} \otimes e_{x_j} = \sum_{k=0}^{3} Z(g|0,0)^{x_k}_{x_i x_j} e_{x_k}.$$

A TQFT is semi-simple if the corresponding Frobenius algebra is semi-simple. We prove semi-simplicity of the TQFT. For $u = 0$ we have

$$Z(0|0,0)^{x_c}_{x_a x_b}|_{u=0} = Z_s(0|0,0)^{x_c}_{x_a x_b} = \begin{cases} T(x_a) & \text{if } a = b = c, \\ 0 & \text{otherwise} \end{cases},$$

so the basis $\{e_{x_i}/T(x_i)\}$ of the Frobenius algebra for $u = 0$ is idempotent. Thus the TQFT is semisimple for $u = 0$. If R is a complete local ring with maximal ideal $\mathfrak{m}$ and A is a Frobenius algebra over R which is free as an R-module, Proposition 2.2 in [5] states that if $A/\mathfrak{m}A$ is a semi-simple Frobenius algebra over $R/\mathfrak{m}$ then A is semi-simple over R. Thus the TQFT is semi-simple. This implies Theorem 3.1.1 (see the proof of Theorem 5.2 in [6]). □

We are now ready to prove Corollary 3.1.1.

Proof. We may check that

$$G^2 = (4\phi^2 v_1 + t_1^2)(4\phi^2 v_2 + t_2^2)I.$$

Taking the trace of this equation and reducing modulo t_1, t_2 we see that

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = 4^2\phi^4 v_1 v_2 Z(g-2|k_1, k_2)|_{t_1=t_2=0}.$$

Then using this equation and the symmetry implied by the (generalized) Kodaira deformation

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = v_1 Z(g|k_1-2, k_2)|_{t_1=t_2=0} = v_2 Z(g|k_1, k_2-2)|_{t_1=t_2=0}$$

(this may also be checked directly from the matrices) we are reduced to checking the following base cases: For $g = 0$,

	$k_2 = 0$	1
$k_1 = 0$	0	0
1	0	ϕ^{-2}

and for $g = 1$,

	$k_2 = 0$	1
$k_1 = 0$	4	0
1	0	0

We conclude,

$$Z(g|k_1, k_2)|_{t_1=t_2=0} = \begin{cases} 4^g \phi^{2g-2} v_1^{\frac{g-1+k_1}{2}} v_2^{\frac{g-1+k_2}{2}} & (g-1) \equiv k_1 \equiv k_2 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Chapter 4: Gromov-Witten Theory of $\mathcal{H}_1$-Bundles

4.1 Summary of Results

Theorem 4.1.1. *Let $a=-1$, so that X is an $\mathcal{H}_1$-bundle over a genus g curve C. Then $Z(g|k_1,k_2) = tr\left(G^{g-1}L_1^{-k_1}L_2^{-k_2}\right)$ where*

$$L_1 = \phi^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & -t_1 \end{bmatrix}\frac{1}{v_1} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix}\frac{1}{v_2}$$

$$+\phi^2\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1(t_1+t_2)} & \frac{1}{t_1(t_1+t_2)} \\ -\frac{1}{t_1t_2} & -\frac{1}{t_1t_2} \end{bmatrix}$$

$$L_2 = \phi^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -(t_1+t_2) & 0 \\ 0 & -t_2 \end{bmatrix}\frac{1}{v_2} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix}\frac{v_1}{v_2^2}$$

$$+\phi\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1+t_2} & 0 \\ 0 & \frac{1}{t_2} \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} t_1(t_1+t_2) & 0 \\ 0 & -t_1t_2 \end{bmatrix} + \phi^2\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{2t_1}{t_1+t_2} & 0 \\ 0 & \frac{-2t_1}{t_2} \end{bmatrix}v_2$$

$$+\phi \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{2(t_1+t_2)^2}{t_1} & \frac{t_2(t_1+2t_2)}{t_1} \\ -\frac{(t_1+2t_2)(t_1+t_2)}{t_1} & -\frac{2t_2^2}{t_1} \end{bmatrix} \frac{v_1}{v_2} + \phi^2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix} \frac{v_1^2}{v_2^2}$$

$$+\phi^3 \begin{bmatrix} \frac{2(2t_1+t_2)}{t_1(t_1+t_2)} & \frac{t_1+2t_2}{t_1(t_1+t_2)} & \frac{-2t_2}{t_1(t_1+t_2)} & \frac{t_1-2t_2}{t_1(t_1+t_2)} \\ \frac{-(t_1+2t_2)}{t_1t_2} & \frac{2(t_1-t_2)}{t_1t_2} & \frac{3t_1+2t_2}{t_1t_2} & \frac{2(t_1+t_2)}{t_1t_2} \\ \frac{2t_2}{t_1(t_1+t_2)} & \frac{3t_1+2t_2}{t_1(t_1+t_2)} & \frac{2(2t_1+3t_2)}{t_1(t_1+t_2)} & \frac{3(t_1+2t_2)}{t_1(t_1+t_2)} \\ \frac{t_1-2t_2}{t_1t_2} & \frac{-2(t_1+t_2)}{t_1t_2} & \frac{-3(t_1+2t_2)}{t_1t_2} & \frac{-2(t_1+3t_2)}{t_1t_2} \end{bmatrix} v_1$$

Remark 4.1.1. *We see from the formula above, that if β is a section class, $Z_\beta(g|k_1,k_2)$ is of the form $p(t_1,t_2)\phi^{2g-2-K_X\cdot\beta}$ where $p(t_1,t_2)$ is a homogeneous polynomial of degree $K_X\cdot\beta$ in t_1,t_2.*

Recall $Z(g|k_1,k_2)|_{t_1=t_2=0}$ denotes the non-equivariant limit which is obtained by setting $t_1=t_2=0$. Its terms correspond to Calabi-Yau section classes. The Calabi-Yau section class partition functions satisfy the following recursions:

Corollary 4.1.1.

$$Z(g|k_1,k_2)|_{t_1=t_2=0} = v_1^2 Z(g|k_1-3,k_2)|_{t_1=t_2=0} + v_1^2 v_2 Z(g|k_1-4,k_2)|_{t_1=t_2=0}$$

$$Z(g|k_1,k_2)|_{t_1=t_2=0} = -\phi^2 v_1^2 v_2^{-2} Z(g-1|k_1,k_2)|_{t_1=t_2=0} + 6\phi^4 v_1^2 v_2^{-1} Z(g-2|k_1,k_2)|_{t_1=t_2=0}$$
$$+(256\phi^8 v_1^2 v_2 + 27\phi^8 v_1^4 v_2^{-2}) Z(g-4|k_1,k_2)|_{t_1=t_2=0}$$

Remark 4.1.2. *The (generalized) Kodaira deformation (Lemma 1.2.1) implies symmetry for the Calabi-Yau section class partition functions $Z(g|k_1,k_2)|_{t_1=t_2=0}$:*

$$Z(g|k_1,k_2)|_{t_1=t_2=0} = v_1 Z(g|k_1-2,k_2+1)|_{t_1=t_2=0}$$

The Kodaira deformation and Corollary 4.1.1 suffice to compute all the Calabi-Yau section class invariants in terms of the following base cases:

	$g=0$	1	2	3
$k_1=0$	0	4	$-\phi^2 v_1^2 v_2^{-2}$	$12\phi^4 v_1^2 v_2^{-1} + \phi^4 v_1^4 v_2^{-4}$
1	ϕ^{-2}	0	$\phi^2 v_1^2 v_2^{-1}$	$16\phi^4 v_1^2 - \phi^4 v_1^4 v_2^{-3}$
2	0	0	$8\phi^2 v_1^2$	$64\phi^4 v_1^2 v_2 + \phi^4 v_1^4 v_2^{-2}$
3	0	$3v_1^2$	$16\phi^2 v_1^2 v_2$	$-\phi^4 v_1^4 v_2^{-1}$

Corollary 4.1.2. *The GW/DT/PT correspondence 5.3.1 holds for the Calabi-Yau section class partition functions when $a=-1$. This will be proved in Chapter 5.*

4.2 Calculations

We will see that the full theory is determined by the following basic partition functions:

$$Z(0|0,0)_\alpha, \quad Z(0|0,0)_{\alpha_1\alpha_2},$$

$$Z(0|1,0)_{\alpha_1\alpha_2}, \qquad Z(0|0,1)_{\alpha_1\alpha_2},$$

$$Z(0|0,0)_{\alpha_1\alpha_2\alpha_3}.$$

Lemma 4.2.1. *The basic partition functions depend on only the following cohomol-*

ogy classes:

$$\begin{array}{ll}
Z(0|0,0)_{\alpha} & s \\
Z(0|0,0)_{\alpha_1\alpha_2} & s,\ s+f_2,\ s+f_1-f_2,\ s+2f_1-2f_2 \\
Z(0|0,0)_{\alpha_1\alpha_2\alpha_3} & s,\ s+f_1,\ s+f_2,\ s+2f_2,\ s+f_1-f_2,\ s+2f_1-2f_2, \\
& s+3f_1-3f_2, s+4f_1-4f_2,\ s+2f_1-f_2 \\
Z(0|1,0)_{\alpha_1\alpha_2} & s,\ s+f_1,\ s+f_2,\ s+2f_2,\ s+f_1-f_2,\ s+2f_1-2f_2, \\
& s+3f_1-3f_2, s+4f_1-4f_2,\ s+2f_1-f_2 \\
Z(0|0,1)_{\alpha_1\alpha_2} & s,\ s+f_1,\ s+f_2,\ s+f_1-f_2,\ s+2f_1-2f_2,\ s+3f_1-3f_2
\end{array}$$

