TECHNICAL RESEARCH REPORT

Tail Probabilities for $M|G|\infty$ Input Processes (I): Preliminary Asymptotics

by M. Parulekar, A.M. Makowski

T.R. 96-41

ISR
INSTITUTE FOR SYSTEMS RESEARCH

Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry

Tail probabilities for $M|G|\infty$ input processes (I): Preliminary asymptotics

MINOTHI PARULEKAR †
minothi@eng.umd.edu
(301) 405-2948

ARMAND M. MAKOWSKI ‡
armand@eng.umd.edu
(301) 405-6648
FAX:(301) 314-9281

Abstract

The infinite server model of Cox with arbitrary service time distribution appears to provide a very large class of traffic models – Pareto and log-normal distributions have already been reported in the literature for several applications. Here we begin the analysis of the large buffer asymptotics for a multiplexer driven by this class of inputs. To do so we rely on recent results by Duffield and O'Connell on overflow probabilities for the general single server queue. In this paper we focus on the key step in this approach which is based on large deviations: The appropriate large deviations scaling is shown to be related to the forward recurrence time for the service time distribution, and a closed form expression is derived for the corresponding generalized limiting log–moment generating function associated with the input process. Two very different regimes are identified. In a companion paper we apply these results to obtain the large buffer asymptotics under a variety of service time distributions.

Key words: Infinite server queue; Large deviations; Tail probabilities; Forward recurrence times; scalings.

†Electrical Engineering Department and Institute for Systems Research, University of Maryland, College Park, MD 20742. The work of this author was supported through NSF Grant NSFD CDR-88-03012

‡Electrical Engineering Department and Institute for Systems Research, University of Maryland, College Park, MD 20742. The work of this author was supported partially through NSF Grant NSFD CDR-88-03012 and through NASA Grant NAGW277S.

1 Introduction

The discrete–time $M|G|\infty$ input processes discussed in this paper can be described as follows: Time is slotted and customers arrive according to a (discrete–time) "Poisson" process. Upon arrival customers are offered to an infinite server group, and the required service times are i.i.d. finite mean rvs – let σ denote the generic service time random variable (expressed in number of time slots). The $M|G|\infty$ input process is then the process $\{b_t,\ t = 0, 1, \ldots\}$ that counts the number of busy servers at the beginning of time slots.

Interest in this class of models stems from the increasing realization that Poisson modeling (and its natural extensions) fails to capture long–range dependence effects, including (asymptotic) self–similarity, which have been detected in traffic measurements for a wide range of networking applications, e.g., Ethernet LANs [9, 13, 23], VBR traffic [3], WAN traffic [10, 22]. These measurements are burstier at many time scales than predicted by Poisson models. This finding has implications for congestion control and traffic performance as already demonstrated in the references [1, 8, 17] for alternative models based on fractional Gaussian noise and fractional Brownian motion.

The class of $M|G|\infty$ traffic models is a versatile one as it accounts for a large range of *positive* auto–correlation structures; in fact the process $\{b_t,\ t = 0, 1, \ldots\}$ can be shown to be associated in the sense that the rvs $b_0, \ldots, b_t$ form a set of associated rvs for all $t = 0, 1, \ldots$ [7]. Interestingly enough, $M|G|\infty$ processes were mentioned by Cox in [4] as an example of a long range dependent process. This occurs when σ has a discrete Pareto distribution with parameter α, $1 < \alpha < 2$, in which case the stationary version of the process $\{b_t,\ t = 0, 1, \ldots\}$ is an asymptotically self–similar process with Hurst parameter $H = (3 - \alpha)/2$. In [15] Likhanov, Tsybakov and Georganas construct an aggregate traffic model by superposing a large number of on–off sources with Pareto distributed activity periods, and show that in the limit the model is nothing else but the $M|G|\infty$ model of Cox. In a different context [22], Paxson and Floyd have found that the $M|G|\infty$ model with an (integer) log–normal service time σ matches reasonably well some wide area applications (e.g., telnet connections [22, p. 235]).

More generally, $M|G|\infty$ input processes can display time dependencies over a wide range of time scales, the extent of which is controlled by the tail behavior of the distribution of σ. In line with the findings of several authors [14] in different contexts, the temporal correlations in $M|G|\infty$ input processes are expected to have a significant impact on queueing performance when such processes are offered to a muliplexer. To gain some insights into this issue we consider a discrete–time single server queue with infinite capacity and constant release rate of c cells/slot under the first–come first–served discipline, as a surrogate for a multiplexer, and feed it with the traffic stream $\{b_t,\ t = 0, 1, \ldots\}$: Let q_t denote the number of cells remaining in the buffer by the end of slot $[t-1, t)$, and let b_{t+1} denote the number of new cells which arrive at the start of time slot $[t, t+1)$. If the multiplexer output link can transmit c cells/slot, then the buffer content sequence $\{q_t,\ t = 0, 1, \ldots\}$ evolves according to the Lindley recursion

$$q_0 = q; \quad q_{t+1} = [q_t + b_{t+1} - c]^+, \quad t = 0, 1, \ldots \tag{1.1}$$

for some initial condition q.

It is well known [16] that the multiplexer will reach statistical equilibrium if $\lambda \mathbf{E}[\sigma] < c$, in which case $q_t \Longrightarrow_t q_\infty$ for some honest rv q_∞ which represents the steady–state buffer content at the multiplexer. Of considerable interest are the tail probabilities $\mathbf{P}[q_\infty > b]$ for large b as a means to estimating buffer overflow probabilities for the corresponding finite buffer system. Such asymptotics are often the first guiding step to size up the buffer at the multiplexer in order to guarantee quality of service requirements.

The rv q_∞ can be represented as

$$q_\infty =_{st} \sup\{S_t - ct,\ t = 0, 1, \ldots\} \tag{1.2}$$

with

$$S_0 = 0; \quad S_t = b_1^\star + \ldots + b_t^\star, \quad t = 1, 2, \ldots \tag{1.3}$$

where $\{b_t^\star,\ t = 0, 1, \ldots\}$ is the *stationary* version of the busy server process. With the representation (1.2) for q_∞ as a point of departure, several authors [6, 11, 12] have derived estimates on the tail probabilities by means of large deviations estimates for the sequence $\{t^{-1}(S_t - ct),\ t = 0, 1, \ldots\}$. Asymptotic lower and upper bounds have both been derived in varying degrees of generality. Invariably the key step consists of finding two monotone increasing $\mathbb{R}_+$–valued sequences $\{v_t,\ t = 0, 1, \ldots\}$ and $\{a_t,\ t = 0, 1, \ldots\}$ increasing at infinity, i.e., $\lim_{t\to\infty} v_t = \lim_{t\to\infty} a_t = \infty$, such that the limit

$$\Lambda(\theta) \equiv \lim_{t\to\infty} \Lambda_t(\theta), \quad \theta \in \mathbb{R} \tag{1.4}$$

exists (possibly as an extended real number), where for each $t = 1, 2, \ldots$, we have set

$$\Lambda_t(\theta) \equiv \frac{1}{v_t} \ln \mathbf{E} \left[\exp \left(\theta \frac{v_t}{a_t} (S_t - ct) \right) \right], \quad \theta \in \mathbb{R}. \tag{1.5}$$

Of course, the limiting function $\Lambda : \mathbb{R} \to [0, \infty]$ is expected to satisfy various properties [6, 11, 12]. Leaving this issue aside for the time being, we focus here on identifying the scaling sequences $\{v_t,\ t = 0, 1, \ldots\}$ and $\{a_t,\ t = 0, 1, \ldots\}$ that lead to a non-trivial limit (1.4). This was done for Pareto service time distributions in [18, 19], but it is by no means obvious at the outset how to identify the scalings for a general service time distribution.

The main results are now described qualitatively; precise statements are available in Section 3: If we choose $a_t = t$, then the appropriate scaling turns out to be given by

$$v_t \equiv -\ln \mathbf{P} [\widehat{\sigma} > t], \quad t = 1, 2, \ldots \tag{1.6}$$

where $\widehat{\sigma}$ is the forward recurrence associated with the service time rv σ, and has distribution

$$\widehat{g}_r \equiv \mathbf{P} [\widehat{\sigma} = r] = \frac{\mathbf{P} [\sigma \geq r]}{\mathbf{E} [\sigma]}, \quad r = 1, 2, \ldots \tag{1.7}$$

To state the results more conveniently, we set

$$\Lambda_{b,t}(\theta) \equiv \frac{1}{v_t} \ln \mathbf{E} \left[\exp(\frac{v_t}{t} \theta S_t) \right], \quad \theta \in \mathbb{R} \tag{1.8}$$

for each $t = 1, 2, \ldots$. Obviously, if the limit

$$\Lambda_b(\theta) \equiv \lim_{t \to \infty} \Lambda_{b,t}(\theta), \quad \theta \in \mathbb{R} \tag{1.9}$$

exists, so does (1.4) with

$$\Lambda(\theta) = \Lambda_b(\theta) - c\theta, \quad \theta \in \mathbb{R} \tag{1.10}$$

and it suffices to concentrate on finding (1.9).

Our first result in that direction [Theorem 3.1] is that we always have

$$\lim_{t \to \infty} \Lambda_{b,t}(\theta) = \infty, \quad \theta > 1. \tag{1.11}$$

For the range $\theta < 1$, the asymptotics depend on whether $v_t = o(t)$ or $v_t = O(t)$.

If $v_t = O(t)$ with $\lim_{t \to \infty} v_t / t = C > 0$, then we show [Theorem 3.2] that

$$\Lambda_b(\theta) \equiv \lim_{t \to \infty} \Lambda_{b,t}(\theta) = \lambda \mathbf{E} [\sigma] \left(\frac{e^{C\theta} - 1}{C} \right) \Sigma(\theta), \quad \theta < 1 \tag{1.12}$$

for some finite quantity $\Sigma(\theta)$ given by (3.3) which depends on G. It is easy to check that the condition $v_t = O(t)$ is tantamount to G having an exponential tail.

