ABSTRACT

Title of dissertation: METHODS OF HARMONIC ANALYSIS APPLIED TO BOSE-EINSTEIN CONDENSATION

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This dissertation studies questions whose origin is in quantum statistical mechanics and is concerned about the evolution of large numbers of quantum spinless, interacting particles. More specifically, we study the analysis of the N-particle linear Schrödinger equation as $N \to \infty$ and discuss rigorously how the 1-body nonlinear Schrödinger equation comes from this limit process. Such problems arise in Bose-Einstein condensation.

In the first part of this dissertation, we consider the 2d and 3d many body Schrödinger equations in the presence of anisotropic switchable quadratic traps. We extend and improve the collapsing estimates in Klainerman-Machedon [29] and Kirkpatrick-Schlein-Staffilani [27]. Combining with an anisotropic version of the generalized lens transform as in Carles [3], we offer a rigorous derivation of the cubic NLS with anisotropic switchable quadratic traps in 2d through an appropriately modified procedure in Elgart-Erdös-Schlein-Yau [12, 13, 14, 15, 16, 17, 18] which is based on a kinetic hierarchy. For the 3d case, we establish the uniqueness of

the corresponding Gross-Pitaevskii hierarchy without the assumption of factorized initial data.

In the second part of this thesis, we consider the Hamiltonian evolution of N weakly interacting Bosons. Assuming triple collisions, its mean field approximation is given by a quintic Hartree equation. We construct a second-order correction to the mean field approximation using a kernel $k(t,x,y)$ describing pair creation and derive an evolution equation for k. We show the global existence for the resulting evolution equation for the correction and establish an apriori estimate comparing the approximation to the exact Hamiltonian evolution. Our error estimate is global and uniform in time. Comparing with the work of Rodnianski and Schlein [35], and Grillakis, Machedon and Margetis [21, 22], where the error estimate grows in time, our approximation tracks the exact dynamics for all time with an error of the order of $O\left(1/\sqrt{N}\right)$.

METHODS OF HARMONIC ANALYSIS APPLIED TO BOSE-EINSTEIN CONDENSATION

by

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Chapter 1

Introduction

1.1 Background

Bose-Einstein condensation (BEC) is the phenomenon in which a large number of particles of integer spin ("Bosons") occupy a macroscopic quantum state. Let $t \in \mathbb{R}$ be the time variable and $\overrightarrow{\mathbf{x}_N} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) \in \mathbb{R}^{nN}$ be the position vector of the N particles in $\mathbb{R}^n$. Then BEC naively means that the N-body wave function $\psi_N(t, \overrightarrow{\mathbf{x}_N})$ satisfies

$$\psi_N(t, \overrightarrow{\mathbf{x}_N}) = \prod_{j=1}^{N} \phi(t, \mathbf{x}_j) \tag{1.1}$$

up to a phase factor solely depending on t, for some one particle state ϕ. In other words, every particle is in the same quantum state. BEC was first predicted theoretically by Einstein for non-interacting particles. The first experimental observation of BEC in an interacting atomic gas did not occur until 1995 using laser cooling technique [1, 11]. E. A. Cornell, W. Ketterle, and C. E. Wieman were awarded the 2001 Nobel Prize in Physics for observing BEC. Many similar successful experiments were performed later on[10, 26, 37]. These observations have stimulated the further study of the theory of many-body Boson systems in the presence of a trap (as explained below).

Gross [23, 24] and Pitaevskii [34], proposed to model the many-body effects by a nonlinear on-site self interaction of a complex order parameter (the "condensate wave function"). The Gross-Pitaevskii equation is given by

$$i\partial_t u = -\triangle u + \sigma |u|^2 u = \left.\frac{\delta \mathcal{E}(u,\bar{u})}{\delta \bar{u}}\right|_u, \; \mathcal{E}(u,\bar{u}) = \int \left(|\nabla u|^2 + \frac{\sigma |u|^4}{2} \right)$$

where $\mathcal{E}$ is the Gross-Pitaevskii energy functional. The Gross-Pitaevskii equation is a phenomenological mean field type equation and its validity needs to be established from the Schrödinger equation with the Hamiltonian given by the pair interaction.

In the laboratory experiments of BEC, the particles are initially confined by traps, e.g., the magnetic fields in [1, 11], then the traps are switched in order to enable measurement or direct observation. To be more precise about the word "switch": in [1, 11] the trap is removed, in [37] the initial magnetic trap is switched to an optical trap, in [10] the trap is turned off in 2 spatial directions to generate a 2d Bose gas. The dynamics during this process are highly nontrivial. To model the evolution, we use a quadratic potential multiplied by a switch function in each spatial direction for analysis. In other words, we assume the external potential to be

$$V_{trap}(t,x) = \sum_{l=1}^{n} \eta_l(t) x_l^2$$

with the switch functions $\eta_l(t)$, $l = 1, ..., n$. This simplified yet reasonably general model is expected to capture the salient features of the actual traps: on the one hand, the quadratic potential varies slowly and tends to ∞ as $|x| \to \infty$; on the other hand, the switch functions describe the space-time anisotropic properties of the confining potential.

In the physics literature, Lieb, Seiringer and Yngvason remarked in [31] that the confining potential is typically $\sim |x|^2$ in the available experiments. Mathematically speaking, the strongest trap we can deal with in the usual regularity setting of the nonlinear Schrödinger equations is the quadratic trap since the work [38] by Yajima and Zhang points out that the ordinary Strichartz estimates start to fail as the trap grows faster than quadratic.

Motivated by the above considerations, we aim to investigate the evolution of a many-body Boson system in anisotropic switchable quadratic traps. The N-body wave function $\psi_N(t, \overrightarrow{\mathbf{x}_N})$ satisfies the N-body Schrödinger equation with anisotropic switchable quadratic traps:

$$i\partial_t \psi_N = \frac{1}{2} H_{\overrightarrow{\mathbf{x}_N}}(t)\psi_N + V_N \psi_N \tag{1.2}$$

with initial data

$$\psi_N(0, \overrightarrow{\mathbf{x}_N}) = \prod_{j=1}^{N} \phi_0(\mathbf{x}_j),$$

where V_N models the interaction between particles, and

$$\begin{aligned} H_{\overrightarrow{\mathbf{x}_N}}(t) \quad &: \quad = \sum_{j=1}^{N} H_{\mathbf{x}_j}(t) := \sum_{j=1}^{N} \left(-\triangle_{\mathbf{x}_j} + V_{trap}(t, \mathbf{x}_j)\right). \\ &: \quad = \sum_{j=1}^{N} \sum_{l=1}^{n} \left(-\frac{\partial^2}{\partial x_{j,l}^2} + \eta_l(t) x_{j,l}^2\right). \end{aligned} \tag{1.3}$$

When the trap is fully on, Lieb, Seiringer, Solovej and Yngvason showed that the ground state of the Hamiltonian exhibits complete BEC [32], provided that the trapping potential $V_{trap}(x)$ satisfies $\inf_{|x|>R} V_{trap}(x) \to \infty$ for $R \to \infty$ and the interaction potential is spherically symmetric. To be more precise, let $\psi_{N,0}$ be the ground state, then

$$\gamma_{N,0}^{(1)} \to |\phi_{GP}\rangle \langle \phi_{GP}| \text{ as } N \to \infty$$

where $\gamma_{N,0}^{(1)}$ is the corresponding one particle marginal density defined via formula (1.4) and ϕ_{GP} minimizes the Gross-Pitaevskii energy functional

$$\int (\left|\nabla\phi\right|^2 + V_{trap}(x) \left|\phi\right|^2 + 4\pi a_0 \left|\phi\right|^4) d\mathbf{x}.$$

Because we are now considering the evolution while the trap is changing, we start with a BEC state / factorized state in equation (1.2).

Though equation (1.2) is linear and the initial data is very special, it is highly nontrivial to see how it is related to BEC, which means the N-body wave function ψ_N is a product of one particles states. On the one hand, ψ_N does not remain a product of one-particle states i.e.

$$\psi_N(t, \overrightarrow{\mathbf{x}_N}) \neq \prod_{j=1}^{N} \phi(t, \mathbf{x}_j), \ t > 0$$

for some one particle state ϕ. On the other hand, it is unrealistic to solve equation (1.2) for large N. Thence, to prove BEC, we need an appropriate mathematical framework to explain how the N-body wave function ψ_N is close to $\prod_{j=1}^{N} \phi(t, \mathbf{x}_j)$ for some one particle state ϕ, where ϕ is expected to solve some nonlinear Schrödinger equation.

Notice that when $\phi \neq \phi'$, we have

$$\left\| \prod_{j=1}^{N} \phi(t, \mathbf{x}_j) - \prod_{j=1}^{N} \phi'(t, \mathbf{x}_j) \right\|_{L^2}^2 \to 2 \ as \ N \to \infty.$$

In other words, our desired limit (the BEC state) is not stable against small perturbations. One way to circumvent this difficulty is to use the concept of the k-particle marginal density $\gamma_N^{(k)}$ associated with ψ_N defined as

$$\gamma_N^{(k)}(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}_k'}) = \int \psi_N(t, \overrightarrow{\mathbf{x}_k}, \overrightarrow{\mathbf{x}_{N-k}}) \overline{\psi_N(t, \overrightarrow{\mathbf{x}_k'}, \overrightarrow{\mathbf{x}_{N-k}})} d\overrightarrow{\mathbf{x}_{N-k}}, \ \overrightarrow{\mathbf{x}_k}, \overrightarrow{\mathbf{x}_k'} \in \mathbb{R}^{nk}, \quad (1.4)$$

and show that

$$\gamma_N^{(k)}(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}) \sim \prod_{j=1}^{k} \phi(t, \mathbf{x}_j)\overline{\phi(t, \mathbf{x}'_j)}.$$

Penrose and Onsager [33] suggested such a formulation in. Another way is to add a second order correction to the mean-field approximation, $\prod_{j=1}^{N} \phi(t, \mathbf{x}_j)$, so that we can approximately solve for ψ_N directly, without taking marginals. The idea of the second-order correction comes from Wu [39, 40], and was rigorously formulated in a slightly different context by Grillakis, Machedon, and Margetis (GMM) [21, 22]. These two methods stand for the two main directions of this thesis. In the following, we use two separate sections to state and discuss briefly our main theorems regarding both directions.

1.2 The Rigorous Derivation of the 2d Cubic Nonlinear Schrödinger Equation with Anisotropic Switchable Quadratic Traps

Consider equation (1.2) when $n = 2$ and let

$$V_N = \frac{1}{N} \sum_{i<j} N^{2\beta} V(N^\beta (\mathbf{x}_i - \mathbf{x}_j)),\ \beta \in \left(0, \frac{3}{4}\right).$$

Notice that this is a "mean field" interaction because of the the factor $1/N$ and V_N is a 2-body interaction which approaches the Dirac delta function as $N \to \infty$. Furthermore, we assume that the switch functions $\eta_l \in C^1(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ in the Hermite-like operator (1.3) satisfy the following conditions.

Condition 1 $\dot{\eta}_l(0) = 0$ *i.e. The trap is not at a switching stage initially.*

Condition 2 $\dot{\eta}_l$ *is supported in* $[0, T_0]$ *and* $T_0\sqrt{\sup_t |\eta_l(t)|} < \frac{\pi}{2}$.

We take the marginal density approach and establish the following theorem.

Theorem 1 *Assume the nonnegative interaction potential V is integrable and belongs to $W^{2,\infty}$ and the switch functions η_l satisfy Conditions 1 and 2. Moreover, suppose the initial data has bounded energy per particle, that is,*

$$\sup_N \frac{1}{N} \langle \psi_N, H_N(t)\psi_N \rangle \bigg|_{t=0} < \infty,$$

where the Hamiltonian $H_N(t)$ is

$$H_N(t) = \frac{1}{2}\sum_{j=1}^{N}\left(\sum_{l=1}^{2}\left(-\frac{\partial^2}{\partial x_{j,l}^2} + \eta_l(t)x_{j,l}^2\right)\right) + \frac{1}{N}\sum_{i<j} N^{2\beta}V(N^{\beta}(\mathbf{x}_i - \mathbf{x}_j)).$$

If $\left\{\gamma_N^{(k)}\right\}$ are the marginal densities associated with ψ_N, the solution of the N-body Schrödinger equation (1.2)*, and ϕ solves the 2d Gross-Pitaevskii equation,*

$$\begin{aligned} i\partial_t\phi - \frac{1}{2}H_{\mathbf{x}}(t)\phi &= b_0 |\phi|^2 \phi \\ \phi(0,\mathbf{x}) &= \phi_0(\mathbf{x}), \end{aligned}$$

where $H_{\mathbf{x}}(t)$ is the operator defined via formula (1.3) *and $b_0 = \int V(x)dx$, then $\forall t \in [0, T_0]$ and $k \geqslant 1$, we have the following convergence:*

$$\left\| \gamma_N^{(k)}(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}) - \prod_{j=1}^{k} \phi(t,\mathbf{x}_j)\overline{\phi(t,\mathbf{x}'_j)} \right\|_{L^2(d\overrightarrow{\mathbf{x}_k} d\overrightarrow{\mathbf{x}'_k})} \to 0 \text{ as } N \to \infty.$$

Example 1 *We give a simple example to explain the switching process we are considering here: say*

$$\begin{aligned} \eta_1(\tau) &= C_1 \text{ when } t \in (-\infty, \tfrac{1}{2}],\ C_2 \text{ when } t \in [1,\infty), \\ \eta_2(\tau) &= C_3 \text{ when } t \in (-\infty, \tfrac{1}{4}],\ C_4 \text{ when } t \in [\tfrac{3}{2},\infty). \end{aligned}$$

Then our switching process contains the cases: turning off / on: $C_2 = 0$ / $C_1 = 0$ and tuning up / down: $C_1 \leqslant C_2$ / $C_2 \leqslant C_1$. As long as $\eta_1(\tau) \in C^1$ and satisfies Condition 2, η_1 can behave as one likes inside $[\frac{1}{2}, 1]$. Same comment applies to η_2. Furthermore, Theorem 1 addresses the time intervals $(-\infty, 0]$ and $[\frac{3}{2}, \infty)$ as well. Since the equation is time translation invariant in these two intervals, we can use Theorem 1 iteratively in each sufficiently small time interval.

Remark 1 *Technically, one should interpret Conditions 1 and 2 in the following way. Due to Condition 1, we have a C^1 even extension of η_l i.e. we define $\eta_l(t) = \eta_l(-t)$ for $\tau < 0$. The fast switching condition 2 in fact ensures that β_l defined via equation* (2.9) *is nonzero in* $[0, T_0]$ *which is crucial in the analysis. See Claim 1 for the proof.*

Remark 2 *We assume $\beta \in \left(0, \frac{3}{4}\right)$ to use the tools from Kirkpatrick-Schlein-Staffilani [27] in which the authors studied the $\eta_l = 0$ case. The case with $\beta = 0$ will yield a Hartree equation instead of the cubic NLS.*

The approach with $\gamma_N^{(k)}$ has been proven to be successful in the $\eta_l = 0$ and $n = 3$ case, which corresponds to the evolution after the removal of the traps, in the fundamental papers [12, 13, 14, 15, 16, 17, 18] by Elgart, Erdös, Schlein, and Yau. Their program, motivated by a kinetic formulaion of Spohn [36], consists of two principal parts: ine one part, they prove that an appropriate limit of the sequence $\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}$ as $N \to \infty$ solves the Gross-Pitaevskii hierarchy

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)\gamma^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(\gamma^{(k+1)}\right), \ k = 1, ..., n, ... \tag{1.5}$$

where $B_{j,k+1}$ are in formula (1.8); in another part, they show that hierarchy (1.5) has a unique solution which is therefore a completely factored state. However, the uniqueness theory for hierarchy (1.5) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. In [29], by assuming a space-time bound, Klainerman and Machedon gave another proof of the uniqueness in [15] through a collapsing estimate and a board game argument. We call the space-time estimates of the solution of Schrödinger equations restricted to a subspace of $\mathbb{R}^n$ "collapsing estimates". We can interpret them as local smoothing estimates for which integrating in time results in a gain of one hidden derivative in the sense of the trace theorem. To be specific, the collapsing estimate of [29] reads: Suppose $u^{(k+1)}$ solves

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_{k+1}}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_{k+1}}}\right) u^{(k+1)} = 0;$$

then, there is $C > 0$, independent of j, k or $u^{(k+1)}(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}})$ s.t.

$$\begin{aligned} &\left\| \left(\prod_{j=1}^{k} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) u^{(k+1)}(t, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \qquad (1.6) \\ \leqslant C &\left\| \left(\prod_{j=1}^{k+1} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) u^{(k+1)}(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}}) \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})}. \end{aligned}$$

Later, the method in Klainerman and Machedon [29] was taken up by Kirkpatrick, Schlein, and Staffilani [27], who studied the corresponding problem in 2d; and by Chen, Pavlović and Tzirakis [4, 5, 6], who considered the 1d and 2d 3-body interaction problem and the general existence theory of hierarchy (1.5).

We are interested in the case $\eta_l \neq 0$. So, we study the Gross-Pitaevskii hierarchy with anisotropic switchable quadratic traps. That is a sequence of functions

$\left\{\gamma^{(k)}(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k})\right\}_{k=1}^{\infty}$, where $\tau\in\mathbb{R}$, $\overrightarrow{\mathbf{x}_k},\overrightarrow{\mathbf{x}'_k}\in\mathbb{R}^{nk}$, which are symmetric, in the sense that $\gamma^{(k)}(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k})=\overline{\gamma^{(k)}}(t,\overrightarrow{\mathbf{x}'_k};\overrightarrow{\mathbf{x}_k})$ and

$$\gamma^{(k)}(t,\mathbf{x}_{\sigma(1)},\mathbf{x}_{\sigma(2)},...,\mathbf{x}_{\sigma(k)};\mathbf{x}'_{\sigma(1)},\mathbf{x}'_{\sigma(2)},...,\mathbf{x}'_{\sigma(k)})=\gamma^{(k)}(t,\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k;\mathbf{x}'_1,\mathbf{x}'_2,...,\mathbf{x}'_k)$$

for any permutation σ, since we focus on Bosons, and satisfy the anisotropic switchable quadratic traps Gross-Pitaevskii infinite hierarchy of equations:

$$\left(i\partial_t-\frac{1}{2}H_{\overrightarrow{\mathbf{x}_k}}(t)+\frac{1}{2}H_{\overrightarrow{\mathbf{x}'_k}}(t)\right)\gamma^{(k)}=b_0\sum_{j=1}^{k}B_{j,k+1}\left(\gamma^{(k+1)}\right). \tag{1.7}$$

In the above, $B_{j,k+1}=B^1_{j,k+1}-B^2_{j,k+1}$ are defined as

$$\begin{aligned}
& B^1_{j,k+1}\left(\gamma^{(k+1)}\right)(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k}) \\
= & \int\int\delta(\mathbf{x}_j-\mathbf{x}_{k+1})\delta(\mathbf{x}_j-\mathbf{x}'_{k+1})\gamma^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})d\mathbf{x}_{k+1}d\mathbf{x}'_{k+1} \\
& B^2_{j,k+1}\left(\gamma^{(k+1)}\right)(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k}) \\
= & \int\int\delta(\mathbf{x}'_j-\mathbf{x}_{k+1})\delta(\mathbf{x}'_j-\mathbf{x}'_{k+1})\gamma^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})d\mathbf{x}_{k+1}d\mathbf{x}'_{k+1}.
\end{aligned} \tag{1.8}$$

These Dirac delta functions in $B_{j,k+1}$ are the reason we consider the collapsing estimates like estimate (1.6).

If the initial data is a BEC / factorized state

$$\gamma^{(k)}(0,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k})=\prod_{j=1}^{k}\phi_0(\mathbf{x}_j)\overline{\phi_0(\mathbf{x}'_j)},$$

hierarchy (1.7) admits one solution

$$\gamma^{(k)}(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k})=\prod_{j=1}^{k}\phi(t,\mathbf{x}_j)\overline{\phi(t,\mathbf{x}'_j)},$$

which is also a BEC state, provided ϕ solves the $n-d$ Gross-Pitaevskii equation

$$\begin{aligned} i\partial_t\phi - \frac{1}{2}H_{\mathbf{x}}(t)\phi &= b_0\left|\phi\right|^2\phi \\ \phi(0,\mathbf{x}) &= \phi_0(\mathbf{x}). \end{aligned}$$

Hence we would like to have uniqueness theorems of hierarchy (1.7). In Chapter 2, we will state the uniqueness theorem and the other tools we need in order to prove Theorem 1.

1.3 Second-order Corrections to the Mean-field Approximation

Throughout this section, we consider $n = 3$ and let

$$\begin{aligned} \eta_l &= 0 \\ V_N &= \frac{1}{N^2}\sum_{i<j<k} v_3(\mathbf{x}_i-\mathbf{x}_j,\mathbf{x}_i-\mathbf{x}_k), \end{aligned}$$

where

$$v_3(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z}) = v_0(\mathbf{x}-\mathbf{y})v_0(\mathbf{x}-\mathbf{z})+v_0(\mathbf{x}-\mathbf{y})v_0(\mathbf{y}-\mathbf{z})+v_0(\mathbf{x}-\mathbf{z})v_0(\mathbf{y}-\mathbf{z}). \quad (1.9)$$

Here, v_3 is built of a nonnegative regular potential, v_0, which decays fast enough away from the origin and has the property

$$v_0(\mathbf{x}) = v_0(-\mathbf{x}).$$

Accordingly, equation (1.2) becomes

$$\begin{aligned} i\partial_t\psi_N &= \left(\sum_{j=1}^{N}\triangle_{\mathbf{x}_j} - \frac{1}{N^2}\sum_{i<j<k} v_3(\mathbf{x}_i-\mathbf{x}_j,\mathbf{x}_i-\mathbf{x}_k)\right)\psi_N \text{ in } \mathbb{R}^{3N+1} \quad (1.10) \\ \psi_N(0,\overrightarrow{\mathbf{x}_N}) &= \prod_{j=1}^{N}\phi_0(\mathbf{x}_j). \end{aligned}$$

Notice that we are now considering the evolution with a 3-body interaction and no external potential. Our goal is to build a second-order correction to the the mean-field approximation based on the Fock space formalism of equation (1.10). This type of second-order correction was used by GMM [21, 22] for the 2-body interaction case. The main motivation for considering the 3-body particle interaction is to point out that when we apply the GMM approximation to the Hamiltonian evolution of many-particle systems equipped with 3-body interactions, the error between GMM approximation and the actual many-body Hamiltonian evolutions can be controlled uniformly in time. (See Knowles and Pickl [30] for another type of uniform error bound.) We will discuss the difference between the 2-body and 3-body cases in the end of this section.

First, we set up the Boson Fock space $\mathcal{F}$ following [21, 22, 35].

Definition 1 *The Hilbert space Boson Fock space $\mathcal{F}$ based on $L^2(\mathbb{R}^3)$ contains vectors of the form $\boldsymbol{\psi} = (\psi_0, \psi_1(\mathbf{x}_1), \psi_2(\mathbf{x}_1, \mathbf{x}_2), \cdots)$ where $\psi_0 \in \mathbb{C}$ and $\psi_n \in L^2_s(\mathbb{R}^{3n})$ are symmetric in $\mathbf{x}_1, \ldots, \mathbf{x}_n$. The Hilbert space structure of $\mathcal{F}$ is given by $(\boldsymbol{\phi}, \boldsymbol{\psi}) = \sum_n \int \phi_n \overline{\psi_n} d\mathbf{x}$.*

Definition 2 *For $f \in L^2(\mathbb{R}^3)$, we define the (unbounded, closed, densely defined) creation operator $a^*(f) : \mathcal{F} \to \mathcal{F}$ and annihilation operator $a(\bar{f}) : \mathcal{F} \to \mathcal{F}$ by*

$$\left(a^*(f)\psi_{n-1}\right)(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(\mathbf{x}_j)\psi_{n-1}(\mathbf{x}_1, \cdots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \cdots, \mathbf{x}_n),$$

$$\left(a(\overline{f})\psi_{n+1}\right)(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \sqrt{n+1} \int \psi_{n+1}(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_n)\overline{f}(\mathbf{x}) d\mathbf{x}.$$

The operator valued distributions $a^*_{\mathbf{x}}$ and $a_{\mathbf{x}}$ are then defined by

$$a^*(f) = \int f(\mathbf{x}) a^*_{\mathbf{x}} d\mathbf{x},$$
$$a(\overline{f}) = \int \overline{f}(\mathbf{x}) a_{\mathbf{x}} d\mathbf{x}.$$

These distributions satisfy the canonical commutation relations

$$[a_{\mathbf{x}}, a^*_{\mathbf{y}}] = \delta(\mathbf{x} - \mathbf{y}), \tag{1.11}$$
$$[a_{\mathbf{x}}, a_{\mathbf{y}}] = [a^*_{\mathbf{x}}, a^*_{\mathbf{y}}] = 0.$$

Thus, we compute

$$\left(\int a^*_{\mathbf{x}} \Delta a_{\mathbf{x}} d\mathbf{x} \right) \psi_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \sum_{j=1}^{n} \Delta_{\mathbf{x}_j} \psi_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$$

and

$$\begin{aligned} &\left(\int v_3(\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{z}) a^*_{\mathbf{x}} a^*_{\mathbf{y}} a^*_{\mathbf{z}} a_{\mathbf{x}} a_{\mathbf{y}} a_{\mathbf{z}} d\mathbf{x} d\mathbf{y} d\mathbf{z} \right) \psi_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \\ = &\sum_{i,j,k} v_3(\mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_k) \psi_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n). \end{aligned}$$

These relations give us the right-hand side of the many-body Schrödinger equation (1.10). So, we define the Fock space Hamiltonian with 3-body interaction to be

$$\begin{aligned} H_N &= \int a^*_{\mathbf{x}} \Delta a_{\mathbf{x}} d\mathbf{x} - \frac{1}{6N^2} \int v_3(\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{z}) a^*_{\mathbf{x}} a^*_{\mathbf{y}} a^*_{\mathbf{z}} a_{\mathbf{x}} a_{\mathbf{y}} a_{\mathbf{z}} d\mathbf{x} d\mathbf{y} d\mathbf{z} \\ &= H_0 - \frac{1}{6N^2} V. \end{aligned} \tag{1.12}$$

Then, of course, we want to apply the evolution operator e^{itH_N} to a BEC / factorized state initial data. To obtain the factorized state initial data in Fock space, define the vacuum state $\Omega \in \mathcal{F}$ and the skew-Hermitian unbounded operator A by

$$\begin{aligned} \Omega &= (1, 0, 0, \cdots) \\ A(\phi) &= a(\overline{\phi}) - a^*(\phi). \end{aligned} \tag{1.13}$$

Then,

$$e^{-\sqrt{N}A(\phi_0)}\Omega = e^{-N\|\phi\|^2/2}\left(1,\cdots,\left(\frac{N^n}{n!}\right)^{1/2}\phi_0(\mathbf{x}_1)\cdots\phi_0(\mathbf{x}_n),\cdots\right),$$

i.e., $e^{-\sqrt{N}A(\phi_0)}\Omega$ renders the Fock space analogue of the initial data

$$\psi_N|_{t=0} = \prod_{i=1}^{N}\phi_0(x_i)$$

in equation (1.10). Whence the Fock space representation of equation (1.10) is the Hamiltonian evolution

$$e^{itH_N}e^{-\sqrt{N}A(\phi_0)}\Omega.$$

Let the one-particle wave function $\phi(t,x)$ solve the quintic Hartree equation

$$i\frac{\partial}{\partial t}\phi + \triangle\phi - \frac{1}{2}\phi\int v_3(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z})\left|\phi(\mathbf{y})\right|^2\left|\phi(\mathbf{z})\right|^2 dydz = 0 \tag{1.14}$$

subject to the initial condition $\phi(0,x) = \phi_0(x)$. Accordingly, then the mean field approximation for $e^{itH_N}e^{-\sqrt{N}A(\phi_0)}\Omega$ is the tensor product of $\phi(t,x)$, or, more specifically,

$$\boldsymbol{\psi}_{MeanField} = e^{-\sqrt{N}A(\phi(t,\cdot))}\Omega. \tag{1.15}$$

A derivation of equation (1.14) is given in Section 3.2.

