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Stochastic Comparisons In
Vacation Models

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STOCHASTIC COMPARISONS
IN VACATION MODELS*

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Abstract

We consider single-server queueing systems, known as vacation models, where at every service completion the server might either serve the next customer from the queue (if any) or take a vacation (i.e., become unavailable to the customers for a random period of time) depending on the *service schedule* of the model. Using coupling arguments, we make stochastic comparisons between quantities of interest of one vacation model to those of another vacation model under a different service schedule. We first establish a stochastic ordering for sequences of service completion epochs, from which stochastic comparisons for waiting time sequences and queue size processes can be deduced. These comparisons are then used to obtain some monotonicity results for vacation models with limited and Bernoulli service schedules.

Keywords: Stochastic Comparison, Coupling, Pathwise Comparison, Vacation Models, Monotonicity.

1. Introduction

A vacation model is a single-server queue where the server sometimes *takes a vacation* (i.e., becomes unavailable to the customers) for a random period of time. The server can start a vacation either at a service completion or at the end of a vacation, and only at these epochs. A vacation is always taken if the queue is empty at either a service or vacation completion. If a customer is waiting in the queue when the server returns from a vacation, the server has to resume its duty. If a customer is waiting in the queue when the server just completes a service, however, the server has a choice of either serving that customer or starting a vacation. In [1], Doshi calls such a system a *multiple-vacation model* to differentiate it from a *single-vacation model* where the server waits for the arrival of the next customer when, upon returning from a vacation, it does not find any customer in the queue.

Whether or not the server can take a vacation upon completing a service is determined by a set of rules called the *service schedule* of the model. The following are some of the service schedules that have appeared in the literature.

1. *Exhaustive schedule.* At every service completion, the server takes a vacation if and only if the queue is empty.

2. *Limited schedule.* At every service completion, the server takes a vacation if and only if the queue is empty or a prespecified number of customers have been served since the server returned from its previous vacation.

3. *Gated schedule.* At every service completion, the server takes a vacation if and only if all those customers that were present in the queue when the server returned from its previous vacation have been served.

4. *Bernoulli schedule.* At every service completion, the server takes a vacation if the queue is empty. Otherwise, with probability p, it serves the next customer and with probability $1 - p$, it takes a vacation.

Vacation models have been analyzed by many authors and the reader is refered to [1] for a survey on the subject. The importance of this class of queueing systems stems from the fact that they arise naturally as models of many real systems. As an example, a computer system in which the processor either performs background jobs or attends to the real-time processes can be modeled by a multiple-vacation model. A machine that needs a maintenance after each busy period can be modeled by a single-vacation model. Vacation models have also been used to study the performance of cyclic-service queueing systems, i.e., systems consisting of a number of queues served in a cyclic

order by a single server.

In this paper, we compare the performance of vacation models under different service schedules. More specifically, we stochastically compare some quantities of interest in a vacation model to those in another vacation model which is either under a different service schedule or under the same schedule but with different schedule parameters. Such stochastic comparisons are useful for obtaining monotonicity results, bounds, and approximations for systems which are too difficult to analyze exactly.

We use a technique called *coupling* to get a stochastic comparison result between the sequence of departure epochs of one vacation model to that of another vacation model. Using this result, other quantities, such as the queue length and the waiting time processes, can be compared as well. Other intuitive results such as the monotonicity of waiting time process as a function of the parameter p in the Bernoulli schedule or as a function of the limit in the limited schedule are also obtained. It should be noted that the comparisons established here are strong in that they are made between the transient versions of the quantities of interest. In the sequel, only multiple-vacation-models will be considered, although similar results can also be established for the single-vacation models.

The rest of the paper is organized as follows. In Section 2, we present the precise description of a multiple-vacation queueing system and recall the definition of a stochastic order between two processes. The comparison results are established in Section 3, and we devote Section 4 to various monotonicity results.

2. Preliminaries

A multiple-vacation model, denoted by Q, is governed by the sequence of random variables $\{\tau_n, \sigma_n, V_n, u_n, n = 1, 2, \ldots\}$ with the following interpretation ($n = 1, 2, \ldots$):

$\tau_n =$ time between the $(n-1)$th and the nth arriving customer.

