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Active Control of Compressor Stall Inception: A Bifurcation-Theoretic Approach

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Abstract

Active control of the onset of stall instabilities in axial flow compressors is pursued using bifurcation analysis of a dynamical model proposed by Moore and Greitzer (1986). This model consists of three ordinary differential equations with state variables being the mass flow rate, pressure rise, and the amplitude of the first harmonic mode of the asymmetric component of the flow. The model is found to exhibit a stationary (pitchfork) bifurcation at the inception of stall, resulting in hysteresis. Using the throttle opening as a control, analysis of the linearized system at stall shows that the critical mode (zero eigenvalue) is unaffected by linear feedback. Hence, nonlinear tools must be used to achieve stabilization. A quadratic feedback control law using measurement of asymmetric dynamics is proposed which stabilizes the bifurcation and eliminates the undesirable hysteretic behavior.

Keywords: Compression systems, Stability, Stabilization, Nonlinear control, Bifurcation

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1. Introduction

Recent years have witnessed an increasing interest in axial flow compressor dynamics, both in terms of analysis of stall phenomena and their control [1]-[6], [11], [12]. This interest is due to the desire for increased performance which is potentially achievable in modern gas turbine jet engines by operation near the maximum pressure rise. This increased performance is at the price of a significantly reduced stability margin. Two basic types of instability are known to occur in compression systems. One, *surge*, is a low frequency, large amplitude oscillation of the mean mass flow rate. The second, *rotating stall*, corresponds to a traveling wave of gas around the annulus of the compressor. This results in very inefficient operation at constant mean mass flow rate and pressure rise. These descriptions are for the fully developed (i.e., post-transient) instabilities.

Greitzer [1] employed a nondimensional fourth-order compression system model and introduced a nondimensional parameter, B, which he found to be a determinant of the nature of post-instability behavior. A global bifurcation of periodic solutions and other bifurcations were found for this model, and were used to explain the observed dependence of the dynamical behavior on the parameter B [3], [4].

Moore and Greitzer [5] extended the previous model to describe the surge and rotating stall phenomena in axial flow compression systems. This model incorporates nonaxisymmetric dynamics, whereas the model of Greitzer [1] reflected only axisymmetric flow dynamics, while employing a nonaxisymmetric (i.e., measured) steady-state compressor characteristic. Also in [5], the general model was specialized to a system of three nonlinear ordinary differential equations. The state variables of this dynamic model are the mass flow rate, pressure rise, and the amplitude of the first harmonic mode of the asymmetric component of the flow. For the case of a cubic axisymmetric compressor characteristic, Moore and Greitzer found it convenient to use the square of the amplitude of the first harmonic mode rather than the amplitude itself as a state variable. McCaughan [6] performed a bifurcation analysis of this model, observing a stationary bifurcation at stall inception. This bifurcation entails the local emergence of a new equilibrium point from the nominal one. This bifurcated equilibrium point is not stable, and results in a jump effect and thus hysteresis. We note that McCaughan [6] also observed a bifurcation to a large amplitude periodic solution for the model of [5]. (Although bifurcations of equilibria are discussed briefly in Section 2 and the Appendix, we refer the reader to texts such as Iooss and Joseph [10] for a detailed treatment.)

Several techniques have been proposed for active control of stall instabilities in axial flow compressors (e.g., [2], [11], [12]). From an analytical point of view, these methods employ linear control to delay the occurrence of stall or to achieve stall avoidance. Of course the physical mechanisms for controller implementation differ among the proposed active control schemes.

The present paper begins with the recognition of the importance of local bifurcations as determinants of the nature of post-instability behavior of axial flow compression systems. The philosophy of the control component of this work is similar to that of Abed and Fu [7]. This entails determining feedback control laws which ensure the stationary bifurcation results in only stable bifurcated solutions. Thus, even though the nominal equilibrium is not stabilizable within the framework of linear theory, it may be possible to stabilize a *neighborhood* of the nominal solution for a range of parameter values including the stall value of the disturbance parameter. It will be seen that an additional outcome of the control laws proposed here is the elimination of a hysteresis loop which occurs in the open-loop system model. The results of this paper apply to models more general than that studied by McCaughan [6], in the sense that the axisymmetric part of the steady state compressor characteristic is not required to be a cubic function, but rather an arbitrary smooth single-valued function of mass flow rate. We do employ a cubic model in demonstrating the results for a particular compression system.

The paper proceeds as follows. In Section 2, bifurcation theory is applied to study the stability of axial flow compression systems in the vicinity of the stall point. A pitchfork bifurcation is observed in the model at the stall point. The dynamical behavior of the compression system near this bifurcation point is found to be strongly dependent on the axisymmetric compressor characteristic. This dependence is exhibited through a formula showing that the stability of bifurcated solutions is influenced by the derivatives of the axisymmetric compressor characteristic at the bifurcation point. In Section 3, a throttle opening control law is given. This control law circumvents the uncontrollability of the zero eigenvalue of the linearized model at stall. A purely quadratic state feedback using measurement of the asymmetric component of the flow is given, and is shown to result in local stabilization of the bifurcation leading to stall. Section 4 contains bifurcation diagrams for the controlled and uncontrolled cases. Concluding remarks are given in Section 5.

