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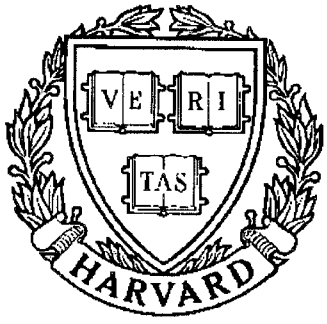

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Some Techniques for Analysis and Design of Robust Controls

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ABSTRACT

Title of Dissertation: Some Techniques for Analysis and Design of Robust Controls

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Department of Electrical Engineering

In this thesis we consider issues concerning the robustness of linear time invariant systems. We first attempt to gain a better understanding of some of the problems involved in the robustness assessment of multivariable systems by analyzing them on a loop by loop basis. Then, we assume norm bounded uncertainty on individual plant elements and explore the robustness of such systems via eigenvalue methods. We obtain results that sometimes are simpler than those obtained using the structured singular value approach. Using spectral analysis and some results from the theory of matrix perturbations we then develop robustness bounds. The bounds provide geometrical insight into the problem and also generate expressions in terms of nominal closed loop maps that can be used for analysis and design.

Another topic covered in this thesis is stability robustness in the presence of parametric uncertainty. Here, we find a parametrization of all the compensators that robustly stabilize the perturbed plant. The parametrization is obtained in terms of the coprime factors of the plants corresponding to extremal values of the uncertain parameters. Necessary and sufficient conditions for the simultaneous stabilizability of a continuum of plants are obtained. The results are restricted to the single variate case and a complete analysis is carried out for a single uncertain parameter.

Some Techniques for Analysis and Design of Robust Controls

By

Mario Jodorkovsky

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Contents

CHAPTER
ONE
INTRODUCTION

The issue of robust control has been extensively studied during the last two decades. Although the need for mantaining performance in the face of model uncertainty has long been recognized (and as a matter of fact motivated H.S. Black's patent in 1927), most studies prior to 1972 concentrated on single-input-single-output systems. It soon became clear that, with the exception of a few very special cases, classical robustness measures for SISO systems cannot be readily applied to the MIMO case. Many techniques for determining robustness of MIMO systems have been and continue to be proposed. These range from Lyapunov based techniques through the recently discovered methods based on Kharitonov's theorem. This thesis will mainly be concerned with robustness measures related to the singular value and the structured singular value.

To cope with structured uncertainty and skewed specifications the singular value approach requires the use of weighting matrices. When these matrices are used, the singular value provides only sufficient conditions for robust performance. Moreover, these sufficient conditions may be extremely conservative. The structured singular value solves this problem by providing necessary and sufficient conditions for robust stability and performance. However, with the exception of a few cases, it cannot be exactly computed. Thus, in practice, only sufficient conditions are available.

The use of weighting matrices in the singular value approach to account for uncertainty and performance structure and the computational complexity of the structured singular value, hinders the capability of the designer to develop the insight and heuristics required in performance specification, preliminary design, testing and troubleshooting analysis stages. Insight and heuristics are fundamental for a first assessment of design tradeoffs and inherent system limitations. It is also not unusual that the designer needs to modify specifications and therefore fast redesigns are required. The use of pure analytical methods is better suited for final design stages and is generally done within optimization frameworks. There is a clear need for simpler, even if less rigorous and precise methods for analysis and design of robust multivariable systems.

Motivated by these arguments we try in this thesis to analyze the robustness of multivariable systems in a more intuitive framework. We do this by tackling the problem from a viewpoint that resembles the scalar, classical approach. In the first part of the thesis we attempt to obtain a better understanding of the multivariable robustness issue and to develop a robustness bound which provides insight into the system. To do this we first try to analyze robustness on a loop by loop basis. We find that, although such an analysis cannot provide a complete robustness assessment, it partially explains the inadequacy of classical robustness tests and the potential lack of robustness properties of design methods based on diagonal dominance. Moreover the approach helps in understanding the way that singular values act to assess robustness. Then, we consider a simple, albeit practical uncertainty representation. We assume that the model of each individual element of the plant transfer matrix can be uncertain. The individual uncertainties are of the unknown but bounded type. With this uncertainty representation a simple robustness theorem can be proved and a partial characterization of the uncertainty which determines the system stability margin can be obtained. It turns out that this characterization is equivalent to a known result of the structured singular value. However our approach allows us to obtain the result in a simpler and more intuitive way.

The main idea behind the development of the robustness bounds is to study the decrease of the system stability margin each time a new uncertainty is added to the system. This allows us to detect combinations of uncertainties which most affect the stability of the system. This recursive approach provides us with a series of necessary conditions for the bound to be less than the limit allowed by stability considerations. The approach leads naturally to the use of some results from the theory of matrix perturbations. A first bound is developed using only matrix norms. This bound proves to be impractical. However, when the spectral norm is used it provides a geometrical interpretation which may help in understanding some problems. A second and more practical bound is developed using spectral analysis. For certain uncertainty structures this bound allows us to obtain expressions involving only individual closed loop maps that provide indications on how these maps should be altered to reduce the robustness bound.

In the second part of this thesis we deal with another aspect of robust control: Design and analysis in the face of parametric uncertainty. Here, we assume that the values of some coefficients of the numerator or denominator of a rational SISO transfer function are only known to lie in some finite, given intervals. The goal is then to find a parametrization of all stabilizing controllers for this infinite family of plants. The main motivation for obtaining such a parametrization is that after stability is guaranteed, the designer can concentrate upon choosing a controller to meet performance requirements. Some optimal control procedures such as the H_2 and the H_∞ methods are based on the this parametrization. Moreover, since the parametrization produces all the stabilizing controllers, analysis of the limiting performance bounds of the system is possible. To tackle this problem we use the fractional approach. Using coprime factorizations of the plants corresponding to the extremal values of the uncertain coefficients we first verify whether the plants are simultaneously stabilizable. Then we find conditions on the controller parameter which are required for the simultaneous stabilizability of the plants. Finally, we obtain necessary and sufficient conditions on the controller parameter for simultaneous stabilizability of the continuum of plants.

The contributions of this work can be summarized as follows:

1. An improved loop by loop test for robustness assessment of multivariable systems is proposed.

2. A theorem providing necessary and sufficient conditions for robust stability is proven.

3. A theorem which partially characterizes perturbations for the computation of the stability margin is proven.

4. Two robustness bounds are developed.

5. A parametrization of all the compensators that simultaneously stabilize a continuum of plants is obtained.

The thesis is organized as follows: In Chapter 2 we describe in some detail the motivations for the development of robustness measures in the frequency domain. We mainly concentrate on the eigenvalue, singular value and structured singular value approaches. In Chapter 3 we present some fundamental stability and robustness theorems which show how the combination of the Multivariable Nyquist Theorem and homotopy arguments leads to the use of singular values for robustness assessment. In the same chapter, we present the loop by loop approach and a theorem that specifies the minimal amount of frequency responses that are required to be carried out in a multivariable system for robustness assessment. In Chapter 4 we discuss some aspects of the element by element uncertainty representation and then devote the rest of the chapter to two theorems. The first theorem deals with stability robustness. The second one characterizes the uncertainty for the calculation of the stability margin. The theorems are presented in two versions to allow comparison. The structured singular value version and the maximal spectral radius one. Chapter 5 deals with the development of robustness bounds. First the bound based on matrix norms is developed and its geometrical interpretation is presented. Then, the bound

based on spectral analysis is developed and various lemmas are proved to show how the bound can be used for rapid robustness assessment. Finally, to illustrate the use of the second bound, the robustness at a single frequency of a loop designed to control the longitudinal dynamics of an airplane is analyzed. Chapter 6 is entirely devoted to the parametrization of all stabilizing controllers for plants with parametric uncertainty. The chapter also includes a literature review and an illustrative example.

CHAPTER
TWO

BACKGROUND

Linear time invariant (LTI) models are undoubtedly the most common mathematical descriptions of systems used for control analysis and synthesis. Indeed, small signal analyses of physical systems usually lead to models that are essentially linear. Moreover, many of these models have parameters whose rate of change is negligible compared to the bandwidth of the controlled (closed loop) system. Thus, the system can also be considered to be time invariant, at least for purposes of controller design.

Inherent to any mathematical description of physical systems and incidentally one of the main reasons for the use of feedback controls, is model uncertainty. Model uncertainty in the framework of LTI systems can be attributed to parameter variations, aggregation or inability to measure accurately. Feedback is often used to balance among several, and possibly conflicting, system performance requirements in the face of model uncertainty. The extent to which feedback achieves performance is limited by the change in the system structure resulting from the application of this particular control scheme. Feedback introduces sensor dynamics and noise and affects the system stability. The real problem in feedback control is then to mantain stability and performance in the face of model uncertainty. While for systems with mild performance requirements it may be easy to meet the specifications for the entire uncertainty range,

it is often impossible to satisfy high performance specifications with conservative control designs.

Robustness is a widely used term to express the capability of a system to maintain stability under model uncertainty. Classical robustness measures for single input–single output systems are the well known gain and phase margins. These margins have also been extensively used as performance indicators due to the close relationship between stability margin, bandwidth, and dynamic response (see e.g. [1]). Their acceptance has been so wide that they are usually incorporated into the specifications of control systems [2]. Still, their strict extension to multivariable systems is possible exclusively on a loop by loop basis. This assertion can be made clear by attempting to derive the definitions of gain and phase margin in a multivariable setting. This is done below following [3] with the addition of homotopy and continuity considerations as in [2].

Consider the $n \times n$ open loop perturbed transfer function matrix,

$$\tilde{G}(s) = G(s)[I + \Delta(s)] \tag{2.1}$$

where $G(s)$ is the nominal (unperturbed) transfer function matrix and $\Delta(s)$ is a dimensionally compatible right multiplicative perturbation matrix.

It is assumed that the perturbed transfer matrix $\tilde{G}(s)$ is stabilizable and detectable, and that the nominal closed loop system obtained by closing a unity gain feedback loop around $G(s)$ is stable.

Subject to the stabilizability and detectability assumptions, the closed loop perturbed system is unstable if and only if there exists a complex frequency $s^\star$ with $Re(s^\star) \geq 0$ such that,

$$det\{I + \tilde{G}(s^\star)\} = det\{I + G(s^\star)[I + \Delta(s^\star)]\} = 0 \tag{2.2}$$

Now consider the homotopy $\hat{G} : \mathbb{C}^{n\times n} \times [0,1] \rightarrow \mathbb{C}^{n\times n}$ between $G(s)$ and $\tilde{G}(s)$ defined by,

$$\hat{G}(s, \epsilon) = G(s)[I + \epsilon\Delta(s)] \tag{2.3}$$

Since the roots of a polynomial are continuous functions of its coefficients and the nominal closed loop system is stable (i.e., all the roots of $det[I + G(s)] = 0$

are in the open left half plane) it follows from (2.2) and (2.3) that there exists a real frequency ω^* and an ϵ^* in $[0,1]$ such that,

$$det\{I + \hat{G}(\jmath\omega^*, \epsilon^*)\} = det\{I + G(\jmath\omega^*)[I + \epsilon^*\Delta(\jmath\omega^*)]\} = 0 \tag{2.4}$$

But the fact that the closed loop nominal system is stable implies that $[I + G(\jmath\omega^*)]$ is invertible. Therefore (2.4) amounts to the condition,

$$det\{I + [I + G(\jmath\omega^*)]^{-1}G(\jmath\omega^*)\epsilon^*\Delta(\jmath\omega^*)\} = 0 \tag{2.5}$$

Denoting the nominal closed loop map by $Q(s) = [I + G(s)]^{-1}G(s)$ and the contracted perturbation by $\hat{\Delta}(s) = \epsilon^*\Delta(s)$, condition (2.5) is equivalent to the existence of an n-vector $x \neq 0$ satisfying,

$$Q(\jmath\omega^*)\hat{\Delta}(\jmath\omega^*)x = -x \tag{2.6}$$

Similar results are obtained for other representations of uncertainties (additive, inverse multiplicative etc.), the only difference being that the closed loop map $Q(s)$ changes.

For SISO systems $Q(s)$ and $\Delta(s)$ are scalars and (2.6) reduces to,

$$\hat{\Delta}(\jmath\omega^*) = -Q^{-1}(\jmath\omega^*) \tag{2.7}$$

The definition of gain margin considers real perturbations of the form $\hat{\Delta}(s) = k - 1$, $k \in \mathbb{R}$ so that condition (2.7) amounts to,

$$k = 1 - Q^{-1}(\jmath\omega^*) = -G^{-1}(\jmath\omega^*) \tag{2.8}$$

i.e., the gain margins are the least $k \geq 1$ and the largest $k \leq 1$ for which there exists a frequency ω^* satisfying (2.8).

Similarly, considering perturbations of the form $\hat{\Delta}(s) = e^{-\jmath\theta} - 1$ in (2.7), the phase margins are defined as the least $0 \leq \theta < 180°$ and the largest $-180° \leq \theta < 0$ satisfying,

$$e^{-\jmath\theta} = 1 - Q^{-1}(\jmath\omega^*) = -G^{-1}(\jmath\omega^*) \tag{2.9}$$

for some ω^* in $\mathbb{R}$.

Expression (2.7) says that the size of the critical perturbation is inversely proportional to the closed loop transfer function. For this reason, it is a common practice in control design to roll-off the frequency response characteristics at high frequencies where the system model is less certain.

From expression (2.6), the difficulties associated with the multivariable generalization of gain and phase margins are apparent. For each nonzero vector in the n-dimensional space there may exist disturbances and real frequencies for which (2.6) is satisfied.

Condition (2.6) reduces to (2.7) when the perturbation matrix has a single nonzero element, in which case the relevant eigenvector x is an elementary vector with a 1 in the row corresponding to the column of the nonzero element of $\Delta(s)$ and zeros everywhere else.

The simplest loop by loop extension of gain and phase margins to multivariable systems [4] considers each feedback loop separately. By opening one feedback loop at a time while keeping the remaining loops closed, the gain and phase margins are determined for each resulting scalar system. The method is very appealing from the practical point of view because of its simplicity. However it suffers from a serious drawback. No simultaneous loop gain or phase variations are considered. We will present later a well known example of a system that exhibits large margins to individual loop gain variations but becomes unstable for small simultaneous gain variations.

A more advanced stability margin definition, the Diagonal[1] Multiloop Gain and Phase Margins [4, 5], can be given by considering Figure 2.1. The diagonal multiloop gain margin is defined as the pair of real numbers $(\underline{l}, \bar{l})$ defining the largest interval for which the system in Figure 2.1 is stable for all combinations of gains satisfying,

$$\underline{l} \leq l_i \leq \bar{l} \qquad i = 1, 2, \cdots, m$$

[1]The original definition does not include the word diagonal. It has been added here to emphasize that it is not a satisfactory definition of MIMO margins.

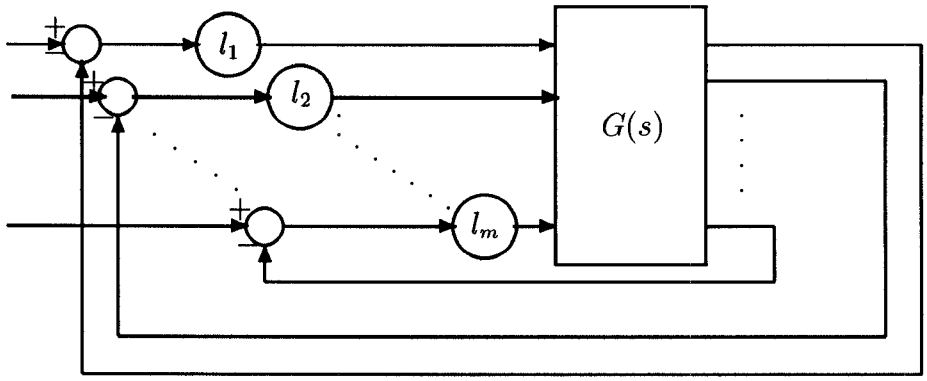


Figure 2.1: Configuration for Diagonal Multiloop Gain and Phase Margin Definition

The diagonal multiloop phase margin is defined anagously.

The diagonal multiloop gain and phase margin definition takes into account simultaneous gain or phase variations. However only the restricted class of diagonal uncertainties is considered. We will show later an example of a closed loop system featuring large margins to diagonal loop variations but arbitrarily small margins to nondiagonal gain variations.

The above methods for measuring multivariable stability margins are easy to implement but, obviously, inadequate robustness tests. Such a test still has practical utility because it will reject many plausible designs that are not robust. The fact that designs that appear robust by these tests may not be robust applies to the classical SISO test as well. It turns out that theoretical robustness assessment based on the Multivariable Nyquist Theorem [6] also fails to check all possible directions as required by expression (2.6). For these reasons, control designs based on extensions of scalar techniques such as loop by loop methods (e.g. see [7]) or the multivariable Nyquist criterion (e.g. the Inverse Nyquist method [6], the Characteristic Loci method [8], etc.) have been criticized [9, 10, 2, 11] for lacking robustness properties.

Very simple multivariable systems can be devised to illustrate the difficulties

in assessing robustness by the aforementioned methods. Consider first an example taken from [9] which consists of a system with open loop transfer function given by,

$$G(s) = \frac{1}{s^2 + 100} \begin{pmatrix} s - 100 & 10(s+1) \\ -10(s+1) & s - 100 \end{pmatrix} \tag{2.10}$$

The closed loop configuration is depicted in Figure 2.2.

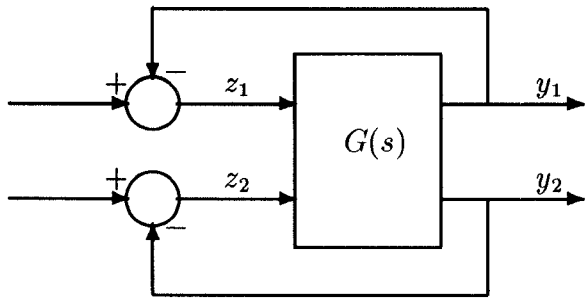


Figure 2.2: Configuration of Doyle's Example

The open loop poles are at $\pm 10j$ and the closed loop exhibits two poles at -1. The transfer function between z_1 and y_1 with the first loop open is $1/s$. The same open loop transfer function is obtained from z_2 to y_2 when the second loop is open and the first is closed. The loop by loop stability margins amount then to infinite (positive) gain margin and 90° phase margin, suggesting a highly robust system. However it turns out that these figures are misleading. For a diagonal multiplicative model perturbation $L(s)$ of the form,

$$L(s) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

the stability regions in the (k_1, k_2) plane depicted in Figure 2.3 are obtained. Thus, a small simultaneous variation of the gain in both loops yields instability. Obviously the multiloop gain margin ideas would capture this lack of robustness.

The inadequacy of the diagonal multiloop gain margin can be illustrated by

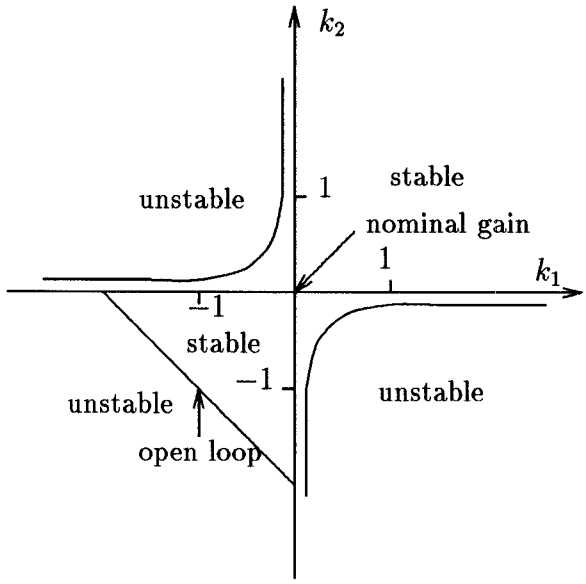


Figure 2.3: Stability Regions in the Gain Plane

an example borrowed from [2]. Consider the open loop transfer matrix,

$$G(s) = \frac{1}{s+1}\begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

The characteristic equation of this system is independent of the off-diagonal term $b_{12}/(s+1)$ therefore the multiloop stability margins correspond to two highly robust decoupled scalar systems. Consider now a non-diagonal perturbation as shown in Figure 2.4. The perturbed open loop transfer function matrix is given by,

$$\tilde{G}(s) = \frac{1}{s+1}\begin{pmatrix} 4 & b_{12} \\ -5/b_{12} & 1 \end{pmatrix} \tag{2.12}$$

The characteristic equation of the perturbed system does not depend on b_{12} as before. However a unity gain feedback applied from each output to its corresponding input yields instability. If b_{12} is large, in which case the unperturbed system (2.11) is poorly-conditioned, a small perturbation suffices to destabilize the system. This small stability margin to non-diagonal model perturbations can neither be detected by the simple loop by loop stability margin test nor

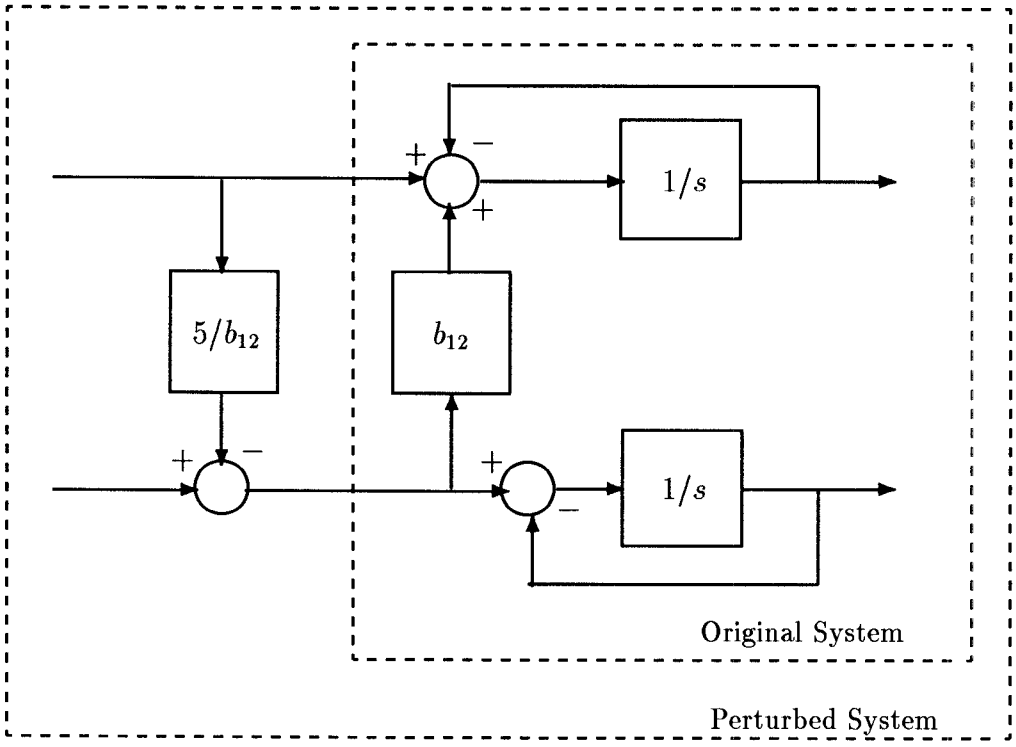


Figure 2.4: Configuration of Lehtomaki's Example

by the diagonal multiloop margin test. An analysis based on the Multivariable Nyquist Criterion will consider the value of the determinant of the return difference matrix $I + G(s)$ at each frequency along a Nyquist contour. This value is $1/(s+1)^2$ and is independent of b_{12}. When evaluated at low frequencies a value near 1 is obtained. However if the return difference matrix evaluated at low frequencies (say, $s = 0$) operates upon the vector $x = (1, \epsilon)^T$ it yields,

$$[I + G(0)]x = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} = \begin{pmatrix} 1 + b_{12}\epsilon \\ \epsilon \end{pmatrix} \tag{2.13}$$

showing that for large b_{12}, a small ϵ renders $I + G(0)$ almost singular. This means by (2.6) that a small low frequency error perturbation suffices to cause closed loop instability and that the direction $(1, \epsilon)^T$ remained undetected by the robustness tests based on extensions of scalar techniques.

From the above examples it seems that the fundamental problem of assessing robustness by extensions of scalar methods is that they rely on the eigenvalues of the open loop transfer function matrix or equivalently, on the determinant of the return difference matrix as measures for rank determination and robustness. Since matrices may not be well conditioned with respect to eigenvalue computations, eigenvalues are not reliable measures of the distance of a matrix from singularity [12, 13].

One possible approach to improve the eigenvalue based methods for robustness assessment is to use classical sensitivity theory. Sensitivity theory would consider not only the magnitude of the eigenvalues as stability measures but also the magnitude of their derivatives with respect to system parameters. However, the use of derivatives involves only infinitesimal variations in the model parameters. Therefore sensitivity theory is not necessarily meaningful for finite model variations.

Another approach is to use matrix norms instead of eigenvalues as measures of stability margins. An explicit characterization of robustness of LTI systems

using matrix norms was developed by Doyle [9]. Doyle, perhaps inspired by fundamental results from the field of numerical linear algebra concerning numerical stability of matrix inversion, used the spectral norm or more generally the singular value decomposition (SVD). Sandell [14] generalized Doyle's results using operator theory in a system input-output stability framework.
An excellent review of the use the SVD in the context of numerical computation in linear algebra and linear systems appears in [13].

The use of matrix norms as robustness measures arises naturally from the fact that the spectral radius is a lower bound on any matrix norm [12]. I.e., if A is any complex square matrix then,

$$\rho(A) \leq \|A\| \tag{2.14}$$

where $\rho(\cdot)$ denotes the spectral radius and $\|\cdot\|$ a matrix norm.
Consider again the perturbed transfer function matrix (2.1) and assume that $G(s)$ is invertible. Then, expression (2.4) can be rewriten as,

$$0 = det\{I + G(\jmath\omega^*)[I + \epsilon^*\Delta(\jmath\omega^*)]\} = det[G(\jmath\omega^*)]det[I + G^{-1}(\jmath\omega^*) + \epsilon^*\Delta(\jmath\omega^*)] \tag{2.15}$$

and therefore condition (2.5) can be replaced by,

$$det\{I + [I + G^{-1}(\jmath\omega^*)]^{-1}\epsilon^*\Delta(\jmath\omega^*)\} = 0 \tag{2.16}$$

Suppose now that for all ω,

$$\rho\{[I + G^{-1}(\jmath\omega)]^{-1}\Delta(\jmath\omega)\} < 1 \tag{2.17}$$

then, there exists no $\epsilon^* < 1$ and no ω^* such that expression (2.16) is satisfied. Thus (2.17) is a sufficient condition for closed loop stability.
By (2.14) a sufficient condition for (2.17) to be satisfied is that for all ω,

$$\|[I + G^{-1}(\jmath\omega)]^{-1}\Delta(\jmath\omega)\| < 1 \tag{2.18}$$

For norms that enjoy the submultiplicative property such as induced norms or the Frobenius norm $\sqrt{tr(A^*A)}$ (also referred to as the Schur norm, or the

Hilbert-Schmidt norm), an allowable upper bound on $\|\Delta(j\omega)\|$ that insures robust stability can be obtained by restricting the value of an upper bound on expression (2.18),

$$\|[I + G^{-1}(j\omega)]^{-1}\| \, \|\Delta(j\omega)\| < 1 \quad \forall\omega \tag{2.19}$$

Specializing to the spectral norm, one obtains Doyle's result,

$$\overline{\sigma}[\Delta(j\omega)] < \underline{\sigma}[I + G^{-1}(j\omega)] \quad \forall\omega \tag{2.20}$$

where $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote the maximal and minimal singular values respectively.

It is worth noting that when the uncertainty under consideration is of the unstructured type [10] i.e., when the uncertainty description consists solely of a (frequency dependent) upper bound on its norm,

$$\|\Delta(j\omega)\| \leq \delta(j\omega) \quad \forall\omega \tag{2.21}$$

with $\delta(j\omega) > 0$ a scalar function, then the sufficient condition (2.17) is also necessary. This follows from the fact that if condition (2.17) is violated then there always exists an uncertainty matrix in the class defined by (2.21) for which expression (2.16) holds.

The choice of the SVD for robustness analysis is not incidental. As emphasized in [13], the SVD is the only generally reliable method to determine rank numerically. The norm of the inverse of a matrix is a reasonable measure of nearness to singularity for square, invertible matrices. For nonsquare matrices, norms cannot be used to determine how close the matrix is to rank deficiency without some further complications. The smallest singular value works for any case.

The SVD also provides qualitative and quantitative information about linear maps. For instance, the SVD is used for the description of certain subspaces associated with the maps and for defining angles between the subspaces [13]. It is also used to describe the internal structure of linear maps in terms of input-output gains and directions [15].

There are other reasons for using the SVD as a measure of the "size" of a matrix besides its computational convenience. As mentioned in [9] and developed in full detail in [10, 15, 16, 3], the SVD provides a suitable notion of multivariable gain. This can be used for defining general feedback properties of multivariable systems as natural extensions of the scalar case. These feedback properties consist not only of stability margins but also of other performance measures such as sensitivity, disturbance rejection, etc..

If $G(s)$ is an $n \times m$ transfer function matrix relating the m-dimensional input vector $u(s)$ to the n-dimensional output vector $y(s)$ through,

$$y(s) = G(s)u(s) \tag{2.22}$$

then the ratio of the l_2 (Euclidean)–norm of the output vector $y(s)$ to the l_2–norm of the input vector $u(s)$ at the complex frequency s is given by,

$$\frac{\|y(s)\|_2}{\|u(s)\|_2} = \left[\frac{u^H(s)G^H(s)G(s)u(s)}{u^H(s)u(s)}\right]^{\frac{1}{2}} \tag{2.23}$$

where $A^H(s)$ denotes the Hermitian transpose of the matrix $A(s)$ (i.e., $A^H(s) = A^T(\bar{s})$, where $\bar{s}$ is the complex conjugate of s).

It turns out that,

$$\underline{\sigma}[G(s)] \leq \frac{\|y(s)\|_2}{\|u(s)\|_2} \leq \overline{\sigma}[G(s)] \tag{2.24}$$

which, when evaluated at $s = j\omega$, means that the ratio between the output and input signal power at this frequency is bounded by the minimal and maximal singular value of the operator respectively.

The maximal and minimal singular values of $G(s)$ are therefore used as multivariable gains and performance trade off analysis and synthesis in the face of model uncertainty can be carried out in a multivariable setting [10] by means of an upper (maximal singular value of $G(s)$) and lower (minimal singular value of $G(s)$) gain.

Notice also that $\text{esssup}_\omega \overline{\sigma}[G(s)]$ is the norm corresponding to the Hardy space H_∞, widely used in modern robust control theory. One of the physical interpretations of this norm is that it is the largest rms value of the output when the

input power is bounded by one.

Using singular values as measures of matrix gain enables formulating system performance specifications in the form,

$$\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)] < 1 \tag{2.25}$$

where $S(\jmath\omega)$ is a closed loop sensitivity function reflecting the ability of the system to track signals, reject disturbances etc., and $\mathrm{w}(\jmath\omega)$ is a (frequency dependent) scalar weighting factor whose inverse determines a desired upper bound on the norm of the sensitivity function. For instance, if one of the system performance specifications refers to plant output disturbance attenuation, then the relevant sensitivity function would be $S(s) = [I + G(s)]^{-1}$ and (2.25) may be expressed as $\underline{\sigma}[I + G(\jmath\omega)] > |\mathrm{w}(\jmath\omega)|$.
From now on, and for simplicity of exposition, we consider only the sensitivity function corresponding to plant output disturbance rejection.

Remembering that the closed loop map $Q(s) = [I + G(s)]^{-1}G(s)$ and assuming that the uncertainty description is given by (2.21), then by (2.18) we obtain that,

$$\overline{\sigma}[Q(\jmath\omega)\Delta(\jmath\omega)] < 1 \qquad \forall\ \omega \in \mathbb{R} \tag{2.26}$$

is a necessary and sufficient condition for robust stability.

Control design however is concerned not only with maintaining stability but also with maintaining performance in the face of model uncertainty. Thus, a good design achieves performance robustness as well as stability robustness. Following Stein and Doyle [17] we have that robust performance is attained if (2.25) is satisfied for the entire family of perturbed transfer matrices (2.1) i.e. if,

$$\overline{\sigma}\{\mathrm{w}(\jmath\omega)[I + \tilde{G}(\jmath\omega)]^{-1}\} = \overline{\sigma}\{\mathrm{w}(\jmath\omega)[I + G(\jmath\omega)(I + \Delta(\jmath\omega))]^{-1}\} =$$
$$\overline{\sigma}\{[I + Q(\jmath\omega)\Delta(\jmath\omega)]^{-1}\mathrm{w}(\jmath\omega)S(\jmath\omega)\} < 1 \tag{2.27}$$

Since by (2.26) $\overline{\sigma}[Q(\jmath\omega)\Delta(\jmath\omega)] < 1$ then,

$$\overline{\sigma}\{[I + Q(\jmath\omega)\Delta(\jmath\omega)]^{-1}\mathrm{w}(\jmath\omega)S(\jmath\omega)\} \leq$$

$$\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)]\overline{\sigma}\{[I+Q(\jmath\omega)\Delta(\jmath\omega)]^{-1}\} = \frac{\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)]}{\underline{\sigma}[I+Q(\jmath\omega)\Delta(\jmath\omega)]} \leq \frac{\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)]}{1-\overline{\sigma}[Q(\jmath\omega)\Delta(\jmath\omega)]} \tag{2.28}$$

and by (2.27) and (2.28) we obtain that a sufficient condition for robust performance is that,

$$\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)] < 1-\overline{\sigma}[Q(\jmath\omega)\Delta(\jmath\omega)] \tag{2.29}$$

which means that the nominal performance specification (2.25) must be tightened up to account for uncertainty in the model.

Suppose now that for some frequency ω a singular value decomposition of the closed loop map is given by $Q = U\Sigma V^H$ with U and V unitary and Σ a diagonal matrix containing the singular values of Q. Also consider the perturbation matrix $\Delta = -\alpha VU^H$ with $0 < \alpha \leq \delta$. Then, Δ satisfies (2.21) and moreover,

$$\overline{\sigma}[(I+Q\Delta)] = \overline{\sigma}[(I-\alpha U\Sigma U^H)] = \overline{\sigma}[U(I-\alpha\Sigma)U^H] = 1-\alpha\underline{\sigma}(Q) = 1-\underline{\sigma}(Q\Delta) \tag{2.30}$$

Now, if (2.27) is satisfied then,

$$\overline{\sigma}(\mathrm{w}S) = \overline{\sigma}[(I+Q\Delta)(I+Q\Delta)^{-1}\mathrm{w}S] \leq \overline{\sigma}(I+Q\Delta)\overline{\sigma}[(I+Q\Delta)^{-1}\mathrm{w}S] < \overline{\sigma}(I+Q\Delta) \tag{2.31}$$

and by (2.30) and (2.31) we obtain that a necessary condition for robust performance is that,

$$\overline{\sigma}[\mathrm{w}(\jmath\omega)S(\jmath\omega)] < 1-\underline{\sigma}[Q(\jmath\omega)\Delta(\jmath\omega)] \tag{2.32}$$

The sufficient and necessary conditions for robust performance (2.29) and (2.32) respectively coalesce into a single condition for SISO systems and for MIMO systems whose closed loop map $Q(\jmath\omega)$ is *spacially round* for all frequencies. Spacially round maps are defined as maps satisfying,

$$\kappa[Q(\jmath\omega)] = \frac{\overline{\sigma}[Q(\jmath\omega)]}{\underline{\sigma}[Q(\jmath\omega)]} = 1 \tag{2.33}$$

Despite the nice properties of the singular values and in general the reliability of some matrix norms as measures of the distance of an operator from

rank deficiency, it must be clear that conceptually their application results in ignoring many aspects of multivariable systems. A single number at each frequency is used to characterize a matrix (see expressions (2.19) and (2.21)) and as such it cannot fully reflect the structural properties of multivariable systems. Moreover, defining performance specifications as in (2.25) does not allow the inclusion of spatial direction information regarding errors, external disturbances, commands, etc.. This may give rise to a high degree of conservatism since stability conditions or other performance measures derived by this method apply equally to a large family of radically different systems. Moreover, performance conditions are restricted to apply equally in all directions.

One way to alleviate this problem [17] is to incorporate structure and directionality information into the system by means of weighting matrices. Instead of the description (2.1) of the perturbed open loop transfer function matrix, a generalized unstructured right multiplicative perturbation is used yielding,

$$\tilde{G}(s) = G(s)[I + W_l^{-1}(s)\hat{\Delta}(s)W_r(s)] \tag{2.34}$$

where W_l and W_r are stable and minimum phase operators, invertible at almost every complex frequency s, and $\hat{\Delta}$ has norm less than unity.

Weighting matrices are also incorporated into the performance specifications. Thus, instead of (2.27) the generalized performance objective is,

$$\overline{\sigma}\{W_e(j\omega)[I + \tilde{G}(j\omega)]^{-1}W_d^{-1}(j\omega)\} < 1 \qquad \forall\, \omega \tag{2.35}$$

where W_e and W_d are also stable, minimum phase and invertible matrices.

With the generalized uncertainty description (2.34) and performance index (2.35) a necessary and sufficient condition for robust stability is given by,

$$\overline{\sigma}[W_r(j\omega)Q(j\omega)W_l^{-1}(j\omega)] < 1 \qquad \forall\, \omega \tag{2.36}$$

Moreover since,

$$W_e[I + \tilde{G}]^{-1}W_d^{-1} = W_e[I + G(I + W_l^{-1}\hat{\Delta}W_r)]^{-1}W_d^{-1} =$$

$$W_e\{(I+G)[I+(I+G)^{-1}GW_l^{-1}\hat{\Delta}W_r]\}^{-1}W_d^{-1} =$$
$$W_e(I+QW_l^{-1}\hat{\Delta}W_r)^{-1}SW_d^{-1} =$$
$$(W_rW_e^{-1})^{-1}[I+(W_rQW_l^{-1})\hat{\Delta}]^{-1}(W_rW_e^{-1})(W_eSW_d^{-1})$$

then, in contrast with (2.29) a sufficient condition for robust performance is,

$$\overline{\sigma}[W_e(j\omega)S(j\omega)W_d^{-1}(j\omega)] < \frac{1-\overline{\sigma}[W_r(j\omega)Q(j\omega)W_l^{-1}(j\omega)]}{\kappa[W_r(j\omega)W_e^{-1}(j\omega)]} \tag{2.37}$$

It turns out that expression (2.37) is the least conservative sufficient condition for robust performance which can be established via singular values. This results from the fact that the tightest possible singular value bound on $W_e[I+\tilde{G}]^{-1}W_d^{-1}$ for all unity norm bounded $\hat{\Delta}$ is $\overline{\sigma}[W_eSW_d^{-1}]\kappa[W_rW_e^{-1}]/(1-\overline{\sigma}[W_rQW_l^{-1}])$. For some design situations (2.36) becomes a necessary condition as well.
Whenever the system specifications are not spacially round the condition number $\kappa[W_rW_e^{-1}]$ may be large and consequently the bound (2.37) will be small. Thus for some skewed design specifications, much tighter (often unachievable) nominal performance is required to assure robust performance. Since (2.37) is the tightest bound that can be produced using singular values we conclude that the use of singular values may lead to very conservative designs even when weights are incorporated into the design parameters and objectives to account for perturbation structure and performance specification directionality.