By assuming that the Hamiltonian H_N is subject to the two body interaction

$$\begin{aligned} H_{N,2} &= \int a_x^*\Delta a_x dx - \frac{1}{2N}\int v_2(x-y)a_x^*a_y^*a_xa_y dxdy \\ &= H_0 - \frac{1}{N}V_2, \end{aligned}$$

by the Fock space formalism of equation (1.10) with the two body interaction, Rodnianski and Schlein [35] derived a cubic Hartree equation for $\phi(t,x)$ (equation (1.17)).

They showed in [35] that the mean-field approximation works (under suitable assumptions on v) in the sense that

$$\begin{aligned}&\frac{1}{N}\|\left(e^{itH_{N,2}}\boldsymbol{\psi}_0,\, a_y^* a_x e^{itH_{N,2}}\boldsymbol{\psi}_0\right) - \left(e^{-\sqrt{N}A(\phi(t,\cdot))}\Omega,\, a_y^* a_x e^{-\sqrt{N}A(\phi(t,\cdot))}\Omega\right)\|_{Tr}\\ &= O(\frac{e^{Ct}}{N}) \qquad N \to \infty\ ;\end{aligned}$$

where $\|\cdot\|_{Tr}$ stands for the trace norm in $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$, and $\boldsymbol{\psi}_0 = e^{-\sqrt{N}A(\phi_0)}\Omega$. For the precise statement of the problem and details of the proof, see Theorem 3.1 of Rodnianski and Schlein [35]. Later, in [21, 22], GMM introduced a second-order correction (GMM type correction) to the mean field approximation of $e^{itH_{N,2}}e^{-\sqrt{N}A(\phi_0)}\Omega$ which greatly improved the error.

Instead of delving into the results in [21, 22], we state our main theorem first. This makes it easier to compare our results with the ones in [21, 22].

Remark 3 *For simplicity, let us write $A(\phi)$ as A, $A(\phi(t,\cdot))$ as $A(t)$, $v_3(x-y,x-z)$ as $v_{3,1-2,1-3}$, and $\phi(y)$ as ϕ_2.*

Theorem 2 *If ϕ_0, the initial datum, satisfies the conditions of*

(i) finite mass:

$$\|\phi_0\|_{L^2_{\mathbf{x}}} = 1,$$

(ii) finite energy:

$$\begin{aligned}E_0 &= \frac{1}{2}\int |\nabla\phi_0|^2\, d\mathbf{x} + \frac{1}{6}\int v_3(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z})\, |\phi_0(\mathbf{x})|^2\, |\phi_0(\mathbf{y})|^2\, |\phi_0(\mathbf{z})|^2\, d\mathbf{x}d\mathbf{y}d\mathbf{z}\\ &\leqslant C_1,\end{aligned}$$

(iii) finite variance:

$$\||\cdot|\,\phi_0\|_{L^2_{\mathbf{x}}} \leqslant C_2,$$

then, based on the mean-field approximation, we can construct $\boldsymbol{\psi}_{GMM}$*, an improved approximation of the wave functionthrough a kernel* $k(t, \mathbf{x}, \mathbf{y})$ *which satisfies an evolution equation derived using the metaplectic representation. Moreover,* $\boldsymbol{\psi}_{GMM}$ *is a second order approximation to the Hamiltonian evolution* $e^{itH_N} e^{-\sqrt{N}A(\phi_0)}\Omega$ *for* H_N *defined in formula* (1.12) *and the following uniform in time error estimate holds*

$$\|\boldsymbol{\psi}_{GMM} - e^{itH_N} e^{-\sqrt{N}A(\phi_0)}\Omega\,\|_{\mathcal{F}} \leqslant \frac{C}{\sqrt{N}},$$

where $\mathcal{F}$ *is the Boson Fock space defined in Definition 1, and* C *depends only on* v*,* C_1 *and* C_2*.*

We remark that Theorem 2 also works when v_0 has a proper singularity at the origin. To be specific, if for some $\varepsilon \in (0, \frac{1}{2})$, we have

$$v_0(x) = \frac{\chi(|x|)}{|x|^{1-\varepsilon}}, \text{ or } G_{2+\varepsilon}(x) \tag{1.16}$$

where $\chi \in C_0^\infty(\mathbb{R}^+ \cup \{0\})$ is nonnegative and decreasing and G_α the kernel of Bessel potential, then Theorems 2 holds. Though we currently do not know the physical meaning for such potentials if $\varepsilon \neq 0$, we would like to understand the analysis when singularities appear since the derivation of the quintic NLS uses an interaction which goes to a delta function when $N \to \infty$. Due to the technicality of treating the singularities, we restrict our analysis to the case of smooth potentials so that the differences between the 2-body and 3-body interactions are easier to see.

1.3.1 Comparison with Results in [21, 22]

In Theorem 2, if we change H_N to $H_{N,2}$, and equation (1.14) to

$$i\frac{\partial}{\partial t}\phi + \triangle\phi - \phi \int v_2(x-y)\left|\phi(y)\right|^2 dy = 0, \tag{1.17}$$

and make the corresponding changes in the construction of $\boldsymbol{\psi}_{GMM}$, then the main theorem in [21, 22] reads

$$\begin{aligned} &\|\boldsymbol{\psi}_{GMM} - e^{itH_{N,2}} e^{-\sqrt{N}A(0)}\Omega\,\|_{\mathcal{F}} \\ &\leqslant \frac{C(1+t)^{\frac{1}{2}}}{\sqrt{N}}, \end{aligned} \tag{1.18}$$

if $v_2(x) = \frac{\chi(|x|)}{|x|}$.

Compared with the above long time estimate, Theorem 2 demonstrates that there is a substantial difference between the 3-body interaction case and 2-body interaction. Technically speaking, the main difference between the 2-body and 3-body interactions lies in the error terms that they produce for the respective many-body wave functions. Though the analysis is more involved even if we assume smooth potential and the formulas are considerably longer, the more complicated error terms in the 3-body interaction case in fact allow more room to play. On the one hand, an error term in the 3-body case carries at least a pair of u, p or ϕ which satisfy some Schrödinger equations; for instance, the term

$$\|v_3(x_1-y_1, x_1-z_1)\bar{u}(t, x_2, x_1)\bar{u}(t, y_1, z_1)\|_{L^1_t L^2}$$

in formula (3.25), can be estimated by Lemma 16. A typical error term in the 2-body case can carry only one term of u, p or ϕ; for example, the term

$$\|v_2(x_1-y_1)u(t, y_1, x_1)\|_{L^1_t([0,T])L^2}$$

implicitly inside formula (47) of [21]. On the other hand, the error estimate in the construction of the second-order correction involves L_t^1, and we have no L_t^1 dispersive estimates for the Schrödinger equation. Therefore, due to the endpoint Strichartz estimates [25], we can construct a $L_t^1(\mathbb{R}^+)$ estimate for the 3-body case which is Lemma 16, without having the $t^{\frac{1}{2}}$ in the 2-body case which is necessary to apply the L_t^2 Kato estimate in [21, 22]. Or in other words, we do Cauchy-Schwarz in time differently.

For the reason stated above, one can not employ the 3-body case error estimate in the 2-body case. Furthermore, the tools of error estimates in the 2-body case [21, 22], do not apply to the 3-body case, regardless of whether v_3 is regular or singular like formula (1.16).

1.4 Organization of the Thesis

The rest of this thesis is devoted to the proofs of Theorems 1 and 2. First, in Chapter 2, we establish Theorem 1. Then, in Chapter 3, we show Theorem 2.

1.5 Conclusion and Further Questions

In this thesis, we have derived rigorously the 2d cubic NLS with anisotropic switchable quadratic traps through a modified Elgart-Erdös-Schlein-Yau procedure. We also derived a 2nd order correction to the mean field approximation subject to 3-body interaction and no external potential.

The main novelty of the work regarding the 2d cubic NLS with anisotropic switchable quadratic traps is that I allow a quadratic trap in the analysis while previous work was done without a trap. I also allow switches on the trap. Even if all the switches are constant 1, it is a new result.

The main novelty of the work regarding 2nd order correction to the mean field approximation is that the error estimate holds uniformly in time.

There are many interesting problems and there is room for refinements in this field. Here are a few of the questions that I am working on and are future avenues for research.

After looking at Theorem 1, it is natural to wonder what we can say about the 3d case. The 3d case is the most physically interesting.

There are also many question to ask on the second-order correction to the mean-field approximation in Theorem 2 since it is fairly new. I would like to continue the study in these two directions:

1) Consider the second-order correction with more singular potentials. It is interesting to use the potential of the form we used in Theorem 1 and ask whether one can derive the cubic or quintic nonlinear Schrödinger equations through the second order correction. The error estimate in [8] shows an unexpected application of the endpoint Strichartz estimate in [25]. We might understand this connection better by considering more singular potentials.

2) Construct the second order correction in the presence of a time-dependent trap and give a deeper explanation of the construction of it. When a time-dependent trap appears in the Hamiltonian like Theorem 1, the evolution is no longer an exponential which is a structure easier to deal with in the Lie algebra settings.

Moreover, the construction in [8] in fact applies to more general initial data other than factorized states.

Chapter 2

Proof of Theorem 1

2.1 Main Auxiliary Theorems

To obtain Theorem 1, we need the auxiliary theorems in this subsection which are of independent interest. We show them in 3d as well. On the one hand, the general idea for the 2d case is derived from the higher dimensional case. On the other hand, the 2d and 3d cases are dramatically different when they are viewed in the context of Theorem 1. We will explain this difference between the 2d and 3d case in Section 2.7. For the moment, notice that the uniqueness theorems in 2d and 3d address two different Gross-Pitaevskii hierarchies which stand for the two sides of the lens transform. Also, we currently do not have a 3d version of the 2d convergence / Theorem 1. We state our auxiliary theorems regarding different dimensions separately for comparison.

First, we have the following collapsing estimates which generalizes estimate 1.6.

Theorem 3 *(3*n-d optimal collapsing estimate) Let* $n = 2$ *or* 3, *write*

$$L_{\mathbf{x}}(t) = \sum_{l=1}^{n} a_l(t) \frac{\partial^2}{\partial x_l^2}, \tag{2.1}$$

where the L^1_{loc} *functions* a_l *satisfy*

$$a_l \geqslant c_0 > 0 \ a.e.$$

Furthermore, assume that $u(t,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2')$ solves the Schrödinger equation

$$\begin{aligned} iu_t + L_{\mathbf{x}_1}(t)u + L_{\mathbf{x}_2}(t)u \pm L_{\mathbf{x}_2'}(t)u &= 0 \text{ in } \mathbb{R}^{3n+1} \\ u(0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2') &= f(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2'). \end{aligned} \tag{2.2}$$

Then,

$$\int_{\mathbb{R}^{n+1}} \left| |\nabla_{\mathbf{x}}|^{\frac{n-1}{2}} u(t,\mathbf{x},\mathbf{x},\mathbf{x})\right|^2 d\mathbf{x}dt \leqslant C \left\| |\nabla_{\mathbf{x}_1}|^{\frac{n-1}{2}} |\nabla_{\mathbf{x}_2}|^{\frac{n-1}{2}} \left|\nabla_{\mathbf{x}_2'}\right|^{\frac{n-1}{2}} f \right\|_2^2 .$$

Theorem 3 is a scale invariant estimate when $a_l = 1$ hence it is optimal. In fact, it holds for all $n \geqslant 2$. The proof is different for $n = 2$ and $n \geqslant 3$. We name the third spatial variables $\mathbf{x}_2'$ to match the uniqueness theorems. We point out that Kirkpatrick, Schlein and Staffilani [27] proved the almost optimal result for the 2d constant coefficient case. Some other collapsing estimates were attained in [7, 20].

2.1.1 2d Auxiliary Theorems

Theorem 3 is the key to show the following uniqueness theorem.

Theorem 4 *(Uniqueness of 2d GP with time-dependent coefficients) Let $L_{\mathbf{x}_k}$ be in formula* (2.1) *and $B_{j,k+1}$ be defined via formula* (1.8)*. Say $\left\{u^{(k)}(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}_k'})\right\}_{k=1}^{\infty}$ solves the Gross-Pitaevskii hierarchy with variable coefficients*

$$\left(i\partial_t + L_{\overrightarrow{\mathbf{x}_{k+1}}}(t) - L_{\overrightarrow{\mathbf{x}_{k+1}'}}(t)\right) u^{(k)} = b_0 \sum_{j=1}^{k} B_{j,k+1}\left(u^{(k+1)}\right),$$

subject to zero initial data and the space-time bound

$$\int_0^T \left\| \prod_{j=1}^{k} \left(|\nabla_{\mathbf{x}_j}|^{\frac{1}{2}} \left|\nabla_{\mathbf{x}_j'}\right|^{\frac{1}{2}} \right) B_{j,k+1} u^{(k+1)}(t,\cdot;\cdot) \right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt \leqslant C^k$$

for some $C > 0$ and all $1 \leqslant j \leqslant k$. Then $\forall k, t \in [0, T]$,

$$\left\| \prod_{j=1}^{k} \left(\left|\nabla_{\mathbf{x}_j}\right|^{\frac{1}{2}} \left|\nabla_{\mathbf{x}'_j}\right|^{\frac{1}{2}} \right) u^{(k)}(t, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} = 0.$$

In contrast to the standard Elgart-Erdös-Schlein-Yau program, we do not need a uniqueness theorem regarding the Gross-Pitaevskii hierarchy with anisotropic switchable quadratic traps (hierarchy (1.7)) to establish Theorem 1. It is enough to have Theorem 4 which has no quadratic potential inside. At a glance, the analysis of the above hierarchy based on the Laplacian is unrelated to the hierarchy (1.7) based on a Hermite like operator $H_{\mathbf{y}}(\tau)$. However, Carles' generalized lens transform [3] links them together. In fact, the generalized lens transform preserves the L^2 critical NLS and thus the 2d Gross-Pitaevskii hierarchies. The specific version of the lens transform we need is provided in Section 2.4.

2.1.2 3d Auxiliary Theorems

As mentioned before, the uniqueness theorem here addresses a different hierarchy from Theorem 4. Of course we can prove a 3d version of Theorem 4. However, the disparity between the 2d and 3d case renders such a theorem of little value because the lens transform does not preserve the 3d cubic NLS. See Section 2.7 for details.

We consider the norm

$$\left\| R_{\tau}^{(k)} \gamma^{(k)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})} \tag{2.3}$$

in which

$$R_{\tau}^{(k)} = \left(\prod_{j=1}^{k} P_{\mathbf{y}_j}(\tau) P_{\mathbf{y}'_j}(-\tau) \right),$$

$$P_{\mathbf{y}}(\tau)=\begin{pmatrix} i\beta_1(\tau)\frac{\partial}{\partial y_1}+\dot{\beta}_1(\tau)y_1 \\ i\beta_2(\tau)\frac{\partial}{\partial y_2}+\dot{\beta}_2(\tau)y_2 \\ i\beta_3(\tau)\frac{\partial}{\partial y_3}+\dot{\beta}_3(\tau)y_3 \end{pmatrix},$$

where β_l solves

$$\ddot{\beta}_l(\tau)+\eta_l(\tau)\beta_l(\tau)=0,\beta_l(0)=1,\dot{\beta}_l(0)=0. \tag{2.4}$$

The operator $i\beta_l(\tau)\frac{\partial}{\partial y_l}+\dot{\beta}_l(\tau)y_l$ was introduced by Carles in [3]. Lemma 3 and relation (2.12) indicate that the norm (2.3) is natural. That is because this operator is in fact the evolution of the momentum operator $-i\nabla$. We will compute it in Section 2.8.

Through a specific generalized lens transform (Proposition 3) we produce the collapsing estimate which is the key estimate to our 3d uniqueness theorem regarding hierarchy (1.7) when $n=3$.

Theorem 5 *Let* $[s,T]\subset[0,T_0]$ *and* β_l *be defined through equation* (2.4), *assume that* $\gamma^{(k+1)}(\tau,\mathbf{y}_{k+1};\mathbf{y}'_{k+1})$ *satisfies the homogeneous equation*

$$\begin{aligned}\left(i\partial_\tau-\frac{1}{2}H_{\overrightarrow{\mathbf{y}_{k+1}}}(\tau)+\frac{1}{2}H_{\overrightarrow{\mathbf{y}'_{k+1}}}(\tau)\right)\gamma^{(k+1)} &= 0 \\ \gamma^{(k+1)}(0,\overrightarrow{\mathbf{y}_{k+1}};\overrightarrow{\mathbf{y}'_{k+1}}) &= \gamma_0^{(k+1)}(\overrightarrow{\mathbf{y}_{k+1}};\overrightarrow{\mathbf{y}'_{k+1}}).\end{aligned} \tag{2.5}$$

Then there exists a $C>0$ *independent of* $\gamma_0^{(k+1)}$, j, k, s, *and* T *s.t.*

$$\begin{aligned}&\left\|R_\tau^{(k)}B_{j,k+1}\left(\gamma^{(k+1)}\right)\right\|^2_{L^2([s,T]\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ \leqslant\ & C\left(\inf_{\tau\in[0,T_0]}\prod_{l=2}^{3}\beta_l^2(\tau)\right)^{-1}\left\|R_\tau^{(k+1)}\gamma^{(k+1)}\right\|^2_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},\end{aligned}$$

where the τ *on the RHS of the above estimate can be chosen freely in* $[s,T]$, From Theorem 5, we can state the following.

Theorem 6 *(Uniqueness of 3d GP with anisotropic switchable quadratic traps) Let* $\left\{\gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k})\right\}_{k=1}^{\infty}$ *solve the 3d Gross-Pitaevskii hierarchy with anisotropic switchable quadratic traps (hierarchy 1.7 when* $n=3$*) subject to zero initial data and the space-time bound*

$$\int_0^{T_0} \left\| R_\tau^{(k)} B_{j,k+1} \gamma^{(k+1)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} d\tau \leqslant C^k \tag{2.6}$$

for some $C>0$ *and all* $1 \leqslant j \leqslant k$*. Then* $\forall k, \tau \in [0, T_0]$,

$$\left\| R_\tau^{(k)} \gamma^{(k)}(\tau, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} = 0.$$

Remark 4 *It is currently unknown how to show directly that the limit of* $\gamma_N^{(k)}$ *in 3d satisfies the space-time bound* (2.6)*.*

2.2 Proof of Theorem 3 when $n = 3$ / 3*3d Collapsing Estimate

We will make use of the lemma.

Lemma 1 *[29]Let* $\boldsymbol{\xi} \in \mathbb{R}^3$ *and* P *be a 2d plane or sphere in* $\mathbb{R}^3$ *with the usual induced surface measure* dS*.*

(1) Say $0 < a$, $b < 2$, $a + b > 2$, *then*

$$\int_P \frac{dS(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^a |\boldsymbol{\eta}|^b} \leqslant \frac{C}{|\boldsymbol{\xi}|^{a+b-2}}.$$

(2) Say $\varepsilon = \frac{1}{10}$, *then*

$$\int_P \frac{dS(\boldsymbol{\eta})}{\left|\frac{\boldsymbol{\xi}}{2}-\boldsymbol{\eta}\right| |\boldsymbol{\xi}-\boldsymbol{\eta}|^{2-\varepsilon} |\boldsymbol{\eta}|^{2-\varepsilon}} \leqslant \frac{C}{|\boldsymbol{\xi}|^{3-2\varepsilon}}.$$

Both constants in the above estimates are independent of P*.*

Proof. See pages 174 - 175 of [29]. ■

By duality, to gain Theorem 3 when $n = 3$, it suffices to prove

$$\left|\int_{\mathbb{R}^{3+1}} |\nabla_{\mathbf{x}}| u(t,\mathbf{x},\mathbf{x},\mathbf{x})h(t,\mathbf{x})d\mathbf{x}dt\right| \leqslant C\|h\|_2 \left\|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2'}f\right\|_2.$$

Let

$$A_t = \begin{pmatrix} \int_0^t a_1(s)ds & 0 & 0 \\ 0 & \int_0^t a_2(s)ds & 0 \\ 0 & 0 & \int_0^t a_3(s)ds \end{pmatrix},$$

then it brings the solution of equation (2.2)

$$u(t,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_2') = \int e^{i(\boldsymbol{\xi}_1^T A_t \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} e^{i\mathbf{x}_1\boldsymbol{\xi}_1} e^{i\mathbf{x}_2\boldsymbol{\xi}_2} e^{i\mathbf{x}_2'\boldsymbol{\xi}_2'} \hat{f}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'.$$

Accordingly, the spatial Fourier transform of $|\nabla_{\mathbf{x}}| u(t,\mathbf{x},\mathbf{x},\mathbf{x})$ is

$$|\boldsymbol{\xi}_1| \int e^{i((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{f}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2',$$

which allows us to compute

$$\begin{aligned}
&\left|\int |\nabla_{\mathbf{x}}| u(t,\mathbf{x},\mathbf{x},\mathbf{x})h(t,\mathbf{x})d\mathbf{x}dt\right|^2 \\
= &\left|\int |\boldsymbol{\xi}_1| e^{i((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{f}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')\right. \\
&\left.\hat{h}(t,\boldsymbol{\xi}_1)dt d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'\right|^2 \text{ (spatial Fourier transform on } h) \\
= &\left|\int \left(\int |\boldsymbol{\xi}_1| e^{i((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{h}(t,\boldsymbol{\xi}_1)dt\right)\right. \\
&\left.\hat{f}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2',\boldsymbol{\xi}_2,\boldsymbol{\xi}_2')d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'\right|^2 \\
\leqslant\ & I(h)\left\|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_2'}f\right\|_{L^2}^2 \text{ (Cauchy-Schwarz)},
\end{aligned}$$

where

$$I(h) = \int \frac{|\boldsymbol{\xi}_1|^2 \left|\int e^{i((\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \hat{h}(t,\boldsymbol{\xi}_1)dt\right|^2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2 |\boldsymbol{\xi}_2'|^2} d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'.$$

So the target of the remainder of this section is to show

$$I(h) \leqslant C \|h\|_{L^2}^2 .$$

Noticing that the integral $I(h)$ is symmetric in $|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|$ and $|\boldsymbol{\xi}_2|$, it suffices that we deal with the region $|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'| > |\boldsymbol{\xi}_2|$ only. We separate this region into two parts, which we refer to as Cases I and II.

When the "$\pm$" sign in equation (2.2) is " $+$ ", Case I is sufficient. To show the estimate for the " $-$ " sign, we need both Cases I and II.

Away from $|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'| > |\boldsymbol{\xi}_2|$, there are other restrictions on the integration regions in Cases I and II. We state the restrictions in the beginning of both Cases I and II. Due to the limited space near "$\int$", we omit the actual region. The reader should bear this in mind during reading.

2.2.1 Case I: $I(h)$ restricted to the region $\left|\boldsymbol{\xi}_2'\right| < |\boldsymbol{\xi}_2|$ with integration order $d\boldsymbol{\xi}_2$ prior to $d\boldsymbol{\xi}_2'$

Write the phase function of the dt integral inside $I(h)$ as

$$\begin{aligned}
& (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2' \\
= & \frac{(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2')}{2} + 2\left(\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2}\right)^T A_t \left(\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2}\right) \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2'.
\end{aligned}$$

The change of variable

$$\boldsymbol{\xi}_{2,new} = \boldsymbol{\xi}_{2,old} - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2} \tag{2.7}$$

leads to

$$
\begin{aligned}
I(h) &= \int \frac{\left|\boldsymbol{\xi}_1\right|^2 \left|\int e^{i(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T A_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')}\hat{h}(t,\boldsymbol{\xi}_1)dt\right|^2}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2} d\xi_1 d\xi_2 d\xi_2' \\
&= \int \frac{\left|\boldsymbol{\xi}_1\right|^2}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2} e^{i(2\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T A_t(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2')} \\
&\quad e^{-i(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T A_{t'}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2}+2\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2 \pm (\boldsymbol{\xi}_2')^T A_{t'} \boldsymbol{\xi}_2')}\hat{h}(t,\boldsymbol{\xi}_1)\overline{\hat{h}(t',\boldsymbol{\xi}_1)}dtdt'd\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2' \\
&= \int d\boldsymbol{\xi}_1 \int J(\overline{\hat{h}})(t,\boldsymbol{\xi}_1)\hat{h}(t,\boldsymbol{\xi}_1)dt
\end{aligned}
$$

where

$$
\begin{aligned}
J(\overline{\hat{h}})(t,\boldsymbol{\xi}_1) &= \int \frac{\left|\boldsymbol{\xi}_1\right|^2 e^{i2\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} e^{-i2\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2} \\
&\quad e^{i(\frac{(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')^T\left(A_t-A_{t'}\right)(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2')}{2} \pm (\boldsymbol{\xi}_2')^T(A_t-A_{t'})\boldsymbol{\xi}_2')}\overline{\hat{h}(t',\boldsymbol{\xi}_1)}dt'd\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'.
\end{aligned}
$$

Assume for the moment that

$$\int \left|J(\overline{\hat{h}})(t,\boldsymbol{\xi}_1)\right|^2 dt \leqslant C\left\|\hat{h}(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2$$

with C independent of h or $\boldsymbol{\xi}_1$, then

$$I(h) \leqslant C\int d\boldsymbol{\xi}_1 \left\|\hat{h}(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2.$$

Hence we end Case I by this proposition.

Proposition 1

$$\int \left|J(f)(t,\boldsymbol{\xi}_1)\right|^2 dt \leqslant C\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}^2$$

where C is independent of f or $\boldsymbol{\xi}_1$.

Remark 5 *To avoid confusing notation in the proof of the proposition, we use $f(t',\boldsymbol{\xi}_1)$ to replace $\overline{\hat{h}(t',\boldsymbol{\xi}_1)}$.*

Proof. Again, by duality, we just need to prove

$$\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \leqslant C\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}\left\|g\right\|_{L_t^2}.$$

For convenience, let

$$\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2') = \frac{\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'\right)^T A_t\left(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'\right)}{2} \pm \left(\boldsymbol{\xi}_2'\right)^T A_t\boldsymbol{\xi}_2'.$$

Then

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right| \\
= &\left|\int dtdt'd\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2' \frac{\left|\boldsymbol{\xi}_1\right|^2 e^{i2\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2} e^{-i2\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2}\left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right)\right. \\
&\left.\left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1)\right)\right| \\
= &\left|\int \frac{\left|\boldsymbol{\xi}_1\right|^2 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2'\right|^2}\left(\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}\left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right)dt\right)\right. \\
&\left.\left(\int e^{-2i\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}\left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1)\right)dt'\right)\right| \\
\leqslant &\int \frac{\left|\boldsymbol{\xi}_1\right|^2 d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_2'\right|^2}\int \frac{d\boldsymbol{\xi}_2}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2} \\
&\left|\int e^{2i\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}\left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t)}\right)dt\right|\left|\int e^{-2i\boldsymbol{\xi}_2^T A_{t'}\boldsymbol{\xi}_2}\left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}f(t',\boldsymbol{\xi}_1)\right)dt'\right|
\end{aligned}$$

To deal with the dt and dt' integrals, for every fixed $\boldsymbol{\xi}_2$, let

$$u(t) = 2\frac{\boldsymbol{\xi}_2^T A_t\boldsymbol{\xi}_2}{\left|\boldsymbol{\xi}_2\right|^2}$$

then

$$\frac{du}{dt} = 2\frac{a_1(t)\xi_{2,1}^2 + a_2(t)\xi_{2,2}^2 + a_3(t)\xi_{2,3}^2}{\left|\boldsymbol{\xi}_2\right|^2} \geqslant 2c_0 > 0$$

which provides a well-defined inverse $t(u)$.