When $v_t = o(t)$ the situation is technically more involved, and additional growth assumptions are required on the scaling sequence $\{v_t,\ t = 1, 2, \ldots\}$. Under the appropriate conditions [Theorem 3.3] we prove that

$$\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta) = \lambda \mathbf{E}[\sigma]\,\theta, \quad \theta < 1. \tag{1.13}$$

The set of conditions under which (1.13) holds are satisfied by many classical distributions. This shown in the companion paper [20] where we apply the results of Theorems 3.2 and 3.3 to obtain asymptotic lower bounds on the tail probabilities $\mathbf{P}[q_\infty > b]$ for large b for a variety of choices of σ, including the geometric, Pareto, log–normal and Weibull distributions (or their natural analogs on $I\!N$).

Several remarks are in order concerning these results: The case $\theta = 1$ appears to depend crucially on the pmf G. In the case $v_t = o(t)$ the limit (1.9) depends on the distribution of σ *only* through its mean, in sharp contrast to the case $v_t = O(t)$ where that limit depends on the *entire* distribution of σ. The temporal correlations of $M|G|\infty$ input processes are controlled by the tail behavior of the distribution of σ. This is made more apparent through the relation

$$\mathrm{cov}[b_t^\star, b_{t+h}^\star] = \lambda \mathbf{E}[\sigma]\, e^{-v_h}, \quad h = 1, 2, \ldots \tag{1.14}$$

for the covariance function of the stationary version $\{b_t^\star,\ t = 0, 1, \ldots\}$ [Lemma 4.1].

The paper is organized as follows: The $M|GI|\infty$ input processes are formally introduced in Section 2 together with some useful facts concerning them. The main results are stated in Section 3, and additional facts concerning the correlation structure of the $M|GI|\infty$ input processes are discussed in Section 4. The analysis begins with the preliminary expressions for (1.8) in Section 5; proofs of the necessary technical steps are available in Sections 9 and 10. The proof of (1.11) is given Section 6; the limits (1.12) and (1.13) are established in Sections 7 and 8, respectively.

A few words on the notation used in this paper: All rvs are defined on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, with $\mathbf{E}$ denoting the corresponding expectation operator. Two rvs X and Y are said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. Weak convergence is denoted by $\Longrightarrow$.

2 $M|GI|\infty$ input processes

We summarize various facts concerning the busy server process $\{b_t,\ t=0,1,\ldots\}$ of a discrete–time $M|G|\infty$ system. Some of these facts are standard while others are discrete–time analogs of properties which are well known for the continuous–time infinite server queue; details are available in [21]:

Consider a system with infinitely many servers. During time slot $[t, t+1)$, β_{t+1} new customers arrive into the system. Customer i, $i = 1, \ldots, \beta_{t+1}$, is presented to its own server and begins service by the start of slot $[t+1, t+2)$; its service time has duration $\sigma_{t+1,i}$. Let b_t denote the number of busy servers, or equivalently of customers still present in the system, at the beginning of slot $[t, t+1)$, with b denoting the number of busy servers initially present in the system at $t = 0$.

The $\mathbb{N}$–valued rvs b, $\{\beta_{t+1},\ t = 0, 1, \ldots\}$ and $\{\sigma_{t,i},\ t = 0, 1, \ldots;\ i = 0, 1, \ldots\}$ satisfy the following assumptions: (i) The rvs are mutually independent; (ii) The rvs $\{\beta_{t+1},\ t = 0, 1, \ldots\}$ are *i.i.d.* Poisson rvs with parameter $\lambda > 0$; (iii) The rvs $\{\sigma_{t,i},\ t = 1, \ldots;\ i = 1, 2, \ldots\}$ are *i.i.d.* with common pmf G on $\{1, 2, \ldots\}$. We denote by σ a generic $\mathbb{N}$–valued rv distributed according to the pmf G. Throughout we shall assume that this pmf G has a finite first moment, or equivalently that $\mathbf{E}[\sigma] < \infty$.

No additional assumptions are made on the rvs $\{\sigma_{0,i},\ i = 1, 2, \ldots\}$ which represent the service durations of the b customers present in the system at the beginning of the slot $[0, 1)$, so that various scenarios can in principle be accommodated: If the initial customers start their service at time $t = 0$, then it is appropriate to assume that the rvs $\{\sigma_{0,i},\ i = 1, 2, \ldots\}$ are also i.i.d. rvs which are distributed according to the pmf G. On the other hand, if we take the viewpoint that the system has been in operation for some time, then these rvs $\{\sigma_{0,i},\ i = 1, 2, \ldots\}$ may be interpreted as the incomplete work (expressed in time slots) that the b "initial" customers require from their respective servers before their service is completed. In general, the statistics of the rvs $\{\sigma_{0,i},\ i = 1, 2, \ldots\}$ cannot be specified in any meaningful way, except for the situation when the system is in steady state.

We note that

$$b_t = b_t^{(0)} + b_t^{(a)}, \quad t = 0, 1, \ldots \tag{2.1}$$

where the rvs $b_t^{(0)}$ and $b_t^{(a)}$ describe the contributions to the number of customers in the system at the beginning of slot $[t, t+1)$ from those initially present (at $t = 0$)

and from the new arrivals, respectively. It is plain that

$$b_t^{(0)} = \sum_{i=1}^{b} \mathbf{1}\left[\sigma_{0,i} > t\right], \quad t = 0, 1, \ldots \tag{2.2}$$

and that the rv $b_t^{(a)}$ can also be interpreted as the number of busy servers in the system at the beginning of slot $[t, t+1)$ given that the system was initially empty (i.e., $b = 0$).

Although the busy server process $\{b_t,\ t = 0, 1, \ldots\}$ is in general *not* a (strictly) stationary process, it does admit a stationary and ergodic version in the sense now stated.

Proposition 2.1 *There exists a stationary and ergodic $\mathbb{N}$-valued process $\{b_t^\star,\ t = 0, 1, \ldots\}$ such that*

$$\{b_{t+k},\ t = 0, 1, \ldots\} \Longrightarrow \{b_t^\star,\ t = 0, 1, \ldots\} \quad (k \to \infty) \tag{2.3}$$

for any choice of the initial condition rv b and of the service times $\{\sigma_{0,i},\ i = 1, 2, \ldots\}$.

It can be shown [21] that this stationary version $\{b_t^\star,\ t = 0, 1, \ldots\}$ can represented through (2.2) with

$$b_t^{(0)} = \sum_{n=1}^{b} \mathbf{1}\left[\widehat{\sigma}_n > t\right], \quad t = 0, 1, \ldots \tag{2.4}$$

where (i) the rvs $\{\widehat{\sigma}_n,\ n = 1, 2, \ldots\}$ are independent of the rv b which is Poisson distributed with parameter $\lambda \mathbf{E}[\sigma]$, and (ii) the rvs $\{\widehat{\sigma}_n,\ n = 1, 2, \ldots\}$ are *i.i.d.* rvs distributed according to the forward recurrence time $\widehat{\sigma}$ associated with σ. This distribution is given by (1.7).

During the analysis we shall find it useful to give the $M|G|\infty$ system an alternative interpretation: Assume that on arrival customers declare the length of their service times. We shall then say that an arriving customer is of type r, $r = 1, 2, \ldots$, if it requires r units of service time (or slots). By keeping track of customer types, we can then view the original Poisson process $\{\beta_{t+1},\ t = 0, 1, \ldots\}$ as being the aggregate of an infinite number of arrival processes, say $\{\beta_{t+1}^r,\ t = 0, 1, \ldots\}$, $r = 1, 2, \ldots$, with β_{t+1}^r denoting the number of customers arriving in time slot $[t, t+1)$ with a service requirement of r slots. Obviously, we have

$$\beta_{t+1} = \sum_{r=1}^{\infty} \beta_{t+1}^r, \quad t = 0, 1, \ldots \tag{2.5}$$

Under the enforced independence assumptions, the arrival processes $\{\beta^r_{t+1},\ t = 0,1,\ldots\}$, $r = 1,2,\ldots$, are mutually independent, and for each $r = 1,2,\ldots$, the rvs $\{\beta^r_{t+1},\ t = 0,1,\ldots\}$ are i.i.d. Poisson rvs with parameter λg_r.

In short, instead of having a single Poisson arrival stream, we now have an infinite number of independent Poisson arrival streams, each feeding into an infinite server queue with a *deterministic* service time. Therefore, if b^r_t denotes the number of type r customers in the system at the beginning of slot $[t, t+1)$, we have

$$b^{(a)}_t = \sum_{r=1}^{\infty} b^r_t \tag{2.6}$$

with

$$b^r_t = \sum_{i=1}^{\min(r,t)} \beta^r_{(t-r)^+ + i}, \quad r = 1,2,\ldots \tag{2.7}$$

The sequence $\{b^r_t,\ t = 0,1,\ldots\}$, $r = 1,2,\ldots$, are mutually independent.

Using these facts it is quite easy to show the following properties of $\{b^\star_t,\ t = 0,1,\ldots\}$ [4, 5, 21].

Proposition 2.2 *The stationary and ergodic version $\{b^\star_t,\ t = 0,1,\ldots\}$ of the busy server process has the the following properties:*

1. *For each $t = 0,1,\ldots$, the rv $b^\star_t$ is a Poisson rv with parameter $\lambda \mathbf{E}[\sigma]$;*
2. *Its covariance structure is given by*

$$\Gamma(h) \equiv \mathrm{cov}[b^\star_t, b^\star_{t+h}] = \lambda \mathbf{E}\left[(\sigma - h)^+\right], \quad t,h = 0,1,\ldots \tag{2.8}$$

3 Main results

We begin with the asymptotics for $\theta > 1$. The result holds without any additional conditions on the scaling v_t, and is established in Section 6.