Notation

θ - angle along circumference

C_{ss} - nondimensional axisymmetric compressor characteristic

A - amplitude of the first harmonic of asymmetric flow

$\dot{m}_C$ - nondimensional compressor mass flow rate

ΔP - nondimensional plenum pressure rise

F - inverse function of nondimensional throttle pressure rise

$D_{xy\ldots}h$ - partial derivative of function h with respect to the variables $x, y, \ldots$

$(\cdot)'$ - denotes differentiation

2. Bifurcation Analysis of Stall Inception

A third-order lumped parameter model, in terms of nondimensional variables, for an axial flow compression system has been introduced by Moore and Greitzer [5]. Using the notation of [3], the model is

$$\frac{dA}{dt} = \frac{\alpha}{\pi W} \int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)\sin\theta d\theta \tag{2.1a}$$

$$\frac{d\dot{m}_C}{dt} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)d\theta \tag{2.1b}$$

$$\frac{d\Delta P}{dt} = \frac{1}{4B^2}\{\dot{m}_C - F(\gamma, \Delta P)\}, \tag{2.1c}$$

where $W, \alpha > 0$ are two constants, γ is the control input, and is associated with the throttle opening, and B is the nondimensional parameter of the same name introduced by Greitzer [1], which is proportional to rotor speed. In the sequel, $B > 0$ and F is a strictly increasing function with respect to each of the variables γ and ΔP.

Taking the compressor characteristic C_{ss} to be a cubic function of $\dot{m}_C$, and using the squared amplitude A^2 (instead of A, say) as a state variable, Moore and Greitzer [5] were able to explicitly evaluate the integrals appearing in (2.1), giving a more convenient dynamic model. McCaughan [6] performed a bifurcation analysis of this model. One conclusion of [6] is that a stationary bifurcation from the nominal equilibrium point occurs as the throttle opening parameter γ is varied. However, a similar analysis for the more general case in which the axisymmetric compressor characteristic is not necessarily a cubic function of $\dot{m}_C$ has not been reported. In this section, we extend the study of [6] to the case in which the axisymmetric compressor characteristic is taken to be a general smooth function of mass flow rate. Note that in Section 4 we also employ a cubic axisymmetric compressor characteristic, by way of illustration.

Suppose C_{ss} is a smooth function of $\dot{m}_C$, and solve for the equilibrium points of (2.1). By Eq. (2.1a), it is easy to see that $A = 0$ always results in $dA/dt = 0$. However, there may be equilibrium points of (2.1) for which $A \neq 0$. First let us consider equilibrium points in the case $A = 0$. Denote such an equilibrium point, the location of which depends on the parameter γ, as $x^0(\gamma) = (0, \dot{m}_C^0(\gamma), \Delta P^0(\gamma))^T$. The values $\dot{m}_C^0(\gamma)$ and $\Delta P^0(\gamma)$ should then satisfy the relationships $\dot{m}_C^0 = F(\gamma, \Delta P^0)$ and $\Delta P^0 = C_{ss}(\dot{m}_C^0)$. Under the assumption $A = 0$, one such equilibrium point $x^0(\gamma)$ is referred to as the *unstalled* or *nominal* equilibrium point for axial flow compression model (2.1). Note that this is the normal operating point of the system, the location of which depends on the throttle control parameter γ. In the following, we consider the stability of the unstalled equilibrium $x^0(\gamma)$ as it depends on γ. Viewing γ as a bifurcation parameter, we study possible bifurcations from $x^0(\gamma)$ at parameter values for which stability is lost. These bifurcations result in new equilibrium points of the model. At these new equilibria, $A \neq 0$, corresponding to stall inception.

Let $X = (x_1, x_2, x_3)^T$ denote the state variation of (2.1) near the unstalled equilibrium point $x^0(\gamma)$, where $x_1 = A$, $x_2 = \dot{m}_C - \dot{m}_C^0(\gamma)$ and $x_3 = \Delta P - \Delta P^0(\gamma)$. The linearization of (2.1) at $x^0(\gamma)$ gives

$$\frac{dX}{dt} = L_0 X, \tag{2.2}$$

with

$$L_0 = \begin{pmatrix} \alpha C'_{ss}(\dot{m}_C^0(\gamma)) & 0 & 0 \\ 0 & C'_{ss}(\dot{m}_C^0) & -1 \\ 0 & \frac{1}{4B^2} & -\frac{1}{4B^2} D_{\Delta P} F(\gamma, \Delta P^0) \end{pmatrix}. \tag{2.3}$$

From (2.2) and the Routh-Hurwitz stability criterion, we have the following stability result.

Lemma 1. The equilibrium point $x^0(\gamma)$ is asymptotically stable for system (2.1) if $C'_{ss}(\dot{m}_C^0(\gamma)) < 0$, while it is unstable if $C'_{ss}(\dot{m}_C^0(\gamma)) > 0$.

The foregoing result is useful in that it classifies the nominal equilibrium as being stable or unstable depending on the value of the parameter γ. Since we are interested in the bahavior of the compression system (2.1) for values of γ for which the nominal equilibrium is unstable, we are led to study the behavior of (2.1) for values of γ near the critical value at which stability is first lost. An equilibrium point depending on a single parameter can lose stability in one of several ways. One possibility is the actual disappearance of the equilibrium in what is known as a saddle node bifurcation. This entails the merging of the nominal equilibrium with another equilibrium as the parameter approaches a value for which the system linearization becomes singular. Although this is the most typical of the so-called statiiionary bifurcations, it is not the type encountered in this paper. In this paper the nominal equilibrium is known to exist both prior to and subsequent to its loss of stability, Two bifurcation mechanisms can occur under this circumstance. One mechanism arises when a real eigenvalue goes from being negative to being positive as the parameter is changed. In this case, the loss of stability coincides with an eigenvalue passing through 0, i.e., crossing the imaginary axis at the origin. The second mechanism occurs when a complex conjugate pair of simple eigenvalues crosses the imaginary axis. Although both scenarios result in loss of stability of the nominal equilibrium, they have very different implications as to system behavior once local stability is lost. In the former case, a *stationary bifurcation* generally occurs, resulting in the appearance of at least one new equilibrium point from the nominal one at criticality. In the latter case, an *Andronov-Hopf bifurcation* occurs, giving rise to small-amplitude periodic solutions near the nominal equilibrium.