Consequently, the integral

$$\int e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)}\right) dt = \overline{\int e^{-iu|\boldsymbol{\xi}_2|^2} \left(e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t(u)) \left|\frac{dt}{du}\right|\right) du},$$

is indeed the Fourier transform of

$$G(u) = e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t(u)) \left|\frac{dt}{du}\right|.$$

This is well-defined since

$$\begin{aligned}
\int_{\mathbb{R}} |G(u)|^2 \, du &= \int_{\mathbb{R}} \left|\overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t(u))} \left|\frac{dt}{du}\right|\right|^2 du = \int_{\mathbb{R}} |g(t)|^2 \left|\frac{dt}{du}\right| dt \\
&\leqslant \frac{1}{2c_0} \|g(\cdot)\|_{L_t^2}^2 .
\end{aligned}$$

Hence

$$\begin{aligned}
& \left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)} dt\right| \\
\leqslant & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{d\boldsymbol{\xi}_2}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2} \\
& \left|\int e^{2i\boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)}\right) dt\right| \left|\int e^{-2i\boldsymbol{\xi}_2^T A_{t'} \boldsymbol{\xi}_2} \left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t',\boldsymbol{\xi}_1)\right) dt'\right| \\
= & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{\left|\overline{\hat{G}(|\boldsymbol{\xi}_2|^2)} \hat{F}(|\boldsymbol{\xi}_2|^2, \boldsymbol{\xi}_1)\right|}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2} d\boldsymbol{\xi}_2 \\
= & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \int \frac{\left|\hat{F}(\rho^2,\boldsymbol{\xi}_1)\overline{\hat{G}(\rho^2)}\right|}{\left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2} \rho^2 d\rho d\boldsymbol{\sigma} \\
& \text{(spherical coordinate in } \boldsymbol{\xi}_2) \\
\leqslant & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_2'|^2} \sup_\rho \left(\int \frac{\rho^2 d\boldsymbol{\sigma}}{\rho \left|\boldsymbol{\xi}_2 - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 \left|\boldsymbol{\xi}_2 + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2}{2}\right|^2}\right) \\
& \left(\int \left|\hat{F}(\rho^2,\boldsymbol{\xi}_1)\right|^2 \rho d\rho\right)^{\frac{1}{2}} \left(\int \left|\hat{G}(\rho^2)\right|^2 \rho d\rho\right)^{\frac{1}{2}} \\
& \text{(Hölder in } \rho)
\end{aligned}$$

$$\leqslant C\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}\left\|g\right\|_{L_t^2}\left\{\int\frac{\left|\boldsymbol{\xi}_1\right|^2}{\left|\boldsymbol{\xi}_2'\right|^2}\sup_{\rho}\left(\int\frac{\rho^2 d\boldsymbol{\sigma}}{\rho\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2}\right)d\boldsymbol{\xi}_2'\right\}$$

However,

$$\begin{aligned}
&\int\frac{\left|\boldsymbol{\xi}_1\right|^2}{\left|\boldsymbol{\xi}_2'\right|^2}\sup_{\rho}\left(\int\frac{\rho^2 d\boldsymbol{\sigma}}{\rho\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2\left|\boldsymbol{\xi}_2+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2}\right)d\boldsymbol{\xi}_2' \\
= &\int\frac{\left|\boldsymbol{\xi}_1\right|^2}{\left|\boldsymbol{\xi}_2'\right|^2}\sup_{\rho}\left(\int\frac{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 d\boldsymbol{\sigma}}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^2\left|\boldsymbol{\xi}_2\right|^2}\right)d\boldsymbol{\xi}_2' \\
&\text{(Reverse the change of variable in formula (2.7).)} \\
= &\left|\boldsymbol{\xi}_1\right|^2\int\frac{d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_2'\right|^{2+2\varepsilon}}\sup_{\rho}\left(\int\frac{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^2 d\boldsymbol{\sigma}}{\left|\boldsymbol{\xi}_2-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'\right|^{2-\varepsilon}\left|\boldsymbol{\xi}_2\right|^{2-\varepsilon}}\right) \\
\leqslant &\ C\left|\boldsymbol{\xi}_1\right|^2\int\frac{d\boldsymbol{\xi}_2'}{\left|\boldsymbol{\xi}_2'\right|^{2+2\varepsilon}\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'\right|^{3-2\varepsilon}}\ \text{(Second part of Lemma 1)} \\
\leqslant &\ C.
\end{aligned}$$

In the above calculation, the $\boldsymbol{\sigma}$ in the first line lives on the unit sphere centered at the origin while the $\boldsymbol{\sigma}$ in the second line is on a unit sphere centered at $\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}$. We use the same symbol because Lebesgue measure is translation invariant.

Thus,

$$\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right|\leqslant C\left\|f(\cdot,\boldsymbol{\xi}_1)\right\|_{L_t^2}\left\|g\right\|_{L_t^2}.$$

■

Remark 6 *Because the integral $I(h)$ is also symmetric in $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_2'$ when the "$\pm$" in equation (2.2) is "$+$", we have acquired the estimate in that case. In Case II, we will assume that "$\pm$" is "$-$".*

2.2.2 Case II: $I(h)$ restricted to the region $\left|\boldsymbol{\xi}_2'\right| > |\boldsymbol{\xi}_2|$ with integration order $d\boldsymbol{\xi}_2'$ prior to $d\boldsymbol{\xi}_2$

This time we write the phase function to be

$$
\begin{aligned}
& (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2')^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2') + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 - (\boldsymbol{\xi}_2')^T A_t \boldsymbol{\xi}_2' \\
= \; & (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) - 2 (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2' + \boldsymbol{\xi}_2^T A_t \boldsymbol{\xi}_2 \\
= \; & \phi(t, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) - 2 (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'.
\end{aligned}
$$

and let

$$
\begin{aligned}
J\left(\hat{h}\right)(t, \boldsymbol{\xi}_1) = & \int \frac{|\boldsymbol{\xi}_1|^2 e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} e^{2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2'}}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2 |\boldsymbol{\xi}_2'|^2} \\
& e^{-i\phi(t', \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} \overline{e^{-i\phi(t, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2')}} \hat{h}(t', \boldsymbol{\xi}_1) dt' d\boldsymbol{\xi}_2' d\boldsymbol{\xi}_2.
\end{aligned}
$$

Again, we want to prove

Proposition 2

$$
\int |J(f)(t, \boldsymbol{\xi}_1)|^2 \, dt \leqslant C \, \|f(\cdot, \boldsymbol{\xi}_1)\|_{L_t^2}^2
$$

where C is independent of f or $\boldsymbol{\xi}_1$.

Proof. We calculate

$$
\begin{aligned}
& \left| \int J(f)(t, \boldsymbol{\xi}_1) \overline{g(t)} dt \right| \\
= & \left| \int \frac{|\boldsymbol{\xi}_1|^2 e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} e^{2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2'}}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2 |\boldsymbol{\xi}_2'|^2} \left(\overline{e^{-i\phi(t, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} g(t)} \right) \right. \\
& \left. \left(e^{-i\phi(t', \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} f(t', \boldsymbol{\xi}_1) \right) dt dt' d\boldsymbol{\xi}_2' d\boldsymbol{\xi}_2 \right| \\
= & \left| \int \frac{|\boldsymbol{\xi}_1|^2 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 |\boldsymbol{\xi}_2|^2 |\boldsymbol{\xi}_2'|^2} \left(\int e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} g(t)} \right) dt \right) \right. \\
& \left. \left(\int e^{2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2} \left(e^{-i\phi(t', \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} f(t', \boldsymbol{\xi}_1) \right) dt' \right) \right|
\end{aligned}
$$

$$
\begin{aligned}
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|^2 \, d\boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2} \int \frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2'|^2 \, |\boldsymbol{\xi}_2'|^2} \left| \int e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} g(t)} \right) dt \right| \\
& \left| \int e^{2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_{t'} \boldsymbol{\xi}_2'} \left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)} f(t', \boldsymbol{\xi}_1) \right) dt' \right|
\end{aligned}
$$

Fix $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_2'$, write

$$
\int e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)} \right) dt = \int e^{-2i|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\omega^T A_t \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)} \right) dt
$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a unit vector in $\mathbb{R}^3$. Without loss of generality, we assume

$$
\max \{|\omega_1|, |\omega_2|, |\omega_3|\} = |\omega_1|
$$

which implies

$$
\frac{1}{\sqrt{3}} \leqslant |\omega_1| \leqslant 1.
$$

Let us further assume that $\omega_1 > 0$ (the proof works exactly the same for the $\omega_1 < 0$ case), then we can write

$$
\begin{aligned}
\boldsymbol{\xi}_2' &= (x, 0, 0) + (0, y_1, y_2) \\
u(t) &= 2\omega_1 \int_0^t a_1(s) ds.
\end{aligned}
$$

Again u is invertible with

$$
\frac{du}{dt} \geqslant \frac{2c_0}{\sqrt{3}} > 0.
$$

So we have

$$
\begin{aligned}
& \int e^{-2i(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^T A_t \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)} \right) dt \\
=\ & \int e^{-2i|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\boldsymbol{\omega}^T A_{t'} \boldsymbol{\xi}_2'} \left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t)} \right) dt \\
=\ & \int e^{-iu(\omega_1 |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| x)} \left(e^{-2i|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)} \overline{e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')} g(t(u))} \left| \frac{dt}{du} \right| \right) du \\
=\ & \overline{\hat{G}(-\omega_1 |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| x)}
\end{aligned}
$$

where

$$G(u)=\overline{e^{-2i|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|(0,\omega_2,\omega_3)^T A_{t(u)}(0,y_1,y_2)}}e^{-i\phi(t(u),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2')}g(t(u))\left|\frac{dt}{du}\right|$$

which still has the property that

$$\int |G(u)|^2\,du \leqslant \frac{\sqrt{3}}{2c_0}\int |g(t)|^2\,dt.$$

Just as in case 1, this procedure hands us

$$\begin{aligned}
&\left|\int J(f)(t,\boldsymbol{\xi}_1)\overline{g(t)}dt\right|\\
\leqslant\ &\int \frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2}{|\boldsymbol{\xi}_2|^2}\int \frac{d\boldsymbol{\xi}_2'}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\int e^{-2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_t\boldsymbol{\xi}_2'}\left(\overline{e^{-i\phi(t,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}g(t)}\right)dt\right|\\
&\left|\int e^{2i(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)^T A_{t'}\boldsymbol{\xi}_2'}\left(e^{-i\phi(t',\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)}f(t',\boldsymbol{\xi}_1)\right)dt'\right|\\
=\ &\int \frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_2|^2}d\boldsymbol{\xi}_2\\
&\left(\int \frac{dx dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\overline{\hat{G}(-\omega_1\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,x)}\hat{F}(-\omega_1\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,x,\boldsymbol{\xi}_1)\right|\right)\\
=\ &\int\left(\int \frac{dx dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\left|\overline{\hat{G}(x)}\hat{F}(x,\boldsymbol{\xi}_1)\right|\right)\frac{|\boldsymbol{\xi}_1|^2}{|\omega_1|\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}d\boldsymbol{\xi}_2\\
\leqslant\ & C\int \frac{|\boldsymbol{\xi}_1|^2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int \frac{dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right)\\
&\left(\int\left|\hat{F}(x,\boldsymbol{\xi}_1)\right|^2dx\right)^{\frac{1}{2}}\left(\int\left|\hat{G}(x)\right|^2dx\right)^{\frac{1}{2}}d\boldsymbol{\xi}_2\ \text{(Hölder in } x\text{)}\\
\leqslant\ & C\,\|f(\cdot,\boldsymbol{\xi}_1)\|_{L_t^2}\,\|g\|_{L_t^2}\int \frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int \frac{dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right)d\boldsymbol{\xi}_2
\end{aligned}$$

The first part of Lemma 1 and the restrictions that $|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|>|\boldsymbol{\xi}_2|$ and $|\boldsymbol{\xi}_2'|<|\boldsymbol{\xi}_2|$ show

$$\begin{aligned}
&\int \frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^2}\left(\sup_x\int \frac{dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^2\,|\boldsymbol{\xi}_2'|^2}\right)d\boldsymbol{\xi}_2\\
\leqslant\ &\int \frac{|\boldsymbol{\xi}_1|^2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^{2+2\varepsilon}}\left(\sup_x\int \frac{dy_1 dy_2}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^{2-\varepsilon}\,|\boldsymbol{\xi}_2'|^{2-\varepsilon}}\right)d\boldsymbol{\xi}_2
\end{aligned}$$

$$\leqslant C\int\frac{|\boldsymbol{\xi}_1|^2\,d\boldsymbol{\xi}_2}{2\,|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^{3-2\varepsilon}\,|\boldsymbol{\xi}_2|^{2+2\varepsilon}}$$
$$\leqslant C,$$

which finishes the proposition. ■

2.3 Proof of Theorem 3 when $n=2$ / 3*2d Collapsing Estimate

By the proof of the $n=3$ case in Section 2.2, we only need to show these two estimates:

Case I Under the restrictions $\left|\boldsymbol{\xi}_1-\boldsymbol{\xi}_{2,old}-\boldsymbol{\xi}_2'\right|>\left|\boldsymbol{\xi}_{2,old}\right|$ and $|\boldsymbol{\xi}_2'|<\left|\boldsymbol{\xi}_{2,old}\right|$, we have

$$\int\frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_2'|}\sup_{\rho}\left(\int\frac{d\boldsymbol{\sigma}(\boldsymbol{\xi}_{2,new})}{\left|\boldsymbol{\xi}_{2,new}-\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|\left|\boldsymbol{\xi}_{2,new}+\frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|}\right)d\boldsymbol{\xi}_2'\leqslant C$$

where $\boldsymbol{\xi}_{2,new}$ and $\boldsymbol{\xi}_{2,old}$ are related by formula (2.7) and we write

$$\boldsymbol{\xi}_{2,new}=\rho\boldsymbol{\sigma}\text{ with }\boldsymbol{\sigma}\in\mathbb{S}^1.$$

Case II Under the restrictions $|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|>|\boldsymbol{\xi}_2|$ and $|\boldsymbol{\xi}_2'|>|\boldsymbol{\xi}_2|$, we have

$$\int\frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|}\left(\sup_x\int\frac{dy}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|\,|\boldsymbol{\xi}_2'|}\right)d\boldsymbol{\xi}_2\leqslant C.$$

where $\boldsymbol{\xi}_2'=(x,y)$.

Lemma 1 plays an important role in giving the corresponding estimates in Section 2.2. In the 2d case, the subsequent lemma provides its replacement.

Lemma 2 *Let $\boldsymbol{\xi}\in\mathbb{R}^2$ and L be a 1d line or circle in $\mathbb{R}^2$ with the usual induced line element dS.*

(1) Say $0 < a,\, b < 1,\, a + b > 1$*, then there exists a* C *independent of* L *s.t.*

$$\int_L \frac{dS(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^a\,|\boldsymbol{\eta}|^b} \leqslant \frac{C}{|\boldsymbol{\xi}|^{a+b-1}}.$$

(2) Let $\varepsilon = \frac{1}{80}$*, then*

$$\sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}\,|\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\boldsymbol{\xi}|^{2-2\varepsilon}}.$$

Proof. We will show the second part in the end of this section. The first part shares the exact same proof with Lemma 2.2 in [29]. ■

2.3.1 Proof of Case I

The change of variable (2.7) turns the restrictions into

$$\begin{aligned}
\left|\boldsymbol{\xi}_{2,new} - \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2}\right| &= \left|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_{2,old} - \boldsymbol{\xi}_2'\right| > \left|\boldsymbol{\xi}_{2,old}\right| > \left|\boldsymbol{\xi}_2'\right|, \\
\left|\boldsymbol{\xi}_{2,new} + \frac{\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2'}{2}\right| &= \left|\boldsymbol{\xi}_{2,old}\right| > \left|\boldsymbol{\xi}_2'\right|.
\end{aligned}$$

Notice that $\boldsymbol{\xi}_{2,new}=\rho\boldsymbol{\sigma}$, we in fact have

$$\begin{aligned}
&\int \frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_2'|}\sup_{\rho}\left(\int \frac{d\boldsymbol{\sigma}\left(\boldsymbol{\xi}_{2,new}\right)}{\left|\boldsymbol{\xi}_{2,new} - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|\left|\boldsymbol{\xi}_{2,new} + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|}\right) d\boldsymbol{\xi}_2' \\
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_2'|^{1+2\varepsilon}}\sup_{\rho}\left(\int_{\mathbb{S}^1} \frac{d\boldsymbol{\sigma}\left(\boldsymbol{\xi}_{2,new}\right)}{\left|\boldsymbol{\xi}_{2,new} - \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^{1-\varepsilon}\left|\boldsymbol{\xi}_{2,new} + \frac{\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'}{2}\right|^{1-\varepsilon}}\right) d\boldsymbol{\xi}_2' \\
\leqslant\ & C\,|\boldsymbol{\xi}_1|\int \frac{1}{|\boldsymbol{\xi}_2'|^{1+2\varepsilon}}\frac{1}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2'|^{2-2\varepsilon}}d\boldsymbol{\xi}_2' \text{ (Second part of Lemma 2)} \\
\leqslant\ & C.
\end{aligned}$$

2.3.2 Proof of Case II

Recall that $\boldsymbol{\xi}_2' = (x, y)$, we estimate

$$\begin{aligned}
&\int \frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|}\left(\sup_x \int \frac{dy}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|\,|\boldsymbol{\xi}_2'|}\right)d\boldsymbol{\xi}_2 \\
\leqslant\ & \int \frac{|\boldsymbol{\xi}_1|}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\,|\boldsymbol{\xi}_2|^{1+2\varepsilon}}\left(\sup_x \int \frac{dy}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2-\boldsymbol{\xi}_2'|^{1-\varepsilon}\,|\boldsymbol{\xi}_2'|^{1-\varepsilon}}\right)d\boldsymbol{\xi}_2 \\
\leqslant\ & C\,|\boldsymbol{\xi}_1| \int \frac{1}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^{2-2\varepsilon}\,|\boldsymbol{\xi}_2|^{1+2\varepsilon}}d\boldsymbol{\xi}_2 \text{ (First part of Lemma 2)} \\
\leqslant\ & C.
\end{aligned}$$

2.3.3 Proof of the Second Part of Lemma 2

Due to

$$|\boldsymbol{\xi}| \leqslant |\boldsymbol{\xi}-\boldsymbol{\eta}| + |\boldsymbol{\xi}+\boldsymbol{\eta}|,$$

we can separate the integral as

$$\begin{aligned}
&\sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}\,|\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}}\right) \\
\leqslant\ & \sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}-\boldsymbol{\eta}|\geqslant \frac{|\boldsymbol{\xi}|}{2}}\right) + \sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}+\boldsymbol{\eta}|\geqslant \frac{|\boldsymbol{\xi}|}{2}}\right).
\end{aligned}$$

We will only show

$$\sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}+\boldsymbol{\eta}|\geqslant \frac{|\boldsymbol{\xi}|}{2}} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}\,|\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\boldsymbol{\xi}|^{2-2\varepsilon}}$$

since the other part is similar. It is clear that

$$\sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}+\boldsymbol{\eta}|\geqslant \frac{|\boldsymbol{\xi}|}{2}} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}\,|\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}}\right) \leqslant \frac{C}{|\boldsymbol{\xi}|^{1-\varepsilon}} \sup_{|\boldsymbol{\eta}|}\left(\int_{\mathbb{S}^1} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}}\right). \quad (2.8)$$

Rotate $\mathbb{S}^1$ such that $\boldsymbol{\xi}$ is on the positive x axis, then write $\boldsymbol{\eta} = \rho e^{i\theta}$ for $(\rho\cos\theta, \rho\sin\theta)$ and observe:

- When $\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$,

$$\left|\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)\right| \geqslant |\boldsymbol{\xi}| \, |\sin\theta|$$

because $|\boldsymbol{\xi}| \, |\sin\theta|$ is the distance between the point $(|\boldsymbol{\xi}|, 0)$ and the line $(angle = \theta)$.

- When $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$,

$$\left|\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)\right| \geqslant |\boldsymbol{\xi}|$$

because $\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)$ is the longest edge in the obtuse triangle which consists of $\rho e^{i\theta}$, $(|\boldsymbol{\xi}|, 0)$ and $\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)$.

Insert these two elementary observations into estimate (2.8), we have

$$\begin{aligned}
& \sup_{|\boldsymbol{\eta}|} \left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}+\boldsymbol{\eta}| \geqslant \frac{|\boldsymbol{\xi}|}{2}} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon} |\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}} \right) \\
\leqslant \; & \frac{C}{|\boldsymbol{\xi}|^{1-\varepsilon}} \sup_{|\boldsymbol{\eta}|} \left(\int_{\mathbb{S}^1} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon}} \right) \\
\leqslant \; & \frac{C}{|\boldsymbol{\xi}|^{1-\varepsilon}} \left[\sup_{\rho} \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{|\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)|^{1-\varepsilon}} \right) + 2 \sup_{\rho} \left(\int_{0}^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta} - (|\boldsymbol{\xi}|, 0)|^{1-\varepsilon}} \right) \right] \\
\leqslant \; & \frac{C}{|\boldsymbol{\xi}|^{1-\varepsilon}} \left[\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{|\boldsymbol{\xi}|^{1-\varepsilon}} \right) + 2 \left(\int_{0}^{\frac{\pi}{2}} \frac{d\theta}{||\boldsymbol{\xi}| \sin\theta|^{1-\varepsilon}} \right) \right] \\
\leqslant \; & \frac{C}{|\boldsymbol{\xi}|^{2-2\varepsilon}}.
\end{aligned}$$

To show the other part, namely

$$\sup_{|\boldsymbol{\eta}|} \left(\int_{\mathbb{S}^1 \text{ and } |\boldsymbol{\xi}-\boldsymbol{\eta}| \geqslant \frac{|\boldsymbol{\xi}|}{2}} \frac{d\boldsymbol{\sigma}(\boldsymbol{\eta})}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{1-\varepsilon} |\boldsymbol{\xi}+\boldsymbol{\eta}|^{1-\varepsilon}} \right) \leqslant \frac{C}{|\boldsymbol{\xi}|^{2-2\varepsilon}},$$

one just needs to notice

$$|\boldsymbol{\xi}+\boldsymbol{\eta}| = \left|(|\boldsymbol{\xi}|, 0) - \rho e^{i(\theta+\pi)}\right|,$$

then one can proceed as above. Therefore we conclude the proof of the second part of Lemma 2.

2.4 The Lens Transform / Preparation for Theorem 5

From now on, we enter the proof of Theorems 5 and 6. We set $n = 3$ until Section 2.7. In this section, we set up the tools involved in the proof of Theorem 5. We build the lens transform we need and state the related properties. For simplicity of notations, we write $U^{(k+1)}(\tau; s)$ to be the solution operator of equation (2.5) and $U_{\mathbf{y}}(\tau; s)$ to be the solution operator of

$$\begin{aligned} \left(i\partial_\tau - \frac{1}{2}H_{\mathbf{y}}(\tau)\right) u &= 0 \\ u(s, \mathbf{y}) &= u_s(\mathbf{y}). \end{aligned}$$

i.e. $U^{(k+1)}(\tau; s)\gamma_0^{(k+1)}$ solves equation (2.5). By definition,

$$U^{(k)}(\tau; s) = \prod_{j=1}^{k} \left(U_{\mathbf{y}_j}(\tau; s) U_{\mathbf{y}'_j}(-\tau; -s) \right).$$

To be specific, we need this version of the generalized lens transform:

Proposition 3 *There is an operator $L_{\mathbf{x}}(t)$ which satisfies the hypothesis in Theorem 3 such that*

$$\begin{aligned} & U^{(k+1)}(\tau; 0)\gamma_0^{(k+1)} \\ = & \prod_{j=1}^{k+1} \left(\prod_{l=1}^{3} \frac{e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)} \frac{\left(|y_{j,l}|^2 - |y'_{j,l}|^2\right)}{2}}}{\beta_l(\tau)} \right) \\ & u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)}, \frac{y_{1,1}}{\beta_1(\tau)}, \frac{y_{1,2}}{\beta_2(\tau)}, \frac{y_{1,3}}{\beta_3(\tau)}, \ldots, \frac{y_{k+1,1}}{\beta_1(\tau)}, \frac{y_{k+1,2}}{\beta_2(\tau)}, \frac{y_{k+1,3}}{\beta_3(\tau)};\right. \\ & \left.\frac{y'_{1,1}}{\beta_1(\tau)}, \frac{y'_{1,2}}{\beta_2(\tau)}, \frac{y'_{1,3}}{\beta_3(\tau)}, \ldots, \frac{y'_{k+1,1}}{\beta_1(\tau)}, \frac{y'_{k+1,2}}{\beta_2(\tau)}, \frac{y'_{k+1,3}}{\beta_3(\tau)}\right) \end{aligned}$$

in $[-T_0,T_0]$*, where* α_l *and* β_l *are defined as in Claim 1, and* $u^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})$ *is the solution of*

$$\begin{aligned}\left(i\partial_t+L_{\overrightarrow{\mathbf{x}_{k+1}}}(t)-L_{\overrightarrow{\mathbf{x}'_{k+1}}}(t)\right)u^{(k+1)} &= 0 \text{ in } \mathbb{R}^{(6k+6)+1}\\ u^{(k+1)}(0,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}}) &= \gamma_0^{(k+1)}.\end{aligned}$$

The proposition will be a corollary of a sequence of claims.

Claim 1 *Assuming Conditions 1 and 2, for* $l=1,2,3$*, the system*

$$\begin{aligned}\ddot{\alpha}_l(\tau)+\eta_l(\tau)\alpha_l(\tau) &= 0,\alpha_l(0)=0,\dot{\alpha}_l(0)=1, \\ \ddot{\beta}_l(\tau)+\eta_l(\tau)\beta_l(\tau) &= 0,\beta_l(0)=1,\dot{\beta}_l(0)=0.\end{aligned} \tag{2.9}$$

defines an odd α_l *and an even* $\beta_l\in C^2(\mathbb{R})$ *with the following properties*

(1) β_l *is nonzero in* $[-T_0,T_0]$;

(2) The Wronskian of α_l *and* β_l *is constant* 1 *i.e.*

$$\dot{\alpha}_l(\tau)\beta_l(\tau)-\alpha_l(\tau)\dot{\beta}_l(\tau)=1;$$

(3) The odd function

$$\upsilon_l(\tau)=\frac{\alpha_l(\tau)}{\beta_l(\tau)}$$

is invertible in $[-T_0,T_0]$ *because*

$$\dot{\upsilon}_l(\tau)=\frac{1}{\left(\beta_l(\tau)\right)^2}>0 \text{ in } [-T_0,T_0].$$

Proof. We show (1) only since all other statements are fairly trivial.

Suppose $\beta_l(\tau_0)=0$ for some τ_0 in $[-T_0,T_0]$ then $\beta_l(-\tau_0)=0$ via β_l is even. Of course $\tau_0\neq 0$ because $\beta_l(0)=1$. Notice that $\cos\left(\tau\sqrt{\sup_\tau|\eta_l(\tau)|}\right)$ is a nontrivial solution of

$$\ddot{v}(\tau) + \sup_{\tau} |\eta_l(\tau)| \, v(\tau) = 0.$$

Since $\cos\left(\tau\sqrt{\sup_{\tau} |\eta_l(\tau)|}\right)$ is not a multiple of β_l, $\cos\left(\tau\sqrt{\sup_{\tau} |\eta_l(\tau)|}\right)$ must have at least one zero in $[-\tau_0, \tau_0]$ due to the Sturm–Picone comparison theorem. But this creates a contradiction. ■

Though Claim 1 is elementary, its consequences lying below make our procedure well-defined.