Proof. In the cases above, X is a toric threefold. By (2.2) for $\beta = s + m_1 f_1 + m_2 f_2$, the degree of $Z_\beta(0|k_1,k_2)_{\alpha_1,\ldots,\alpha_j}$ as polynomial in t_1,t_2 is given by

$$N = \sum_{i=1}^{j} \deg(\alpha_i) - \operatorname{vdim} \overline{\mathcal{M}}^{\bullet}_h(X/\vec{F},\beta) = 2j - (-2k_1 - k_2 + 2 + 3m_1 + 2m_2).$$

Since X is compact the degree is nonnegative. This provides an upper bound on m_1, m_2. To obtain a lower bound, recall the toric cone theorem implies the Mori cones are generated by the torus-invariant curves. For levels $(0,0), (1,0)$, and $(0,1)$ the cones are generated by $\langle s, f_2, f_1 - f_2\rangle$. □

Lemma 4.2.2. *The partition functions for the degree s, level $(0,0)$ cap, tube, and pants are given by*

$$Z_s(0|0,0)_{x_a} = 1$$

$$Z_s(0|0,0)_{x_a x_b} = \begin{cases} T(x_a) & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$Z_s(0|0,0)_{x_a x_b x_c} = \begin{cases} T(x_a)^2 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

where $a, b, c \in \{0, 1, 2, 3\}$.

Proof. This follows by the same argument as for Lemma 3.2.2,

$\square$

Lemma 4.2.3. *The partition functions for the degree* $s + f_2, s + f_1 - f_2$*, and* $s + 2f_1 - 2f_2$ *level* $(0, 0)$ *tubes vanish*

$$Z_{s+f_2}(0|0,0)^{x_a}_{x_b} = 0$$

$$Z_{s+f_1-f_2}(0|0,0)^{x_a}_{x_b} = 0$$

$$Z_{s+2f_1-2f_2}(0|0,0)^{x_a}_{x_b} = 0$$

Proof. This follows similarly to Lemma 3.2.4.

$\square$

Lemma 4.2.4. *The partition functions for the tubes of degree* s*, level* $(1, 0)$ *and* $(0, 1)$ *are given by*

$$Z_s(0|1,0)^{x_a}_{x_b} = \begin{cases} \phi^{-2} t_1(t_1 + t_2) & a = b = 0 \\ 0 & \textit{otherwise} \end{cases}$$

$$Z_s(0|0,1)^{x_a}_{x_b} = \begin{cases} \phi^{-1}(t_1 + t_2) & a = b = 0 \\ \phi^{-1} t_2 & a = b = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. First let the level be $(1, 0)$. In this case, any $\mathbb{T}$-fixed stable relative map representing the class s must have image $C_0[\vec{L}]$ (see Figure 1). The curve C_0 has normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ (see (1.1)). Applying Lemma 2.2.1 and (1.2) we compute

$$Z_s(0|1,0)^{x_0}_{x_0} = Z^{\mathrm{loc}}(-1,-1)_{0,\infty}(t_1, t_1+t_2)T(x_0) = \phi^{-2}t_1(t_1+t_2),$$

$$Z_s(0|1,0)^{x_a}_{x_b} = 0 \text{ if } (x_a,x_b)\neq(x_0,x_0).$$

Next let the level be $(0,1)$. In this case, any $\mathbb{T}$-fixed stable relative map representing the class s must have image $C_0[\vec{L}]$ or $C_1[\vec{L}]$ (see Figure 1). The curves C_0 and C_1 each have normal bundle $\mathcal{O}\oplus\mathcal{O}(-1)$ (see (1.1)). Applying Lemma 2.2.1 and 1.2 we compute

$$Z_s(0|0,1)^{x_0}_{x_0} = Z^{\mathrm{loc}}(0,-1)_{0,\infty}(t_1,t_1+t_2)T(x_0) = \left(\phi^{-1}\frac{1}{t_1}\right)t_1(t_1+t_2) = \phi^{-1}(t_1+t_2),$$

$$Z_s(0|0,1)^{x_1}_{x_1} = Z^{\mathrm{loc}}(0,-1)_{0,\infty}(-t_1,t_2)T(x_1) = \left(-\phi^{-1}\frac{1}{t_1}\right)(-t_1t_2) = \phi^{-1}t_2,$$

$$Z_s(0|0,1)^{x_a}_{x_b} = 0 \text{ if } (x_a,x_b)\neq(x_0,x_0),(x_1,x_1).$$

□

Lemma 4.2.5. *The partition function for the level* $(0,1)$*, degree* $s+f_2$ *tube is given by*

$$[Z_{s+f_2}(0,1)^{x_a}_{x_b}] = \phi\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\otimes\begin{bmatrix}\frac{1}{t_1+t_2} & 0\\ 0 & \frac{1}{t_2}\end{bmatrix}$$

Proof. By geometric constraints (similar to the proof of Lemma 3.2.6) the following invariants vanish

$$Z_{s+f_2}(0|0,1)^{x_1}_{x_0} = Z_{s+f_2}(0|0,1)^{x_3}_{x_0} = Z_{s+f_2}(0|0,1)^{x_2}_{x_1} = Z_{s+f_2}(0|0,1)^{x_3}_{x_2} = 0.$$

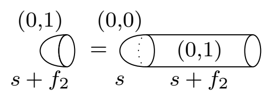
If the image of a $\mathbb{T}$-fixed stable relative map meets the transforms of F_1 and F_2 at x_2, the image must be $C_2[\vec{L}]$ (see Figure 1). The curve C_2 has normal bundle $\mathcal{O} \oplus \mathcal{O}(1)$ (see 1.1). Applying Lemma 2.2.1 and (1.2) we conclude

$$Z_{s+f_2}(0|0,1)_{x_2}^{x_2} = Z^{\mathrm{loc}}(0,1)_{0,\infty}(t_1, -t_1 - t_2)T(x_2) = \frac{-\phi}{t_1 + t_2}.$$

Similarly,

$$Z_{s+f_2}(0|0,1)_{x_3}^{x_3} = Z^{\mathrm{loc}}(0,1)_{0,\infty}(-t_1, -t_2)T(x_3) = \frac{-\phi}{t_2}.$$

Now we attach the $(0,0)$ cap to the $(0,1)$ tube



and since the degree $s + f_2$, $(0,1)$ cap vanishes by dimension constrants we get the relations

$$0 = \sum_{c=0}^{3} Z_{s+f_2}(0|0,1)_{x_a}^{x_c}$$

from which we can solve for the other invariants. $\square$

Lemma 4.2.6. *The partition function for the level* $(1,0)$*, degree* $s+f_2$ *tube is given by*

$$[Z_{s+f_2}(0|1,0)_{x_b}^{x_a}] = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1}{t_1+t_2} & 0 \\ 0 & 0 \end{bmatrix}$$

Proof. The invariants $Z_{s+f_2}(0|1,0)_{x_b}^{x_a}$ vanish unless $a, b \in \{0, 2\}$. To see this recall that $X[\vec{L}]$ is a chain of varieties given by a union of X and copies of $\mathbb{P}^1 \times \mathcal{H}_1$. Since the intersection of the image $q(C')$ of the stable map with the component X is $\mathbb{T}$-fixed, we have $q(C') \cap X$ is either C_0 (possibly with an attached fiber) which has class s or C_2 which has class $s + f_2$ (see Figure 1). In the second case, $q(C') = C_2[\vec{L}]$.

In the first case, by considering the projection to $\mathcal{H}_1$ we see a curve in class $s + f_2$ in $\mathbb{P}^1 \times \mathcal{H}_1$ may join the point x_0 in one fiber to x_0 or x_2 in another fiber, but not to x_1 or x_3.