The Jacobian matrix of (2.1) at the nominal equilibrium point is given by L_0 as in (2.3). For this matrix to have a pair of pure imaginary eigenvalues, it is not difficult to see that $C'_{ss}(\dot{m}^0_C(\gamma))$ must be positive. (This is seen by examining the trace of the lower right 2×2 submatrix of (2.3).) However, if this were the case then the matrix L_0 would have $\alpha C'_{ss}(\dot{m}^0_C(\gamma))$ as a positive real eigenvalue, and the equilibrium would therefore be unstable. This means that a *stationary bifurcation must occur before an Andronov-Hopf bifurcation* for the nominal equilibrium x^0 of the model (2.1). In the remainder of this paper, therefore, we focus on stationary bifurcations from x^0 and their stabilization.

From (2.3), the linearization of (2.1) has one zero eigenvalue and two stable eigenvalues when $C'_{ss}(\dot{m}^0_C(\gamma)) = 0$. This implies a stationary bifurcation may occur from the equilibrium point x^0 for some value of γ^0. To analyze the bifurcation behavior of the model, we employ a result from bifurcation analysis recalled in Appendix A. This result allows us to derive conditions for existence and stability of a bifurcation from the nominal equilibrium.

Let x^0 be the equilibrium point at which $C'_{ss}(\dot{m}^0_C(\gamma)) = 0$ for some $\gamma = \gamma^0$. The Taylor series expansion of (2.1) at the point (x^0, γ^0) is given by

$$\frac{dX}{dt} = L_0 X + Q_0(X,X) + C_0(X,X,X) + (\gamma - \gamma^0) L_1 X + \cdots \tag{2.4}$$

Here L_0 is as in (2.3) and

$$Q_0(X,X) = \begin{pmatrix} \alpha C''_{ss}(\dot{m}^0_C) x_1 x_2 \\ \frac{1}{4} C''_{ss}(\dot{m}^0_C)(W^2 x_1^2 + 2x_2^2) \\ -\frac{1}{8B^2} D_{(\Delta P)^2} F(\gamma^0, \Delta P^0) x_3^2 \end{pmatrix} \tag{2.5}$$

$$C_0(X,X,X) = \begin{pmatrix} \frac{\alpha}{8} C'''_{ss}(\dot{m}^0_C)(W^2 x_1^3 + 4x_1 x_2^2) \\ C'''_{ss}(\dot{m}^0_C)(\frac{1}{4} W^2 x_1^2 x_2 + \frac{1}{6} x_2^3) \\ -\frac{1}{24B^2} D_{(\Delta P)^3} F(\gamma^0, \Delta P^0) x_3^3 \end{pmatrix} \tag{2.6}$$

$$L_1 = \begin{pmatrix} \alpha\phi_1 C''_{ss}(\dot{m}^0_C) & 0 & 0 \\ 0 & \phi_1 C''_{ss}(\dot{m}^0_C) & 0 \\ 0 & 0 & -\frac{\phi_2}{4B^2} \end{pmatrix}, \tag{2.7}$$

where

$$\phi_1 = D_\gamma F(\gamma^0, \Delta P^0) \quad \text{and} \quad \phi_2 = D_{\Delta P \gamma} F(\gamma^0, \Delta P^0). \tag{2.8}$$

Set $l = (1, 0, 0)$ and $r = l^T$, the left and right eigenvectors, respectively, corresponding to the zero eigenvalue of L_0. The dynamical behavior of (2.1) with respect to the variation of γ near the unstalled point x^0 is analyzed next.

To check the transversality condition $lL_1 r \neq 0$ (see Appendix A), we compute

$$lL_1 r = \alpha \phi_1 C''_{ss}(\dot{m}^0_C). \tag{2.9}$$

The bifurcation stability coefficients are calculated, using Eqs. (A.3)-(A.4), as

$$\beta_1 = lQ_0(r, r) = 0 \tag{2.10}$$

and

$$\begin{aligned} \beta_2 &= 2l\{2Q_0(r, \delta) + C_0(r, r, r)\} \\ &= \frac{\alpha W^2}{4}\{2D_{\Delta P}F(\gamma^0, \Delta P^0)(C''_{ss}(\dot{m}^0_C))^2 + C'''_{ss}(\dot{m}^0_C)\}. \end{aligned} \tag{2.11}$$

The next result follows readily from Lemma A.1 and the discussions above. The term "pitchfork bifurcation" used in the theorem statement refers to a situation in which two new equilibrium points emerge from a given one, and both occur locally on the same side of criticality (i.e., either for $\gamma > \gamma_0$ or for $\gamma < \gamma_0$). It is important to note that the stability characteristics of the bifurcated equilibria determine whether or not hysteresis and jump phenomena result from the bifurcation.

Theorem 1. Suppose $C''_{ss}(\dot{m}^0_C) \neq 0$ and that F is strictly increasing in each of its variables. Suppose also that the stability coefficient β_2 given in (2.11) is nonzero. Then the system (2.1) exhibits a pitchfork bifurcation with respect to small variations of γ at the point (x^0, γ^0) where $C'_{ss}(\dot{m}^0_C) = 0$. Moreover, if $\beta_2 < 0$ (resp. $\beta_2 > 0$) the local bifurcated solutions near x^0 will be asymptotically stable (resp. unstable).