A more realistic and less conservative description of uncertainty is to assign to each element of the plant a (possibly different) upper bound. I.e., if Δ_{ij} denotes the ij-th element of Δ then the plant uncertainty description is given by the bounds,

$$|\Delta_{ij}(j\omega)| \leq \delta_{ij}(\omega) \tag{2.38}$$

with $\delta_{ij}(\omega) \geq 0$. Alternatively, denoting by $|\Delta|$ the matrix Δ with all its elements Δ_{ij} replaced by their absolute values $|\Delta_{ij}|$ and by $\overline{\Delta}$ the matrix with elements δ_{ij}, then (2.38) can be written as,

$$|\Delta(j\omega)| \leq \overline{\Delta}(\omega) \tag{2.39}$$

where the inequality applies element by element. This *element by element* or *multidimensional* uncertainty description has been used in [18, 19, 20] to derive sufficient conditions for robust stability using nonnegative and M-matrices.
A well known fact in matrix theory is that the spectral radius of a square matrix A is given by,

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \tag{2.40}$$

where $\|\cdot\|$ is any submultiplicative matrix norm (see e.g. [12]). Also, by the definition of the spectral norm $\|\cdot\|_2$ we have that,

$$|A| \leq |B| \Longrightarrow \|A\|_2 \leq \|B\|_2 \tag{2.41}$$

for any complex matrices A and B of the same dimensions. Moreover, since $\|\cdot\|_2$ is an absolute norm then,

$$\|A\|_2 = \||A|\|_2 \tag{2.42}$$

Thus, if $|A| \leq B$ we obtain by (2.41) and (2.42) that,

$$\|A^m\|_2^{1/m} \leq \||A|^m\|_2^{1/m} \leq \|B^m\|_2^{1/m} \tag{2.43}$$

Letting $m \to \infty$ in (2.43) we obtain by (2.40),

$$\rho(A) \leq \rho(|A|) \leq \rho(B) \tag{2.44}$$

From (2.44) it follows that,

$$\rho[Q(j\omega)\Delta(j\omega)] \leq \rho[|Q(j\omega)\Delta(j\omega)|] \leq \rho[|Q(j\omega)||\Delta(j\omega)|] = \rho[|Q(j\omega)|\overline{\Delta}] \tag{2.45}$$

Then if $Q = (I + G^{-1})^{-1}$ the condition,

$$\rho[|Q(j\omega)|\overline{\Delta}] < 1 \tag{2.46}$$

is by (2.17) sufficient for insuring stability. Since the matrix $|Q(j\omega)|\overline{\Delta}$ is nonnegative, then $\rho[|Q(j\omega)|\overline{\Delta}]$ is its Perron root, which enjoys many computational advantages. Using M-matrices and taking advantage of their numerous equivalent characterizations [21] other sufficient conditions can be derived [18].

Condition (2.46) may be extremely conservative as well. However due to the uncertainty description (2.39) upon which it is based it generally yields less conservative results than the singular value approach [19].

The use of different matrix norms and matrix measures to reduce conservatism was investigated by Barrett [22]. In this work, a hierarchy of robustness tests which utilize more and more information about the uncertainty is defined. Weighted norms based on equivalent transformations are also used to further reduce conservatism. However no systematic method for choosing the weights and the norms is provided.

The work by Lehtomaki *et al.* [23, 24] introduces various possible structures for the modeling error matrix and also incorporates partial structure knowledge into the stability conditions. This is done by means of robustness tests which rule out uncertainty directions which are known not to occur.

The introduction of completely structured uncertainty into robustness tests based on singular values is first found in the works of Safonov [25] and Doyle [26]. The starting and fundamental point of their work is, as already used in an earlier work by Safonov and Athans [27] in the context of LQG, the transformation of the plant with structured uncertainty into one characterized by a (block) diagonal uncertainty structure. This particular structure enables many algebraic manipulations (for example, matrices with the same block diagonal structure always commute). Once the system has a block diagonal uncertainty structure, robustness measures based on the multivariable Nyquist necessary and sufficient condition are defined. By definition, the robustness measures provide necessary and sufficient conditions for robust stability. The ease of algebraic manipulation coupled with the assumption of complex uncertainty provide the robustness measures with several properties that are eventually used to aid in their computation.

A simple procedure to bring the system to a block diagonal uncertainty form is reported in [28]. Assuming for simplicity that the uncertainties are of the

additive type only, with the matrix Δ_k denoting the uncertainty in the transfer matrix P_k, the procedure consists of deriving all the transfer function matrices M_{ij} defined by the relationship $r_i = M_{ij} z_j$, where z_j is a vector of signals injected at the output of P_j and r_i is the vector of signals measured at the input of P_i. The transfer function matrices are computed with all the loops closed and with no uncertainties. The loop signals used for the determination of M_{ij} are depicted in Figure 2.5 where the transfer matrices P_j and P_i are shown explicitly. The

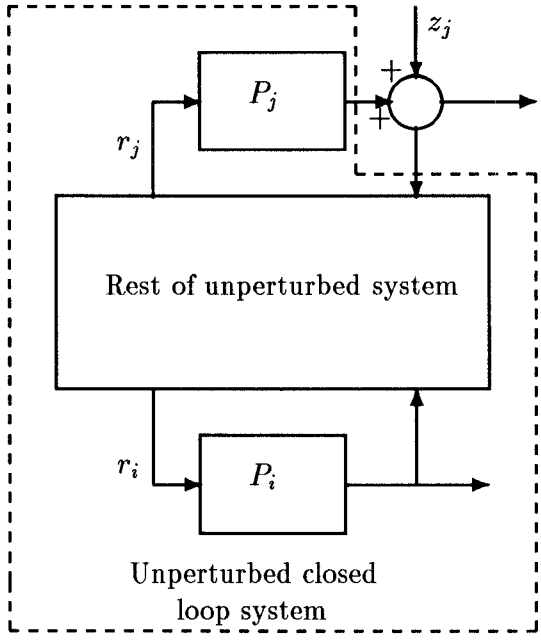


Figure 2.5: Determination of the matrices M_{ij}

matrix $M = \{M_{ij}\}$ represents all the interactions of the feedback loop with its uncertainties and the equivalent system representation exhibits a block diagonal uncertainty structure $\Delta_d = diag[\Delta_1, \Delta_2, ..., \Delta_n]$ as depicted in Figure 2.6.

When the element by element type of uncertainty is considered, a more formal procedure for bringing the system into diagonal uncertainty structure is suggested by Daniel *et al.* [29]. The original structured, nondiagonal uncertainty

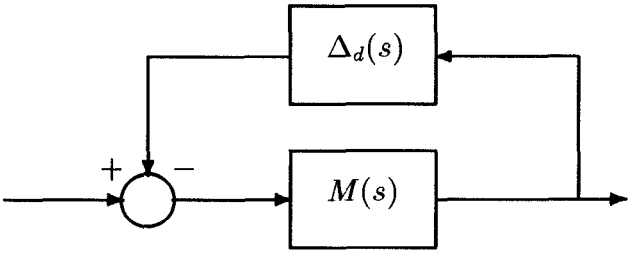


Figure 2.6: System with Block Diagonal Uncertainty Structure

matrix $\Delta = \{\Delta_{ij}\}$ is written as,

$$\Delta = \sum_{i,j=1}^{n} \Delta_{ij} e_i {e_j}^T = \sum_{k=1}^{n^2} \Delta_k (E_1)_k {(E_2^T)_k}^T \tag{2.47}$$

where e_i is the n-elementary vector with a 1 in the i-th position and zeros everywhere else, $(E_1)_k$ is the k-th column of an $n \times n^2$ matrix E_1 and ${(E_2^T)_k}^T$ is the k-th row of an $n^2 \times n$ matrix E_2. Δ_k, $(E_1)_k$ and ${(E_2^T)_k}^T$ may be chosen as $\Delta_k = \Delta_{ij}$, $(E_1)_k = e_i$ and $(E_2^T)_k = e_j$ for $k = (i-1)n + j \quad i, j = 1, 2, ..., n$ so that,

$$\Delta = E_1 \Delta_d E_2$$

with (2.48)

$$\Delta_d = diag[\Delta_1, \Delta_2, ..., \Delta_{n^2}]$$

If an element of Δ is zero then the corresponding dyad in the sum (2.47) is dropped and the dimensions of E_1, Δ_d and E_2 are reduced accordingly. The transformation (2.48) can be utilized to define a new system M with diagonal uncertainty structure Δ_d and with the same set of nonzero eigenvalues as the original system. To see this notice that by using (2.48), the determinant of the return difference matrix of the original system can be written as,

$$det(I + Q\Delta) = det(I + QE_1\Delta_d E_2) = det(I + E_2 Q E_1 \Delta_d) \tag{2.49}$$

Therefore the new system is obtained by defining $M = E_2 Q E_1$. Clearly the new system may be of much larger dimension than the original one. For example

in the extreme case where $\Delta_{ij} \neq 0$ for all $i, j \in [1, n]$ then M has dimension $n^2 \times n^2$.

In [25] Safonov defines the multiloop stability margin (MSM) $k_m(\omega)$ for each frequency as,

$$k_m(\omega) = \min_{\Delta \in \Delta_d} \{k \in [0, \infty] \ : \ det(I - k\Delta M) = 0\} \tag{2.50}$$

where each uncertainty element $\Delta_i \in \Delta$ belongs to a certain domain $\mathcal{D}_i$. According to the definition (2.50) of the MSM, robustness is assured if and only if $(1/k_m)\Delta_{i_{actual}} \in \mathcal{D}_i \ \ \forall i$ and $\forall \omega$. If the uncertainty elements are norm bounded, suitable (frequency dependent) scaling matrices can be used to normalize the bounds. Since these matrices can be absorbed into the interconnection structure M then the necessary and sufficient condition for robust stability amounts to $k_m(\omega) > 1 \ \ \forall \omega$.
In [30] an algorithm based on a mapping theorem by Zadeh and Desoer [31] is proposed for the exact computation of the MSM for the case of real domains $\mathcal{D}_i \in \mathbb{R} \quad i = 1, 2, ..., n$. The algorithm can also be used for certain types of complex uncertainties.

In [26], Doyle defines the 2n-tuple of positive integers,

$$\mathcal{K} = (m_1, m_2, ..., m_n, l_1, l_2, ..., l_n) \tag{2.51}$$

and also the integer,

$$N = \sum_{j=1}^{n} m_j l_j \tag{2.52}$$

to describe the structure of the block diagonal uncertainty matrix. If $\mathcal{M}(l)$ denotes the algebra of complex $l \times l$ matrices then the block uncertainty matrix with structure corresponding to $\mathcal{K}$ in (2.51) is given by,

$$\Delta_d(\delta) = \{diag(\overbrace{\Delta_1, \Delta_1, ..., \Delta_1}^{m_1}, \overbrace{\Delta_2, \Delta_2, ..., \Delta_2}^{m_2}, ..., \Delta_{n-1}, \overbrace{\Delta_n, \Delta_n, ..., \Delta_n}^{m_n})$$
$$: \ \Delta_j \in \mathcal{M}(l_j) \text{ and } \overline{\sigma}(\Delta_j) \leq \delta \quad \text{for each} \quad j = 1, 2, ..., n\} \tag{2.53}$$

With the above notation Doyle defines the structured singular value (SSV) as,

$$\mu(M) = \begin{cases} 0 & \text{if no } \Delta \in \Delta_d(\infty) \text{ solves } det(I + M\Delta) = 0 \\ \left(\min_{\Delta \in \Delta_d(\infty)} \{\overline{\sigma}(\Delta) \ : \ det(I + M\Delta) = 0\}\right)^{-1} & \text{otherwise} \end{cases} \tag{2.54}$$

or equivalently, $\mu(M)$ is the positive real number such that,

$$\begin{gathered} det(I + M\Delta) \neq 0 \qquad \forall\, \Delta \in \Delta_d(\delta) \\ \text{if and only if} \\ \delta(M)\mu < 1 \end{gathered} \tag{2.55}$$

Notice that the definition of the SSV makes μ a function of $\mathcal{K}$. Thus, in this thesis, Doyle's definition will be used with the notation $\mu_{\mathcal{K}}(M)$.
Assuming as in the MSM case above that the perturbations are normalized, then by (2.55) robust stability is attained if and only if,

$$\sup_{\omega} \mu_{\mathcal{K}}(M) < 1 \tag{2.56}$$

For the same uncertainty structure, the MSM and the SSV represent equivalent robustness measures and they are related by,

$$\mu_{\mathcal{K}}(M) = {k_m}^{-1}(M) \tag{2.57}$$

It is readily verified from (2.53) that in the extremal cases where $n = 1,\ l_1 = 1$ and $n = 1,\ m_1 = 1$ then, $\mu_{\mathcal{K}}(M) = \rho(M)$ and $\mu_{\mathcal{K}}(M) = \overline{\sigma}(M)$ respectively.
The definition of the SSV does not provide means for its computation. Moreover, except for a few special cases there exists no method for the exact computation of $\mu_{\mathcal{K}}(M)$. Bounds were developed to compute it. From the definition (2.54) the following bounds are implied,

$$\rho(M) \leq \mu_{\mathcal{K}}(M) \leq \overline{\sigma}(M) \tag{2.58}$$

However the gap between the bounds (2.58) may be arbitrarily large. Better bounds are obtained by defining a set $\mathcal{U}$ consisting of all unitary matrices U (i.e.

$UU^H = U^H U = I$) with structure $\mathcal{K}$ and a set $\mathcal{D}$ comprising all positive real diagonal matrices with underlying structure $\mathcal{K}$ i.e.,

$$\mathcal{U} = \{diag(\overbrace{U_1, U_1, \ldots, U_1}^{m_1}, \overbrace{U_2, U_2, \ldots, U_2}^{m_2}, \ldots, \overbrace{U_n, U_n, \ldots, U_n}^{m_n} : U_i \in \mathbb{C}^{l_i \times l_i} \; ; \; U_i {U^H}_i = I_{m_i}\}$$

and (2.59)

$$\mathcal{D} = \{diag(d_1 I_{l_1}, d_2 I_{l_1}, \ldots, d_{m_1} I_{l_1}, d_{m_1+1} I_{l_2}, \ldots, d_{m_n} I_{l_n}) \; ; \; d_i \in (0, \infty)\}$$

where I_j denotes the $j \times j$ identity matrix.
With these definitions the improved bounds are given by [26],

$$\max_{U \in \mathcal{U}} \rho(MU) \leq \mu_{\mathcal{K}}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}) \tag{2.60}$$

A main result in [26] concerning the computation of the SSV is that the left hand inequality in (2.60) is actually an equality. In the same paper Doyle also proves that when there are three or fewer nonrepeated uncertainty blocks i.e., $n \leq 3$ and $m_j = 1 \; \forall j$, the right hand inequality in (2.60) is an equality as well. More properties of μ as a matrix function can be found in [26, 28, 32, 33].

Based on the interpretation of modeling errors appearing in [4] and the ability of the SSV to deal with simultaneous, block structured uncertainties Doyle *et al.* [28] suggest the introduction of fictitious uncertainties at different locations in the loop to meet performance requirements. The addition of these uncertainties along with stability requirements enforces performance specifications. Thus, in contrast with the limitations of the singular value approach described earlier, the SSV provides necessary and sufficient conditions for robust stability and for robust performance as well.

Despite the theoretical capability of the SSV to provide a framework for nonconservative control design, its computation is so difficult that to date it has been rarely used.
As already reported in [26], the use of the lower bound in (2.60) for the computation of μ is not convenient since the optimization problem involves nonglobal

maxima. In [34], Fan and Tits suggest a new expression for the SSV which has the form of a smooth optimization problem with smooth constraints. The result of this reformulation of μ is a fast lower bound procedure for its computation. However, the algorithm does not guarantee that μ is achieved. Another computation scheme for lower bounds based on a power algorithm can be found in [35].

Alternatively, as mentioned in [26], the upper bound in (2.60) could be used for obtaining an approximation of μ. The advantage of using the upper bound is that since $\overline{\sigma}(e^D M e^{-D})$ is convex in D and the mapping $e^D \rightarrow D$ is monotone. Then, $\overline{\sigma}(DMD^{-1})$ has only global minima. Some work related to the proof of the convexity of $\overline{\sigma}(e^D M e^{-D})$ appears in [36, 32, 37]. In [26] Doyle proposed a procedure based on descent directions to compute the upper bound and in [38] Fan and Tits studied an alternative upper bound for μ based on the generalized numerical range.

In practice, as suggested in [39] it is desirable to use both upper and lower bounds for μ since, by computing the upper bound, it may be easier to recognize whether the lower bound local maxima are global or not.

A nice geometric framework for the computation of μ in the case of element by element uncertainties was proposed by Kouvaritakis and Latchman [40]. In this paper it is shown that a necessary and sufficient condition for $\rho(M\Delta) = \overline{\sigma}(M)\overline{\sigma}(\Delta)$ is that the (normalized) right and left singular vectors corresponding to the maximal singular value of M are aligned to the left and right singular vectors corresponding to the maximal singular value of Δ. In a paper by Daniel *et al.* [29] it is shown that the above alignment, referred to as the *major principal direction alignment* (MPDA) is attainable by similarity scaling, irrespective of the number of uncertainty blocks, in the case where the maximal singular value of DMD^{-1} remains distinct for all D. However it should be emphasized that this case is not generic [34].

It is important to note that in the development of the bounds (2.58) and (2.60) a key assumption is that the uncertainties are complex. It turns out that

for the *mixed* real parametric and complex uncertainty case these bounds are no longer true. Other bounds exist for this case [41]. However their computation is much more difficult compared to the pure complex case. The computation of an upper bound has been studied in [41] based on the minimization of the eigenvalues of an Hermitian matrix. In [42] a lower bound is computed by solving a real eigenvalue maximization problem using a power algorithm.

The framework of the SSV has been further extended by incorporating phase information into the uncertainty. Some preliminary results are reported in [43].

CHAPTER
THREE

ROBUSTNESS FUNDAMENTALS

In this chapter we present some fundamental theorems regarding the robustness of multivariable systems. In the first section a simple version of the Multivariable Nyquist Theorem is stated and Lehtomaki's Fundamental Robustness Theorem, which introduces the use of homotopies, is stated and proved. Then, we show how these theorems lead to practical robustness tests based on singular values. We explore a loop by loop approach for assessing robustness of multivariable systems in the second section. The approach explains, in very basic and intuitive terms, reasons for the inadequacy of some classical stability margin measures and for the potential lack of robustness of synthesis methods based on diagonal dominance. The approach also helps understand the fact that singular value based methods provide a complete albeit potentially conservative robustness assessment.

3.1 Basic Multivariable Stability and Robustness Theorems

Consider the multivariable, linear time invariant feedback system of Figure 3.1 where $P(s) = [p_{ij}(s)]$ is the $n_i \times n_o$ transfer function matrix of the plant

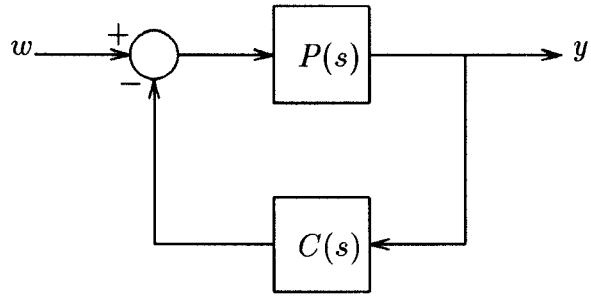


Figure 3.1: Multivariable LTI Feedback System

model and $C(s) = [c_{ij}(s)]$ is the $n_o \times n_i$ transfer function matrix of the compensator.

Let the triple (A, B, F) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_i}$ and $F \in \mathbb{R}^{n_o\times n}$ be a state space realization of the open loop transfer matrix $G(s) = P(s)C(s)$. I.e., if $\mathcal{L}(A)$ denotes the spectrum of A then, $G(s) = F(sI - A)^{-1}B$ for all $s \in \mathbb{C} - \mathcal{L}(A)$. The system open loop and closed loop characteristic polynomials $\phi_{ol}(s)$ and $\phi_{cl}(s)$ respectively are given by,

$$\phi_{ol}(s) = det(sI - A)$$

and (3.1)

$$\phi_{cl}(s) = det(sI - A + BF)$$

Using the formulas for the determinant of block matrices (see e.g. [44]) we have,

$$det\begin{pmatrix} sI - A & -B \\ F & I \end{pmatrix} = \begin{cases} det(sI - A + BF) & \forall\, s \in \mathbb{C} \\ det(sI - A)det[I + F(sI - A)^{-1}B] & \forall\, s \in \mathbb{C} - \mathcal{L}(A) \end{cases} \tag{3.2}$$

Combining (3.1) and (3.2) one obtains,

$$det[I + G(s)] = det[I + P(s)C(s)] = \frac{\phi_{cl}(s)}{\phi_{ol}(s)} \quad \forall\, s \in \mathbb{C} - \mathcal{L}(A) \tag{3.3}$$

Thus, the determinant of the return difference matrix equals the ratio between the closed loop and open loop characteristic polynomials. This result is basic for the proof of the Multivariable Nyquist Theorem and Lehtomaki's Fundamental Robustness Theorem.

To state a version of the Multivariable Nyquist Theorem we denote by $\mathcal{N}(\Omega, f(s), \mathcal{C})$ the number of clockwise encirclements of the point Ω by the locus of $f(s)$, as s traverses clockwise around the closed contour $\mathcal{C}$ in the complex plane and by p the number of closed right half plane zeros of $\phi_{ol}(s)$.
Let Γ_R be the (standard) Nyquist contour depicted in Figure 3.2. Γ_R encloses all the closed right half plane zeros of $\phi_{ol}(s)$ and avoids the imaginary zeros of $\phi_{ol}(s)$ by identations (to the left) of radius $1/R$. Then, a simple version of the

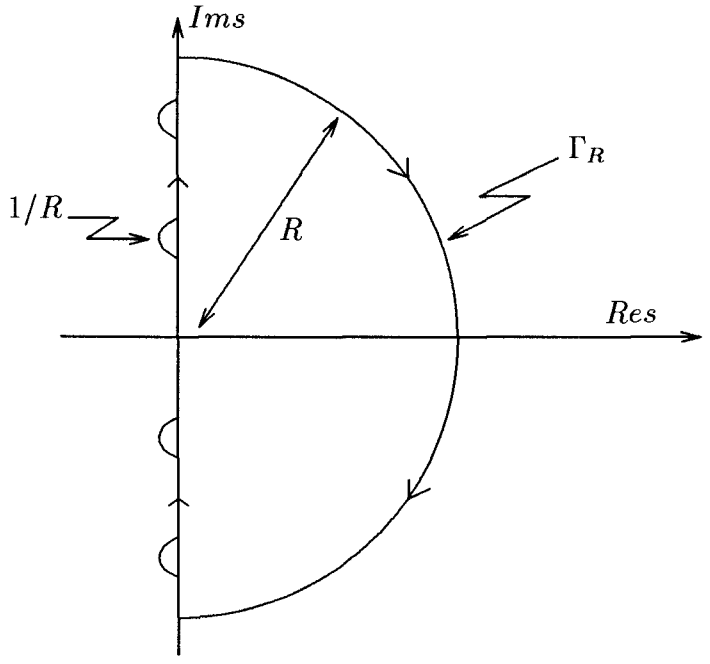


Figure 3.2: The Contour Γ_R

Multivariable Nyquist Theorem [2, 4] can be stated as follows,

Theorem 3.1 *The system of Figure 3.1 is closed loop asymptotically stable (i.e., all the roots of $\phi_{cl}(s) = 0$ are in the open left half complex plane) if and only if,*

$$\mathcal{N}(0, det[I + G(s)], \Gamma_R) = -p \tag{3.4}$$

for all R sufficiently large. •

As mentioned in [2, 4], Theorem 3.1 does not require any controllability or observability assumptions to test for internal stability. This follows from the fact that the roots under consideration are the zeros of $\phi_{ol}(s)$ which are (A, B, F) invariants. Notice also that, in contrast with the SISO case, the dependence of $det[I + G(s)]$ on the compensator $C(s)$ implicit in $G(s)$ may not be simple, and may not be easily depicted with a frequency plot. This makes the Multivariable Nyquist Theorem difficult to use.

To deal with robustness consider the closed loop system of Figure 3.1 with a perturbed plant $\tilde{P}(s)$. Let $(\tilde{A}, \tilde{B}, \tilde{F})$ be a realization of the perturbed open loop transfer matrix $\tilde{G}(s) = C(s)\tilde{P}(s)$. Then the perturbed system open loop and closed loop characteristic polynomials are given by,

$$\tilde{\phi}_{ol}(s) = det(sI - \tilde{A})$$

and (3.5)

$$\tilde{\phi}_{cl}(s) = det(sI - \tilde{A} + \tilde{B}\tilde{F})$$

In principle, Theorem 3.1 could be applied individually to $G(s)$ and $\tilde{G}(s)$ to test for stability of the nominal and perturbed systems. However such a test, in addition to requiring a complete description of the model perturbation, is only applicable to cases where the perturbations belong to a small, discrete set. More realistic robustness tests would allow finding a whole family of perturbations with some common characteristic, for which the closed loop system remains stable. They would also require only a partial characterization of the model uncertainty as in general the knowledge of the perturbation is very limited.
A theorem, fundamental for the development of realistic methods for robustness assessment is Lehtomaki's Fundamental Robustness Theorem [2, 4, 23, 24, 45]. The theorem uses the concept of homotopy which will be now defined.

Definitions [45]: Let $f : X \rightarrow Y$ be a function from X to Y and let $S \subset X$ be a subset of X. Then the restriction of f to S, $f|S : S \rightarrow Y$ is the function defined by $(f|S)(x) = f(x) \ \ \forall\, x \in S$.

Let $f, g : X \to Y$ be continuous maps and let $\times$ denote the Cartesian product. Then, a homotopy in Y between f and g is a continuous map $H : X \times [0,1] \to Y$ such that $H|X \times \{0\} = f$ and $H|X \times \{1\} = g$. If such a homotopy exists, then f and g are homotopic in Y.

Consider now the contour of Figure 3.2 and assume that a homotopy $G_R : \Gamma_R \times [0,1] \to \mathbb{C}^{n_o \times n_o}$ between $G|\Gamma_R$ and $\tilde{G}|\Gamma_R$ exists. Then, Lehtomaki's Fundamental Robustness Theorem is the following,

Theorem 3.2 *The polynomial $\tilde{\phi}_{cl}(s)$ has no closed right half plane zeros and hence the perturbed feedback system is stable if the following conditions hold,*

1. *$\phi_{ol}(s)$ and $\tilde{\phi}_{ol}(s)$ have the same number of closed right half plane zeros.*
2. *If for some $\omega_o \in \mathbb{R}$ $\tilde{\phi}_{ol}(j\omega_o) = 0$ then $\phi_{ol}(j\omega_o) = 0$.*
3. *$\phi_{cl}(s)$ has no closed right half plane zeros.*
4. *$det[I + G_R(s,t)] \neq 0$ for all $(s,t) \in \Gamma_R \times [0,1]$, and all R sufficiently large.*

Proof: By virtue of condition (*2*) and the identation construction of Γ_R, for any $t \in [0,1]$ and for all R sufficiently large the contour Γ_R will enclose all closed right half plane zeros of $\phi_{ol}(s)$ and $\tilde{\phi}_{ol}(s)$. From Theorem 3.1 and condition (*3*) we conclude that,

$$\mathcal{N}(0, det[I + G_R(s,0)], \Gamma_R) = -p \tag{3.6}$$

where p is the number of closed right half plane zeros of $\phi_{ol}(s)$ (and also of $\tilde{\phi}_{ol}(s)$ by condition (*1*)).

Suppose now that as t is varied continuously from zero to one, the number of encirclements $\mathcal{N}(0, det[I + G_R(s,t)], \Gamma_R)$ changes. Since $det[I + G_R(s,t)]$ is a continuous function in s and t in $\Gamma_R \times [0,1]$, its locus on Γ_R forms a closed bounded contour in the complex plane for any $t \in [0,1]$. The only way to change

the number of encirclements of the critical point $(0,0)$ is for the locus to pass through the critical point for some $t \in [0,1]$. That is, for some $t_o \in [0,1]$,

$$det[I + G_R(s,t_o)] = 0 \quad \text{for some } s \in \Gamma_R \tag{3.7}$$

Condition (*4*) eliminates the possibility that $det[I+G_R(s,t_o)] = 0$. This prevents $\mathcal{N}(0, det[I + G_R(s,t)], \Gamma_R)$ from changing as t is varied from zero to one and thus it must be true that it remains constant at $-p$ for all t. This implies that $\mathcal{N}(0, det[I+G_R(s,1)] = \mathcal{N}(0, det[I+\tilde{G}(s)] = -p$ and thus by condition (*1*) and Theorem 3.1, $\hat{\phi}_{cl}(s)$ has no closed right half plane zeros. •

Condition (*2*) of Theorem 3.2 may be relaxed by employing a modified Nyquist contour. A less restrictive version of the theorem can be found in [45].

The idea behind Theorem 3.2 is that of continuously deforming the Nyquist plot corresponding to $G(s)$ into the actual system $\tilde{G}(s)$ without changing the number of encirclements of the critical point. This idea turns out to be fundamental to the applicability of singular values as tools for multivariable robustness assessment. To illustrate this, consider the simple case of a system whose open loop transfer function matrix $G(s)$ is perturbed by stable additive model perturbations $\Delta(s)$ as depicted in Figure 3.3. In order to apply Theorem 3.2 we first define the homotopy,

$$G_R(s,t) = G(s) + t\Delta(s) \tag{3.8}$$

which trivially satisfies $G_R(s,0) = G(s)$ and $G_R(s,1) = \tilde{G}(s) = G(s) + \Delta(s)$. Denote by $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ the functions which take maximal and minimal singular values respectively. Then, a direct application of Theorem 3.2, which leads naturally to the use of singular values as tools for multivariable robustness assessment, is given by the following theorem [4, 23, 24, 46, 47],

Theorem 3.3 *The closed loop characteristic polynomial $\tilde{\phi}_{cl}(s)$ of the perturbed system of Figure 3.3 has no zeros in the closed right half plane if the following conditions hold,*

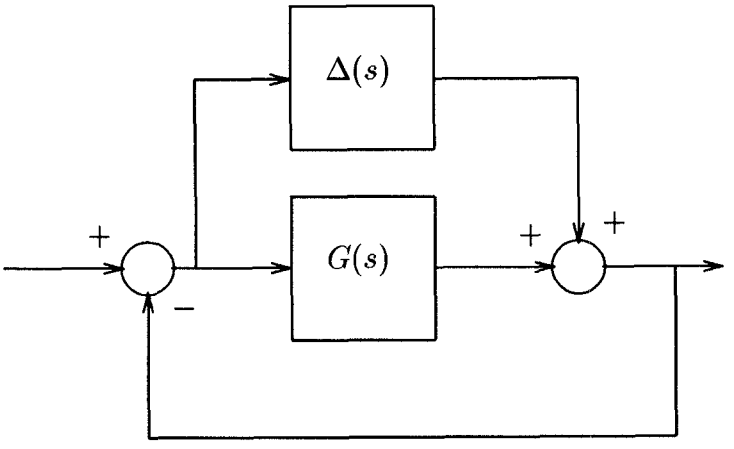


Figure 3.3: System with Additive Model Perturbations

1. *Conditions (1), (2) and (3) of Theorem 3.2 hold.*

2. $\underline{\sigma}[I + G(s)] > \overline{\sigma}[\Delta(s)]$ *for all* $s \in \Gamma_R$.

Proof: From (3.8) it follows that,

$$I + G_R(s,t) = I + G(s) + t\Delta(s) \tag{3.9}$$

If for some frequency s and some t, $I + G_R(s,t)$ in (3.9) is rank defficient, then there must exist an n_o–vector x with $\|x\| = 1$ (where $\|\cdot\|$ denotes the Euclidean norm) such that,

$$[I + G(s) + t\Delta(s)]x = 0 \tag{3.10}$$

From (3.10), the definition of singular values, and since $t \in [0,1]$ it then follows that,

$$\underline{\sigma}[I+G(s)] \leq \|[I+G(s)]x\| = \|t\Delta(s)x\| \leq t\|\Delta(s)\| \leq \|\Delta(s)\| = \overline{\sigma}[\Delta(s)] \tag{3.11}$$

Therefore (3.11) implies that if,

$$\underline{\sigma}[I + G(s)] > \overline{\sigma}[\Delta(s)] \quad \text{for all} \quad s \in \Gamma_R \tag{3.12}$$

then expression (3.9) is nonsingular for all $s \in \Gamma_R$ and all $t \in [0,1]$. Thus, condition (*4*) of Theorem 3.2 is satisfied and the theorem is proven. •

Theorem 3.3 assures stability of the closed loop system for all possible stable, additive perturbations whose norm is bounded by condition (*2*). It is not difficult to see that for perturbations which have arbitrary structure (referred to as unstructured perturbations) and that are characterized only by a bound on their norm, the conditions of Theorem 3.3 are not just sufficient but also necessary. This topic will be further discussed in the next chapter.

3.2 A Loop by Loop Approach

In Section 3.1 we presented three basic theorems regarding the stability and robustness of multivariable systems. The robustness theorems assumed model perturbations for which the open loop unperturbed and actual transfer matrices $G(s) = C(s)P(s)$ and $\tilde{G}(s) = C(s)\tilde{P}(s)$ respectively are homotopic in $\mathbb{C}^{n_o \times n_o}$. The robustness test consisted in checking the condition $det[I + G_R(s,t)] \neq 0$ for all $(s,t) \in \Gamma_R \times [0,1]$ and all R sufficiently large, where the map G_R was defined to be a homotopy between $G|\Gamma_R$ and $\tilde{G}|\Gamma_R$.

This section is aimed at obtaining further insight into the structure of multivariable systems, at examining some practical means to test for robustness, and at better understanding the role of the singular values in the robustness assessment of these systems. We will try to accomplish this by exploring the possibility of assessing robustness on a loop by loop basis.
Throughout this section, the explicit dependence of the various maps on the complex variable s will be dropped for notational convenience.
First, we show a compensator matrix representation which allows us to display the loop locations where the feedback paths are to be opened for loop by loop analysis.

Consider again the system of Figure 3.1. The closed loop map between the input vector w and the output vector y is given by,

$$y = P(I + CP)^{-1}w \tag{3.13}$$

The signals at the outputs of the individual compensator elements may be displayed by decomposing the compensator matrix as the product of two matrices $C = C_2 \cdot C_1$, with C_1 an $(n_i n_o) \times n_i$ matrix of the form,

$$C_1 = (\hat{C}_1 \ \ldots \ \hat{C}_{n_o})^T \quad \hat{C}_i = diag[c_{ij}] \quad j = 1, 2, \cdots, n_i \quad i \in [1, n_o]$$

and C_2 a $n_o \times (n_i n_o)$ matrix of the form,

$$C_2 = \text{blockdiag}(\overbrace{1\ 1\ \cdots\ 1}^{n_i \text{ times}}, \cdots, \overbrace{1\ 1\ \cdots\ 1}^{n_i \text{ times}})_{n_i n_o \text{ times}}$$

With this decomposition of the compensator matrix, the feedback system looks as depicted in Figure 3.4, where the output z_{ij} of C_1 is the signal after the

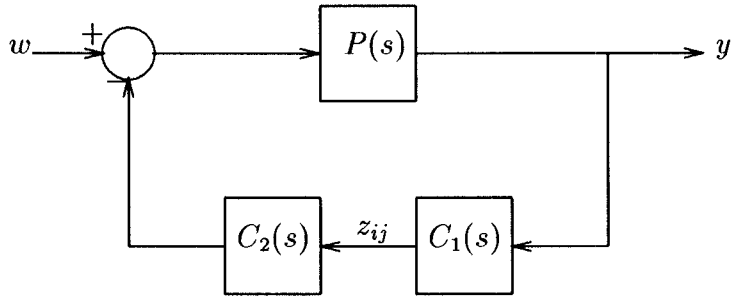


Figure 3.4: Feedback Loop with Compensator Decomposition

component c_{ij} of the compensator $C(s)$. To illustrate this, a detailed decomposition of $C(s)$ for a 2×2 plant is shown in Figure 3.5. For this example the corresponding matrices C_1 and C_2 are given by,

$$\begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{12} \\ c_{21} & 0 \\ 0 & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix} = C_2 \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix}$$

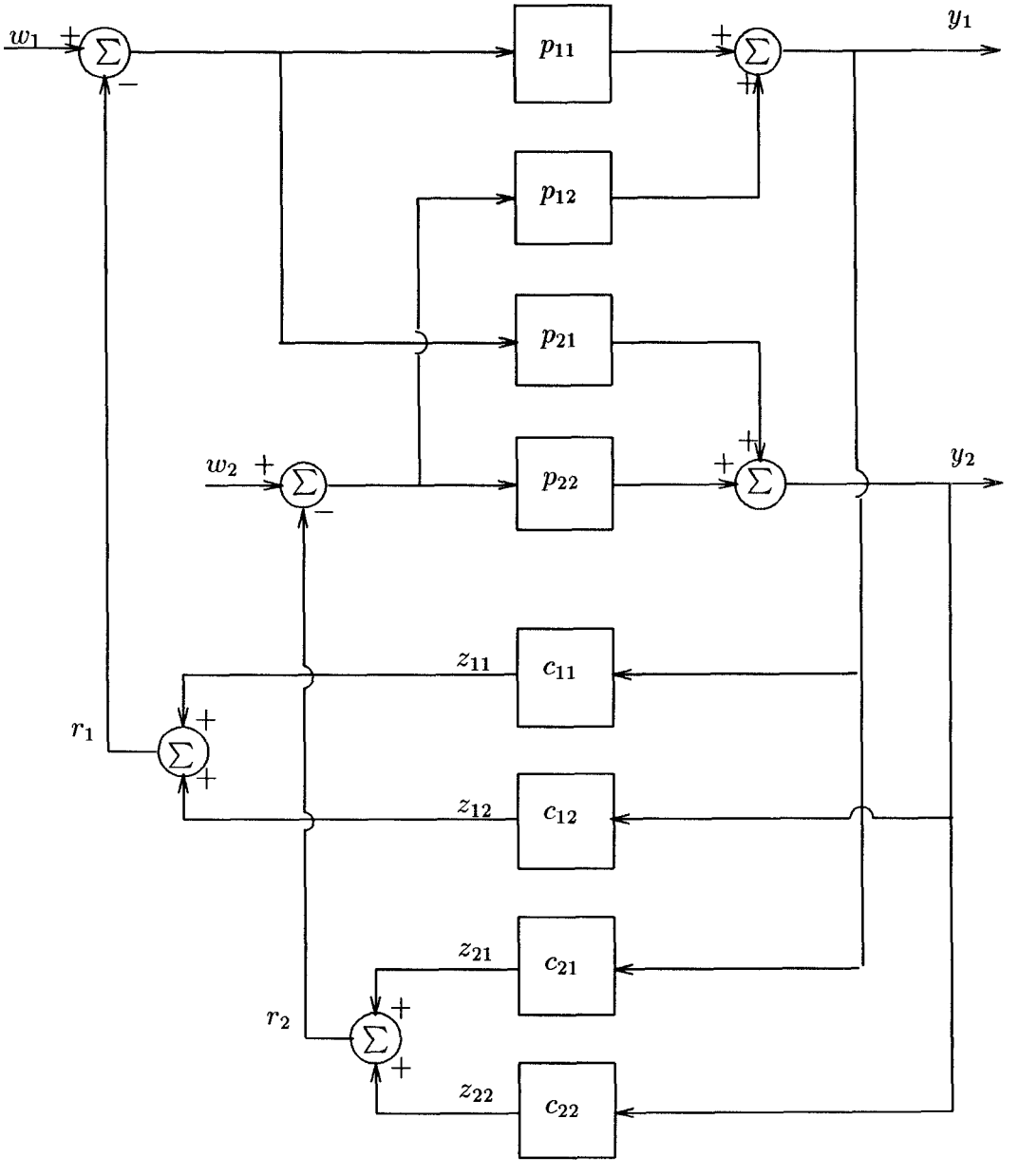


Figure 3.5: Decomposition of Compensator Matrix

To facilitate analysis of the system when the loop is opened after one of the compensator elements we will use the following notation,

C^{mn} the matrix $C = [c_{ij}]$ with all its elements replaced by zeros except the mn element.