Theorem 3.1 *Under no additional assumptions, we always have*

$$\lim_{t\to\infty} \Lambda_{b,t}(\theta) = \infty, \quad \theta > 1. \tag{3.1}$$

In view of Theorem 3.1 it remains only to consider the case $\theta \leq 1$. However, for that range the result depends crucially on whether $v_t = O(t)$ or $v_t = o(t)$ as should be apparent from Theorems 3.2 and 3.3 below; their proofs can be found in Sections 7 and 8, respectively.

Theorem 3.2 *Assume $v_t = O(t)$ with $\lim_{t\to\infty} v_t/t = C > 0$. Then, for each $\theta \neq 1$ in $\mathbb{R}$, the limit $\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta)$ exists and is given by*

$$\Lambda_b(\theta) = \begin{cases} \lambda \mathbf{E}[\sigma] \left(\frac{e^{C\theta}-1}{C}\right) \Sigma(\theta) & \text{if } \theta < 1 \\ \infty & \text{if } \theta > 1, \end{cases} \tag{3.2}$$

where

$$\Sigma(\theta) = 1 + \left(1 - e^{-C\theta}\right) \left(\sum_{r=1}^{\infty} \exp\left(r(\theta C - \frac{v_r}{r}) \right) \right), \quad \theta \in \mathbb{R}. \tag{3.3}$$

Moreover, $\Sigma(\theta) < \infty$ for $\theta < 1$.

We say that the sequence $\{v_t/t,\ t = 1,2,\ldots\}$ is monotone decreasing (resp. increasing) in the limit if there exists a finite integer T such that the tail $\{v_t/t,\ t = T+1, T+2, \ldots\}$ is monotone decreasing (resp. increasing).

Theorem 3.3 *Assume $v_t = o(t)$ with $\{v_t/t,\ t = 1,2,\ldots\}$ monotone decreasing in the limit. Assume further that there exists a mapping $\Gamma : \mathbb{N} \to \mathbb{N}$ such that (i) $\Gamma(t) < t$ for all $t = 1,2,\ldots$, (ii) $\lim_{t\to\infty} v_t \frac{\Gamma(t)}{t} = \infty$ and (iii) $\lim_{t\to\infty} \frac{v_t}{t} \frac{\Gamma(t)}{v_{\Gamma(t)}} = 0$. Then, for each $\theta \neq 1$ in $\mathbb{R}$, the limit $\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta)$ exists and is given by*

$$\Lambda_b(\theta) = \begin{cases} \lambda \mathbf{E}[\sigma]\, \theta & \text{if } \theta < 1 \\ \infty & \text{if } \theta > 1. \end{cases} \tag{3.4}$$

No general result appears to hold for at the boundary point $\theta = 1$, but we suspect from various examples that $\theta \to \Lambda_b(\theta)$ is either left– or right– continuous [21]. We conclude with a result that complements Theorems 3.2 at the boundary point $\theta = 1$.

Theorem 3.4 *Under the assumptions of Theorem 3.2, we also have (3.1) for $\theta = 1$ if either (i) $v_t \leq Ct$ infinitely often or (ii) $v_t > Ct$ for $t = T, T+1, \ldots$ for some finite T and $\limsup_{t\to\infty}(v_t - Ct) = K$ for some finite $K \geq 0$.*

Conditions (i) and (ii) are non overlapping, and do cover most distributions of interest. However, Theorem 3.4 does not cover the situation in (ii) with $\limsup_{t\to\infty}(v_t - Ct) = \infty$. Indeed, with $C = 1$, for $v_t = t + \sqrt{t}$ we find $\Lambda_b(1) = \infty$, while for $v_t = t + \frac{t}{\ln t}$, we have $\Lambda_b(1) < \infty$. Details are available in [21].

4 Correlation structure

Before establishing these results, we make a slight detour discussing the correlation structure of the $M|G|\infty$ process. The first indication that the rvs $\{b_t,\ t=0,1,\ldots\}$ exhibit some form of dependence can already be traced to the fact that these rvs are indeed positively correlated in a strong sense: For all $t=0,1,\ldots$, we write $b^t \equiv (b_0, b_1, \ldots, b_t)$.

Proposition 4.1 *For any choice of the initial condition rv b and of the service times $\{\sigma_{0,i},\ i=1,2,\ldots\}$, the rvs $\{b_t,\ t=0,1,\ldots\}$ are associated, in that for any $t=0,1,\ldots$ and any pair of non–decreasing mappings $f,g : \mathbb{N}^{t+1} \to \mathbb{R}$,*

$$\mathbf{E}\left[f(b^t)g(b^t)\right] \geq \mathbf{E}\left[f(b^t)\right]\mathbf{E}\left[g(b^t)\right] \tag{4.1}$$

provided the expectations exist and are finite.

The notion of association was introduced by Esary, Proschan and Walkup in [7], and the reader is invited to consult this reference for additional material on the topic.

Proof. Recall that the collections of rvs $\{b_t^{(0)},\ t=0,1,\ldots\}$ and $\{b_t^{(a)},\ t=0,1,\ldots\}$ are independent. Hence, in view of (2.1), we need only show the association (4.1) for each of these two collections [7, (P2), p. 1467].

Fix $i=1,2,\ldots$. For each $t=0,1,\ldots$, we have $\mathbf{1}\left[\sigma_{0,i}>t\right] = f_t(\sigma_{0,i})$ for some non–decreasing mapping $f_t : \mathbb{R} \to \mathbb{R}$. It is now plain that the rvs $\{\mathbf{1}\left[\sigma_{0,i}>t\right],\ t=0,1,\ldots\}$ are associated as we recall that the rv $\sigma_{0,i}$ is associated by itself [7, (P4), p. 1467]. By independence, for each $n=1,2,\ldots$, the rvs $\{\sum_{i=1}^n \mathbf{1}\left[\sigma_{0,i}>t\right],\ t=0,1,\ldots\}$ are therefore associated [7, (P2),(P4), p. 1467], or to put it differently, the rvs $\{\sum_{i=1}^b \mathbf{1}\left[\sigma_{0,i}>t\right],\ t=0,1,\ldots\}$ are conditionally associated given b.

Next, for any $t=0,1,\ldots$ and any pair of non–decreasing mappings $f,g : \mathbb{N}^{t+1} \to \mathbb{R}$, we find from this last remark that

$$\begin{aligned}
\mathbf{E}\left[f(b^{(0),t})g(b^{(0),t})\right] &= \mathbf{E}\left[\mathbf{E}\left[f(b^{(0),t})g(b^{(0),t})|b\right]\right] \\
&\geq \mathbf{E}\left[\mathbf{E}\left[f(b^{(0),t})|b\right]\mathbf{E}\left[g(b^{(0),t})|b\right]\right] \\
&\geq \mathbf{E}\left[\mathbf{E}\left[f(b^{(0),t})|b\right]\right]\mathbf{E}\left[\mathbf{E}\left[g(b^{(0),t})|b\right]\right] \qquad (4.2)
\end{aligned}$$

and the desired conclusion on $\{b_t^{(0)},\ t=0,1,\ldots\}$ follows. The passage to (4.2) is a consequence of the fact that the rv b is associated, and of the non–decreasing character of the mappings $n \to \mathbf{E}\left[f(b^{(0),t})|b=n\right]$ and $n \to \mathbf{E}\left[g(b^{(0),t})|b=n\right]$.

For each $r = 1, 2, \ldots$, the rvs $\{b_t^r,\ t = 0, 1, \ldots\}$ are clearly associated by virtue of (2.7) and of the independence of the rvs $\{\beta_{t+1}^r,\ t = 0, 1, \ldots\}$ [7, (P4), Thm 2.1, p. 1467]. In view of (2.6) and property (P4) in [7, p. 1467], the association of the rvs $\{b_t^{(a)},\ t = 0, 1, \ldots\}$ now follows from that of the independent collections $\{b_t^r,\ t = 0, 1, \ldots\}$, $r = 1, 2, \ldots$ [7, (P2), p. 1467]. ■

From (4.1), we already get

$$\mathrm{cov}[b_t, b_{t+h}] \geq 0, \quad t, h = 0, 1, \ldots \tag{4.3}$$

On the other hand, by suitably selecting the initial conditions (2.4) we see that Proposition 4.1 holds for the stationary version $\{b_t^\star,\ t = 0, 1, \ldots\}$, and the expression (2.8) is clearly compatible with (4.3).

The strength of the positive correlation exhibited by the sequence $\{b_t^\star,\ t = 0, 1, \ldots\}$ can be formalized as follows: We say that the sequence $\{b_t^\star,\ t = 0, 1, \ldots\}$ exhibits *short range dependence* if

$$\sum_{h=0}^{\infty} \Gamma(h) < \infty. \tag{4.4}$$

Otherwise, the sequence $\{b_t^\star,\ t = 0, 1, \ldots\}$ is said to be *long range dependent* [2, 3]. As we now show, for $M|G|\infty$ processes this dependence can be partially characterized through the scaling $\{v_t,\ t = 1, 2, \ldots\}$.