$\sigma_n =$ length of the nth service.

$V_n =$ length of the nth vacation period.

$u_n =$ server's decision at the end of the nth service, with $u_n = 1$ (resp. $u_n = 0$) if the server decides to serve the next customer (resp. to take a vacation).

From the random variables mentioned above, define the following quantities ($n = 1, 2, \ldots$).

$A_n =$ arrival time of the nth arriving customer ($= \sum_{j=1}^{n} \tau_j$).

$D_n =$ departure (i.e., service completion) time of the nth departing customer.

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r_n = number of vacations completed up to time D_n (not including the one that might be started at time D_n).

q_n = number of customers left behind by the nth departing customer.

s_n = number of customers (including the nth departing customer) that have been served up to time D_n since the end of the r_nth vacation.

c_n = number of customers in the queue at the end of the r_nth vacation.

W_n = waiting time (the period from the arrival time to the start of service) of the nth arriving customer.

$N(t)$ = number of customers in the system at time $t \geq 0$.

Notice that the nth arriving customer might not be identical to the nth departing customer since we do not limit ourselves to the first-come-first-serve (FCFS) discipline. We will see that most of the comparison results obtained here hold true irrespective of the order in which the customers are served. However, we will need the following assumptions (A1)–(A5), where

(A1) Once a customer enters the system, it does not leave until its service is completed.

(A2) Once a service is started, it is carried out to completion, i.e., there is no service preemption.

(A3) All service and vacation times are strictly positive with probability 1, i.e., their pdf's have no atom at the origin.

(A4) A customer arrives at time $t = 0$ to an empty queue and receives service immediately.

(A5) Since a vacation has to be taken when the queue is empty at a service completion, $u_n = 0$ whenever $q_n = 0$, $n = 1, 2, \ldots$.

A multiple-vacation queueing system can be described as follows. At time D_n the server completes a service of length σ_n. If $u_n = 1$, a new service of length σ_{n+1} begins. However, if $u_n = 0$, the server starts a vacation of length V_{r_n+1}. At the end of this vacation (i.e., at time $D_n + V_{r_n+1}$), the server starts a service of length σ_{n+1} if the queue is not empty. Otherwise, it takes additional vacations until the next customer arrives.

Since the server is always doing one of two things, i.e., either serving a customer or taking a vacation, we observe that

$$D_n = \sum_{j=1}^{n} \sigma_j + \sum_{j=1}^{r_n} V_j, \qquad n = 1, 2, \ldots \tag{2.1}$$

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and from the description of the system, we easily see that

$$D_{n+1} = \begin{cases} D_n + (1 - u_n)V_{r_n+1} + \sigma_{n+1} & \text{if } q_n > 0; \\ D_n + \displaystyle\sum_{r_n < j \le r_{n+1}} V_j + \sigma_{n+1} & \text{if } q_n = 0, \end{cases} \qquad n = 1, 2, \ldots \qquad (2.2)$$

If $q_n = 0$, the server immediately starts a vacation and continues to take additional vacations until the next customer arrives. Because of assumptions (A1) and (A2) the next customer is the $(n+1)$st arriving customer and thus

$$r_{n+1} = \min\{k > r_n : D_n + \sum_{r_n < j \le k} V_j \ge A_{n+1}\}. \qquad (2.3)$$

In terms of the random variables described above, the service schedules mentioned in Section 1 can be more precisely defined as follows.

1. The exhaustive schedule:

$$u_n = \begin{cases} 1 & \text{if } q_n \ge 1; \\ 0 & \text{otherwise.} \end{cases} \qquad (2.4)$$

2. The limited schedule with parameter L, $L = 1, 2, \ldots$:

$$u_n = \begin{cases} 1 & \text{if } q_n \ge 1 \text{ and } 1 \le s_n \le L - 1; \\ 0 & \text{otherwise.} \end{cases} \qquad (2.5)$$

3. The gated schedule:

$$u_n = \begin{cases} 1 & \text{if } 1 \le s_n \le c_n - 1; \\ 0 & \text{otherwise.} \end{cases} \qquad (2.6)$$

4. The Bernoulli schedule with parameter p, $0 \le p \le 1$:

$$P^n[u_n = 1] = \begin{cases} p & \text{if } q_n \ge 1; \\ 0 & \text{otherwise} \end{cases} \qquad (2.7)$$

where P^n is the probability conditioned on all the information prior to the nth decision.