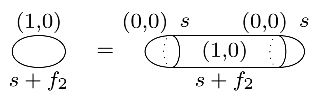
Since C_2 has normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(1)$, by the above discussion and Lemma 2.2.1 and (1.2) we get

$$Z_{s+f_2}(0|1,0)_{x_2}^{x_2} = Z^{\mathrm{loc}}(-1,1)_{0,\infty}(t_1, -t_1 - t_2)T(x_2) = -\frac{t_1}{t_1 + t_2}.$$

By dimension constraints, the partition function vanishes for the absolute geometry

$$Z_{s+f_2}(0|1,0) = 0.$$

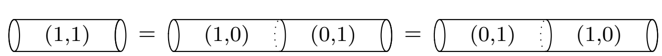
We glue two $(0,0)$ caps to the $(1,0)$ tube



to produce the relation

$$0 = \sum_{b=0}^{3}\left(\sum_{a=0}^{3} Z_s(0|0,0)_{x_a} Z_{s+f_2}(0|1,0)_{x_b}^{x_a}\right) Z_s(0|0,0)^{x_b}.$$

which implies $Z_{s+f_2}(0|1,0)_{x_0}^{x_0} = 2Z_{s+f_2}(0|1,0)_{x_2}^{x_0} - \frac{t_1}{t_1+t_2}$. We may produce a $(1,1)$ tube by gluing a $(1,0)$ tube to a $(0,1)$ tube, but we have the freedom to swap the order of the tubes.



This gives us the equation

$$Z_{s+f_2}(0|1,1)_{x_b}^{x_a} = \sum_{c=0}^{3}\left(Z_s(0|1,0)_{x_c}^{x_a} Z_{s+f_2}(0|0,1)_{x_b}^{x_c} + Z_{s+f_2}(0|1,0)_{x_c}^{x_a} Z_s(0|0,1)_{x_b}^{x_c}\right)$$

$$= \sum_{c=0}^{3}\left(Z_s(0|0,1)_{x_c}^{x_a} Z_{s+f_2}(0|1,0)_{x_b}^{x_c} + Z_{s+f_2}(0|0,1)_{x_c}^{x_a} Z_s(0|1,0)_{x_b}^{x_c}\right)$$

which by Lemma 4.2.4 and Lemma 4.2.5 implies

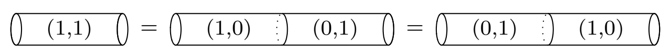
$$Z_{s+f_2}(0|1,1)^{x_0}_{x_2} = t_1 = (t_1+t_2)Z_{s+f_2}(0|1,0)^{x_0}_{x_2}.$$

□

Lemma 4.2.7. *The partition function for the level* $(1,0)$*, degree* $s+2f_2$ *tube vanishes*

$$Z_{s+2f_2}(1,0)^{x_a}_{x_b} = 0$$

Proof. By geometric constraints (similar to the proof of Lemma 4.2.6), $Z_{s+2f_2}(0|1,0)^{x_a}_{x_b}$ vanishes unless $a,b \in \{0,2\}$. We may produce a $(1,1)$ tube by gluing a $(1,0)$ tube to a $(0,1)$ tube, but we have the freedom to swap the order of the tubes.



This gives us the equation

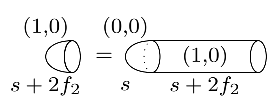
$$Z_{s+2f_2}(0|1,1)^{x_a}_{x_b} = \sum_{c=0}^{3}\left(Z_{s+f_2}(0|1,0)^{x_a}_{x_c}Z_{s+f_2}(0|0,1)^{x_c}_{x_b} + Z_{s+2f_2}(0|1,0)^{x_a}_{x_c}Z_s(0|0,1)^{x_c}_{x_b}\right)$$

$$= \sum_{c=0}^{3}\left(Z_s(0|0,1)^{x_a}_{x_c}Z_{s+2f_2}(0|1,0)^{x_c}_{x_b} + Z_{s+f_2}(0|0,1)^{x_a}_{x_c}Z_{s+f_2}(0|1,0)^{x_c}_{x_b}\right)$$

which by Lemma 4.2.4, Lemma 4.2.5, and Lemma 4.2.6 implies

$$Z_{s+2f_2}(0|1,1)^{x_0}_{x_2} = 0 = (t_1+t_2)Z_{s+2f_2}(0|1,0)^{x_0}_{x_2}.$$

We attach the $(0,0)$ cap to the $(1,0)$ tube



and since the degree $s+2f_2$, $(1,0)$ cap vanishes by dimension constrants we get the relations

$$0 = \sum_{c=0}^{3} Z_{s+2f_2}(0|1,0)^{x_c}_{x_a}$$

from which we see all the invariants vanish.

∎

Lemma 4.2.8. *Let $n > 0$. The partition function for the level $(1,0)$ degree $s + n(f_1 - f_2)$ tube vanishes*

$$Z_{s+n(f_1-f_2)}(0|1,0)^{x_a}_{x_b} = 0 \qquad \textit{if } n > 0$$

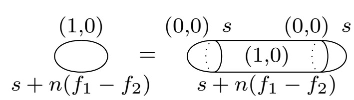
Proof. The image of a $\mathbb{T}$-fixed stable relative map representing the class $s+n(f_1-f_2)$ must be the (disconnected) union of $C_0[\vec{L}]$ and a collection of fibers with total class $n(f_1 - f_2)$ (see Figure 1). Since the map is admissible it meets the transforms of F_1 and F_2 at the point x_0. Thus

$$Z_{s+n(f_1-f_2)}(0|1,0)^{x_a}_{x_b} = 0 \qquad \text{if } (a,b) \neq (0,0).$$

By dimension constraints, the partition function vanishes for the absolute geometry

$$Z_{s+n(f_1-f_2)}(0|1,0) = 0.$$

We glue two $(0,0)$ caps to the $(1,0)$ tube



to produce the relation

$$0 = \sum_{b=0}^{3} \left(\sum_{a=0}^{3} Z_s(0|0,0)_{x_a} Z_{s+n(f_1-f_2)}(0|1,0)^{x_a}_{x_b} \right) Z_s(0|0,0)^{x_b}.$$

We conclude

$$Z_{s+n(f_1-f_2)}(0|1,0)^{x_0}_{x_0} = 0.$$

∎

Lemma 4.2.9. *The partition function for the level* $(1,0)$ *degree* $s+f_1$ *tube is given by*

$$[Z_{s+f_1}(0|1,0)^{x_a}_{x_b}] = \phi \begin{bmatrix} \frac{2t_1+t_2}{t_1(t_1+t_2)} & \frac{1}{t_1} & \frac{1}{t_1+t_2} & \frac{1}{t_1+t_2} \\ -\frac{(t_1+t_2)}{t_1t_2} & -\frac{1}{t_1} & 0 & 0 \\ -\frac{1}{t_1+t_2} & 0 & \frac{t_2}{t_1(t_1+t_2)} & \frac{t_2}{t_1(t_1+t_2)} \\ \frac{1}{t_2} & 0 & -\frac{1}{t_1} & -\frac{1}{t_1} \end{bmatrix}$$

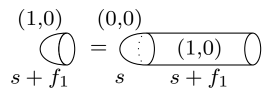
Proof. The space $X[\vec{L}]$ is a chain of varieties given by the union of X and copies of $\mathcal{H}_1 \times \mathbb{P}^1$. The image $q(C')$ of the $\mathbb{T}$-fixed stable relative map in X may be any of the C_i, possibly with attached fibers. Assume that there are no attached fibers in the X component (the analysis is similar and simpler in the other case). If $q(C') \cap X$ is C_1 or C_3 which have class $s+f_1$, then $q(C')$ is $C_1[\vec{L}]$ or $C_3[\vec{L}]$ respectively (see Figure 1). If $q(C') \cap X$ is C_2 which has class $s+f_2$, then the image of q in one of the copies of $\mathcal{H}_1 \times \mathbb{P}^1$ has class $s+f_1-f_2$; such a curve can join x_2 in one fiber to x_2 or x_3 in another fiber. If $q(C') \cap X$ is C_0 which has class s, then there are two possibilities. In the first case the image of q in one copy of $\mathcal{H}_1 \times \mathbb{P}^1$ has class $s+f_1$; such a curve can join x_0 in one fiber, to any of the other fixed points in another fiber. In the second case, the image of $q(C')$ in one copy has class $s+f_2$ and the image in another copy has class $s+f_1-f_2$. Such a chain can join x_0 in one fiber with x_0, x_2, or x_3 in another fiber.