Lemma 4.1 *We have*

$$\Gamma(h) = \lambda \mathbf{E}[\sigma]\, e^{-v_h}, \quad h = 1, 2, \ldots \tag{4.5}$$

Proof. Fix $h = 1, 2, \ldots$, and note that

$$\begin{aligned}
\Gamma(h) &= \lambda \mathbf{E}\left[(\sigma - h)^+\right] \\
&= \lambda \sum_{r=0} \mathbf{P}\left[(\sigma - h)^+ > r\right] \\
&= \lambda \sum_{r=0} \mathbf{P}\left[\sigma > h + r\right] \\
&= \lambda \sum_{r=h+1} \mathbf{P}\left[\sigma \geq r\right] \\
&= \lambda \mathbf{E}[\sigma] \sum_{r=h+1} \mathbf{P}\left[\widehat{\sigma} = r\right] \\
&= \lambda \mathbf{E}[\sigma]\, \mathbf{P}\left[\widehat{\sigma} > h\right]
\end{aligned}$$

and (4.5) follows from (1.6). ■

Consequently, if $v_t = O(t)$, then the process $\{b_t^\star,\ t = 0, 1, \ldots\}$ is short range dependent; in that case, using the fact

$$\mathbf{P}[\sigma > t] = \mathbf{E}[\sigma]\left(e^{-v_t} - e^{-v_{t+1}}\right), \quad t = 1, 2, \ldots \tag{4.6}$$

derived from (1.6)–(1.7), we readily see that G has an exponential tail. On the other hand, if $v_t = o(t)$, the situation is not as clear cut and the process $\{b_t^\star,\ t = 0, 1, \ldots\}$ can be either short or long range dependent. As we now show, this ambiguity is resolved through the finiteness of $\mathbf{E}\left[\sigma^2\right]$.

Proposition 4.2 *We have the relation*

$$\sum_{h=0}^{\infty} \Gamma(h) = \lambda \mathbf{E}[\sigma]\, \mathbf{E}[\widehat{\sigma}] = \frac{\lambda}{2} \mathbf{E}[\sigma(\sigma + 1)], \tag{4.7}$$

so that the stationary sequence $\{b_t^\star,\ t = 0, 1, \ldots\}$ is short range dependent (resp. long range dependent) if and only if $\mathbf{E}\left[\sigma^2\right]$ is finite (resp. infinite).

Proof. From (4.5), we see that

$$\begin{aligned}
\sum_{h=0}^{\infty} \Gamma(h) &= \lambda \mathbf{E}[\sigma] \sum_{h=0}^{\infty} \mathbf{P}[\widehat{\sigma} > h] \\
&= \lambda \mathbf{E}[\sigma]\, \mathbf{E}[\widehat{\sigma}] \\
&= \lambda \mathbf{E}[\sigma] \sum_{r=1}^{\infty} r \mathbf{P}[\widehat{\sigma} = r] \\
&= \lambda \mathbf{E}[\sigma] \left(\mathbf{E}[\sigma]\right)^{-1} \sum_{r=1}^{\infty} r \mathbf{P}[\sigma \geq r] \\
&= \lambda \sum_{r=1}^{\infty} r \sum_{t=r}^{\infty} \mathbf{P}[\sigma = t] \\
&= \lambda \sum_{t=1}^{\infty} \mathbf{P}[\sigma = t] \left(\sum_{r=1}^{t} r\right) \\
&= \frac{\lambda}{2} \sum_{t=1}^{\infty} t(t+1) \mathbf{P}[\sigma = t]
\end{aligned}$$

and the conclusion (4.7) is now immediate. ■

5 Evaluation of $\Lambda_{b,t}(\theta)$ $(t = 1, 2, \ldots,\ \theta \in \mathbb{R})$

For each $t = 1, 2, \ldots$, we set

$$L_{b,t}(\theta) = \ln \mathbf{E}\left[\exp(\theta \sum_{s=1}^{t} b_s^\star)\right] \quad \theta \in \mathbb{R} \tag{5.1}$$

where $\{b_t^\star,\ t = 0, 1, \ldots\}$ is the stationary and ergodic version of the busy server process. In that case the rvs $b_t^{(0)}$ and $b_t^{(a)}$ are given by (2.4) and (2.6)–(2.7), respectively. Our interest in (5.1) stems from the fact that

$$\Lambda_{b,t}(\theta) = \frac{1}{v_t} L_{b,t}(\theta_t) \tag{5.2}$$

with the notation

$$\theta_t \equiv \frac{v_t}{t}\theta, \quad \theta \in \mathbb{R},\ t = 1, 2, \ldots \tag{5.3}$$

being used throughout the discussion.

From (2.1) and the independence of the rvs $b_t^{(0)}$ and $b_t^{(a)}$, we get

$$L_{b,t}(\theta) = L_t^{(0)}(\theta) + L_t^{(a)}(\theta), \quad \theta \in \mathbb{R} \tag{5.4}$$

where we have set

$$L_t^{(0)}(\theta) = \ln \mathbf{E}\left[\exp(\theta \sum_{s=1}^{t} b_s^{(0)})\right], \quad \theta \in \mathbb{R} \tag{5.5}$$

and

$$L_t^{(a)}(\theta) = \ln \mathbf{E}\left[\exp(\theta \sum_{s=1}^{t} b_s^{(a)})\right], \quad \theta \in \mathbb{R} \tag{5.6}$$

Therefore, in order to evaluate $\Lambda_{b,t}(\theta)$ as given by (1.8) it suffices to evaluate $L_t^{(0)}(\theta)$ and $L_t^{(a)}(\theta)$.

Lemma 5.1 *For each $t = 1, 2, \ldots$, we have the expressions*

$$L_t^{(0)}(\theta) = -\lambda \mathbf{E}[\sigma]\left(1 - \mathbf{E}[\exp(\theta \min(t, \widehat{\sigma} - 1))]\right) \tag{5.7}$$

for all θ in $\mathbb{R}$.

Proof. Fix θ in $\mathbb{R}$ and $t = 1, 2, \ldots$, and recall that the i.i.d. rvs $\{\widehat{\sigma}_n,\ n = 1, 2, \ldots\}$

are independent of the rv b. From these facts we readily conclude that

$$\begin{aligned} L_t^{(0)}(\theta) &= \ln \mathbf{E}\left[\exp\left(\theta \sum_{s=1}^{t} b_s^{(0)}\right)\right] \\ &= \ln \mathbf{E}\left[\exp\left(\theta \sum_{s=1}^{t} \sum_{n=1}^{b} \mathbf{1}\left[\widehat{\sigma}_n > s\right]\right)\right] \\ &= \ln \mathbf{E}\left[\mathbf{E}\left[\exp\left(\theta \sum_{n=1}^{b} \sum_{s=1}^{t} \mathbf{1}\left[\widehat{\sigma}_n > s\right]\right) \mid b\right]\right] \\ &= \ln \mathbf{E}\left[\Gamma(t,\theta)^b\right] \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} \Gamma(t,\theta) &= \mathbf{E}\left[\exp\left(\theta \sum_{s=1}^{t} \mathbf{1}\left[\widehat{\sigma} > s\right]\right)\right] \\ &= \mathbf{E}\left[\exp\left(\theta \min(t, \widehat{\sigma} - 1)\right)\right]. \end{aligned} \tag{5.9}$$

Finally, because the rv b is Poisson with parameter $\lambda \mathbf{E}[\sigma]$, we get

$$\ln \mathbf{E}\left[\Gamma(t,\theta)^b\right] = -\lambda \mathbf{E}[\sigma]\left(1 - \Gamma(t,\theta)\right), \tag{5.10}$$

and the desired result is an immediate consequence of (5.8)–(5.10). ∎

The next lemma evaluates the contribution due to arrivals; its proof is given in Section 9.

Lemma 5.2 *For each $t = 1, 2, \ldots$, we have the expressions*

$$\begin{aligned} L_t^{(a)}(\theta) = & - \lambda t + \lambda \mathbf{E}\left[(t-\sigma)^+ e^{\theta\sigma}\right] \\ & + \lambda\left(1 - e^{-\theta}\right)^{-1} \mathbf{E}\left[e^{\theta \min(t,\sigma)} - 1\right] \end{aligned} \tag{5.11}$$

for all θ in $\mathbb{R}$.

In the remainder of this section, we seek to simplify the expressions (5.7) and (5.11), and to do so we find it useful to define several auxiliary quantities: For each $\beta \geq 0$ and for all $t = 1, 2, \ldots$, we set

$$\widehat{F}_\beta(t,\theta) \equiv \sum_{r=1}^{t} r^\beta e^{\theta r} \mathbf{P}\left[\widehat{\sigma} > r\right], \quad \theta \in \mathbb{R}. \tag{5.12}$$

Lemma 5.3 *For all* $t = 1, 2, \ldots$, *we have*

$$L_t^{(0)}(\theta) = \lambda \mathbf{E}[\sigma] \left(1 - e^{-\theta}\right) \widehat{F}_0(t, \theta), \quad \theta \in \mathbb{R}. \tag{5.13}$$

Proof. Fix $t = 1, 2, \ldots$ and θ in $\mathbb{R}$. Starting with (5.7) we get

$$\begin{aligned} L_t^{(0)}(\theta) &= \lambda \mathbf{E}[\sigma] \left(\sum_{r=1}^{t} e^{\theta(r-1)} \widehat{g}_r + e^{\theta t} \mathbf{P}[\widehat{\sigma} > t] - 1 \right) \\ &= \lambda \mathbf{E}[\sigma] \left(\sum_{r=1}^{t} e^{\theta(r-1)} (\mathbf{P}[\widehat{\sigma} > r-1] - \mathbf{P}[\widehat{\sigma} > r]) + e^{\theta t} \mathbf{P}[\widehat{\sigma} > t] - 1 \right) \\ &= \lambda \mathbf{E}[\sigma] \left(\sum_{r=0}^{t-1} e^{\theta r} \mathbf{P}[\widehat{\sigma} > r] - \sum_{r=1}^{t} e^{\theta(r-1)} \mathbf{P}[\widehat{\sigma} > r] + e^{\theta t} \mathbf{P}[\widehat{\sigma} > t] - 1 \right) \end{aligned}$$

and the conclusion (5.13) follows by simple algebra as we note $\mathbf{P}[\widehat{\sigma} > 0] = 1$.

■

The expression (5.11) can also be simplified and the final result is stated in the following lemma, the proof of which can be found in Section 10.