From Stoyan [5, p. 26], we borrow the following definitions.

Definition 1. *An $\mathbb{R}^k$-valued random variable X^1 is stochastically smaller than another $\mathbb{R}^k$-valued random variable X^2, denoted $X^1 \le_{st} X^2$, if*

$$E[f(X^1)] \le E[f(X^2)]$$

for every monotone non-decreasing function $f : \mathbb{R}^k \to \mathbb{R}$ for which the expectations are well defined. Here a monotone non-decreasing function $f : \mathbb{R}^k \to \mathbb{R}$ is understood as a function f with the property that $f(x) \le f(y)$ whenever $x^i \le y^i, 1 \le i \le k$.

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This definition can be extended to random sequences and stochastic processes as follows.

Definition 2. *Let $X^i = \{X^i(t), t \in T\}$, $i = 1, 2$, be two families of $\mathbb{R}^l$-valued random variables with $T \subseteq [0, \infty)$. Then we say $X^1 \leq_{st} X^2$ if $\left(X^1(t_1), \ldots, X^1(t_n)\right) \leq_{st} \left(X^2(t_1), \ldots, X^2(t_n)\right)$ for all $n = 1, 2, \ldots$ and $t_1, \ldots, t_n \in T$.*

The following result is a special case of Proposition 1.10.4 in Stoyan [5, p. 28], the proof of which can be found in Kamae et al. [2].

Lemma 1. *Let $X^i = \{X^i(t), t \in T\}$, $i = 1, 2$, be two random sequences with $T = \{0, 1, \ldots\}$, or two stochastic processes with sample paths which are right continuous with left limits with $T = [0, \infty)$. Then $X^1 \leq_{st} X^2$ if and only if there exist two stochastic processes $\{\hat{X}^i(t), t \in T\}$, $i = 1, 2$, defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that*

$$\hat{X}^1(t) \leq \hat{X}^2(t), \qquad t \in T \tag{2.8}$$

and

$$\{\hat{X}^i(t), t \in T\} =_{st} \{X^i(t), t \in T\}, \qquad i = 1, 2$$

where $=_{st}$ denotes equivalence in probability law. Furthermore, $X^1 =_{st} X^2$ if and only if we have an equality in (2.8).

The method of showing that two stochastic processes satisfy Definition 2 through the use of Lemma 1 is known as *coupling.*

3. Comparison Results

We first prove the following basic result.

Theorem 1. *Let Q^i, $i = 1, 2$, be multiple-vacation queueing models each under a service schedule where the decision process $\{u_n^i, n = 1, 2, \ldots\}$ is such that, for all $n = 1, 2, \ldots$, u_n^i is fully determined by q_n^i, i.e., $u_n^i = \gamma_n^i(q_n^i)$. If*

$$\gamma_n^1(q) \geq \gamma_n^2(q), \qquad q = 0, 1, \ldots; n = 1, 2, \ldots \tag{3.1}$$

and

$$\{\tau_n^1, \sigma_n^1, V_n^1, n = 1, 2, \ldots\} =_{st} \{\tau_n^2, \sigma_n^2, V_n^2, n = 1, 2, \ldots\}, \tag{3.2}$$

then the stochastic comparison

$$\{D_n^1, n = 1, 2, \ldots\} \leq_{st} \{D_n^2, n = 1, 2, \ldots\} \tag{3.3}$$

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holds true.