Definition 3 *(A reminder of the norm) Let* β_l *be defined via equation* (2.9). *We define*

$$P_{\mathbf{y}}(\tau) = \begin{pmatrix} i\beta_1(\tau)\frac{\partial}{\partial y_1} + \dot{\beta}_1(\tau)y_1 \\ i\beta_2(\tau)\frac{\partial}{\partial y_2} + \dot{\beta}_2(\tau)y_2 \\ i\beta_3(\tau)\frac{\partial}{\partial y_3} + \dot{\beta}_3(\tau)y_3 \end{pmatrix}$$

and

$$R_{\tau}^{k} = \textstyle\prod_{j=1}^{k} P_{\mathbf{y}_j}(\tau) P_{\mathbf{y}'_j}(-\tau).$$

Lemma 3 $P_{\mathbf{y}}(\tau)$ *commutes with the linear operator*

$$i\partial_\tau - \frac{1}{2}\left(-\triangle_{\mathbf{y}_k} + \eta(\tau)\left|\mathbf{y}_k\right|^2\right).$$

Moreover,

$$P_{\mathbf{y}}(\tau) U_{\mathbf{y}}(\tau; s) f = U_{\mathbf{y}}(\tau; s) P_{\mathbf{y}}(s) f.$$

Lemma 4 *Say* $K_1(t, x_0, y_0)$ *is the Green's function of the 1d free Schrödinger equation*

$$\left(i\partial_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) v = 0,$$

then

$$U_{\mathbf{y}}(\tau;0)u_0 = \left(\prod_{l=1}^{3} \frac{e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)}\frac{y_l^2}{2}}}{(\beta_l(\tau))^{\frac{1}{2}}}\right) \int \left(\prod_{l=1}^{3} K_1(\frac{\alpha_l(\tau)}{\beta_l(\tau)}, \frac{y_l}{\beta_l(\tau)}, y_{0l})\right) u_0(y_{01}, y_{02}, y_{03})dy_{01}dy_{02}dy_{03}, \tag{2.10}$$

valid in the interval $[-T, T]$ *in which* η_l *are Lipschitzian and* $\beta_l(\tau) \neq 0$.

Proof. Carles computed the isotropic case of formula (2.10) in [3]. We include a proof of Lemmas 3 and 4 using the metaplectic representation in Section 2.8. ■

We can now prove Proposition 3. On the one hand, via Claim 1, we can invert

$$t(\tau) = \upsilon_1(\tau) = \frac{\alpha_1(\tau)}{\beta_1(\tau)} \text{ in } [-T_0, T_0].$$

Therefore, the integral part of formula (2.10)

$$\phi(t, \mathbf{x}) = \int \left(K_1(t, x_1, y_{01})K_1(\upsilon_2(\upsilon_1^{-1}(t)), x_2, y_{02})K_1(\upsilon_3(\upsilon_1^{-1}(t)), x_3, y_{03})\right) u_0(y_{01}, y_{02}, y_{03})dy_{01}dy_{02}dy_{03}$$

in fact solves

$$\left(i\partial_t + \widetilde{L_{\mathbf{x}}}(t)\right)\phi = 0 \text{ in } \mathbb{R}^3 \times [-\upsilon_1^{-1}(T_0), \upsilon_1^{-1}(T_0)]$$

$$\phi(0, \mathbf{x}) = u_0,$$

where

$$\widetilde{L_{\mathbf{x}}}(t) = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\frac{\beta_1^2(\upsilon_1^{-1}(t))}{\beta_2^2(\upsilon_1^{-1}(t))}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}\frac{\beta_1^2(\upsilon_1^{-1}(t))}{\beta_3^2(\upsilon_1^{-1}(t))}\frac{\partial^2}{\partial x_3^2}.$$

On the other hand, plugging $-\tau$ into formula (2.10) yields

$$U_{\mathbf{y}}(-\tau;0)u_0 = \left(\prod_{l=1}^{3} \frac{e^{-i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)}\frac{y_l^2}{2}}}{(\beta_l(\tau))^{\frac{1}{2}}}\right) \int \left(\prod_{l=1}^{3} K_1(-\frac{\alpha_l(\tau)}{\beta_l(\tau)}, \frac{y_l}{\beta_l(\tau)}, y_{0l})\right) u_0(y_{01}, y_{02}, y_{03})dy_{01}dy_{02}dy_{03}$$

because α_l and $\dot{\beta}_l$ are odd while β_l are even.

Whence in $[-T_0, T_0]$

$$\begin{aligned}
U^{(k+1)}(\tau;0)\gamma_0^{(k+1)} &= \prod_{j=1}^{k+1}\left(U_{\mathbf{y}_j}(\tau;0)U_{\mathbf{y}'_j}(-\tau;0)\right)\gamma_0^{(k+1)} \\
&= \prod_{j=1}^{k+1}\left(\prod_{l=1}^{3}\frac{e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)}\frac{\left(\left|y_{j,l}\right|^2-\left|y'_{j,l}\right|^2\right)}{2}}}{\beta_l(\tau)}\right) \\
&\quad u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)},\frac{y_{1,1}}{\beta_1(\tau)},\frac{y_{1,2}}{\beta_2(\tau)},\frac{y_{1,3}}{\beta_3(\tau)},...,\frac{y_{k+1,1}}{\beta_1(\tau)},\frac{y_{k+1,2}}{\beta_2(\tau)},\frac{y_{k+1,3}}{\beta_3(\tau)};\right. \\
&\quad \left.\frac{y'_{1,1}}{\beta_1(\tau)},\frac{y'_{1,2}}{\beta_2(\tau)},\frac{y'_{1,3}}{\beta_3(\tau)},...,\frac{y'_{k+1,1}}{\beta_1(\tau)},\frac{y'_{k+1,2}}{\beta_2(\tau)},\frac{y'_{k+1,3}}{\beta_3(\tau)}\right)
\end{aligned}$$

if $u^{(k+1)}(t,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}})$ solves

$$\begin{aligned}
\left(i\partial_t+\widetilde{L_{\overrightarrow{\mathbf{x}_{k+1}}}}(t)-\widetilde{L_{\overrightarrow{\mathbf{x}'_{k+1}}}}(t)\right)u^{(k+1)} &= 0 \text{ in } \mathbb{R}^{6k+6}\times[-\upsilon_1^{-1}(T_0),\upsilon_1^{-1}(T_0)] \\
u^{(k+1)}(0,\overrightarrow{\mathbf{x}_{k+1}};\overrightarrow{\mathbf{x}'_{k+1}}) &= \gamma_0^{(k+1)}.
\end{aligned}$$

At long last, define

$$L_{\mathbf{x}}(t)=\{\begin{array}{c}\widetilde{L_{\mathbf{x}}}(t),\text{ when } t\in[-\upsilon_1^{-1}(T_0),\upsilon_1^{-1}(T_0)] \\ \widetilde{L_{\mathbf{x}}}(\upsilon_1^{-1}(T_0)),\text{ when } t\geqslant\upsilon_1^{-1}(T_0)\text{ or } t\leqslant-\upsilon_1^{-1}(T_0)\end{array}$$

then we obtain the desired variant of the generalized lens transform i.e. Proposition 3.

2.5 Proof of Theorem 5

Without loss of generality, we show Theorem 5 for $B^1_{j,k+1}$ in $B_{j,k+1}$ when j is taken to be 1. This corresponds to the estimate:

$$\int_s^T d\tau \int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}} \left| R_\tau^{(k)} \gamma^{(k+1)}(\tau, \overrightarrow{\mathbf{y}_k}, \mathbf{y}_1; \overrightarrow{\mathbf{y}'_k}, \mathbf{y}_1) \right|^2 d\overrightarrow{\mathbf{y}_k} d\overrightarrow{\mathbf{y}'_k} \tag{2.11}$$
$$\leqslant C \left(\inf_{\tau\in[0,T_0]} \prod_{l=2}^{3} \beta_l^2(\tau) \right)^{-1} \int_{\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)}} \left| R_\tau^{(k+1)} \gamma^{(k+1)}(\tau, \overrightarrow{\mathbf{y}_{k+1}}; \overrightarrow{\mathbf{y}'_{k+1}}) \right|^2 d\overrightarrow{\mathbf{y}_{k+1}} d\overrightarrow{\mathbf{y}'_{k+1}},$$

$\forall \tau \in [s, T]$, if $\gamma^{(k+1)}$ satisfies equation (2.5).

By Proposition 3, we compute

$$R_\tau^{(k)} \gamma^{(k+1)}(\tau, \overrightarrow{\mathbf{y}_k}, \mathbf{y}_1; \overrightarrow{\mathbf{y}'_k}, \mathbf{y}_1) \tag{2.12}$$
$$= \left(\prod_{l=1}^{3} \frac{1}{\beta_l(\tau)} \right) \prod_{j=1}^{k} \left(\prod_{l=1}^{3} \frac{e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)} \frac{\left(\left| y_{j,l} \right|^2 - \left| y'_{j,l} \right|^2 \right)}{2}}}{\beta_l(\tau)} \right)$$
$$\left(\left(\prod_{j=1}^{k} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) u^{(k+1)}\left(\frac{\alpha_1(\tau)}{\beta_1(\tau)}, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1 \right) \right),$$

if we let

$$x_{j,l} = \frac{y_{j,l}}{\beta_l(\tau)} \text{ and } x'_{j,l} = \frac{y'_{j,l}}{\beta_l(\tau)},$$

because of the relations

$$i\beta_l(\tau) \frac{\partial}{\partial y_{j,l}} \left(e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)} \frac{\left| y_{j,l} \right|^2}{2}} \right) + \dot{\beta}_l(\tau) y_{j,l} \left(e^{i\frac{\dot{\beta}_l(\tau)}{\beta_l(\tau)} \frac{\left| y_{j,l} \right|^2}{2}} \right) = 0,$$
$$\beta_l(\tau) \frac{\partial}{\partial y_{j,l}} = \frac{\partial}{\partial x_{j,l}}.$$

Consequently,

$$\int_s^T d\tau \int_{\mathbb{R}^{3k}\times\mathbb{R}^{3k}} \left| R_\tau^{(k)} \gamma^{(k+1)}(\tau, \overrightarrow{\mathbf{y}_k}, \mathbf{y}_1; \overrightarrow{\mathbf{y}'_k}, \mathbf{y}_1) \right|^2 d\overrightarrow{\mathbf{y}_k} d\overrightarrow{\mathbf{y}'_k}$$

$$
\begin{aligned}
&= \int_s^T d\tau \int_{\mathbb{R}^{6k}} \Bigg| \left(\prod_{l=1}^{3} \frac{1}{\beta_l(\tau)}\right)^{k+1} \\
&\quad \left(\prod_{j=1}^{k}\left(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j}\right)\right) u^{(k+1)}\big(\frac{\alpha_1(\tau)}{\beta_1(\tau)}, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1\big)\Bigg|^2 d\overrightarrow{\mathbf{y}_k} d\overrightarrow{\mathbf{y}'_k} \\
&= \int_s^T \frac{d\tau}{(\beta_1(\tau))^2} \int_{\mathbb{R}^{6k}} \left(\prod_{l=2}^{3} \frac{1}{\beta_l(\tau)}\right)^2 \\
&\quad \Bigg| \left(\prod_{j=1}^{k}\left(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j}\right)\right) u^{(k+1)}\big(\frac{\alpha_1(\tau)}{\beta_1(\tau)}, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1\big)\Bigg|^2 d\overrightarrow{\mathbf{x}_k} d\overrightarrow{\mathbf{x}'_k} \\
&\leqslant \left(\inf_{\tau\in[0,T_0]} \prod_{l=2}^{3} \beta_l^2(\tau)\right)^{-1} \int_s^T \frac{d\tau}{(\beta_1(\tau))^2} \\
&\quad \int_{\mathbb{R}^{6k}} \Bigg| \left(\prod_{j=1}^{k}\left(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j}\right)\right) u^{(k+1)}\big(\frac{\alpha_1(\tau)}{\beta_1(\tau)}, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1\big)\Bigg|^2 d\overrightarrow{\mathbf{x}_k} d\overrightarrow{\mathbf{x}'_k} \\
&\leqslant \left(\inf_{\tau\in[0,T_0]} \prod_{l=2}^{3} \beta_l^2(\tau)\right)^{-1} \\
&\quad \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^{6k}} \Bigg| \left(\prod_{j=1}^{k}\left(\nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}'_j}\right)\right) u^{(k+1)}(t, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1)\Bigg|^2 d\overrightarrow{\mathbf{x}_k} d\overrightarrow{\mathbf{x}'_k}
\end{aligned}
$$

where we used the fact that the Wronskian of α_l and β_l is constant 1, i.e.

$$\frac{dt}{d\tau} = \frac{\dot{\alpha}_1(\tau)\beta_1(\tau) - \alpha_1(\tau)\dot{\beta}_1(\tau)}{(\beta_1(\tau))^2} = \frac{1}{(\beta_1(\tau))^2}$$

as shown in Claim 1.

A corollary of Theorem 3 tells us that

Corollary 1 *Let $L_{\mathbf{x}}(t)$ be the same as in Theorem 3 and $u^{(k+1)}$ verify*

$$\left(i\partial_t + L_{\overrightarrow{\mathbf{x}_{k+1}}}(t) - L_{\overrightarrow{\mathbf{x}'_{k+1}}}(t)\right) u^{(k+1)} = 0.$$

Then there is a $C > 0$, independent of j, k, and $u^{(k+1)}$ s.t.

$$\begin{aligned}
& \left\| \left(\prod_{j=1}^{k} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) \left(B^1_{j,k+1} u^{(k+1)} \right) (t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}'_k}) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\
= & \left\| \left(\prod_{j=1}^{k} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) u^{(k+1)}(t, \overrightarrow{\mathbf{x}_k}, \mathbf{x}_1; \overrightarrow{\mathbf{x}'_k}, \mathbf{x}_1) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\
\leqslant & \; C \left\| \left(\prod_{j=1}^{k+1} \left(\nabla_{\mathbf{x}_j} \nabla_{\mathbf{x}'_j} \right) \right) u^{(k+1)}(0, \overrightarrow{\mathbf{x}_{k+1}}; \overrightarrow{\mathbf{x}'_{k+1}}) \right\|_{L^2(\mathbb{R}^{3(k+1)}\times\mathbb{R}^{3(k+1)})},
\end{aligned}$$

Whence inequality 2.11 follows.

2.6 The Uniqueness of Hierarchy 1.7

To get Theorem 6, we of course use the Klainerman-Machedon board game argument to group the terms. For convenience, we assume $b_0 = 1$ here.

Lemma 5 *One can express $\gamma^{(1)}(\tau_1, \cdot; \cdot)$ in the Gross-Pitaevskii hierarchy 1.7 as a sum of at most 4^n terms of the form*

$$\int_D J(\underline{\tau}_{n+1}, \mu_m) d\underline{\tau}_{n+1},$$

or in other words,

$$\gamma^{(1)}(\tau_1, \cdot; \cdot) = \sum_m \int_D J(\underline{\tau}_{n+1}, \mu_m) d\underline{\tau}_{n+1}. \tag{2.13}$$

Here $\underline{\tau}_{n+1} = (\tau_2, \tau_3, ..., \tau_{n+1})$, $D \subset [s, \tau_1]^n$, μ_m are a set of maps from $\{2, ..., n+1\}$ to $\{1, ..., n\}$ satisfying $\mu_m(2) = 1$ and $\mu_m(j) < j$ for all j, and

$$\begin{aligned}
J(\underline{\tau}_{n+1}, \mu_m) \;=\; & U^{(1)}(\tau_1; \tau_2) B_{1,2} U^{(2)}(\tau_2; \tau_3) B_{\mu_m(3),2} \cdots \\
& U^{(n)}(\tau_n; \tau_{n+1}) B_{\mu_m(n+1),n+1}(\gamma^{(n+1)}(\tau_{n+1}, \cdot; \cdot)).
\end{aligned}$$

Proof. The RHS of formula (2.13) is in fact a Duhamel principle. This lemma follows from the proof of Theorem 3.4 in [29] which uses a board game inspired by the Feynman graph argument in [15]. One just needs to replace $e^{i(t_1-t_2)\triangle_y}$ by $U_{\mathbf{y}}(t_1;t_2)$, and $e^{i(t_1-t_2)\triangle^{(k)}}$ by $U^{(k)}(t_1;t_2)$. ■

Let $D_{\tau_2} = \{(\tau_3,...,\tau_{n+1}) \,|\, (\tau_2,\tau_3,...,\tau_{n+1}) \in D\}$ where D is as in Lemma 5. Assuming that we have already verified

$$\left\| R_s^{(1)}\gamma^{(1)}(s,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0,$$

applying Lemma 5 to $[s,\tau_1]\subset[0,T_0]$, we have

$$\begin{aligned}
& \left\| R_{\tau_1}^{(1)}\gamma^{(1)}(\tau_1,\cdot)\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
= & \left\| R_{\tau_1}^{(1)}\int_D U^{(1)}(\tau_1;\tau_2)B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...d\tau_2...d\tau_{n+1}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
= & \left\| \int_s^{\tau_1} U^{(1)}(\tau_1;\tau_2) \right. \\
& \left. \left(\int_{D_{\tau_2}} R_{\tau_2}^{(1)}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...d\tau_3...d\tau_{n+1}\right)d\tau_2\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
& \text{(Lemma 3)} \\
\leqslant & \int_s^{\tau_1}\left\|\int_{D_{\tau_2}} R_{\tau_2}^{(1)}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...d\tau_3...d\tau_{n+1}\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} d\tau_2 \\
\leqslant & \int_{[s,\tau_1]^n}\left\|R_{\tau_2}^{(1)}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...\right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} d\tau_2 d\tau_3...d\tau_{n+1} \\
\leqslant & (\tau_1-s)^{\frac{1}{2}}\int_{[s,\tau_1]^{n-1}} d\tau_3...d\tau_{n+1} \\
& \left\|R_{\tau_2}^{(1)}B_{1,2}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...\right\|_{L^2(\tau_2\in[s,\tau_1]\times\mathbb{R}^3\times\mathbb{R}^3)} \\
\leqslant & C(\tau_1-s)^{\frac{1}{2}}\int_{[s,\tau_1]^{n-1}}\left\|R_{\tau_2}^{(2)}U^{(2)}(\tau_2;\tau_3)B_{\mu_m(3),2}...\right\|_{L^2(\mathbb{R}^6\times\mathbb{R}^6)} d\tau_3...d\tau_{n+1} \\
& \text{(Theorem 5)} \\
& \text{(Same procedure } n-2 \text{ times)}
\end{aligned}$$

$$\leqslant \quad C\left(C\left(\tau_1 - s\right)\right)^{\frac{n-1}{2}} \int_s^{\tau_1} \left\| R^{(n)}_{\tau_{n+1}} B_{\mu_m(n+1),n+1} \gamma^{(n+1)}(\tau_{n+1}, \cdot) \right\|_{L^2(\mathbb{R}^{3n}\times\mathbb{R}^{3n})} d\tau_{n+1}$$
$$\leqslant \quad C\left(C\left(\tau_1 - s\right)\right)^{\frac{n-1}{2}}.$$

Let $(\tau_1 - s)$ be sufficiently small, and $n \to \infty$, we infer that

$$\left\| R^{(1)}_{\tau_1} \gamma^{(1)}(\tau_1, \cdot) \right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0 \text{ in } [s, \tau_1].$$

Similar arguments show that $\left\| R^{(k)}_{\tau} \gamma^{(k)}(\tau, \cdot) \right\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} = 0$, $\forall k, \tau \in [0, T_0]$. Hence we have attained Theorem 6.

2.7 Derivation of the 2d Cubic NLS with Anisotropic Switchable Quadratic Traps / Proof of Theorem 1

For a more comprehensible presentation, let us suppose

$$H_{\mathbf{y}}(\tau) = \sum_{l=1}^{n} \left(-\frac{\partial^2}{\partial y_{j,l}^2} + \eta_l(\tau) y_{j,l}^2 \right)$$

is the ordinary Hermite operator

$$H_{\mathbf{y}} = -\triangle_{\mathbf{y}} + |\mathbf{y}|^2$$

in this section to make formulas shorter and more explicit. We will add two remarks in the proof to address the small modifications needed for the general case.

We start by reviewing the standard Elgart-Erdös-Schlein-Yau program in this setting.

Step A. Observe that, by definition, $\left\{\gamma_N^{(k)}\right\}$ solves the quadratic trap Bogoliubov–

Born–Green–Kirkwood–Yvon (BBGKY) hierarchy

$$\begin{aligned}
&\left(i\partial_\tau-\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}}+\left|\overrightarrow{\mathbf{y}_k}\right|^2\right)+\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}}+\left|\overrightarrow{\mathbf{y}'_k}\right|^2\right)\right)\gamma_N^{(k)} \qquad (2.14)\\
=\;& \frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left(V_N(\mathbf{y}_i-\mathbf{y}_j)-V_N(\mathbf{y}'_i-\mathbf{y}'_j)\right)\gamma_N^{(k)}\\
&+\frac{N-k}{N}\sum_{j=1}^{k}\int dy_{k+1}[(V_N(\mathbf{y}_i-\mathbf{y}_{k+1})-V_N(\mathbf{y}'_i-\mathbf{y}_{k+1}))\\
&\gamma_N^{(k+1)}(\tau,\overrightarrow{\mathbf{y}_k},\mathbf{y}_{k+1};\overrightarrow{\mathbf{y}'_k},\mathbf{y}_{k+1})]
\end{aligned}$$

where $V_N(\mathbf{x})=N^{n\beta}V\left(N^\beta\mathbf{x}\right)$. It converges (at least formally) to the quadratic trap Gross-Pitaevskii infinite hierarchy

$$\begin{aligned}
&\left(i\partial_\tau-\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}}+\left|\overrightarrow{\mathbf{y}_k}\right|^2\right)+\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}}+\left|\overrightarrow{\mathbf{y}'_k}\right|^2\right)\right)\gamma^{(k)} \qquad (2.15)\\
=\;& b_0\sum_{j=1}^{k}B_{j,k+1}\left(\gamma^{(k+1)}\right).
\end{aligned}$$

Prove rigorously that the sequence $\left\{\gamma_N^{(k)}\right\}$ is compact with respect to the weak* topology on the trace class operators and every limit point $\left\{\gamma^{(k)}\right\}$ satisfies hierarchy 2.15.

Step B. Utilize a suitable uniqueness theorem of hierarchy 2.15 to conclude that

$$\gamma^{(k)}(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k})=\prod_{j=1}^{k}\phi(\tau,\mathbf{y}_j)\overline{\phi(\tau,\mathbf{y}'_j)},$$

where ϕ solves the 2d quadratic trap cubic NLS

$$i\partial_\tau\phi=\frac{1}{2}\left(-\triangle+|\mathbf{y}|^2\right)\phi+b_0\phi\left|\phi\right|^2.$$

So the compact sequence $\left\{\gamma_N^{(k)}\right\}$ has only one limit point, i.e.

$$\gamma_N^{(k)}\to\prod_{j=1}^{k}\phi(\tau,\mathbf{y}_j)\overline{\phi(\tau,\mathbf{y}'_j)}$$

in the weak* topology. Since $\gamma^{(k)}$ is an orthogonal projection, the convergence in the weak* topology is equivalent to the convergence in the trace norm topology.

We modify this procedure to show Theorem 1. We remark that the main additional tool is the lens transform. When $H_{\mathbf{y}}(\tau)$ is the Hermite operator, $\alpha_l = \sin\tau$, $\beta_l = \cos\tau$ and $T_0 < \frac{\pi}{2}$ i.e. the lens transform and its inverse reads as follow.

Definition 4 *We define the lens transform* $T_l : L^2(d\overrightarrow{\mathbf{x}_k}d\overrightarrow{\mathbf{x}_k'}) \to L^2(d\overrightarrow{\mathbf{y}_k}d\overrightarrow{\mathbf{y}_k'})$ *and its inverse by*

$$
\begin{aligned}
\left(T_l u^{(k)}\right)(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}_k'}) &= \frac{e^{-i\frac{\tan\tau}{2}\left(\left|\overrightarrow{\mathbf{y}_k}\right|^2 - \left|\overrightarrow{\mathbf{y}_k'}\right|^2\right)}}{(\cos\tau)^{nk}} u^{(k)}(\tan\tau, \frac{\overrightarrow{\mathbf{y}_k}}{\cos\tau}; \frac{\overrightarrow{\mathbf{y}_k'}}{\cos\tau}) \\
\left(T_l^{-1}\gamma^{(k)}\right)(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}_k'}) &= \frac{e^{\frac{it}{2(1+t^2)}\left(\left|\overrightarrow{\mathbf{x}_k}\right|^2 - \left|\overrightarrow{\mathbf{x}_k'}\right|^2\right)}}{(1+t^2)^{\frac{nk}{2}}} \gamma^{(k)}(\arctan t, \frac{\overrightarrow{\mathbf{x}_k}}{\sqrt{1+t^2}}; \frac{\overrightarrow{\mathbf{x}_k'}}{\sqrt{1+t^2}}).
\end{aligned}
$$

T_l is unitary by definition and the variables are related by

$$\tau = \arctan t, \ \mathbf{y}_k = \frac{\mathbf{x}_k}{\sqrt{1+t^2}} \text{ and } \mathbf{y}_k' = \frac{\mathbf{x}_k'}{\sqrt{1+t^2}}.$$

Remark 7 *For the general anisotropic case, we still need the 2d version of Proposition 3.*

Let us write

$$
\begin{aligned}
\left(T_l^{-1}\gamma^{(k)}\right)(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}_k'}) &= \gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}_k'}) \frac{e^{\frac{it}{2(1+t^2)}\left(|\mathbf{x}_k|^2 - \left|\mathbf{x}_k'\right|^2\right)}}{(1+t^2)^{\frac{nk}{2}}} \\
&= \gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}_k'}) h_n^{(k)}(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}_k'}),
\end{aligned}
$$

then we have a more explicit version of Proposition 3.

Proposition 4

$$\begin{aligned}
&\left(i\partial_t+\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}}-\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)\left(T_l^{-1}\gamma^{(k)}\right)(t,\overrightarrow{\mathbf{x}_k};\overrightarrow{\mathbf{x}'_k})\\
=&\ \frac{h_n^{(k)}}{1+t^2}\left[\left(i\partial_\tau-\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}_k}}+\left|\overrightarrow{\mathbf{y}_k}\right|^2\right)+\frac{1}{2}\left(-\triangle_{\overrightarrow{\mathbf{y}'_k}}+\left|\overrightarrow{\mathbf{y}'_k}\right|^2\right)\right)\gamma^{(k)}(\tau,\overrightarrow{\mathbf{y}_k};\overrightarrow{\mathbf{y}'_k})\right]
\end{aligned}$$

Proof. This is a direct computation. ∎

Via this proposition, we understand how the lens transform acts on hierarchies 2.14 and 2.15.

Lemma 6 *(Gross-Pitaevskii hierarchy under the lens transform)* $\left\{\gamma^{(k)}\right\}$ *solves the quadratic trap Gross-Pitaevskii hierarchy 2.15 if and only if* $\left\{u^{(k)}=T_l^{-1}\gamma^{(k)}\right\}$ *solves the infinite hierarchy*

$$\left(i\partial_t+\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}}-\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)u^{(k)}=\frac{(1+t^2)^{\frac{n}{2}}}{1+t^2}b_0\sum_{j=1}^{k}B_{j,k+1}\left(u^{(k+1)}\right). \tag{2.16}$$

In particular, when $n=2$, the lens transform preserves the Gross-Pitaevskii hierarchy.

Lemma 7 *(BBGKY hierarchy under the lens transform)* $\left\{\gamma_N^{(k)}\right\}$ *solves the quadratic trap BBGKY hierarchy 2.14 if and only if* $\left\{u_N^{(k)}=T_l^{-1}\gamma_N^{(k)}\right\}$ *solves the hierarchy*

$$\begin{aligned}
&\left(i\partial_t+\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}}-\frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right)u_N^{(k)} \qquad (2.17)\\
=&\ \frac{1}{N}\frac{1}{1+t^2}\sum_{1\leqslant i<j\leqslant k}\left(V_N\left(\frac{\mathbf{x}_i-\mathbf{x}_j}{\sqrt{1+t^2}}\right)-V_N\left(\frac{\mathbf{x}'_i-\mathbf{x}'_j}{\sqrt{1+t^2}}\right)\right)u_N^{(k)}\\
&+\frac{N-k}{N}\frac{1}{1+t^2}\sum_{j=1}^{k}\int d\mathbf{x}_{k+1}\left[\left(V_N\left(\frac{\mathbf{x}_i-\mathbf{x}_{k+1}}{\sqrt{1+t^2}}\right)-V_N\left(\frac{\mathbf{x}'_i-\mathbf{x}_{k+1}}{\sqrt{1+t^2}}\right)\right)\right.\\
&\left.u_N^{(k+1)}(t,\overrightarrow{\mathbf{x}_k},\mathbf{x}_{k+1};\overrightarrow{\mathbf{x}'_k},\mathbf{x}_{k+1})\right],
\end{aligned}$$

We can now prove Theorem 1.