From the above discussion we conclude that the invariants $Z_{s+f_1}(0|1,0)^{x_1}_{x_2}$ and $Z_{s+f_1}(0|1,0)^{x_1}_{x_3}$ vanish because the image of the stable map cannot join these points. The curves C_1, C_3 each have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}$ (see (1.1)), so by Lemma 2.2.1, (1.2), and the above discussion we calculate

$$Z_{s+f_1}(0|1,0)_{x_1}^{x_1} = Z^{\mathrm{loc}}(1,0)_{0,\infty}(-t_1,t_2)T(x_1) = -\frac{1}{t_1},$$

$$Z_{s+f_1}(0|1,0)_{x_3}^{x_3} = Z^{\mathrm{loc}}(1,0)_{0,\infty}(-t_1,-t_2)T(x_3) = -\frac{1}{t_1}.$$

We attach the $(0,0)$ cap to the $(1,0)$ tube



to deduce the relations

$$\sum_{c=0}^{3} Z_{s+f_1}(0|1,0)_{x_a}^{x_c} = 0.$$

Using the above relations we can write $[Z_{s+f_1}(0|1,0)_{x_b}^{x_a}]$ as a matrix in two unknowns A, B:

$$[Z_{s+f_1}(0|1,0)_{x_b}^{x_a}] = \phi \begin{bmatrix} A + \frac{B(t_1+t_2)}{t_2} & \frac{1}{t_1} & A & -B + \frac{1}{t_1} \\ -\frac{(t_1+t_2)}{t_1 t_2} & -\frac{1}{t_1} & 0 & 0 \\ -A & 0 & -A + \frac{B(t_1+t_2)}{t_2} & B \\ \frac{(-B+\frac{1}{t_1})(t_1+t_2)}{t_2} & 0 & -\frac{B(t_1+t_2)}{t_2} & -\frac{1}{t_1} \end{bmatrix}.$$

By dimension constraints, the invariants $Z_{s+3f_1}(0|3,0)_{x_b}^{x_a}$ vanish. Thus the gluing formula implies the cube of the matrix vanishes:

$$[Z_{s+f_1}(0|1,0)_{x_b}^{x_a}]^3 = 0.$$

This yields the equations

$$A = \frac{t_2(2t_1+t_2) - t_1(t_1+t_2)^2 B}{t_1 t_2 (t_1+t_2)},$$

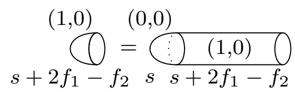
$$B = \frac{t_2}{t_1(t_1+t_2)}.$$

□

Lemma 4.2.10. *The partition function for the level* $(1,0)$ *degree* $s+2f_1-f_2$ *tube vanishes*

$$Z_{s+2f_1-f_2}(0|1,0)^{x_a}_{x_b} = 0$$

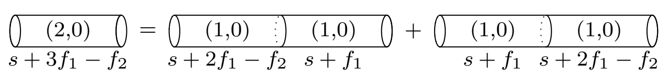
Proof. By geometric constraints (similar to the proof of Lemma 4.2.9), the invariants $Z_{s+2f_1-f_2}(0|1,0)^{x_1}_{x_2}, Z_{s+2f_1-f_2}(0|1,0)^{x_1}_{x_3}$ vanish. The invariants for the level $(1,0)$ degree $s+2f_1-f_2$ cap vanish by dimension constraints. We attach the $(0,0)$ cap to the $(1,0)$ tube



to deduce the relations

$$\sum_{c=0}^{3} Z_{s+2f_1-f_2}(0|1,0)^{x_c}_{x_a} = 0.$$

By dimension constraints the invariants $Z_{s+3f_1-f_2}(0|2,0)^{x_a}_{x_b}$ vanish. Gluing two $(1,0)$ tubes



we get the relations

$$0 = Z_{s+3f_1-f_2}(0|2,0)^{x_a}_{x_b} = \sum_{c=0}^{3} \left(Z_{s+2f_1-f_2}(0|1,0)^{x_a}_{x_c} Z_{s+f_1}(0|1,0)^{x_c}_{x_b} \right.$$
$$\left. + Z_{s+f_1}(0|1,0)^{x_a}_{x_c} Z_{s+2f_1-f_2}(0|1,0)^{x_c}_{x_b} \right).$$

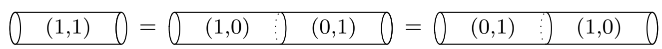
One may check these equations imply the desired vanishing.

□

Lemma 4.2.11. *Let* $n > 0$*. The partition function for the level* $(0,1)$ *degree* $s+n(f_1-f_2)$ *tube vanishes*

$$Z_{s+n(f_1-f_2)}(0|0,1)^{x_a}_{x_b} = 0 \qquad \textit{if } n > 0$$

Proof. The image of a stable relative map in class $s + n(f_1 - f_2)$ is the (disconnected) union of $C_0[\vec{L}]$ or $C_1[\vec{L}]$ and fibers with class $n(f_1 - f_2)$ (see Figure 1). Thus $Z_{s+n(f_1-f_2)}(0|0,1)^{x_a}_{x_b} = 0$ unless $a = b = 0$ or $a = b = 1$. We produce a $(1,1)$ tube by gluing a $(1,0)$ tube to a $(0,1)$ tube and use the freedom to swap the order of the tubes



to get the equation

$$
\begin{aligned}
&Z_{s+f_1+n(f_1-f_2)}(0|1,1)^{x_a}_{x_b} = \\
&\sum_{c=0}^{3} \left(Z_{s+f_1}(0|1,0)^{x_a}_{x_c} Z_{s+n(f_1-f_2)}(0|0,1)^{x_c}_{x_b} + Z_{s+f_2}(0|1,0)^{x_a}_{x_c} Z_{s+(n+1)(f_1-f_2)}(0|0,1)^{x_c}_{x_b} \right) \\
&= \sum_{c=0}^{3} \left(Z_{s+n(f_1-f_2)}(0|0,1)^{x_a}_{x_c} Z_{s+f_1}(0|1,0)^{x_c}_{x_b} + Z_{s+(n+1)(f_1-f_2)}(0|0,1)^{x_a}_{x_c} Z_{s+f_2}(0|1,0)^{x_c}_{x_b} \right)
\end{aligned}
$$

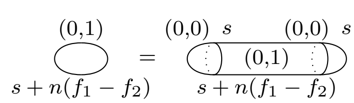
from which we see

$$Z_{s+f_1+n(f_1-f_2)}(0|1,1)^{x_0}_{x_3} = \frac{1}{t_1+t_2} Z_{s+n(f_1-f_2)}(0|0,1)^{x_0}_{x_0} = 0.$$

By dimension constraints, the partition function vanishes for the absolute geometry

$$Z_{s+n(f_1-f_2)}(0|0,1) = 0.$$

We glue two $(0,0)$ caps to the $(0,1)$ tube



to produce the relation

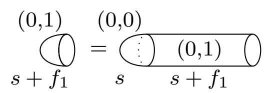
$$0 = \sum_{b=0}^{3} \left(\sum_{a=0}^{3} Z_s(0|0,0)_{x_a} Z_{s+n(f_1-f_2)}(0|0,1)^{x_a}_{x_b} \right) Z_s(0|0,0)^{x_b}.$$

which implies $Z_{s+n(f_1-f_2)}(0|0,1)^{x_1}_{x_1} = 0$. $\square$

Lemma 4.2.12. *The partition function for the level* $(0,1)$ *degree* $s+f_1$ *tube is given by*

$$[Z_{s+f_1}(0|0,1)_{x_a}^{x_b}] = \phi^2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1(t_1+t_2)} & \frac{1}{t_1(t_1+t_2)} \\ -\frac{1}{t_1 t_2} & -\frac{1}{t_1 t_2} \end{bmatrix}$$

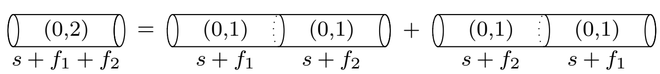
Proof. The invariants for the level $(0,1)$ degree $s+f_1$ cap vanish by dimension constraints. We attach the $(0,0)$ cap to the $(0,1)$ tube



to deduce the relations

$$\sum_{c=0}^{3} Z_{s+f_1}(0|0,1)_{x_a}^{x_c} = 0.$$

The invariants $Z_{s+f_1+f_2}(0|0,2)_{x_b}^{x_a}$ vanish by dimension constraints. Gluing two $(0,1)$ tubes



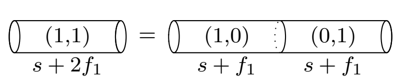
we get the relations

$$0 = \sum_{c=0}^{3} \left(Z_{s+f_1}(0|0,1)_{x_c}^{x_a} Z_{s+f_2}(0|0,1)_{x_b}^{x_c} + Z_{s+f_2}(0|0,1)_{x_c}^{x_a} Z_{s+f_1}(0|0,1)_{x_b}^{x_c} \right).$$

Using these relations we may write $[Z_{s+f_1}(0|0,1)_{x_a}^{x_b}]$ as a matrix in three unknowns

$$[Z_{s+f_1}(0|0,1)_{x_b}^{x_a}] = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} U & -\frac{V t_2}{t_1+t_2} \\ V & W \end{bmatrix}.$$

By dimension constraints the invariants $Z_{s+2f_1}(0|1,1)_{x_b}^{x_a}$ vanish. Gluing a $(1,0)$ tube to a $(0,1)$ tube



we get the relations

$$0 = \sum_{c=0}^{3} Z_{s+f_1}(0|1,0)_{x_c}^{x_a} Z_{s+f_1}(0|0,1)_{x_b}^{x_c}$$

from which we see

$$Z_{s+2f_2}(0|1,1)_{x_0}^{x_1} = 0 = -\frac{t_1+t_2}{t_1t_2}U - \frac{1}{t_1}W$$

$$Z_{s+2f_2}(0|1,1)_{x_1}^{x_1} = 0 = \frac{1}{t_1}V - \frac{1}{t_1}W$$

and

$$[Z_{s+f_1}(0|0,1)_{x_b}^{x_a}] = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} -\frac{Wt_2}{t_1+t_2} & -\frac{Wt_2}{t_1+t_2} \\ W & W \end{bmatrix}.$$