Proof: By Lemma 1, (3.2) implies that there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ on which $\hat{Q}^i$ (i.e., $\{\hat{\tau}_n^i, \hat{\sigma}_n^i, \hat{V}_n^i, n = 1, 2, \ldots\}$), $i = 1, 2$, are defined such that $\hat{Q}^i$ is stochastically equivalent to Q^i for each $i = 1, 2$ and

$$\hat{\tau}_n^1 = \hat{\tau}_n^2 = \hat{\tau}_n \quad \hat{\sigma}_n^1 = \hat{\sigma}_n^2 = \hat{\sigma}_n \quad \hat{V}_n^1 = \hat{V}_n^2 = \hat{V}_n, \qquad n = 1, 2, \ldots \tag{3.4}$$

For simplicity, we shall drop all the hats from our notation by assuming without loss of generality that Q^1 and Q^2 are such that for $n = 1, 2, \ldots$, $\tau_n^1 = \tau_n^2 = \tau_n$, $\sigma_n^1 = \sigma_n^2 = \sigma_n$, and $V_n^1 = V_n^2 = V_n$.

To prove (3.3) it suffices to show that the pathwise comparison

$$D_n^1 \leq D_n^2, \qquad n = 1, 2, \ldots \tag{3.5}$$

holds almost everywhere in Ω.

To that end, we use an argument by induction. By assumption (A4), $D_1^1 = \sigma_1 = D_1^2$ and so (3.5) holds for $n = 1$. The induction hypothesis postulates that (3.5) holds for $n = 1, \ldots, m$ for some $m \geq 1$, and the induction step will be completed by showing

$$D_{m+1}^1 \leq D_{m+1}^2. \tag{3.6}$$

We consider the following three cases and show that (3.6) holds in each case.

Case 1: $q_m^2 > 0$ and $D_m^1 = D_m^2 = D_m$.

By (2.1) and assumption (A3), the condition $D_m^1 = D_m^2$ implies $r_m^1 = r_m^2 = r_m$ a.s. Furthermore, since both Q^1 and Q^2 have the same arrival process, the number of arrivals up to time D_m in both queue are identical (denoted by $A(D_m)$). Since the number of departures up to and including time D_m in each queue is m, we have by assumption (A1) that $q_m^1 = A(D_m) - m = q_m^2 > 0$, in which case we have by (2.2) that

$$D_{m+1}^i = D_m + (1 - u_m^i)V_{r_m+1} + \sigma_{m+1}, \qquad i = 1, 2. \tag{3.7}$$

Owing to (3.1) and to the fact that $q_m^1 = q_m^2 = q_m$, we have $u_m^1 = \gamma_m^1(q_m) \geq \gamma_m^2(q_m) = u_m^2$ and (3.6) thus follows from (3.7).

Case 2: $q_m^2 > 0$ and $D_m^1 < D_m^2$.

By (2.1) and assumption (A3), $D_m^1 < D_m^2$ implies $r_m^2 > r_m^1$ a.s. and

$$D_m^2 = D_m^1 + \sum_{j=r_m^1+1}^{r_m^2} V_j. \tag{3.8}$$

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If $q_m^1 > 0$, then the server in Q^1 takes at most one vacation before serving the next customer, i.e.,

$$D_{m+1}^1 \leq D_m^1 + V_{r_m^1+1} + \sigma_{m+1}.$$

But, we see from (3.8) that

$$D_m^1 + V_{r_m^1+1} \leq D_m^2$$

and so

$$D_{m+1}^1 \leq D_m^2 + \sigma_{m+1} \leq D_{m+1}^2$$

where the last inequality follows from (2.2).

If $q_m^1 = 0$, then the server in Q^1 takes a number of vacations until the $(m+1)$st customer arrives. Since $q_m^2 > 0$, this customer must have arrived before time D_m^2 but after time D_m^1, i.e., $D_m^1 < A_{m+1} \leq D_m^2$. From (3.8) we thus have $A_{m+1} \leq D_m^1 + \sum_{j=r_m^1+1}^{r_m^2} V_j$ and in view of (2.3), we conclude that $r_m^2 \geq r_{m+1}^1$. Therefore,

$$D_{m+1}^1 \leq D_m^1 + \sum_{j=r_m^1+1}^{r_m^2} V_j + \sigma_{m+1} = D_m^2 + \sigma_{m+1} \leq D_{m+1}^2$$

where again the last inequality follows from (2.2).