C_{kl} the matrix $C = [c_{ij}]$ with its kl element replaced by a zero.

h_{ij} transfer function between w_i and y_j with the loop open at z_{ij}.

e_j elementary column vector with a 1 in the jth place and zero everywhere else.

$\mathcal{I}$ the set of all index pairs ij, $i \in [1, n_i]$, $j \in [1, n_o]$.

With this notation the transfer function between the input vector component w_j and the output vector component y_i when the loop is open at z_{ji} is given by,

$$h_{ij} = e_j{}^T P(I + C_{ij}P)^{-1} e_i \tag{3.14}$$

Recall that the stability and the robustness theorems of Section 3.1 were based on the determinant of the return difference matrix of the system. Using expression (3.14) we can express the determinant of the system return difference matrix in terms of transfer functions obtained when one of the loops is opened, as shown in the following Lemma,

Lemma 3.1 *For any index pair* $ij \in \mathcal{I}$,

$$det(I + CP) = (1 + c_{ij}h_{ij})det(I + C_{ij}P) \tag{3.15}$$

Proof For any index pair $ij \in \mathcal{I}$ we can write $C = C_{ij} + C^{ij}$ therefore,

$$det(I + CP) = det(I + C_{ij}P + C^{ij}P) = \\ det[I + C^{ij}P(I + C_{ij}P)^{-1}]det(I + C_{ij}P) \tag{3.16}$$

But the matrix $C^{ij}P(I + C_{ij}P)^{-1}$ has at most rank 1 and moreover can be written as $c_{ij}e_i{e_j}^T P(I + C_{ij}P)^{-1}$. Therefore,

$$det[I + C^{ij}P(I + C_{ij}P)^{-1}] = \\ det[I + c_{ij}e_i{e_j}^T P(I + C_{ij}P)^{-1}] = 1 + c_{ij}{e_j}^T P(I + C_{ij}P)^{-1}e_i \tag{3.17}$$

and from (3.14), (3.15) and (3.17) the result follows. •

Lemma 3.1 allows us to state the following theorem which provides necessary and sufficient conditions for loop by loop robustness analysis of multivariable systems,

Theorem 3.4 *A necessary and sufficient condition for* $det(I + CP) \neq 0$ *is that* $0 \neq |1 + c_{ij}h_{ij}| < \infty$ *for all index pairs* $ij \in \mathcal{I}$.

Proof Necessity follows directly from Lemma 3.1. To prove sufficiency suppose $0 \neq |1 + c_{ij}h_{ij}| < \infty$ for all pairs $ij \in \mathcal{I}$, and $det(I + CP) = 0$. By Lemma 3.1 this implies that $det(I + C_{ij}P) = 0$ for all pairs $ij \in \mathcal{I}$.
It is simpler to show contradiction in the above hypothesis by using Mason's rule for the loop transmittance. Recall (see e.g. [1]) that by Mason's rule the transfer function T between an input-output pair is given by,

$$T = \frac{\sum T_k \Delta_k}{\Delta} \tag{3.18}$$

where,
T_k is the transmittance of the k-th forward path between the input and the output,
Δ is the graph determinant given by $\Delta = 1 - \sum l_1 + \sum l_2 - \cdots$, with l_i the product of the transmittances of i nontouching loops and,

Δ_k is the cofactor of T_k i.e., the determinant of the remaining subgraph when the path which produces T_k is removed.

Now consider an index pair $ij \in \mathcal{I}$ and the transfer function $T = h_{ij}$. Notice that $\Delta = det(I + C_{ij}P)$. Assuming $c_{ij} < \infty$, in order for $|1 + c_{ij}h_{ij}|$ to be finite it is required by (3.18) that $\Delta_k = 0$ for all k since by hypothesis $\Delta = 0$. This in turn implies, by the definition of Δ_k, that there exist two sub-graphs, say L_1 and L_2 with L_1 containing the input w_i and the output y_j, such that there is no closed path which passes through both of them and such that the graph determinant of L_2, $\Delta(L_2) = 0$.
Repeating the same arguments for the sub-graphs L_2 and any index pair belonging to it we conclude inductively that there exist an index pair, say $k\ell \in \mathcal{I}$ such that,

$$|1 + c_{k\ell}h_{k\ell}| = |1 + c_{k\ell}p_{\ell k}| = 0 \tag{3.19}$$

in contradiction with the hypothesis. •

From Theorem 3.4 we conclude that the roots of the determinant of the return difference matrix of a multivariable system consisting of a $n_i \times n_o$ plant and a $n_o \times n_i$ compensator are a subset of the roots of the return difference scalar functions corresponding to $n_i \cdot n_o$ single variable loops. Therefore, the application of Theorem 3.2 on a loop by loop basis requires testing $n_i \cdot n_o$ scalar functions (at most) along the Nyquist contour for robustness assessment. Intuitively, the theorem expresses the fact that the system may be composed of various subsystems with no closed paths between them. Thus, in order to assess robustness, each one of the subsystems should be tested. As an extreme example, consider a square $n \times n$ multivariable system consisting of n individual single input–single output subsytems with no connection between them.

As an illustration of Theorem 3.4, we will consider again the example used in Chapter 2 to show the inadequacy of the multiloop gain margin. The closed loop system is depicted in Figure 3.6. The system block diagram of Figure 3.6

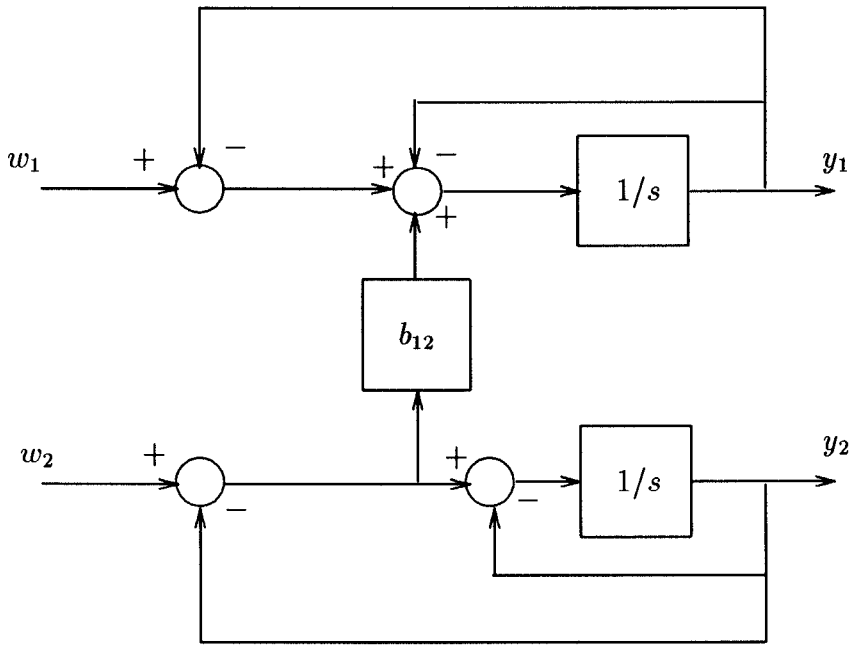


Figure 3.6: Loop by Loop Classical Stability Margins Example

can be easily transformed into one of the forms of Figure 3.5 by considering Figure 3.5 with the following transfer functions,

$$p_{11}(s) = (s+1)^{-1} \quad p_{12}(s) = b_{12}(s+1)^{-1} \qquad c_{11}(s) = 1 \quad c_{12}(s) = 0$$

$$p_{21}(s) = 0 \qquad p_{22}(s) = (s+1)^{-1} \qquad c_{21}(s) = 0 \quad c_{22}(s) = 1$$

Opening the loop at z_{11} and measuring the transfer function $h_{11} = y_1/w_1$ or doing the same for $h_{22} = y_2/w_2$ with the loop opened at z_{22} yields extremely high margins, in accordance to the results obtained by the multiloop stability margin test of Chapter 2. However, Theorem 3.4 dictates that more loops have to be checked to correctly assess robustness. Thus, if the loop is opened at z_{21} while assuming some nonzero $c_{21}(s)$, and the transfer function h_{21} between the input w_2 and the output y_1 is evaluated, then the following characteristic equation is obtained,

$$1 + c_{21}(s)h_{21}(s) = 0 \iff s^2 + 2s + c_{21}(s)b_{12}s + c_{21}(s)b_{12} + 4 = 0 \qquad (3.20)$$

From (3.20) it is easily seen that if b_{12} is large then a small model perturbation equivalent to a compensator coupling term of the form $c_{21} = -2/b_{12}$ suffices to destabilize the system. For such a disturbance, the characteristic equation (3.20) will exhibit a root at $s = 0$.
In practice, the robustness assessment for the above example will be obtained by carrying out 4 frequency responses, to cover all possible input-output pairs.

Theorem 3.4 shows that one of the requirements for a complete robustness assessment is the consideration of $n_i \times n_o$ stability margins. This explains in a formal but simple way one of the reasons for the lack of robustness properties of control design methods based on diagonal dominance. As these methods are based on shaping the characteristic loci of the diagonal maps only, they fail to consider all the loops as required by the theorem. For the same reason, classical multivariable stability margin measures such as the loop by loop gain and phase margins or the diagonal multiloop gain and phase margins mentioned in Chapter 2 do not provide a complete robustness assessment.

Theorem 3.4 helps also to understand the way the singular value operates to provide a complete robustness assessment. According to the theorem, some uncertainty directions may remain undetected by Nyquist based tests if the closed loop system including uncertainties (which will be represented in the homotopy of the actual test), contains subsystems with no closed path connection between them. However from the homotopy (3.8) defined for Theorem 3.3 and from Figure 3.3, we realize that all norm bounded matrices Δ are taken into account and therefore all possible loops are considered. As the singular value test of Theorem 3.3 provides sufficient conditions for expression (3.9) to be nonsingular all possible directions are detected and therefore the test not only satisfies the requirements of Theorem 3.4 but also considers all possible combinations of simultaneous uncertainty.

As mentioned in Chapter 2, the fact that the singular value approach detects all uncertainty directions makes it potentially very conservative. For systems

exhibiting highly structured uncertainty the singular value may lead to overdesign. The use of weighting matrices on the uncertainty matrix may alleviate the problem. However if the system specifications are skewed robust performance may not be achievable. For these reasons we consider, in the next chapter, an alternative framework that allows us to consider structured uncertainty.

CHAPTER
FOUR

ROBUSTNESS ASSESSMENT FOR SYSTEMS WITH STRUCTURED, ELEMENT BY ELEMENT COMPLEX UNCERTAINTY

This chapter deals with necessary and sufficient conditions for robust stability of multivariable systems in the face of structured uncertainty. Two approaches are presented. The first involves the use of the Structured Singular Value and deals with general block complex uncertainty. The second uses basic eigenvalue arguments and considers only element by element complex uncertainty. The second approach simplifies the proofs of some robustness theorems for the element by element uncertainty case.

4.1 Introduction

The Structured Singular Value (SSV) [26] has proven to be a powerful system theoretic tool. Originally devised for the robustness analysis of multivariable LTI systems with structured uncertainties, it may be identified as a general linear algebra tool for the analysis of linear fractional matrix perturbation problems [32].

The general idea behind the SSV, as described in detail in Chapter 2, is to give the matrix representing the uncertainty a specific structure (block diagonal). Then a singular value analysis on the transformed system is performed. The block diagonal structure enables matrix manipulations which are basic for the

proofs of the SSV properties.

However, it seems that as an analysis tool the SSV is too general to fit the spirit of this thesis. That is, the SSV is not appropiate for the analysis of multivariable systems in the vein of classical, scalar techniques. To allow such analysis, it is first required to restrict the type of uncertainty to one that resembles scalar perturbations. Then, it is worth considering more fundamental tools for stability analysis.

It is expected that the simpler uncertainty representation and the more basic analysis tools will lead to more intuitive and simpler proofs of robustness theorems. It is also hoped that this simpler approach will assist in finding practical means to assess robustness. This issue will be discussed in the next chapter.

This chapter is essentially devoted to two theorems. The first is a robustness theorem for systems with structured uncertainty. The second is a theorem which shows how to compute the stability margin obtained from the first theorem. Alternatively, the second theorem may be thought of as providing a partial characterization of the uncertainty defining the stability margin.

The theorems will be presented in two versions. The first version, corresponding to the SSV approach, will be presented in Section 4.3 along with proofs of some properties of the SSV. The uncertainty considered in this section will be of general block structure. However, to apply the SSV it will be first necessary to transform the system into one with block diagonal uncertainty structure. This results in general in a system of high dimensions. In Section 4.4, the theorems will be stated and proved using the basic eigenvalue (or maximal spectral radius) approach. However in this section, the uncertainty will be restricted to be of the element by element type. With this approach no prior transformation of the system is required for analysis.

In Section 4.2 some characteristics and some limitations of the element by element complex uncertainty representation are discussed.

4.2 Uncertainty Description

One possible way to describe uncertainty in the mathematical model of a plant is to assign individual perturbations to each one of the plant elements. Depending on the nature of the uncertainty, the individual perturbations may be modelled as either multiplicative or additive. Multiplicative perturbations are perhaps the most common type of uncertainty since they represent relative model errors. If $p_{ij}(s)$, the nominal ij element of the plant $P(s)$, is subjected to a multiplicative perturbation $\Delta_{ij}(s)$, then the actual plant element is,

$$\tilde{p}_{ij}(s) = p_{ij}(s)[1 + \Delta_{ij}(s)] \tag{4.1}$$

Denoting by $\|\cdot\|_\infty$ the infinity operator norm and assuming that both the perturbation matrix and the plant $\Delta(s), P(s) \in \mathbb{RH}_\infty$, then from (4.1) we obtain,

$$\|\tilde{p}_{ij}(s) - p_{ij}(s)\|_\infty = \|p_{ij}(s)\Delta_{ij}(s)\|_\infty \leq \|p_{ij}(s)\|_\infty \|\Delta_{ij}(s)\|_\infty \tag{4.2}$$

I.e., multiplicative perturbations are suitable for describing frequency dependent model errors which are proportional to the nominal plant element magnitude.

Additive perturbations, although less commonly used, are also necessary. For instance, consider the case where the nominal plant element is zero but a model perturbation exists. Then, only additive perturbations can be used. For the case of an additive, individual plant perturbation, the actual plant element is,

$$\tilde{p}_{ij}(s) = p_{ij}(s) + \Delta_{ij}(s) \tag{4.3}$$

Notice that (4.1) can also be written as,

$$\tilde{p}_{ij}(s) = p_{ij}(s)[1 + \Delta_{ij}(s)] = p_{ij}(s) + \hat{\Delta}_{ij}(s) \tag{4.4}$$

where $\hat{\Delta}_{ij}(s) = p_{ij}(s)\Delta_{ij}(s)$. As the nominal plant model is perfectly known, multiplicative perturbations can be represented as additive ones. Thus, additive perturbations are the most general. For this reason, these additive perturbations will be assumed from now on.

As mentioned in Chapter 2, the element by element type of perturbation has been extensively used [25, 18, 19, 48, 40] as a natural way to represent structured uncertainty. In [25] Safonov used it to represent uncertainty only in the diagonal plant elements. In [19] Lunze suggested the use of multidimensional uncertainty description to cope with the conservatism inherent in the use of the unstructured singular value for robustness assessment. In [48] Kouvaritakis and Latchman showed that the set of element by element structured perturbations is a proper subset of the set containing the class of unstructured perturbations. Although the element by element perturbation assignment is a most general method to describe model uncertainty when the perturbations are characterized only by a bound on their norm, a certain degree of conservatism is introduced by the current methods of robustness analysis. For example, it is not very uncommon that model uncertainty exists in a sensor. To represent sensor uncertainty on a plant-element-by-element basis, it is necessary to assign independent uncertainty to each one of the elements whose output is measured by the sensor. For illustration, consider the case where the model of the sensor measuring the first output of a 2×2 plant $P(s)$ is uncertain. This may be represented as a left multiplicative perturbation $\Delta(s)$ on the plant $P(s)$ of the form,

$$\Delta(s) = \begin{pmatrix} \Delta_{11}(s) & 0 \\ 0 & 0 \end{pmatrix}$$

The perturbed plant $\tilde{P}(s)$ is therefore given by,

$$\tilde{P}(s) = [I + \Delta(s)]P(s) = \begin{pmatrix} [1 + \Delta_{11}(s)]p_{11}(s) & [1 + \Delta_{11}(s)]p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix} \quad (4.5)$$

The current methods for robustness analysis (and also those used in this work) compute the magnitude and phase of the individual uncertainties in the plant elements which cause the closed loop system to be as close as possible to instability at each frequency. In general, different magnitudes and phases of the individual uncertainties are obtained. However, as the uncertainty source is a single one, it should be represented by only one magnitude and phase pair (as

in (4.5)). Thus, the results may be pessimistic. The same arguments apply to actuator uncertainty.

A way to circumvent this problem is obviously to allow uncertainty representation by multiplicative matrix perturbations. In this case however, the method used for robustness analysis must be able to handle multiplicative uncertainty representation. For example, the framework for robustness analysis used by the SSV (i.e., transforming the system into an equivalent one with block diagonal uncertainty structure) allows for matrix multiplicative uncertainty.

In the following two sections it will be assumed that some of the plant individual transfer functions are uncertain. No considerations regarding the source of the uncertainty will be made. Therefore we will not address the conservatism mentioned before. Only a bound on the norms of the uncertainties will be assumed. Thus, the phases of the uncertainties will be unrestricted. This type of uncertainty, known as complex uncertainty, arises when the model (in contrast to the model parameters) is uncertain.

4.3 The Structured Singular Value Approach

In Chapter 2, two procedures for bringing uncertain feedback systems into a form where the uncertainty matrix is block diagonal were outlined. The first procedure [28] dealt with general block diagonal forms while the second [29] specialized to element by element uncertainties. This block diagonal uncertainty matrix structure is the basis for Doyle's generalization of the singular value approach to assess robustness in the face of structured uncertainty.

Two main assumptions will be made in the robustness analysis of this chapter. First, it will be assumed that the elements of the perturbation matrix belong to $\mathbb{RH}_\infty$ i.e., the uncertainty can be modelled as a linear time invariant proper and stable system. The second assumption is that the compensator, whose design was based on the unperturbed plant model, renders the nominal closed loop system stable.

The transformed system can be represented in a block diagram as depicted in Figure 2.6 where the matrix $M(s) \in \mathbb{RH}_\infty$, whose elements are closed loop maps, represents all the interactions of the feedback loop with its uncertainties and $\Delta_d(s) \in \mathbb{RH}_\infty$ is a block diagonal uncertainty matrix. The matrix $M(s)$ is a generalized version of the closed loop map $Q(s)$ of Chapter 2. While the map $Q(s)$ corresponds to perturbations at a single location in the loop, the matrix $M(s)$ corresponds to perturbations at arbitrary locations in the loop. The stability assumptions on the nominal closed loop system and on the perturbation elements make the Multivariable Nyquist Theorem easy to apply. It suffices to verify that the determinant of the return difference matrix $I + M(s)\Delta_d(s)$ does not vanish along the Nyquist contour, for any permissible perturbation.

The description of the uncertainty matrix of Figure 2.6 is given in Chapter 2 by expressions (2.51), (2.52) and (2.53). Notice that the common upper bound δ on the individual uncertainties in (2.53) can be always obtained by suitably scaling the matrix $M(s)$.

The definition of the structured singular value, also given in Chapter 2, will be repeated here for convenience.

Given $\mathcal{K}$, the structured singular value (SSV) $\mu_{\mathcal{K}}(M)$ is defined as [26],

$$\mu_{\mathcal{K}}(M) = \begin{cases} 0 & \text{if no } \Delta \in \Delta_d(\infty) \text{ solves } det(I + M\Delta) = 0 \\ \left(\min_{\Delta \in \Delta_d(\infty)} \{\overline{\sigma}(\Delta) \ : \ det(I + M\Delta) = 0\}\right)^{-1} & \text{otherwise} \end{cases} \tag{4.6}$$

The definition of the SSV in (4.6) is not limited to matrices arising from feedback systems but applies to general complex matrices. In the context of robustness analysis it is understood that the definition holds for the matrices evaluated at each point in an appropiate Nyquist contour. Assuming that $\lim_{s\to\infty} \|M(s)\Delta(s)\| = 0$ for each permissible $\Delta(s)$ then, in the Multivariable Nyquist Theorem, only the portion of the contour Γ_R on the imaginary axis has to be considered. In what follows we will assume that this is the case. Therefore definition (4.6) will hold for all $\omega \in \mathbb{R}$.

As a function mapping square matrices to the real line the SSV enjoys several

properties [26, 28, 32, 49]. Some of them were mentioned in Chapter 2. Here we will present and prove only those which are required for the proofs of the Stability Robustness Theorem (or the Small μ Theorem) and a theorem regarding the attainability of a lower bound of the SSV. This will be done in the form of a few lemmas.

The proofs of the first two lemmas are based on inclusion properties of uncertainty matrix sets with different block structure.

Lemma 4.1 *Denote by $\rho(M)$, $\overline{\sigma}(M)$ and $\mu_{\mathcal{K}}(M)$ the spectral radius, the maximal singular value and the structured singular value of the complex square matrix M with underlying uncertainty structure $\mathcal{K}$. Then,*

$$\rho(M) \leq \mu_{\mathcal{K}}(M) \leq \overline{\sigma}(M) \tag{4.7}$$

Proof: Consider the perturbation set $\hat{\Delta} = \{\delta I\}$ (this corresponds in (2.53) to an uncertainty structure $\mathcal{K}_1$ with $m_1 = n$, $m_2 = \ldots = m_n = 0$ and $l_1 = 1$). Then $\Delta \in \hat{\Delta}$ satisfies $det(I + M\Delta) = det(I + \delta M) = 0$ if and only if $\delta = -\lambda^{-1}(M)$ where λ is an eigenvalue of M. The largest eigenvalue yields the smallest δ. I.e.,

$$\Delta \in \hat{\Delta} \Longleftrightarrow \mu_{\mathcal{K}_1}(M) = \rho(M) \tag{4.8}$$

Now suppose the perturbations are not necessarily diagonal but belong to the general set of complex matrices $\tilde{\Delta} = \{\Delta \in \mathbb{C}^{N \times N}\}$ (in (2.53) this means an uncertainty structure $\mathcal{K}_2$ with $m_1 = 1$, $l_1 = N$, where N is given by (2.52)). If $\overline{\sigma}(\Delta) < \overline{\sigma}^{-1}(M)$ then, $\overline{\sigma}(M\Delta) < 1$ and by (4.6) $\mu_{\mathcal{K}_2}(M) \leq \overline{\sigma}(M)$. Moreover, if we consider a singular value decomposition $M = Y\Sigma U^H$ where U and Y are unitary, then the perturbation $\Delta = \delta e^{-\gamma s} U Y^H$ with $\gamma \in \mathbb{R}$ is in $\tilde{\Delta}$ and the matrix $M\Delta = \delta e^{-\gamma s} Y \Sigma Y^H$ is normal. As a normal matrix, $M\Delta$ satisfies $\rho(M\Delta) = \overline{\sigma}(M\Delta)$. By choosing $\delta = \overline{\sigma}^{-1}(M)$ and an appropiate γ such that $\lambda_{max}(M\Delta) = -1$ we obtain $det(I + M\Delta) = 0$. Therefore by (4.6) $\mu_{k_2}(M) \geq \overline{\sigma}^{-1}(\Delta) = \delta^{-1} = \overline{\sigma}(M)$. Hence $\mu_{\mathcal{K}_2}(M) \geq \overline{\sigma}(M)$. Combining both results yields,

$$\Delta \in \tilde{\Delta} \Longleftrightarrow \mu_{\mathcal{K}_2}(M) = \overline{\sigma}(M) \tag{4.9}$$

Now consider a general block diagonal matrix complex perturbation set Δ_d with underlying structure $\mathcal{K}$. Clearly,

$$\hat{\Delta} \subset \Delta_d \subset \tilde{\Delta} \tag{4.10}$$

By (4.10) and the definition of the SSV (4.6) we obtain,

$$\mu_{\mathcal{K}_1}(M) \leq \mu_{\mathcal{K}}(M) \leq \mu_{\mathcal{K}_2}(M)$$

Therefore by (4.8) and (4.9) the result (4.7) follows. •

Lemma 4.2 *With the notation of Lemma 4.1 consider two complex square matrices A and B of the same dimensions. Let $\hat{\mathcal{K}}$ and $\tilde{\mathcal{K}}$ denote the structures corresponding to the matrix perturbation sets $\hat{\Delta}$ and $\hat{\Delta}A$ respectively. Then,*

$$\mu_{\hat{\mathcal{K}}}(AB) \leq \overline{\sigma}(A)\mu_{\tilde{\mathcal{K}}}(B) \tag{4.11}$$

Proof: Suppose that the matrices Δ_1 and Δ_2 with structure $\hat{\mathcal{K}}$ satisfy,

$$\mu_{\hat{\mathcal{K}}}(AB) = \frac{1}{\overline{\sigma}(\Delta_1)} \tag{4.12}$$

and

$$\mu_{\tilde{\mathcal{K}}}(B) = \frac{1}{\overline{\sigma}(\Delta_2 A)} \tag{4.13}$$

From the definition (4.6) of the SSV, expressions (4.12) and (4.13) mean that Δ_1 and $\Delta_2 A$ are the matrices of largest spectral norm satisfying,

$$det(I + AB\Delta_1] = 0 \tag{4.14}$$

and

$$det(I + B\Delta_2 A) = 0 \tag{4.15}$$

Since A, B and Δ_2 are square matrices of the same dimensions, expression (4.15) is equivalent to,

$$det(I + AB\Delta_2) = 0 \tag{4.16}$$

From (4.12) and (4.16) we have,

$$\frac{1}{\overline{\sigma}(\Delta_1)} \leq \frac{1}{\overline{\sigma}(\Delta_2)} \tag{4.17}$$

Combining (4.12), (4.13) and (4.17) we obtain,

$$\frac{1}{\overline{\sigma}(A)}\mu_{\tilde{\mathcal{K}}}(AB) = \frac{1}{\overline{\sigma}(A)} \cdot \frac{1}{\overline{\sigma}(\Delta_1)} \leq \frac{1}{\overline{\sigma}(A)} \cdot \frac{1}{\overline{\sigma}(\Delta_2)} \leq \frac{1}{\overline{\sigma}(\Delta_2 A)} = \mu_{\tilde{\mathcal{K}}}(B)$$

and the Lemma is proven. •

Consider now the sets $\mathcal{U}$ and $\mathcal{D}$ defined by (2.59). Then we have the following lemma,

Lemma 4.3 *Let the sets $\mathcal{U}$ and $\mathcal{D}$ be as in (2.59). Then, with the notation of Lemma 4.1 the structured singular value of the matrix M is bounded by,*

$$\max_{U \in \mathcal{U}} \rho(MU) \leq \mu_{\mathcal{K}}(M) \leq \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}) \tag{4.18}$$

Proof: Clearly the matrices $D \in \mathcal{D}$ and $\Delta \in \Delta_d(\delta)$ commute. Thus, $det(I + M\Delta) = det(I + MD^{-1}D\Delta) = det(I + MD^{-1}\Delta D) = det(I + DMD^{-1}\Delta)$ and $\mu_{\mathcal{K}}(M) = \mu_{\mathcal{K}}(DMD^{-1})$. By Lemma 4.1 it then follows that $\mu_{\mathcal{K}}(M) \leq \overline{\sigma}(DMD^{-1})$ and the upper bound in (4.18) holds.
To prove the lower bound notice that since $U \in \mathcal{U}$ is unitary then $det(I+M\Delta) = 0$ if and only if $det(I+MUU^H\Delta) = 0$. Also, $U^H\Delta \in \Delta_d(\delta)$ and $\overline{\sigma}(U^H\Delta) = \overline{\sigma}(\Delta)$. Therefore $\mu_{\mathcal{K}}(MU) = \mu_{\mathcal{K}}(M)$ and by Lemma 4.1 $\rho(MU) \leq \mu_{\mathcal{K}}(M)$. •

Using the results of the above lemmas the Structured Singular Value Stability Robustness Theorem or Small μ Theorem can be proved [28, 49]. For convenience we assume that the matrix M and the perturbations have been scaled so that $\delta = 1$ in (2.53).

Theorem 4.1 *The system of Figure 2.6 remains stable for all perturbations $\Delta \in \Delta_d(1)$ with structure $\mathcal{K}$ if and only if,*

$$\mu_c = \sup_{\omega \in \mathbb{R}} \mu_{\mathcal{K}}[M(j\omega)] < 1 \tag{4.19}$$

Proof: By Lemmas 4.1 and 4.2 and since $\overline{\sigma}(\Delta) \leq 1$ we have,

$$\sup_{\omega \in \mathbb{R}} \rho[\Delta(j\omega)M(j\omega)] \leq \sup_{\omega \in \mathbb{R}} \mu_{\mathcal{K}}[\Delta(j\omega)M(j\omega)] \leq$$
$$\sup_{\omega \in \mathbb{R}} \mu_{\mathcal{K}}[M(j\omega)] \cdot \overline{\sigma}[\Delta(j\omega)] \leq \mu_c < 1$$

So if $\mu_c < 1$ then $\rho[\Delta(j\omega)M(j\omega)] < 1$ for all $\Delta \in \Delta_d(1)$ and for all ω in the Nyquist contour (in this case assumed to be the indented imaginary axis only). Since $M(s), \Delta(s) \in \mathbb{R}\mathrm{H}_\infty$ then by the Multivariable Nyquist Theorem the system remains stable.

Conversely, suppose $\mu_c \geq 1$. That is, there exists a real frequency ω_o such that $\mu_{\mathcal{K}}[M(j\omega_o)] \geq 1$. Then, by the definition (4.6) of the SSV, there exists a perturbation $\Delta_o \in \Delta(\infty)$ such that $det[I + M(j\omega_o)\Delta_o(j\omega_o)] = 0$ and $\overline{\sigma}[\Delta(j\omega_o)] \leq 1$. That is, the system is unstable and $\Delta_o \in \Delta_d(1)$. •

As remarked in Chapter 2 and in contrast with the singular value case, the definition of the SSV does not provide means for its computation. Thus, the relationship between the SSV and its computable bounds (4.18) should be examined. It turns out that when the uncertainties are assumed complex then, the SSV equals its lower bound. Moreover, when the number of uncertainty blocks is $n \leq 3$, then the SSV equals its upper bound as well.

The prove that the SSV equals its lower bound requires a lemma concerning the solutions of a polynomial in several variables. Denote by $\|\cdot\|_\infty$ the usual infinity norm on $\mathbb{C}^k$ i.e., if $z \in \mathbb{C}^k$ then $\|z\|_\infty = \max_{i \leq k} |z_i|$. Also, if $p : \mathbb{C}^k \to \mathbb{C}$ is a polynomial, let β denote the norm of the solution of $p(z) = 0$ with minimum $\|\cdot\|_\infty$ norm i.e., $\beta = \min\{\|z\|_\infty \ : \ p(z) = 0\}$. Then, the required lemma [26, 32] is,

Lemma 4.4 *Let p be a polynomial from $\mathbb{C}^k \to \mathbb{C}$. Then, there exists a $z \in \mathbb{C}^k$ such that $p(z) = 0$ and $|z_i| = \beta$ for each i.*

Proof: The proof is based on the fact that the polynomial $p(z)$ can always be decomposed as,

$$p(z) = \sum_{i=0}^{n} p_i(z_1, z_2, \ldots, z_{r-1}, z_{r+1}, \ldots, z_k) z_r^i$$

where the p_i are polynomials in $k-1$ variables (all the variables except z_r). Let $\hat{z}$ be a minimizing solution i.e., $p(\hat{z}) = 0$ and $\|\hat{z}\|_\infty = \beta$. Denote also by $\hat{p}_i$ the polynomial p_i evaluated at $\hat{z}$ (notice that $\hat{p}_i$ does not depend on $\hat{z}_r$). If $|\hat{z}_i| = \beta$ for all i, then $\hat{z}$ satisfies the lemma. Assume then that one of the components of $\hat{z}$, say $\hat{z}_r$, satisfies $\hat{z}_r < \beta$. Three situations must be considered:

1. $\hat{p}_i = 0$ for all i. Then, $p(\hat{z}) = 0$ regardless of the value of $\hat{z}_r$. By choosing $\hat{z}_r = \beta$, the lemma is satisfied.

2. $\hat{p}_0 \neq 0$, but $\hat{p}_i = 0$ for $i \geq 1$. Clearly in this case $p(\hat{z}) \neq 0$. Therefore this situation is not possible.

3. $\hat{p}_i \neq 0$ for some indices $i \geq 1$. In this case $\hat{z}_r$ can be viewed as a zero of the nontrivial polynomial $q(z_r) = \sum_{i=0}^{n} \hat{p}_i z_r^i$. Consider now an $\epsilon > 0$ such that $|\hat{z}_r| + \epsilon < \beta$. By continuity of the roots of a polynomial, an $\eta > 0$ can always be found such that if $|\zeta_i - \hat{z}_i| < \eta$ for all indices except r, then there exists a $\bar{z}_r$ with $|\bar{z}_r - \hat{z}_r| < \epsilon$ such that,

$$\sum_{i=0}^{n} p_i(\zeta_1, \ldots, \zeta_{r-1}, \zeta_{r+1}, \ldots, \zeta_k)\bar{z}_r^i = 0$$

In particular the variables ζ_i can be chosen such that for each i, $|\zeta_i| < |\hat{z}_i|$. Thus, the vector $\zeta = (\zeta_1, \ldots, \zeta_{r-1}, \bar{z}_r, \zeta_{r+1}, \ldots, \zeta_k)^T \in \mathbb{C}^k$ satisfies $p(\zeta) = 0$ however $\|\zeta\|_\infty < \beta$. This contradicts the definition of β. Hence this situation cannot occur, implying that $\hat{p}_i = 0$ for all $i \geq 1$. From case (2), $\hat{p}_0 = 0$ as well so that $\hat{z}_r$ may be chosen to satisfy $|\hat{z}_r| = \beta$. Therefore the lemma is proven. •

Using Lemma 4.4, the equality between the structured singular value and its lower bound can now be proven [26, 32] by means of the following theorem,

Theorem 4.2 *Let Δ have a given block structure $\mathcal{K}$ and let $\mathcal{U}$ be as in (2.59). Then,*

$$\max_{U \in \mathcal{U}} \rho(UM) = \mu_{\mathcal{K}}(M) \tag{4.20}$$

Proof: The case $\mu(M) = 0$ is trivial. So assume $\mu(M) > 0$. Let Δ be a minimizing solution in the definition (4.6) of the SSV. I.e., $det(I + M\Delta) = 0$ and $\overline{\sigma}(\Delta) = \mu_{\mathcal{K}}^{-1}(M)$. Suppose that a singular value decomposition is carried out for each block that makes up Δ. The process will generate matrices $V, Y \in \mathcal{U}$ and a diagonal $\hat{\Sigma}$ satisfying,

$$det(I + MV\hat{\Sigma}Y^H) = 0$$

Define a matrix,

$$\Sigma = diag(z_1, z_2, \ldots, z_n) \text{ with } z_i \in \mathbb{C} \quad i = 1, 2, \ldots, n$$

Then, $det(I + MV\Sigma Y^H)$ can be viewed as a polynomial in the diagonal elements of Σ. By hypothesis, a minimum norm root of this polynomial has an infinity norm of $\gamma = \mu_{\mathcal{K}}^{-1}(M)$. Let $\bar{\Sigma}$ be the diagonal matrix whose diagonal elements have equal magnitude γ. By Lemma 4.4 $det(I + MV\bar{\Sigma}Y^H) = 0$. The matrix $\bar{\Sigma}$ can be written as $\bar{\Sigma} = \gamma\Psi$ for some $\Psi \in \mathcal{U}$. With this notation we have,

$$det(I + \gamma MV\Psi Y^H) = 0$$

therefore, $\rho(MV\Psi Y^H) \geq 1/\gamma = \mu_{\mathcal{K}}(M)$. Since $V\Psi Y^H \in \mathcal{U}$, by Lemma 4.3 the reverse inequality also holds. Therefore the theorem is proven. •

4.4 The Maximal Spectral Radius Approach

In Section 4.3 the structured singular value was defined, some of its properties were presented and two fundamental theorems were stated and proved. The first theorem provided necessary and sufficient conditions, in terms of the SSV, for robust stability of systems with norm bounded uncertainty. The second theorem showed that for complex uncertainty, the SSV equals a computable expression based on the spectral radius.

In this section we consider the same issues as in Section 4.3. However, we tackle them from a more fundamental viewpoint: the eigenvalue approach. As we will show, the application of basic eigenvalue arguments is possible due to the element by element nature of the perturbations and also since the uncertainty is assumed complex. The proofs here are not based on a specific structure of the uncertainty (as in the SSV approach where the block diagonal structure of the perturbation is critical for the proofs). Therefore there is no need to bring the system into the form of Figure 2.6. Moreover, the proofs are simpler and more intuitive than those based on the SSV.

In Section 4.2 we described the element by element (also referred to as multidimensional) uncertainty type. By appropiate scaling of the plant model, the individual uncertainties $\Delta_{ij}(s)$ can be normalized so that $\|\Delta_{ij}(s)\|_\infty \leq 1$, where $\|\cdot\|_\infty$ denotes the infinity operator norm. A perturbation matrix is then characterized only by its structure i.e., by the location of its nonzero entries. To characterize the uncertainty we then use the following definition:

An $n_i \times n_o$ *indicator* matrix $T = \{t_{ij}\}$ is a matrix whose entries are either $t_{ij} = 0$ or $t_{ij} = 1$. An $n_i \times n_o$ perturbation matrix $\Delta = \{\Delta_{ij}\}$ has indicator T if and only if the only elements of the uncertainty matrix with nonzero bounds are those corresponding to the nonzero entries of T.

To denote the number of uncertain elements of a plant we also define:

The *cardinality* of the indicator matrix, denoted by $\alpha(T)$, is the number of nonzero entries of T.

Consider now the nominally stable, perturbed, multivariable closed loop system depicted in Figure 4.1, with $P(s)$ an $n_i \times n_o$ plant, $C(s)$ an $n_o \times n_i$ dynamic compensator and $\Delta(s)$ an $n_i \times n_o$ stable perturbation matrix with indicator T and bounded elements $\|\Delta_{ij}(s)\|_\infty \leq 1$. As mentioned before it is assumed that

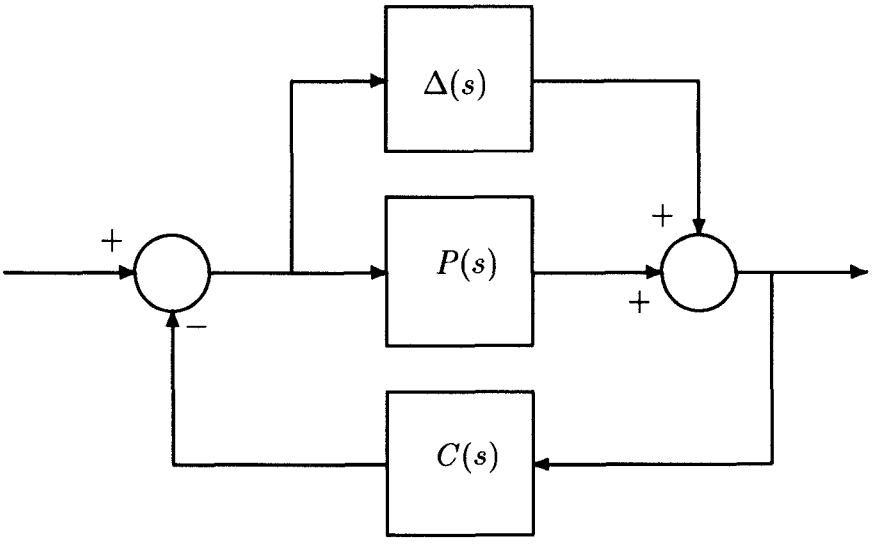


Figure 4.1: Perturbed Multivariable System

the normalized bounds on the perturbation elements are obtained after appropiate scaling i.e., the plant $P(s)$ includes normalization factors.