The level creation operators U_1, U_2 are now determined up to the unknown W. The commutator $[U_1, U_2]$ vanishes if and only if $W = -\frac{1}{t_1t_2}$. □

We now have what we need to compute the level creation operators

$$U_1 := \left[Z(0|1,0)_{x_b}^{x_a}\right], \qquad U_2 := \left[Z(0|0,1)_{x_b}^{x_a}\right].$$

$$U_1 = \phi^{-2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} t_1(t_1+t_2) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1}{t_1+t_2} & 0 \\ 0 & 0 \end{bmatrix} v_2$$

$$+\phi \begin{bmatrix} \frac{2t_1+t_2}{t_1(t_1+t_2)} & \frac{1}{t_1} & \frac{1}{t_1+t_2} & \frac{1}{t_1+t_2} \\ -\frac{(t_1+t_2)}{t_1t_2} & -\frac{1}{t_1} & 0 & 0 \\ -\frac{1}{t_1+t_2} & 0 & \frac{t_2}{t_1(t_1+t_2)} & \frac{t_2}{t_1(t_1+t_2)} \\ \frac{1}{t_2} & 0 & -\frac{1}{t_1} & -\frac{1}{t_1} \end{bmatrix} v_1$$

$$U_2 = \phi^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} t_1+t_2 & 0 \\ 0 & t_2 \end{bmatrix} + \phi \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1+t_2} & 0 \\ 0 & \frac{1}{t_2} \end{bmatrix} v_2$$

$$+\phi^2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1(t_1+t_2)} & \frac{1}{t_1(t_1+t_2)} \\ -\frac{1}{t_1 t_2} & -\frac{1}{t_1 t_2} \end{bmatrix} v_1$$

We may invert these to recover the level annihilation operators

$$L_1 := \left[Z(0|-1,0)^{x_a}_{x_b} \right], \qquad L_2 := \left[Z(0|0,-1)^{x_a}_{x_b} \right]$$

$$L_1 = \phi^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & -t_1 \end{bmatrix} \frac{1}{v_1} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix} \frac{1}{v_2}$$

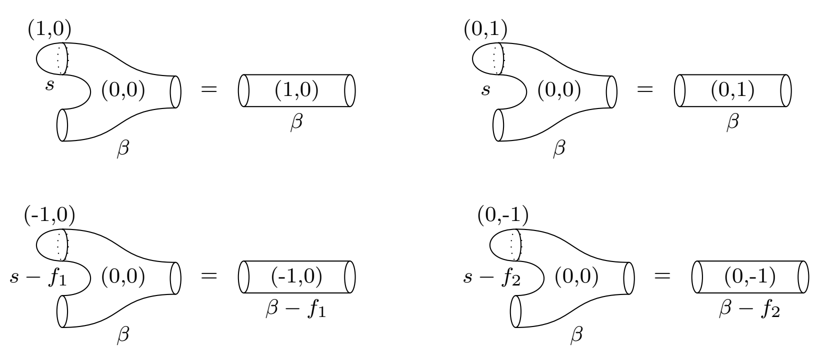
$$+\phi^2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1(t_1+t_2)} & \frac{1}{t_1(t_1+t_2)} \\ -\frac{1}{t_1 t_2} & -\frac{1}{t_1 t_2} \end{bmatrix}$$

$$L_2 = \phi^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -(t_1+t_2) & 0 \\ 0 & -t_2 \end{bmatrix} \frac{1}{v_2} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix} \frac{v_1}{v_2^2}$$

$$+\phi \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{t_1+t_2} & 0 \\ 0 & \frac{1}{t_2} \end{bmatrix}$$

Lemma 4.2.13. *The partition functions for the* $(0,0)$ *pair of pants* $Z(0|0,0)^{x_a}_{x_b}$ *are determined by the invariants computed thus far.*

Proof. Let β be any section class. We apply the gluing formula to the following diagrams



to get the following relations

$$\phi^{-2}t_1(t_1+t_2)Z_\beta(0|0,0)_{x_a x_b x_0}T(x_0)^{-1} = Z_\beta(0|1,0)_{x_a x_b}$$

$$\phi^{-1}(t_1+t_2)Z_\beta(0|0,0)_{x_a x_b x_0}T(x_0)^{-1} + \phi^{-1}t_2 Z_\beta(0|0,0)_{x_a x_b x_1}T(x_1)^{-1} = Z_\beta(0|0,1)_{x_a x_b}$$

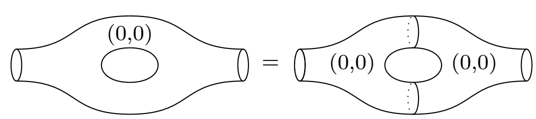
$$-\phi^{-1}t_1 Z_\beta(0|0,0)_{x_a x_b x_1}T(x_1)^{-1} - \phi^{-1}t_1 Z_\beta(0|0,0)_{x_a x_b x_3}T(x_3)^{-1} = Z_\beta(0|-1,0)_{x_a x_b}$$

$$-\phi^{-1}(t_1+t_2)Z_\beta(0|0,0)_{x_a x_b x_2}T(x_2)^{-1} - \phi^{-1}t_2 Z_\beta(0|0,0)_{x_a x_b x_3}T(x_3)^{-1} = Z_\beta(0|0,-1)_{x_a x_b}$$

The first equation determines the invariants $Z(0|0,0)_{x_a x_b x_c}$ where at least one of a, b, c is 0. The second equation then determines the invariants where at least one of a, b, c is 1, and so on.

□

We may calculate the genus raising operator $G = \left[Z(1|0,0)^{x_a}_{x_b}\right]$ by gluing two pairs of pants at two points and applying the gluing formula:



$$Z(1|0,0)^{x_a}_{x_b} = Z(0|0,0)_{x_b x_c x_d} Z(0|0,0)^{x_a x_c x_d}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} t_1(t_1+t_2) & 0 \\ 0 & -t_1 t_2 \end{bmatrix} + \phi^2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{2t_1}{t_1+t_2} & 0 \\ 0 & \frac{-2t_1}{t_2} \end{bmatrix} v_2$$

$$+\phi \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{2(t_1+t_2)^2}{t_1} & \frac{t_2(t_1+2t_2)}{t_1} \\ -\frac{(t_1+2t_2)(t_1+t_2)}{t_1} & -\frac{2t_2^2}{t_1} \end{bmatrix} \frac{v_1}{v_2} + \phi^2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} \frac{t_1+t_2}{t_1} & \frac{t_2}{t_1} \\ -\frac{t_1+t_2}{t_1} & -\frac{t_2}{t_1} \end{bmatrix} \frac{v_1^2}{v_2^2}$$

$$+\phi^3 \begin{bmatrix} \frac{2(2t_1+t_2)}{t_1(t_1+t_2)} & \frac{t_1+2t_2}{t_1(t_1+t_2)} & \frac{-2t_2}{t_1(t_1+t_2)} & \frac{t_1-2t_2}{t_1(t_1+t_2)} \\ \frac{-(t_1+2t_2)}{t_1t_2} & \frac{2(t_1-t_2)}{t_1t_2} & \frac{3t_1+2t_2}{t_1t_2} & \frac{2(t_1+t_2)}{t_1t_2} \\ \frac{2t_2}{t_1(t_1+t_2)} & \frac{3t_1+2t_2}{t_1(t_1+t_2)} & \frac{2(2t_1+3t_2)}{t_1(t_1+t_2)} & \frac{3(t_1+2t_2)}{t_1(t_1+t_2)} \\ \frac{t_1-2t_2}{t_1t_2} & \frac{-2(t_1+t_2)}{t_1t_2} & \frac{-3(t_1+2t_2)}{t_1t_2} & \frac{-2(t_1+3t_2)}{t_1t_2} \end{bmatrix} v_1$$

Now Theorem 4.1.1 follows similarly to Theorem 3.1.1. We now prove Corollary 4.1.1.