Case 3. $q_m^2 = 0$

Since $D_m^1 \leq D_m^2$ by the induction hypothesis, and both Q^1 and Q^2 have the same arrival process, we see that $q_m^1 \leq q_m^2$, whence $q_m^1 = 0$ since $q_m^2 = 0$.

If $D_m^1 = D_m^2 = D_m$, then as in Case 1 we have $r_m^1 = r_m^2 = r_m$. From (2.3) we have

$$r_{m+1}^1 = \min\{k > r_m : D_m + \sum_{j=r_m+1}^{k} V_j \geq A_{m+1}\} = r_{m+1}^2 = r_{m+1}$$

and so

$$D_{m+1}^1 = D_m + \sum_{j=r_m+1}^{r_{m+1}} V_j + \sigma_{m+1} = D_{m+1}^2.$$

If $D_m^1 < D_m^2$, then as in Case 2 we have $r_m^1 < r_m^2$. Again from (2.3) we have

$$r_{m+1}^2 = \min\{k > r_m^2 : D_m^2 + \sum_{j=r_m^2+1}^{k} V_j \geq A_{m+1}\}$$

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and so

$$r_{m+1}^2 > r_m^2 > r_m^1 \tag{3.9}$$

and

$$A_{m+1} \le D_m^2 + \sum_{j=r_m^2+1}^{r_{m+1}^2} V_j$$

$$= D_m^1 + \sum_{j=r_m^1+1}^{r_m^2} V_j + \sum_{j=r_m^2+1}^{r_{m+1}^2} V_j \tag{3.10}$$

$$= D_m^1 + \sum_{j=r_m^1+1}^{r_{m+1}^2} V_j$$

where (3.8) was used to obtain the first equality. Furthermore, r_{m+1}^2 is the smallest integer satisfying (3.9) and (3.10). Hence, by the definition of r_{m+1}^1, we see that $r_{m+1}^1 = r_{m+1}^2$ and thus $D_{m+1}^1 = D_{m+1}^2$. $\qquad\square$

Notice that we have proved Theorem 1 without using any knowledge of the order in which the customers in each of Q^1 and Q^2 are served.

Using Theorem 1, comparison of other quantities can be readily made. For $i = 1, 2$ and $t \ge 0$, the number $N^i(t)$ of customers in Q^i at time t is simply $N^i(t) = A(t) - D^i(t)$ where $A(t) = \max\{k : A_k \le t\}$ and $D^i(t) = \max\{k : D_k^i \le t\}$. From (3.5), it is immediate that $D^1(t) \ge D^2(t)$ for all $t \ge 0$ and so $N^1(t) \le N^2(t)$ for all $t \ge 0$. Furthermore, if the service discipline is FCFS, we have $W_n^i = D_n^i - \sigma_n - A_n$ for $n = 1, 2, \ldots$ and $i = 1, 2$. Again, (3.5) implies that $W_n^1 \le W_n^2$ for all $n = 1, 2, \ldots$. We summarize these facts in

Corollary 1. *For Q^1 and Q^2 as in Theorem 1, we have $\{N^1(t), t \ge 0\} \le_{st} \{N^2(t), t \ge 0\}$. Furthermore, if the service discipline in each system is FCFS then $\{W_n^1, n = 1, 2, \ldots\} \le_{st} \{W_n^2, n = 1, 2, \ldots\}$.*

If Q^1 and Q^2 in Theorem 1 are such that for each $i = 1, 2$ $N^i(t)$ goes to N^i in distribution as t goes to infinity, then obviously we have $N^1 \le_{st} N^2$. The same can be said about the waiting times for the case of FCFS discipline. If the discipline is not FCFS, however, we can only compare the limiting averages of the customer sojourn times, as stated in the following corollary which can be shown using Little's result [4].

Corollary 2. *Let Q^1 and Q^2 be as in Theorem 1 and $S_j^i = W_j^i + \sigma_j^i$ be the sojourn time of the jth*

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arriving customer in Q^i. If for each $i = 1, 2$ the limit

$$\lambda^i = \lim_{t \to \infty} \frac{1}{t} \int_0^t A^i(x)\, dx$$

a.s. exists as a finite constant and the limits

$$\bar{N}^i = \lim_{t \to \infty} \frac{1}{t} \int_0^t N^i(x)\, dx, \qquad \bar{S}^i = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n S_j^i$$

a.s. exist as finite random variables, then $\bar{S}^1 \leq_{\text{st}} \bar{S}^2$.