Note that the formula (2.11) shows that the stability of bifurcated solutions is influenced by the derivatives of the axisymmetric compressor characteristic at the bifurcation point. Theorem 1 quantifies the local dependence of post-stall behavior of the compression system on the axisymmetric compressor characteristic (i.e., on the function C_{ss}). Bifurcation diagrams for an axial flow compression system with a particular form for C_{ss} is given in Section 4.

3. Control of Stall Inception

From Lemma 1, the unstalled equilibrium point becomes unstable after the parameter γ passes through the critical value γ^0. The point (x^0, γ^0) is therefore called the *stall point.* Moreover, according to Theorem 1 the local bifurcated solutions near the stall point might not be stable. In this case the bifurcation is said to be subcritical. If such a condition occurs, the compression system may exhibit a *jump* from the stable nominal equilibrium when the parameter γ crosses the critical value γ^0. This results in a *hysteresis loop* in the dynamics of the system with respect to the parameter γ near the stall point. An example of such behavior will be given in Section 4.

In this section, we seek feedback control laws governing the throttle setting near the stall point which prevent the occurrence of this hysteresis or jump behavior. The control design begins by observing that jump behavior occurs because there is no nearby stable equilibrium after the nominal equilibrium loses stability. A small-energy throttle control is applied to the system, which is equivalent to replacing the throttle parameter γ in (2.1) by $\gamma + u$ where u is a control signal which is of small-amplitude near the nominal equilibrium. By examining the Jacobian matrix L_0 of the system as given in Eq. (2.3), we find that throttle control cannot affect the system eigenvalue that passes through zero as γ varies through the critical value γ^0. (This is simply the $(1,1)$-element of L_0, due to the block triangular structure of L_0.) Given these circumstances, we seek nonlinear feedbacks which transform the subcritical bifurcation at γ^0 into a supercritical bifurcation, i.e., to give rise to stable equilibrium points locally. These equilibrium points serve as alternative steady-states, so that a jump to a distant equilibrium will no longer occur.

Note that such a control law, although obtained through local analysis, mitigates an undesirable global effect, namely hysteresis. As in Section 2, denote by γ^0 the critical value of the throttle control parameter at which stability of is first lost. the design a control law ensuring that the local bifurcated solutions near the stall point are stable. An important consequence of this is that the controlled system will not exhibit jump or hysteresis phenomena near the nominal equilibrium.

Denote the stall point by (x^0, γ^0), and let $\gamma := \gamma^0 + u$, where u is the control input.

Then we can rewrite system (2.1) as the nonlinear control system

$$\frac{dA}{dt} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)\sin\theta d\theta \tag{3.1a}$$

$$\frac{d\dot{m}_C}{dt} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C + WA\sin\theta)d\theta \tag{3.1b}$$

$$\frac{d\Delta P}{dt} = \frac{1}{4B^2}\{\dot{m}_C - F(\gamma^0 + u, \Delta P)\}. \tag{3.1c}$$

In this section these equations and their control are considered without specifying parameter values or the form of the steady-state axisymmetric compressor and throttle characteristics. In Section 4 an example is given where these are specified.

It is observed from (3.1a) that $A = 0$ is an invariant submanifold of (3.1) regardless of the choice of control input u, implying that (3.1) is uncontrollable. Moreover, it is not difficult to check that system (3.1) possesses an uncontrollable zero eigenvalue at the stall point. This means that we cannot extend the stable region of of the nominal equilibrium point for a broader range of values of the parameter γ by using a linear state feedback.

Next, we design a control law ensuring that the local bifurcated solutions near the stall point are stable. An important consequence of this is that the controlled system will not exhibit jump or hysteresis phenomena near the nominal equilibrium. To make the selection of a control law tractable and systematic, we begin by restricting the control to belong to a parametrized family of smooth feedback control laws. Formulas (A.3) and (A.4) show that, in general, only terms up to cubic order in the state affect the values of the stability coefficients β_1 and β_2 at a stationary bifurcation point. This leads us to limit the search to control laws containing only linear and quadratic terms in the state. Next, we notice that linear terms in the feedback control might affect eigenvalues and eigenvectors of the system linearization at the bifurcation point. (However, recall the zero eigenvalue, being uncontrollable, would be unaffected by linear feedback terms.) The analysis would become cumbersome, however, if eigenvalues and eigenvectors were not fixed, as can be seen from an examination of formulas (A.3) and (A.4). Thus, we are led to seek a feedback control law of the following form:

$$u = q_1 A^2 + q_2 A(\dot{m}_C - \dot{m}_C^0) + q_3 A(\Delta P - \Delta P^0) + q_4(\dot{m}_C - \dot{m}_C^0)^2$$

$$+ q_5(\dot{m}_C - \dot{m}_C^0)(\Delta P - \Delta P^0) + q_6(\Delta P - \Delta P^0)^2. \tag{3.2}$$

where the q_i are constants. We seek values of the control gains q_i such that, for system (3.1) with u given by (3.2), β_2 at the equilibrium x^0 for $\gamma = \gamma^0$ will be rendered negative.

Denote by β_1^* and β_2^* the two stability coefficients of the controlled model (3.1) at x^0 for for $\gamma = \gamma^0$. Formulas (A.3) and (A.4) are also valid for the controlled system. Using these formulas, it is straightforward to express β_1^* and β_2^* in terms of β_1 and β_2, the stability coefficients of the uncontrolled version of (3.1). Indeed, we find that $\beta_1^* = \beta_1 = 0$ and $\beta_2^* = \beta_2 + 2\alpha q_1 D_\gamma F(\gamma^0, \Delta P^0) C''_{ss}(\dot{m}_C^0)$. Note that $\beta_1^* = 0$ regardless of the choice of control gains. By Lemma A.1 in the Appendix, this implies that if $\beta_2^* \neq 0$, then a pitchfork bifurcation must also occur in the controlled system. From the expression above for β_2^*, it is apparent that only the quadratic component of the feedback, $q_1 A^2$, contributes to the determination of the sign of β_2^*. Because of this, we obtain a very simple quadratic feedback controller for the system (3.1), as summarized in the next theorem.