Consider also the following definitions,

$$Q(s) \stackrel{\text{def}}{=} C(s)[I + P(s)C(s)]^{-1}$$

$$\tilde{\Delta}(s) \stackrel{\text{def}}{=} \{\Delta(s) : \Delta(s) \text{ is stable, has indicator } T \text{ and } \|\Delta_{ij}(s)\|_\infty \leq 1\}$$

$$\hat{\rho}_{\tilde{\Delta}}(\omega) \stackrel{\text{def}}{=} \max_{\Delta \in \tilde{\Delta}} \rho[\Delta(j\omega)Q(j\omega)]$$

As in Section 4.2 we assume that $\lim_{s\to\infty} \|Q(s)\Delta(s)\| = 0$ for each permissible $\Delta(s)$ so that when applying the Multivariable Nyquist Theorem to the system in Figure 4.1, only the portion of the Nyquist contour Γ_R on the imaginary axis will be considered.

Now we state the robust stability theorem,

Theorem 4.3 *Consider the system of Figure 4.1. Assume that the unperturbed closed loop system is asymptotically stable. Then the closed loop system remains asymptotically stable for all model perturbations* $\Delta(s) \in \tilde{\Delta}(s)$ *if and only if*

$$\hat{\rho}_{\tilde{\Delta}}(\omega) < 1 \qquad \forall\, \omega \in \mathbb{R} \tag{4.21}$$

Proof Applying the Multivariable Nyquist Theorem at the output of the perturbation matrix under the assumptions of a stable nominal closed loop system and stable perturbations, each member in the family of perturbed systems is asymptotically stable if and only if,

$$det[I + \Delta(j\omega)Q(j\omega)] \neq 0 \qquad \forall\, \omega \in \mathbb{R} \quad \forall\, \Delta \in \tilde{\Delta} \tag{4.22}$$

But $det[I + \Delta(j\omega)Q(j\omega)] = 0$ is equivalent to the existence of a vector $x(\omega) \neq 0$ such that $[I + \Delta(j\omega)Q(j\omega)]x(\omega) = 0$ or equivalently, if $\lambda(\cdot)$ denotes an eigenvalue then there exists one with value $\lambda[\Delta(j\omega)Q(j\omega)] = -1$. Thus, from expression (4.22) we conclude that the perturbed system is stable if and only if,

$$\lambda[\Delta(j\omega)Q(j\omega)] \neq -1 \qquad \forall\, \omega \in \mathbb{R} \quad \forall\, \Delta \in \tilde{\Delta} \tag{4.23}$$

Noting that $\rho[\Delta(j\omega)Q(j\omega)] < 1 \implies \lambda[\Delta(j\omega)Q(j\omega)] \neq -1$, the sufficiency of (4.21) is apparent since by definition,

$$\rho[\Delta(j\omega)Q(j\omega)] \leq \hat{\rho}_{\tilde{\Delta}}(\omega) < 1$$

To show necessity suppose that there exists a frequency $\omega^* \in \mathbb{R}$ and a perturbation $\Delta^* \in \tilde{\Delta}$ for which,

$$\hat{\rho}_{\tilde{\Delta}}(\omega^*) = \rho[\Delta^*(\omega^*)Q(\omega^*)] \geq 1$$

and consider the perturbation matrix,

$$\hat{\Delta}(s) = \alpha\Delta^*(s)e^{-\gamma s}$$

with $0 \leq \alpha \leq 1$ and $\gamma \in \mathbb{R}$ arbitrary. Clearly $\hat{\Delta}$ is admissible i.e., $\hat{\Delta}(s) \in \tilde{\Delta}(s)$ and since eigenvalues are homogeneous there exists an $\alpha = \alpha^*$ such that,

$$\rho[\hat{\Delta}(\omega^*)Q(\omega^*)] = \rho[\alpha^*\Delta^*(\omega^*)Q(\omega^*)] = 1$$

Denoting by λ_{max} the eigenvalue corresponding to the spectral radius, a γ can always be found which yields,

$$\lambda_{max}[\alpha^*\Delta^*(\omega^*)Q(\omega^*)] = -1$$

Therefore (4.23) is violated and (4.21) is also necessary. •

Remarks:

1. The assumption of stable perturbations is made for simplicity and from the fact that most uncertainties are indeed stable. However the stability theorem can be modified to allow certain unstable perturbations by using homotopy arguments (cf. Chapter 3).

2. The sufficient conditions in Theorem 4.3 can also be easily obtained by a straigthforward application of the Small Gain Theorem. Notice that the necessary conditions are strongly based on the assumption of complex uncertainty.

Theorem 4.3 provides the necessary and sufficient condition

$$\max_{\Delta \in \tilde{\Delta}} \rho[\Delta(\jmath\omega)Q(\jmath\omega)] < 1 \qquad \forall\, \omega \in \mathbb{R}$$

for robust stability. By computing

$$\hat{\rho}_{\tilde{\Delta}}(\omega) = \rho[\Delta^*(\jmath\omega)Q(\jmath\omega)] = \max_{\Delta \in \tilde{\Delta}} \rho[\Delta(\jmath\omega)Q(\jmath\omega)] \tag{4.24}$$

at each frequency ω a measure of robustness of the system is obtained. The smaller the value of (4.24) is at each frequency, the more robust is the system. For each fixed frequency it is then necessary to find the *worst* permissible perturbation $\Delta^* \in \tilde{\Delta}$ such that (4.24) is attained.

Another viewpoint is possible. Since design usually involves trade-offs it will usually be true that improving robustness degrades performance. If the designer has a very accurate measure of the uncertainty bounds then $\hat{\rho}_{\tilde{\Delta}}(\omega)$ can be allowed to get close to one as long as it remains less than one.

From now on we will consider (4.24) at a fixed frequency. Therefore the dependence of the various functions and matrices on the frequency will be omitted.

The following theorem provides a partial characterization of the worst permissible, complex and element by element bounded uncertainty.

Theorem 4.4 *From among all the uncertainty matrices* $\Delta^* \in \tilde{\Delta}$ *satisfying (4.24) there is always one with* $|\Delta^*_{ij}| = 1$ *for all index pairs* ij *corresponding to the nonzero entries of the indicator* T.

Proof Suppose first that the cardinality $\alpha(T) = 1$. Then the matrix ΔQ has rank 1, at most, and therefore has no more than one nonzero eigenvalue. Let ij be the index pair corresponding to the nonzero entry of the indicator T. If for some $|\Delta_{ij}| > 0$ all the eigenvalues of ΔQ are zero, then for any $|\Delta_{ij}| > 0$ all the eigenvalues are zero and a solution to (4.24) is obtained with $|\Delta_{ij}| = 1$. Assume that there exists a nonzero eigenvalue λ. Then,

$$det(\lambda I - \Delta Q) = 0 \Longleftrightarrow det(I - \frac{\Delta}{\lambda} Q) = 0 \tag{4.25}$$

and from (4.25) an eigenvalue of maximal modulus is obtained for $|\Delta_{ij}| = 1$. To see this more clearly notice that if e_i denotes the n_i-vector with a 1 in the i-th row and zeros everywhere else, e_j is the n_o-vector with a 1 in the j-th row and zeros everywhere else and q_{ji} is the ji-th element of the matrix Q then, for the case $\alpha(T) = 1$ when some eigenvalue λ is not zero we have,

$$det(I - \frac{\Delta}{\lambda} Q) = det(I - e_i e_j^T \frac{\Delta_{ij}}{\lambda} Q) = det(I - e_j^T \frac{\Delta_{ij}}{\lambda} Q e_i) = 1 - \frac{\Delta_{ij}}{\lambda} q_{ji} \tag{4.26}$$

and by (4.26) since $q_{ji} \neq 0$ or all the eigenvalues of ΔQ would be zero, the unique nonzero eigenvalue of ΔQ is,

$$\lambda = \Delta_{ij} q_{ji} \tag{4.27}$$

Suppose now that $\alpha(T) = k + 1$ with $k > 0$ an integer and that Δ^*_{k+1} is a permissible perturbation satisfying (4.24). Assume further that there exists an element of Δ^*_{k+1}, say the ij-th element, such that $|\Delta^*_{ij}| < 1$. By the element by element nature of the uncertainties, the perturbation matrix Δ^*_{k+1} can be expressed as,

$$\Delta^*_{k+1} = \Delta_k + \Delta_1 \tag{4.28}$$

where the matrices Δ_k and Δ_1 are permissible, correspond to the indicators T_k and T_1 with cardinalities $\alpha(T_k) = k$ and $\alpha(T_1) = 1$ respectively and the only

nonzero element of Δ_1 is Δ^*_{ij}, (the ij-th element of Δ^*_{k+1}).

Denote by λ^*_{k+1} the eigenvalue corresponding to the spectral radius of $\Delta^*_{k+1}Q$. λ^*_{k+1} solves the polynomial equation,

$$0 = det(\lambda^*_{k+1}I - \Delta^*_{k+1}Q) = det(\lambda^*_{k+1}I - \Delta_k Q - \Delta_1 Q) \tag{4.29}$$

and assuming momentarily that λ^*_{k+1} is not an eigenvalue of $\Delta_k Q$ then (4.29) can be written as,

$$0 = det(I - \Delta_1 Q[\lambda^*_{k+1}I - \Delta_k Q]^{-1})det(\lambda^*_{k+1}I - \Delta_k Q) \tag{4.30}$$

Since we have assumed that $det(\lambda^*_{k+1}I - \Delta_k Q) \neq 0$ and since Δ_1 has rank 1 expression (4.30) yields,

$$0 = det(I - \Delta_1 Q[\lambda^*_{k+1}I - \Delta_k Q]^{-1}) = 1 - \Delta_{ij} e_j^T Q(\lambda^*_{k+1}I - \Delta_k Q)^{-1} e_i \tag{4.31}$$

with e_i and e_j defined as before.

From (4.31) and since $\Delta_{ij} \neq 0$ we obtain,

$$e_j^T Q(\lambda^*_{k+1}I - \Delta_k Q)^{-1} e_i = \frac{1}{\Delta_{ij}} \tag{4.32}$$

Denote by λ_k the eigenvalues of $\Delta_k Q$ and consider the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ defined as $f(\lambda) = |e_j^T Q(\lambda I - \Delta_k Q)^{-1} e_i|$. The function $f(\lambda)$ has the following properties,

P1 $\lim_{\lambda \to \lambda_k} f(\lambda) = \infty$

P2 $\lim_{|\lambda| \to \infty} f(\lambda) = 0$

P3 $f(\lambda)$ is continuous in λ except at its isolated singularities.

Property **P3** follows from the fact that the operator valued function $(\lambda I - \Delta_k Q)^{-1}$ is meromorphic in $\mathbb{C}$ [50].

By the properties of the function f and owing to the fact that there is no restriction on the phase of Δ_{ij}, there exists a $\hat{\lambda}^*_{k+1}$ and a $\hat{\Delta}_{ij}$ satisfying expression (4.32) such that $|\hat{\lambda}^*_{k+1}| > |\lambda^*_{k+1}|$ and $|\hat{\Delta}_{ij}| = 1$. Thus, Δ^*_{k+1} does not satisfy (4.24)

in contradiction with the assumption. The same argument applied to the other elements of the uncertainty matrix proves the assertion.

It remains to consider the case where λ^* is also an eigenvalue of $\Delta_k Q$ and therefore no conclusions can be drawn from expressions (4.30) and (4.31). But this case occurs when the eigenvalue λ^* of $\Delta^*_{k+1} Q$ is inherited from $\Delta_k Q$ i.e., there exists an eigenvector x common to $\Delta^*_{k+1} Q$ and $\Delta_k Q$ which is also an eigenvector corresponding to a zero eigenvalue of $\Delta_1 Q$ namely,

$$\Delta^*_{k+1} Q x = \Delta_k Q x + \Delta_1 Q x = \Delta_k Q x = \lambda^* x \tag{4.33}$$

and the perturbation Δ_1 may be replaced by one with $|\Delta_{ij}| = 1$ without violating (4.24). •

By Theorem 4.4, we can always find perturbation matrices that solve the optimization problem (4.24) and whose nonzero elements have the form,

$$\Delta_{kl} = e^{\jmath\theta_{kl}} \qquad 0 \leq \theta_{kl} < 2\pi \tag{4.34}$$

By defining the following sets,

$$\mathcal{I} \stackrel{\text{def}}{=} \{kl : t_{kl} = 1\}$$

and (4.35)

$$\Theta_{\mathcal{I}} \stackrel{\text{def}}{=} \{\theta_{kl} : kl \in \mathcal{I};\ 0 \leq \theta_{kl} < 2\pi\}$$

and by virtue of (4.34), the maximization problem (4.24) can be reformulated as,

$$\hat{\rho}_{\tilde{\Delta}}(\omega) = \rho[\Delta^*(\jmath\omega) Q(\jmath\omega)] = \max_{\theta_{kl} \in \Theta_{\mathcal{I}}} \rho[\Delta(\jmath\omega) Q(\jmath\omega)] \tag{4.36}$$

Notice that when the perturbation matrix Δ is diagonal, without any zero diagonal element, or when the system is transformed to such a form (as required by the SSV approach) then the optimization problem (4.36) can be restated as a maximization one over the set of diagonal unitary matrices. I.e., the same result as in Theorem 4.2 is obtained. Thus, for the case of element by element uncertainty, an alternative and perhaps simpler proof of the equality between the SSV and its lower bound is obtained.

CHAPTER
FIVE

A PERTURBATION APPROACH TO THE DEVELOPMENT OF ROBUSTNESS BOUNDS

In this chapter we develop bounds upon the Maximal Spectral Radius for the purpose of making it's application practical. The bounds are intended to assure stability robustness while providing insight into the system. The approach used is based on perturbation methods and computes the decrease of the system stability margin each time an additional model uncertainty is assumed. In this way, recursive bounds are obtained which generate necessary conditions for maintaining the bound within its allowed limits at each recursion step . The last step also provides sufficient conditions. The evaluation of the first bound developed relies entirely on matrix norms. This bound provides geometrical insight into the robustness problem. The second bound, based on spectral analysis, enables the study of certain subspaces associated with the matrices involved. The analysis of these subspaces renders robustness conditions in the form of expressions comprising only individual closed loop maps. The expressions may assist in preliminary control design or may be incorporated into design specifications. An application example concerning the control of the longitudinal dynamics of an aircraft is also presented.

5.1 Introduction

In Chapter 4 , Theorems 4.3–4.4, we showed that the evaluation of the stability margin of multivariable systems whose models are characterized by element by element, norm bounded uncertainties requires the computation of the uncertainty phases which yield the maximal value of $\rho(\Delta Q)$. This computation has to be carried out at each frequency along the punctured imaginary axis. In general, the relationships between the closed loop individual maps q_{ij} and the individual uncertainties Δ_{kl} arising from the expression for $\rho(\Delta Q)$ are extremely complex. Thus, little insight is obtained on how the closed loop maps may be altered to meet system specifications while maintaining stability.

An exception occurs when the uncertainty matrix has a single nonzero column or row. We call this, for the obvious reason, a rank one perturbation matrix. Recall that an indicator matrix $T = \{t_{ij}\}$ is a matrix whose entries are either zero or one and that the only elements with nonzero bounds of the uncertainty matrix $\Delta = \{\Delta_{ij}\}$ with indicator T are those corresponding to the nonzero entries of T. Recall also that the set of indeces corresponding to the nonzero entries of T is denoted by $\mathcal{I}$ i.e., $\mathcal{I} = \{kl : \ t_{kl} = 1\}$.

The following theorem characterizes the maximal spectral radius for the above type of rank one perturbation matrices,

Theorem 5.1 *Consider a system characterized by a perturbation indicator matrix $T = \{t_{ij}\}$ with a single nonzero column (row). Suppose that the nonzero column (row) is the k-th (l-th) one. Then,*

$$\hat{\rho}_{\tilde{\Delta}} = \max_{\Delta \in \tilde{\Delta}} \rho(\Delta Q) = \sum_{i:ik \in \mathcal{I}} |q_{ki}| \quad (= \sum_{j:lj \in \mathcal{I}} |q_{jl}|)$$

Proof The proof will be carried out for the column case. The row case is treated similarly.

By Theorem 4.4 the nonzero elements of the perturbation matrix Δ can be

assumed to be of the form $e^{\jmath\theta_{ij}}$, $ij \in \mathcal{I}$ and the maximal spectral radius problem can be formulated as,

$$\hat{\rho}_{\tilde{\Delta}} = \max_{\theta_{ik} \in \Theta_{\mathcal{I}}} \rho(\Delta Q)$$

where $\Theta_{\mathcal{I}} \stackrel{\text{def}}{=} \{\theta_{ik} : ik \in \mathcal{I};\ 0 \leq \theta_{ik} < 2\pi\}$.

Denoting by $\Delta_{\cdot k} = (\Delta_{1k}\ \Delta_{2k}\ \ldots\ \Delta_{n_i k})^T$ the column of the matrix Δ containing nonzero elements (of the form $\Delta_{ik} = e^{\jmath\theta_{ik}}$) and by $q_{k\cdot} = (q_{k1}\ q_{k2}\ \ldots\ q_{kn_i})$ the k-th row of Q we have that the unique nonzero eigenvalue of ΔQ is given by,

$$det(\lambda I - \Delta Q) = det(\lambda I - \Delta_{\cdot k} q_{k\cdot}) = 0 \Longleftrightarrow \lambda = q_{k\cdot}\Delta_{\cdot k}$$

The above expression implies,

$$\hat{\rho}_{\tilde{\Delta}} = \max_{\theta_{ik} \in \Theta_{\mathcal{I}}} q_{k\cdot}\Delta_{\cdot k} = \max_{\theta_{ik} \in \Theta_{\mathcal{I}}} \sum_{i:ik \in \mathcal{I}} q_{ki} e^{\jmath\theta_{ik}} = \sum_{i:ik \in \mathcal{I}} |q_{ki}|$$

and the assertion is proved. •

Combining Theorems 4.3 and 5.1 we obtain the following result. For systems with indicators having a single nonzero column or row, necessary and sufficient conditions for robust stability amount to maintaining the sum of the absolute values of certain individual closed loop maps less than one. Clearly, this simple condition can be easily incorporated into closed loop specifications and used in control design.

Notice that a special case of the rank one indicator is the single uncertainty case.

For cases where the indicator matrix has rank greater than one, little insight is obtained from the expressions resulting from the maximal spectral radius computation. To overcome this problem the designer may appeal to design frameworks which integrate optimal control methods with robustness tools. For example, the μ-synthesis design framework [39] encompasses H_∞ optimization and the structured singular value. These design frameworks however are best used in final design stages or as a first stage in very complicated problems, where insight is not otherwise attainable. Clearly, other situations such as preliminary system evaluation, specification definition, preliminary design, troubleshooting

analysis etc., require frameworks much closer to classical control methods, based mainly on a combination of insight and engineering skills. Notice that in these cases one is less concerned with the conservatism of the results than with the understanding of the system.

This chapter is aimed at developing a method that uses the maximal spectral radius to ensure stability robustness of multivariable systems and to obtain insight into the system. The main idea is to analyze the decrease of the stability margin each time a new uncertainty is added to the system. This perturbation approach enables the detection of subsystems which, when subjected to model uncertainty, most affect the system stability margin. It also enables the study of how the interaction of different model perturbations affects stability. The method leads recursively to an upper bound upon the maximal spectral radius. Each recursion step generates necessary conditions to mantain the bound under the limit allowed by stability considerations. The necessary conditions appear as expressions relating individual closed loop maps. The information obtained from these conditions can therefore be incorporated into design specifications. We will not address here the question of the conservativeness of the bound. Our main interest is to obtain expressions that can be utilized in preliminary design stages without requiring the solution of an optimization problem. Notice however that as the method developed here is concerned with structured uncertainty we may expect in many cases less conservative results than those obtained by the singular value approach.

The chapter is organized as follows:
Section 5.2 deals with the diagonalizability of the matrices ΔQ. The diagonalizability of these matrices is a basic requirement for the development of the bounds. It is shown that these matrices are generically diagonalizable. We then use the genericity of the results to assume that the matrices have been (infinitesimaly) perturbed so that they become similar to diagonal matrices.

Since the idea of adding one perturbation at a time strongly resembles a matrix perturbation problem we devote Section 5.3 to develop a bound upon the maximal spectral radius based solely on matrix norms. The bound shows that as the eigenvectors of certain matrices deviate from orthogonality, the system becomes prone to instability under model perturbations. However, the information obtained is too general to allow analysis. The bound seems to be useful, as is the singular value analysis, only for spatially round systems. A geometrical interpretation of the bound which arises naturally when the spectral norm is used is also shown.

In Section 5.4 we develop a procedure which leads to an upper bound on the maximal spectral radius, based on spectral analysis. The bound, in a way that resembles the structured singular value, distinguishes between the invariant subspaces of the matrix ΔQ corresponding to the zero and nonzero eigenvalues. The nonzero eigenvalues yield an expression that still requires the solution of an optimization problem for its computation. However the dimensionality of the problem reduces to the size of the nonzero rows (or columns) of the uncertainty matrix. The expressions resulting from the zero eigenvalue do not depend on the uncertainty matrix. Therefore in addition to being trivial to compute they display a clear relationship between individual closed loop maps and the stability bound. Necessary and sufficient conditions for the bound to be small are then generated without any computational requirements.

Finally, in Section 5.5, a practical application of the bound developed in Section 5.4 is shown by analyzing the robustness of a loop designed by the Characteristic Locus method to control the vertical plane dynamics of an aircraft.

Let $\langle \cdot, \cdot \rangle$ denote the usual (Euclidean) inner product in $\mathbb{C}^n$ and let A be a matrix in $\mathbb{C}^{m \times n}$.

The following notation will be used throughout this chapter,

$\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\}$ the range space of A.

$\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ the null space of A.

$\mathcal{R}^{\perp}(A)$ is the orthogonal complement of $\mathcal{R}(A)$.

$\mathcal{N}^{\perp}(A)$ is the orthogonal complement of $\mathcal{N}(A)$.

A^H is the adjoint of A (i.e., the matrix satisfying $\langle Ax, y\rangle = \langle x, A^H y\rangle$ $\forall x \in \mathbb{C}^n, \forall y \in \mathbb{C}^m$).

$rank(A)$ is the dimension of $\mathcal{R}(A)$.

5.2 Diagonalization Lemmas

A fundamental requirement in the development of bounds on the maximal spectral radius by the perturbation methods of this chapter is the diagonalizability of the matrices ΔQ. In this section we show that generically, these matrices are diagonalizable. The isolated frequencies where the matrices may not be diagonalizable correspond to branch points of characteristic functions of the transfer matrix ΔQ. By the Extended Principle of the Argument [51] used in the Multivariable Nyquist Theorem [52], these points must be avoided by the Nyquist contour Γ_R. Hence, the above frequencies are of no interest in subsequent analyses and can be disregarded. At all other points in the complex plane, genericity allows us to assume infinitesimal perturbations on the matrices to make them diagonalizable.

As matrices of rank one play a crucial role in all the developments we consider them in some detail. Consider first a general complex square matrix of rank one. The following lemma proves that rank one matrices can be represented by a dyad.

Lemma 5.1 *Consider a matrix $A \in \mathbb{C}^{n\times n}$. Then $rank(A) \leq 1$ if and only if there exist two vectors $a, b \in \mathbb{C}^n$ such that,*

$$A = ab^H \tag{5.1}$$

Proof: First notice that $rank(A) = 0$ if and only if $A = 0$. Therefore for this case the assertion holds trivially.

Suppose $rank(A) = 1$. Then $\mathcal{R}(A)$ spans a one dimensional subspace. Let $a \in \mathbb{C}^n$ be a basis for $\mathcal{R}(A)$. Then any vector $y \in \mathcal{R}(A)$ can be expressed as $y = c(y)a$ with $c(y) \in \mathbb{C}$. Consider now the elementary basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$, where $e_i \in \mathbb{C}^n$ has a 1 in the i-th place and zero everywhere else. Then A can be written,

$$A = (Ae_1 \; Ae_2 \; \cdots \; Ae_n) = (\bar{b}_1 a \; \bar{b}_2 a \; \cdots \; \bar{b}_n a) = ab^H$$

with $b = (b_1 \; b_2 \; \ldots \; b_n)^T \in \mathbb{C}^n$.

Conversely, assume that $A = ab^H$. Then for each $y \in \mathbb{C}^n$ we have $Ay = ab^H y = c(y)a$ with $c(y) = b^H y \in \mathbb{C}$. That is, A spans a one dimensional subspace. Therefore $rank(A) = 1$. •

A second lemma is concerned with the diagonalizability of matrices of rank one.

Lemma 5.2 *Suppose the matrix $A \in \mathbb{C}^{n \times n}$ is of rank one and has a nonzero eigenvalue. Then A is diagonalizable by similarity transformation.*

Proof: A matrix is diagonalizable by similarity transformation if and only if its Jordan blocks have order 1. Since A has rank 1, by Lemma 5.1 there exist two vectors $a, b \in \mathbb{C}^n$ such that $A = ab^H$.

Denote $a = (a_1 \; a_2 \; \ldots \; a_n)^T$ and $b = (b_1 \; b_2 \; \ldots \; b_n)^T$. Then A has exactly two distinct eigenvalues: zero (of algebraic multiplicity $n - 1$) and $\sum_{i=1}^n a_i \bar{b}_i$ which by hypothesis is nonzero. Since A has two distinct eigenvalues its minimal polynomial has order of at least 2. Consider now the second order polynomial,

$$p(\lambda) = \lambda(\lambda - \sum_{i=1}^{n} a_i \bar{b}_i)$$

It can be easily verified that $p(\lambda)$ annihilates A. Therefore it must be its minimal polynomial and therefore the Jordan blocks of A have order 1. •

Consider now a matrix $A \in \mathbb{C}^{n \times n}$ of rank $k \leq n$. This matrix has k linearly independent rows (and columns). Therefore, the other $n - k$ rows can be written

as a linear combination of these. This means that A can be expressed as a sum of k matrices of rank one. Lemma 5.2 may be applied now to each one of the matrices in the sum. We then conclude that a matrix $A \in \mathbb{C}^{n\times n}$ of rank k may always be written in a dyadic expansion of the form,

$$A = \sum_{i=1}^{k} a^{(i)} b^{(i)H}$$

where $a^{(i)}, b^{(i)} \in \mathbb{C}^n \quad i = 1, \ldots, k$.

A fourth lemma deals with the relationship between the rank of a matrix and the dimension of its null space. Recall that an eigenvalue is nondefective if its algebraic and geometric multiplicities are the same.

Lemma 5.3 *Suppose that the algebraic multiplicity of the zero eigenvalue of a rank deficient matrix $M \in \mathbb{C}^{n\times n}$ equals n minus the number of its linearly independent rows. Then, the zero eigenvalue is nondefective.*

Proof: By definition, the invariant subspace corresponding to an eigenvalue λ of M is $\mathcal{N}(\lambda I - M)$. Therefore the invariant subspace corresponding to the zero eigenvalue of M is $\mathcal{N}(M)$ and its geometric multiplicity is $dim\{\mathcal{N}(M)\}$. Since any closed subspace $\mathcal{M}$ of a Hilbert space is complemented by $\mathcal{M}^{\perp}$ [53, page 205] then,

$$\mathcal{N}(M) \oplus \mathcal{N}^{\perp}(M) = \mathbb{C}^{n\times n} \tag{5.2}$$

where $\oplus$ denotes direct sum. From (5.2) we obtain,

$$dim\{\mathcal{N}(M)\} + dim\{\mathcal{N}^{\perp}(M)\} = n \tag{5.3}$$

But $\mathcal{N}^{\perp}(M) = \mathcal{R}(M^H)$ and $dim\{\mathcal{R}(M^H)\} = dim\{\mathcal{R}(M)\} = rank(M)$, therefore by (5.3),

$$dim\{\mathcal{N}(M)\} = n - rank(M) \tag{5.4}$$

Thus the geometric multiplicity of the zero eigenvalue of M is $n - rank(M)$ which by assumption is equal to the algebraic multiplicity of the zero eigenvalue.

Therefore the algebraic and geometric multiplicities of the zero eigenvalue are the same. That is, it is nondefective.•

Now we define the notion of *genericity*,

Definition Let $\mathcal{X}$ denote a topological space. Then a property $\mathcal{P}$ of an element $x \in \mathcal{X}$ is *generic* if the following statements are true:

1. If x has the property $\mathcal{P}$, then there exists a neighborhood of x such that every point in this neighborhood also has the property $\mathcal{P}$.

2. If x does not have the property $\mathcal{P}$ then every neighborhood of x contains an element that has the property $\mathcal{P}$.

In the following Lemmas we consider the topological space $\mathbb{C}^{n\times n}$ with a topology induced by any matrix norm $\|\cdot\|$.

Lemma 5.4 *Let A be a matrix in $\mathbb{C}^{n\times n}$. Then the property A has distinct eigenvalues is generic.*

Proof: Suppose first that A has distinct eigenvalues. Since norms and eigenvalues are continuous functions of the matrix entries [12] there always exists a $\delta > 0$ such that for each matrix $E \in \mathbb{C}^{n\times n}$ with $\|E\| < \delta$ the eigenvalues of the matrix $A+E$ remain distinct. Thus, statement (1) of the definition of genericity holds.

Now suppose that A has repeated eigenvalues. Let $S \in \mathbb{C}^{n\times n}$ be a nonsingular matrix satisfying,

$$A = SJS^{-1}$$

where J is the Jordan form of A. Consider also the perturbation matrix,

$$F = S\Lambda_F S^{-1} \qquad \text{with} \qquad \Lambda_F = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$$

Clearly, for any $\delta > 0$ we can always choose $\epsilon_i \quad i = 1, \ldots, n$ such that the eigenvalues of $A + F$ are distinct and $\|F\| = \|S\Lambda_F S^{-1}\| \leq \|S\|\|\Lambda_F\|\|S^{-1}\| < \delta$.

Thus, statement (2) of the definition of genericity also holds and the lemma is proven. •

Lemma 5.5 *Consider the same topological space as in Lemma 5.4. Let B be a matrix in $\mathbb{C}^{n\times n}$. Then B is generically nonsingular.*

Proof: The proof is similar to that of Lemma 5.4. If B is nonsingular or equivalently has no zero eigenvalues, by the continuity property of norms and eigenvalues we can always find a neighborhood of B composed of matrices with nonzero eigenvalues. Conversely, if B is singular, we can show using its Jordan form that a perturbation matrix F of arbitrary norm such that $B + F$ is nonsingular always exists. •

The following lemma is concerned with the diagonalizability of the matrix ΔQ. For convenience and without losing generality, we will assume that the number of nonzero columns of $\Delta(s)$ is larger than or equal to the number of its nonzero rows. Otherwise, the lemma and all the subsequent analysis can be carried out for the matrices $Q\Delta$ with the obvious changes.

Lemma 5.6 *The matrix $M(s) = \Delta(s)Q(s)$ is generically diagonalizable.*

Proof Since the input-output pairs of the plant $P(s)$ can be arbitrarily numbered, we may assume without loss of generality that the first m rows of $M(s)$ correspond to the nonzero rows of the indicator T and the remaining $n_i - m$ rows are zero. Therefore $M(s)$ can be decomposed as,

$$M(s) = \begin{pmatrix} \overbrace{M_1(s)}^{m} & \overbrace{M_2(s)}^{n_i-m} \\ 0 & 0 \end{pmatrix} \begin{matrix} \}\, m \\ \}\, n_i - m \end{matrix} \tag{5.5}$$

The eigenvalues of $M(s)$ are the solutions of the characteristic equation,

$$det[\lambda(s)I - M(s)] = 0 \tag{5.6}$$

and since the matrix $\lambda(s)I - M(s)$ is block diagonal, expression (5.6) can be written as,

$$\lambda^{n_i-m}(s)det[\lambda(s)I - M_1(s)] = 0 \tag{5.7}$$

First, consider the matrix $M(s)$ evaluated at a fixed frequency s^*. By Lemma 5.5, the submatrix $M_1(s^*)$ is generically of full rank. Therefore, the algebraic multiplicity of the zero eigenvalue of $M(s^*)$ is (generically) $n_i - rank\{M(s^*)\}$. By Lemma 5.3 we conclude that the zero eigenvalue of $M(s^*)$ is nondefective. Moreover, it is well known (see e.g. [12, page 47]) that distinct eigenvalues necessarily yield linearly independent eigenvectors. Applying Lemma 5.4 we conclude that as $M_1(s^*)$ has generically distinct eigenvalues the matrix $M(s^*)$ is diagonalizable.

It remains to show that as s varies the matrix $M(s)$ remains diagonalizable. If the matrix $\Delta(s)$ is of normal rank one, then $M(s)$ will also be of normal rank one and by virtue of Lemma 5.2, $M(s)$ is diagonalizable for all frequencies in the punctured s plane.

When $\Delta(s)$ is of normal rank greater than one we prove that ΔQ is diagonalizable almost everywhere by following the development of characteristic functions in [52]. Note that from expression (5.7) one obtains a characteristic equation that can be expressed in the general form,

$$det[\lambda(s)I - M_1(s)] = f_1(\lambda, s)f_2(\lambda, s)\dots f_k(\lambda, s) = 0 \tag{5.8}$$

where $\{f_i(\lambda, s) \quad i = 1, 2, \dots, k\}$ are polynomials irreducible over the field of rational functions in s. Each one of the irreducible factors in (5.8) has the form,

$$f_i(\lambda, s) = \lambda_i^{m_i}(s) + a_{i1}(s)\lambda_i^{m_i-1}(s) + \dots + a_{im_i}(s) \quad i = 1, 2, \dots, k \tag{5.9}$$

where m_i is the degree of the i-th irreducible polynomial and the coefficients $\{a_{ij}(s) \;\; i = 1, 2, \dots, k \;\; j = 1, 2, \dots, m_i\}$ are rational functions in s. Denoting by $b_{i0}(s)$ the least common denominator of the coefficients $\{a_{ij}(s) \;\; j = 1, 2, \dots, m_i\}$ in expression (5.9), the characteristic equation (5.8) yields,

$$b(s, \lambda_i) = b_{i0}(s)\lambda_i^{m_i}(s) + b_{i1}(s)\lambda_i^{m_i-1}(s) + \dots + b_{im_i}(s) = 0 \quad i = 1, 2, \dots, k \tag{5.10}$$

where now the coefficients $\{b_{ij}(s) \;\; i = 1, 2, \ldots, k \;\; j = 1, 2, \ldots, m_i\}$ are polynomials in s. Each function $\lambda_i(s) \;\; i = 1, 2, \ldots, k$ defined by (5.10) is an algebraic function [55].

For complex polynomials such as $b(s, \lambda_i)$ in (5.10), a scalar valued function of its coefficients and the coefficients of the derivative of the polynomial with respect to the independent variable s is defined. This function, denoted by $D(\cdot)$ is referred to as the discriminant of the polynomial [55, 52, 56]. Bliss [55] and Barnett [56] showed that a polynomial has multiple roots if and only if its discriminant is zero. Bliss also showed that the discriminant of a polynomial vanishes at a finite number of points in the complex plane. Applying these arguments to the polynomials (5.10) we obtain algebraic functions $\lambda_i(s)$ that when evaluated at each frequency s in the complex plane yield distinct eigenvalues except for a set of frequencies of measure zero. Thus, $M_1(s)$ has distinct eigenvalues. Moreover, it can be also shown [52] that characteristic functions have zero solutions only at a finite number of points in the complex plane. Therefore, $M_1(s)$ has full normal rank and by Lemma 5.3 the zero eigenvalue of $M(s)$ is nondefective. This completes the proof. •

In the sequel we will assume that the frequencies where the various matrices are evaluated do not correspond to branch points of characteristic functions of $\Delta(s)Q(s)$ and that if necessary, the matrices have been infinitesimaly perturbed so that ΔQ is diagonalizable.

5.3 A Bound on the Maximal Spectral Radius Based on Matrix Norms

In this section we develop an upper bound for the maximal spectral radius $\hat{\rho}_{\tilde{\Delta}}$ based on a perturbation approach using only matrix norms. The bound is developed for a fixed frequency. Therefore the explicit dependence of the various matrices and functions on the frequency will be dropped. Also, as mentioned at

the end of Section 5.2, all the matrices of the form ΔQ are assumed diagonalizable.