2.7.1 Proof of Theorem 1

Step 1. Let $n = 2$, consider $\left\{u_N^{(k)} = T_l^{-1}\gamma_N^{(k)}\right\}$ which solves hierarchy 2.17.

Step 2. Write

$$\widetilde{V}(\mathbf{x}) = \frac{1}{1+t^2}V(\frac{\mathbf{x}}{\sqrt{1+t^2}}),$$

then

$$\frac{1}{(1+T^2)^{1-\frac{1}{p}}}\left\|V\right\|_p \leqslant \left\|\widetilde{V}\right\|_p \leqslant \left\|V\right\|_p \text{ when } T < \infty \text{ and } p \geqslant 1.$$

Therefore we can employ the proof in Kirkpatrick-Schlein-Staffilani [27] to show that the sequence $\left\{u_N^{(k)}\right\}$ is compact with respect to the weak* topology on the trace class operators and every limit point $\left\{u^{(k)}\right\}$ satisfies the Gross-Pitaevskii hierarchy 2.16. Moreover, based on a fixed time trace theorem argument as in [27], for $\alpha < 1$, we have

$$\int_0^T dt \left\|\prod_{j=1}^{k}\left(\left\langle\nabla_{\mathbf{x}_j}\right\rangle^{\alpha}\left\langle\nabla_{\mathbf{x}_j}\right\rangle^{\alpha}\right) B_{j,k+1}\left(u^{(k+1)}\right)\right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} \leqslant C^k.$$

for every limit point $\left\{u^{(k)}\right\}$. To be more precise, the proof in [27] involves a smooth approximation. We omit this detail here.

Remark 8 *The auxiliary Hamiltonian*

$$\widetilde{H_N}(t) = \frac{1}{2}\sum_{j=1}^{N} L_{\mathbf{x}_j}(t) + \frac{1}{N}\sum_{i<j} N^{2\beta}\tilde{V}(N^{\beta}\left(\mathbf{x}_i - \mathbf{x}_j\right)).$$

which corresponds to the anisotropic quadratic potential case does not lead to the conservation of the quantity

$$\left\langle \psi_N, \left(\widetilde{H_N}(t)\right)^k \psi_N \right\rangle.$$

On the other hand, the following estimate controls the energy.

$$\begin{aligned}
\frac{d}{dt}\left\langle \psi_N, \left(\widetilde{H_N}(t)\right)^k \psi_N \right\rangle &= \left\langle \psi_N, \left[\frac{d}{dt}, \left(\widetilde{H_N}(t)\right)\right]\left(\widetilde{H_N}(t)\right)^{k-1}\psi_N \right\rangle + ... \\
&\quad + \left\langle \psi_N, \left(\widetilde{H_N}(t)\right)^{k-1}\left[\frac{d}{dt}, \left(\widetilde{H_N}(t)\right)\right]\psi_N \right\rangle \\
&\leqslant Ck\left\langle \psi_N, \left(\widetilde{H_N}(t)\right)^k \psi_N \right\rangle
\end{aligned}$$

since a_1 and a_2, the coefficients of $L_{\mathbf{X}}$, are C^1 in the context of Theorem 1. Thus Gronwall's inequality takes care of the problem for us as long as we are considering finite time.

Step 3. By Theorem 4 (2d uniqueness) or Theorem 7.1 in [27], we deduce that

$$u^{(k)}(t, \overrightarrow{\mathbf{x}_k}; \overrightarrow{\mathbf{x}_k'}) = \prod_{j=1}^{k} \tilde{\phi}(t, \mathbf{x}_j)\overline{\tilde{\phi}(t, \mathbf{x}_j')}$$

where $\tilde{\phi}$ solves the 2d cubic NLS

$$i\partial_t\tilde{\phi} = -\frac{1}{2}\triangle\tilde{\phi} + b_0\tilde{\phi}\left|\tilde{\phi}\right|^2.$$

Hence the compact sequence $\left\{u_N^{(k)}\right\}$ has only one limit point, so

$$u_N^{(k)} \to \prod_{j=1}^{k} \tilde{\phi}(t, \mathbf{x}_j)\overline{\tilde{\phi}(t, \mathbf{x}_j')}$$

in the weak* topology. Since $u^{(k)}$ is an orthogonal projection, the convergence in the weak* topology is equivalent to the convergence in the trace norm topology.

Remark 9 *It is necessary to use Theorem 4 in this paper for the general anisotropic quadratic traps case.*

Step 4. Let ϕ solve the 2d quadratic trap cubic NLS

$$i\partial_\tau \phi = \frac{1}{2}\left(-\triangle + |\mathbf{y}|^2\right)\phi + b_0\phi\,|\phi|^2,$$

then the lens transform of $u^{(k)}$ is

$$\gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k}) = \prod_{j=1}^{k} \phi(\tau, \mathbf{y}_j)\overline{\phi(\tau, \mathbf{y}'_j)},$$

due to the fact that the lens transform preserves mass critical NLS, which is the cubic NLS in 2d.

Step 5. The convergence

$$u_N^{(k)} \to u^{(k)}$$

in the trace norm indicates the convergence in the Hilbert-Schmidt norm. But the lens transform

$$T_l : L^2(d\overrightarrow{\mathbf{x}}\, d\overrightarrow{\mathbf{x}}') \to L^2(d\overrightarrow{\mathbf{y}}\, d\overrightarrow{\mathbf{y}}')$$

is unitary (so preserves the norm) and thus

$$\gamma_N^{(k)} = T_l u_N^{(k)} \to T_l u^{(k)} = \gamma^{(k)}.$$

Thence we conclude that $\gamma_N^{(k)}$ converges to

$$\gamma^{(k)}(\tau, \overrightarrow{\mathbf{y}_k}; \overrightarrow{\mathbf{y}'_k}) = \prod_{j=1}^{k} \phi(\tau, \mathbf{y}_j)\overline{\phi(\tau, \mathbf{y}'_j)},$$

in the Hilbert-Schmidt norm, which is Theorem 1.

2.7.2 Comments about the 3d case

It is natural to wonder what we can say about the 3d case using the above method. Visiting Lemma 6 again yields the hierarchy

$$\left(i\partial_t + \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}_k}} - \frac{1}{2}\triangle_{\overrightarrow{\mathbf{x}'_k}}\right) u^{(k)} = \left(1+t^2\right)^{\frac{1}{2}} b_0 \sum_{j=1}^{k} B_{j,k+1}\left(u^{(k+1)}\right). \tag{2.18}$$

Due to the factor $(1+t^2)^{\frac{1}{2}}$, it is difficult to see of what use a 3d version of Theorem 4 might be. We can certainly give a uniqueness theorem regarding hierarchy 2.18 with the techniques in this paper. But it is unknown how to verify the space-time bound when $n = 3$ as stated earlier,

Another possibility to attack the 3d case is the standard Elgart-Erdos-Schlein-Yau procedure, but we presently know very little about the analysis of the Hermite like operator $H_{\mathbf{y}}(\tau)$.

Finally, we remark that it is not clear whether the Feynman diagrams argument, the key to the uniqueness theorem in [15] on which [14, 15, 16, 17, 18] are based, leads to a 3d uniqueness theorem of hierarchy 1.7 or 2.18, which represent the two sides of the lens transform.

2.8 the Generalized Lens Transform and the Metaplectic Representation

In this section, we prove Lemmas 3 and 4 via the metaplectic representation. The 3d anisotropic case drops out once we show the 1d case. Before we delve into the proof, we remark that we currently do not have an explanation away from direct computations for Proposition 4 or for the fact that the generalized lens transform preserves L^2 critical NLS. The group theory proof presented in this section only shows the linear case: Lemmas 3 and 4.

Through out this section, we consider the metaplectic representation

$$\mu : Sp\,(2, \mathbb{R}) \to \textit{Unitary Operators on } L^2(\mathbb{R}).$$

which has the property:

$$d\mu \left(\begin{pmatrix} 0 & 1 \\ -\eta(\tau) & 0 \end{pmatrix} \right) = i \left(-\frac{1}{2}\partial_y^2 + \eta(\tau)\frac{y^2}{2} \right).$$

For more information regarding μ and $d\mu$, we refer the readers to Folland's monograph [19]. We comment that μ is not a well-defined group homomorphism on all of $Sp\,(2, \mathbb{R})$, but the fact that it is well-defined in a neighborhood of the identity of $Sp\,(2, \mathbb{R})$ is good enough for our purpose here.

2.8.1 Proof of Lemma 4 / the Generalized Lens Transform

Proposition 5 *Define α and β through the system*

$$\begin{aligned} \ddot{\alpha}(\tau) + \eta(\tau)\alpha(\tau) &= 0, \alpha(0) = 0, \dot{\alpha}(0) = 1, \\ \ddot{\beta}(\tau) + \eta(\tau)\beta(\tau) &= 0, \beta(0) = 1, \dot{\beta}(0) = 0, \end{aligned}$$

and let

$$B(\tau) = \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot{\beta}(\tau) & \dot{\alpha}(\tau) \end{pmatrix}.$$

Assume β is nonzero in some time interval $[0,T]$, then $\mu(B(\tau))f$ solves the Schrödinger equation with switchable quadratic trap:

$$\begin{aligned} i\partial_\tau u &= \left(-\frac{1}{2}\partial_y^2+\eta(\tau)\frac{y^2}{2}\right)u \text{ in } \mathbb{R}\times[0,T] \\ u(0,y) &= f(y)\in L^2(\mathbb{R}). \end{aligned} \tag{2.19}$$

Proof. We calculate

$$\begin{aligned} \partial_\tau|_{\tau=0}\mu(B(\tau_0+\tau))f &= (\partial_\tau|_{\tau=0}\mu(B(\tau_0+\tau)))f \\ &= \left(\partial_\tau|_{\tau=0}\mu\left(B(\tau_0+\tau)B^{-1}(\tau_0)B(\tau_0)\right)\right)f \\ &= \left(\partial_\tau|_{\tau=0}\mu\left(B(\tau_0+\tau)B^{-1}(\tau_0)\right)\right)\mu(B(\tau_0))f \\ &= d\mu(B'(\tau_0)B^{-1}(\tau_0))\mu(B(\tau_0))f. \end{aligned}$$

where

$$\begin{aligned} & B'(\tau_0)B^{-1}(\tau_0) \\ =& \begin{pmatrix} \dot\beta(\tau_0) & -\dot\alpha(\tau_0) \\ -\ddot\beta(\tau_0) & \ddot\alpha(\tau_0) \end{pmatrix}\begin{pmatrix} \dot\alpha(\tau_0) & \alpha(\tau_0) \\ \dot\beta(\tau_0) & \beta(\tau_0) \end{pmatrix} \\ =& \begin{pmatrix} \dot\beta(\tau_0) & -\dot\alpha(\tau_0) \\ \eta(\tau_0)\beta(\tau_0) & -\eta(\tau_0)\alpha(\tau_0) \end{pmatrix}\begin{pmatrix} \dot\alpha(\tau_0) & \alpha(\tau_0) \\ \dot\beta(\tau_0) & \beta(\tau_0) \end{pmatrix} \\ =& \begin{pmatrix} 0 & \dot\beta(\tau_0)\alpha(\tau_0)-\dot\alpha(\tau_0)\beta(\tau_0) \\ \eta(\tau_0)\left(\dot\alpha(\tau_0)\beta(\tau_0)-\dot\beta(\tau_0)\alpha(\tau_0)\right) & 0 \end{pmatrix}. \end{aligned}$$

Notice that the Wronskian of α and β is constant 1 i.e.

$$\dot\alpha(\tau)\beta(\tau)-\alpha(\tau)\dot\beta(\tau)=1.$$

So

$$\begin{aligned} d\mu(B'(\tau_0)B^{-1}(\tau_0)) &= d\mu\left(\begin{pmatrix} 0 & -1 \\ \eta(\tau_0) & 0 \end{pmatrix}\right) \\ &= -\frac{i}{2}\left(-\partial_y^2 + \eta(\tau_0)y^2\right). \end{aligned}$$

In other words,

$$\partial_\tau\left(\mu\left(B\left(\tau\right)\right)f\right) = -\frac{i}{2}\left(-\partial_y^2 + \eta(\tau)y^2\right)\left(\mu\left(B\left(\tau\right)\right)f\right).$$

Before we end the proof, we remark that $\beta \neq 0$ is required for the metaplectic representation to be well-defined. ■

Through the LDU decomposition of the matrix B, we derive the generalized lens transform. The LDU decomposition of the matrix B is

$$\begin{aligned} B(\tau) &= \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot{\beta}(\tau) & \dot{\alpha}(\tau) \end{pmatrix} \\ &= \begin{pmatrix} \beta(\tau) & -\alpha(\tau) \\ -\dot{\beta}(\tau) & \alpha(\tau)\frac{\dot{\beta}(\tau)}{\beta(\tau)} + \frac{1}{\beta(\tau)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{\dot{\beta}(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence we have

$$\mu\left(B(\tau)\right)f = \mu\left(\begin{pmatrix} 1 & 0 \\ -\frac{\dot{\beta}(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\right)\mu\left(\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\right)\mu\left(\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}\right)f, \quad (2.20)$$

where

$$\begin{aligned}
\mu\left(\begin{pmatrix} 1 & 0 \\ -\frac{\dot\beta(\tau)}{\beta(\tau)} & 1 \end{pmatrix}\right) f(y) &= e^{i\frac{\dot\beta(\tau)}{\beta(\tau)}\frac{y^2}{2}} f(y) \text{ by } (4.25) \text{ in } [19] \\
\mu\left(\begin{pmatrix} \beta(\tau) & 0 \\ 0 & \frac{1}{\beta(\tau)} \end{pmatrix}\right) f(y) &= \frac{1}{(\beta(\tau))^{\frac{1}{2}}} f(\frac{y}{\beta(\tau)}) \text{ by } (4.24) \text{ in } [19] \\
\mu\left(\begin{pmatrix} 1 & -\frac{\alpha(\tau)}{\beta(\tau)} \\ 0 & 1 \end{pmatrix}\right) f(y) &= e^{i\frac{\alpha(\tau)}{\beta(\tau)}\frac{\partial_y^2}{2}} f \text{ by } (4.54) \text{ in } [19].
\end{aligned}$$

Due to the definition of μ, equality 2.20 in fact holds up to a " $\pm$ " sign which depends on the time interval. However, the LHS and the RHS of equality 2.20 agree for sufficiently small τ. By continuity, they must agree on the time interval $[0, T]$ where $\beta \neq 0$. So we conclude the following lemma concerning the generalized lens transform.

Lemma 8 *[3] Assume β is nonzero in the time interval $[0, T]$, then the solution of the Schrödinger equation with switchable quadratic trap (equation (2.19)) in $[0, T]$ is given by*

$$u(\tau, y) = \frac{e^{i\frac{\dot\beta(\tau)}{\beta(\tau)}\frac{y^2}{2}}}{(\beta(\tau))^{\frac{1}{2}}} v(\frac{\alpha(\tau)}{\beta(\tau)}, \frac{y}{\beta(\tau)}),$$

if $v(t, x)$ solves the free Schrördinger equation

$$\begin{aligned}
i\partial_t v &= -\frac{1}{2}\partial_x^2 v \text{ in } \mathbb{R}^{1+1} \\
v(0, x) &= f(x) \in L^2(\mathbb{R}).
\end{aligned}$$

The anisotropic case, Lemma 4, follows from the above lemma.

2.8.2 Proof of Lemma 3 / Evolution of Momentum

Using the metaplectic representation, we can also compute the evolution of momentum and position.

Lemma 9 *The evolution of momentum and position is given by*

$$\begin{aligned} P(\tau) &= \mu\left(B(\tau)\right) \circ \left(-i\partial_y\right) \circ \left(\mu\left(B(\tau)\right)\right)^{-1} = -i\beta(\tau)\partial_y - \dot{\beta}(\tau)y \\ Y(\tau) &= \mu\left(B(\tau)\right) \circ y \circ \left(\mu\left(B(\tau)\right)\right)^{-1} = i\alpha(\tau)\partial_y + \dot{\alpha}(\tau)y. \end{aligned}$$

Proof. Let us only compute the momentum, position can be obtained similarly.

$$\begin{aligned} & \mu\left(B(\tau)\right)\left(-i\partial_y\right)\left(\mu\left(B(\tau)\right)\right)^{-1} \\ = \; & \mu\left(B(\tau)\right)\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -i\partial_y \\ y \end{pmatrix}\left(\mu\left(B(\tau)\right)\right)^{-1} \\ = \; & \begin{pmatrix} 1 & 0 \end{pmatrix}\left(B(\tau)\right)^T\begin{pmatrix} -i\partial_y \\ y \end{pmatrix} \quad \text{(Theorem 2.15 in [19])} \\ = \; & \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \beta(\tau) & -\dot{\beta}(\tau) \\ -\alpha(\tau) & \dot{\alpha}(\tau) \end{pmatrix}\begin{pmatrix} -i\partial_y \\ y \end{pmatrix} \\ = \; & -i\beta(\tau)\partial_y - \dot{\beta}(\tau)y \end{aligned}$$

■

Remark 10 *We select $-i\partial_y$ to be the momentum to match the canonical commutation relations in Folland [19] which is*

$$[-i\partial_y, y] = -iI.$$

The above lemma reproduces the following result in Carles [3].

Lemma 10 *[3] The operators $P(\tau)$ and $Y(\tau)$ commute with the linear operator*

$$i\partial_\tau + \frac{1}{2}\partial_y^2 - \eta(\tau)\frac{y^2}{2}$$

Moreover,

$$\begin{aligned} P(\tau)U(\tau;s) &= U(\tau;s)P(s) \\ Y(\tau)U(\tau;s) &= U(\tau;s)Y(s) \end{aligned}$$

if we let $U_y(\tau;s)$ be the solution operator of

$$\begin{aligned} i\partial_\tau u &= \left(-\frac{1}{2}\partial_y^2 + \eta(\tau)\frac{y^2}{2}\right) u \text{ in } \mathbb{R}^{1+1} \\ u(s,y) &= u_s(y) \in L^2(\mathbb{R}), \end{aligned}$$

or in other words

$$U_y(\tau;s) = \mu\left(B(\tau)\right)\mu\left(B(s)\right)^{-1}.$$

Thence we have shown Lemma 3.

2.9 Conclusion of Chapter 2

In this chapter, we have derived rigorously the 2d cubic NLS with anisotropic switchable quadratic traps through a modified Elgart-Erdös-Schlein-Yau procedure. We have attained partial results in 3d as well. Unfortunately, when $n = 3$, we still have unsolved problems as stated in Section 2.7.2.

Chapter 3

Proof of Theorem 2

3.1 Outline of the Proof of Theorem 2

We prove Theorem 2 via Theorems 7 and 8 stated below. They deal with the construction of $\boldsymbol{\psi}_{GMM}$ and the error estimate separately. However, it is worth pointing out that Theorem 2 is a special case of Theorems 7 and 8, which apply to a more general setting beyond initial data of the form $e^{-\sqrt{N}A(\phi_0)}\Omega$.

Theorem 7 *Let ϕ be a sufficiently smooth solution of the quintic Hartree equation*

$$i\frac{\partial}{\partial t}\phi + \triangle\phi - \frac{1}{2}\phi\int v_3(x-y,x-z)\left|\phi(y)\right|^2\left|\phi(z)\right|^2 dydz = 0 \tag{3.1}$$

with initial data ϕ_0 and the 3-body interaction potential v_3 being symmetric in x, y, and z. Assume the following:

(1) Let a complex kernel $k(t,x,y) \in L^2_s(dxdy)$ for almost all t, solve the equation

$$iu_t + ug^T + gu - (I+p)m = (ip_t + [g,p] + u\overline{m})\left(I+p\right)^{-1}u, \tag{3.2}$$

with

$$u(t,x,y) := \sinh(k) := k + \frac{1}{3!}k\overline{k}k + \dots\ ,$$
$$\cosh(k) := I + p(t,x,y) := \delta(x-y) + \frac{1}{2!}k\overline{k} + \dots\ ,$$

$$
\begin{aligned}
g(t,x,y) \;&:\; = -\triangle\delta(x-y) + \left(\int v_3(x-y,x-z)\left|\phi(z)\right|^2 dz\right)\overline{\phi}(x)\overline{\phi}(y) \\
&\quad +\frac{1}{2}\left(\int v_3(x-y,x-z)\left|\phi(y)\right|^2\left|\phi(z)\right|^2 dydz\right)\delta(x-y), \\
m(t,x,y) \;&:\; = -\left(\int v_3(x-y,x-z)\left|\phi(z)\right|^2 dz\right)\overline{\phi}(x)\overline{\phi}(y),
\end{aligned}
$$

where the products ug^T, $k\overline{k}$ etc. stand for compositions of operators.

(2) For V defined as in formula (1.12)*, the functions,*

$$
\|e^B V e^{-B}\Omega\|_{\mathcal{F}}, \|e^B[A,V]e^{-B}\Omega\|_{\mathcal{F}}, \|e^B[A,[A,V]]e^{-B}\Omega\|_{\mathcal{F}}, \|e^B[A,[A,[A,V]]]e^{-B}\Omega\|_{\mathcal{F}},
$$

are locally integrable in time, where

$$
B(t) := \frac{1}{2}\int \left(k(t,x,y)a_x a_y - \overline{k}(t,x,y)a_x^* a_y^*\right) dxdy. \tag{3.3}
$$

(3) $\int d(t,x,x)\ dx$ is also locally integrable in time, where

$$
\begin{aligned}
d(t,x,y) := &\left(i\sinh(k)_t + \sinh(k)g^T + g\sinh(k)\right)\overline{\sinh(k)} \\
&-\left(i\cosh(k)_t + [g,\cosh(k)]\right)\cosh(k) \\
&-\sinh(k)\overline{m}\cosh(k) - \cosh(k)m\overline{\sinh(k)}.
\end{aligned} \tag{3.4}
$$

Then we define

$$
\boldsymbol{\psi}_{GMM} := e^{-\sqrt{N}A(\phi(t,\cdot))}e^{-B(t)}e^{-i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Omega
$$

where

$$
\begin{aligned}
\chi_0(t) \;&:\; = -\frac{1}{3}\int v_3(x-y,x-z)\left|\phi(x)\right|^2\left|\phi(y)\right|^2\left|\phi(z)\right|^2 dxdydz, \\
\chi_1(t) \;&:\; = \frac{1}{2}\int d(t,x,x)dx.
\end{aligned}
$$

This definition of $\boldsymbol{\psi}_{GMM}$ *yields the error estimate*

$$\begin{aligned}
&\|\boldsymbol{\psi}_{GMM} - e^{itH_N} e^{-\sqrt{N}A(\phi_0)} e^{-B(0)}\Omega\,\|_{\mathcal{F}} \\
&\leqslant \frac{\int_0^t \|e^B V e^{-B}\Omega\|_{\mathcal{F}} ds}{6N^2} + \frac{\int_0^t \|e^B [A, V] e^{-B}\Omega\|_{\mathcal{F}} ds}{6N^{\frac{3}{2}}} \\
&+ \frac{\int_0^t \|e^B [A, [A, V]] e^{-B}\Omega\|_{\mathcal{F}} ds}{12N} + \frac{\int_0^t \|e^B [A, [A, [A, V]]] e^{-B}\Omega\|_{\mathcal{F}} ds}{36N^{\frac{1}{2}}}.
\end{aligned}$$

Theorem 8 *Assume* $v_3(x-y, x-z) = v(x-y, x-z)$ *i.e. equation* (3.1) *becomes*

$$i\frac{\partial}{\partial t}\phi + \triangle\phi - \frac{1}{2}\phi \int v(x-y, x-z)\,|\phi(y)|^2\,|\phi(z)|^2\,dydz = 0. \tag{3.5}$$

If ϕ_0*, the initial data of quintic Hartree equation* (3.5)*, satisfies (i), (ii), and (iii), then the hypotheses in Theorem 7 are satisfied globally in time. Moreover, we have the error estimate uniformly in time that*

$$\|\boldsymbol{\psi}_{GMM} - e^{itH_N} e^{-\sqrt{N}A(\phi_0)} e^{-B(0)}\Omega\,\|_{\mathcal{F}} \leqslant \frac{C}{\sqrt{N}}$$

where C *depends only on* v*,* C_1*,* C_2 *and* $\|u(0, \cdot, \cdot)\|_{L^2_{(x,y)}}$.

We deduce Theorem 2 from Theorems 7 and 8 by setting

$$k(0, x, y) = 0.$$

The proof of Theorem 8 relies on the following theorem regarding the long time behavior of the solution to the Hartree equation.

Theorem 9 *If* ϕ *solve the Hartree equation* (3.5) *subject to (i), (ii), and (iii), then*

$$\|\phi\|_{L^6_x} \leqslant \frac{C}{t}, \; for\ t \geqslant 1,$$

where C *is a function of* v*,* C_1 *and* C_2 *only.*

3.2 The Derivation of 2nd Order Corrections / Proof of Theorem 7

3.2.1 Derivation of The Quintic Hartree equation

We first derive the quintic Hartree equation (3.1) for the one-particle wave function ϕ as needed in Theorem 7.

Lemma 11 *The following commutating relations hold, where A denotes $A(\phi)$, and A , V are defined by formulas* (1.13) *and* (1.12)*:*

$$\begin{aligned}
&[A,V] \\
=\ & 3\int v_3(x-y,x-z)(\overline{\phi}(x)a_y^*a_z^*a_xa_ya_z+\phi(x)a_x^*a_y^*a_z^*a_ya_z)dxdydz
\end{aligned}$$

$$\begin{aligned}
&[A,[A,V]] \\
=\ & 6\int v_3(x-y,x-z)(\overline{\phi}(x)\overline{\phi}(y)a_z^*a_xa_ya_z+2\phi(x)\overline{\phi}(y)a_x^*a_z^*a_ya_z \\
&+\phi(x)\phi(y)a_x^*a_y^*a_z^*a_z)dxdydz \\
&+6\int v_3(x-y,x-z)\left|\phi(x)\right|^2a_y^*a_z^*a_ya_zdxdydz
\end{aligned}$$

$$\begin{aligned}
&[A,[A,[A,V]]] \\
=\ & 36\int v_3(x-y,x-z)\left|\phi(x)\right|^2(\overline{\phi}(y)a_z^*a_ya_z+\phi(y)a_y^*a_z^*a_z)dxdydz \\
&+6\int v_3(x-y,x-z)(\overline{\phi}(x)\overline{\phi}(y)\overline{\phi}(z)a_xa_ya_z+\phi(x)\phi(y)\phi(z)a_x^*a_y^*a_z^*)dxdydz \\
&+18\int v_3(x-y,x-z)(\overline{\phi}(x)\overline{\phi}(y)\phi(z)a_z^*a_xa_y+\phi(x)\phi(y)\overline{\phi}(z)a_x^*a_y^*a_z)dxdydz
\end{aligned}$$

$$
\begin{aligned}
& [A,[A,[A,[A,V]]]] \\
= \; & 72\int v_3(x-y,x-z)\,|\phi(x)|^2\,(\overline{\phi}(y)\overline{\phi}(z)a_y a_z+\phi(y)\phi(z)a_y^* a_z^*)dxdydz \\
& +144\int v_3(x-y,x-z)\,|\phi(x)|^2\,\overline{\phi}(y)\phi(z)a_z^* a_y dxdydz \\
& +72\int v_3(x-y,x-z)\,|\phi(x)|^2\,|\phi(y)|^2\,a_z^* a_z dxdydz
\end{aligned}
$$

$$
\begin{aligned}
& [A,[A,[A,[A,[A,V]]]]] \\
= \; & 360\int v_3(x-y,x-z)\,|\phi(x)|^2\,|\phi(y)|^2\,(\overline{\phi}(z)a_z+\phi(z)a_z^*)dxdydz
\end{aligned}
$$

$$
\begin{aligned}
& [A,[A,[A,[A,[A,[A,V]]]]]] \\
= \; & 720\int v_3(x-y,x-z)\,|\phi(x)|^2\,|\phi(y)|^2\,|\phi(z)|^2\,dxdydz
\end{aligned}
$$

Proof. This is a direct calculation using the canonical commutation relation (1.11). ■

Now, we write $\Psi_0(t)=e^{\sqrt{N}A(t)}e^{itH_N}e^{-\sqrt{N}A(0)}e^{-B(0)}\Omega$ for which we carry out the calculation in the spirit of equation (3.7) in Rodnianski and Schlein [35].