Lemma 5.4 *For all* $t = 1, 2, \ldots$, *we have*

$$L_t^{(a)}(\theta) = \lambda \mathbf{E}[\sigma] \left(e^{\theta} - 1\right) \Sigma(t, \theta), \quad \theta \in \mathbb{R} \tag{5.14}$$

where

$$\Sigma(t, \theta) = t + \left(t(1 - e^{-\theta}) - e^{-\theta}\right) \widehat{F}_0(t, \theta) - \left(1 - e^{-\theta}\right) \widehat{F}_1(t, \theta). \tag{5.15}$$

By combining Lemmas 5.3 and 5.4 via (5.4), and grouping like terms, we obtain the following compact expression.

Lemma 5.5 *For each* $t = 1, 2, \ldots$, *we have*

$$L_{b,t}(\theta) = \lambda \mathbf{E}[\sigma]\, t(e^{\theta} - 1) \left(1 + (1 - e^{-\theta}) \left[\widehat{F}_0(t, \theta) - \frac{\widehat{F}_1(t, \theta)}{t} \right] \right), \quad \theta \in \mathbb{R}. \tag{5.16}$$

The easy calculations leading to (5.16) are omitted in the interest of brevity.

6 Asymptotics for $\theta > 1$

We can now start proving the main results of this paper, namely, Theorems 3.1, 3.2 and 3.3. In this section we present asymptotics which are common to both regimes, viz. $v_t = O(t)$ and $v_t = o(t)$. Proposition 6.1 provides the key towards establishing the asymptotic behavior for $\theta > 1$ described in Theorem 3.1.

Proposition 6.1 *We always have*

$$\lim_{t\to\infty} \frac{1}{v_t} L_t^{(0)}(\theta_t) = \infty, \quad \theta > 1. \tag{6.1}$$

Proof. Going back to the proof of Lemma 5.3, we find that

$$L_t^{(0)}(\theta) \geq \lambda \mathbf{E}[\sigma] \left(e^{\theta t} \mathbf{P}[\widehat{\sigma} > t] - 1 \right), \quad \theta \in \mathbb{R} \tag{6.2}$$

for each $t = 1, 2, \ldots$, so that

$$\begin{aligned} \frac{1}{v_t} L_t^{(0)}(\theta_t) &\geq \frac{1}{v_t} \lambda \mathbf{E}[\sigma] \left(e^{(\theta-1)v_t} - 1 \right) \\ &= \lambda \mathbf{E}[\sigma] \left(\frac{e^{(\theta-1)v_t}}{v_t} - \frac{1}{v_t} \right), \quad \theta \in \mathbb{R}. \end{aligned} \tag{6.3}$$

The stated result (6.1) is now immediate once we note that

$$\lim_{t\to\infty} \frac{e^{(\theta-1)v_t}}{v_t} = \infty, \quad \theta > 1. \tag{6.4}$$

■

A proof of Theorem 3.1. Fix $t = 1, 2, \ldots$. By Jensen's inequality we have

$$\frac{1}{v_t} L_t^{(a)}(\theta_t) \geq \theta \frac{1}{t} \sum_{s=1}^{t} \mathbf{E}\left[b_s^{(a)} \right], \quad \theta \in \mathbb{R} \tag{6.5}$$

and in Section 9 we show that the arguments leading to Lemma 5.2 also imply

$$\sum_{s=1}^{t} \mathbf{E}\left[b_s^{(a)} \right] = \lambda \sum_{s=1}^{t} \mathbf{E}[\min(\sigma, s)]. \tag{6.6}$$

Because $\lim_{t\to\infty} \mathbf{E}[\min(\sigma, t)] = \mathbf{E}[\sigma]$, we conclude that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{t} \mathbf{E}\left[b_s^{(a)} \right] = \lambda \mathbf{E}[\sigma] \tag{6.7}$$

by Cesaro convergence – as expected of course from the ergodic properties of the process $\{b_t,\ t = 0, 1, \ldots\}$. In short, we have

$$\liminf_{t\to\infty} \frac{1}{v_t} L_t^{(a)}(\theta_t) \geq \lambda \mathbb{E}\,[\sigma]\,\theta, \quad \theta \in \mathbb{R} \tag{6.8}$$

and the conclusion (3.1) follows from (6.1). ∎

By Theorem 3.1 we need only consider the case $\theta \leq 1$. However the analysis for that range depends crucially on whether $v_t = O(t)$ or $v_t = o(t)$. We shall take on these two cases separately in the next two sections. To prepare for this analysis, we note that

$$\begin{aligned} \widehat{F}_\beta(t, \theta_t) &= \sum_{r=1}^{t} r^\beta e^{\theta_t r} e^{-v_r} \\ &= \sum_{r=1}^{t} r^\beta \exp\left(r(\theta \frac{v_t}{t} - \frac{v_r}{r})\right), \quad \theta \in \mathbb{R} \end{aligned} \tag{6.9}$$

for each $t = 1, 2, \ldots$.

7 Asymptotics when $v_t = O(t)$

We assume $v_t = O(t)$ with $\lim_t \frac{v_t}{t} = C > 0$, and for each $\beta \geq 0$, we set

$$\Sigma_\beta(\theta) \equiv \sum_{r=1}^{\infty} r^\beta \exp\left(r(\theta C - \frac{v_r}{r})\right), \quad \theta \in \mathbb{R}. \tag{7.1}$$

The following preliminary result plays a key role in establishing Theorem 3.2.

Lemma 7.1 *For each $\beta \geq 0$, the quantity $\Sigma_\beta(\theta)$ is finite (resp. infinite) if $\theta < 1$ (resp. $\theta > 1$). Moreover, we have*

$$\lim_{t\to\infty} \widehat{F}_\beta(t, \theta_t) = \Sigma_\beta(\theta) < \infty, \quad \theta < 1. \tag{7.2}$$

Proof. A simple application of Cauchy's convergence criterion already yields the fact that $\Sigma_\beta(\theta)$ is finite (resp. infinite) if $\theta < 1$ (resp. $\theta > 1$).

Fix $\theta < 1$. By the finiteness of $\Sigma_\beta(\theta)$, the conclusion (7.2) follows if we show that

$$\lim_{t\to\infty} \sum_{r=1}^{t} r^\beta \left| \exp\left(r(\theta C - \frac{v_r}{r})\right) - \exp\left(r(\theta \frac{v_t}{t} - \frac{v_r}{r})\right) \right| = 0. \tag{7.3}$$

Upon making use of the fact that

$$|e^b - e^a| = |\int_a^b e^x dx| \le |b-a| e^{\max(a,b)}, \quad a, b \in \mathbb{R}, \tag{7.4}$$

we conclude that

$$\begin{aligned} & \sum_{r=1}^{t} r^\beta \left| \exp\left(r(\theta C - \frac{v_r}{r}) \right) - \exp\left(r(\theta \frac{v_t}{t} - \frac{v_r}{r}) \right) \right| \\ \le \quad & |\theta||C - \frac{v_t}{t}| \sum_{r=1}^{\infty} r^{\beta+1} e^{-v_r} \exp\left(r \max(C\theta, \frac{v_t}{t}\theta) \right), \quad t = 1, 2, \ldots \end{aligned} \tag{7.5}$$

Hence, for t large enough we have

$$\begin{aligned} & \lim_{r\to\infty} \left(r^{\beta+1} e^{-v_r} \exp\left(r \max(C\theta, \frac{v_t}{t}\theta) \right) \right)^{\frac{1}{r}} \\ = \quad & \lim_{r\to\infty} \left(r^{\beta+1} \right)^{\frac{1}{r}} \exp\left(-\frac{v_r}{r} + \max(C\theta, \frac{v_t}{t}\theta) \right) \\ = \quad & \exp\left(-C + \max(C\theta, \frac{v_t}{t}\theta) \right) < 1, \end{aligned} \tag{7.6}$$

and Cauchy's convergence criterion again implies

$$\sum_{r=1}^{\infty} r^{\beta+1} e^{-v_r} \exp\left(r \max(C\theta, \frac{v_t}{t}\theta) \right) < \infty. \tag{7.7}$$

The conclusion (7.3) follows now from (7.5) and (7.7). ■

The results in Theorem 3.2 are all straigthforward consequences of the following simple observation.

Lemma 7.2 *Fix θ in $\mathbb{R}$. The limit $\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta)$ exists (resp. exists and is finite) if and only if the limit*

$$L(\theta) \equiv \lim_{t\to\infty} \left(\widehat{F}_0(t, \theta_t) - \frac{\widehat{F}_1(t, \theta_t)}{t} \right) \tag{7.8}$$

exists (resp. exists and is finite), in which case

$$\Lambda_b(\theta) = \lambda \mathbf{E}[\sigma] \left(\frac{e^{C\theta} - 1}{C} \right) \left(1 + \left(1 - e^{-C\theta} \right) L(\theta) \right). \tag{7.9}$$

Proof. Fix θ in $\mathbb{R}$ and $t = 1, 2, \ldots$. It is plain from (5.16) and (5.2)–(5.3) that

$$\Lambda_{b,t}(\theta) = \lambda \mathbf{E}[\sigma] \frac{t}{v_t}(e^{\theta_t} - 1)\left(1 + (1 - e^{-\theta_t})\left[\widehat{F}_0(t, \theta_t) - \frac{\widehat{F}_1(t, \theta_t)}{t}\right]\right). \tag{7.10}$$

The conclusion (7.8)–(7.9) now follows immediately from the assumption $\lim_{t\to\infty} \frac{v_t}{t} = C$, so that $\lim_{t\to\infty} e^{\pm\theta_t} = e^{\pm C\theta}$. ∎

A proof of Theorems 3.2 and 3.4: In view of Theorem 3.1, we need only consider the case $\theta \leq 1$. For $\theta < 1$, Lemma 7.1 yields the existence of the limit (7.8), with

$$L(\theta) = \lim_{t\to\infty}\left(\widehat{F}_0(t, \theta_t) - \frac{\widehat{F}_1(t, \theta_t)}{t}\right) = \lim_{t\to\infty} \widehat{F}_0(t, \theta_t) = \Sigma_0(\theta) < \infty. \tag{7.11}$$

Therefore, by Lemma 7.2, $\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta)$ exists, is finite and given by (3.3).