We start with a basic matrix perturbation theorem. Let A be a complex square matrix and $\tilde{A} = A + E$ a perturbation of A, where E is an arbitrary complex matrix dimensionally compatible to A. Denote by $\mathcal{L}(\tilde{A})$ the spectrum of $\tilde{A}$. Recall also that a consistent matrix norm is one that enjoys the submultiplicative property. Then we have the following theorem by Bauer and Fike [57],

Theorem 5.2 *Let B be nonsingular and let $\|\cdot\|$ be a consistent matrix norm. If $\tilde{\lambda} \in \mathcal{L}(\tilde{A})$ is not an eigenvalue of A, then*

$$\|B^{-1}(A - \tilde{\lambda}I)^{-1}B\|^{-1} \leq \|B^{-1}EB\| \tag{5.11}$$

Proof: Write,

$$B^{-1}(\tilde{A} - \tilde{\lambda}I)B = B^{-1}[(A - \tilde{\lambda}I) + E]B =$$
$$B^{-1}(A - \tilde{\lambda}I)B\{I + [B^{-1}(A - \tilde{\lambda}I)B]^{-1}[B^{-1}EB]\} \tag{5.12}$$

Since the matrix $\tilde{A} - \tilde{\lambda}I$ is singular then by (5.12) one obtains,

$$1 \leq \|[B^{-1}(A - \tilde{\lambda}I)B]^{-1}[B^{-1}EB]\| \leq \|B^{-1}(A - \tilde{\lambda}I)^{-1}B\|\|B^{-1}EB\|$$

which is equivalent to (5.11). •

Now consider the decomposition,

$$\Delta_r Q = \Delta_{r-1} Q + \Delta_1 Q \tag{5.13}$$

Assuming that no eigenvalues of $\Delta_r Q$ are inherited from $\Delta_{r-1}Q$, Theorem 5.2 can be applied to the matrix $\Delta_{r-1}Q$ perturbed by $\Delta_1 Q$ in (5.13) yielding,

$$\|(\Delta_{r-1}Q - \tilde{\lambda}I)^{-1}\|^{-1} \leq \|\Delta_1 Q\|$$

or by taking inverses,

$$\|(\Delta_{r-1}Q - \tilde{\lambda}I)^{-1}\| \geq \|\Delta_1 Q\|^{-1} \tag{5.14}$$

By the assumptions, the matrix $\Delta_{r-1}Q$ is diagonalizable by similarity transformation. Therefore there exists a nonsingular matrix Z and a diagonal matrix Λ_{r-1} similar to $\Delta_{r-1}Q$ such that,

$$\Delta_{r-1}Q = Z\Lambda_{r-1}Z^{-1} \tag{5.15}$$

Substituting (5.15) into (5.14) yields,

$$\begin{aligned}\|(\Delta_{r-1}Q - \tilde{\lambda}I)^{-1}\| = \|(Z\Lambda_{r-1}Z^{-1} - \tilde{\lambda}I)^{-1}\| = \\ \|Z(\Lambda_{r-1} - \tilde{\lambda}I)^{-1}Z^{-1}\| \geq \|\Delta_1 Q\|^{-1}\end{aligned} \tag{5.16}$$

Define the *condition number* of a nonsingular matrix A with respect to the norm $\|\cdot\|$ as $\kappa(A) = \|A\|\|A^{-1}\|$. Then by (5.16) and the consistency of the norm $\|\cdot\|$ we have,

$$\|(\Lambda_{r-1} - \tilde{\lambda}I)^{-1}\| \geq \|Z\|^{-1}\|Z^{-1}\|^{-1}\|\Delta_1 Q\|^{-1} = \kappa^{-1}(Z)\|\Delta_1 Q\|^{-1} \tag{5.17}$$

Suppose momentarily that $\tilde{\lambda}$ is positive, real and satisfies $\tilde{\lambda} > \rho(\Delta_{r-1}Q)$. Then,

$$\|(\Lambda_{r-1} - \tilde{\lambda}I)^{-1}\| \leq [\tilde{\lambda} - \rho(\Delta_{r-1}Q)]^{-1} \tag{5.18}$$

From (5.17) and (5.18) one obtains,

$$[\tilde{\lambda} - \rho(\Delta_{r-1}Q)]^{-1} \geq \kappa^{-1}(Z)\|\Delta_1 Q\|^{-1}$$

or equivalently,

$$\tilde{\lambda} \leq \rho(\Delta_{r-1}Q) + \kappa(Z)\|\Delta_1 Q\| \tag{5.19}$$

Recall from Chapter 4 that the maximal spectral radius was defined as,

$$\hat{\rho}(\Delta Q) \stackrel{\text{def}}{=} \max_{\Delta \in \tilde{\Delta}} \rho(\Delta Q)$$

where $\tilde{\Delta}$ denotes the set of permissible perturbations in the class defined by the indicator matrix T. Using this definition the bound (5.19) can be modified to,

$$\tilde{\lambda} \leq \hat{\rho}(\Delta_{r-1}Q) + \kappa(Z)\|\Delta_1 Q\| \tag{5.20}$$

Since the perturbations under consideration are complex, the matrix Δ_r can always be rotated (by multiplying it by $e^{\jmath\phi}$ for some $\phi \in \mathbb{R}$), without affecting the eigenvectors and the absolute values of the eigenvalues of $\Delta_r Q$. Therefore, without loss of generality, the eigenvalue of $\Delta_r Q$ with the largest absolute value can be assumed positive real. Hence (5.20) can be specialized to the spectral radius and we can write,

$$\rho(\Delta_r Q) \leq \hat{\rho}(\Delta_{r-1} Q) + \kappa(Z)\|\Delta_1 Q\| \tag{5.21}$$

By (5.21) the maximal spectral radius for the system with r uncertainties is therefore bounded by,

$$\hat{\rho}(\Delta_r Q) \leq \hat{\rho}(\Delta_{r-1} Q) + \sup_{\Delta_r \in \hat{\Delta}} \|\Delta_1 Q\| \kappa(Z) = \sup_{\Delta_{r-1} \in \hat{\Delta}} \|\Delta_1 Q\| \sup_{\Delta_1 \in \hat{\Delta}} \kappa(Z) \tag{5.22}$$

Since matrix norms are real valued continuous functions and the set $\hat{\Delta}$ is compact, the suprema in (5.22) is attained. Therefore, if Δ_{ij} is the unique nonzero element of the matrix Δ_1 and denoting by $Q_{\Delta_{ij}}$ the matrix whose i-th row is the j-th row of the matrix Q and all its other elements are zero then, expression (5.22) can be written,

$$\hat{\rho}(\Delta_r Q) \leq \hat{\rho}(\Delta_{r-1} Q) + \|Q_{\Delta_{ij}}\| \max_{\Delta_{r-1} \in \hat{\Delta}} \kappa(Z) \tag{5.23}$$

Expression (5.23) was obtained assuming $\tilde{\lambda} > \rho(\Delta_{r-1} Q)$ in (5.18). Clearly if $\tilde{\lambda}$ is the eigenvalue of $\Delta_r Q$ corresponding to the maximal spectral radius then $\tilde{\lambda} \geq \hat{\rho}(\Delta_{r-1} Q)$. The case $\tilde{\lambda} = \hat{\rho}(\Delta_{r-1} Q)$ occurs when the maximal spectral radius of $\Delta_r Q$ is inherited from $\Delta_{r-1} Q$ i.e., $\hat{\rho}(\Delta_r Q) = \hat{\rho}(\Delta_{r-1} Q)$.

From expression (5.23) we learn that each time a new model perturbation is added to the system, the maximal spectral radius may be increased by an amount amplified by the condition number of the matrix of eigenvectors of $\Delta_{r-1} Q$. Notice that the condition number of the matrix of eigenvectors of a matrix is independent of the condition number of the matrix itself. Thus, it is possible to have a perfectly conditioned matrix $\Delta_{r-1} Q$ with an ill conditioned matrix of eigenvectors.

It is interesting to have some geometrical interpretation of the bound (5.23). It turns out that when the spectral norm is used, an interpretation can be obtained through the singular values.

We first cite two theorems [58, 12] concerning Hermitian matrices which will be required for understanding the geometry of the bound. Recall that an $r \times r$ principal submatrix of an $n \times n$ matrix A is a matrix obtained by deleting $n - r$ rows and their corresponding columns from the matrix A. Also, label the eigenvalues of an Hermitian matrix according to increasing size $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

The theorems are as follows,

Theorem 5.3 (The Inclusion Principle) *Let $A \in \mathbb{C}^{n \times n}$ be an Hermitian matrix, let r be an integer with $1 \leq r \leq n$, and let A_r denote any $r \times r$ principal submatrix of A. Then, for each integer k such that $1 \leq k \leq r$ we have,*

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A)$$

Proof: A proof based on the Courant-Fischer minimax properties of the eigenvalues of Hermitian matrices can be found in [12]. •

Theorem 5.4 (Poincaré Separation Theorem) *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian. Let r be a given integer with $1 \leq r \leq n$, and let $u_1, \ldots, u_r \in \mathbb{C}^n$ be r given orthonormal vectors. Let $B_r \in \mathbb{C}^{r \times r}$ be the matrix whose ij-th element is $u_i^H A u_j$. If the eigenvalues of A and B_r are arranged in increasing order then,*

$$\lambda_k(A) \leq \lambda_k(B_r) \leq \lambda_{k+n-r}(A) \qquad k = 1, 2, \ldots, r$$

Proof [12]: Choose $n-r$ additional vectors $u_{r+1}, \ldots, u_n$ so that an orthonormal set $\{u_1, u_2, \ldots, u_n\}$ is obtained. Define the matrix $U = (u_1 \; u_2 \; \cdots \; u_n) \in \mathbb{C}^{n \times n}$. The matrix U is unitary, therefore $U^H A U$ has the same eigenvalues as A. The

matrix B_r is a principal submatrix of $U^H AU$ obtained by deleting the last $n-r$ rows and columns. By Theorem 5.3 the assertion follows. •

Following [12] we now show a development which leads to an inequality by Wielandt. The inequality will provide the geometrical interpretation for the upper bound.

Consider a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ whose singular values are $\sigma_1 \geq \cdots \geq \sigma_n > 0$. Denote $B = AA^H$ and let $x, y \in \mathbb{C}^n$ be any pair of orthonormal vectors. Define $C = (x\ y)^H B(x\ y) \in \mathbb{C}^{2\times 2}$ and denote the eigenvalues of C by $0 < \gamma_1 \leq \gamma_2$. By the Poincaré Separation Theorem 5.4 we have,

$$\sigma_n^2 \leq \gamma_1 \leq \gamma_2 \leq \sigma_1^2 \tag{5.24}$$

It is not difficult to verify the following relationships,

$$\begin{aligned} 1 - \frac{|x^H By|^2}{(x^H Bx)(y^H By)} &= 4\frac{(x^H Bx)(y^H By) - |x^H By|^2}{(x^H Bx + y^H By)^2 - (x^H Bx - y^H By)^2} \\ &= \frac{4det(C)}{(trC)^2 - (x^H Bx - y^H By)^2} = \frac{4\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2 - (x^H Bx - y^H By)^2} \\ &\geq \frac{4\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2} \end{aligned} \tag{5.25}$$

From (5.25) we obtain,

$$\frac{|x^H By|^2}{(x^H Bx)(y^H By)} \leq 1 - \frac{4\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2} = \left(\frac{\gamma_2/\gamma_1 - 1}{\gamma_2/\gamma_1 + 1}\right)^2 \tag{5.26}$$

The upper bound in (5.26) is a monotonically increasing function of γ_2/γ_1. Moreover, from (5.24) we have that $\gamma_2/\gamma_1 \leq \sigma_1^2/\sigma_n^2 = \kappa^2(A)$, where $\kappa(\cdot)$ denotes the spectral condition number. Hence from (5.26) we have,

$$\frac{|x^H By|^2}{(x^H Bx)(y^H By)} \leq \left(\frac{\sigma_1^2/\sigma_2^2 - 1}{\sigma_1^2/\sigma_2^2 + 1}\right)^2 = \left(\frac{\kappa^2(A) - 1}{\kappa^2(A) + 1}\right)^2 \tag{5.27}$$

Defining the angle θ in the first quadrant by $\cot(\theta/2) = \kappa$ yields,

$$\frac{\kappa^2(A) - 1}{\kappa^2(A) + 1} = \frac{\cot^2(\theta/2) - 1}{\cot^2(\theta/2) + 1} = \cos\theta \tag{5.28}$$

From (5.27) and (5.28) we obtain,

$$\frac{|x^H By|^2}{(x^H Bx)(y^H By)} \leq \cos^2\theta \tag{5.29}$$

To state Wielandt's inequality denote the usual (Euclidean) inner product in $\mathbb{C}^n$ by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $\| \cdot \|_2$. Then we have the following theorem [12],

Theorem 5.5 (Wielandt Inequality) *Let $A \in \mathbb{C}^{n \times n}$ be a given nonsingular matrix with spectral condition number κ, and define the angle θ in the first quadrant by $\cot(\theta/2) = \kappa$. Then,*

$$|\langle Ax, Ay \rangle| \leq \cos\theta \|Ax\|_2 \|Ay\|_2 \tag{5.30}$$

for every pair of orthogonal vectors $x, y \in \mathbb{C}^n$. Moreover, there exists an orthonormal pair of vectors $x, y \in \mathbb{C}^n$ for which equality holds in (5.30).

Proof: Expression (5.30) is obtained by simply substituting $B = A^H A$ in (5.29). Now consider two orthonormal eigenvectors of B, $u_1, u_n \in \mathbb{C}^n$ corresponding to the eigenvalues σ_1^2 and σ_n^2 respectively (these vectors always exist since B is Hermitian). Define $x = (u_1 + u_n)/\sqrt{2}$ and $y = (u_1 - u_n)/\sqrt{2}$. Then $\{x, y\}$ is an orthonormal set satisfying

$$x^H B x = y^H B y = (\sigma_1^2 + \sigma_n^2)/2 \quad \text{and} \quad x^H B y = (\sigma_1^2 - \sigma_n^2)/2$$

For this case, equality is attained in (5.27) and the result is proven. •

Wielandt's inequality provides an attainable upper bound for the spectral condition number of a matrix. Thus, $\theta(A)$ may be geometrically interpreted as the minimal angle between Ax and Ay as x and y range over all possible orthonormal pairs of vectors.

To see how Wielandt's inequality gives a geometrical interpretation to the bound (5.23), let us first consider the case where the plant includes a single uncertain element $\tilde{p}_{ij} = p_{ij} + \Delta_{ij}$ and it is required to assess how an additional uncertainty would affect stability. Recall that the dimensions of the plant transfer matrix P and of the compensator transfer matrix C are $n_i \times n_o$ and $n_o \times n_i$ respectively. Consequently, the dimensions of the closed loop map $Q = C(I + PC)^{-1}$

are $n_o \times n_i$.

The new perturbation affects the bound upon the maximal spectral radius (5.23) through an amplification $\kappa(Z)$, where Z is the similarity transformation matrix which diagonalizes,

$$\Delta_{r-1}Q = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \Delta_{ij}q_{j1} & \cdots & \Delta_{ij}q_{ji} & \cdots & \Delta_{ij}q_{jn_i} \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \tag{5.31}$$

where we have kept the notation Δ_{r-1} to avoid confusion with the perturbation matrix Δ_1. We have denoted $Q = \{q_{kl}\}$ and in (5.31) the only nonzero row is the i-th row. By the results of Section 5.2 the matrix $\Delta_{r-1}Q$ in (5.31) has (generically) a single nonzero eigenvalue $\Delta_{ij}q_{ji}$ whose corresponding eigenvector is the n_i-elementary vector e_i. Moreover, the matrix $\Delta_{r-1}Q$ is generically diagonalizable therefore there exist $n_i - 1$ eigenvectors spanning $\mathcal{N}(\Delta_{r-1}Q)$ which together with e_i form a linearly independent set of vectors. Denote by $x^{(k)} = (x_1^{(k)}\ x_2^{(k)}\ \cdots\ x_{n_i}^{(k)})^T \quad k = 1, \ldots, n_i - 1$ the eigenvectors corresponding to the zero eigenvalue and by $q^{(i)^T}$ the i-th row of $\Delta_{r-1}Q$. Then, the eigenvectors corresponding to the zero eigenvalue satisfy,

$$q^{(i)^T}x^{(k)} = q_{j1}x_1^{(k)} + q_{j2}x_2^{(k)} + \cdots + q_{jn_i}x_{n_i}^{(k)} = 0 \qquad k = 1, \ldots, n_i - 1 \tag{5.32}$$

The dimension of the subspace $span\{q^{(i)}, e_i\}$ is at most 2. Therefore we can always choose the $n_i - 2$ (say) first eigenvectors $x^{(1)}\ x^{(2)}\ \cdots\ x^{(n_i-2)}$ corresponding to the zero eigenvalue such that the subspace $span\{q^{(i)}, e_i\}$ is orthogonal to the subspace spanned by these eigenvectors and moreover the eigenvectors form an orthogonal set i.e.,

$$span\{q^{(i)}, e_i\} \ \perp \ span\{x^{(1)}\ x^{(2)}\ \cdots\ x^{(n_i-2)}\}$$

and (5.33)

$$x^{(k)} \perp x^{(l)} \qquad k,l \in [1,n_i] \quad k \neq l$$

To form the similarity transformation matrix Z it remains to choose one additional eigenvector $x^{(n_i-1)}$ corresponding to the zero eigenvalue. This eigenvector must satisfy (5.32) and also be linearly independent of e_i and the eigenvectors $\{x^{(1)}\ x^{(2)}\ \cdots\ x^{(n_i-2)}\}$.

Suppose we choose $x^{(n_i-1)} \perp span\{x^{(1)}\ x^{(2)}\ \cdots\ x^{(n_i-2)}\}$. Then, disregarding the trivial case where the dimension of $span\{q^{(i)}, e_i\}$ is one (and therefore $\Delta_{r-1}Q$ is diagonal) we have that since the subspaces $span\{q^{(i)}, e_i\}$ and $span\{x^{(1)}\ x^{(2)}\ \cdots\ x^{(n_i-2)}\}$ are complementary then $x^{(n_i-1)} \in span\{q^{(i)}, e_i\}$.

Now consider the standard basis $\{e_1\ e_2, \ldots, e_{n_i}\}$ for $\mathbb{C}^{n_i}$. Considering each pair of vectors from this basis as pairs of orthogonal vectors in the application of Theorem 5.5 to the matrix Z, we obtain that the angle θ is not larger than the smaller angle between the columns of the matrix Z. From the construction of Z we conclude that the smallest angle between its columns is the angle between $x^{(n_i-1)}$ and e_i or for this simple case, the angle complementary to the angle between $q^{(i)}$ and e_i. Thus, the smallest the angle between $q^{(i)}$ and e_i, the closer to one is the condition number of Z.

In the extreme case mentioned before where the dimension of the subspace $span\{q^{(i)}, e_i\}$ is one, then $\theta = \pi/2$ and therefore $\kappa(Z) = 1$ as expected (since $Z = I$).

The generalization of the geometric interpretation for the case where a perturbation is added to a system that already has more than a single uncertain element involves the consideration of the angle between subspaces of higher dimensions. In addition, these subspaces rotate with the phases of the uncertainties. Specifically suppose that the matrix $\Delta_{r-1}Q$ has (in contrast to (5.31)) r nonzero rows denoted by $\Delta_1 q^{(1)}\ \Delta_2 q^{(2)}, \cdots, \Delta_r q^{(r)}$. By the results of Section 5.2 this matrix has r nonzero eigenvalues to which correspond a set of linearly independent eigenvectors, say $\{y^{(1)}\ y^{(2)}, \ldots, y^{(r)}\}$. By the same construction of

the matrix Z as in the simpler case of two uncertainties we conclude that an upper bound for the angle θ can be obtained by computing the angle between the subspaces $span\{\Delta_1 q^{(1)}\ \Delta_2 q^{(2)}, \cdots, \Delta_r q^{(r)}\}$ and $span\{y^{(1)}\ y^{(2)}, \ldots, y^{(r)}\}$ (for the definition and computation of the angles between subspaces see [59]). Notice that since all the vectors depend now on the uncertainty phases the angle between the subspaces depends also on these phases, the maximal spectral radius problem can be stated as one of finding the uncertainty phases which yield the largest angle.

As a final remark in this section note that the similarity transformation matrix Z in (5.23) is by no means unique since $\Delta_{r-1}Q$ has repeated eigenvalues. Hence, a smaller upper bound (5.23) may be obtained by finding a set of eigenvectors of $\Delta_{r-1}Q$ that minimizes $\kappa(Z)$.

5.4 A Bound on the Maximal Spectral Radius Based on Spectral Analysis

The bound on the maximal spectral radius based on matrix norms developed in Section 5.3 reflects some fundamental results of matrix perturbation theory. The results state that large deviations (in some sense) of a matrix from normality may result in high susceptibility of its eigenvalues to perturbations of individual elements of the matrix. The bound also has a geometrical interpretation in terms of the angle between certain subspaces, when the spectral norm is used. However it is clear that these results are more conceptual than practical since it is difficult to obtain directly from expression (5.23) relationships between the individual closed loop maps that can be used to minimize the bound.
In this section we want to take advantage of the special structure of the matrices $\Delta_{r-1}Q$ to develop a bound upon the maximal spectral radius which potentially provides direct information for its minimization.
Each time an additional perturbation is added to the system a nonzero row

vector is generated. The matrix $\Delta_{r-1}Q$ is converted to $\Delta_r Q$ by either the replacement of a zero row by the vector or by adding this vector to an existing nonzero row. We will show that the projection of the vector on the null space of $\Delta_{r-1}Q$ provides valuable information in terms of the individual closed loop maps regarding the minimization of a bound on the maximal spectral radius. The information appears in the form of algebraic expressions which are independent of the elements of the uncertainty matrix. To actually compute the bound it is still required to evaluate the condition number of a matrix which depends on the uncertainties. However the matrix is of much lower dimensions that the one corresponding to the bound based solely on matrix norms.

Consider a system whose uncertainty structure is defined by the indicator T with cardinality $\alpha(T) = r$. As in the last section the matrix $\Delta_r Q$ can be decomposed as,

$$\Delta_r Q = \Delta_{r-1}Q + \Delta_1 Q \tag{5.34}$$

and we can consider the matrix $\Delta_{r-1}Q$ as being perturbed by the rank one matrix $\Delta_1 Q$. Since we are dealing with the maximal spectral radius, we can apply Theorem 4.4 or alternatively consider equations (4.34)-(4.36) to restrict the elements of the uncertainty to be of unit magnitude. Thus, the elements of the $n_i \times n_o$ matrix Δ_1 in (5.34) are all zero with the exception of the lk entry which is given by $\Delta_{lk} = e^{\jmath\theta_{lk}}$. The matrix $\Delta_1 Q$ has therefore all its rows zero except for the l-th row which is $(e^{\jmath\theta_{lk}}q_{k1}\ e^{\jmath\theta_{lk}}q_{k2}\ \cdots\ e^{\jmath\theta_{lk}}q_{kn_i})$.

One way to deal with the maximal spectral radius of an operator A is to study its spectrum using the operator-valued function $\lambda \longrightarrow R_A(\lambda)$ referred to as the *resolvent* of A and defined by [50],

$$R_A(\lambda) = (A - \lambda I)^{-1}$$

Note that each eigenvalue of A is a singularity of $R_A(\lambda)$.
The resolvent has been used to study the effect of one dimensional perturbations on the spectrum of symmetric operators acting on infinite- dimensional Hilbert

spaces [60]. Following a similar procedure to that in [60] we now show a development which leads to an algebraic equation relating the spectrum of $\Delta_r Q$ to the spectrum of $\Delta_{r-1}Q$. The equation will eventually be used to obtain a bound upon the maximal spectral radius.

Denote by the resolvent of $\Delta_k Q$ by R_k. Recalling that Δ_k is $n_i \times n_o$ and Q is $n_o \times n_i$ we have that if u and v are two n_i-dimensional vectors satisfying $R_r u = v$ then by (5.34),

$$u = R_r^{-1} v = \Delta_r Q v - \lambda v = (\Delta_{r-1} Q + \Delta_1 Q) v - \lambda v$$

Therefore,

$$(\Delta_{r-1} Q - \lambda I) v = u - \Delta_1 Q v \tag{5.35}$$

From the results of Section 5.2 we know that the matrix $\Delta_1 Q$ is generically of rank one. Therefore it is diagonalizable. That is, there exists a nonsingular matrix S such that,

$$\Delta_1 Q = S \Lambda_1 S^{-1} \tag{5.36}$$

where Λ_1 has all its elements zero except for one in the main diagonal which is $q_{kl} e^{j\theta_{lk}}$. Without loss of generality we may assume that the columns of the matrix S are ordered such that the unique nonzero element of Λ_1 is in the ll-th place.

Notice that since $\Delta_1 Q$ has only one nonzero row the similarity transformation matrix S does not depend on the uncertainty angle θ_{lk} but only on the elements of the k-th row of Q.

Substituting (5.36) into (5.35) yields,

$$(\Delta_{r-1} Q - \lambda I) v = u - S \Lambda_1 S^{-1} v \tag{5.37}$$

By premultiplying (5.37) by S^{-1} we obtain,

$$(S^{-1} \Delta_{r-1} Q S S^{-1} - \lambda S^{-1}) v = S^{-1} u - \Lambda_1 S^{-1} v \tag{5.38}$$

Defining the vectors $\hat{v} = S^{-1} v$ and $\hat{u} = S^{-1} u$, expression (5.38) can be written,

$$(S^{-1} \Delta_{r-1} Q S - \lambda I) \hat{v} = \hat{u} - \Lambda_1 \hat{v} \tag{5.39}$$

Let $\langle \cdot, \cdot \rangle$ denote the usual (Euclidean) inner product in $\mathbb{C}^{n_i}$ and let e_l denote the elementary n_i-vector with a 1 in the l-th place and zeros everywhere else. Since the only nonzero element of Λ_1 is given by $q_{kl}e^{\jmath\theta_{lk}}$ and it is located in the ll-th place then, the vector $\Lambda_1\hat{v}$ may be written in the form,

$$\Lambda_1\hat{v} = e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{v}\rangle e_l$$

Denoting

$$\hat{R}_{r-1} = (S^{-1}\Delta_{r-1}QS - \lambda I)^{-1} \tag{5.40}$$

and premultiplying (5.39) by $\hat{R}_{r-1}$ we obtain,

$$\hat{v} = \hat{R}_{r-1}\hat{u} - e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{v}\rangle \hat{R}_{r-1}e_l \tag{5.41}$$

Forming the inner product of (5.41) with e_l gives,

$$\langle e_l, \hat{v}\rangle = \langle e_l, \hat{R}_{r-1}\hat{u}\rangle - e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{v}\rangle\langle e_l, \hat{R}_{r-1}e_l\rangle \tag{5.42}$$

and from (5.42) we obtain,

$$\langle e_l, \hat{v}\rangle = \frac{\langle e_l, \hat{R}_{r-1}\hat{u}\rangle}{1 + e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{R}_{r-1}e_l\rangle} \tag{5.43}$$

Now define $\hat{R}_r = S^{-1}R_rS = (S^{-1}\Delta_rQS - \lambda I)^{-1}$. Then (5.41) and (5.43) yield,

$$\hat{R}_r\hat{u} = \hat{v} = \hat{R}_{r-1}\hat{u} - e^{\jmath\theta_{lk}}q_{kl}\frac{\langle e_l, \hat{R}_{r-1}\hat{u}\rangle}{1 + e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{R}_{r-1}e_l\rangle}\hat{R}_{r-1}e_l \tag{5.44}$$

The eigenvalues of Δ_rQ are the singularities of R_r or equivalently of $\hat{R}_r$. Except for those inherited from $\hat{R}_{r-1}$, the singularities of $\hat{R}_r$ are by (5.44) the solutions of,

$$\phi(\lambda) = 1 + e^{\jmath\theta_{lk}}q_{kl}\langle e_l, \hat{R}_{r-1}e_l\rangle = 0 \tag{5.45}$$

We now concentrate on the singularities of R_r not inherited from R_{r-1} or in other words, on the solutions of equation (5.45). The case of inherited eigenvalues will be dwelt upon later.

By the results of Section 5.2, there exists generically a nonsingular matrix Z such that,

$$\Delta_{r-1}Q = Z\Lambda_{r-1}Z^{-1} \tag{5.46}$$

where Λ_{r-1} is diagonal and similar to $\Delta_{r-1}Q$.

Notice that in contrast to the case $r \leq 2$, when $r > 2$ the elements of the matrix Z depend on the elements of the uncertainty matrix Δ_{r-1}. For this reason, the computation of the maximal spectral radius (or for block uncertainties, the structured singular value) involves the solution of an optimization problem where the optimization variables are the phases of the uncertainties.

Using (5.40) and (5.46) we can write,

$$\hat{R}_{r-1} = (S^{-1}\Delta_{r-1}QS - \lambda I)^{-1} = (S^{-1}Z\Lambda_{r-1}Z^{-1}S - \lambda I)^{-1} = \\ S^{-1}Z(\Lambda_{r-1} - \lambda I)^{-1}Z^{-1}S = W(\Lambda_{r-1} - \lambda I)^{-1}W^{-1} \tag{5.47}$$

where we have defined the matrix $W = S^{-1}Z$.

Substituting (5.47) into (5.45) yields,

$$\phi(\lambda) = 1 + e^{\jmath\theta_{lk}}q_{kl}\langle e_l, W(\Lambda_{r-1} - \lambda I)^{-1}W^{-1}e_l\rangle = \\ 1 + e^{\jmath\theta_{lk}}q_{kl}\langle W^H e_l, (\Lambda_{r-1} - \lambda I)^{-1}W^{-1}e_l\rangle = 0 \tag{5.48}$$

where W^H is the adjoint (the Hermitian transpose) of W.

By (5.48) we conclude that the eigenvalues of $\Delta_r Q$ that are not inherited from $\Delta_{r-1}Q$ must satisfy the equation,

$$e^{\jmath\theta_{lk}}q_{kl}\langle W^H e_l, (\lambda I - \Lambda_{r-1})^{-1}W^{-1}e_l\rangle = 1 \tag{5.49}$$

Now let by $x_{\cdot l} = (x_{1l}\ x_{2l}\ \ldots\ x_{n_i l})^T$ denote the l-th column of W^{-1} and $w_{l\cdot} = (w_{l1}\ w_{l2}\ \ldots\ w_{ln_i})$ the l-th row of W. Then, equation (5.49) can be written as,

$$e^{\jmath\theta_{lk}}q_{kl}w_{l\cdot}(\lambda I - \Lambda_{r-1})^{-1}x_{\cdot l} = 1 \tag{5.50}$$

or equivalently, if $\{\lambda_1, \lambda_2, \ldots, \lambda_{n_i}\}$ are the diagonal elements of Λ_{r-1} ordered according to expression (5.46),

$$e^{\jmath\theta_{lk}}q_{kl}\sum_{i=1}^{n_i}\frac{w_{li}x_{il}}{\lambda - \lambda_i} = 1 \tag{5.51}$$

Suppose that the number of nonzero rows of the matrix Δ_{r-1} is m. Clearly, $m \leq r-1$ with equality holding if and only if the perturbation matrix is diagonal.

The matrix $\Delta_{r-1}Q$ has therefore m nonzero eigenvalues and $n_i - m$ zero ones. Without loss of generality we may assume that the columns of the matrix Z in (5.46) are ordered such that $\lambda_{m+1} = \lambda_{m+2} = \ldots \lambda_{n_i} = 0$. Then the sum in (5.51) can be decomposed into,

$$e^{j\theta_{lk}} q_{kl} \left\{ \sum_{i=1}^{m} \frac{w_{li} x_{il}}{\lambda - \lambda_i} + \frac{1}{\lambda} \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right\} = 1 \tag{5.52}$$

with $\lambda_i \neq 0 \quad i = 1, 2, \ldots, m$.

Expression (5.52) provides a relationship between the distinct spectra of $\Delta_r Q$ and $\Delta_{r-1}Q$. It shows that the part of the spectrum of $\Delta_r Q$ not inherited from $\Delta_{r-1}Q$ is obtained from the solution of an algebraic equation whose order is at most the order of the minimal polynomial of the matrix.

It may be possible that the part of the spectrum of $\Delta_r Q$ inherited from $\Delta_{r-1}Q$ contains the eigenvalue corresponding to the spectral radius of $\Delta_r Q$. This situation is reflected in (5.52) by having $w_{li} x_{il} = 0$ where i is the index corresponding to the eigenvalue λ_i of $\Delta_{r-1}Q$ of maximal modulus. However we will not consider this case separately since, as will become clear later, the bound on the maximal spectral radius to be developed in this section does not require it. An illustration of such a situation will be presented in the example at the end of this section.

It turns out that owing to the special structure of the matrices involved, by studying the geometry of certain subspaces associated with the matrices, some properties of expression (5.52) can be found. These properties play an important role in the development of a bound for the maximal spectral radius. For the sake of clarity the properties will be stated in the form of a series of lemmas.

The first lemma reveals a special feature of the similarity transformation inverse matrix S^{-1} resulting from the fact that the only nonzero element of the matrix Δ_1 is Δ_{lk}. This feature will be widely used in further developments. Recall that in (5.36) we assumed that the columns of the matrix S were ordered such that the unique nonzero element of Λ_1 is in the ll-th place. Under this

assumption, and if $q_{k\cdot}$ denotes the k-th row of the matrix Q, then we have the following result,

Lemma 5.7 *The l-th row of the similarity transformation inverse matrix S^{-1} in (5.36) is given by $\alpha q_{k\cdot}$ where $\alpha \in \mathbb{C}$ is a normalization factor.*

Proof According to the decomposition of the matrix $\Delta_r Q$ in (5.34), the l-th row of S^{-1} is the left (or row) eigenvector corresponding to the unique nonzero eigenvalue of $\Delta_1 Q$. This eigenvalue is given by $e^{\jmath\theta_{lk}} q_{kl}$. Moreover the unique nonzero row of the matrix $\Delta_1 Q$ is $(e^{\jmath\theta_{lk}} q_{k1}\ e^{\jmath\theta_{lk}} q_{k2}\ \cdots\ e^{\jmath\theta_{lk}} q_{kn_i})$. Therefore, if $t_{l\cdot} = (t_{l1}\ t_{l2}\ \cdots\ t_{ln_i})$ denotes the left eigenvector of $\Delta_1 Q$ corresponding to the nonzero eigenvalue, then each one of its components satisfies,

$$t_{ll} e^{\jmath\theta_{lk}} q_{ki} = e^{\jmath\theta_{lk}} q_{kl} t_{li} \quad i = 1, 2, \ldots, n_i$$

Hence $t_{li} = \alpha q_{ki}$ $i = 1, 2, \ldots, n_i$ with $\alpha \in \mathbb{C}$. Since $SS^{-1} = I$, the normalization factor α satisfies $\langle t_{l\cdot}^H, s_{\cdot l} \rangle = 1$ where $s_{\cdot l}$ is the right (column) eigenvector of $\Delta_1 Q$ corresponding to the nonzero eigenvalue. •

The next lemma provides a simplification for expression (5.52) by choosing an appropiate basis for the null space of the matrix $\Delta_{r-1} Q$.

Lemma 5.8 *The similarity transformation matrix Z in (5.46) can be chosen such that in (5.52) $w_{li} = 0$ for $i = m+2, m+3, \ldots, n_i$.*

Proof The right invariant subspace corresponding to the zero eigenvalue of the matrix $\Delta_{r-1} Q$ is $\mathcal{N}(\Delta_{r-1} Q)$. By virtue of the basic assumptions we have that $dim\{\mathcal{N}(\Delta_{r-1} Q)\} = n_i - m$.
Now, choose a basis $\{\phi_{m+1}, \phi_{m+2}, \ldots, \phi_{n_i}\}$ for $\mathcal{N}(\Delta_{r-1} Q)$ with the following property: if the projection $\sum_{i=m+1}^{n_i} \langle q_{k\cdot}^H, \phi_i \rangle \phi_i$ of $q_{k\cdot}^T$ into $\mathcal{N}(\Delta_{r-1} Q)$ is nonzero then the basis forms an orthogonal set and one of the basis vectors, say ϕ_{m+1}, is the projection itself i.e.,

$$\phi_{m+1} = \sum_{i=m+1}^{n_i} \langle q_{k\cdot}^H, \phi_i \rangle \phi_i \tag{5.53}$$

If the projection of $q_{k\cdot}^T$ into $\mathcal{N}(\Delta_{r-1}Q)$ is zero then choose an arbitrary (not necessarily orthogonal) basis.

Notice that since we are dealing with finite dimensional spaces the existence of orthogonal bases is assured [12, 53].

Since the basis vectors $\{\phi_{m+1}, \phi_{m+2}, \ldots, \phi_{n_i}\}$ are eigenvectors corresponding to the zero eigenvalue of $\Delta_{r-1}Q$ they can be assigned as the last $n_i - m$ columns of the similarity transformation matrix Z. By Lemma 5.7 the l-th row of S^{-1} is $\alpha q_{k\cdot}$ with $\alpha \in \mathbb{C}$ a normalization constant. Therefore for each of the cases above the last $n_i - (m+1)$ elements of the l-th row of $W = S^{-1}Z$ are zero i.e., $w_{l(m+2)} = w_{l(m+3)} \ldots = w_{ln_i} = 0$ and the lemma is proven. •

Lemma 5.8 merely expresses the fact that at least $n_i-(m+1)$ zero eigenvalues of $\Delta_{r-1}Q$ are inherited by $\Delta_r Q$. When the projection of $q_{k\cdot}^T$ into $\mathcal{N}(\Delta_{r-1}Q)$ is nonzero then exactly $n_i - (m+1)$ zero eigenvalues are inherited and expression (5.52) can be written in simplified form,

$$e^{j\theta_{lk}} q_{kl} \left\{ \sum_{i=1}^{m} \frac{w_{li} x_{il}}{\lambda - \lambda_i} + \frac{w_{l(m+1)} x_{(m+1)l}}{\lambda} \right\} = 1 \tag{5.54}$$

When the above projection is zero then all the $n_i - m$ zero eigenvalues of $\Delta_{r-1}Q$ are inherited by $\Delta_r Q$. In this case $w_{l(m+1)} = 0$ as well and expression (5.54) is further simplified leading to an algebraic equation of order m at most. The circumstances leading to this case will be examined later on.

As mentioned before, when $r > 2$ the elements of the similarity transformation matrix Z depend on the elements of the uncertainty matrix Δ_{r-1} and consequently, the computation of the maximal spectral radius (or the structured singular value) becomes difficult. It turns out that for certain perturbation structures, a careful examination of the left and right invariant subspaces corresponding to the zero eigenvalue of the matrix $\Delta_{r-1}Q$ leads to a useful property which some coefficients of expression (5.54) have. First, we consider a theorem from [57] concerning the characterization of invariant subspaces that is required for proving the result.

Theorem 5.6 *Let the columns of X form a linearly independent set and let the columns of Y span $\mathcal{R}^{\perp}(X)$. Then $\mathcal{R}(X)$ is an invariant subspace of A if and only if,*

$$Y^H AX = 0 \tag{5.55}$$

In this case $\mathcal{R}(Y)$ is an invariant subspace of A^H.

Proof: By definition, $\mathcal{R}(X)$ is an invariant subspace of A if and only if $A\mathcal{R}(X) \subset \mathcal{R}(X)$. But,

$$\begin{aligned} A\mathcal{R}(X) \subset \mathcal{R}(X) &\iff A\mathcal{R}(X) \perp \mathcal{R}^{\perp}(X) \\ &\iff \mathcal{R}(AX) \perp \mathcal{R}(Y) \\ &\iff Y^H AX = 0 \end{aligned}$$

which establishes (5.55). Writing (5.55) in the form $X^H A^H Y$, we see that $\mathcal{R}(Y)$ must be an invariant subspace of A^H. •

Denote now by T_i the indicator of the uncertainty matrix Δ_i. Using the genericity arguments of Section 5.2 and Theorem 5.6 we can prove the following lemma,

Lemma 5.9 *Assume that the number of nonzero rows and columns of the indicator matrix T_{r-1} is the same. Then there exist bases for the right and the left invariant subspaces corresponding to the zero eigenvalue of $\Delta_{r-1}Q$ which depend only on the elements of the matrix $T_{r-1}Q$.*

Proof Form a matrix $\hat{Q}$ as follows. If the i-th row of Δ_{r-1} is zero then the i-th row of $\hat{Q}$ is also zero. All the other rows of $\hat{Q}$ are identical to the corresponding rows of Q. By the special structure of T_{r-1} we have that,

$$\mathcal{N}(\hat{Q}) \subset \mathcal{N}(\Delta_{r-1}Q) \tag{5.56}$$

Since the matrix Δ_{r-1} is assumed to have no more zero columns than zero rows it follows by the genericity arguments of Section 5.2 that,

$$dim\{\mathcal{N}(\hat{Q})\} = dim\{\mathcal{N}(\Delta_{r-1}Q)\} = n_i - m \tag{5.57}$$

Combining (5.56)-(5.57) yields $\mathcal{N}(\Delta_{r-1}Q) = \mathcal{N}(\hat{Q})$. Therefore there exist bases for the right invariant subspace corresponding to the zero eigenvalue of $\Delta_{r-1}Q$ which depend only on the elements of the matrix $\hat{Q}$ or equivalently, on the elements of the matrix $T_{r-1}Q$.

To prove the claim for the left invariant subspace notice that if the columns of a matrix X form a basis for $\mathcal{N}(\hat{Q})$ then $\hat{Q}X = 0$ and therefore by Theorem 5.6 any matrix Y whose columns span $\mathcal{R}^{\perp}(X)$ is a left invariant subspace of $\hat{Q}$. As X depends only on the elements of $T_{r-1}Q$, the matrix Y depends only on these elements as well. Moreover $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ are orthogonal complements in $\mathbb{C}^{(n_i-m)\times(n_i-m)}$ therefore the left invariant subspaces corresponding to the zero eigenvalue of $\hat{Q}$ and $\Delta_{r-1}Q$ are the same and the lemma is proven. •

From Lemma 5.9 we conclude that when the number of nonzero columns and rows of T_{r-1} is the same the matrices Z (Z^{-1}) can be chosen such that the columns (rows) corresponding to the zero eigenvalue of $\Delta_{r-1}Q$ depend only on the elements of $T_{r-1}Q$. Since the matrices S and S^{-1} do not depend on the phase of the elements of the uncertainty matrix Δ_1 then, the last $n_i - m$ components of the vectors $x_{.l}$ and $w_{l.}^T$ depend only on elements of the closed loop map Q. Thus, the coefficient of $1/\lambda$ in (5.52) or in (5.54) is a function of the elements of the closed loop map matrix Q only. This result is crucial for obtaining conditions on the individual closed loop maps to minimize the value of an upper bound on the maximal spectral radius.

As mentioned before, there exist circumstances where the coefficient of $1/\lambda$ in (5.54) is zero. This corresponds to the case when the whole null space of $\Delta_{r-1}Q$ is inherited by $\Delta_r Q$. The following lemma describes a common situation where this happens.