A careful examination of the proof of Theorem 1 reveals that (3.6) was shown to hold for Cases 2 and 3 without making use of any knowledge about the service schedules used for the systems. As a consequence, if Q^1 is under the exhaustive schedule then for Case 1 we have $1 = \hat{u}_m^1 \geq \hat{u}_m^2$ no matter what schedule (even one with a randomized decision process) is employed in $\hat{Q}^2$. The same can be said when Q^2 is under a limited schedule with threshold value one (single schedule) and Q^1-under any service schedule since in this case we have $\hat{u}_m^1 \geq \hat{u}_m^2 = 0$ for Case 1. Thus we have the following theorem.

Theorem 2. *Let Q^{ex} and Q^{si} be multiple-vacation queueing models under the exhaustive and single schedules, respectively, and let Q be another multiple-vacation queueing system under any schedule. If*

$$\{\tau_n^{ex}, \sigma_n^{ex}, V_n^{ex}, n = 1, 2, \ldots\} =_{\text{st}} \{\tau_n, \sigma_n, V_n, n = 1, 2, \ldots\} =_{\text{st}} \{\tau_n^{si}, \sigma_n^{si}, V_n^{si}, n = 1, 2, \ldots\},$$

then the stocahstic comparisons

$$\{D_n^{ex}, n = 1, 2, \ldots\} \leq_{\text{st}} \{D_n, n = 1, 2, \ldots\} \leq_{\text{st}} \{D_n^{si}, n = 1, 2, \ldots\}$$

hold true.

4. Monotonicity in The Limited and The Bernoulli Schedules

In this section, we show that the departure times, the number of customers in the system, and the waiting times (for the case of FCFS discipline) are monotonically decreasing (stochastically) with respect to the threshold parameter in the limited schedule, and with respect to the parameter p in the Bernoulli schedule. To that end, we generalize Theorem 1 to include the limited and Bernoulli schedules in Theorem 3 and 4, respectively.

Theorem 3. *Let* Q^i, $i = 1, 2$, *be multiple-vacation models each under a service schedule where the decision process* $\{u_n^i, n = 1, 2, \ldots\}$ *is such that, for all* $n = 1, 2, \ldots$, u_n^i *is fully determined by* q_n^i *and* s_n^i, *i.e.,* $u_n^i = \eta_n^i(q_n^i, s_n^i)$. *Suppose that, for each* $i = 1, 2$ *and* $n = 1, 2, \ldots$, η_n^i *is such that*

$$s \leq s' \quad \text{implies} \quad \eta_n^i(q, s) \geq \eta_n^i(q, s'), \qquad q = 0, 1, \ldots \tag{4.1}$$

and that for each $n = 1, 2, \ldots$,

$$\eta_n^1(q, s) \geq \eta_n^2(q, s), \qquad q = 0, 1, \ldots; s = 1, 2, \ldots \tag{4.2}$$

If

$$\{\tau_n^1, \sigma_n^1, V_n^1, n = 1, 2, \ldots\} =_{\text{st}} \{\tau_n^2, \sigma_n^2, V_n^2, n = 1, 2, \ldots\},$$

then the stochastic comparison

$$\{D_n^1, n = 1, 2, \ldots\} \leq_{\text{st}} \{D_n^2, n = 1, 2, \ldots\}$$

holds true.

Proof: The proof is similar to that of Theorem 1 and in view of the remark preceding Theorem 2, we need only show that here (3.6) holds for Case 1 where $D_m^1 = D_m^2 = D_m$ with $q_m^2 > 0$.

If we can show that $s_m^1 \leq s_m^2$, then by (4.1) and (4.2), we have

$$u_m^1 = \eta_m^1(q_m, s_m^1) \geq \eta_m^1(q_m, s_m^2) \geq \eta_m^2(q_m, s_m^2) = u_m^2$$

and so we obtain (3.6) from (3.7).