Proof. We may check that U_1 satisfies the following equation

$$U_1^4 = t_1(t_1+t_2)\phi^{-2}U_1^3 + \left((t_2+2t_1)\phi^{-1}v_1 + t_1^2\phi^{-2}v_2\right)U_1^2 + (v_1^2+2t_1\phi^{-1}v_1v_2)U_1 + v_1^2v_2I$$

Taking the specialization $t := t_1 = t_2$ we may check that G satisfies the following equation

$$G^4 = (-\phi^2v_1^2 + 6t\phi v_1v_2)v_2^{-2}G^3 + (6\phi^4v_1^2v_2 - 24t\phi^3v_1v_2^2 + 8t^2\phi^2v_2^3 + 3t^2\phi^2v_1^2$$

$$-18t^3\phi v_1v_2 + 5t^4v_2^2)v_2^{-2}G^2 + (256\phi^8v_1^2v_2^3 + 27\phi^8v_1^4 - 216t\phi^7v_1^3v_2 + 312t^2\phi^6v_1^2v_2^2$$

$$+96t^3\phi^5v_1v_2^3 - 16t^4\phi^4v_2^4 - 24t^4\phi^4v_1^2v_2 + 132t^5\phi^3v_1v_2^2 - 20t^6\phi^2v_2^3 - 4t^6\phi^2v_1^2$$

$$+24t^7\phi v_1v_2 - 4t^8v_2^2)v_2^{-2}I$$

Then the desired recursions

$$Z(g|k_1,k_2)|_{t_1=t_2=0} = v_1^2Z(g|k_1-3,k_2)|_{t_1=t_2=0} + v_1^2v_2Z(g|k_1-4,k_2)|_{t_1=t_2=0}$$

$$Z(g|k_1,k_2)|_{t_1=t_2=0} = -\phi^2v_1^2v_2^{-2}Z(g-1|k_1,k_2)|_{t_1=t_2=0} + 6\phi^4v_1^2v_2^{-1}Z(g-2|k_1,k_2)|_{t_1=t_2=0}$$

$$+(256\phi^8v_1^2v_2 + 27\phi^8v_1^4v_2^{-2})Z(g-4|k_1,k_2)|_{t_1=t_2=0}$$

follow by taking traces of matrix equations and reducing mod t_1, t_2. □

Chapter 5: Donaldson-Thomas and Stable Pairs Invariants

In this chapter we use the notation Z^{GW} for the Gromov-Witten partition function.

5.1 Donaldson-Thomas Theory

The moduli space $I_n(X, \beta)$ parameterizes ideal sheaves $\mathcal{J}_Z$ such that

$$\mathrm{ch}(\mathcal{J}_Z) = (1, 0, -\beta, n).$$

The moduli space has a $\mathbb{T}$-equivariant perfect obstruction theory and virtual fundamental class coming from the deformation theory of ideal sheaves. For the fibers $F_1, \ldots, F_r$ over the points $p_1, \ldots, p_r$ there is a relative moduli space of ideal sheaves $I_n(X/\vec{F}, \beta)$ [15,18] parameterizing ideal sheaves $\mathcal{J}_Z$ on some $X[\vec{L}]$ (see Section 2.1) such that

(i.) If Y is a component of the singular locus of $X[\vec{L}]$ or the transform of one of the divisors F_i, then $\mathcal{O}_Z$ is normal to Y, that is $\mathrm{Tor}_1^{\mathcal{O}_{X[\vec{L}]}}(\mathcal{O}_Z, \mathcal{O}_Y) = 0$.

(ii.) $\mathrm{Aut}_{X[\vec{L}]}(\mathcal{J}_X)$ is finite

The moduli space has a $\mathbb{T}$-equivariant perfect obstruction theory and virtual fundamental class. When β is a section class, the support Z meets the transforms of each of the divisors F_i at a single point, and there are boundary maps

$$\epsilon_i : I_n(X/\vec{F}, \beta) \to \mathcal{H}_a.$$

For $\alpha_1, \ldots, \alpha_r \in \mathcal{B}$ the relative Donaldson-Thomas partition function is defined by

$$Z^{\mathrm{DT}}_\beta(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r} = \sum_{n \in \mathbb{Z}} q^{n+\frac{1}{2}K_X \cdot \beta} \int_{[I_n(X/\vec{F}, \beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \epsilon_i^*(\alpha_i)$$

where the integral denotes equivariant pushforward to a point. The reduced relative partition function is defined by

$$Z^{\mathrm{DT,red}}_\beta(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r} = \frac{Z^{\mathrm{DT}}_\beta(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r}}{Z^{\mathrm{DT}}_0(g|k_1, k_2)}$$

The partition functions of section class, relative, reduced Donaldson-Thomas invariants are given by

$$Z^{\mathrm{DT,red}}(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r} = \sum_{m_1, m_2} Z^{\mathrm{DT,red}}_{s+m_1 f_1 + m_2 f_2}(g|k_1, k_2)_{\alpha_1 \ldots \alpha_r} v_1^{m_1} v_2^{m_2}.$$

The degeneration formula for Donaldson-Thomas invariants [15] implies we have the gluing formula

$$Z^{\mathrm{DT,red}}(g|k_1, k_2)_{\alpha_1 \ldots \alpha_s \gamma_1 \ldots \gamma_t} = \sum_{\lambda \in \mathcal{B}} Z^{\mathrm{DT,red}}(g'|k_1', k_2')_{\alpha_1 \ldots \alpha_s \lambda} Z^{\mathrm{DT,red}}(g''|k_1'', k_2'')^{\lambda}_{\gamma_1 \ldots \gamma_t}$$

for $k_1 = k_1' + k_1'', k_2 = k_2' + k_2'', g = g' + g''$ and

$$Z^{\mathrm{DT,red}}(g|k_1, k_2)_{\alpha_1 \ldots \alpha_s} = \sum_{\lambda \in \mathcal{B}} Z^{\mathrm{DT,red}}(g-1|k_1, k_2)^{\lambda}_{\alpha_1 \ldots \alpha_s \lambda}$$

where the invariants with raised indices are defined as in 2.3.

5.2 Stable Pairs

A stable pair is a two-term complex of coherent sheaves

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F}$$

where $\mathcal{F}$ is pure with one-dimensional support C' and s has a zero dimensional cokernel Q. The moduli space $P_n(X, \beta)$ parameterizes stable pairs where $\chi(\mathcal{F}) = n$ and $[C'] = \beta$. It has a two-term $\mathbb{T}$-equivariant deformation/obstruction theory and a virtual fundamental class coming from the deformation theory of complexes in the derived category $D^b(X)$ [29]. For the fibers $F_1, \ldots, F_r$ over the points $p_1, \ldots, p_r$ there is a relative moduli space of stable pairs $P_n(X/\vec{F}, \beta)$ [29] parameterizing stable pairs

$$\mathcal{O}_{X[\vec{L}]} \xrightarrow{s} \mathcal{F}$$

on some $X[\vec{L}]$ (see Section 2.1) such that the support of $\mathcal{F}$ pushes down to the class $\beta \in H_2(X, \mathbb{Z})$ and

(i.) $\mathcal{F}$ is pure with finite locally free resolution

(ii.) the higher derived functors of the restriction of $\mathcal{F}$ to the singular locus of $X[\vec{L}]$ and the transforms of the F_i vanish

(iii.) the section s has a zero dimensional cokernel with support disjoint from the singular loci of $X[\vec{L}]$

(iv.) the pair has only finite many automorphisms covering the automorphisms of $X[\vec{L}]/X$.

The moduli space has a $\mathbb{T}$-equivariant perfect obstruction theory and virtual fundamental class. When β is a section class, the support C' meets the transforms of each of the divisors F_i at a single point, and there are boundary maps

$$\epsilon_i : P_n(X/\vec{F}, \beta) \to \mathcal{H}_a.$$

For $\alpha_1, \ldots, \alpha_r \in \mathcal{B}$ the relative stable pairs (or Pandharipande-Thomas) partition function is defined by

$$Z_\beta^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1\dots\alpha_r} = \sum_{n\in\mathbb{Z}} q^{n+\frac{1}{2}K_X\cdot\beta}\int_{[P_n(X/\vec{F},\beta)]^{\mathrm{vir}}}\prod_{i=1}^r \epsilon_i^*(\alpha_i).$$

The partition functions of section class, relative, stable pairs invariants are given by

$$Z^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1\dots\alpha_r} = \sum_{m_1,m_2} Z^{\mathrm{PT}}_{s+m_1f_1+m_2f_2}(g|k_1,k_2)_{\alpha_1\dots\alpha_r} v_1^{m_1}v_2^{m_2}.$$

The degeneration formula for stable pairs [15] implies we have the gluing formula

$$Z^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1\dots\alpha_s\gamma_1\dots\gamma_t} = \sum_{\lambda\in\mathcal{B}} Z^{\mathrm{PT}}(g'|k_1',k_2')_{\alpha_1\dots\alpha_s\lambda} Z^{\mathrm{PT}}(g''|k_1'',k_2'')^{\lambda}_{\gamma_1\dots\gamma_t}$$

for $k_1 = k_1'+k_1'', k_2 = k_2'+k_2'', g = g'+g''$ and

$$Z^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1\dots\alpha_s} = \sum_{\lambda\in\mathcal{B}} Z^{\mathrm{PT}}(g-1|k_1,k_2)^{\lambda}_{\alpha_1\dots\alpha_s\lambda}.$$

5.3 GW/DT/PT Correspondence

The MNOP conjecture [18] is a conjectural correspondencce between Gromov-Witten invariants and reduced Donaldson-Thomas invariants. There is also a conjectural correspondence between Gromov-Witten invariants and stable pairs invariants [15, 20, 29]. These conjectures have been proved in the toric case [19, 27]. However, unless $C = \mathbb{P}^1$, they are unknown for our geometry.