Theorem 2. Let $C''_{ss}(\dot{m}_C^0) \neq 0$ at the point (x^0, γ^0). Then the stationary bifurcation of (3.1) at (x^0, γ^0) can be rendered a supercritical pitchfork bifurcation by a purely quadratic feedback control of the form $u = q_1 A^2$.

4. Illustrative Example

In the foregoing, we have considered the analysis and control of stationary bifurcation and hysteresis for the third-order Moore-Greitzer model [5]. In this section we illustrate the results for a particular compression system model employing, for simplicity, a cubic axisymmetric compressor characteristic. As noted in Section 2, the results of this paper are applicable to compression system models with non-cubic compressor characteristics as well.

In this section we employ the numerical continuation and bifurcation analysis package AUTO [8] to illustrate the compression system behavior for the chosen example. The behavior of both the uncontrolled and controlled systems is considered. The compression system example considered next involves the cubic axisymmetric compressor characteristic given in (4.1) below, as well as the throttle characteristic given in (4.2).

Let the axisymmetric compressor characteristic $C_{ss}(\dot{m}_C)$ be

$$C_{ss}(\dot{m}_C) = 1.56 + 1.5(\dot{m}_C - 1) - 0.5(\dot{m}_C - 1)^3. \tag{4.1}$$

Let F (the inverse of the throttle pressure rise map) be given by

$$F(\gamma, \Delta P) = \gamma\sqrt{\Delta P}. \tag{4.2}$$

Choose parameter values $\alpha = 0.4114$, $W = 1.0$ and $B = 0.5$.

It is easy to check that there are two values of γ for which the compression system (2.1) has an equilibrium point $(A, \dot{m}_C, \Delta P)$ with $C'_{ss}(\dot{m}_C) = 0$. Either using the analytical results of the preceding sections, or numerically, we can show that these are pitchfork bifurcations. One pitchfork bifurcation occurs for a large value of pressure rise ΔP, while the other occurs for small pressure rise. The former pitchfork bifurcation is found to be subcritical (i.e., it gives rise to unstable equilibria) while the latter is supercritical (i.e., it gives rise to stable equilibria). These observations are depicted in Figure 1. Solid lines in this figure indicate stable equilibria, while dotted lines indicate unstable equilibria. From this figure it is clear that the pitchfork bifurcation of most relevance in a practical setting, i.e., the one occurring for a larger ΔP, is subcritical. Thus hysteresis is expected to occur in the compression system, and a hysteresis loop is discernible in Figure 1.

Each of the graphs in Figure 1 shows the effects of the pitchfork bifurcations occurring in the system on a system state variable. Figure 1(c) is perhaps the most clear. From this figure, we see that a subcritical pitchfork bifurcation occurs from the nominal equilibrium as γ is decreased. A second, supercritical pitchfork bifurcation occurs at $\gamma = 0$. Note that Figures 1(a, b, d) also reflect these pitchfork bifurcations, but symmetry in the other state variables results in their being double-valued for the bifurcated solution. Hence these figures seem to indicate only one bifurcated equilibrium, rather than two.

Now, following Theorem 2 of Section 3, we choose the control input $u = 1.0A^2$, which is quadratic in the asymmetric flow amplitude A. In Figure 2, bifurcation diagrams for the system with this control law in effect are given. From this figure, it is clear that hysteresis loop of the stable system equilibria of Figure 1 has been eliminated. The system will tend to operate at the stabilized bifurcated equilibria after stall occurs.

5. Conclusions

The control of axial flow compressors at the initiation of stall has been studied using the Moore-Greitzer model. The state variables of this model are pressure rise, mass flow rate, and the first harmonic of the asymmetric component of the flow. It was found that control based solely on the system linearization could not ensure stable operation past the stall point, due to uncontrollability of the system eigenvalue which causes instability as the throttle parameter is varied. The bifurcation behavior of the model was studied and used as a basis for nonlinear control design. A simple bifurcation analysis showed that quadratic feedback of the first harmonic of the asymmetric component of the flow could locally stabilize the bifurcation, and hence eliminate the global hysteresis loop.

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Appendix A. Bifurcation Stability Coefficients

In this appendix we recall a bifurcation-theoretic result on stability and bifurcation of one-parameter families of nonlinear systems.

Consider a one-parameter family of autonomous systems

$$\dot{x} = f(x, \mu), \tag{A.1}$$

where $x \in I\!R^n$, $\mu \in I\!R$ and the vector field f is assumed to be sufficiently smooth in x and μ. Suppose that (A.1) possesses a nominal equilibrium point that depends smoothly on μ for all values of μ in some interval of interest. (This means we are considering *bifurcation from known solutions* [14].) For simplicity of notation, suppose $f(0, \mu) = 0$ for all μ. Let the Jacobian matrix $D_x f(0,0)$ possess a simple zero eigenvalue with all remaining eigenvalues in the open left half complex plane. The parameter μ is referred to as the *bifurcation parameter.*

The formulas and results given below remain valid if the nominal equilibrium is not the origin and the critical parameter value is not $\mu = 0$; simply evaluate all quantities (including partial derivatives) at the adjusted equilibrium and parameter value.