Proposition 6 *Let ϕ solve the Hartree equation*

$$
i\frac{\partial}{\partial t}\phi+\triangle\phi-\frac{1}{2}\phi\int v_3(x-y,x-z)\,|\phi(y)|^2\,|\phi(z)|^2\,dydz=0
$$

then $\Psi_0(t)$ satisfies

$$
\begin{aligned}
\frac{1}{i}\frac{\partial}{\partial t}\Psi_0(t)=\Bigg(&H_0-\frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]]-\frac{1}{6}N^{-2}V-\frac{1}{6}N^{-3/2}[A,V] \\
&-\frac{1}{12}N^{-1}[A,[A,V]]-\frac{1}{36}N^{-\frac{1}{2}}[A,[A,[A,V]]] \\
&+\frac{N}{3}\int v_3(x-y,x-z)\,|\phi(x)|^2\,|\phi(y)|^2\,|\phi(z)|^2\,dxdydz\Bigg)\Psi_0(t)\;.
\end{aligned}
\tag{3.6}
$$

Proof. Applying the formulas

$$\left(\frac{\partial}{\partial t}e^{C(t)}\right)\left(e^{-C(t)}\right)=\dot{C}+\frac{1}{2!}[C,\dot{C}]+\frac{1}{3!}\left[C,[C,\dot{C}]\right]+\dots$$

$$e^{C}He^{-C}=H+[C,H]+\frac{1}{2!}\left[C,[C,H]\right]+\dots\ .$$

to $C=\sqrt{N}A$ and $H=H_N$, we obtain

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi_0(t)=L_0\Psi_0\ , \tag{3.7}$$

where

$$\begin{aligned}
L_0=&\frac{1}{i}\left(\frac{\partial}{\partial t}e^{\sqrt{N}A(t)}\right)e^{-\sqrt{N}A(t)}+e^{\sqrt{N}A(t)}H_Ne^{-\sqrt{N}A(t)}\\
=&\frac{1}{i}\left(N^{1/2}\dot{A}+\frac{N}{2}[A,\dot{A}]\right)+H_0+N^{1/2}[A,H_0]+\frac{N}{2!}[A,[A,H_0]]-\frac{1}{6}\Big(N^{-2}V\\
&+N^{-3/2}[A,V]+\frac{N^{-1}}{2!}[A,[A,V]]+\frac{N^{-\frac{1}{2}}}{3!}[A,[A,[A,V]]]\\
&+\frac{1}{4!}[A,[A,[A,[A,V]]]]+\frac{N^{\frac{1}{2}}}{5!}[A,[A,[A,[A,[A,V]]]]]\\
&+\frac{N}{6!}[A,[A,[A,[A,[A,[A,V]]]]]]\Big)\ .
\end{aligned}$$

The Hartree equation (3.1) is equivalent to setting terms of order $\sqrt{N}$ to zero i.e.

$$\frac{1}{i}\dot{A}+[A,H_0]-\frac{1}{6}\frac{1}{5!}[A,[A,[A,[A,[A,V]]]]]=0\ .$$

Or more explicitly, the above equation is

$$\begin{aligned}
0\ =\ & a(\overline{i\phi_t}+\overline{\triangle\phi}-\frac{1}{2}\overline{\phi}\int v_{3,1-2,1-3}\left|\phi_2\right|^2\left|\phi_3\right|^2dydz)\\
&+a^*(i\phi_t+\triangle\phi-\frac{1}{2}\phi\int v_{3,1-2,1-3}\left|\phi_2\right|^2\left|\phi_3\right|^2dydz),
\end{aligned}$$

via Lemma 11 and the fact that $[\Delta_x a_x,a_y^*]=(\Delta\delta)(x-y)$.

Thus

$$\frac{1}{i}[A,\dot{A}] + [A,[A,H_0]] - \frac{1}{5!}\frac{1}{6}[A,[A,[A,[A,[A,[A,V]]]]]] = 0 \ ,$$

i.e. equation (3.7) simplifies to

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi_0(t) = \Big(H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]] - \frac{1}{6}N^{-2}V - \frac{1}{6}N^{-3/2}[A,V]$$
$$- \frac{1}{12}N^{-1}[A,[A,V]] - \frac{1}{36}N^{-\frac{1}{2}}[A,[A,[A,V]]]$$
$$+ \frac{N}{3}\int v_{3,1-2,1-3}\left|\phi_1\right|^2\left|\phi_2\right|^2\left|\phi_3\right|^2 dxdydz\Big)\Psi_0(t) \ .$$

which is equation (3.6). ■

Because $\int v_{3,1-2,1-3}\left|\phi_1\right|^2\left|\phi_2\right|^2\left|\phi_3\right|^2 dxdydz$ only contributes a phase when ϕ_0 is sufficiently smooth, we write

$$\frac{N}{3}\int v_3(x-y,x-z)\left|\phi(x)\right|^2\left|\phi(y)\right|^2\left|\phi(z)\right|^2 dxdydz := -N\chi_0 \ .$$

Then the first two terms on the right-hand side of equation (3.6) are the main ones we need to consider, since the next four terms are at most $O\left(1/\sqrt{N}\right)$.

In order to kill the terms involving "only creation operators" i.e. $a_x^* a_y^*$ in $\frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]]$, we introduce B (see 3.3) and denote

$$\Psi = e^B\Psi_0 = e^B e^{\sqrt{N}A(t)} e^{itH_N} e^{-\sqrt{N}A(0)} e^{-B(0)}\Omega.$$

Hence we have

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi = L\Psi \ ,$$

where

$$
\begin{aligned}
L &= \frac{1}{i}\left(\frac{\partial}{\partial t}e^{B}\right)e^{-B} + e^{B}L_0e^{-B} \\
&= L_Q - \frac{1}{6}N^{-2}e^{B}Ve^{-B} - \frac{1}{6}N^{-3/2}e^{B}[A,V]e^{-B} - \frac{1}{12}N^{-1}e^{B}[A,[A,V]]e^{-B} \\
&- \frac{1}{36}N^{-\frac{1}{2}}e^{B}[A,[A,[A,V]]]e^{-B} - N\chi_0 \ ,
\end{aligned}
$$

and the quadratic terms

$$
L_Q = \frac{1}{i}\left(\frac{\partial}{\partial t}e^{B}\right)e^{-B} + e^{B}\left(H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]]\right)e^{-B} \ . \tag{3.8}
$$

At this point, we proceed to seek a equation for k s.t. the coefficient of $a_x^* a_y^*$ in $\frac{1}{4!}\frac{1}{6}e^{B}[A,[A,[A,[A,V]]]]e^{-B}$ is eliminated.

Remark 11 *One might be concerned of the pure creation $a_x^* a_y^* a_z^*$ in $[A,[A,[A,V]]]$. Lemma 16 and the factor $1/\sqrt{N}$ will take care of that. Note that [21, 22] do not have terms like this.*

3.2.2 Equation for k

3.2.2.1 The infinitesimal metaplectic representation[21]

Let sp be the infinite dimensional Lie algebra of matrices of the form

$$
S(d,k,l) = \begin{pmatrix} d & k \\ l & -d^{T} \end{pmatrix}
$$

where k and l are symmetric, and $Quad$ be the Lie algebra consisting of homogeneous quadratics of the form

$$
\begin{aligned}
Q(d,k,l) :=& \frac{1}{2}\begin{pmatrix} a_x & a_x^* \end{pmatrix}\begin{pmatrix} d & k \\ l & -d^T \end{pmatrix}\begin{pmatrix} -a_y^* \\ a_y \end{pmatrix} \\
=& -\int d(x,y)\frac{a_x a_y^* + a_y^* a_x}{2}dxdy + \frac{1}{2}\int k(x,y)a_x a_y dxdy \\
& -\frac{1}{2}\int l(x,y)a_x^* a_y^* dxdy
\end{aligned}
$$

equipped with Poisson bracket. In the spirit of page 185, Folland [19], we define the infinitesimal metaplectic representation: a Lie algebra isomorphism $\mathcal{I} : sp \to Quad$ by $Q(d,k,l) = \mathcal{I}(S(d,k,l))$. Then we see that

$$
B = \mathcal{I}(K) \ ,
$$

for

$$
K = \begin{pmatrix} 0 & k(t,x,y) \\ \overline{k}(t,x,y) & 0 \end{pmatrix}, \tag{3.9}
$$

and it follows that

(i)

$$
\mathcal{I}\left(e^S C e^{-S}\right) = e^{\mathcal{I}(S)}\mathcal{I}\left(C\right)e^{-\mathcal{I}(S)}
$$

if $\mathcal{I}\left(C\right) \in sp$.

(ii)

$$
\mathcal{I}\left(\left(\frac{\partial}{\partial t}e^S\right)e^{-S}\right) = \left(\frac{\partial}{\partial t}e^{\mathcal{I}(\mathcal{S})}\right)e^{-\mathcal{I}(\mathcal{S})}
$$

if $\mathcal{I}(\mathcal{S})$ is skew-Hermitian.

(iii)

$$
e^{\mathcal{I}(\mathcal{S})}\begin{pmatrix} a_x & a_x^* \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}e^{-\mathcal{I}(\mathcal{S})} = \begin{pmatrix} a_x & a_x^* \end{pmatrix}e^S\begin{pmatrix} f \\ g \end{pmatrix}
$$

if $\mathcal{I}(\mathcal{S})$ is skew-Hermitian.

Remark 12 *Properties (i) and (ii) will be used below. (iii) will be used in Section 3.4.*

3.2.2.2 Derivation of Equation (3.2)

Use the simplifications noted in Remark 3, recall that

$$\begin{aligned} & \frac{1}{4!6}[A,[A,[A,[A,V]]]] \\ = & \frac{1}{2}\int v_{3,1-2,1-3}\left|\phi_1\right|^2(\overline{\phi}_2\overline{\phi}_3 a_y a_z+\phi_2\phi_3 a_y^* a_z^*)dxdydz \\ & +\int v_{3,1-2,1-3}\left|\phi_1\right|^2\overline{\phi}_2\phi_3 a_z^* a_y dxdydz \\ & +\frac{1}{2}\int v_{3,1-2,1-3}\left|\phi_1\right|^2\left|\phi_2\right|^2 a_z^* a_z dxdydz\ , \end{aligned}$$

and

$$H_0=\int a_x^*\Delta a_x dx$$

we write

$$G=\begin{pmatrix} g & 0 \\ 0 & -g^T \end{pmatrix} \qquad \text{and} \qquad M=\begin{pmatrix} 0 & m \\ -\overline{m} & 0 \end{pmatrix}$$

with

$$g=-\triangle\delta_{1-2}+\left(\int v_{3,1-2,1-3}\left|\phi_3\right|^2 dz\right)\overline{\phi}_1\phi_2+\frac{1}{2}\left(\int v_{3,1-2,1-3}\left|\phi_2\right|^2\left|\phi_3\right|^2 dydz\right)\delta_{1-2}$$

and

$$m=-\left(\int v_{3,1-2,1-3}\left|\phi_3\right|^2 dz\right)\overline{\phi}_1\overline{\phi}_2.$$

Of course, we would like to be able to write

$$H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]] = \mathcal{I}\left(G\right) + \mathcal{I}\left(M\right) \ .$$

Unfortunately, the above equality is not true. For example

$$\mathcal{I}\left(\left(\begin{matrix} -\triangle\delta_{1-2} & 0 \\ & \\ 0 & \triangle\delta_{1-2} \end{matrix}\right)\right) = \int \frac{a_x^*\triangle a_x + a_x\triangle a_x^*}{2}dx.$$

However, the commutators of $\mathcal{I}\left(G\right), \mathcal{I}\left(M\right)$ and $H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]]$ with B are the same as in the discussion in page 287 in [21]. The same idea applies here.

Split

$$H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]] = H_G + \mathcal{I}(\mathcal{M})$$

where

$$H_G = H_0 - \int v_{3,1-2,1-3}\left|\phi_3\right|^2 \overline{\phi}_2\phi_1 a_x^* a_y dxdydz - \frac{1}{2}\int v_{3,1-2,1-3}\left|\phi_2\right|^2\left|\phi_3\right|^2 a_x^* a_x dxdydz$$

which has the property that

$$[H_G, B] = [\mathcal{I}\left(G\right), B].$$

Now, L_Q from formula (3.8) reads

$$\begin{aligned}
L_Q &= \frac{1}{i}\left(\frac{\partial}{\partial t}e^B\right)e^{-B} + e^B\left(H_0 - \frac{1}{4!}\frac{1}{6}[A,[A,[A,[A,V]]]]\right)e^{-B} \\
&= \mathcal{I}\left(\left(\frac{1}{i}\frac{\partial}{\partial t}e^K\right)e^{-K}\right) + e^B H_G e^{-B} + \mathcal{I}\left(e^K M e^{-K}\right) \\
&= \mathcal{I}\left(\left(\frac{1}{i}\frac{\partial}{\partial t}e^K\right)e^{-K}\right) + H_G + \left(e^B H_G e^{-B} - H_G\right) + \mathcal{I}\left(e^K M e^{-K}\right) \\
&= H_G + \mathcal{I}\left(\left(\frac{1}{i}\frac{\partial}{\partial t}e^K\right)e^{-K}\right) + [e^B, H_G]e^{-B} + \mathcal{I}\left(e^K M e^{-K}\right) \\
&= H_G + \mathcal{I}\left(\left(\frac{1}{i}\frac{\partial}{\partial t}e^K\right)e^{-K} + [e^K, G]e^{-K} + e^K M e^{-K}\right) \\
&= H_G + \mathcal{I}(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3) \ .
\end{aligned}$$

Then by the definition of the isomorphism $\mathcal{I}$, the coefficient of $a_x a_y$ is $-(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{12}$, and the coefficient of $a_x^* a_y^*$ is $(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{21}$. To write it explicitly:

$$\begin{aligned}
&-(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{12} \\
&= \overline{(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{21}} \\
&= (i \sinh(k)_t + \sinh(k) g^T + g \sinh(k)) \overline{\cosh(k)} \\
&- (i \cosh(k)_t - [\cosh(k), g] \sinh(k) \\
&- \sinh(k) \overline{m} \sinh(k) - \cosh(k) m \overline{\cosh(k)} \ .
\end{aligned} \tag{3.10}$$

Setting formula (3.10) to 0 confers equation (3.2). This implies that

$$\begin{aligned}
\mathcal{I}(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3) &= -\int (\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{11} \frac{a_x a_y^* + a_y^* a_x}{2} dx dy \\
&= -\int d(t, x, y) \frac{a_x a_y^* + a_y^* a_x}{2} dx dy \\
&= -\int d(t, x, y) a_y^* a_x dx dy - \frac{1}{2} \int d(t, x, x) dx
\end{aligned}$$

where $d(t, x, y)$ is given by formula (3.4).

Remark 13 *$(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)_{ij}$ means the entry on the ith row and the jth column of the matrix $(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)$.*

We summarize the computations we have done so far in this proposition:

Proposition 7 *If ϕ and k solve equations (3.1) and (3.2), then the coefficients of $a_x a_y$ and $a_x^* a_y^*$ in $e^B [A, [A, [A, [A, V]]]] e^{-B}$ are 0 and L_Q becomes*

$$\begin{aligned}
L_Q = & H_0 - \int v_3(x - y, y - z) \left|\phi(z)\right|^2 \overline{\phi}(y) \phi(x) a_x^* a_y dx dy dz \\
& - \frac{1}{2} \int v_3(x - y, y - z) \left|\phi(y)\right|^2 \left|\phi(z)\right|^2 a_x^* a_x dx dy dz \\
& - \int d(t, x, y) a_y^* a_x dx dy - \frac{1}{2} \int d(t, x, x) dx \ .
\end{aligned}$$

Recall that $\Psi = e^B \Psi_0 = e^B e^{\sqrt{N}A(t)} e^{itH_N} e^{-\sqrt{N}A(0)} e^{-B(0)} \Omega$ *solves*

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi = L\Psi.$$

We can now write out

$$\begin{aligned}
L &= L_Q - \frac{1}{6}N^{-2}e^B V e^{-B} - \frac{1}{6}N^{-3/2}e^B[A,V]e^{-B} - \frac{1}{12}N^{-1}e^B[A,[A,V]]e^{-B} \\
&\quad -\frac{1}{36}N^{-\frac{1}{2}}e^B[A,[A,[A,V]]]e^{-B} - N\chi_0 \\
&= H_0 - \int v_3(x-y,y-z)\left|\phi(z)\right|^2 \overline{\phi}(y)\phi(x)a_x^* a_y dxdydz \\
&\quad -\frac{1}{2}\int v_3(x-y,y-z)\left|\phi(y)\right|^2\left|\phi(z)\right|^2 a_x^* a_x dxdydz \\
&\quad -\int d(t,x,y)a_y^* a_x dxdy - \frac{1}{6}N^{-2}e^B V e^{-B} - \frac{1}{6}N^{-3/2}e^B[A,V]e^{-B} \\
&\quad -\frac{1}{12}N^{-1}e^B[A,[A,V]]e^{-B} - \frac{1}{36}N^{-\frac{1}{2}}e^B[A,[A,[A,V]]]e^{-B} - \frac{1}{2}\int d(t,x,x)dx \\
&\quad -N\chi_0 \\
&= \tilde{L} - \chi_1 - N\chi_0
\end{aligned}$$

if we write

$$\begin{aligned}
\chi_0(t) &= -\frac{1}{3}\int v_3(x-y,x-z)\left|\phi(x)\right|^2\left|\phi(y)\right|^2\left|\phi(z)\right|^2 dxdydz \\
\chi_1(t) &= \frac{1}{2}\int d(t,x,x)dx.
\end{aligned}$$

Remark 14 *Note that* $\left(\tilde{L}\right)^* = \tilde{L}$ *and* $\tilde{L}$ *commutes with functions of time. This is needed in the proof of Theorem 7 which is below.*

3.2.2.3 The proof of Theorem 7

Applying the above proposition, we can give the proof of Theorem 7 at this point.

$$
\begin{aligned}
& \|e^{-\sqrt{N}A(t)}e^{-B(t)}e^{-i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Omega - e^{itH_N}e^{-\sqrt{N}A(0)}e^{-B(0)}\Omega\,\|_{\mathcal{F}} \\
= & \|\Omega - e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}e^{B(t)}e^{\sqrt{N}A(t)}e^{itH_N}e^{-\sqrt{N}A(0)}e^{-B(0)}\Omega\,\|_{\mathcal{F}} \\
= & \|\Omega - e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi\,\|_{\mathcal{F}}
\end{aligned}
$$

since $e^{-\sqrt{N}A(t)}$ and $e^{-B(t)}$ are unitary.

But

$$
\begin{aligned}
& \frac{\partial}{\partial t}\|\Omega - e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi\,\|_{\mathcal{F}}^2 \\
= & \; 2\,\mathrm{Re}\left(\frac{\partial}{\partial t}\left(e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\right), e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\right) \\
= & \; 2\,\mathrm{Re}\left(\left(\frac{\partial}{\partial t} - i\tilde{L}\right)\left(e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\right), e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\right) \\
= & \; 2\,\mathrm{Re}\left(i\tilde{L}\Omega, e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\right) \\
\leqslant & \; 2\|\tilde{L}\Omega\,\|_{\mathcal{F}}\|e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega\,\|_{\mathcal{F}}
\end{aligned}
$$

due to the fact that

$$
\left(\frac{1}{i}\frac{\partial}{\partial t} - \tilde{L}\right)(e^{i\int_0^t(N\chi_0(s)+\chi_1(s))ds}\Psi - \Omega) = \tilde{L}\Omega.
$$

Notice that

$$
\begin{aligned}
\widetilde{L}\Omega = & -\left(\frac{1}{6N^2}e^{B}Ve^{-B} + \frac{1}{6}N^{-3/2}e^{B}[A,V]e^{-B} + \right. \\
& \left. \frac{1}{12}N^{-1}e^{B}[A,[A,V]]e^{-B} + \frac{1}{36}N^{-\frac{1}{2}}e^{B}[A,[A,[A,V]]]e^{-B}\right)\Omega,
\end{aligned}
$$

we reach

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\Omega - e^{i\int_0^t (N\chi_0(s)+\chi_1(s))ds}\Psi\|_{\mathcal{F}} \\
\leqslant\ & \frac{\|e^B V e^{-B}\Omega\|_{\mathcal{F}}}{6N^2} + \frac{\|e^B[A,V]e^{-B}\Omega\|_{\mathcal{F}}}{6N^{\frac{3}{2}}} \\
&+\frac{\|e^B[A,[A,V]]e^{-B}\Omega\|_{\mathcal{F}}}{12N} + \frac{\|e^B[A,[A,[A,V]]]e^{-B}\Omega\|_{\mathcal{F}}}{36N^{\frac{1}{2}}}
\end{aligned}
$$

Whence we complete the proof of Theorem 7 because $e^{itH_N}e^{-\sqrt{N}A(0)}e^{-B(0)}\Omega$ and $e^{-\sqrt{N}A(t)}e^{-B(t)}e^{-i\int_0^t (N\chi_0(s)+\chi_1(s))ds}\Omega$ share the same initial data $e^{-\sqrt{N}A(0)}e^{-B(0)}\Omega$.

3.3 Solving Equation (3.2) / Proof of Theorem 8 (Part I)

Starting from this section, we begin the proof of Theorem 8. In other words, we are assuming that

$$v_3(x-y, x-z) = v(x-y, x-z)$$

where v is defined in formula (1.9).

We first study equation (3.2). We prove an apriori estimate for $u = \sinh(k)$ and use it in a Duhamel iteration argument to show global existence. Finally we verify that $\int d(t,x,x)dx$ is locally integrable in time.

Written in the notations in Remark 3, equation (3.2) reads

$$\left(iu_t + ug^T + gu - (I+p)m\right) = \left(ip_t + [g,p] + u\overline{m}\right)(I+p)^{-1} u,$$

where

$$
\begin{aligned}
u(t,x,y) = \sinh(k) &= k + \frac{1}{3!}k\overline{k}k + \dots \ , \\
\cosh(k)(t,x,y) = I + p(t,x,y) &= \delta_{1-2} + \frac{1}{2!}k\overline{k} + \dots \ ,
\end{aligned}
$$

$$g(t,x,y) = -\triangle\delta_{1-2} + \left(\int v_{1-2,1-3}\left|\phi_3\right|^2 dz\right)\overline{\phi}_1\phi_2$$
$$+\frac{1}{2}\left(\int v_{1-2,1-3}\left|\phi_2\right|^2\left|\phi_3\right|^2 dydz\right)\delta_{1-2}$$
$$m(t,x,y) = -\left(\int v_{1-2,1-3}\left|\phi_3\right|^2 dz\right)\overline{\phi}_1\overline{\phi}_2.$$

As mentioned in Theorem 7, we write composition of kernels as products in the above e.g.

$$k\overline{k}(x,y) = \int k(x,z)\overline{k}(z,y)dz.$$

Observe that $g(t,x,y) = \overline{g(t,y,x)}$, i.e. $g^* = g$; and $m(t,x,y) = m(t,y,x)$, i.e. $m^T = m$. Moreover, $u^T = u$, $p^* = p$ because $k \in L^2_s(dxdy)$.

Via $e^K e^{-K} = I$ with K defined in formula (3.9), we obtain the trigonometric identity

$$\begin{aligned} u\overline{u} &= \cosh(k)\overline{\cosh(k)} - I \\ &= 2p + p^2 \end{aligned}$$

which is a relation between u and p.

3.3.1 An Apriori Estimate of u

Theorem 10 *Let $v_3(x-y,x-z) = v(x-y,x-z)$. If $u = \sinh(k)$ is a solution of equation (3.2) on some time interval $[0,T]$, then there exists a $C \geqslant 0$, independent of T, s.t.*

$$\|u(T)\|_{L^2_{(x,y)}} \leqslant C\left(1 + \|u(0)\|_{L^2_{(x,y)}}\right).$$

The major observation is the following lemma which is also the cornerstone to showing Theorem 11.

Lemma 12 *[22]From equation* (3.2)*, we deduce*

$$(ip_t + [g,p] + u\overline{m})(I+p)^{-1} = -(I+p)^{-1}(ip_t + [g,p] - m\bar{u}) \tag{3.11}$$

and consequently

$$i(u\overline{u})_t + [g, u\overline{u}] = m\bar{u}(I+p) - (I+p)u\overline{m}. \tag{3.12}$$

Proof. Multiply equation (3.2) on the right by $\overline{u}$, it reads

$$\left(iu_t + ug^T + gu\right)\overline{u} - (I+p)m\overline{u} = (ip_t + [g,p] + u\overline{m})(I+p)^{-1}u\overline{u}. \tag{3.13}$$

Take the adjoint in the operator kernel sense of equation (3.2), multiply on the left by u, i.e.

$$u\left(-i\bar{u}_t + g^T\bar{u} + \bar{u}g\right) - u\overline{m}(I+p) = u\overline{u}(I+p)^{-1}(-ip_t - [g,p] + m\overline{u}). \tag{3.14}$$

Subtracting equations (3.13) and (3.14), we have

$$\begin{aligned} & i(u\overline{u})_t + [g, u\overline{u}] - (I+p)m\overline{u} + u\overline{m}(I+p) \qquad (3.15)\\ = \; & (ip_t + [g,p] + u\overline{m})(I+p)^{-1}u\overline{u} - u\overline{u}(I+p)^{-1}(-ip_t - [g,p] + m\overline{u}) \end{aligned}$$

With $u\overline{u} = \cosh(k)\overline{\cosh(k)} - I$ and $u\overline{u} = 2p + p^2$, we compute

$$(I+p)^{-1}u\overline{u} - (I+p) = (I+p)^{-1} = u\overline{u}(I+p)^{-1} - (I+p)$$

and

$$(I+p)^{-1}u\overline{u} = (I+p)^{-1}p + p = u\overline{u}(I+p)^{-1}$$

which transform equation (3.15) to

$$\begin{aligned} & i(2p+p^2)_t + [g, 2p+p^2] + u\overline{m}(I+p)^{-1} - (I+p)^{-1}m\overline{u} \\ = \; & (ip_t + [g,p])((I+p)^{-1}p + p) - ((I+p)^{-1}p + p)(-ip_t - [g,p]) \end{aligned}$$

i.e.

$$\begin{aligned} & 2(ip_t + [g,p]) + u\overline{m}(I+p)^{-1} - (I+p)^{-1}m\overline{u} \\ = \quad & (ip_t + [g,p])(I+p)^{-1}p + (I+p)^{-1}p\,(ip_t + [g,p]) \end{aligned}$$

which is equation (3.11) due to $I - (I+p)^{-1}p = (I+p)^{-1}$.