Conjecture 5.3.1 (Gromov-Witten/Donaldson-Thomas/Stable Pairs Correspondence)**.** *After the change of variables* $q = -e^{iu}$

$$Z_\beta^{GW}(g|k_1,k_2)_{\alpha_1,\dots,\alpha_r} = Z_\beta^{DT,red}(g|k_1,k_2)_{\alpha_1,\dots,\alpha_r} = Z_\beta^{PT}(g|k_1,k_2)_{\alpha_1,\dots,\alpha_r}$$

This change of variables requires some explanation. The rationality conjecture 5.3.2 implies the Laurent series $Z_\beta^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1,\dots,\alpha_r}$ can be analytically continued to a meromorphic function. Thus we can expand the function at $q=-1$ and express it in terms of u by the change of variables $q=-e^{iu}$. The functional equation conjecture 5.3.1 states that $Z_\beta^{\mathrm{PT}}(g|k_1,k_2)_{\alpha_1,\dots,\alpha_r}$ is invariant under $q \leftrightarrow 1/q$ which implies the coefficients in the expansion in u don't involve i.

We prove the conjecture when β is a section class and $a=0,-1$. First we need to see the correspondence holds for the degree one theory of local curves. Under the change of variables we have

$$\phi = 2\sin\left(\frac{u}{2}\right) = q^{-1/2}(1+q).$$

Define $Z^{\mathrm{PT,loc}}$ and $Z^{\mathrm{DT,loc}}$ similarly to $Z^{\mathrm{GW,loc}}$. Then by Theorem 3 in [21] we have

$$Z^{\mathrm{DT,loc,red}}(n_1,n_2)_{p_1\dots p_r} = Z^{\mathrm{GW,loc}}(n_1,n_2)_{p_1\dots p_r}. \tag{5.1}$$

Then we need to show the following lemma which shows the stable pairs partition function agrees with 2.5 after the appropriate shift and change of variables:

Lemma 5.3.1.

$$Z^{PT,loc}(n_1,n_2)_{p_1\dots p_r}(t_1,t_2) = \phi^{n_1+n_2}\frac{1}{t_1t_2}t_1^{-n_1}t_2^{-n_2}$$

Proof. The result follows from standard calculations, but we did not find an immediate reference. Applying localization to compute the partition function for the level $(0,0)$ absolute geometry we have

$$Z^{\mathrm{PT,loc}}(0,0) = W(1,\emptyset,\emptyset)_{|s,t_1,t_2}\cdot W^{(0,0)}_{(1)}\cdot W(1,\emptyset,\emptyset)_{|-s,t_1,t_2}.$$

From section 4.9 in [30] we have the calculation of the stable pairs vertex

$$W(1,\emptyset,\emptyset)_{|s,t_1,t_2} = (1+q)^{\frac{t_1+t_2}{s}}.$$

The edge weight

$$W_{(1)}^{(0,0)} = \frac{1}{t_1t_2}$$

may be calculated from the edge character in section 4.6 of [30]. Applying localization to compute the level $(0,0)$ cap we have

$$Z^{\mathrm{PT,loc}}(0,0)_0 = W(1,\emptyset,\emptyset)_{|s,t_1,t_2} \cdot W_{(1)}^{(0,0)} \cdot S_{(1)}^{(1)}|_{-s,t_1,t_2}.$$

where $S_{(1)}^{(1)}$ is the rubber term. By the gluing formula for local curves [21] we have

$$Z^{\mathrm{PT,loc}}(0,0) = t_1t_2\left(Z^{\mathrm{PT,loc}}(0,0)_0\right)^2,$$

from which we conclude

$$Z^{\mathrm{PT,loc}}(0,0)_0 = \frac{1}{t_1t_2}$$

and $S_{(1)}^{(1)}|_{s,t_1,t_2} = (1+q)^{\frac{t_1+t_2}{s}}$. We apply localization to compute the level $(0,0)$ tube,

$$Z^{\mathrm{PT,loc}}(0,0)_{0,\infty} = S_{(1)}^{(1)}|_{s,t_1,t_2} \cdot W_{(1)}^{(-1,0)} \cdot S_{(1)}^{(1)}|_{-s,t_1,t_2} = \frac{1}{t_1t_2}.$$

We apply localization to compute the level $(-1,0)$ tube,

$$Z^{\mathrm{PT,loc}}(-1,0)_{0,\infty} = q^{1/2} \cdot S_{(1)}^{(1)}|_{s,t_1-s,t_2} \cdot W_{(1)}^{(-1,0)} \cdot S_{(1)}^{(1)}|_{-s,t_1,t_2}.$$

The edge weight $W_{(1)}^{(-1,0)} = \frac{1}{t_2}$ may be calculated from the edge character in section 4.6 of [30]. We get $Z^{\mathrm{PT,loc}}(-1,0)_{0,\infty} = q^{1/2}(1+q)^{-1}\frac{1}{t_2} = \phi^{-1}\frac{1}{t_2}$. Similarly, $Z^{\mathrm{PT,loc}}(0,-1)_{0,\infty} = \phi^{-1}\frac{1}{t_1}$. Then the other degree one invariants of local curves are determined via the gluing formula for local curves. $\square$

Then to prove Corollaries 3.1.2 and 4.1.2, one can go through the proofs of Theorems 3.1.1 and 4.1.1 and see that the same arguments are valid for stable pairs theory and Donaldson-Thomas theory. We sketch the types of arguments used to see that they hold in the stable pairs context:

(i.) Certain invariants vanish by dimension constraints

The virtual dimensions of the three moduli spaces $\overline{\mathcal{M}}^{\bullet}_h(X/\vec{F},\beta), P_n(X/\vec{F},\beta)$ and $I_n(X/\vec{F},\beta)$ agree, and the degrees of the partition functions must be non-negative because X is compact.

(ii.) Certain invariants vanish by geometric constraints

In several places in the argument we show that the image of a relative stable map representing a particular cohomology class cannot satisfy certain relative conditions. These arguments hold in the stable pairs context if we consider the support of the coherent sheaf $\mathcal{F}$ rather than the image $q(C')$ of a stable map. The arguments hold in the Donaldson-Thomas theory context if we consider instead the support Z of $\mathcal{O}_{X[\vec{L}]}/\mathcal{J}_Z$. The transversality conditions imply that no roaming points (zero-dimensional connected components of Z) may intersect the transforms of the divisors $F_1,\ldots,F_r$. Therefore the roaming points do not affect the analysis of whether Z may satisfy the relative conditions. Invariants of non-effective classes vanish for all three theories.

(iii.) Certain invariants may be calculated from the gluing formula

The gluing formula is valid for all three theories, and respects the correspondence 5.3.1.

(iv.) Certain invariants may be calculated from the local curve invariants

The degree one invariants of local curves satisfy the correspondence 5.3.1 by Lemma 5.3.1 and 5.1. Let the base curve C be $\mathbb{P}^1$. Suppose that C_i has normal bundle $\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$. Then X is a $\mathbb{T}$-equivariant compactification of

$$N = \mathrm{Tot}\left(\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)\right).$$

The obstruction theory on $P_n(N,1) \subset P_n(X,\beta)$ is obtained by restriction. Consider $\mathbb{T}$-fixed relative stable pairs $\mathcal{O}_{X[\vec{L}]} \xrightarrow{s} \mathcal{F}$ such that the support of $\mathcal{F}$ pushes down to the section class β and meets the transforms of the divisors $F_1, \ldots, F_r$ at the point x_i. If for all such stable pairs, the support of $\mathcal{F}$ is $C_i[\vec{L}]$ then we may compute $Z_\beta^{\mathrm{PT}}(0|k_1,k_2)_{x_i \ldots x_i}$ from the invariants of local curves. The obstruction theory on $I_n(N,1) \subset I_n(X,\beta)$ is obtained by restriction. Let $Z_0^{\mathrm{DT}}(\mathcal{N}_{C_j|X})$ denote the degree zero Donaldson-Thomas partition function of the local curve corresponding to C_j. Then relative localization satisfies a factorization rule (see the discussion in [21] Section 4.6):

$$Z_\beta^{\mathrm{DT}}(0|k_1,k_2)_{\alpha_1 \ldots \alpha_r} = T(x_i)^r Z^{\mathrm{DT,loc}}(n_1,n_2)_{p_1 \ldots p_r}(w_1(x_i), w_2(x_i)) \cdot \prod_{j \neq i} Z_0^{\mathrm{DT}}(\mathcal{N}_{C_j|X})$$

$$Z_0^{\mathrm{DT}}(0|k_1,k_2) = \prod_j Z_0^{\mathrm{DT}}(\mathcal{N}_{C_j|X})$$

and we conclude

$$Z_\beta^{\mathrm{DT,red}}(0,k_1,k_2)_{\alpha_1 \ldots \alpha_r} = T(x_i)^r Z^{\mathrm{DT,loc,red}}(n_1,n_2)_{p_1 \ldots p_r}(w_1(x_i), w_2(x_i))$$

Remark 5.3.1. *This argument applies as well to see the GW/DT/PT holds for Calabi-Yau section classes for $\mathbb{P}^2$-bundles over a smooth, complex, projective curve C, by the proof in [9].*