The equilibrium points of system (A.1) are the solutions of $f(x, \mu) = 0$, and thus depend on the value of the parameter μ. With the assumption that $D_x f(0,0)$ is singular, system (A.1) may possess several equilibrium paths $x(\mu)$ emanating from the origin for μ near 0. Such a situation is known as a *stationary* (or *static*) *bifurcation*, and the point $x = 0$, $\mu = 0$ is a *bifurcation point.* The solutions $x(\mu) \neq 0$ of (A.1) emanating from the origin for μ near 0 are called *bifurcated solutions.*

Following the notation of [7], we write the Taylor series expansion of (A.1) at $(0,0)$ as

$$\begin{aligned}\dot{x} &= f(x, \mu)\\ &= L_0 x + Q_0(x,x) + C_0(x,x,x) + \cdots\\ &\qquad + \mu(L_1 x + Q_1(x,x) + \cdots) + \mu^2(L_2 x + \cdots) + \cdots,\end{aligned} \tag{A.2}$$

where $Q_k(x,x) := \frac{1}{2!k!} D_{\mu^k xx} f(x,\mu)$, $C_k(x,x,x) := \frac{1}{3!k!} D_{\mu^k xxx} f(x,\mu)$, etc., are the quadratic terms, cubic terms, etc., of $f(x,\mu)$. Here, the quadratic and high order terms in the expansion are written as symmetric forms. For instance, $Q_k(x,y) = Q_k(y,x)$ for each $k \geq 0$

and any $x, y \in I\!R^n$. Moreover, in (A.2) $L_k := \frac{1}{k!} D_{\mu^k x} f(0,0)$ for $k \geq 1$ and the Jacobian matrix L_0 is assumed to have only a simple zero eigenvalue with the remaining eigenvalues stable.

Denote by l and r the left (row) and right (column) eigenvectors, respectively, of the matrix L_0 corresponding to the simple zero eigenvalue. Set the first component of r to 1 (possibly after a reordering of state variables) and choose the left eigenvector l so that $lr = 1$. Denote

$$\beta_1 := lQ_0(r, r) \tag{A.3}$$

and

$$\beta_2 := 2l\{2Q_0(r, \delta) + C_0(r, r, r)\} \tag{A.4}$$

where δ is the unique solution to

$$L_0 \delta = -Q_0(r, r), \tag{A.5}$$

$$l\delta = 0. \tag{A.6}$$

We are now in a position to recall the following stability criterion for system (A.2). This criterion applies only in the case $\beta_1 = 0$, and is useful for studying pitchfork bifurcation. Thus this result is sufficient for the present paper. If it happens that $\beta_1 \neq 0$ then another type of stationary bifurcation, a *transcritical bifurcation*, would be expected to occur. For details, see [7].

Lemma A.1. Let the assumptions above hold, and assume the transversality condition $lL_1 r \neq 0$. Then (A.2) undergoes a stationary bifurcation from $x = 0$ at $\mu = 0$. If $\beta_1 = 0$ and $\beta_2 < 0$, the bifurcated equilibrium points of (A.2) for μ near 0 are asymptotically stable. If, on the other hand, $\beta_1 = 0$ and $\beta_2 > 0$, then the bifurcated solutions of (A.2) for μ near 0 are unstable. In either case, the bifurcation is a pitchfork bifurcation, and the bifurcated equilibria locally occur only for $\mu(lL_1 r)/\beta_2 < 0$.

Figure Captions

Figure 1. Compression system open-loop operating points

Figure 2. Compression system closed-loop operating points

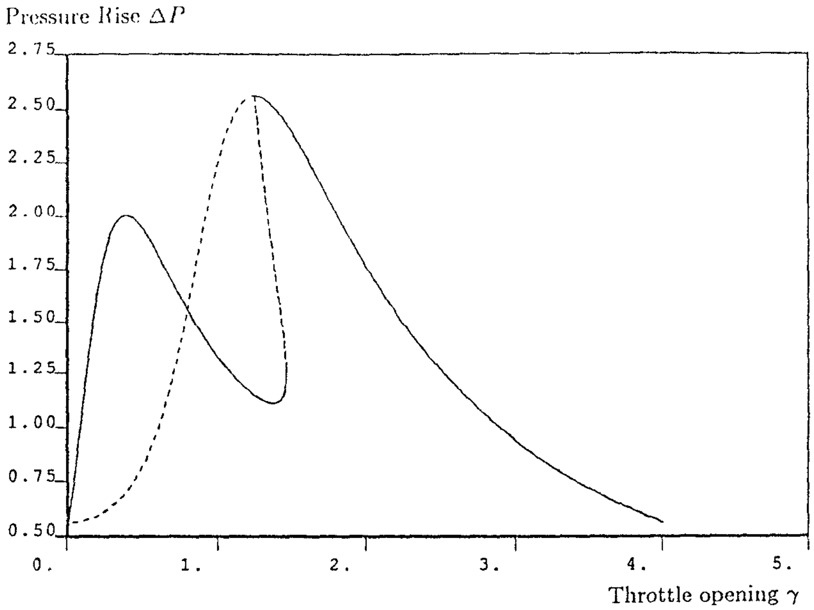

Figure 1 (a)