Suppose that $s_m^1 > s_m^2$. Note that for $i = 1, 2$, the definition of s_m^i implies that the $(m - s_m^i + 1)$st up to the mth departures in Q^i are not interrupted by any vacation, so that

$$D_k^i = D_m - \sum_{j=k+1}^{m} \sigma_j, \qquad m - s_m^i + 1 \leq k \leq m - 1 \tag{4.3}$$

and, in particular, with $i = 1$ and $k = m - s_m^2 (> m - s_m^1)$, we find

$$D_{m-s_m^2}^1 = D_m - \sum_{j=m-s_m^2+1}^{m} \sigma_j. \tag{4.4}$$

From the definition of s_m^2 we see that the $(m - s_m^2)$th and the $(m - s_m^2 + 1)$st departure epochs in Q^2 are separated by the $(m - s_m^2 + 1)$st service time plus at least one vacation period and therefore

$$D_{m-s_m^2+1}^2 - D_{m-s_m^2}^2 > \sigma_{m-s_m^2+1}. \tag{4.5}$$

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Combining (4.4) and (4.5), we see that

$$D^2_{m-s^2_m} < D^2_{m-s^2_m+1} - \sigma_{m-s^2_m+1} = D_m - \sum_{j=m-s^2_m+1}^{m} \sigma_j = D^1_{m-s^2_m},$$

where the first equality comes from (4.3) with $i = 2$ and $k = m - s^2_m + 1$. This contradicts the induction hypothesis, and consequently $s^1_m \leq s^2_m$. $\qquad\square$

Let Q^1 and Q^2 be two multiple-vacation queueing models under the limited schedule with threshold parameters L^1 and L^2, respectively. From (2.5), we can easily see that the schedules satisfy condition (4.1) in Theorem 3. If $L^1 \geq L^2$ then (4.2) is also satisfied. Hence,

Corollary 3. *For a multiple-vacation model under a limited service schedule with threshold parameter L, the processes $\{D_n, n = 1, 2, \ldots\}$ and $\{N(t), t \geq 0\}$ are monotonically decreasing with L. Furthermore, if the service discipline is FCFS, so is $\{W_n, n = 1, 2, \ldots\}$.*

We have seen that in Theorem 2 we allow one decision process to be random if one of the schedules is either exhaustive or limited with threshold parameter one. In the next theorem, Theorem 1 will be somewhat generalized by allowing both decision processes to be random.

Theorem 4. *Let Q^i, $i = 1, 2$, be a multiple vacation queueing model under a service schedule where the decision process $\{u^i_n, n = 1, 2, \ldots\}$ is such that*

$$P[u^i_n = 1 | \tau^i_j, \sigma^i_j, V^i_j, j = 1, 2, \ldots; u^i_k, 1 \leq k \leq n - 1] = P[u^i_n = 1 | q^i_n], \qquad n = 1, 2, \ldots \qquad (4.6)$$

Suppose that for all $n = 1, 2, \ldots$, we have

$$P[u^1_n = 1 | q^1_n = q] \geq P[u^2_n = 1 | q^2_n = q], \qquad q = 0, 1, \ldots \qquad (4.7)$$

If

$$\{\tau^1_n, \sigma^1_n, V^1_n, n = 1, 2, \ldots\} =_{st} \{\tau^2_n, \sigma^2_n, V^2_n, n = 1, 2, \ldots\},$$

then the stochastic comparison

$$\{D^1_n, n = 1, 2, \ldots\} \leq_{st} \{D^2_n, n = 1, 2, \ldots\}$$

holds true.

Proof: We want a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ on which $\hat{Q}^1$ and $\hat{Q}^2$ are defined such that, for each $i = 1, 2$, $\hat{Q}^i$ is stochastically equivalent to Q^i and (3.5) holds almost everywhere on $\hat{\Omega}$. We use a construction technique similar to the one used by Sonderman in [3].