Lemma 5.10 *Suppose that the decomposition of the uncertainty matrix* $\Delta_r = \Delta_{r-1} + \Delta_1$ *is such that the column of* Δ_{r-1} *corresponding to the nonzero column of* Δ_1 *is also nonzero (i.e., if* Δ_{lk} *is the single nonzero element of* Δ_1 *then there*

exists an element Δ_{nk} *in* Δ_{r-1} *with* $n \neq l$ *which is also nonzero). In this case,*

$$\mathcal{N}(\Delta_{r-1}Q) = \mathcal{N}(\Delta_r Q)$$

Proof The number of nonzero rows of $\Delta_r Q$ is not less than the number of nonzero rows of Δ_{r-1}. Therefore by the genericity assumptions,

$$dim\{\mathcal{N}(\Delta_r Q)\} \leq dim\{\mathcal{N}(\Delta_{r-1}Q)\} \tag{5.58}$$

Consider now a set of vectors $\{\phi_{m+1}, \phi_{m+2}, \ldots, \phi_{n_i}\}$ such that for each nonzero row j of Δ_{r-1} and for each $i = m+1, m+2, \ldots, n_i$, $\langle q_{j\cdot}^H, \phi_i \rangle = 0$. Since we are assuming that the rank of Δ_{r-1} is determined by its number of rows and by the genericity arguments, the above set of vectors form a basis for $\mathcal{N}(\Delta_{r-1}Q)$. This basis also satisfies $\langle q_{k\cdot}^H, \phi_i \rangle = 0 \;\; i = m+1, m+2, \ldots, n_i$, therefore any vector in $\mathcal{N}(\Delta_{r-1}Q)$ is an eigenvector of $\Delta_r Q = \Delta_{r-1}Q + \Delta_1 Q$ corresponding to the zero eigenvalue i.e.,

$$\mathcal{N}(\Delta_{r-1}Q) \subset \mathcal{N}(\Delta_r Q) \tag{5.59}$$

By (5.58) and (5.59) we obtain $\mathcal{N}(\Delta_{r-1}Q) = \mathcal{N}(\Delta_r Q)$ therefore the assertion is proven. •

Under the hypotheses of Lemma 5.10, expression (5.54) reduces to,

$$e^{j\theta_{lk}} q_{kl} \sum_{i=1}^{m} \frac{w_{li} x_{il}}{\lambda - \lambda_i} = 1 \tag{5.60}$$

Now we proceed to develop a bound on the maximal spectral radius. The bound is obtained by manipulating expression (5.54) as follows,

$$\begin{aligned}
1 = |q_{kl}| \left| \sum_{i=1}^{m} \frac{w_{li} x_{il}}{\lambda - \lambda_i} + \frac{1}{\lambda} \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \leq \\
|q_{kl}| \left\{ \left| \sum_{i=1}^{m} \frac{w_{li} x_{il}}{\lambda - \lambda_i} \right| + \frac{1}{|\lambda|} \left| \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \right\} \leq \\
|q_{kl}| \left\{ \sum_{i=1}^{m} \frac{|w_{li} x_{il}|}{|\lambda - \lambda_i|} + \frac{1}{|\lambda|} \left| \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \right\} \leq \\
|q_{kl}| \left\{ \sum_{i=1}^{m} \frac{|w_{li} x_{il}|}{|\lambda| - |\lambda_i|} + \frac{1}{|\lambda|} \left| \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \right\}
\end{aligned} \tag{5.61}$$

Consider now an uncertainty matrix Δ_r^* satisfying $\rho(\Delta_r^* Q) = \hat{\rho}(\Delta_r Q)$. By (5.61) the spectral radius of $\Delta_r^* Q$ satisfies,

$$1 \leq |q_{kl}| \left\{ \sum_{i=1}^{m} \frac{|w_{li} x_{il}|}{\rho(\Delta_r^* Q) - |\lambda_i|} + \frac{1}{\rho(\Delta_r^* Q)} \left| \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \right\} \tag{5.62}$$

where in (5.62) the eigenvalues λ_i and the coefficients $w_{li}, x_{il} \quad i = 1, 2, \ldots, m$ correspond to the uncertainty $\Delta_{r-1}^* = \Delta_r^* - \Delta_1^*$.

By definition we have,

$$\hat{\rho}(\Delta_{r-1} Q) \geq \rho(\Delta_{r-1}^* Q) \geq |\lambda_i| \quad i = 1, 2, \ldots, m \tag{5.63}$$

and since obviously $\hat{\rho}(\Delta_r Q) \geq \hat{\rho}(\Delta_{r-1} Q)$ expressions (5.62) and (5.63) yield,

$$1 \leq |q_{kl}| \left\{ \frac{1}{\hat{\rho}(\Delta_r Q) - \hat{\rho}(\Delta_{r-1} Q)} \sum_{i=1}^{m} |w_{li} x_{il}| + \frac{1}{\hat{\rho}(\Delta_r Q)} \left| \sum_{j=m+1}^{n_i} w_{lj} x_{jl} \right| \right\} \tag{5.64}$$

From now on, without loss of generality we assume, as in Lemma 5.6, that the first m rows of $\Delta_{r-1} Q$ correspond to the nonzero rows of the topology T. Then, we can partition the $n_i \times n_o$ matrix Δ_{r-1} as follows,

$$\Delta_{r-1} = \begin{pmatrix} \Delta_1 & \Delta_2 \\ 0 & 0 \end{pmatrix} \tag{5.65}$$

where Δ_1 is $m \times m$ and Δ_2 is $m \times (n_o - m)$.

A conformal partition of the closed loop $n_o \times n_i$ matrix Q will result in,

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \tag{5.66}$$

where the dimensions of the submatrices Q_1, Q_2, Q_3 and Q_4 are $m \times m$, $m \times (n_i - m)$, $(n_o - m) \times m$ and $(n_o - m) \times (n_i - m)$ respectively.

Combining (5.65) and (5.66) yields the following partition on the $n_i \times n_i$ matrix $\Delta_{r-1} Q$,

$$\Delta_{r-1} Q = \begin{pmatrix} \Delta_1 Q_1 + \Delta_2 Q_3 & \Delta_1 Q_2 + \Delta_2 Q_4 \\ 0 & 0 \end{pmatrix} \tag{5.67}$$

From (5.67) we obtain that the corresponding conformal partitions of the similarity transformation $n_i \times n_i$ matrices Z and its inverse $V = Z^{-1}$ are,

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \tag{5.68}$$

where Z_{11} is $m \times m$, Z_{12} is $m \times (n_i - m)$, Z_{21} is $(n_i - m) \times m$ and Z_{22} is $(n_i - m) \times (n_i - m)$.

It is easy to see by analyzing the structure of the eigenvectors corresponding to the nonzero eigenvalues of the matrix $\Delta_{r-1}Q$ in (5.67) that both matrices Z and V in (5.68) are block upper triangular i.e., $Z_{21} = V_{21} = 0$.

Recall from (5.49) that the eigenvalues of $\Delta_r Q$ that are not inherited from $\Delta_{r-1}Q$ satisfy the equation,

$$e^{\jmath\theta_{lk}} q_{kl} \langle W^H e_l, (\lambda I - \Lambda_{r-1})^{-1} W^{-1} e_l \rangle = 1 \tag{5.69}$$

where $W = S^{-1}Z$. The l-th column of S is the eigenvector corresponding to the unique nonzero eigenvalue of $\Delta_1 Q$. This eigenvector if, normalized, is e_l. Consequently by Lemma 5.7, the l-th row of S^{-1} is

$$t_{l\cdot} = (\frac{q_{k1}}{q_{kl}} \; \frac{q_{k2}}{q_{kl}} \; \cdots \; \frac{q_{kn_i}}{q_{kl}}) \tag{5.70}$$

Using (5.70) we may write expression (5.69) as,

$$e^{\jmath\theta_{lk}} q_{kl} \langle Z^H t_{l\cdot}^H, (\lambda I - \Lambda_{r-1})^{-1} Z^{-1} e_l \rangle = 1 \tag{5.71}$$

From expression (5.71) and partitions (5.68) we conclude that the coefficient $\sum_{i=1}^{m} |w_{li} x_{il}|$ in (5.64) is given by,

$$\sum_{i=1}^{m} |w_{li} x_{il}| = |\hat{t}_{l\cdot} Z_{11}| |V_{12} e_1| \tag{5.72}$$

where $\hat{t}_{l\cdot}$ denotes the row vector $t_{l\cdot}$ truncated to its first m components and $|a|$ denotes the vector a whose components a_i have been replaced by $|a_i|$.

Since $ZV = I$ it follows that $V_{12} = -Z_{11}^{-1} Z_{12} V_{22}$ and $V_{22} = Z_{22}^{-1}$. Therefore from (5.72) we obtain,

$$\sum_{i=1}^{m} |w_{li} x_{il}| = |\hat{t}_{l\cdot} Z_{11}||V_{12} e_1| \leq |\hat{t}_{l\cdot}||Z_{11}||V_{12} e_1| =$$
$$|\hat{t}_{l\cdot}||Z_{11}||Z_{11}^{-1} Z_{12} V_{22} e_1| \leq |\hat{t}_{l\cdot}||Z_{11}||Z_{11}^{-1}||Z_{12} Z_{22}^{-1} e_1| \leq$$
$$\|\hat{t}_{l\cdot}\|\|Z_{11}\|\|Z_{11}^{-1}\|\|Z_{12} Z_{22}^{-1} e_1\| = \kappa(Z_{11})\|\hat{t}_{l\cdot}\|\|Z_{12} Z_{22}^{-1} e_1\| \qquad (5.73)$$

where $\|\cdot\|$ denotes any matrix norm induced by an absolute vector norm [12, pages 310, 365].

In a similar way the coefficient $\left|\sum_{j=m+1}^{n_i} w_{lj} x_{jl}\right|$ in (5.64) can be expressed in terms of the partition (5.68) as,

$$\left|\sum_{j=m+1}^{n_i} w_{lj} x_{jl}\right| = |t_{l\cdot} \begin{pmatrix} Z_{12} \\ Z_{22} \end{pmatrix} Z_{22}^{-1} e_1| = |t_{l\cdot} \begin{pmatrix} Z_{12} Z_{22}^{-1} \\ I \end{pmatrix} e_1| \qquad (5.74)$$

Combining (5.64) (5.73) and (5.74) we obtain,

$$1 \leq |q_{kl}| \left\{ \frac{\kappa(Z_{11})\|\hat{t}_{l\cdot}\|\|Z_{12} Z_{22}^{-1} e_1\|}{\hat{\rho}(\Delta_r Q) - \hat{\rho}(\Delta_{r-1} Q)} + \frac{1}{\hat{\rho}(\Delta_r Q)} |t_{l\cdot} \begin{pmatrix} Z_{12} Z_{22}^{-1} \\ I \end{pmatrix} e_1| \right\} \qquad (5.75)$$

As a consequence of Lemma 5.9 the similarity transformation matrices may be chosen such that the coefficients $w_{li}, x_{il} \quad i = m+1, m+2, \ldots, n_i$ depend only on (some of) the elements of the closed loop map matrix Q. In terms of the partitions (5.68) this means that the eigenvectors of the matrix $\Delta_{r-1} Q$ may be chosen such that the submatrices Z_{12} and Z_{22} do not depend on the uncertainties. However the coefficients w_{li} and $x_{il} \quad i = 1, 2, \ldots, m$ or equivalently the submatrices Z_{11} and Z_{21} depend in general on the uncertainty matrix.

Expression (5.75) provides a relationship between the maximal spectral radius (or the structured singular value for the case of complex, element by element uncertainty) corresponding to the $r-1$ and r uncertainties cases. This relationship generates recursively an upper bound for the stability margin. Each recursion step provides necessary conditions for the upper bound to assure robust stability. The last step corresponds to the bound and therefore generates

also sufficient conditions. The whole set of conditions can then be used to impose values upon the individual closed loop maps that will guarantee robust stability.

Notice that a lower bound is also obtained from (5.75). However it is trivial (i.e. it is nonpositive).

To illustrate the series of conditions generated by expression (5.75) we will consider a 3×3 closed loop map Q. The analysis will be carried for a few steps. At each step a new uncertainty will be added to the system. The only restriction concerning the addition of uncertainties is that the number of nonzero columns and rows of the uncertainty matrix should be always kept equal (except for the last step). This restriction, which results from the assumptions of Lemma 5.9, requires a careful numbering of the plant input-output pairs. Some uncertainty structures however may require the introduction of small, fictituous uncertainty elements in order to comply with the assumption of Lemma 5.9.

In the example it will be understood that for robust stability all the conditions obtained should hold for each frequency in an appropiately indented Nyquist contour.

The closed loop map will be represented by the matrix,

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \tag{5.76}$$

where each q_{ij} is a complex scalar.

Step 1: $r = 1$. The uncertainty matrix is given by,

$$\Delta_1 = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.77}$$

In the single uncertainty case the matrix $\Delta_1 Q$ has only one nonzero eigenvalue therefore,

$$\hat{\rho}(\Delta_1 Q) = |\Delta_{11} q_{11}| = |q_{11}| \tag{5.78}$$

and the condition for robustness is given by,

$$|q_{11}| < 1 \tag{5.79}$$

Clearly in this case the bound is tight and condition (5.79) is necessary and sufficient as well.

Step 2: r=2. The uncertainty matrix is,

$$\Delta_2 = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.80}$$

and Δ_{r-1} is given by (5.77), (here we keep the notation Δ_{r-1} to avoid confusion with the perturbation matrix Δ_1). By Lemma 5.7 the second row of the matrix S^{-1} is,

$$t_{2\cdot} = \frac{1}{q_{22}}(\; q_{21} \quad q_{22} \quad q_{23} \;) \tag{5.81}$$

To compute Z_{12} and Z_{22} we have to find two vectors that span $\mathcal{N}(\Delta_{r-1}Q)$. One choice is the vectors $(-q_{12} \; q_{11} \; 0)^T$ and $(0 \; -q_{13} \; q_{12})^T$. Therefore the matrix Z looks,

$$Z = \begin{pmatrix} \mathrm{x} & -q_{12} & 0 \\ 0 & q_{11} & -q_{13} \\ 0 & 0 & q_{12} \end{pmatrix} \tag{5.82}$$

where x is used here to denote a complex number that we do not need to compute right now. Note that in this case $Z_{11} = \mathrm{x}$.

From (5.82) we obtain,

$$Z_{12}Z_{22}^{-1} = \frac{1}{q_{11}}(\; -q_{21} \quad -q_{13} \;) \tag{5.83}$$

Substituting (5.81) and (5.83) into (5.74) yields,

$$\left| \sum_{j=m+1}^{n_i} w_{lj}x_{jl} \right| = \left| 1 - \frac{q_{12}q_{21}}{q_{11}q_{22}} \right| \tag{5.84}$$

Since Z_{11} is a scalar then $\kappa(Z_{11}) = 1$. Moreover it is easy to verify that,

$$\|\hat{t}_{2\cdot}\| = \left| \frac{q_{21}}{q_{22}} \right| \qquad \|Z_{12}Z_{22}^{-1}e_1\| = \left| \frac{q_{12}}{q_{11}} \right| \tag{5.85}$$

therefore substituting (5.85) into (5.73) we obtain,

$$\sum_{i=1}^{m} |w_{li} x_{il}| = \left| \frac{q_{12} q_{21}}{q_{11} q_{22}} \right| \tag{5.86}$$

The result of Step 1 and expressions (5.84) and (5.86) can now be substituted into (5.68) to yield,

$$1 \leq |q_{22}| \left\{ \frac{1}{\hat{\rho}(\Delta_2 Q) - |q_{11}|} \left| \frac{q_{12} q_{21}}{q_{11} q_{22}} \right| + \frac{1}{\hat{\rho}(\Delta_2 Q)} \left| 1 - \frac{q_{12} q_{21}}{q_{11} q_{22}} \right| \right\} \tag{5.87}$$

Notice that when $q_{12} q_{21} = 0$ expression (5.87) yields $\hat{\rho}(\Delta_2 Q) \leq |q_{22}|$. The other eigenvalue is inherited from Step 1 therefore the maximal spectral radius bound is given by $\max\{|q_{11}|, |q_{22}|\}$.

In the case where $q_{11} q_{22} = q_{12} q_{21}$ then again one of the eigenvalues is inherited from Step 1 (the zero eigenvalue) and the maximal spectral radius bound is given by the unique solution of (5.87) which is $\hat{\rho}(\Delta_2 Q) \leq |q_{11}| + |q_{22}|$.

Step 3: The uncertainty matrix is,

$$\Delta_3 = \begin{pmatrix} \Delta_{11} & \Delta_{12} & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.88}$$

and $\Delta_{r-1} = \Delta_2$ is given by (5.80). The first row of S^{-1} is given by,

$$t_{1\cdot} = \frac{1}{q_{21}} \begin{pmatrix} q_{21} & q_{22} & q_{23} \end{pmatrix} \tag{5.89}$$

To find a basis for $\mathcal{N}(\Delta_2 Q)$ it suffices by Lemma 5.9 to find a vector $\bar{z} = (z_{13}\ z_{23}\ z_{33})^T$ such that,

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \end{pmatrix} \hat{z} = 0$$

By choosing $z_{33} = -1$ we obtain,

$$\begin{pmatrix} z_{13} \\ z_{23} \\ z_{33} \end{pmatrix} = \frac{1}{q_{11} q_{22} - q_{12} q_{21}} \begin{pmatrix} q_{22} q_{13} - q_{12} q_{23} \\ q_{11} q_{23} - q_{21} q_{13} \\ -q_{11} q_{22} + q_{12} q_{21} \end{pmatrix}$$

The above vector is the third column of the matrix Z which looks,

$$Z = \begin{pmatrix} \mathrm{x} & \mathrm{x} & z_{13} \\ \mathrm{x} & \mathrm{x} & z_{23} \\ 0 & 0 & z_{33} \end{pmatrix} \tag{5.90}$$

Notice that Z_{11} is the 2×2 leading principal submatrix in (5.90).

By Lemma 5.10 we know that $\sum_{j=m+1}^{n_i} w_{lj} x_{jl} = 0$. Substituting (5.89) and (5.90) into (5.73) yields,

$$\sum_{i=1}^{m} |w_{li} x_{il}| = \kappa(Z_{11}) \left\| \begin{pmatrix} 1 & \frac{q_{22}}{q_{21}} \end{pmatrix} \right\| \left\| \frac{1}{q_{11}q_{22} - q_{12}q_{21}} \begin{pmatrix} q_{22}q_{13} - q_{12}q_{23} \\ q_{11}q_{23} - q_{21}q_{13} \end{pmatrix} \right\| \tag{5.91}$$

Finally, the bound is obtained by substituting (5.91) into (5.68),

$$1 \leq |q_{21}| \left\{ \frac{1}{\hat{\rho}(\Delta_3 Q) - \hat{\rho}(\Delta_2 Q)} \kappa(Z_{11}) \cdot \right. \\ \left. \|(1 \; q_{22}/q_{21})\| \left\| \frac{1}{q_{11}q_{22} - q_{12}q_{21}} \begin{pmatrix} q_{22}q_{13} - q_{12}q_{23} \\ q_{11}q_{23} - q_{21}q_{13} \end{pmatrix} \right\| \right\} \tag{5.92}$$

Summary of Conditions:

1. According to expression (5.79) a necessary condition for the robustness bound to be small is that $|q_{11}| < 1$. This is the well known necessary and sufficient condition for robustness against individual perturbations. Similar robustness conditions against individual perturbations in other plant elements follow from the bounds corresponding to simultaneous perturbations.

2. From expression (5.87) we conclude that as the ratio $|q_{12}q_{21}/q_{11}q_{22}|$ decreases, the bound for the maximal spectral radius $\hat{\rho}(\Delta_2 Q)$ gets smaller. In the extremal case where the ratio is zero the bound is $\hat{\rho}(\Delta_2 Q) \leq \max\{|q_{11}|, |q_{22}|\}$ i.e. we also require $|q_{22}| < 1$.

3. From expression (5.92) we infer that we can obtain a small robustness bound by keeping $|q_{21}|$ small, the condition number of the matrix Z_{11},

which is the matrix of eigenvectors of

$$\begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},$$

as close to 1 as possible and the minors of Q corresponding to the rows (1,2) and columns (1,3) and (2,3) small. The actual values depend on the specific application. Notice that the smaller the norms of the elements q_{12} and q_{21}, the closer the matrix Z_{11} is to normal. Therefore, the closer $\kappa(Z_{11})$ is to 1.

5.5 A Practical Application of the Robustness Bound

In this section we illustrate the use of the recursive bound developed in Section 5.4 by assessing at a single frequency the robustness of a loop designed to control the vertical dynamics of an aircraft. The model to be used is a three input, three output and five state linearized model at datum flight conditions. The model can be found in [61, 62, 49].

The inputs are denoted by,

ψ spoiler angle, measured in tenths of a degree,

a_x forward acceleration due to engine thrust, in m/sec^2,

δ elevator angle, in degrees,

whereas the notation for the states is,

h altitude relative to some reference point, in meters,

v_x forward speed, in m/sec,

θ pitch angle, in degrees,

$\dot{\theta}$ pitch rate, in deg/sec,

$\dot{h}$ vertical speed, in $m/sec.$

The outputs to be controlled are the first three states h, v_x and θ.

For almost all aircraft and in most flight conditions, the longitudinal dynamics exhibit two characteristic modes. The first mode is characterized by a short period and a relatively high damping factor. This first mode is referred to as the *short period mode.* The second mode, referred to as the *phugoid mode* has a long period relative to the first mode and very light damping. The phugoid mode is even unstable sometimes.

The short period mode transient response consists of variations in the angle of attack, in the vertical acceleration and in the pitch angle of the aircraft. The forward velocity however experiences little change.

The phugoid mode transient is characterized by almost no change in the angle of attack. However the pitch angle, the vertical acceleration and the forward velocity vary.

The vertical acceleration amplitudes in the phugoid mode are only slightly larger than in the short period mode. However, due to the large differences in the frequencies, the altitude excursions at the phugoid frequency are much higher than those at the short period frequency.

As pointed out in [63], the phugoid mode can be thought of as an exchange of potential and kinetic energy. The aircraft makes a sinusoidal flight path in the vertical plane. When going from the highest to the lowest point in the path, the aircraft picks up speed, lift is therefore increased and the flight path is curved until the aircraft starts climbing again and the velocity decreases.

A state space description (A, B, C) of the aircraft model (there is no direct input-output transmission) is given by [61, 62, 49],

$$A = \begin{pmatrix} 0 & 0 & 1.132 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -0.120 & 1.000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.575 & 0 & -0.0732 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

From the state space description we obtain the following elements $p_{ij}(s)$ of the plant transfer matrix $P(s) = C(sI - A)^{-1}B$,

$$\begin{aligned}
p_{11}(s) &= \frac{h(s)}{\psi(s)} = \frac{-1.6(s+1.4)(s-0.66)(s-0.055)}{s(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{12}(s) &= \frac{h(s)}{a_x(s)} = \frac{0.3(s^2+2\cdot 0.38\cdot 1.13\cdot s+1.13^2)}{s(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{13}(s) &= \frac{h(s)}{\delta(s)} = \frac{0.07(s-4.5)(s+3.6)(s+0.017)}{s(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{21}(s) &= \frac{v_x(s)}{\psi(s)} = \frac{-0.12(s+0.47)(s^2+2\cdot 0.035\cdot 2.1\cdot s+2.1^2)}{(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{22}(s) &= \frac{v_x(s)}{a_x(s)} = \frac{s}{(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{23}(s) &= \frac{v_x(s)}{\delta(s)} = \frac{-0.005(s-31)(s+1.12)}{(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{31}(s) &= \frac{\theta(s)}{\psi(s)} = \frac{4.42(s+0.26)(s+0.12)}{(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{32}(s) &= \frac{\theta(s)}{a_x(s)} = \frac{0.048(s+6.76)}{(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)} \\
p_{33}(s) &= \frac{\theta(s)}{\delta(s)} = \frac{-1.66(s+0.6)(s+0.09)}{(s^2+2\cdot 0.6\cdot 1.3\cdot s+1.3^2)(s^2+2\cdot 0.096\cdot 0.18\cdot s+0.18^2)}
\end{aligned} \tag{5.93}$$

From (5.93) we realize that the short period and the phugoid modes are characterized by the frequencies $\omega_{sp} = 1.3$ rad/sec (0.2 Hz) and $w_p = 0.18$ rad/sec (0.0286 Hz) and damping factors $\zeta_{sp} = 0.6$ and $\zeta_p = 0.096$ respectively.

The control design to be tested, taken from [49], is based on the Characteristic Locus method [8]. The Characteristic Locus method consists basically of constructing a compensator which, for the frequency region of interest, approximately commutes with the plant. As two commuting matrices share the same set of eigenvectors, it is not difficult to see that the eigenvalues of the system formed by the series connection of the plant and the compensator are the product of the plant and compensator eigenvalues. Thus, such a compensator enables the designer to individually shape the eigenvalues or characteristic functions of the plant.

Since the eigenvectors of the plant generally have elements which are not rational functions of s, it is impractical to attempt to build a compensator that commutes exactly with the plant. The Characteristic Locus method then assumes that the eigenvectors of the plant do not change too fast with frequency and builds a series of approximately commutative controllers at selected frequencies. In addition to this, a constant gain compensator is generally also designed to reduce interaction at high frequencies (where the gains of the plant are low). The complete compensator is then realized by combining all the designs.

The design shown in [49] consists of a combination of three compensators: a high frequency decoupling compensator K_h, a middle frequency compensator $K_m(s) = UM(s)V$ and a low frequency compensator $K_l(s)$. The compensator matrices are,

$$K_h = \begin{pmatrix} -71.535 & 0.0036 & -3.669 \\ -8.5375 & 9.9984 & -0.5376 \\ -189.44 & -0.0065 & -69.378 \end{pmatrix}$$

$$U = \begin{pmatrix} 0.2426 & -0.2077 & -0.0016 \\ -0.0087 & 0.0079 & 0.9999 \\ 0.6151 & 0.9656 & 0.0010 \end{pmatrix} \qquad V = \begin{pmatrix} 2.1937 & 0.0031 & 0.5587 \\ -1.39 & -0.0017 & 0.6491 \\ 0.0278 & 1.0 & 0.0014 \end{pmatrix}$$

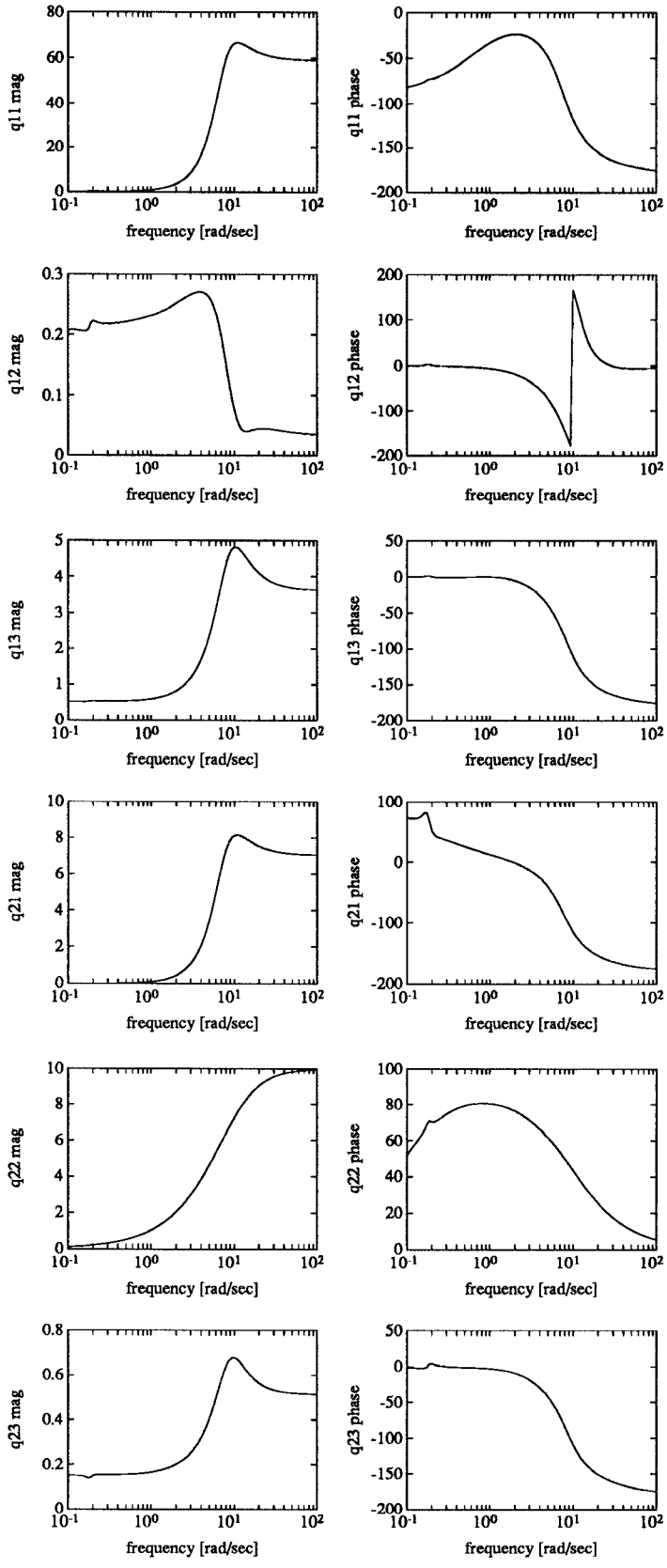
$$M(s) = \begin{pmatrix} m(s) & 0 & 0 \\ 0 & m(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad m(s) = \frac{0.0933s + 0.2175}{0.0933s + 1}$$

$$K_l(s) = \frac{1 + 2s}{2s} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The complete compensator $C(s)$ is therefore given by,

$$C(s) = K_h U M(s) V K_l(s)$$

Frequency response plots of the individual elements $q_{ij}(s)$ of the closed loop transfer matrix $C(s)[I + P(s)C(s)]^{-1}$ (which as shown in Chapter 4 is the closed loop map corresponding to the case of additive perturbations) are shown in Figure 5.1. Notice that the units of the various individual closed loop maps are, q_{11}–$[(deg/m) \cdot 10^{-1}]$, q_{12}–$[(deg/m/sec) \cdot 10^{-1}]$, q_{13}– $[(deg/deg) \cdot 10^{-1}]$, q_{21}– $[1/sec^2]$, q_{22}– $[1/sec]$, q_{23}– $[m/sec^2/deg]$, q_{31}– $[deg/m]$, q_{32}– $[deg/m/sec]$ and q_{33}– $[deg/deg]$.


q11 mag
q11 phase
q12 mag
q12 phase
q13 mag
q13 phase
q21 mag
q21 phase
q22 mag
q22 phase
q23 mag
q23 phase
frequency [rad/sec]

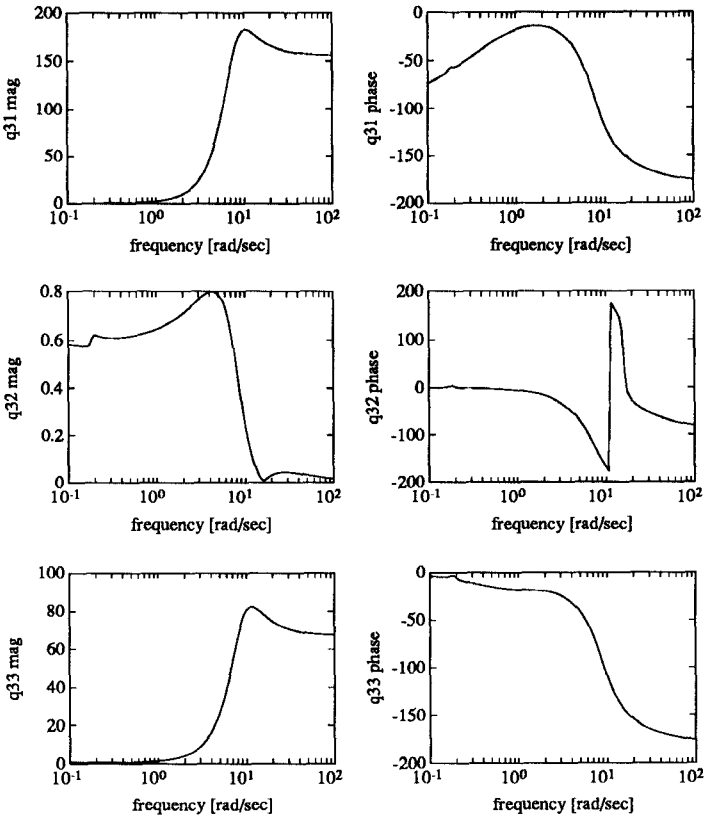


Figure 5.1: Frequency Responses of the Elements of $C(s)[I + P(s)C(s)]^{-1}$

To analyze the closed loop transient response, a digital simulation of the system was implemented using Grumman's *PROTOBLOCK*. This is a program that creates a graphic environment for *MATLAB* and *ACSL*. A general block diagram of the simulated system is depicted in Figure 5.2. where h_c, v_{x_c} and θ_c are the altitude, forward speed and pitch angle commands respectively.

Figure 5.3 depictes the response of the outputs h, v_x and θ of the unperturbed system to a unit step in the input commands h_c, v_{x_c} and θ_c.

Figure 5.4 shows again the response of the output h to a unit step in the input h_c but now the plot extends up to 500 seconds and the vertical axis has a high resolution scale. Notice that after approximately 10 seconds all the transients have died out and the system remains in steady state.

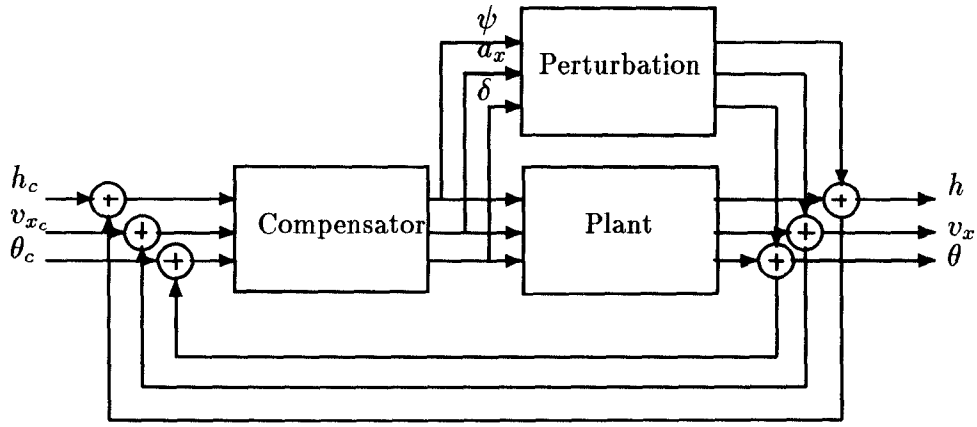


Figure 5.2: Simulation Block Diagram

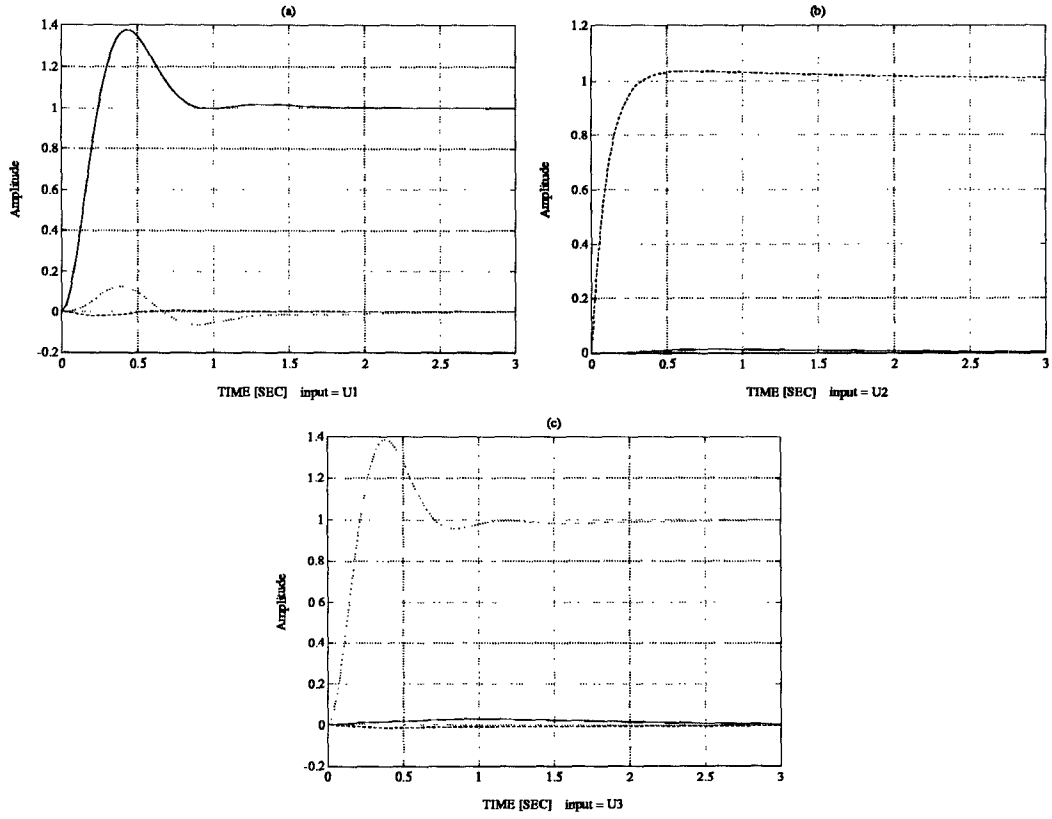


Figure 5.3: Responses of h (solid line), v_x (dashed line) and θ (dotted line) to unit step inputs in (a) h_c (b) v_x and (c) θ_c. The system is unperturbed.

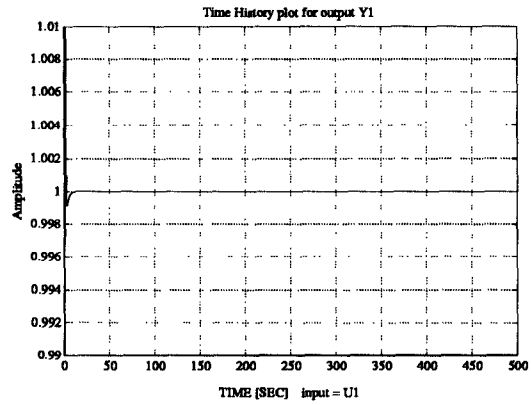


Figure 5.4: Long term transient response of h to a unit step input in h_c. The system model is unperturbed.

The application of the robustness bound will be carried out at a single frequency and under the assumption that only the diagonal elements of the plant matrix are uncertain. A frequency will be chosen where the system exhibits potential for robustness problems. According to the robustness analysis appearing in [64], under certain coupling conditions the robustness of a feedback system with an ill conditioned plant may be poor with respect to simultaneous model perturbations. This situation is possible even for systems exhibiting large stability margins against individual perturbations. Therefore, we will choose a frequency for which the plant is ill conditioned.

A plot of the spectral condition number of the plant (5.93) is depicted in Figure 5.5. The numbers shown in the plot of Figure 5.5 suggest an extremely ill conditioned plant. However care must be exercised when interpreting these numbers since the condition number depends on the units chosen.

We will choose the frequency of the phugoid mode $\omega_p = 0.18$ rad/sec since at this frequency the relative condition number is the highest.