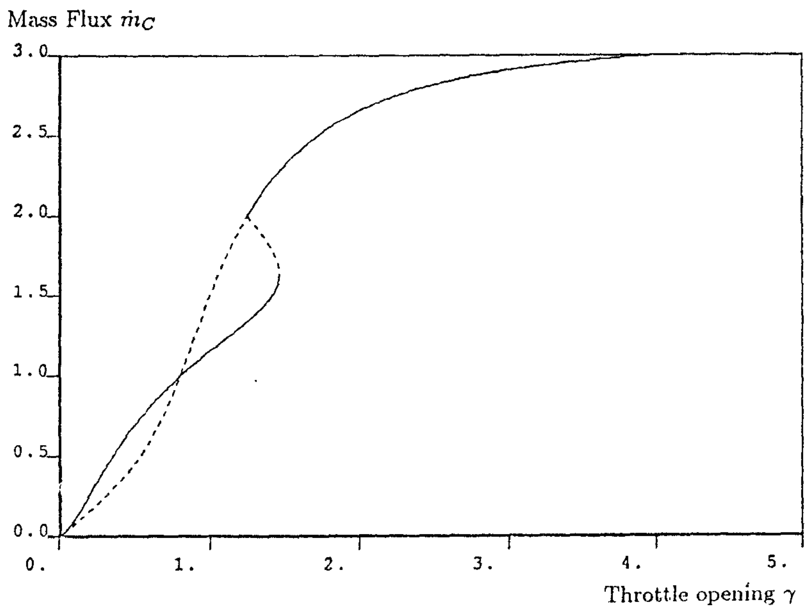

Figure 1 (b)

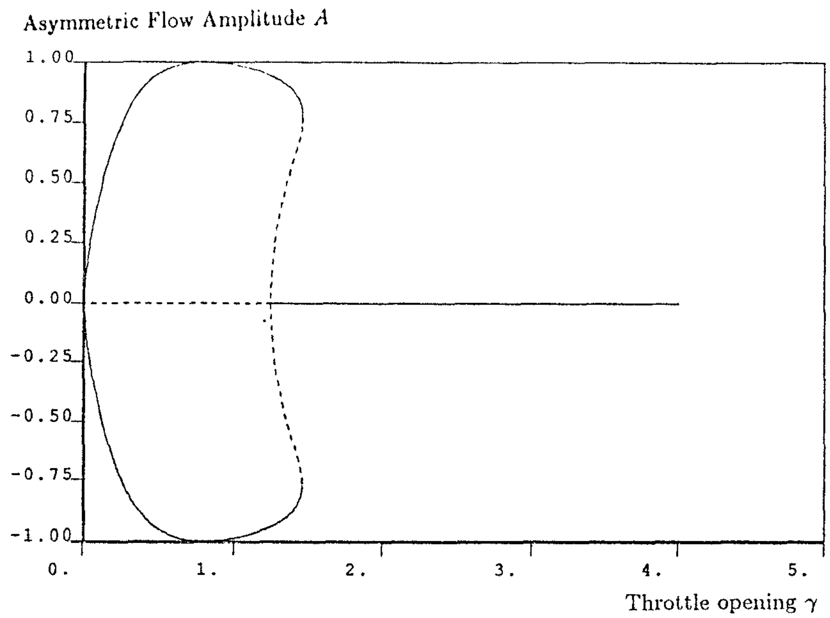

Figure 1 (c)

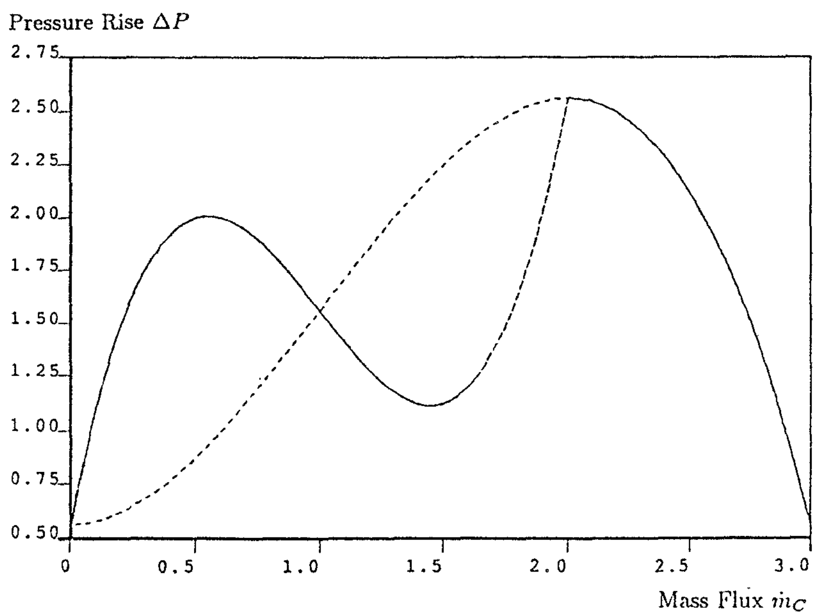

Figure 1 (d)

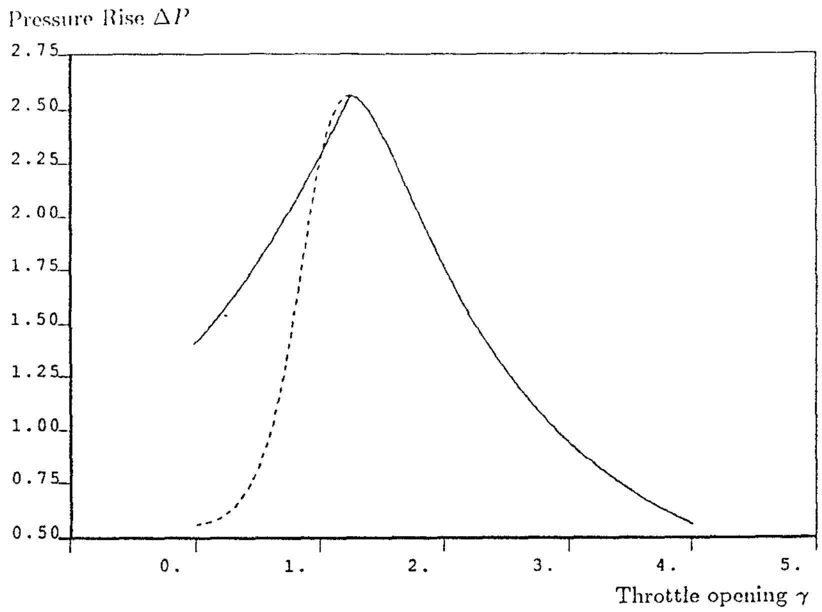

Figure 2 (a)