Since ϕ is invariant under the transformation $q \leftrightarrow q^{-1}$, as a corollary we verify the functional equation conjecture (see [22] Conjecture 4) for our geometry:

Corollary 5.3.1 (Functional Equation)**.** *Let $a = 0$ or $a = -1$. For a section class β, $Z_\beta^{PT}(g|k_1, k_2) = Z_\beta^{DT,red}(g|k_1, k_2)$ is invariant under the transformation $q \leftrightarrow q^{-1}$.*

Next we prove that rationality conjecture (see [22] Conjecture 3) for our geometry:

Corollary 5.3.2 (Rationality)**.** *Let $a = 0$ or $a = -1$. For a section class β, $q^{-\frac{1}{2}K_X\cdot\beta} Z_\beta^{PT}(g|k_1, k_2) = q^{-\frac{1}{2}K_X\cdot\beta} Z_\beta^{DT,red}(g|k_1, k_2)$ is the Laurent series expansion in q of a rational function in $\mathbb{Q}(q, t_1, t_2)$.*

Proof. By Remarks 3.1.1 and 4.1.1 and 5.3.1, $Z_\beta^{\mathrm{PT}}(g|k_1, k_2) = p(t_1, t_2)\phi^{2g-2-K_X\cdot\beta}$ where $p(t_1, t_2)$ is a homogeneous polynomial of degree $K_X\cdot\beta$ in t_1, t_2. Thus the power of $q^{1/2}$ in the unshifted partition function $q^{-\frac{1}{2}K_X\cdot\beta} Z_\beta^{\mathrm{PT}}(g|k_1, k_2)$ is $2g - 2 - 2K_X \cdot \beta$ which is even. We see the invariants are proportional to a power of $1 + q$ times a power of q. □

5.4 BPS Invariants

We recall the Gopakumar-Vafa BPS state counts as defined in [29]. BPS counts are hoped to be integers underlying the Gromov-Witten invariants of threefolds, avoiding multiple cover and degenerate contributions. The connected stable pairs invariants F_β^{PT} are *defined* by the equation

$$\sum_{\beta\neq 0} F_\beta^{\mathrm{PT}}(q)v^\beta = \log\left(1 + \sum_{\beta\neq 0} Z_\beta^{\mathrm{PT}} v^\beta\right).$$

Let $a \leq 0$. An effective curve of degree zero in s is of the form $n_1(f_1 + af_2) + n_2 f_2$ where n_1, n_2 are non-negative integers, which are not both zero. In this case the virtual dimension

$$\text{vdim } P_n(X, \beta) = (2 + a)n_1 + 2n_2$$

is positive if $a > -2$ and the degree zero stable pair invariants vanish. Thus for the section class invariants computed in 3.1.1,4.1.1 the connected invariants agree with the ordinary ones. For a Calabi-Yau section class β we can write

$$Z_\beta^{\text{PT}} = F_\beta^{\text{PT}} = \sum_{g' > -\infty} n_{g',\beta} \phi^{2g'-2}.$$

Following [29], the BPS state counts $n_{g',\beta}$ are defined by this equation.

For a Calabi-Yau section class, Corollary 3.1.1, Corollary 4.1.1, and Lemma 1.2.1, imply the following strengthening of the rationality conjecture 5.3.2:

Corollary 5.4.1 (Rationality, BPS Refinement)**.** *Let $a = 0$ or $a = -1$. For a Calabi-Yau section class β, $n_{g',\beta} = 0$ unless $g' = g$ where g is the genus of C. In particular, the vanishing conjecture of [29] holds:*

$$n_{g',\beta} = 0$$

for $g' < 0$.

We now restate our main results for Calabi-Yau section classes in terms of BPS counts. Let X be a level (k_1, k_2), $\mathcal{H}_a$-bundle over a smooth, connected, complex, projective curve C of genus g and let β be a Calabi-Yau section class. By Corollary 5.4.1 above, this has a unique potentially non-vanishing BPS invariant corresponding to $g' = g$. Denote this invariant by $n_{g,\beta}(k_1, k_2)$. Then we define a generating function of BPS counts

$$n_g(k_1, k_2) := \sum_{\substack{\beta = s + m_1 f_1 + m_2 f_2 \\ \beta \cdot K_X = 0}} n_{g,\beta}(k_1, k_2) v_1^{m_1} v_2^{m_2}.$$

Now our main results for Calabi-Yau section class partition functions may be restated as follows:

Corollary 5.4.2. *Let $a = 0$, and let m_1, m_2 be integers. If $\beta = s + m_1 f_1 + m_2 f_2$ is a Calabi-Yau section class then*

$$n_{g,\beta}(k_1, k_2) = \begin{cases} 4^g & m_1 = \frac{g-1+k_1}{2}, m_2 = \frac{g-1+k_2}{2} \\ 0 & \textit{otherwise} \end{cases}$$

Corollary 5.4.3. *Let $a = -1$. Then the BPS counts for Calabi-Yau section classes are determined by the following recursions*

$$n_g(k_1, k_2) = n_g(k_1 - 2, k_2 + 1),$$

$$n_g(k_1, k_2) = v_1^2 n_g(k_1 - 3, k_2) + v_1^2 v_2 n_g(k_1 - 4, k_2),$$

$$n_g(k_1, k_2) = -v_1^2 v_2^{-2} n_{g-1}(k_1, k_2) + 6 v_1^2 v_2^{-1} n_{g-2}(k_1, k_2) + (256 v_1^2 v_2 + 27 v_1^4 v_2^{-2}) n_{g-4}(k_1, k_2),$$

and the following base cases

	$g = 0$	1	2	3
$k_1 = 0$	0	4	$-v_1^2 v_2^{-2}$	$12 v_1^2 v_2^{-1} + v_1^4 v_2^{-4}$
1	1	0	$v_1^2 v_2^{-1}$	$16 v_1^2 - v_1^4 v_2^{-3}$
2	0	0	$8 v_1^2$	$64 v_1^2 v_2 + v_1^4 v_2^{-2}$
3	0	$3 v_1^2$	$16 v_1^2 v_2$	$-v_1^4 v_2^{-1}$.

We can verify the following corollary:

Corollary 5.4.4 (BPS integrality). *Let $a = 0$ or $a = -1$. For a Calabi-Yau section class β, the invariants $n_{g,\beta}(k_1, k_2)$ are integers.*

Proof. This follows from the GW/DT/PT correspondence 5.3.1 and the integrality of stable pairs invariants. We may also see this directly from Corollaries 5.4.2 and 5.4.3. For $a = 0$ this follows from the formula. For $a = 1$, the base cases are integers and the recursions have integer coefficients. Note that we may also use the second recursion to find negative values of k_1 using

$$n_g(k_1 - 4, k_2) = v_1^{-2} v_2^{-1} n_g(k_1, k_2) - v_2^{-1} n_g(k_1 - 3, k_2)$$

□

(Equivariant integrality) Let $a = 0$ or $a = -1$. By Remarks 3.1.1 and 4.1.1, for a section class β, $Z_\beta(g|k_1, k_2) = p(t_1, t_2)\phi^{2g-2-K_X \cdot \beta}$ where $p(t_1, t_2)$ is a homogeneous polynomial of degree $K_X \cdot \beta$ in t_1, t_2. The expression $\phi^{2g-2-K_X \cdot \beta}$ occurs in Pandharipande's calculation of the contribution of a nonsingular embedded curve of genus g representing an infinitesimally isolated solution to incidence conditions [23].

Corollary 5.4.5. *The polynomial $p(t_1, t_2)$ has integer coefficients.*

Proof. For $g > 0$ this follows immediately from Theorems 3.1.1 and 4.1.1 since the entries in the matrices have integer coefficients. For $g = 0$, we can check the claim for the base cases $0 \leq k_1, k_2 \leq 3$ and then use the recursions on $Z(g|k_1, k_2) = tr(G^{g-1} U_1^{k_1} U_2^{k_2})$ given by taking the traces of the matrix equations

$$U_1^2 = t_1 \phi^{-1} U_1 + v_1 I,$$

$$U_2^2 = t_2\phi^{-1}U_2 + v_2 I,$$

for $a = 0$ and

$$U_1^4 = t_1(t_1+t_2)\phi^{-2}U_1^3+\left((2t_1+t_2)\phi^{-1}v_1 + t_1^2 v_2\phi^{-2}\right)U_1^2+\left(v_1^2 + 2t_1\phi^{-1}v_1v_2\right)U_1+v_1^2v_2 I,$$

$$U_2^4 = (t_1+2t_2)\phi^{-1}U_2^3+\left(2v_2 - t_2(t_1+t_2)\phi^{-2}\right)U_2^2+\left(v_1 - t_1\phi^{-1}v_2 - 2t_2\phi^{-1}v_2\right)U_2-v_2^2 I,$$

for $a = -1$. Since the coefficients of these recursions have integer coefficients, the claim follows. $\square$

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