The model perturbations to be considered here will be of the element by element multiplicative type. I.e., if the ij element of the plant is perturbed then its transfer function $\tilde{p}_{ij}$ will be represented by,

$$\tilde{p}_{ij}(s) = p_{ij}(s)[1 + \eta_{ij}(s)] \tag{5.94}$$

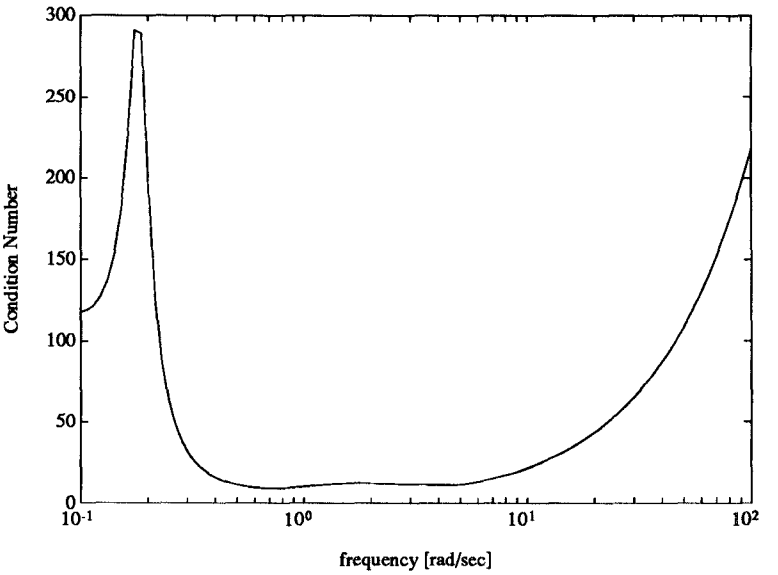


Figure 5.5: Plant Spectral Condition Numbers

where $p_{ij}(s)$ is the nominal element and $\eta_{ij}(s)$ is a proper and stable, norm bounded transfer function. Notice that as mentioned in Section 2 of Chapter 4 this type of uncertainty can be alternatively represented as an additive perturbations by writing,

$$\tilde{p}_{ij}(s) = p_{ij}(s) + \Delta_{ij}(s) \tag{5.95}$$

with $\Delta_{ij}(s) = p_{ij}(s)\eta_{ij}(s)$.

The robustness analysis will be performed in several steps. In the first step we will compute the maximal magnitude of additive uncertainty allowed by the design in each one of the individual diagonal plant elements (without considering simultaneous uncertainties). In the second step, we will devise some models for the uncertainties in the plant elements using real rational proper and stable functions. In this step it will be assumed that only individual model perturbations were considered in the design process and that some margins were left to account for possible simultaneous perturbations. In all the other steps, we will compute robustness margins at the frequency of the phugoid mode predicted by the bound on the maximal spectral radius and we will compare them with simulation results.

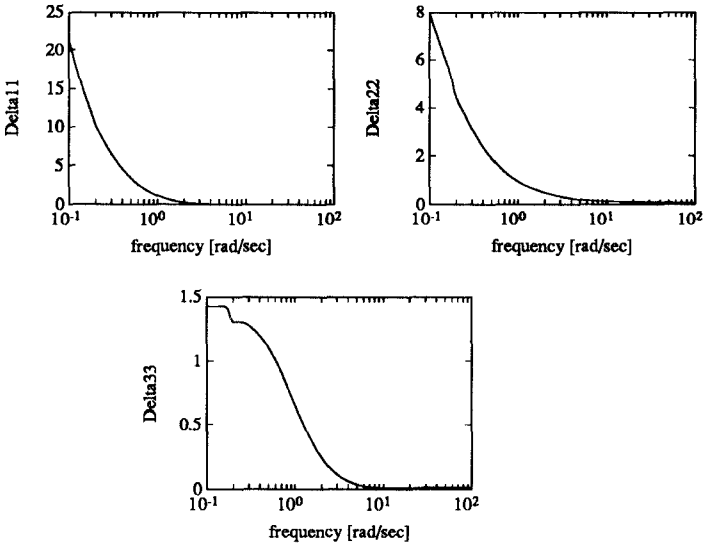


Figure 5.6: Additive Perturbation Bounds

Step1: In this step we first compute, for each diagonal element of the plant, the maximal additive perturbation for which the closed loop system remains stable. To do this assume that the additive perturbation matrix has all its elements zero except the element in the ii-th place. Assume also that the ii-th perturbation element is a proper and stable real rational function denoted by $\Delta_{ii}(s)$. The nonzero element of the perturbation matrix can be normalized to unity by considering the modified closed loop map $\hat{Q}(s) = |\Delta_{ii}(s)|Q(s)$. By Theorems 4.3 and 5.1 we obtain that robust stability requires that $\hat{\rho}_{\tilde{\Delta}} = \max_{\Delta \in \tilde{\Delta}} \rho(\Delta \hat{Q}) = |\Delta_{ii}||q_{ii}| = |\hat{q}_{ii}| < 1$. Therefore the maximal individual perturbation is bounded by,

$$|\Delta_{ii}(s)|_{max} < \frac{1}{|q_{ii}(s)|} \tag{5.96}$$

Plots of the additive perturbation bounds (5.96) as a function of frequency are depicted in Figure 5.6.

The bounds (5.96) can be translated into bounds on the individual nonzero plant elements multiplicative perturbations (5.94) by using the relationship $\eta_{ii}(s) = \Delta_{ii}(s)/p_{ii}(s)$.

Step 2: In this step we will assume models for the individual multiplicative perturbations.

From (5.96) we have that, at each complex frequency s, the maximal individual multiplicative uncertainty allowed by the design is given by,

$$|\eta_{ii}(s)|_{max} = \frac{|\Delta_{ii}(s)|_{max}}{|p_{ii}(s)|} = \frac{1}{|q_{ii}(s)p_{ii}(s)|} \tag{5.97}$$

We will denote the actual individual multiplicative perturbations by $f_{ii}(s)$, $i = 1,2,3$. The perturbation models will be chosen so that they satisfy $|f_{ii}(s)| < |\eta_{ii}(s)|_{max} = 1/|q_{ii}(s)p_{ii}(s)|$ and have the form,

$$f_{ii}(s) = k_{ii}\frac{\nu_{ii}s+1}{\tau_{ii}s+1}B_{ii}(s)$$

with (5.98)

$$k_{ij} \approx \frac{1}{2}\min_{\omega\in\mathbb{R}}\frac{1}{|q_{ii}(\omega)p_{ii}(\omega)|} \quad \text{and} \quad k_{ii}\frac{\nu_{ii}}{\tau_{ii}} = 1$$

In (5.98), $B_{ii}(s)$ is a stable all pass transfer function used to determine the uncertainty phase at a desired frequency. The parameters of $B_{ii}(s)$ can be varied to find, at the frequency where the analysis is being performed, the uncertainty phase which yields the worst stability margin. This reflects the fact that since we are assuming complex perturbations the phase is unrestricted.

The choice of k_{ii} in (5.98) provides at least 6 db stability margin against individual perturbations at all frequencies. This is the margin left to cope with simultaneous uncertainty effects. The relationship between the gain and the ratio between the time constants produces relative errors at high frequencies up to 100%. This choice is based on the fact that the higher the frequency the less the model is certain.

Table 5.1 shows the uncertainty parameters chosen,

Figure 5.7 depicts plots of the magnitude of the maximal multiplicative perturbations $|\eta_{ii}|_{max}$ along with the uncertainty models (5.98) evaluated with the parameters of Table 5.1. Notice that since the perturbations are multiplicative they are dimensionless. From Figure 5.7 it is apparent that the phugoid mode

Index Pair	k	ν	τ
11	0.04	1	0.04
22	0.1	1	0.1
33	0.035	1	0.035

Table 5.1: Perturbation Parameters

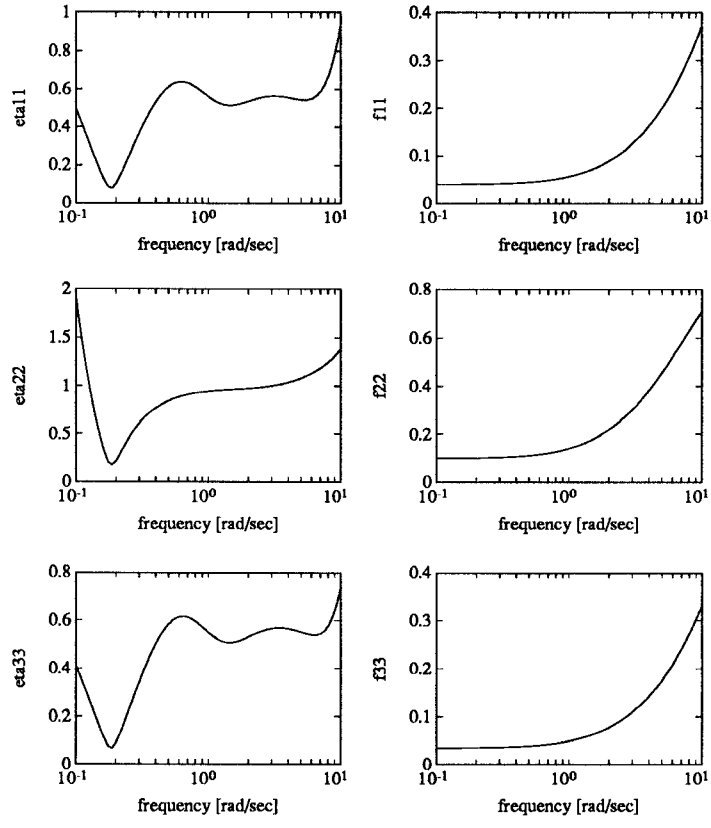


Figure 5.7: Multiplicative Perturbations and Uncertainty Models

determines the dc gains of the uncertainty models.
The values of the all pass function parameters are not relevant here. We will consider them later in the simulation stages.

In the sequel we will deal with robustness analysis at the frequency of the phugoid mode only. Therefore all matrices and functions will be evaluated at this frequency.

Step 3: In this step we analyze by means of the bound developed in Section 5.4 the robustness of the system at the phugoid frequency ω_p when the only uncertain elements in the plant are the diagonal elements p_{11}, p_{22}, and p_{33}. First, we normalize the uncertainty by making a transformation on the matrix of closed loop maps Q such that the uncertainty elements at ω_p will be of magnitude one. From (5.95) we know that the multiplicative perturbations f_{11}, f_{22} and f_{33} on the diagonal elements can be expressed as additive perturbations Δ_{11}, Δ_{22} and Δ_{33} by computing,

$$\Delta_{ii} = p_{ii} f_{ii} \qquad i = 1, 2, 3$$

Consider now a diagonal invertible matrix W of the form,

$$W = \begin{pmatrix} 1/|p_{11} f_{11}| & 0 & 0 \\ 0 & 1/|p_{22} f_{22}| & 0 \\ 0 & 0 & 1/|p_{33} f_{33}| \end{pmatrix}$$

Since we can always write,

$$\rho[\Delta Q] = \rho[W^{-1} W \Delta Q] = \rho[W \Delta Q W^{-1}] = \rho[\hat{\Delta}\hat{Q}]$$

where $\hat{Q} = QW^{-1}$ and $\hat{\Delta} = W\Delta$ then the computation of the maximal spectral radius may be carried out considering perturbation matrices $\hat{\Delta}$ with elements of unit magnitude and a modified closed loop map $\hat{Q}$.

At the phugoid frequency the matrix Q is given by,

$$Q = \begin{pmatrix} 0.080e^{-\jmath 74.0^\circ} & 0.200e^{\jmath 3.50^\circ} & 0.5200e^{\jmath 1.40^\circ} \\ 0.005e^{\jmath 81.50^\circ} & 0.190e^{\jmath 69.0^\circ} & 0.140e^{-\jmath 0.66^\circ} \\ 0.120e^{-\jmath 58.6^\circ} & 0.580e^{\jmath 3.50^\circ} & 0.710e^{-\jmath 3.4^\circ} \end{pmatrix}$$

and the weighting matrix W is,

$$W = \begin{pmatrix} 0.175 & 0 & 0 \\ 0 & 0.393 & 0 \\ 0 & 0 & 1.565 \end{pmatrix}$$

The matrix $\hat{Q}$ is therefore given by,

$$\hat{Q} = QW^{-1} = \begin{pmatrix} 0.470e^{-\jmath 74.0^\circ} & 0.530e^{\jmath 3.50^\circ} & 0.3300e^{\jmath 1.40^\circ} \\ 0.029e^{\jmath 81.50^\circ} & 0.476e^{\jmath 69.0^\circ} & 0.088e^{-\jmath 0.66^\circ} \\ 0.680e^{-\jmath 58.6^\circ} & 1.470e^{\jmath 3.50^\circ} & 0.450e^{-\jmath 3.4^\circ} \end{pmatrix} \tag{5.99}$$

The maximal spectral radii corresponding to individual uncertainties in the diagonal elements of the plant are given by the diagonal elements of the matrix $\hat{Q}$ in (5.99). As expected from our choice of uncertainty models the margins are approximately 6 db.

To compute the bound on the maximal spectral radius corresponding to the case of two simultaneous uncertainties we use $\hat{Q}$ in (5.99) as the matrix of closed loop maps for the evaluation of expression (5.75) or expression (5.87) with the appropiate index pairs. Denoting by $\hat{\rho}_{ij}$ the bound on the maximal spectral radius when the plant elements p_{ii} and p_{jj} are uncertain we obtain the following results,

$$\hat{\rho}_{12} = 0.615 \qquad \hat{\rho}_{13} = 1.045 \qquad \hat{\rho}_{23} = 0.955 \tag{5.100}$$

From the bounds (5.100) we conclude that the control design yields a fairly robust system at the frequency corresponding to the phugoid mode, against simultaneous uncertainties of the form (5.98) in the plant elements p_{11} and p_{22}. When only a multiplicative uncertainty $\eta_{11}(s)$ in the element p_{11} exists, the

system stability margin amounts to $1/0.47 = 2.13$ or 6.56 db. By adding a second uncertainty in the element p_{22}, the margin bound is reduced only to $1/0.615 = 1.63$ or 4.22 db.

The situation is much worse for the case of simultaneous uncertainty in the elements p_{22} and p_{33} and even worse when the uncertainty appears simultaneously in the elements p_{11} and p_{33}. In the former case the stability margin bound is reduced from $1/0.47 = 2.13$ (6.56 db) to $1/0.955 = 1.047$ (0.043 db) so that the system may be very close to instability. In the latter case the addition of a second uncertainty yields a negative stability margin bound (-0.38 db) which means that the system may be unstable.

Step 4: In this step we verify by means of digital simulation how conservative the results predicted by the bounds (5.100) are. The simulations were performed using the configuration of Figure 5.2 with perturbations of the form (5.98) with the parameters of Table 5.1. The all pass functions in (5.98) were chosen to yield the smallest possible stability margins at the phugoid frequency. This functions were determined by exhaustive simulation search.

For the case of perturbations in the plant elements p_{11} and p_{22} the all pass functions are,

$$B_{11}(s) = \frac{0.18 - s}{0.18 + s} \quad \text{and} \quad B_{22}(s) = \frac{s - 0.23}{s + 0.23}$$

Figure 5.8 depicts the same responses to step commands as in Figure 5.3 but now with the plant elements p_{11} and p_{22} perturbed. A comparison between Figures 5.3 and 5.8 can be done to see how the perturbations in the plant elements p_{11} and p_{22} affect the closed loop fast modes of large residue. In general the response in (a) becomes more oscillatory, in (b) more damped and the response in (c) changes very slightly. However we are not dealing here with these fast modes. Only the phugoid mode is under consideration. The phugoid mode has a very long relative period and a small closed loop residue. Therefore to observe it we need to let the faster modes decay and also use a high resolution scale in the plots. Figure 5.9 shows the phugoid mode decay of each of the outputs

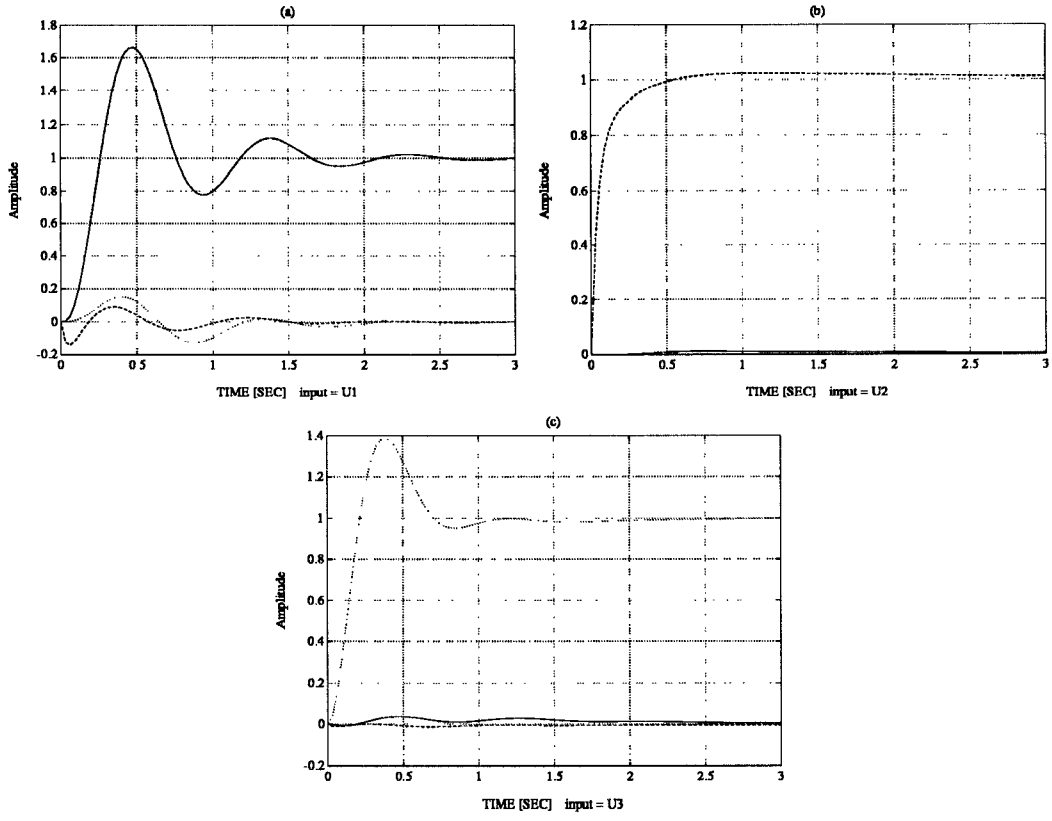


Figure 5.8: Responses of h (solid line), v_x (dashed line) and θ (dotted line) to unit step inputs in (a) h_c (b) v_{x_c} and (c) θ_c. The plant elements p_{11} and p_{22} are perturbed.

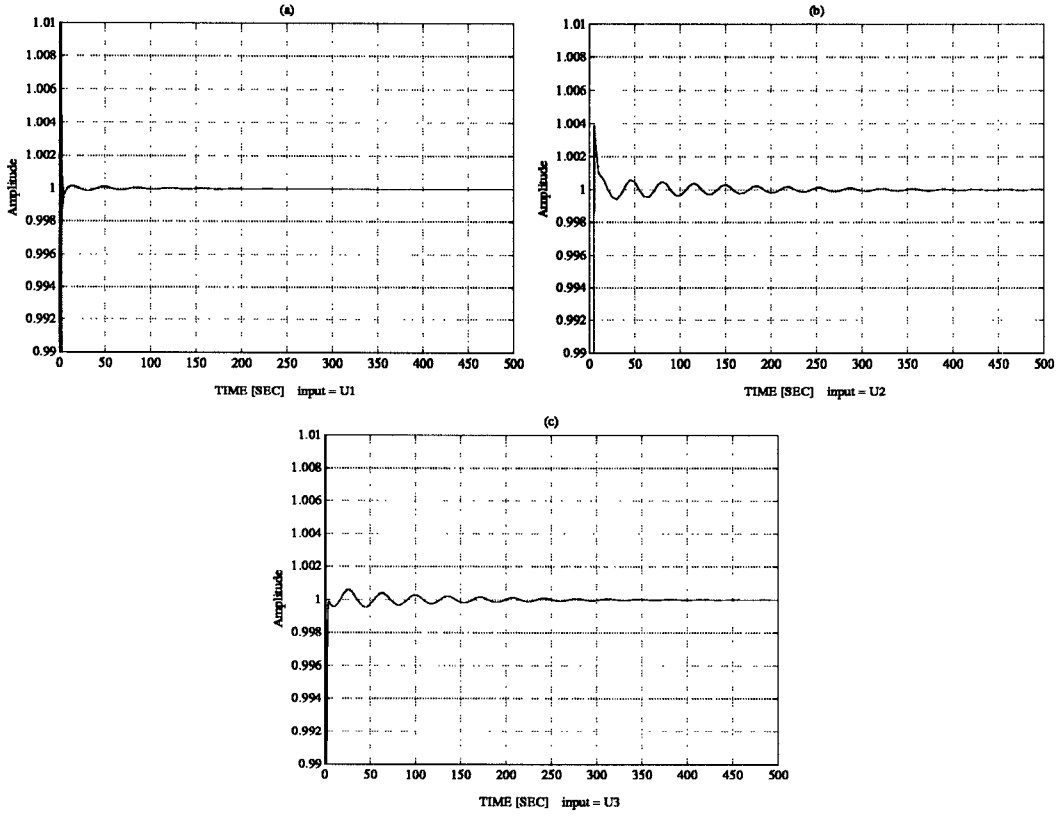


Figure 5.9: Transient response time of (a) h, (b) v_x and (c) θ in response to unit steps in their corresponding commands. p_{11} and p_{22} are perturbed.

in response to a step in the corresponding command. The decaying oscillation obtained matches the result predicted by the bound on the maximal spectral radius (a stability margin of 4.22 db at this frequency).

When the system is subjected to perturbations in the plant elements p_{22} and p_{33}, the all pass functions yielding the smallest stability margins at the phugoid frequency are,

$$B_{22}(s) = \frac{s - 0.23}{s + 0.23} \quad \text{and} \quad B_{33}(s) = \frac{s - 0.25}{s + 0.25}$$

The long term response of the system to step inputs is shown in Figure 5.10. From Figure 5.10 we conclude that the system remains stable for the perturbations in the elements p_{22} and p_{33}. However comparing the responses to those

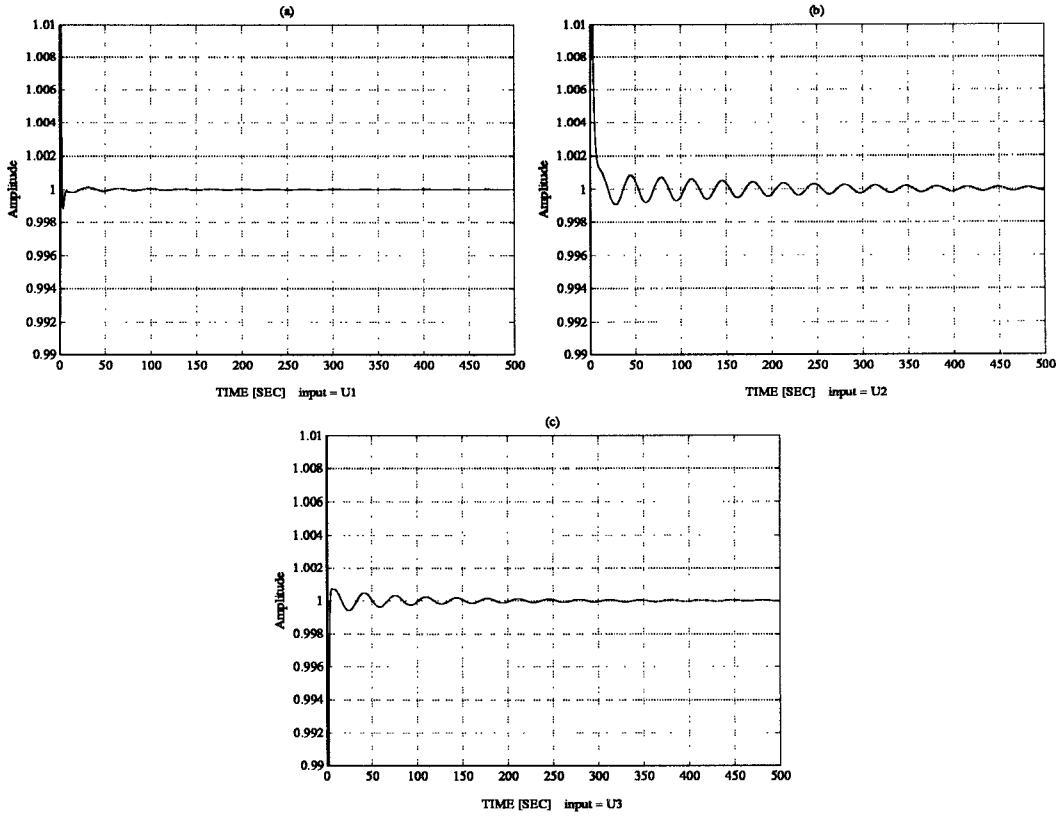


Figure 5.10: Transient response time of (a) h, (b) v_x and (c) θ in response to unit steps in their corresponding commands. p_{22} and p_{33} are perturbed.

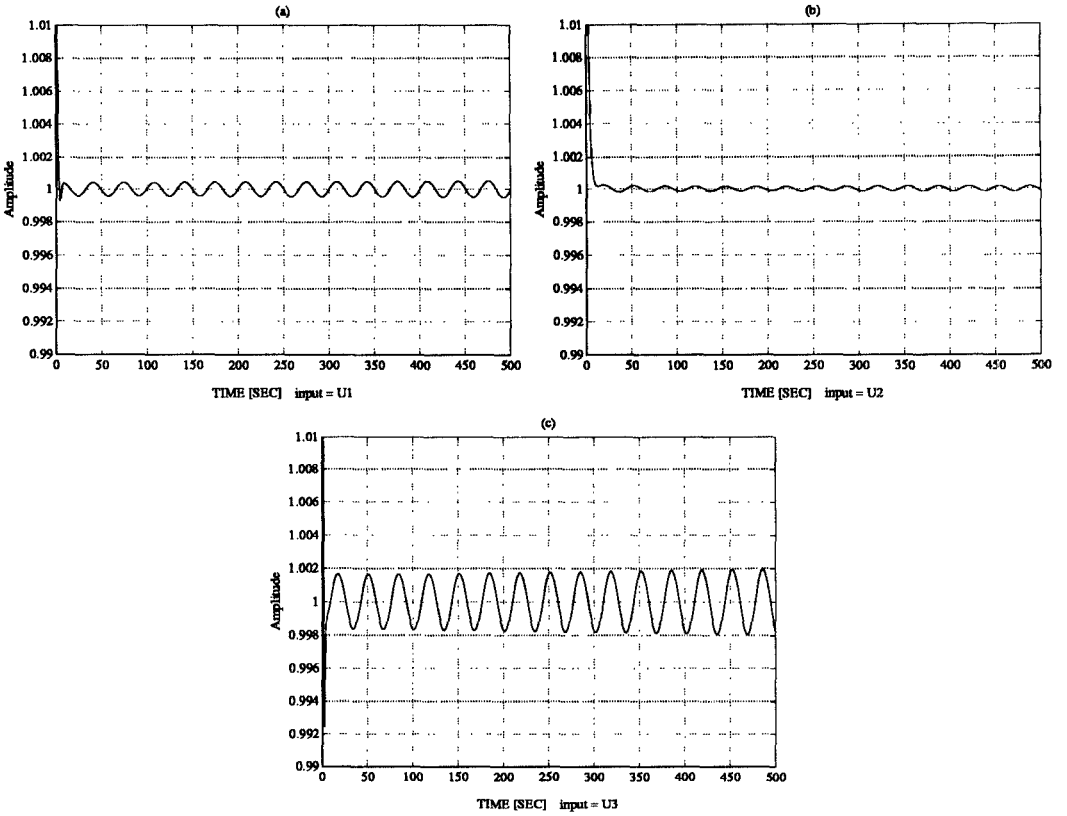


Figure 5.11: Transient response time of (a) h, (b) v_x and (c) θ in response to unit steps in their corresponding commands. p_{11} and p_{33} are perturbed.

corresponding to perturbations in the plant elements p_{11} and p_{22} (Figure 5.9) we notice that the system is less damped. The oscillation at the phugoid frequency decays in a slower manner. This qualitative conclusion is also predictable from the bounds (5.100).

The all pass functions calculated for the case where the plant elements p_{11} and p_{33} are subjected to perturbations are,

$$B_{11}(s) = \frac{8.46 - s}{8.46 + s} \quad \text{and} \quad B_{33}(s) = -\frac{s^2 - 8.48s + 0.17}{s^2 + 8.48s + 0.17}$$

The response of the system to step inputs is shown in Figure 5.11. As predicted by the bound $\hat{\rho}_{13}$ in (5.100), the closed loop system has turned unstable. The phugoid mode oscillates with a slowly growing amplitude as expected from the small negative stability margin.

Expression (5.75) may be used to define a new set of specifications upon the closed loop maps at the phugoid frequency to guarantee robust stability against simultaneous perturbations.

Despite the fact that in the particular example above the bound on the maximal spectral radius succeded in predicting the behavior of the closed loop system in the face of uncertainty it is clear that in other cases the predictions may be much more conservative. Moreover, when the number of uncertainties is greater than two and the perturbation matrix has rank greater than one the computation of the condition number in (5.75) makes the evaluation of the bound more cumbersome and the result more conservative. However, as pointed out earlier, the main use of the bound is in obtaininig expressions involving the individual closed loop maps which provide indications of how the maps should be altered to improve the robustness of the system.

CHAPTER
SIX

A PARAMETRIZATION OF ALL STABILIZING CONTROLLERS FOR PLANTS WITH PARAMETRIC UNCERTAINTY

In this chapter we obtain a parametrization of all (internally) stabilizing controllers for rational SISO plants characterized by uncertainty in the coefficients of the numerator and denominator polynomials. The parametrization has a rational function as the free variable. The case of a single uncertain coefficient is analyzed and necessary and sufficient conditions for robust stability are derived. The results suggest the possibility of using this approach for synthesis of controllers for uncertain plants. The conditions in the multiparameter uncertainty case require more advanced tools for stability verification. A simple, illustrative example is provided. Most of the results of this chapter can be found in [65].

6.1 Introduction

A particular class of model uncertainty, which frequently arises in control design practice, consists of one or more parameters whose values are only known to lie in some finite interval. Due to this uncertainty, the controlled system should satisfy the system specifications for all values of the uncertain parameters. Thus, the control design goal is to achieve robust stability as well as robust performance. For a variety of reasons, which may arise from either practical or

theoretical considerations, the control system may be required to consist of a single, fixed controller.

One possible approach to this problem is to define a set of nominal values of the uncertain parameters and to consider deviations from these nominal values as model perturbations. Most of the methods for analysis and synthesis of control systems are based on this deviation from nominal, or perturbation approach. These methods can be roughly divided into frequency domain methods and parameter space methods. Comprehensive surveys of the frequency domain and parameter space methods can be found in Dorato [66] and Šiljak [67] respectively.

Another possible approach would be to consider each combination of the uncertain parameters as defining a different plant. If the uncertain parameters take values only in discrete sets then a finite number of plants may be considered. This would be the case of systems characterized by multiple modes of operation or of plants linearized at several equilibria. However, if one of the parameter values lies on a continuous interval, we are faced with an uncountably infinite number of plants. The control design aim is therefore to find a common stabilizing compensator for all the plants. This approach has been termed the Simultaneous Stabilization Problem and was first adopted for stabilizing systems at different modes of operation (including failure modes), with a single compensator. Both, the SISO and the MIMO cases appear in the literature. Some works on the subject are found in Vidyasagar and Viswanadham [68], Saeks *et al* [69], Saeks and Murray [70] Vidyasagar [71], Debowski and Kurylowicz [72], Emre [73], Ghosh and Byrnes [74], etc.. Nevertheless, the Simultaneous Stabilization Problem has been addressed mainly for a finite number of plants. For an infinite number of plants, Barmish and Wei [75], Wei and Barmish [76], Wei and Yedavalli [77] and Kwakernaak [78], have obtained conditions for simultaneous stabilization by a single compensator for certain classes of SISO plants, as well as some generalizations to the MIMO case. In [77], the problem of robust stabilizability of plants with both unstructured and parameter uncertainty is considered. In [79], using interpolation and conformal mapping techniques,

Tannenbaum has demonstrated a procedure for finding a stable compensator which stabilizes a SISO plant with uncertainty in the gain factor.

While all the above studies deal with the problem of finding a single compensator for simultaneous stabilization, very little attention has been paid to the problem of finding the set of all controllers which robustly stabilize a plant with uncertainty in its coefficients. The utility of knowing this set and of having a simple parametrization for its elements is apparent: Since stability is guaranteed for all values of the uncertain parameters one can concentrate upon choosing a controller which yields the required system performance. Methods for finding the controller parameters which optimize some system measures may be devised. For instance, some of the H_∞ sensitivity minimization procedures are based on such a parametrization. An additional use of the parametrization is in analysis of control systems. The limiting performance bounds of a given plant may be found by searching over the set of all stabilizing compensators.

This chapter is concerned with a parametrization of all stabilizing controllers for SISO plants having uncertain coefficients taking values in given intervals. As a first attempt, a complete analysis has been carried out for the case of a single uncertain coefficient. This relatively simple case enables us to obtain insight into the different problems involved in this approach and to estimate the feasibility of further generalizations and applications.

The control configuration considered here is depicted in Figure 6.1, where

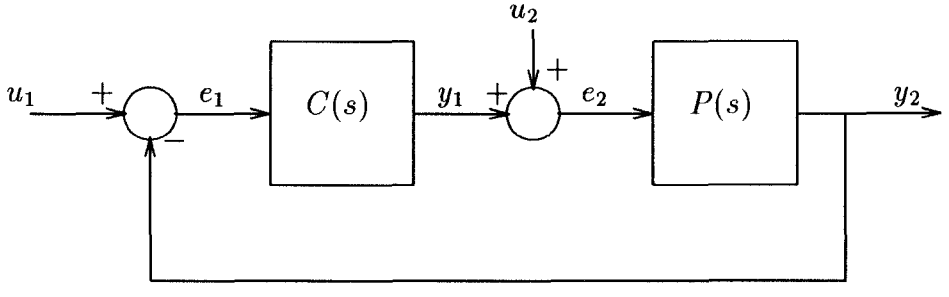


Figure 6.1: Control Configuration

the plant $P(s)$ is modelled by the real rational and proper transfer function,

$$P(s) = \frac{n(s)}{d(s)} = K\frac{s^m + a_1 s^{m-1} + \ldots + a_{m-1}s + a_m}{s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n} \tag{6.1}$$

We assume that $m \leq n$, only one coefficient is uncertain, and its value is known to lie in a given interval. There are three cases to be considered:

Case 1: one numerator coefficient is uncertain i.e., there is an $i \in [1, m]$ such that $a_i \in [\underline{a}_i, \overline{a}_i]$.

Case 2: one denominator coefficient is uncertain i.e., there is a $j \in [1, n]$ such that $b_j \in [\overline{b}_j, \underline{b}_j]$.

Case 3: the gain is uncertain i.e., $K \in [\underline{K}, \overline{K}]$.

The first issue to be considered in this chapter is the conditions for stabilizability of two plants–those corresponding to the extremal values of the uncertain coefficient. It turns out that numerous classes of pairs of plants admit simultaneous stabilization [69], [71]. The set of all compensators which simultaneously stabilize the plants corresponding to the extremal values of the uncertain coefficient will be obtained. The set is parametrized by a function which must be a unit in the ring of stable and proper real rational functions (that is, a function which is chosen from a collection of functions having all their poles and zeros in the open left half plane and all of which have the same number of finite poles and zeros) and has to satisfy certain interpolation and degree conditions. Next, we will derive necessary and sufficient conditions on the controller free parameter for simultaneous stabilization of the plant at all values of the uncertain plant coefficient. The conditions require that a certain characteristic equation based on the controller parameter be Hurwitz for all positive values of a gain. This condition amounts to a simple Root-Locus type test.

The chapter is organized as follows: Section 6.2 deals with the simultaneous stabilization of the pair of plants corresponding to the extremal values of the uncertain plant coefficient. The section closely follows the paper of Saeks *et al.* [69] and presents the parametrization of all stabilizing controllers along with the conditions on the controller free parameter. Section 6.3 is entirely

devoted to the development of necessary and sufficient conditions for robust stabilization. Section 6.4 shows some results of the multiparameter uncertainty case and points out some of the difficulties encountered in this generalization. Section 6.5 depicts the geometry of the closed loop maps which are obtained with the controller parametrization. It is shown that by choosing specific coefficient values, the closed loop maps can be made affine in the controller parameter, as required in some of the H_∞ methods. Section 6.6 presents a simple example which illustrates the ideas of the preceding sections.

6.2 Simultaneous Stabilization of Two Plants

Denote the plant by $P(s,\alpha)$ with α representing a coefficient about which all that is known is that it lies in the interval $[\underline{\alpha},\overline{\alpha}]$ and s being the usual complex variable. For fixed $\alpha = \hat{\alpha}$, $P(s,\hat{\alpha})$ is a standard transfer function. We assume throughout that $P(s,\alpha)$ is SISO. Our goal in this section is to obtain the parametrized set of all controllers that simultaneously stabilize the plants $P(s,\underline{\alpha})$ and $P(s,\overline{\alpha})$.

We assume that $P(s,\underline{\alpha})$ and $P(s,\overline{\alpha})$ are simultaneously stabilizable, i.e., there exists at least one common implementable compensator $C(s)$ which stabilizes both plants.

The compensator $C(s)$ is implementable if it is proper, i.e., analytic at $s = \infty$.

Several theorems dealing with conditions for simultaneous stabilizability of plants can be found in [69], [72], [71]. In [71] it is shown that the problem of simultaneously stabilizing $l+1$ multivariable plants is equivalent to that of simultaneously stabilizing l plants with a stable compensator. Moreover, provided that the plants have at least $l+1$ inputs or outputs, the collection of $l+1$ plants is generically simultaneously stabilizable.

For the SISO case simple conditions for simultaneous stabilization exist and large classes of plants satisfy these conditions. For illustration, consider the class of plants which are either stable (but possibly nonminimum phase) or minimum

phase (but possibly unstable). To show that any two plants in one of the above categories are always simultaneously stabilizable we use two theorems from [69] and [72] repeated here for convenience in a unified form.

Theorem 6.1 *Let $\tilde{P}(s)$ and $\hat{P}(s)$ be distinct SISO plants with coprime fractional representations over $\mathbb{RH}_\infty$,*

$$\tilde{P}(s) = N_{\tilde{p}}(s)D_{\tilde{p}}^{-1}(s) \qquad \hat{P}(s) = N_{\hat{p}}(s)D_{\hat{p}}^{-1}(s)$$

Then there exists a compensator that simultaneously stabilizes both plants if and only if either,

(i) $D_{\tilde{p}}(s)$ and $D_{\hat{p}}(s)$ are coprime and $\tilde{P}(s)$ and $\hat{P}(s)$ taken together have an even number of poles between every pair of positive real axis zeros of $N_{\tilde{p}}(s)D_{\hat{p}}(s) - N_{\hat{p}}(s)D_{\tilde{p}}(s)$.

(ii) $N_{\tilde{p}}(s)$ and $N_{\hat{p}}(s)$ are coprime and $\tilde{P}(s)$ and $\hat{P}(s)$ taken together have an even number of zeros between every pair of positive real axis zeros of $N_{\tilde{p}}(s)D_{\hat{p}}(s) - N_{\hat{p}}(s)D_{\tilde{p}}(s)$. •

For stable plants, $D_{\tilde{p}}(s)$ and $D_{\hat{p}}(s)$ are coprime over $\mathbb{RH}_\infty$ and there are no poles in the right half complex plane. Thus condition (i) of theorem 6.1 is satisfied. Analogously for minimum phase plants, $N_{\tilde{p}}(s)$ and $N_{\hat{p}}(s)$ are coprime, there are no zeros in the right half complex plane and therefore condition (ii) of Theorem 6.1 is trivially satisfied. In the sequel we will make reference to this class of plants for further illustrations.