Consider now the boundary case $\theta = 1$. By monotonicity we note that for all $t = 1, 2, \ldots$, the inequality $\Lambda_{b,t}(\theta) \leq \Lambda_{b,t}(1)$ holds whenever $\theta < 1$, whence $\Lambda_b(\theta) \leq \liminf_{t\to\infty} \Lambda_{b,t}(1)$ for $\theta < 1$. Next, letting θ go to 1, we see that

$$\lim_{\theta\uparrow 1} \Lambda_b(\theta) \leq \liminf_{t\to\infty} \Lambda_{b,t}(1) \tag{7.12}$$

where

$$\lim_{\theta\uparrow 1} \Lambda_b(\theta) = \lambda \mathbf{E}[\sigma]\left(\frac{e^C - 1}{C}\right)\left(1 + (1 - e^{-C})\Sigma_0(1)\right), \tag{7.13}$$

as we note that

$$\lim_{\theta\uparrow 1} \Sigma_0(\theta) = \Sigma_0(1) = \sum_{r=1}^{\infty} \exp\left(r(C - \frac{v_r}{r})\right) \tag{7.14}$$

by monotone convergence.

From (7.12)–(7.14) it is clear that if we establish the fact $\Sigma_0(1) = \infty$, then $\liminf_{t\to\to\infty} \Lambda_{b,t}(1) = \infty$, and the limit $\Lambda_b(1) \equiv \lim_{t\to\infty} \Lambda_{b,t}(1)$ thus exists and is infinite. To carry out this last step, we consider separately the two sets of conditions stated in Theorem 3.4: Under (i), the set $R \equiv \{t = 1, 2, \ldots : v_t \leq Ct\}$ is countably infinite, so that

$$\Sigma_0(1) \geq \sum\nolimits_{r\in R} \exp\left(r(C - \frac{v_r}{r})\right) \geq \sum\nolimits_{r\in R} 1 = \infty. \tag{7.15}$$

Under (ii), the condition $\limsup_{r\to\infty}(v_r - Cr) = K$ for some finite $K \geq 0$ implies for any $\varepsilon > 0$, the existence of an integer $t^\star = t^\star(\varepsilon)$ such that $0 \leq v_r - Cr \leq K + \varepsilon$

for all $r \geq t^\star$, whence

$$\Sigma_0(1) \geq \sum_{r=t^\star}^{\infty} e^{-(K+\varepsilon)} = \infty. \tag{7.16}$$

■

We note in passing that it is also possible to show that

$$\lim_{t\to\infty} \frac{1}{v_t} L_t^{(0)}(\theta_t) = 0, \quad \theta < 1 \tag{7.17}$$

and

$$\lim_{t\to\infty} \frac{1}{v_t} L_t^{(a)}(\theta_t) = \lambda \mathbf{E}[\sigma] \left(\frac{e^{C\theta} - 1}{C} \right) \Sigma(\theta), \quad \theta < 1 \tag{7.18}$$

where $\Sigma(\theta)$ is given by (3.3). These limiting results are easy consequences of Lemmas 5.3 and 5.4, with the details left to the interested reader.

8 Asymptotics when $v_t = o(t)$

We assume $v_t = o(t)$ with $\lim_t \frac{v_t}{t} = 0$, so that now

$$\lim_{t\to\infty} \frac{1 - e^{-\theta_t}}{\theta_t} = 1, \quad \theta \in \mathbb{R}. \tag{8.1}$$

Moreover, the sequence $\{v_t/t,\ t = 1, 2, \ldots\}$ is monotone decreasing in the limit, and conditions (i)–(iii) are enforced. The counterpart to Lemma 7.1 is first presented.

Lemma 8.1 *We have*

$$\lim_{t\to\infty} \frac{v_t}{t} \widehat{F}_0(t, \theta_t) = 0, \quad \theta < 1 \tag{8.2}$$

and

$$\lim_{t\to\infty} \frac{v_t}{t^2} \widehat{F}_1(t, \theta_t) = 0, \quad \theta < 1. \tag{8.3}$$

Proof. The conclusion (8.3) is an easy consequence of (8.2) once we note the obvious inequality

$$\widehat{F}_1(t, \theta) \leq t \widehat{F}_0(t, \theta), \quad \theta \in \mathbb{R} \tag{8.4}$$

for all $t = 1, 2, \ldots$.

We now turn to the proof of (8.2). In the interest of clarity, we discuss only the case when the sequence $\{v_t/t,\ t = 1, 2, \ldots\}$ is monotone decreasing, and leave it to the reader to extend the arguments to the asymptotically monotone case, an easy but tedious exercise. Moreover, for each $t = 1, 2, \ldots$, $\widehat{F}_0(t, \theta_t)$ is a non–decreasing function of θ, so that we need only establish (8.2) in the range $0 < \theta < 1$.

Fixing θ in the interval $(0, 1)$ and $t = 1, 2, \ldots$, we begin with the decomposition

$$\frac{v_t}{t}\widehat{F}_0(t, \theta_t) = \frac{v_t}{t}\sum_{r=1}^{\Gamma(t)} e^{\theta\frac{v_t}{t}r - v_r} + \frac{v_t}{t}\sum_{r=\Gamma(t)+1}^{t} e^{\theta\frac{v_t}{t}r - v_r} \tag{8.5}$$

where $\Gamma(t)$ is as described in Theorem 3.3. The analysis successively considers the two terms in this last expression.

We first discuss the second term of (8.5): From the monotonicity of the sequence $\{v_t/t,\ t = 1, 2, \ldots\}$, we get

$$\theta\frac{v_t}{t}r - v_r = (\theta - 1)\frac{v_t}{t}r - (\frac{v_r}{r} - \frac{v_t}{t})r \leq (\theta - 1)\frac{v_t}{t}r, \quad r = 1, \ldots, t \tag{8.6}$$

and it is now plain that

$$\begin{aligned}
\frac{v_t}{t}\sum_{r=\Gamma(t)+1}^{t} e^{\theta\frac{v_t}{t}r - v_r} &\leq \frac{v_t}{t}\sum_{r=\Gamma(t)+1}^{t} e^{(\theta-1)\frac{v_t}{t}r} \\
&= \frac{v_t}{t}\left(\frac{e^{(\theta-1)v_t\frac{(t+1)}{t}} - e^{(\theta-1)\frac{v_t}{t}\Gamma(t)}}{e^{(\theta-1)\frac{v_t}{t}} - 1}\right) \\
&= \left(\frac{e^{(\theta-1)\frac{v_t}{t}} - 1}{\frac{v_t}{t}}\right)^{-1}\left(e^{(\theta-1)v_t\frac{(t+1)}{t}} - e^{(\theta-1)\frac{v_t}{t}\Gamma(t)}\right).
\end{aligned}$$

Using (8.1) and condition (ii) of Theorem 3.3, we readily conclude

$$\lim_{t\to\infty}\frac{v_t}{t}\sum_{r=\Gamma(t)+1}^{t} e^{\theta\frac{v_t}{t}r - v_r} = 0. \tag{8.7}$$

Next, going back to the first term of (8.5), we note for $0 < \theta < 1$ that

$$\theta\frac{v_t}{t}r - v_r \leq \theta\frac{v_r}{r}r - v_r \leq (\theta - 1)\frac{v_{\Gamma(t)}}{\Gamma(t)}r, \quad r = 1, \ldots, \Gamma(t) \tag{8.8}$$

by the monotonicity of the sequence $\{v_t/t,\ t = 1, 2, \ldots\}$. Therefore,

$$\begin{aligned}
\frac{v_t}{t}\sum_{r=1}^{\Gamma(t)} e^{\theta\frac{v_t}{t}r - v_r} &\leq \frac{v_t}{t}\sum_{r=1}^{\Gamma(t)} e^{(\theta-1)\frac{v_{\Gamma(t)}}{\Gamma(t)}r} \\
&= \frac{v_t}{t}e^{(\theta-1)\frac{v_{\Gamma(t)}}{\Gamma(t)}}\left(\frac{e^{(\theta-1)v_{\Gamma(t)}} - 1}{e^{(\theta-1)\frac{v_{\Gamma(t)}}{\Gamma(t)}} - 1}\right) \\
&= e^{(\theta-1)\frac{v_{\Gamma(t)}}{\Gamma(t)}}\left(\frac{e^{(\theta-1)\frac{v_{\Gamma(t)}}{\Gamma(t)}} - 1}{\frac{v_{\Gamma(t)}}{\Gamma(t)}}\right)^{-1}\frac{v_t}{t}\frac{\Gamma(t)}{v_{\Gamma(t)}}\left(e^{(\theta-1)v_{\Gamma(t)}} - 1\right).
\end{aligned}$$

This time, (8.1) and condition (iii) of Theorem 3.3 lead to

$$\lim_{t\to\infty} \frac{v_t}{t} \sum_{r=1}^{\Gamma(t)} e^{\theta \frac{v_t}{t} r - v_r} = 0. \tag{8.9}$$

Combining (8.5), (8.7) and (8.9) readily yields (8.2). ■

Here, the counterpart to Lemma 7.2 is the following observation which depends only on the fact that $v_t = o(t)$.