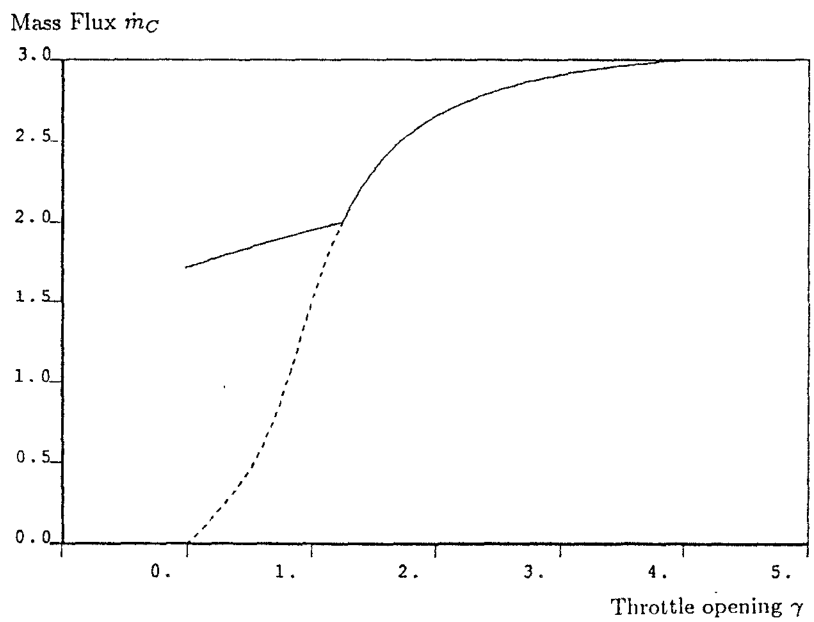

Figure 2 (b)

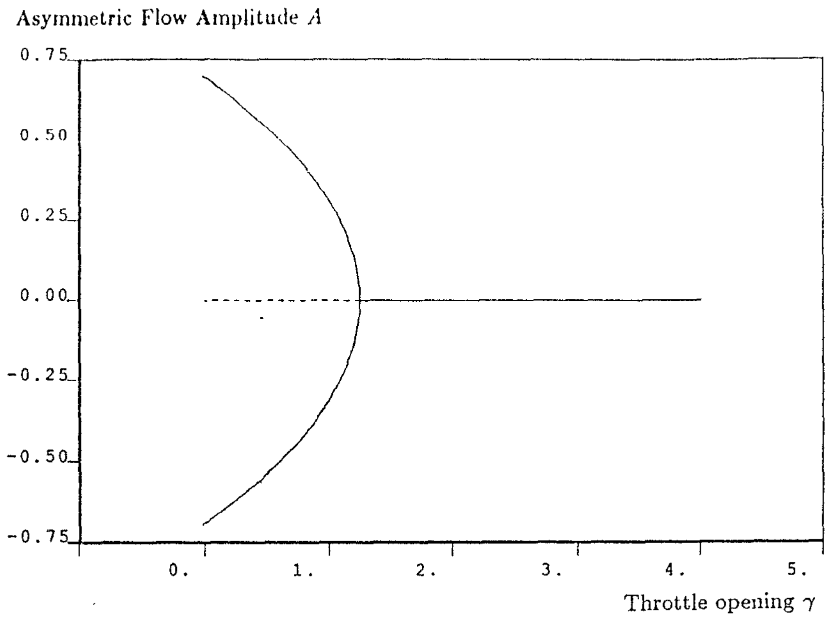

Figure 2 (c)

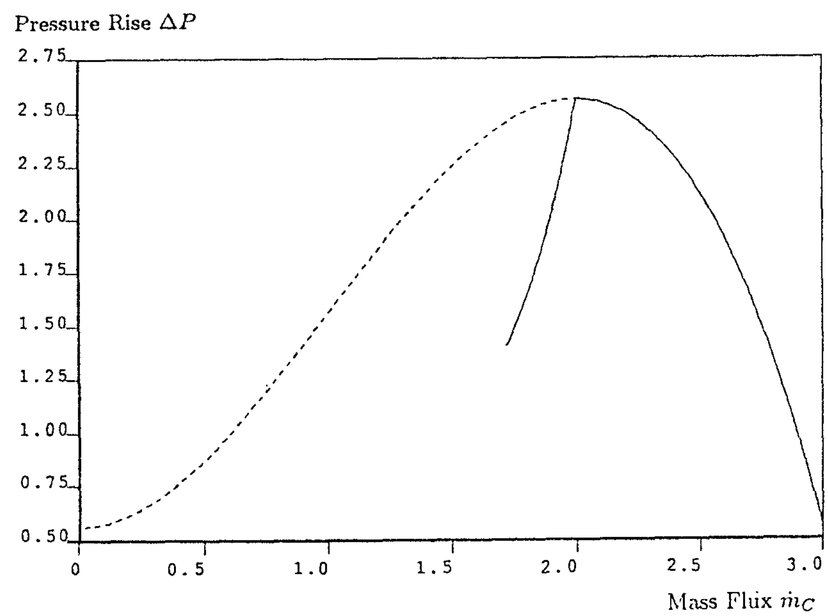

Figure 2 (d)