A coprime factorization of $P(s,\alpha)$ is given by,

$$P(s,\alpha) = N_p(s,\alpha)D_p^{-1}(s,\alpha) \qquad X(s,\alpha)N_p(s,\alpha) + Y(s,\alpha)D_p(s,\alpha) = 1$$
$$N_p(s,\alpha), D_p(s,\alpha), X(s,\alpha), Y(s,\alpha) \in \mathbb{RH}_\infty \tag{6.2}$$

and the set of all controllers that stabilize $P(s,\alpha)$ for fixed α is given by [71],

$$\mathbf{C}(s,\alpha) = \left\{ C(s,\alpha) = \frac{X(s,\alpha) + R(s,\alpha)D_p(s,\alpha)}{Y(s,\alpha) - R(s,\alpha)N_p(s,\alpha)}; \right.$$
$$\left. R(s,\alpha) \in \mathbb{RH}_\infty;\ Y(s,\alpha) - R(s,\alpha)N_p(s,\alpha) \not\equiv 0 \right\} \tag{6.3}$$

A controller $C(s,\underline{\alpha},\overline{\alpha})$ will simultaneously stabilize $P(s,\underline{\alpha})$ and $P(s,\overline{\alpha})$ if and only if,

$$C(s,\underline{\alpha},\overline{\alpha}) \in \mathbf{C}(s,\underline{\alpha}) \cap \mathbf{C}(s,\overline{\alpha}) \tag{6.4}$$

Any controller $C(s,\underline{\alpha},\overline{\alpha})$ satisfying (6.4) may be written in coprime fractional representation based on either $P(s,\underline{\alpha})$ or $P(s,\overline{\alpha})$, i.e.,

$$\begin{aligned} C(s,\underline{\alpha},\overline{\alpha}) &= N_c(s,\underline{\alpha})D_c^{-1}(s,\underline{\alpha}) \\ \text{or} & \\ C(s,\underline{\alpha},\overline{\alpha}) &= N_c(s,\overline{\alpha})D_c^{-1}(s,\overline{\alpha}) \end{aligned} \tag{6.5}$$

with $N_c(s,\cdot), D_c(s,\cdot) \in \mathbb{RH}_\infty$ coprime. Clearly (6.5) implies that there exists a unit $M(s) = m_n(s)m_d^{-1}(s) \in \mathbb{RH}_\infty$ (also $M^{-1}(s) \in \mathbb{RH}_\infty$) such that,

$$N_c(s,\underline{\alpha}) = N_c(s,\overline{\alpha})M(s) \qquad D_c(s,\underline{\alpha}) = D_c(s,\overline{\alpha})M(s) \tag{6.6}$$

Expressions (6.6) enable us to express the controller parameters $R_{\underline{\alpha}}(s,\alpha)$ and $R_{\overline{\alpha}}(s,\alpha)$ corresponding to $C(s,\underline{\alpha})$ and $C(s,\overline{\alpha})$ respectively as a function of the unit $M(s)$. Combining (6.3) and (6.6) yields,

$$\begin{aligned} X(s,\underline{\alpha}) + R(s,\underline{\alpha})D_p(s,\underline{\alpha}) &= [X(s,\overline{\alpha}) + R(s,\overline{\alpha})D_p(s,\overline{\alpha})]M(s) \\ \text{and} & \\ Y(s,\underline{\alpha}) - R(s,\underline{\alpha})N_p(s,\underline{\alpha}) &= [Y(s,\overline{\alpha}) - R(s,\overline{\alpha})N_p(s,\overline{\alpha})]M(s) \end{aligned} \tag{6.7}$$

From expressions (6.7), and using the commutativity of the functions (this is a SISO problem), we obtain,

$$R(s,\underline{\alpha}) = \frac{X(s,\underline{\alpha})N_p(s,\overline{\alpha}) + Y(s,\underline{\alpha})D_p(s,\overline{\alpha}) - M(s)}{N_p(s,\underline{\alpha})D_p(s,\overline{\alpha}) - D_p(s,\underline{\alpha})N_p(s,\overline{\alpha})} \tag{6.8}$$

The unit $M(s)$ in (6.8) has to be such that $R(s,\underline{\alpha}) \in \mathbb{RH}_\infty$, but otherwise is a free parameter. Assuming temporarily that such a unit exists we can substitute $R(s,\underline{\alpha})$ in (6.8) into expression (6.3) evaluated at $\alpha = \underline{\alpha}$ to obtain,

$$\mathbf{C}(s,\underline{\alpha},\overline{\alpha}) = \left\{ C(s) = \frac{M(s)D_p(s,\underline{\alpha}) - D_p(s,\overline{\alpha})}{N_p(s,\overline{\alpha}) - M(s)N_p(s,\underline{\alpha})}; \right.$$
$$\left. M(s), M^{-1}(s) \in \mathbb{RH}_\infty;\ M(s) \not\equiv \frac{N_p(s,\overline{\alpha})}{N_p(s,\underline{\alpha})} \right\} \tag{6.9}$$

Expression (6.9) represents the set of all compensators, parametrized by the unit $M(s)$, which simultaneously stabilize $P(s,\underline{\alpha})$ and $P(s,\overline{\alpha})$.

To derive the conditions on $M(s)$ such that $R(s,\underline{\alpha}) \in \mathbb{RH}_\infty$ we consider the following fractional coprime representation of the plant $P(s,\alpha)$,

$$N_p(s,\alpha) = \frac{n(s,\alpha)}{p(s)}; \qquad D_p(s,\alpha) = \frac{d(s,\alpha)}{p(s)} \tag{6.10}$$

with $n(s,\alpha)$ equal to the numerator and $d(s,\alpha)$ the denominator in (6.1). $p(s)$ is any strictly Hurwitz polynomial satisfying $\delta[p(s)] = \delta[d(s)]$ where $\delta[\cdot]$ denotes the degree of a polynomial. Using the Bezout identity in (6.2) and the fact that we are dealing with scalars we can rewrite (6.8) as,

$$R(s,\underline{\alpha}) = \frac{X(s,\underline{\alpha})[N_p(s,\overline{\alpha})D_p(s,\underline{\alpha}) - N_p(s,\underline{\alpha})D_p(s,\overline{\alpha})] + D_p(s,\overline{\alpha}) - D_p(s,\underline{\alpha})M(s)}{D_p(s,\underline{\alpha})[N_p(s,\underline{\alpha})D_p(s,\overline{\alpha}) - D_p(s,\underline{\alpha})N_p(s,\overline{\alpha})]}$$

or (6.11)

$$R(s,\underline{\alpha}) = \frac{Y(s,\underline{\alpha})[N_p(s,\underline{\alpha})D_p(s,\overline{\alpha}) - N_p(s,\overline{\alpha})D_p(s,\underline{\alpha})] + N_p(s,\overline{\alpha}) - N_p(s,\underline{\alpha})M(s)}{N_p(s,\underline{\alpha})[N_p(s,\underline{\alpha})D_p(s,\overline{\alpha}) - D_p(s,\underline{\alpha})N_p(s,\overline{\alpha})]}$$

Substituting (6.10) into expressions (6.11) yields,

$$R(s,\underline{\alpha}) =$$
$$p(s)\frac{X(s,\underline{\alpha})[n(s,\overline{\alpha})d(s,\underline{\alpha}) - n(s,\underline{\alpha})d(s,\overline{\alpha})] + [d(s,\overline{\alpha}) - d(s,\underline{\alpha})M(s)]p(s)}{d(s,\underline{\alpha})[n(s,\underline{\alpha})d(s,\overline{\alpha}) - d(s,\underline{\alpha})n(s,\overline{\alpha})]}$$
or (6.12)
$$R(s,\underline{\alpha}) =$$
$$p(s)\frac{Y(s,\underline{\alpha})[n(s,\underline{\alpha})d(s,\overline{\alpha}) - n(s,\overline{\alpha})d(s,\underline{\alpha})] + [n(s,\overline{\alpha}) - n(s,\underline{\alpha})M(s)]p(s)}{n(s,\underline{\alpha})[n(s,\underline{\alpha})d(s,\overline{\alpha}) - d(s,\underline{\alpha})n(s,\overline{\alpha})]}$$

Now we consider individually each of the possible uncertain coefficients.

Case 1: $\alpha = a_i, \quad i \in [1,m]$

In this case we have,

$$n(s,\alpha) = n(s,a_i); \quad d(s,\alpha) = d(s) \quad \text{and} \quad n(s,\overline{a}_i) - n(s,\underline{a}_i) = Ks^{m-i}(\overline{a}_i - \underline{a}_i) \tag{6.13}$$

and substituting (6.13) into (6.12) yields,

$$R(s,\underline{a}_i) = \frac{p(s)}{d(s)}\frac{KX(s,\underline{a}_i)s^{m-i}(\overline{a}_i - \underline{a}_i) + [1 - M(s)]p(s)}{Ks^{m-i}(\underline{a}_i - \overline{a}_i)} \tag{6.14}$$

Denoting $X(s,\underline{a}_i) = x_n(s,\underline{a}_i)x_d^{-1}(s,\underline{a}_i)$ we obtain the conditions on $M(s)$ for $R(s,\underline{a}_i)$ to be stable from (6.14),

$$m_n(s) - m_d(s) = s^{m-i}q(s) \tag{6.15}$$

with $q(s)$ a polynomial satisfying

$$Kx_n(s^*,\underline{a}_i)m_d(s^*)[\overline{a}_i - \underline{a}_i] + q(s^*)x_d(s^*,\underline{a}_i)p(s^*) = 0$$

for all s^* with Real $(s^*) \geq 0$ such that $d(s^*) = 0$ (including multiplicities).

Stability conditions like (6.15) will be referred to as interpolation conditions [69]. For $R(s,\underline{a}_i)$ to be proper we must extend its stability properties to the extended right half complex plane. This is accomplished by imposing degree conditions on $M(s)$ in (6.14),

$$\delta[m_d(s)] - \delta[m_n(s) - m_d(s)] \geq n - m + i \tag{6.16}$$

Case 2: $\alpha = b_j, \quad j \in [1, n]$

In this case we have,

$$n(s,\alpha) = n(s); \quad d(s,\alpha) = d(s,b_j) \quad \text{and} \quad d(s,\overline{b}_j) - d(s,\underline{b}_j) = s^{n-j}(\overline{b}_j - \underline{b}_j) \tag{6.17}$$

Substituting (6.17) into (6.12) gives,

$$R(s,\underline{b}_j) = \frac{p(s)}{n(s)} \frac{Y(s,\underline{b}_j)s^{n-j}(\overline{b}_j - \underline{b}_j) + [1 - M(s)]p(s)}{s^{n-j}(\overline{b}_j - \underline{b}_j)} \tag{6.18}$$

Denoting $Y(s,\underline{\alpha}) = y_n(s,\underline{\alpha})y_d^{-1}(s,\underline{\alpha})$, the interpolation conditions that follow from (6.18) are,

$$m_n(s) - m_d(s) = s^{n-j}q(s) \tag{6.19}$$

with $q(s)$ a polynomial satisfying,

$$y_n(s^*,\underline{\alpha})m_d(s^*)[\overline{b}_j - \underline{b}_j] + q(s^*)y_d(s^*,\underline{\alpha})p(s^*) = 0$$

for all s^* with Real $(s^*) \geq 0$ such that $n(s^*) = 0$ (including multiplicities).

The degree condition is,

$$\delta[m_d(s)] - \delta[m_n(s) - m_d(s)] \geq n - m + j \tag{6.20}$$

Case 3: $\alpha = K$

In this case we have,

$$n(s,\alpha) = n(s,K); \quad d(s,\alpha) = d(s)$$

and (6.21)

$$n(s,\overline{K}) - n(s,\underline{K}) = (\overline{K} - \underline{K})(s^m + a_1 s^{m-1} + \ldots + a_{m-1}s + a_m)$$

Substituting (6.21) into (6.12) yields,

$$R(s,\underline{K}) = \frac{p(s)}{d(s)} \frac{X(s,\underline{K})(\overline{K} - \underline{K})n(s,\overline{K}) + [1 - M(s)]p(s)\overline{K}}{(\underline{K} - \overline{K})n(s,\overline{K})} \tag{6.22}$$

From (6.22) we obtain the interpolation conditions,

$$m_n(s) - m_d(s) = q(s) \tag{6.23}$$

with $q(s)$ a polynomial satisfying

$$x_n(s^*, \underline{K})n(s^*, \overline{K})m_d(s^*)[\overline{K} - \underline{K}] + \overline{K}q(s^*)x_d(s^*, \underline{K})p(s^*) = 0$$

for all s^* with Real $(s^*) \geq 0$ such that $n(s^*)d(s^*) = 0$ (including multiplicities). The degree condition amounts to,

$$\delta[m_d(s)] - \delta[m_n(s) - m_d(s)] \geq n - m \tag{6.24}$$

Since we have assumed simultaneous stabilizability of $P(s, \underline{\alpha})$ and $P(s, \overline{\alpha})$, the existence of at least one unit $M(s)$ satisfying the interpolation and degree conditions is guaranteed.

One more condition on $M(s)$ is required for also assuring properness of the compensator i.e., $C(\infty, \underline{\alpha}, \overline{\alpha}) < \infty$. Since the numerator and the denominator of $C(s)$ in (6.9) are coprime, they cannot be zero simultaneously at $s = \infty$. Therefore the requirement for properness is,

$$M(\infty) \neq \frac{N_p(\infty, \overline{\alpha})}{N_p(\infty, \underline{\alpha})} \tag{6.25}$$

Condition (6.25) is irrelevant when the plant is strictly proper i.e., $P(\infty, \alpha) = 0$. In this case $N_p(\infty, \alpha) = 0$ and $Y(\infty, \alpha) \neq 0$ since $N_p(s, \alpha)$ and $Y(s, \alpha)$ are coprime, therefore $C(\infty, \alpha) < \infty$ in (6.3).

For illustration consider two strictly proper and stable plants $P(s, \underline{a}_i)$ and $P(s, \overline{a}_i)$. A wise choice for $p(s)$ in (6.10) is $p(s) = d(s)$. This leads to $N_p(s, a_i) = P(s, a_i)$; $D_p(s, a_i) = 1$; $X(s, a_i) = 0$; $Y(s, a_i) = 1$. Expression (6.14) simplifies to,

$$R(s, \underline{a}_i) = \frac{[1 - M(s)]d(s)}{K s^{m-i}(\underline{a}_i - \overline{a}_i)} \tag{6.26}$$

Since $X(s, a_i) = 0$ we have $x_n(s, a_i) = 0$ and we may choose $x_d(s, a_i) = 1$. Therefore the interpolation conditions (6.15) reduce to,

$$m_n(s) - m_d(s) = s^{m-i}q(s) \quad \text{with } q(s) \text{ an arbitrary polynomial} \tag{6.27}$$

The degree conditions on $M(s)$ are given by (6.16). Since the plants are strictly proper the stabilizing controllers will be implementable.

As a final remark in this section note that by writing $M(s)$ in a polynomial fractional representation,

$$M(s) = \frac{m_n(s)}{m_d(s)} = \frac{x_0 s^m + x_1 s^{m-1} + \ldots + x_{m-1} s + x_m}{y_0 s^m + y_1 s^{m-1} + \ldots + y_{m-1} s + y_m} \tag{6.28}$$

interpolation conditions for particular cases can be translated to requirements on the coefficients of (6.28). For example, (6.27) becomes,

$$m_n(s) - m_d(s) = s^{m-i} q(s) \Longleftrightarrow y_j = x_j \text{ for all } j \geq i+1 \tag{6.29}$$

For all cases, the degree conditions translate to,

$$\delta[m_d(s)] - \delta[m_d(s) - m_n(s)] \geq \ell \Longleftrightarrow y_j = x_j \text{ for all } j \leq \ell - 1 \tag{6.30}$$

with $\ell \leq m$.

6.3 Stabilization for All Values of the Uncertain Parameter

So far we have shown a parametrization of all the controllers that simultaneously stabilize $P(s, \underline{\alpha})$ and $P(s, \overline{\alpha})$. Simultaneous stabilizability imposes certain interpolation and degree conditions upon the controller parameter $M(s)$ which also must be a unit in $\mathbb{R}\mathrm{H}_\infty$. For proper but not strictly proper plants an additional condition on $M(\infty)$ is required for insuring properness of the compensators.

Our goal in this section is to extend the results of Section 6.2 to simultaneous stabilization of an infinite set of plants $P(s, \alpha)$ with $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. As expected, further conditions on the controller parameter $M(s)$ are required. However, it turns out these conditions are mild and easy to verify. We summarize the additional conditions for robust stability in the following theorem.

Theorem 6.2 *Consider the family of rational proper plants (6.1). Assume that one of the coefficients α is only known to lie in a given interval $[\underline{\alpha}, \overline{\alpha}]$. Consider*

also the set of all proper controllers (6.9) that simultaneously stabilize $P(s,\underline{\alpha})$ *and* $P(s,\overline{\alpha})$*, parametrized by a unit* $M(s)$ *satisfying the interpolation, degree and properness conditions. Then, the controllers stabilize* $P(s,\alpha)$ *for all* $\alpha \in [\underline{\alpha},\overline{\alpha}]$ *if and only if, the characteristic equation,*

$$1 + \eta \cdot M(s) = 0 \tag{6.31}$$

is strictly Hurwitz for all $\eta \geq 0$.

Proof: Let $N_c(s,\underline{\alpha}) = X(s,\underline{\alpha}) + R(s,\underline{\alpha})D_p(s,\underline{\alpha})$ and $D_c(s,\underline{\alpha}) = Y(s,\underline{\alpha}) - R(s,\underline{\alpha})N_p(s,\underline{\alpha})$. Then, by (6.3), (6.4), (6.5) and (6.6), the set of all proper compensators that simultaneously stabilize $P(s,\underline{\alpha})$ and $P(s,\overline{\alpha})$ is given by

$$C(s) = N_c(s,\underline{\alpha}) \cdot D_c^{-1}(s,\underline{\alpha}) \tag{6.32}$$

where, $N_c(s,\underline{\alpha})$ and $D_c(s,\underline{\alpha})$ are coprime, $R(s,\underline{\alpha})$ is given by (6.8) with $M(s)$ a unit in $\mathbb{RH}_\infty$ satisfying all the assumptions. It is well known [69], [71] that for $C(s)$ in (6.32) to internally stabilize $P(s,\alpha) = N_p(s,\alpha)D_p^{-1}(s,\alpha)$ with $N_p(s,\alpha)$ and $D_p(s,\alpha)$ coprime, a necessary and sufficient condition is that,

$$U(s,\alpha) = N_c(s,\underline{\alpha})N_p(s,\alpha) + D_c(s,\underline{\alpha})D_p(s,\alpha) \tag{6.33}$$

is a unit in $\mathbb{RH}_\infty$ for all $\alpha \in [\underline{\alpha},\overline{\alpha}]$. Replacing the expressions for $N_c(s,\underline{\alpha})$ and $D_c(s,\underline{\alpha})$ with $R(s,\underline{\alpha})$ given by (6.8) in (6.33) yields,

$$\begin{aligned} U(s,\alpha) = \{D_p(s,\underline{\alpha})N_p(s,\overline{\alpha}) - D_p(s,\overline{\alpha})N_p(s,\underline{\alpha})\}^{-1}\{M(s)[D_p(s,\underline{\alpha})N_p(s,\alpha) - \\ D_p(s,\alpha)N_p(s,\underline{\alpha})] + [D_p(s,\alpha)N_p(s,\overline{\alpha}) - D_p(s,\overline{\alpha})N_p(s,\alpha)]\} \end{aligned} \tag{6.34}$$

For $\alpha = a_i,\ i \in [1,m]$ we substitute (6.10) and (6.13) into (6.34) to obtain,

$$U(s,a_i) = \frac{\overline{a}_i - a_i}{\overline{a}_i - \underline{a}_i}[1 + \frac{a_i - \underline{a}_i}{\overline{a}_i - a_i}M(s)] \tag{6.35}$$

Denoting $\eta = (a_i - \underline{a}_i)(\overline{a}_i - a_i)^{-1}$, we realize that as a_i varies in the interval $[\underline{a}_i,\overline{a}_i]$, η takes all values in the interval $[0,\infty]$. Therefore from (6.35), an equivalent condition for $U(s,a_i)$ to be a unit in $\mathbb{RH}_\infty$ for all $a_i \in [\underline{a}_i,\overline{a}_i]$ is,

$$1 + \eta \cdot M(s) = 0$$

strictly Hurwitz for all $\eta \geq 0$.

Using the same arguments, the other two cases lead to the same condition. •

When dealing with a single controller which is known to stabilize $P(s, \underline{\alpha})$ and $P(s, \overline{\alpha})$ a Root-Locus plot of the system with α as the parameter will provide the answer to whether the compensator also stabilizes $P(s, \alpha)$ for all values of $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. In the case of all stabilizing controllers a similar Root-Locus condition imposed upon the unit $M(s)$ is obtained through (6.31). Since $M(s)$ is a unit, condition (6.31) is satisfied by numerous families of functions, e.g., units that are also passive (positive) functions, functions whose imaginary part does not change sign at all frequencies, functions which exhibit certain pole-zero interlacing properties, etc..

6.4 Generalization to Several Unknown Parameters

So far uncertainty in only one parameter has been considered. For various classes of plants the results seem to be simple enough to be used in control optimization procedures (especially when the interpolation conditions are not too restrictive). It turns out that a further generalization of the results can be obtained, at the expense of the simplicity of the conditions on the controller free parameter.

To illustrate the multiparameter case we will consider a particular class of plants mentioned in Section 6.2. The class consists of plants (6.1) which are stable (but possibly nonminimum phase) and that have a vector $\mathbf{a}$ of unknown parameters composed of elements $\{a_i, i \in \mathcal{I}\}$ with $\mathcal{I} = \{i; i \in [1, m]\}$, such that $n(s) = n(s, \mathbf{a})$. By Theorem 6.1, every pair of plants from this class is simultaneously stabilizable. As in Section 6.2 we choose $p(s) = d(s)$ in the coprime representation (6.10) so as to obtain,

$$N_p(s, \mathbf{a}) = P(s, \mathbf{a}); \quad D_p(s) = 1; \quad X(s) = 0; \quad Y(s) = 1 \tag{6.36}$$

To obtain interpolation and degree conditions on the unit $M(s)$ we first note that,

$$n(s,\overline{\mathbf{a}}) - n(s,\underline{\mathbf{a}}) = K\sum_{i\in\mathcal{I}}[\overline{a}_i - \underline{a}_i]s^{m-i} \tag{6.37}$$

Therefore expression (6.26) will be,

$$R(s,\underline{\mathbf{a}}) = \frac{[1-M(s)]d(s)}{K\sum\limits_{i\in\mathcal{I}}[\overline{a}_i - \underline{a}_i]s^{m-i}} \tag{6.38}$$

From (6.38) the interpolation conditions are,

$$m_n(s^*) - m_d(s^*) = 0 \tag{6.39}$$

for all s^* with Real $(s^*) \geq 0$ such that $\sum\limits_{i\in\mathcal{I}}[\overline{a}_i - \underline{a}_i]s^{*^{m-i}} = 0$ (including multiplicities).

The degree conditions are,

$$\delta[m_d(s)] - \delta[m_d(s) - m_n(s)] \geq n - m + i^* \tag{6.40}$$

with $i^* = \min\{i; i \in \mathcal{I}\}$.

To obtain the conditions for robust stabilization we substitute (6.36) and (6.37) into (6.34) to obtain,

$$U(s,\mathbf{a}) = \frac{M(s)\sum\limits_{i\in\mathcal{I}}[a_i - \underline{a}_i]s^{m-i} + \sum\limits_{i\in\mathcal{I}}[\overline{a}_i - a_i]s^{m-i}}{\sum\limits_{i\in\mathcal{I}}[\overline{a}_i - \underline{a}_i]s^{m-i}} \tag{6.41}$$

For $U(s,\mathbf{a})$ to be a unit, both, the numerator and the denominator of (6.41) must be strictly Hurwitz. While the roots of the denominator polynomial are easily calculated, the stability of the numerator polynomial has to be verified for all possible combinations of the uncertain parameter values. Perhaps some version of Kharitonov's Theorem [80] can be used to check condition (6.41).

6.5 Geometry of the Closed Loop Maps

The set of closed loop maps which can be achieved by means of the stabilizing controllers (6.9) exhibits a very simple geometry. By properly choosing a

specific value of the unknown parameter of the plant for which control design will be carried out, closed loop maps will be affine in the controller parameter. This simple geometry is used in H_∞ design procedures to transform a minimizing problem into a model matching one [81].

Consider the compensators (6.9) and the plant $P(s,\alpha)$. It is straightforward to show that the closed loop transfer matrix $H(s)$ between the input vector $(u_1\ u_2)^T$ and the output vector $(e_1\ e_2)^T$ in Figure 6.1 is given by,

$$H(s) = \begin{pmatrix} [1+P(s,\alpha)C(s)]^{-1} & -P(s,\alpha)[I+C(s)P(s,\alpha)]^{-1} \\ C(s)[1+P(s,\alpha)C(s)]^{-1} & [I+C(s)P(s,\alpha)]^{-1} \end{pmatrix} \tag{6.42}$$

Substituting (6.2) and (6.9) into (6.42) yields,

$$H(s) = 1/U(s,\alpha)\cdot$$

$$\begin{pmatrix} D_p(s,\alpha)[N_p(s,\overline{\alpha}) - M(s)N_p(s,\underline{\alpha})] & -N_p(s,\alpha)[N_p(s,\overline{\alpha}) - M(s)N_p(s,\underline{\alpha})] \\ D_p(s,\alpha)[M(s)D_p(s,\underline{\alpha}) - D_p(s,\overline{\alpha})] & D_p(s,\alpha)[N_p(s,\overline{\alpha}) - M(s)N_p(s,\underline{\alpha})] \end{pmatrix} \tag{6.43}$$

where $U(s,\alpha)$ is given by expression (6.34). Since by (6.34) $U(s,\underline{\alpha}) = 1$, substituting $\alpha = \underline{\alpha}$ in (6.43) leads to closed loop maps affine in the controller parameter $M(s)$. The same conclusions are obtained when considering the maps from the inputs to the outputs $(y_1\ y_2)^T$ and when the maps are evaluated at $\alpha = \overline{\alpha}$.

6.6 Application Example

The ideas presented in this chapter will be illustrated by means of a design example. The example is intentionally simple to avoid calculations which may obscure the main ideas. However, it represents a realistic design case.

Consider an aerodynamic surface whose angle of attack is controlled by means of a suitable actuator-sensor pair. The surface is exposed to various aerodynamic conditions which cause the location of its center of pressure to change. However for all flight conditions the location of the center of pressure relative to the axis of rotation is such that the surface is aerodynamically unstable.

Under several simplifying assumptions, the equation of motion of the surface is,

$$J\ddot{\alpha}(t) = T(t) + k\alpha(t) \tag{6.44}$$

with

α —the surface angle of attack

J —the surface moment of inertia about its axis of rotation

T —torque exerted by the actuator

k —aerodynamic moment coefficient

The aerodynamic moment coefficient k is proportional to the distance between the center of pressure and the axis of rotation (x_{cp}), the air density (ρ) and the square of the wind velocity (v^2). At various flight conditions, x_{cp}, ρ and v are such that k may lie anywhere within the range of values $[\underline{k}, \overline{k}] = [1, 10]$. However the rate of change of the aerodynamic conditions is so slow that for control purposes the coefficients are assumed constant.

A single controller is to be designed so that the following specifications will be met for every flight condition,

(i) Tracking errors after sinusoidal inputs in the frequency range $[0, 1]$ rad/sec should be less than 5%.

(ii) Gain and phase margins should be at least $6db$ and about 40° respectively.

A block diagram of the closed loop system is depicted in Figure 6.2.

Applying the Laplace Transform to equation (6.44) and assuming for simplicity $J = 1$, the family of plants to be controlled is obtained,

$$P(s, k) = \frac{\alpha(s)}{T(s)} = \frac{1}{s^2 - k} \tag{6.45}$$

The range of k is determined by its known variation values and the required minimal gain margins. Therefore,

$$k \in [\hat{k}_1, \hat{k}_2] = [0.5\underline{k}, 2\overline{k}] = [0.5, 20] \tag{6.46}$$

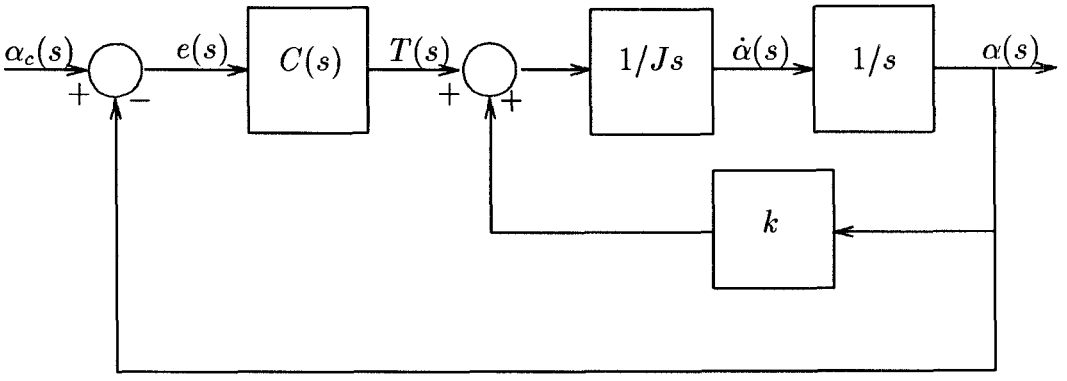


Figure 6.2: Surface's Closed Loop System

Notice that the k-span is 1:40.

By Theorem 6.1, any pair of distinct plants of the form (6.45) are simultaneously stabilizable. A fractional coprime representation of the plants (6.45) is,

$$\begin{aligned} &P(s,k) = N_p(s)D_p^{-1}(s,k) \\ &\text{with} \\ &N_p(s) = \frac{n(s)}{p(s)} = \frac{1}{p(s)}; \quad D_p(s,k) = \frac{d(s,k)}{p(s)} = \frac{s^2-k}{p(s)} \end{aligned} \tag{6.47}$$

where $p(s)$ is any stable polynomial with $\delta[p(s)] = 2$.

The parametrized controllers for robust stabilization are obtained by substituting (6.47) into (6.9),

$$C(s) = \frac{M(s)[s^2 - \hat{k}_1] - [s^2 - \hat{k}_2]}{1 - M(s)} \tag{6.48}$$

For simultaneous stabilization of $P(s,\hat{k}_1)$ and $P(s,\hat{k}_2)$ with a proper controller, $M(s)$ has also to satisfy the interpolation and degree conditions (6.19) and (6.20). Since for the plants (6.45) $n(s) = 1$ and $j = n = 2$, the only requirement resulting from these conditions is,

$$\delta[m_d(s)] - \delta[m_n(s) - m_d(s)] \geq 2 \tag{6.49}$$

Necessary and sufficient conditions for simultaneous stabilization of all plants $P(s,k) \quad k \in [\hat{k}_1, \hat{k}_2]$ are given by Theorem 6.2. I.e., it is required that the

characteristic equation corresponding to,

$$1 + \eta \cdot M(s) = 0 \tag{6.50}$$

is strictly Hurwitz for all $\eta \geq 0$.

To summarize, if we can find a unit $M(s)$ satisfying (6.49) and (6.50), then the $C(s)$ given by (6.48) will stabilize $P(s,k)$ for all $k \in [\hat{k}_1, \hat{k}_2]$.

This is as far as the results of the chapter can take us. To illustrate their utility we next use them in designing a controller for this uncertain plant.

One possible approach to the design problem is to minimize a constrained sensitivity function of the form,

$$S_W(s) = \frac{e(s)}{\alpha_c(s)} + W(s)\frac{T(s)}{\alpha_c(s)} = [1 + C(s)P(s)]^{-1} + W(s)C(s)[1 + C(s)P(s)]^{-1} \tag{6.51}$$

so as to obtain an implementable controller which stabilizes the set of plants (6.45) for all $k \in [0.5, 20]$ and leads to closed loop systems satisfying specifications (i) and (ii). Closed loop stability is guaranteed provided the controller is of the form (6.48) with $M(s) \not\equiv 1$ a unit in $\mathbb{RH}_\infty$ satisfying (6.49) and (6.50).

In (6.51), $W(s)$ represents a (frequency dependent) weighting factor between the closed loop sensitivity function and the control effort. This weight determines the bandwidth and the low frequency gains of the loop required for tracking as well as the phase margin of the system. $W(s)$ may also be devised to make the system robust against high frequency unstructured model uncertainties. This will be the case if, beyond crossover, the weight imposes fast frequency roll-off on the open loop system.

In the process of optimal control design one would define the weight $W(s)$ in (6.51) according to the system specifications and then apply a particular algorithm to choose a controller $C(s)$ which minimizes some norm of $S_W(s)$. In our case we would expect such an algorithm to choose from among all units $M(s) \not\equiv 1$ in $\mathbb{RH}_\infty$ satisfying (6.49) and (6.50) one which minimizes (6.51) in some sense. However there exists no algorithm yet for doing this. Therefore,

for illustration purposes we will design the system as follows: First, we will assume that a complete decoupling between the input $\alpha_c(s)$ and the weighted output $e(s) + W(s)T(s)$ is possible i.e., $S_W(s) = 0$ may be attained for all complex frequencies s. Second, we will choose a weight $W(s)$ which together with the complete decoupling condition results in a controller producing closed loop systems satisfying specifications (i) and (ii). Finally we will verify that the corresponding controller parameter $M(s)$ is a unit in $\mathbb{RH}_\infty$ satisfying conditions (6.49) and (6.50) i.e., the controller indeed stabilizes the set of all plants (6.45) with $k \in [0.5, 20]$.

Before proceeding with the design we show an interesting feature resulting from the complete decoupling assumption for general (scalar) plants and performance measure (6.51). The set of all stabilizing controllers for a general scalar plant with coprime representation (6.2) is given by (6.3). Substituting (6.2) and (6.3) into the constrained sensitivity function (6.51) yields,

$$S_W(s) = \{Y(s) + W(s)X(s) - R(s)[N(s) - W(s)D(s)]\}D(s) \tag{6.52}$$

For complete decoupling between the inputs $\alpha_c(s)$ and the weighted output $e(s) + W(s)T(s)$ it is required that $S_W(s) = 0$ for all complex frequencies s. By (6.52) this is attained when the controller parameter,

$$R(s) = \frac{Y(s) + W(s)X(s)}{N(s) - W(s)D(s)} \tag{6.53}$$

is in $\mathbb{RH}_\infty$. Substituting (6.53) into the expression for all stabilizing controllers (6.3) gives,

$$C(s) = -\frac{1}{W(s)} \tag{6.54}$$

Therefore the weighting function yields the same controller for all plants that admit a coprime factorization for which $R(s) \in \mathbb{RH}_\infty$ in (6.53).

In the case of simultaneous stabilization we need $S_W(s)$ in (6.51) or (6.52) to be affine in the unit $M(s)$. By the results of Section 6.5 this is attained for $\alpha = \underline{\alpha}$. Equating (6.48) and (6.54) gives,

$$M(s) = \frac{W(s)(s^2 - \hat{k}_2) - 1}{W(s)(s^2 - \hat{k}_1) - 1} \tag{6.55}$$

With the controller (6.54), the sensitivity function $S(s) = e(s)/\alpha_c(s)$ for the family of plants (6.45) is given by,

$$S(s) = \frac{1}{1 + P(s,k)C(s)} = \frac{W(s)(s^2 - k)}{W(s)(s^2 - k) - 1} \tag{6.56}$$

To meet the sensitivity requirements corresponding to specification (i) we impose,

$$|S(j\omega)| = \left| \frac{W(j\omega)(\omega^2 + k)}{W(j\omega)(\omega^2 + k) + 1} \right| \leq 0.05 \quad \omega \in [0,1]\frac{\text{rad}}{\text{sec}}, \quad k \in [1,10] \tag{6.57}$$

Taking $|W(j\omega)(\omega^2 + k)| \ll 1$ in the above frequency range and considering the worst case, $k = 10$, we obtain from (6.57),

$$|W(j\omega)| = 0.005 \qquad \omega \in [0,1]\frac{\text{rad}}{\text{sec}} \tag{6.58}$$

Suppose now $|W(j\omega)| = 0.005$ for all real frequencies ω. Then we can evaluate the magnitude of the system's open loop transfer function as a function of the parameter k,

$$|P(j\omega,k)C(j\omega)| = \frac{1}{0.005(\omega^2 + k)} \tag{6.59}$$

Letting $|P(j\omega,k)C(j\omega)| = 1$ in (6.59) we obtain the range of the crossover frequencies $\omega_{c.o.}(k)$,

$$\omega_{c.o.}(k) \in [13.78, 14.1]\frac{\text{rad}}{\text{sec}} \qquad k \in [1,10] \tag{6.60}$$

From (6.60) we realize that the open loop crossover frequency range is very small. This enables us to add some dynamics to the weighting function $W(s)$ to improve the poor phase margin that is obtained by setting $W(s) =$ constant. To satisfy specification (ii) we choose,

$$W(s) = -0.005\frac{s/200 + 1}{s/14.5 + 1} \tag{6.61}$$

Substituting (6.61) and into (6.55) gives,

$$M(s) = \frac{s^3 + 200s^2 + 2739s + 36000}{s^3 + 200s^2 + 2758s + 39900} \tag{6.62}$$

It can be easily checked that $M(s)$ in (6.62) is a unit in $\mathbb{RH}_\infty$. With the aid of the alternative degree condition (6.30) we may also verify that (6.49) is satisfied. Moreover, it is not difficult to check that the condition for robust stability (6.50) is also satified. Thus, the controller $C(s) = -1/W(s)$ robustly stabilizes the family of plants (6.45) with $k \in [\hat{k}_1, \hat{k}_2] = [0.5, 20]$. Due to our choice of $W(s)$, specifications (i) and (ii) are satisfied for all $k \in [\underline{k}, \overline{k}] = [1, 10]$.

CHAPTER
SEVEN

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

In this thesis we have studied some issues in robust control. First we have explored a loop by loop approach for robustness assessment. We have found that this approach can partially explain the inadequacy of some robustness tests, the potential lack of robustness of designs based on diagonal dominance, and the way singular values operate to provide a complete robustness assessment. We believe that this approach should be further explored since clearly it is fundamental for understanding relationships between directionality and robustness and also for devising practical methods for measuring multivariable stability margins. The development of robust control design methods based on a loop by loop approach is also highly desirable especially for systems with a small number of inputs and outputs (such as those found in aerospace applications). Second, we have considered norm bounded structured uncertainty. We have used an approach which is oriented to provide a better understanding of the subject and to facilitate robustness analysis and design at several stages of systems' development. We have used a relatively simple representation of uncertainty and basic analysis tools. We have tackled the problem using a perturbation approach which allows us to exploit properties of matrices of rank one. We have shown that using this approach simpler and more intuitive proofs of robustness theorems can sometimes

be obtained and that some practical, recursive robustness bounds can be developed. Due to the simple representation of uncertainty considered, we have been able to perform analysis using the original system structure. There is no need to transform the system, as analysis in the structured singular value framework requires, into diagonal uncertainty form. Thus, the dimensionality of the problem does not increase. It will be interesting to try this approach for other representations of uncertainty. This may lead to simpler proofs of other results of the structured singular value. The explicit derivation of the analytical expressions involving only individual closed loop maps which result from the bounds on the maximal spectral radius should also be carried out. These expressions should be obtained for cases involving several simultaneous uncertainties. One way to do this is by using symbolic manipulation routines. The use of the perturbation approach for the exact computation of the maximal spectral radius, at least for some cases, should also be investigated. Finally, ways of incorporating the robustness information provided by the bounds into control design procedures should be explored.

A second issue that has been considered in this thesis is the parametrization of all stabilizing controllers for plants with uncertain parameters. We have used the fractional approach to study this problem. The single uncertain parameter SISO case has been covered completely. Also, the complications expected in the multiparameter case have been mentioned. A complete analysis for the multiparameter case still needs to be carried out and the generalization of the method for MIMO systems should be explored. Finally, it remains to find how to modify optimization routines so that they take into account the constraints imposed by the simultaneous stabilizability conditions on the controller parameter.

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