Lemma 8.2 *Fix θ in $\mathbb{R}$. The limit $\Lambda_b(\theta) \equiv \lim_{t\to\infty} \Lambda_{b,t}(\theta)$ exists (resp. exists and is finite) if and only if the limit*

$$K(\theta) \equiv \lim_{t\to\infty} \frac{v_t}{t} \left(\widehat{F}_0(t,\theta_t) - \frac{\widehat{F}_1(t,\theta_t)}{t} \right) \tag{8.10}$$

exists (resp. exists and is finite), in which case

$$\Lambda_b(\theta) = \lambda \mathbf{E}[\sigma]\, \theta \left(1 + \theta K(\theta)\right). \tag{8.11}$$

Proof. Fix θ in $\mathbb{R}$ and $t = 1, 2, \ldots$. It is plain from (5.16) and (5.2)–(5.3) that

$$\Lambda_{b,t}(\theta) = \lambda \mathbf{E}[\sigma]\, \theta \left(\frac{e^{\theta_t} - 1}{\theta_t} \right) \left(1 + \left(\frac{1 - e^{-\theta_t}}{\theta_t} \right) \theta_t \left[\widehat{F}_0(t,\theta_t) - \frac{\widehat{F}_1(t,\theta_t)}{t} \right] \right). \tag{8.12}$$

The conclusion (8.10) and the expression (8.11) follow immediately from (8.1) and (8.12). ■

A proof of Theorem 3.3: Here too, in view of Theorem 3.1, we need only consider the case $\theta \leq 1$. For $\theta < 1$, we get from Lemma 8.1 that the limit (8.10) exists with

$$K(\theta) = \lim_{t\to\infty} \frac{v_t}{t} \left(\widehat{F}_0(t,\theta_t) - \frac{\widehat{F}_1(t,\theta_t)}{t} \right) = 0 \tag{8.13}$$

and the desired conclusion (3.4) follows from (8.11). ■

More precise information can be obtained for the various contributions to the limit (3.4). In particular, by relying on Lemmas 5.3 and 5.4, we can easily derive the limiting results

$$\lim_{t\to\infty} \frac{1}{v_t} L_t^{(0)}(\theta_t) = 0, \quad \theta < 1. \tag{8.14}$$

and

$$\lim_{t\to\infty}\frac{1}{v_t}L_t^{(a)}(\theta_t)=\lambda\mathbf{E}\left[\sigma\right]\theta,\quad \theta<1. \tag{8.15}$$

9 A proof of Lemma 5.2

Fix $t=1,2,\ldots$ and θ in $\mathbb{R}$. Using (2.6) and the fact that the rvs $\{b_t^r,\ t=1,2,\ldots;r=1,2,\ldots\}$ are mutually independent, we have

$$\begin{aligned}
L_t^{(a)}(\theta) &= \ln\mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}b_s^{(a)}\right)\right]\\
&= \ln\mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}\sum_{r=1}^{\infty}b_s^r\right)\right]\\
&= \sum_{r=1}^{\infty}\ln\mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}b_s^r\right)\right]\\
&= \sum_{r=1}^{\infty}\ln\mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}\sum_{i=1}^{\min(r,s)}\beta^r_{(s-r)^++i}\right)\right].
\end{aligned} \tag{9.1}$$

Fixing $r=1,2,\ldots$, we carry on with the analysis by considering two separate cases:

a. Assume $t\le r$: We note that

$$\sum_{s=1}^{t}\sum_{i=1}^{\min(r,s)}\beta^r_{(s-r)^++i}=\sum_{s=1}^{t}\sum_{i=1}^{s}\beta_i^r=\sum_{i=1}^{t}(t+1-i)\beta_i^r \tag{9.2}$$

and by the independence of the rvs involved, we get

$$\begin{aligned}
\ln\mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}\sum_{i=1}^{\min(r,s)}\beta^r_{(s-r)^++i}\right)\right] &= \sum_{i=1}^{t}\ln\mathbf{E}\left[\exp\left(\theta(t+1-i)\beta_i^r\right)\right]\\
&= \sum_{i=1}^{t}\ln\left(e^{-\lambda g_r(1-e^{\theta(t+1-i)})}\right)\\
&= -\lambda g_r\left(t-e^\theta\sum_{i=0}^{t-1}e^{\theta i}\right)\\
&= -\lambda g_r\left(t-\frac{e^{t\theta}-1}{e^\theta-1}e^\theta\right).
\end{aligned} \tag{9.3}$$

b. Assume $t>r$: Set $h=t-r$. Elementary algebra yields

$$\sum_{s=1}^{t}\sum_{i=1}^{\min(r,s)}\beta^r_{(s-r)^++i} = \sum_{s=1}^{r}\sum_{i=1}^{s}\beta_i^r+\sum_{s=r+1}^{t}\sum_{i=1}^{r}\beta^r_{s-r+i}$$

$$
\begin{aligned}
&= \sum_{s=1}^{r}\sum_{i=1}^{s}\beta_i^r + \sum_{s=1}^{t-r}\sum_{i=1}^{r}\beta_{s+i}^r \\
&= \sum_{s=1}^{r}\sum_{i=1}^{s}\beta_i^r + \sum_{i=1}^{r}\sum_{j=i+1}^{h+i}\beta_j^r \\
&= \sum_{s=1}^{r}\sum_{i=1}^{s}\beta_i^r + \sum_{i=1}^{r}(\sum_{j=1}^{h+i}\beta_j^r - \sum_{j=1}^{i}\beta_j^r) \\
&= \sum_{s=1}^{r}\sum_{i=1}^{s}\beta_i^r + \sum_{i=1}^{r}\sum_{j=1}^{h+i}\beta_j^r - \sum_{i=1}^{r}\sum_{j=1}^{i}\beta_j^r \\
&= \sum_{i=1}^{r}\sum_{j=1}^{h+i}\beta_j^r \\
&= \sum_{i=1}^{r}\sum_{j=1}^{h}\beta_j^r + \sum_{i=1}^{r}\sum_{j=h+1}^{h+i}\beta_j^r \\
&= \sum_{i=1}^{r}\sum_{j=1}^{h}\beta_j^r + \sum_{i=1}^{r}\sum_{j=1}^{i}\beta_{h+j}^r \\
&= \sum_{i=1}^{r}\sum_{j=1}^{h}\beta_j^r + \sum_{j=1}^{r}(r+1-j)\beta_{h+j}^r \\
&= r\sum_{j=1}^{h}\beta_j^r + \sum_{j=1}^{r}(r-j+1)\beta_{h+j}^r. \qquad (9.4)
\end{aligned}
$$

Therefore, again making use of the independence of the rvs involved, we see that

$$
\begin{aligned}
&\ln \mathbf{E}\left[\exp\left(\theta\sum_{s=1}^{t}\sum_{i=1}^{\min(r,s)}\beta_{(s-r)^++i}^r\right)\right] \\
&= \sum_{j=1}^{h}\ln \mathbf{E}\left[\exp\left(\theta r\beta_j^r\right)\right] + \sum_{j=1}^{r}\ln \mathbf{E}\left[\exp(\theta(r+1-j)\beta_{j+h}^r)\right] \\
&= \sum_{j=1}^{h} -\lambda g_r(1-e^{\theta r}) + \sum_{j=1}^{r} -\lambda g_r(1-e^{\theta(r+1-j)}) \\
&= -\lambda g_r\left(h(1-e^{\theta r}) + r - e^{\theta(r+1)}\sum_{j=1}^{r}e^{-\theta j}\right) \\
&= -\lambda g_r\left((t-r)(1-e^{r\theta}) + r - \frac{e^{r\theta}-1}{e^{\theta}-1}e^{\theta}\right). \qquad (9.5)
\end{aligned}
$$

We can now combine (9.3) and (9.5) into a single expression

$$\begin{aligned}
& \ln \mathbf{E}\left[\exp\left(\theta \sum_{s=1}^{t} \sum_{i=1}^{\min(r,s)} \beta^{r}_{(s-r)^{+}+i}\right)\right] \\
= & -\lambda g_r \left(t - (t-r)^{+} e^{r\theta} - \frac{e^{\theta \min(t,r)} - 1}{1 - e^{-\theta}}\right) \qquad (9.6)
\end{aligned}$$

where the last equality used the fact that $(t-r)^{+} + \min(t,r) = t$, and the desired conclusion immediately follows from (9.1) and (9.6). ■

Before closing this section we note from (2.6) that

$$\begin{aligned}
\mathbf{E}\left[b_t^{(a)}\right] &= \sum_{r=1}^{\infty} \sum_{i=1}^{\min(r,t)} \mathbf{E}\left[\beta^{r}_{(t-r)^{+}+i}\right] \\
&= \sum_{r=1}^{\infty} \lambda g_r \min(r,t) \\
&= \lambda \mathbf{E}\left[\min(\sigma, t)\right], \quad t = 1, 2, \ldots \qquad (9.7)
\end{aligned}$$

and the relation (6.6) holds.