Multiplying equation (3.11) on the right and left by $(I+p)$ produces

$$(I+p)(ip_t + [g,p] + u\overline{m}) = -\left(ip_t + [g,p] - m\bar{u}\right)(I+p)$$

i.e. equation (3.12):

$$i(u\overline{u})_t + [g, u\overline{u}] = m\bar{u}(I+p) - (I+p)u\overline{m}$$

because $u\overline{u} = 2p + p^2$. ■

Taking the trace in formula (3.12) yields

$$\frac{d}{dt}\|u\|_{L^2}^2 = Tr\left[(1/i)\big(m\overline{u}(1+p) - (1+p)u\overline{m}\big)\right] \ .$$

Note that

$$\begin{aligned} \|u\|_{L^2}^2 &= Tr\,(u\overline{u}) \\ &= 2Tr\,(p) + Tr\left(p^2\right) \\ &\geqslant \|p\|_{L^2}^2 \end{aligned}$$

because $p(t,x,y) = \frac{1}{2!}k\overline{k} + \ldots$ must have a nonnegative trace. So

$$\begin{aligned} \frac{d}{dt}\|u\|_{L^2}^2 &\leqslant 2\left(\|m\|_{L^2}\|u\|_{L^2} + \|m\|_{L^2}\|u\|_{L^2}\|p\|_{L^2}\right) \\ &\leqslant 2\left(\|m\|_{L^2}\|u\|_{L^2} + \|m\|_{L^2}\|u\|_{L^2}^2\right). \end{aligned}$$

By a Gronwall's inequality, we deduce

$$\|u(T)\|_{L^2_{(x,y)}} \leqslant \left(\int_0^T \|m\|_{L^2_{(x,y)}} dt + \|u(0)\|_{L^2_{(x,y)}}\right) \exp\left(\int_0^T \|m\|_{L^2_{(x,y)}} dt\right).$$

The following lemma gives us Theorem 10.

Lemma 13 *If* $v_3(x-y, x-z) = v(x-y, x-z)$, *then*

$$\|m\|_{L^1_t(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C < \infty$$

Proof. Because

$$\begin{aligned} & v(x-y, x-z) \\ = \; & v_0(x-y)v_0(x-z) + v_0(x-y)v_0(y-z) + v_0(x-z)v_0(y-z), \end{aligned}$$

we have

$$\begin{aligned} \|m\|^2_{L^2_{(x,y)}} &= \int \left(\int v(x-y, x-z) \left|\phi_3\right|^2 dz\right)^2 |\phi_1|^2 |\phi_2|^2 dx dy \\ &\leqslant C \int |\phi_1|^2 |\phi_2|^2 v_0^2(x-y) \left(\int v_0(x-z) \left|\phi_3\right|^2 dz\right)^2 dx dy \\ &\quad + C \int |\phi_1|^2 |\phi_2|^2 v_0^2(x-y) \left(\int v_0(y-z) \left|\phi_3\right|^2 dz\right)^2 dx dy \\ &\quad + C \int |\phi_1|^2 |\phi_2|^2 \left(\int v_0(x-z) v_0(y-z) \left|\phi_3\right|^2 dz\right)^2 dx dy \\ &= I + II + III. \end{aligned}$$

A combination of Hölder and interpolation gives the following estimates

$$\begin{aligned} I + II &= 2C \int |\phi_1|^2 \left(\int v_0^2(x-y) |\phi_2|^2 dy\right) \left(\int v_0(x-z) \left|\phi_3\right|^2 dz\right)^2 dx \\ &\leqslant C \left\|\phi\right\|^2_{L^6} \left\|\int v_0^2(\cdot - y) |\phi_2|^2 dy\right\|_{L^\infty} \left\|\int v_0(\cdot - z) \left|\phi_3\right|^2 dz\right\|^2_{L^3} \\ &\leqslant C \left\|\phi\right\|^2_{L^6} \left\|\phi_0\right\|^2_{L^2} \left\|\phi\right\|^4_{L^6} \leqslant C \left\|\phi_0\right\|^2_{L^2} \left\|\phi\right\|^6_{L^6}, \end{aligned}$$

$$
\begin{aligned}
III &= C\int v_0(x-z_1)v_0(y-z_1)v_0(x-z_2)v_0(y-z_2) \\
&\quad |\phi_1|^2|\phi_2|^2\left|\phi(z_1)\right|^2\left|\phi(z_2)\right|^2 dxdydz_1dz_2 \\
&\leqslant C\int dz_1dz_2\left|\phi(z_1)\right|^2\left|\phi(z_2)\right|^2\left(\int v_0^2(x-z_1)v_0^2(y-z_1)|\phi_1|^2|\phi_2|^2dxdy\right)^{\frac{1}{2}} \\
&\quad \left(\int v_0^2(x-z_2)v_0^2(y-z_2)|\phi_1|^2|\phi_2|^2dxdy\right)^{\frac{1}{2}} \\
&= C\left(\int dz\left|\phi(z)\right|^2\left(\int v_0^2(x-z)|\phi_1|^2dx\right)\right)^2 \\
&\leqslant C\left\||\phi|^2\right\|_{L^3}^2\left\|\int v_0^2(x-z)|\phi_1|^2dx\right\|_{L^{\frac{3}{2}}}^2 \\
&\leqslant C\left\|\phi\right\|_{L^6}^4\left\|\phi\right\|_{L^3}^4 \leqslant C\left\|\phi_0\right\|_{L^2}^2\left\|\phi\right\|_{L^6}^6 .
\end{aligned}
$$

i.e. $\|m\|_{L^2_{(x,y)}} \leqslant C\left\|\phi\right\|_{L^6}^3 \leqslant Ct^{-3}$, for $t \geqslant 1$, by Theorem 9. So we conclude the lemma. ■

Remark 15 *Theorem 10 also has consequences on* p *because* $\|p\|_{L^2} \leqslant \|u\|_{L^2}$.

3.3.2 The Existence of u

Because equation (3.2)

$$
\left(iu_t + ug^T + gu - (I+p)m\right) = \left(ip_t + [g,p] + u\overline{m}\right)(I+p)^{-1}u,
$$

is fully nonlinear in k, it is not easy to solve for k directly from the equation. However, if we put in

$$
I + p = \cosh(k) = \sqrt{I + u\overline{u}}
$$

in the operator sense, equation (3.2) becomes a quasilinear NLS equation in $u = \sinh(k)$. In fact, written out explicitly, the left hand side of equation (3.2) is

$$
\begin{aligned}
iu_t + ug^T + gu \quad = \quad & \left(i\frac{\partial}{\partial t} - \Delta_x - \Delta_y\right) u(t,x,y) \qquad (3.16)\\
& + \overline{\phi}_1 \int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz\right) \phi(y_1) u(t, y_1, y) dy_1\\
& + \overline{\phi}_2 \int u(t, x, y_1) \left(\int v(y_1 - y, x-z) \left|\phi_3\right|^2 dz\right) \phi(y_1) dy_1\\
& + \frac{1}{2}\left(\int v_{\cdot-2,\cdot-3} \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz\right)(x) u(t,x,y)\\
& + \frac{1}{2}\left(\int v_{\cdot-2,\cdot-3} \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz\right)(y) u(t,x,y)
\end{aligned}
$$

and the main term of the right hand side

$$
\begin{aligned}
ip_t + [g,p] \quad = \quad & \left(i\frac{\partial}{\partial t} - \Delta_x + \Delta_y\right) p(t,x,y) \qquad (3.17)\\
& + \overline{\phi}_1 \int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz\right) \phi(y_1) p(t, y_1, y) dy_1\\
& - \overline{\phi}_2 \int p(t, x, y_1) \left(\int v(y_1 - y, x-z) \left|\phi_3\right|^2 dz\right) \phi(y_1) dy_1\\
& + \frac{1}{2}\left(\int v_{\cdot-2,\cdot-3} \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz\right)(x) p(t,x,y)\\
& - \frac{1}{2}\left(\int v_{\cdot-2,\cdot-3} \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz\right)(y) p(t,x,y).
\end{aligned}
$$

For our purpose, obtaining some reasonable estimates of u and $p = \cosh(k) - I$ is enough. So we would like to get around solving for k and go to u directly.

But at first, we ask the following: k certainly determines u, but does u determine k? The proof of Theorem 7 actually needs a well-defined k.

We answer the above question by the following lemma:

Lemma 14 *[22]The map*

$$k \mapsto u = \sinh(k)$$

is one to one, onto, continuous, with a continuous inverse, from symmetric Hilbert-Schmidt kernels k onto symmetric Hilbert-Schmidt kernels u.

Proof. The proof of this lemma is in [22]. ■

Now we consider the existence of u satisfying equation (3.2). As asserted, equation (3.2) is a quasilinear NLS of u. However, we can transform it into a semilinear equation which is easier to deal with, through the following lemma.

Lemma 15 *[22]The following equations are equivalent for a symmetric, Hilbert-Schmidt u:*

$$\begin{aligned} iu_t + ug^T + gu &= (I+p)m + (ip_t + [g,p] + u\overline{m})(I+p)^{-1}u \\ iu_t + ug^T + gu &= (I+p)m + \frac{1}{2}\left((I+p)^{-1}m\overline{u} + u\overline{m}(I+p)^{-1}\right)u \\ &\quad + \frac{1}{2}\left[ip_t + [g,p], (I+p)^{-1}\right]u \qquad (3.18) \\ iu_t + ug^T + gu &= (I+p)m + \frac{1}{2}\left((I+p)^{-1}m\overline{u} + u\overline{m}(I+p)^{-1}\right)u \\ &\quad + \frac{1}{2}\left[W, (I+p)^{-1}\right]u \qquad (3.19) \end{aligned}$$

if we set

$$\begin{aligned} W &:= \frac{1}{2\pi i}\int_\Gamma (u\overline{u} - z)^{-1} F (u\overline{u} - z)^{-1}\sqrt{I+z}\,dz \\ F &:= m\overline{u}(I+p) - (I+p)u\overline{m} \end{aligned}$$

Here, Γ is a contour enclosing the spectrum of the non-negative Hilbert-Schmidt operator $u\overline{u}$.

Proof. (Sketch) Equation (3.18) is the same as equation (3.2), suitably re-written. The keystone of the proof is

$$ip_t + [g,p] = W.$$

But

$$
\begin{aligned}
ip_t + [g,p] &= i\left(\sqrt{I+u\overline{u}}\right)_t + [g, \sqrt{I+u\overline{u}}] \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma} (u\overline{u}-z)^{-1}\left(i(u\overline{u})_t + [g,u\overline{u}]\right)(u\overline{u}-z)^{-1}\sqrt{I+z}dz
\end{aligned}
$$

because

$$
\begin{aligned}
\sqrt{I+u\overline{u}} &= -\frac{1}{2\pi i}\int\limits_{\Gamma} (u\overline{u}-z)^{-1}\sqrt{I+z}dz \\
i\left((u\overline{u}-z)^{-1}\right)_t + [g,(u\overline{u}-z)^{-1}] &= -(u\overline{u}-z)^{-1}\left(i(u\overline{u})_t + [g,u\overline{u}]\right)(u\overline{u}-z)^{-1}.
\end{aligned}
$$

The result follows from equation (3.12)

$$
i(u\overline{u})_t + [g,u\overline{u}] = F = m\overline{u}(I+p) - (I+p)u\overline{m}.
$$

■

Whence, we only need to show the existence for equation (3.19) which is of the form

$$
iu_t + ug^T + gu = m + N(u)
$$

where the nonlinear part $N(u)$ involves no derivatives of u. Via the ordinary iteration procedure, we conclude the following existence theorem:

Theorem 11 *[22]Given $u_0 \in L^2_{(x,y)}(\mathbb{R}^6)$ symmetric, there exists ε_0 such that if*

$$
\|m\|_{L^1_t([0,T])L^2_{(x,y)}} \leqslant \varepsilon_0
$$

then there exists $u \in L^\infty_t([0,T])L^2_{(x,y)}$ solving equation (3.19) *and hence equation* (3.2) *with prescribed initial condition $u(0,x,y) = u_0(x,y) \in L^2_{(x,y)}(\mathbb{R}^6)$.*

Since we have shown $\|m\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} < \infty$ in Lemma 13, we can divide $\mathbb{R}^+$ into countably many time intervals $[T_n, T_{n+1}]$ such that $\|m\|_{L_t^1([T_n,T_{n+1}])L^2_{(x,y)}} \leqslant \varepsilon_0$. So the above existence theorem in fact implies the global existence of u and thus p.

Via Theorem 10, we have

$$\|u\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C,$$

which implies

$$\|p\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C.$$

Moreover, the following estimates hold.

Theorem 12 *Let $u \in L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}$ be the solution of equation* (3.2) *subject to $u_0 \in L^2_{(x,y)}(\mathbb{R}^6)$ described in Theorem 11. Then u satisfies the following additional properties:*

$$\left\| \left(i\frac{\partial}{\partial t} - \Delta_x - \Delta_y \right) u \right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C \tag{3.20}$$

$$\left\| \left(i\frac{\partial}{\partial t} - \Delta_x + \Delta_y \right) p \right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C \tag{3.21}$$

where C only depends on v, C_1, C_2 and $\|u_0\|_{L^2_{(x,y)}}$. See Theorem 2 for C_1 and C_2.

Proof. We will only show estimate 3.20. Estimate 3.21 can be shown similarly from

$$ip_t + [g, p] = W.$$

The proof is separated into 2 parts.

On the one hand we show

$$\|iu_t + ug^T + gu\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant \|m\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} + \|N(u)\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C_\varepsilon.$$

On the other hand we control the terms in $iu_t + ug^T + gu$ different from $(i\frac{\partial}{\partial t} - \Delta_x - \Delta_y)u$, namely

$$\int \left(\int v(x - y_1, x - z) \left|\phi_3\right|^2 dz \right) \overline{\phi}(x)\phi(y_1)u(y_1, y)dy_1$$

and

$$\frac{1}{2}\left(\int v(x - y, x - z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right) u.$$

One sees the above two terms from formula (3.16).

Part I. Recall that

$$N(u) = pm + \frac{1}{2}\left((I + p)^{-1} m\overline{u} + u\overline{m}\,(I + p)^{-1} \right) u + \frac{1}{2}\left[W, (I + p)^{-1} \right] u.$$

We have proven

$$\|p\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}} \leqslant \|u\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C_\varepsilon.$$

Together with the fixed time estimate:

$$\|kl\|_{H-S} \leqslant \|k\|_{op} \|l\|_{H-S} \tag{3.22}$$

these take care of most of the terms in $N(u)$ because $(I + p)^{-1}$ and $(u\overline{u} - z)^{-1}|_{z\in\Gamma}$ have uniformly bounded operator norms. In inequality 3.22, $\|\cdot\|_{H-S}$ stands for the Hilbert-Schmidt norm and $\|\cdot\|_{op}$ stands for the operator norm. We only need to account for W. However, the fact that $|z| \leqslant C\|u\|^2_{L^2_{(x,y)}}$ on Γ implies

$$\|W\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C\left(1 + \|u\|^6_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}}\right)\|m\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C.$$

i.e. $\|N(u)\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C.$

Part II.

Using Hölder, it is not difficult to see the estimate

$$
\begin{aligned}
& \left\| \int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz \right) \overline{\phi}(x)\phi(y_1)u(y_1,y)dy_1 \right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \\
= & \left\| \left\{ \int \left| \int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz \right) \overline{\phi}(x)\phi(y_1)u(y_1,y)dy_1 \right|^2 dxdy \right\}^{\frac{1}{2}} \right\|_{L_t^1(\mathbb{R}^+)} \\
\leqslant & \left\| \left(\int \left|\overline{\phi}(x)\right|^2 \left(\int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz \right)^2 \left|\phi(y_1)\right|^2 dy_1 \right) \right. \right. \\
& \left. \left. \left(\int \left|u(y_1,y)\right|^2 dy_1 \right) dxdy \right)^{\frac{1}{2}} \right\|_{L_t^1(\mathbb{R}^+)} \\
= & \left\| \left\{ \int \left(\int v(x-y_1, x-z) \left|\phi_3\right|^2 dz \right)^2 \left|\overline{\phi}(x)\right|^2 \left|\phi(y_1)\right|^2 dxdy_1 \right\}^{\frac{1}{2}} \left\|u\right\|_{L^2_{(x,y)}} \right\|_{L_t^1(\mathbb{R}^+)} \\
\leqslant & \left\|m\right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \left\|u\right\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}} \\
\leqslant & C.
\end{aligned}
$$

It remains to show:

$$
\left\| \left(\int v(x-y, x-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right) u \right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C. \tag{3.23}
$$

Write

$$
\begin{aligned}
& \left\| \left(\int v(x-y, x-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right) u \right\|_{L^2_{(x,y)}} \\
= & \left(\int \left|u(t,x,y)\right|^2 \left(\int v(x-y, x-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right)^2 dxdy \right)^{\frac{1}{2}} \\
\leqslant & C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-y)v_0(x-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right)^2 dxdy \right)^{\frac{1}{2}} \\
& +C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-y)v_0(y-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right)^2 dxdy \right)^{\frac{1}{2}} \\
& +C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-z)v_0(y-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right)^2 dxdy \right)^{\frac{1}{2}}
\end{aligned}
$$

$$= I + II + III.$$

According to the estimate

$$\left| \int v_0(x-y) \left|\phi(y)\right|^2 dy \right| \leqslant C \left\|\phi\right\|_{L^6}^2 \leqslant Ct^{-2}$$

we acquire, for $t \geqslant 1$,

$$I = C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-y) \left|\phi_2\right|^2 dy \right)^4 dxdy \right)^{\frac{1}{2}} \leqslant Ct^{-4} \|u\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}}$$

$$\begin{aligned} II + III &= 2C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-y)v_0(y-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right)^2 dxdy \right)^{\frac{1}{2}} \\ &\leqslant 2C \left(\int \left|u(t,x,y)\right|^2 \left(\int v_0(x-y) \left|\phi_2\right|^2 dy \right)^2 C \left\|\phi\right\|_{L^6}^4 dxdy \right)^{\frac{1}{2}} \\ &\leqslant Ct^{-4} \|u\|_{L_t^\infty(\mathbb{R}^+)L^2_{(x,y)}}. \end{aligned}$$

i.e. estimate 3.23

$$\left\| \left(\int v(x-y,x-z) \left|\phi_2\right|^2 \left|\phi_3\right|^2 dydz \right) u \right\|_{L_t^1(\mathbb{R}^+)L^2_{(x,y)}} \leqslant C.$$

■

3.3.3 The Trace $\int d(t,x,x)\ dx$

Recall that

$$\begin{aligned} d(t,x,y) = &\left(i \sinh(k)_t + \sinh(k) g^T + g \sinh(k) \right) \overline{\sinh(k)} \\ &- \left(i \cosh(k)_t + [g, \cosh(k)] \right) \cosh(k) \\ &- \sinh(k) \overline{m} \cosh(k) - \cosh(k) m \overline{\sinh(k)}. \end{aligned}$$

defined by Formula (3.4). Rewrite it as

$$\begin{aligned} d(t,x,y) &= \left(iu_t + ug^T + gu\right)\overline{u} - \left(ip_t + [g,p]\right)(I+p) \\ & -u\overline{m}(I+p) - (I+p)\, m\overline{u} \end{aligned}$$

because I commutes with everything and $I_t = 0$.

Notice that if $k_1(x,y) \in L^2_{(x,y)}$ and $k_2(x,y) \in L^2_{(x,y)}$ then

$$\int |k_1k_2|(x,x)dx = \int |\int k_1(x,y)k_2(y,x)dy|dx \leqslant \|k_1\|_{L^2_{(x,y)}}\|k_2\|_{L^2_{(x,y)}}.$$

At this point, we have already shown that $m, iu_t + ug^T + gu,\ ip_t + [g,p]$ and $u\overline{m} \in L^1_t(\mathbb{R}^+)L^2_{(x,y)}$ and $u,\, p \in L^\infty_t(\mathbb{R}^+)L^2_{(x,y)}$. So except $\left(ip_t + [g,p]\right) I$, all traces in Formula (3.4) are well-defined and integrable on $\mathbb{R}^+$.

However,

$$ip_t + [g,p] = W,$$

for

$$\begin{aligned} W &= \frac{1}{2\pi i}\int_\Gamma \left(u\overline{u} - z\right)^{-1} F \left(u\overline{u} - z\right)^{-1} \sqrt{I+z}dz \\ F &= m\overline{u}(I+p) - (I+p)u\overline{m}. \end{aligned}$$

Inside the contour integral of W, since $\left(u\overline{u} - z\right)^{-1}|_{z\in\Gamma}$ has uniformly bounded operator norm and $\left|\sqrt{I+z}\right| \leqslant C\left(1 + \|u\|_{L^\infty_t(\mathbb{R}^+)L^2_{(x,y)}}\right)$, we are in fact dealing with

$$(Bounded)(H-S)(H-S)(Bounded)$$

where $H-S$ stands for Hilbert-Schmidt. But $(Bounded)(H-S)$ is Hilbert-Schmidt. So we are looking at $(H-S)(H-S)$ which has a trace well-defined and locally integrable in time.

3.4 Error Estimates / Proof of Theorem 8 (Part II)

We finish the proof of Theorem 8 with the proposition below whose proof consists of classical techniques.

Proposition 8 *Let ϕ to be the solution of the Hartree equation subject to (i), (ii), and (iii). Assume we have*

$$\begin{aligned} \| \left(i\frac{\partial}{\partial t} + \Delta_x \right) \phi \|_{L_t^1(\mathbb{R}^+)L_x^2} &\leqslant C_3 \ , \\ \| \left(i\frac{\partial}{\partial t} - \Delta_x - \Delta_y \right) u \|_{L_t^1(\mathbb{R}^+)L_{(x,y)}^2} &\leqslant C_4 \ , \\ \| \left(i\frac{\partial}{\partial t} - \Delta_x + \Delta_y \right) p \|_{L_t^1(\mathbb{R}^+)L_{(x,y)}^2} &\leqslant C_5 \ , \end{aligned}$$

then we have the error estimates:

$$\begin{aligned} \int \|e^B V e^{-B}\Omega\|_{\mathcal{F}}\, dt &\leqslant C \\ \int \|e^B [A, V] e^{-B}\Omega\|_{\mathcal{F}}\, dt &\leqslant C \\ \int \|e^B [A, [A, V]] e^{-B}\Omega\|_{\mathcal{F}}\, dt &\leqslant C \\ \int \|e^B [A, [A, [A, V]]] e^{-B}\Omega\|_{\mathcal{F}}\, dt &\leqslant C \end{aligned}$$

where C only depends on v, ϕ, C_3, C_4, C_5, and $\|u_0\|_{L^2_{(x,y)}}$.

Remark 16 *We can prove*

$$\| \left(i\frac{\partial}{\partial t} + \Delta_x \right) \phi \|_{L_t^1(\mathbb{R}^+)L_x^2} \leqslant C.$$

with the same method to show estimate 3.23.

Remark 17 *Theorem 12 shows that C_4, C_5 depends only on v, C_1, C_2 and $\|u_0\|_{L^2_{(x,y)}}$. So C here is determined by v, C_1, C_2 and $\|u_0\|_{L^2_{(x,y)}}$.*

Remark 18 *For Theorem 2, we take $k(0,x,y)=0$ i.e. $u_0=0$.*

Ideally, we would like to prove Proposition 8 in complete details. However,

$$\begin{aligned}
e^B a^*_{x_0} e^{-B} &= e^B \begin{pmatrix} a_x & a^*_x \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-B} = \begin{pmatrix} a_x & a^*_x \end{pmatrix} e^K \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \int \left(u(x_1,x_0) a_{x_0} + \overline{\cosh(k)}(x_1,x_0) a^*_{x_0} \right) dx_1, \\
e^B a_{x_0} e^{-B} &= e^B \begin{pmatrix} a_x & a^*_x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-B} = \begin{pmatrix} a_x & a^*_x \end{pmatrix} e^K \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \int \left(\cosh(k)(x_2,x_0) a_{x_2} + \overline{u}(x_2,x_0) a^*_{x_2} \right) dx_2,
\end{aligned}$$

and

$$\cosh(k)(x,y) = \delta(x-y) + p(x,y),$$

their products generate a large number of terms. The fact that we will always commute the annihilations to the right, e.g. $a^*_{x_1} a_{y_2} a^*_{z_2} = \delta(y_2 - z_2) a^*_{x_1} + a^*_{x_1} a^*_{z_2} a_{y_2}$, to avoid $k(x,x)$ or related traces, produces even more terms. Hence it is impractical to list every single term in $e^B V e^{-B}\Omega$ etc., instead, we prove a key lemma and do a typical estimate.

Lemma 16 *(Key Lemma) Let $x_1, y_1, y_2 \in \mathbb{R}^3$, $x_2 \in \mathbb{R}^{n_1}$, $y_3 \in \mathbb{R}^{n_2}$ with the possibility that n_1 or n_2 is zero. Assume f, g satisfy*

$$\begin{aligned}
\left\| \left(i\frac{\partial}{\partial t} \pm \Delta_{x_1} \pm \Delta_{x_2} \right) f(t,x_1,x_2) \right\|_{L^1_t(\mathbb{R}^+)L^2_x} &\leqslant C, \\
\left\| \left(i\frac{\partial}{\partial t} \pm (\Delta_{y_1} + \Delta_{y_2}) \pm \Delta_{y_3} \right) g(t,y_1,y_2,y_3) \right\|_{L^1_t(\mathbb{R}^+)L^2_y} &\leqslant C.
\end{aligned}$$

Moreover suppose $f|_{t=0}$, $g|_{t=0} \in L^2$.

Then

$$\int dt \left(\int v^2(x_1 - y_1, x_1 - y_2) \left| f(t, x_1, x_2) \right|^2 \left| g(t, y_1, y_2, y_3) \right|^2 dx_1 dx_2 dy_1 dy_2 dy_3 \right)^{\frac{1}{2}} \leqslant C.$$

Remark 19 *Specializing to the case $n_1, n_2 = 0, 3, or\ 6$, we will apply Lemma 16 to prove Proposition 8.*

In addition to the endpoint Strichartz estimates [25] which are necessary, we need the following estimate to prove Lemma 16.

Claim 2

$$\left\| \int v_0^2(x-y) v_0^2(x-z) f(y,z) dydz \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3, dx)} \leqslant C \left\| f \right\|_{L^{\frac{3}{2}}(\mathbb{R}^6, dydz)}$$

Proof.

$$\begin{aligned} & \left\| \int v_0^2(x-y) v_0^2(x-z) f(y,z) dydz \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3, dx)} \\ \leqslant & \left\| \int v_0^2(x-y) \left\| v_0^2 \right\|_{L^3} \left\| f(y, \cdot) \right\|_{L^{\frac{3}{2}}} dy \right\|_{L^{\frac{3}{2}}} \\ \leqslant & \left\| v_0^2 \right\|_{L^1} \left\| v_0^2 \right\|_{L^3} \left\| f \right\|_{L^{\frac{3}{2}}} = C \left\| f \right\|_{L^{\frac{3}{2}}}. \end{aligned}$$

■

We can prove Lemma 16 now.

Proof. By Duhamel's principle, it suffices to prove

$$\int dt \Bigg(\int \left| e^{it(\pm \triangle_{x_1} \pm \triangle_{x_2})} f(x_1, x_2) \right|^2 \left| e^{it(\pm(\Delta_{y_1} + \Delta_{y_2}) \pm \Delta_{y_3})} g(y_1, y_2, y_3) \right|^2$$
$$v^2(x_1 - y_1, x_1 - y_2) dx_1 dx_2 dy_1 dy_2 dy_3 \Bigg)^{\frac{1}{2}} \leqslant C \left\| f \right\|_{L^2} \left\| g \right\|_{L^2}$$

because we have

$$
\begin{aligned}
& \| \left(i\frac{\partial}{\partial t} \pm \Delta_{x_1} \pm \Delta_{x_2} \pm (\Delta_{y_1} + \Delta_{y_2}) \pm \Delta_{y_3} \right) f(t,x_1,x_2) g(t,y_1,y_2,y_3) \|_{L^1_t(\mathbb{R}^+) L^2_{(x,y)}} \\
\leqslant \;& C
\end{aligned}
$$

with $f|_{t=0}, g|_{t=0} \in L^2$ which also guarantees $f, g \in L^\infty_t L^2_x$ by the energy estimate.

The proof is divided into two steps.