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Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space where $\{\tilde{\tau}_n^i, \tilde{\sigma}_n^i, \tilde{V}_n^i, n = 1, 2, \ldots\}$, $i = 1, 2$, are defined such that for each $i = 1, 2$, $\{\tilde{\tau}_n^i, \tilde{\sigma}_n^i, \tilde{V}_n^i, n = 1, 2, \ldots\} =_{st} \{\tau_n^i, \sigma_n^i, V_n^i, n = 1, 2, \ldots\}$ and, for all $n = 1, 2, \ldots$, $\tilde{\tau}_n^1 = \tilde{\tau}_n^2$, $\tilde{\sigma}_n^1 = \tilde{\sigma}_n^2$, and $\tilde{V}_n^1 = \tilde{V}_n^2$.

For each pair (n, q) with $n = 1, 2, \ldots$ and $q = 0, 1, \ldots$, let $(\Omega_{nq}, \mathcal{F}_{nq}, P_{nq})$ be a probability space where u_{nq}^i, $i = 1, 2$, are defined and such that, for $i = 1, 2$,

$$P_{nq}[u_{nq}^i = x] = P[u_n^i = x | q_n^i = q], \qquad x = 0, 1$$

and

$$u_{nq}^1 \geq u_{nq}^2$$

everywhere on Ω_{nq}. This is made possible by condition (4.7).

Finally, define $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ as the product space of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $(\Omega_{nq}, \mathcal{F}_{nq}, P_{nq})$, $n = 1, 2, \ldots$ and $q = 0, 1, \ldots$ On this space, define $\{\hat{\tau}_n^i, \hat{\sigma}_n^i, \hat{V}_n^i, n = 1, 2, \ldots\}$, $i = 1, 2$, by setting

$$\hat{\tau}_n^i(\hat{\omega}) = \tilde{\tau}_n^i(\tilde{\omega}), \quad \hat{\sigma}_n^i(\hat{\omega}) = \tilde{\sigma}_n^i(\tilde{\omega}), \quad \hat{V}_n^i(\omega) = \tilde{V}_n^i(\tilde{\omega}), \quad \hat{u}_{nq}^i(\omega) = u_{nq}^i(\omega_{nq}).$$

for each $i = 1, 2$, $\hat{\omega} \in \hat{\Omega}$, $n = 1, 2, \ldots$, and $q = 0, 1, \ldots$. Also, for each $i = 1, 2$, $\hat{Q}^i$ is defined such that at the end of the nth service, the server's decision is based on the random variable $\hat{u}_{n\hat{q}_n^i}^i$. Then, by (4.6), $\hat{Q}^i$ is stochastically equivalent to Q^i.

The rest of the proof proceeds exactly like the proof of Theorem 1 with $u_n^i = \gamma_n^i(q_n^i)$ replaced by $\hat{u}_{n\hat{q}_n^i}^i$ for all $i = 1, 2$; $n = 1, 2, \ldots$. $\qquad\square$

The following corollary directly follows from Theorem 4 and Corollary 1.

Corollary 4. *For a multiple-vacation model under a Bernoulli service schedule with parameter p, the processes $\{D_n, n = 1, 2, \ldots\}$ and $\{N(t), t \geq 0\}$ are monotonically decreasing with p. Furthermore, if the service discipline is FCFS, so is $\{W_n, n = 1, 2, \ldots\}$.*

References

[1] B.T. Doshi, "Queueing Systems with Vacations—A Survey," *Queueing Systems* 1, pp. 29–66 (1986).

[2] T. Kamae, U. Krengel, and G.L. O'Brien, "Stochastic Inequalities on Partially Ordered Spaces," *Ann. Prob.* 5, pp. 899–912 (1977).

[3] D. Sonderman, "Comparing Multi-Server Queues with Finite Waiting Rooms, II: Different Numbers of Servers," *Adv. Appl. Prob.* 11, pp. 448–455 (1979).

[4] S. Stidham, Jr., "A Last Word on $L = \lambda W$," *Oper. Res.* 22, pp. 417–421 (1974).

[5] D. Stoyan, *Comparison Methods for Queues and Other Stochastic Models*, English Translation (D.J. Daley, ed.), J. Wiley & Sons, New York (1983).