Step I: Write the partial Fourier transform to be

$$f'_{\xi_2}(x_1) = \int e^{ix_1\xi_1} \hat{f}(\xi_1, \xi_2) d\xi_1,$$

then we have

$$
\begin{aligned}
& \int dx_2 \left| e^{it(\pm\triangle_{x_1} \pm \triangle_{x_2})} f(x_1, x_2) \right|^2 \\
= & \int dx_2 \int d\xi_1 d\xi'_1 d\xi_2 d\xi'_2 e^{ix_1(\xi_1 - \xi'_1)} e^{it(\pm 1)(|\xi_1|^2 - |\xi'_1|^2)} e^{ix_2(\xi_2 - \xi'_2)} e^{it(\pm 1)(|\xi_2|^2 - |\xi'_2|^2)} \\
& \hat{f}(\xi_1, \xi_2) \overline{\hat{f}(\xi'_1, \xi'_2)} \\
= & \int d\xi_1 d\xi'_1 d\xi_2 d\xi'_2 e^{ix_1(\xi_1 - \xi'_1)} e^{it(\pm 1)(|\xi_1|^2 - |\xi'_1|^2)} \delta(\xi_2 - \xi'_2) e^{it(\pm 1)(|\xi_2|^2 - |\xi'_2|^2)} \\
& \hat{f}(\xi_1, \xi_2) \overline{\hat{f}(\xi'_1, \xi'_2)} \\
= & \int d\xi_2 \int d\xi_1 d\xi'_1 e^{ix_1(\xi_1 - \xi'_1)} e^{it(\pm 1)(|\xi_1|^2 - |\xi'_1|^2)} \hat{f}(\xi_1, \xi_2) \overline{\hat{f}(\xi'_1, \xi_2)} \\
= & \int d\xi_2 \left| e^{\pm it\triangle_{x_1}} f'_{\xi_2}(x_1) \right|^2 .
\end{aligned}
$$

Step II: Let ξ_2, η_3 be the phase variables corresponding to x_2, y_3. Utilizing Hölder and Claim 2, we get

$$
\begin{aligned}
& \int dt (\int \left| e^{it(\pm\triangle_{x_1} \pm \triangle_{x_2})} f(x_1, x_2) \right|^2 \left| e^{it(\pm\left(\Delta_{y_1} + \Delta_{y_2}\right) \pm \Delta_{y_3})} g(y_1, y_2, y_3) \right|^2 \\
& v^2(x_1 - y_1, x_1 - y_2) dx_1 dx_2 dy_1 dy_2 dy_3)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 3\int dt\bigg(\int \left|e^{\pm it\triangle_{x_1}} f'_{\xi_2}(x_1)\right|^2 \left|e^{\pm it\left(\Delta_{y_1}+\Delta_{y_2}\right)} g'_{\eta_3}(y_1,y_2)\right|^2 \\
&\quad v_0^2(x_1-y_1)v_0^2(x_1-y_2)dx_1dy_1dy_2d\xi_2d\eta_3\bigg)^{\frac{1}{2}} \\
&\leqslant C\int\left(\int d\xi_2\left\|\left|e^{\pm it\triangle_{x_1}} f'_{\xi_2}(x_1)\right|^2\right\|_{L^3_{x_1}}\right)^{\frac{1}{2}} \\
&\quad\left(\int d\eta_3\left\|\int\left|e^{\pm it\left(\Delta_{y_1}+\Delta_{y_2}\right)} g'_{\eta_3}(y_1,y_2)\right|^2 v_0^2(x_1-y_1)v_0^2(x_1-y_2)dy_1dy_2\right\|_{L^{\frac{3}{2}}_{x_1}}\right)^{\frac{1}{2}} dt \\
&\leqslant C\int dt\left(\int d\xi_2\left\|e^{\pm it\triangle_{x_1}} f'_{\xi_2}(x_1)\right\|^2_{L^6_{x_1}}\right)^{\frac{1}{2}} \\
&\quad\left(\int d\eta_3\left\|e^{\pm it\left(\Delta_{y_1}+\Delta_{y_2}\right)} g'_{\eta_3}(y_1,y_2)\right\|^2_{L^3_{(y_1,y_2)}}\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int dt\int d\xi_2\left\|e^{\pm it\triangle_{x_1}} f'_{\xi_2}(x_1)\right\|^2_{L^6_{x_1}}\right)^{\frac{1}{2}} \\
&\quad\left(\int dt\int d\eta_3\left\|e^{\pm it\left(\Delta_{y_1}+\Delta_{y_2}\right)} g'_{\eta_3}(y_1,y_2)\right\|^2_{L^3_{(y_1,y_2)}}\right)^{\frac{1}{2}} \\
&\leqslant C\left\|f\right\|_{L^2}\left\|g\right\|_{L^2}\quad\text{(endpoint Strichartz [25])}
\end{aligned}
$$

The endpoint Strichartz estimates we used in the last line are the 3d $L_t^2L_x^6$ and the 6d $L_t^2L_x^3$ estimates. ■

3.4.1 Error term $e^BVe^{-B}\Omega$, An Example

Write

$$
\begin{aligned}
& e^BVe^{-B} \\
= & \int dx_0dy_0dz_0v(x_0-y_0,x_0-z_0) \\
& e^Ba^*_{x_0}e^{-B}e^Ba^*_{y_0}e^{-B}e^Ba^*_{z_0}e^{-B}e^Ba_{x_0}e^{-B}e^Ba_{y_0}e^{-B}e^Ba_{z_0}e^{-B} \qquad (3.24)
\end{aligned}
$$

$$\begin{aligned}
= & \int dx_0 dy_0 dz_0 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 v(x_0 - y_0, x_0 - z_0) \\
& \left(u(x_1, x_0) a_{x_1} + \overline{\cosh(k)}(x_1, x_0) a^*_{x_1}\right) \left(u(y_1, y_0) a_{y_1} + \overline{\cosh(k)}(y_1, y_0) a^*_{y_1}\right) \\
& \left(u(z_1, z_0) a_{z_1} + \overline{\cosh(k)}(z_1, z_0) a^*_{z_1}\right) \left(\cosh(k)(x_2, x_0) a_{x_2} + \overline{u}(x_2, x_0) a^*_{x_2}\right) \\
& \left(\cosh(k)(y_2, y_0) a_{y_2} + \overline{u}(y_2, y_0) a^*_{y_2}\right) \left(\cosh(k)(z_2, z_0) a_{z_2} + \overline{u}(z_2, z_0) a^*_{z_2}\right)
\end{aligned}$$

Because we are applying $e^B V e^{-B}$ to Ω, we neglect the terms in product 3.24 which have more annihilation operators than creation operators. It is also unnecessary to consider terms ending with a_{z_2} or $a_{x_2} a_{y_2} a^*_{z_2}$. These facts imply that $e^B V e^{-B}\Omega$ has nonzero elements solely in its 0th, 2nd, 4th and 6th Fock space slots. To exemplify the use of Lemma 16, we estimate two typical terms: the order 6 term

$$\begin{aligned}
& \int dx_0 dy_0 dz_0 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \\
& v(x_0 - y_0, x_0 - z_0) \overline{\cosh(k)}(x_1, x_0) \overline{\cosh(k)}(y_1, y_0) \overline{\cosh(k)} \\
& (z_1, z_0) \overline{u}(x_2, x_0) \overline{u}(y_2, y_0) \overline{u}(z_2, z_0) a^*_{x_1} a^*_{y_1} a^*_{z_1} a^*_{x_2} a^*_{y_2} a^*_{z_2}
\end{aligned}$$

which contributes to the 6th Fock space slot of $e^B V e^{-B}\Omega$ as

$$\begin{aligned}
& \psi_6(x_1, y_1, z_1, x_2, y_2, z_2) \\
= & \int dx_0 dy_0 dz_0 v(x_0 - y_0, x_0 - z_0) \\
& \overline{\cosh(k)}(x_1, x_0) \overline{\cosh(k)}(y_1, y_0) \overline{\cosh(k)}(z_1, z_0) \\
& \overline{u}(x_2, x_0) \overline{u}(y_2, y_0) \overline{u}(z_2, z_0),
\end{aligned}$$

and an order 4 term

$$\begin{aligned}
&\int dx_0 dy_0 dz_0 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 v(x_0 - y_0, x_0 - z_0) \\
&\overline{\cosh(k)}(x_1, x_0)\overline{\cosh(k)}(y_1, y_0)\overline{\cosh(k)}(z_1, z_0)\overline{u}(x_2, x_0)\cosh(k)(y_2, y_0)\overline{u}(z_2, z_0) \\
&a^*_{x_1} a^*_{y_1} a^*_{z_1} a^*_{x_2} a_{y_2} a^*_{z_2} \\
= &\int dx_0 dy_0 dz_0 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 v(x_0 - y_0, x_0 - z_0) \\
&\overline{\cosh(k)}(x_1, x_0)\overline{\cosh(k)}(y_1, y_0)\overline{\cosh(k)}(z_1, z_0)\overline{u}(x_2, x_0)\cosh(k)(y_2, y_0)\overline{u}(z_2, z_0) \\
&(\delta(y_2 - z_2) a^*_{x_1} a^*_{y_1} a^*_{z_1} a^*_{x_2} + a^*_{x_1} a^*_{y_1} a^*_{z_1} a^*_{x_2} a^*_{z_2} a_{y_2})
\end{aligned}$$

which contributes to the 4th Fock space slot of $e^B V e^{-B}\Omega$ as

$$\begin{aligned}
\psi_4(x_1, y_1, z_1, x_2) \quad = \quad &\int dx_0 dy_0 dz_0 dy_2 v(x_0 - y_0, x_0 - z_0) \\
&\overline{\cosh(k)}(x_1, x_0)\overline{\cosh(k)}(y_1, y_0)\overline{\cosh(k)}(z_1, z_0) \\
&\cosh(k)(y_2, y_0)\overline{u}(x_2, x_0)\overline{u}(y_2, z_0)
\end{aligned}$$

neglecting symmetrization and normalization.

3.4.1.1 Estimate of ψ_6, a triple product involving one u

Via the fact that

$$\cosh(k)(x, y) = \delta(x - y) + p(x, y)$$

we write out the product in ψ_6 as

$$\psi_6 = \psi_{6,\delta\delta\delta} + \psi_{6,p\delta\delta} + \psi_{6,pp\delta} + \psi_{6,ppp}$$

according to the factors of cosh carried in each term i.e.

$$\begin{aligned}\psi_{6,\delta\delta\delta} &= \int v(x_0-y_0,x_0-z_0)\delta(x_1-x_0)\delta(y_1-y_0)\delta(z_1-z_0)\\ &\quad \overline{u}(x_2,x_0)\overline{u}(y_2,y_0)\overline{u}(z_2,z_0)dx_0dy_0dz_0\\ &= v(x_1-y_1,x_1-z_1)\overline{u}(x_2,x_1)\overline{u}(y_2,y_1)\overline{u}(z_2,z_1)\end{aligned}$$

and

$$\begin{aligned}\psi_{6,ppp} &= \int v(x_0-y_0,x_0-z_0)\overline{p}(x_1,x_0)\overline{p}(y_1,y_0)\overline{p}(z_1,z_0)\\ &\quad \overline{u}(x_2,x_0)\overline{u}(y_2,y_0)\overline{u}(z_2,z_0)dx_0dy_0dz_0\end{aligned}$$

etc. We proceed to estimate the worst term:

$$\begin{aligned}&\int dt\left(\int\left|\psi_{6,\delta\delta\delta}\right|^2 dx_1dy_1dz_1dx_2dy_2dz_2\right)^{\frac{1}{2}}\\ =&\int dt\left(\int\left|v(x_1-y_1,x_1-z_1)\overline{u}(x_2,x_1)\overline{u}(y_2,y_1)\overline{u}(z_2,z_1)\right|^2 dx_1dy_1dz_1dx_2dy_2dz_2\right)^{\frac{1}{2}}\\ \leqslant& C\end{aligned}$$

where $\overline{u}(y_2,y_1)\overline{u}(z_2,z_1)$ takes the place of g in Lemma 16.

For terms in ψ_6 involving p, we deal with them as the following: By Cauchy-Schwarz on $dx_0dy_0dz_0$, we obtain

$$\begin{aligned}&\int\left(\int\left|\psi_{6,ppp}\right|^2 dx_1dy_1dz_1dx_2dy_2dz_2\right)^{\frac{1}{2}}dt\\ \leqslant&\sup_t\left(\int\left|p(x_1,x_0)p(y_1,y_0)p(z_1,z_0)\right|^2 dx_0dy_0dz_0dx_1dy_1dz_1\right)^{\frac{1}{2}}\\ &\int\left(\int\left|v(x_0-y_0,x_0-z_0)u(x_2,x_0)u(y_2,y_0)u(z_2,z_0)\right|^2 dx_0dy_0dz_0dx_2dy_2dz_2\right)^{\frac{1}{2}}dt\end{aligned}$$

where the first integral is majorized by the energy estimate of p, the second integral is the same as the one appearing in $\psi_{6,\delta\delta\delta}$ and can be taken care of by Lemma 16.

Remark 20 *In the estimate regarding $\psi_{6,ppp}$, we can do Cauchy-Schwarz in another way:*

$$\begin{aligned}
& \int \left(\int \left| \psi_{6,ppp} \right|^2 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \right)^{\frac{1}{2}} dt \\
\leqslant & \sup_t \left(\int \left| p(x_1, x_0) u(y_2, y_0) u(z_2, z_0) \right|^2 dx_0 dy_0 dz_0 dx_1 dy_2 dz_2 \right)^{\frac{1}{2}} \\
& \int \left(\int \left| v(x_0 - y_0, x_0 - z_0) u(x_2, x_0) p(y_1, y_0) p(z_1, z_0) \right|^2 dx_0 dy_0 dz_0 dy_1 dz_1 dx_2 \right)^{\frac{1}{2}} dt
\end{aligned}$$

which also works by Lemma 16. Because $\|u\| \geqslant \|p\|$

3.4.1.2 Estimate of ψ_4, a double product involving one u

$$\begin{aligned}
\psi_4(x_1, y_1, z_1, x_2) &= \int dx_0 dy_0 dz_0 dy_2 v(x_0 - y_0, x_0 - z_0) \\
& \quad \overline{\cosh(k)}(x_1, x_0) \overline{\cosh(k)}(y_1, y_0) \overline{\cosh(k)}(z_1, z_0) \\
& \quad \cosh(k)(y_2, y_0) \overline{u}(x_2, x_0) \overline{u}(y_2, z_0) \\
&= \psi_{4,\delta\delta\delta\delta} + ... + \psi_{4,pppp}
\end{aligned}$$

where the worst term is

$$\begin{aligned}
\psi_{4,\delta\delta\delta\delta} &= \int dx_0 dy_0 dz_0 dy_2 v(x_0 - y_0, x_0 - z_0) \\
& \quad \delta(x_1 - x_0) \delta(y_1 - y_0) \delta(z_1 - z_0) \delta(y_2 - y_0) \overline{u}(x_2, x_0) \overline{u}(y_2, z_0) \\
&= v(x_1 - y_1, x_1 - z_1) \bar{u}(x_2, x_1) \overline{u}(y_1, z_1).
\end{aligned} \tag{3.25}$$

Letting $\overline{u}(y_1, z_1)$ be g in Lemma 16, we derive the desired estimate

$$\int dt \left(\int dx_1 dy_1 dz_1 dx_2 \left| \psi_{4,\delta\delta\delta\delta} \right|^2 \right)^{\frac{1}{2}} \leqslant C.$$

3.4.2 Remark for all other error terms

At a glance, we can handle all terms using Lemma 16, except

$$\int v(x-y,x-z)\phi(x)\phi(y)\phi(z)a_x^*a_y^*a_z^*dxdydz$$

in $[A,[A,[A,V]]]$, since all other terms end with a instead of a^*. This observation allows the application of Lemma 16. But Lemma 16 also applies to

$$e^B\left(\int v(x-y,x-z)\phi(x)\phi(y)\phi(z)a_x^*a_y^*a_z^*dxdydz\right)e^{-B}\Omega.$$

because we can let $\phi(x_1)$ be $f(x_1)$, $\phi(y_1)\phi(y_2)$ be $g(y_1,y_2)$

Therefore we have established Proposition 8 and thus Theorem 8.

3.5 The Long Time Behavior of The Hartree Equation / Proof of Theorem 9

In this section, we discuss the Hartree equation (3.5)

$$i\frac{\partial}{\partial t}\phi+\triangle\phi-\frac{1}{2}\phi\int v(x-y,x-z)\left|\phi(y)\right|^2\left|\phi(z)\right|^2dydz=0$$

where

$$v(x-y,x-z)=v_0(x-y)v_0(x-z)+v_0(x-y)v_0(y-z)+v_0(x-z)v_0(y-z).$$

We assume the nonnegative regular potential v_0 decays fast enough away from the origin and has the property that

$$v_0(x)=v_0(-x).$$

Throughout this section, we write

$$\begin{aligned}
A &= \int v_0(x-y)v_0(x-z)\,|\phi(y)|^2\,|\phi(z)|^2\,dydz \\
B &= \int v_0(x-y)v_0(y-z)\,|\phi(y)|^2\,|\phi(z)|^2\,dydz \\
C &= \int v_0(x-z)v_0(y-z)\,|\phi(y)|^2\,|\phi(z)|^2\,dydz,
\end{aligned}$$

for convenience i.e. equation (3.5) becomes

$$i\frac{\partial}{\partial t}\phi + \triangle\phi - \frac{1}{2}\left(A\phi - B\phi - C\phi\right) = 0. \tag{3.26}$$

So (ii) becomes

$$E(t)|_{t=0} = \left(\frac{1}{2}\int|\nabla\phi|^2 + \frac{1}{6}\int(A+B+C)\,|\phi|^2\right)|_{t=0} < \infty\ .$$

and (i)-(iii) implies

$$E_c(t) = \int t^2\left(\left|\nabla(e^{i\frac{|x|^2}{4t}}\phi)\right|^2 + \frac{1}{6}(A+B+C)\,|\phi|^2\right) < \infty$$

To prove Theorem 9, we are going to argue that

$$\dot{E}_c(t) \leqslant 0 \text{ for } t \geqslant 1$$

which leads to

$$\|\phi\|_{L^6} \leqslant C\left\|\nabla(e^{i\frac{|x|^2}{4t}}\phi)\right\|_{L^2} \leqslant C\left(\frac{E_c(t)}{t^2}\right)^{\frac{1}{2}} \leqslant \frac{C}{t} \text{ for } t \geqslant 1.$$

Here are the details of Theorem 9.

3.5.1 Conservation of Mass, Momentum, and Energy

First, it is not difficult to see the conservation law of the L^2 mass

$$\partial_t\rho - \nabla_j p^j = 0 \tag{3.27}$$

where

$$\rho := \frac{1}{2}\left|\phi\right|^2$$

and

$$p_j := \frac{1}{2i}(\phi\nabla_j\bar{\phi} - \bar{\phi}\nabla_j\phi)$$

because equation (3.26) is of the form

$$i\frac{\partial}{\partial t}\phi + \triangle\phi = F(\left|\phi\right|^2)\phi.$$

Times $-\bar{\phi}$ to equation (3.26), we acquire

$$-p_0 + \frac{1}{2}\sigma - \triangle\rho + (A+B+C)\rho = 0$$

where

$$\begin{aligned} p_0 \quad &: \quad = \frac{1}{2i}(\phi\bar{\phi}_t - \bar{\phi}\phi_t) \\ \sigma \quad &: \quad = tr(\sigma_{jk}) = tr(\nabla_j\bar{\phi}\nabla_k\phi + \nabla_k\bar{\phi}\nabla_j\phi) \end{aligned}$$

Moreover, letting

$$\begin{aligned} \lambda \quad &: \quad = (-p_0 + \frac{1}{2}\sigma + \frac{1}{3}(A+B+C)\rho) = \triangle\rho - \frac{2}{3}(A+B+C)\rho \\ e \quad &: \quad = \frac{1}{2}\sigma + \frac{1}{3}(A+B+C)\rho \end{aligned}$$

produces the conservation law of energy

$$\partial_t e - \nabla_j\sigma_0^j + l_0 = 0 \tag{3.28}$$

where

$$\begin{aligned} \sigma_0^j \quad &: \quad = \phi_t\nabla_j\bar{\phi} + \bar{\phi}_t\nabla_j\phi \\ l_0 \quad &= \quad \frac{2}{3}(A+B+C)\rho_t - \frac{1}{3}(A+B+C)_t\rho. \end{aligned}$$

A direct computation shows that

$$\begin{aligned}
\int A_t\rho &= \int v_0(x-y)v_0(x-z)(|\phi(y)|^2|\phi(z)|^2)_t(\frac{|\phi(x)|^2}{2})dxdydz \\
&= \int v_0(x-y)v_0(y-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz \\
&\quad + \int v_0(x-z)v_0(y-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz
\end{aligned}$$

and

$$\begin{aligned}
\int B_t\rho &= \int v_0(x-y)v_0(x-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz \\
&\quad + \int v_0(x-z)v_0(y-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz
\end{aligned}$$

$$\begin{aligned}
\int C_t\rho &= \int v_0(x-y)v_0(y-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz \\
&\quad + \int v_0(x-y)v_0(x-z)|\phi(y)|^2|\phi(z)|^2(\frac{|\phi(x)|^2}{2})_t dxdydz
\end{aligned}$$

i.e.

$$\int (A+B+C)_t\rho = 2\int (A+B+C)\rho_t$$

which implies $\int l_0 = 0$ i.e. the conservation of energy

$$E(t) = \frac{1}{2}\int |\nabla\phi|^2 + \frac{1}{6}\int (A+B+C)|\phi|^2$$

Similarly, we derive the conservation law of momentum:

$$\partial_t p_j - \nabla_k\left\{\sigma_j^j - \delta_j^k\lambda\right\} + l_j = 0 \tag{3.29}$$

where

$$l_j := \frac{2}{3}(A+B+C)\rho_j - \frac{1}{3}(A+B+C)_j\rho.$$

3.5.2 Conformal Identity

At this point, if we multiply conservation law 3.27 by $\frac{|x|^2}{2}$, 3.29 by tx^j and 3.28 by t^2 and add the resulting identities, we obtain the conformal identity:

$$\partial_t e_c - \nabla_j \tau^j + r = 0$$

where

$$\begin{aligned}
e_c \quad &: \quad = (\frac{|x|^2}{2})\rho + tx^j p_j + t^2 e = t^2 \left(\left| \nabla (e^{i\frac{|x|^2}{4t}}\phi) \right|^2 + \frac{1}{3}(A+B+C)\rho \right) \\
\tau^j \quad &: \quad = (\frac{|x|^2}{2})p^j + tx^k \sigma^j_k + tx^j \left(-\triangle\rho + \frac{2}{3}(A+B+C)\rho \right) + t^2 \sigma^j_0 \\
r \quad &: \quad = t^2 l_0 + tx^j l_j - nt\triangle\rho + t(n-1)\frac{2}{3}(A+B+C)\rho.
\end{aligned}$$

This suggests

$$\dot{E}_c + R_c = 0 \tag{3.30}$$

where

$$R_c := t\int \left((n-1)\frac{2}{3}(A+B+C)\rho + x^j l_j \right) dx.$$

To determine $\dot{E}_c$, we calculate

$$\begin{aligned}
& \int x^j l_j dx \\
= \quad & \frac{8}{3}\int x^j v(x-y,x-z)(\rho_1)_j \rho_2\rho_3 - \frac{8}{3}\int x^j v(x-y,x-z)\rho_1(\rho_2)_j\rho_3 \\
= \quad & \frac{8}{3}\int \rho_3 v(x-y,x-z)x^j[(\rho_1)_j\rho_2 - \rho_1(\rho_2)_j] \\
= \quad & \frac{16}{3}\int \rho_3 v(x-y,x-z)x\cdot\nabla_{1-2}(\rho_1\rho_2) \\
= \quad & \frac{8}{3}\int \rho_3 v(x-y,x-z)(x+y)\cdot\nabla_{1-2}(\rho_1\rho_2) \\
& + \frac{8}{3}\int \rho_3 v(x-y,x-z)(x-y)\cdot\nabla_{1-2}(\rho_1\rho_2)
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\frac{8}{3}\int \rho_1\rho_2\rho_3\nabla_{1-2}v(x-y,x-z)\cdot(x+y) \\
& -\frac{8}{3}\int \rho_1\rho_2\rho_3\nabla_{1-2}v(x-y,x-z)\cdot(x-y) \\
& -\frac{8}{3}n\int \rho_1\rho_2\rho_3 v(x-y,x-z)
\end{aligned}
$$

where $\nabla_{1-2} = \nabla_{x-y} = \frac{1}{2}(\nabla_x - \nabla_y)$.

Insert formula (1.9)

$$
v(x-y,x-z) = v_0(x-y)v_0(x-z) + v_0(x-y)v_0(y-z) + v_0(x-z)v_0(y-z)
$$

to the above computation, it is

$$
\begin{aligned}
& \int x^j l_j dx + \frac{2}{3}n\int (A+B+C)\rho \\
= \quad & -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(x-z)\left(\nabla_{1-2}v_0(x-y)\right)\cdot(x+y) \\
& -\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-y)\left(\nabla_x v_0(x-z)\right)\cdot(x+y) \\
& -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\nabla_{1-2}v_0(x-y)\right)\cdot(x+y) \\
& +\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-y)\left(\nabla_y v_0(y-z)\right)\cdot(x+y) \\
& -\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\nabla_x v_0(x-z)\right)\cdot(x+y) \\
& +\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-z)\left(\nabla_y v_0(y-z)\right)\cdot(x+y) \\
& -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(x-z)\left(\nabla_{1-2}v_0(x-y)\right)\cdot(x-y) \\
& -\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-y)\left(\nabla_x v_0(x-z)\right)\cdot(x-y) \\
& -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\nabla_{1-2}v_0(x-y)\right)\cdot(x-y) \\
& +\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-y)\left(\nabla_y v_0(y-z)\right)\cdot(x-y) \\
& -\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\nabla_x v_0(x-z)\right)\cdot(x-y) \\
& +\frac{4}{3}\int \rho_1\rho_2\rho_3 v_0(x-z)\left(\nabla_y v_0(y-z)\right)\cdot(x-y).
\end{aligned}
$$

Notice that, in the above calculation.

$$\begin{aligned} 1st + 3rd &= 0 \\ (4th + 11th) + (6th + 13th) &= 0 \\ (2nd + 9th) + (5th + 12th) &= -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\nabla_x v_0(x-z)\right)\cdot(x-z). \end{aligned}$$

So $\int x^j l_j dx$ simplifies to

$$\begin{aligned} \int x^j l_j dx &= -\frac{2}{3}n\int (A+B+C)\rho \\ &\quad -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(x-z)\left(\left(\nabla v_0\right)(x-y)\right)\cdot(x-y) \\ &\quad -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\left(\nabla v_0\right)(x-y)\right)\cdot(x-y) \\ &\quad -\frac{8}{3}\int \rho_1\rho_2\rho_3 v_0(y-z)\left(\left(\nabla v_0\right)(x-z)\right)\cdot(x-z). \end{aligned}$$

Hence

$$\begin{aligned} R_c &= t\int\left((n-1)\frac{2}{3}(A+B+C)\rho + x^j l_j\right)dx \\ &= -\frac{8}{3}t\int \rho_1\rho_2\rho_3 v_0(x-z)\left\{v_0(x-y) + \left(\left(\nabla v_0\right)(x-y)\right)\cdot(x-y)\right\} \\ &\quad -\frac{8}{3}t\int \rho_1\rho_2\rho_3 v_0(y-z)\left\{v_0(x-y) + \left(\left(\nabla v_0\right)(x-y)\right)\cdot(x-y)\right\} \\ &\quad -\frac{8}{3}t\int \rho_1\rho_2\rho_3 v_0(y-z)\left\{v_0(x-z) + \left(\left(\nabla v_0\right)(x-z)\right)\cdot(x-z)\right\}. \end{aligned}$$

When v_0 decays fast enough, we have

$$R_c \geqslant 0,$$

or in other words

$$\dot{E}_c \leqslant 0 \text{ for } t \geqslant 1,$$

which implies $E_c(t)$ does not increase as claimed.

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