10 A proof of Lemma 5.4

For each $\beta \geq 0$ and for all $t = 1, 2, \ldots$, we set

$$\Phi_\beta(t,\theta) \equiv \sum_{r=1}^{t} r^{\beta} g_r e^{\theta r}, \quad \theta \in \mathbb{R}. \qquad (10.1)$$

Fix θ in $\mathbb{R}$ and $t = 1, 2, \ldots$. In order to establish (5.14) we begin by rewriting (5.11) as

$$\begin{aligned}
L_t^{(a)}(\theta) &= -\lambda t + \lambda \sum_{r=1}^{t} (t-r) g_r e^{\theta r} \\
&\quad + \lambda \left(1 - e^{-\theta}\right)^{-1} \left(\sum_{r=1}^{t} g_r e^{\theta r} + e^{\theta t} \mathbf{P}\left[\sigma > t\right] - 1\right) \\
&= \lambda \left(t(\Phi_0(t,\theta) - 1) - \Phi_1(t,\theta)\right) \\
&\quad + \lambda (1 - e^{-\theta})^{-1} \left(\Phi_0(t,\theta) - 1 + e^{\theta t} \mathbf{E}\left[\sigma\right] \widehat{g}_{t+1}\right). \qquad (10.2)
\end{aligned}$$

The basic idea behind the passage from (10.2) to (5.14) consists in expressing the quantities $\Phi_\beta(t,\theta)$ in terms of $\widehat{F}_\beta(t,\theta)$. To do so, we first note from (1.7) that the relations

$$\begin{aligned} g_r &= \mathbf{P}[\sigma > r-1] - \mathbf{P}[\sigma > r] \\ &= \mathbf{E}[\sigma](\widehat{g}_r - \widehat{g}_{r+1}) \\ &= \mathbf{E}[\sigma](\mathbf{P}[\widehat{\sigma} > r-1] + \mathbf{P}[\widehat{\sigma} > r+1] - 2\mathbf{P}[\widehat{\sigma} > r]) \end{aligned} \tag{10.3}$$

hold for all $r = 1, 2, \ldots$. Therefore, substituting for (10.3) in (10.1), for each $\beta \geq 0$, we have

$$\begin{aligned} &\Phi_\beta(t,\theta) \\ = \;& \mathbf{E}[\sigma] \sum_{r=1}^{t} r^\beta (\mathbf{P}[\widehat{\sigma} > r-1] + \mathbf{P}[\widehat{\sigma} > r+1] - 2\mathbf{P}[\widehat{\sigma} > r]) e^{\theta r} \\ = \;& \mathbf{E}[\sigma]\left(e^\theta \sum_{r=0}^{t-1}(r+1)^\beta \mathbf{P}[\widehat{\sigma} > r] e^{\theta r} + e^{-\theta}\sum_{r=2}^{t+1}(r-1)^\beta \mathbf{P}[\widehat{\sigma} > r] e^{\theta r} - 2\widehat{F}_\beta(t,\theta)\right) \\ = \;& \mathbf{E}[\sigma]\left(e^\theta \sum_{r=1}^{t}(r+1)^\beta \mathbf{P}[\widehat{\sigma} > r] e^{\theta r} + e^{-\theta}\sum_{r=1}^{t}(r-1)^\beta \mathbf{P}[\widehat{\sigma} > r] e^{\theta r} - 2\widehat{F}_\beta(t,\theta)\right) \\ & + \mathbf{E}[\sigma]\left(e^\theta - \mathbf{1}[\beta = 0]\mathbf{P}[\widehat{\sigma} > 1] - (t+1)^\beta e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma} > t] + t^\beta e^{\theta t}\mathbf{P}[\widehat{\sigma} > t+1]\right). \end{aligned}$$

Next, we specialize this last relation for $\beta = 0$ and $\beta = 1$: For $\beta = 0$, we get

$$\begin{aligned} \Phi_0(t,\theta) &= \mathbf{E}[\sigma](e^\theta + e^{-\theta} - 2)\widehat{F}_0(t,\theta) \\ &\quad + \mathbf{E}[\sigma]\left(e^\theta - \mathbf{P}[\widehat{\sigma} > 1] + e^{\theta t}\mathbf{P}[\widehat{\sigma} > t+1] - e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma} > t]\right). \end{aligned}$$

Hence, using the fact

$$\mathbf{P}[\widehat{\sigma} > 1] = 1 - \widehat{g}_1 = 1 - (\mathbf{E}[\sigma])^{-1} \tag{10.4}$$

in this last expression for $\Phi_0(t,\theta)$, we conclude that

$$\begin{aligned} \Phi_0(t,\theta) - 1 &= \mathbf{E}[\sigma](e^\theta - 1)(1 - e^{-\theta})\widehat{F}_0(t,\theta) - 1 \\ &\quad + \mathbf{E}[\sigma]\left(e^{\theta t}\mathbf{P}[\widehat{\sigma} > t+1] - e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma} > t] + e^\theta - (1 - \mathbf{E}[\sigma]^{-1})\right) \\ &= \mathbf{E}[\sigma](e^\theta - 1)(1 - e^{-\theta})\widehat{F}_0(t,\theta) \\ &\quad + \mathbf{E}[\sigma]\left(e^{\theta t}\mathbf{P}[\widehat{\sigma} > t+1] - e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma} > t] + e^\theta - 1\right). \end{aligned} \tag{10.5}$$

Similarly, for $\beta = 1$ we now find

$$\begin{aligned}
&\Phi_1(t,\theta)\\
= \;& \mathbf{E}[\sigma]\left(e^\theta\left(\widehat{F}_1(t,\theta)+\widehat{F}_0(t,\theta)\right)+e^{-\theta}\left(\widehat{F}_1(t,\theta)-\widehat{F}_0(t,\theta)\right)-2\widehat{F}_1(t,\theta)\right)\\
&+\mathbf{E}[\sigma]\left(e^\theta + te^{\theta t}\mathbf{P}[\widehat{\sigma}>t+1]-(t+1)e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]\right)\\
= \;& \mathbf{E}[\sigma]\left((e^\theta-1)(1-e^{-\theta})\widehat{F}_1(t,\theta)+(e^\theta-e^{-\theta})\widehat{F}_0(t,\theta)\right)\\
&+\mathbf{E}[\sigma]\left(te^{\theta t}\mathbf{P}[\widehat{\sigma}>t+1]-(t+1)e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]+e^\theta\right) \qquad (10.6)
\end{aligned}$$

Injecting (10.5) and (10.6) into (10.2), we get

$$\begin{aligned}
&t(\Phi_0(t,\theta)-1)-\Phi_1(t,\theta)\\
= \;& \mathbf{E}[\sigma]\left(t(e^\theta-1)(1-e^{-\theta})\widehat{F}_0(t,\theta)+t(e^\theta-1)-te^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]\right)\\
&+\mathbf{E}[\sigma]\left(te^{\theta t}\mathbf{P}[\widehat{\sigma}>t+1]-(e^\theta-1)(1-e^{-\theta})\widehat{F}_1(t,\theta)-(e^\theta-e^{-\theta})\widehat{F}_0(t,\theta)\right)\\
&+\mathbf{E}[\sigma]\left(-te^{\theta t}\mathbf{P}[\widehat{\sigma}>t+1]+(t+1)e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]-e^\theta\right)\\
= \;& \mathbf{E}[\sigma](e^\theta-1)\left(t-(1-e^{-\theta})\widehat{F}_1(t,\theta)+(t(1-e^{-\theta})-e^{-\theta}-1)\widehat{F}_0(t,\theta)\right)\\
&+\mathbf{E}[\sigma]\left(e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]-e^\theta\right) \qquad (10.7)
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_0(t,\theta)-1+e^{\theta t}\mathbf{E}[\sigma]\widehat{g}_{t+1}\\
= \;& \mathbf{E}[\sigma]\left((e^\theta-1)(1-e^{-\theta})\widehat{F}_0(t,\theta)+(e^\theta-1)\right)\\
&+\mathbf{E}[\sigma]\left(e^{\theta t}\mathbf{P}[\widehat{\sigma}>t+1]-e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]+e^{\theta t}(\mathbf{P}[\widehat{\sigma}>t]-\mathbf{P}[\widehat{\sigma}>t+1])\right)\\
= \;& \mathbf{E}[\sigma](1-e^{-\theta})\left((e^\theta-1)\widehat{F}_0(t,\theta)+e^\theta-e^{\theta(t+1)}\mathbf{P}[\widehat{\sigma}>t]\right). \qquad (10.8)
\end{aligned}$$

We are now ready to conclude: Substitute (10.7) and (10.8) into (10.2) we get (5.14) after some simple algebra. ■

References

[1] R.G. Addie, M. Zukerman and T. Neame, "Fractal traffic: Measurements, modeling and performance evaluation in *Proceedings of Infocom '95*, Boston (MA), April 1995, pp. 985–992.

[2] J. Beran, *Statistics for Long-Memory Processes*, Chapman and Hall, New York (NY), 1994.

[3] J. Beran, R. Sherman, M. S. Taqqu and W. Willinger "Long-range dependence in variable bit-rate video traffic," *IEEE Transactions on Communications* **COM–43** (1995), pp. 1566–1579.

[4] D. R. Cox, "Long–Range Dependence: A Review," *Statistics: An Appraisal*, H. A. David and H. T. David, Eds., The Iowa State University Press, Ames (IA), 1984, pp. 55–74.

[5] D. R. Cox and V. Isham, *Point Processes*, Chapman and Hall, New York (NY), 1980.

[6] N. G. Duffield and N. O'Connell, "Large deviations and overflow probabilities for the general single server queue, with applications," *Proceedings of the Cambridge Philosophical Society* **118** (1995), pp. 363–374.

[7] J.D. Esary, F. Proschan, and D.W. Walkup, "Association of random variables, with applications," *Annals of Mathematical Statistics* **38** (1967), pp. 166–1474.

[8] A. Erramilli, O. Narayan and W. Willinger, "Experimental queuing analysis with long–range dependent packet traffic," *IEEE/ACM Transactions on Networking* **4** (1996), pp. 209–223.

[9] H. J. Fowler and W. E. Leland, "Local area network traffic characteristics, with implications for broadband network congestion management," *IEEE Journal on Selected Areas in Communications* **JSAC–9** (1991), pp. 1139–1149.

[10] M. Garrett and W. Willinger, "Analysis, modeling and generation of self–similar VBR video traffic," *Proceedings of SIGCOMM '94*, September 1994, pp. 269–280.

[11] P.W. Glynn and W. Whitt, "Logarithmic asymptotics for steady–state tail probabilities in a single–server queue," *Journal of Applied Probability* **31** (1994), pp. 131–159.

[12] G. Kesidis, J. Walrand and C.S. Chang, "Effective bandwidths for multiclass Markov fluids and other ATM sources," *IEEE/ACM Transactions on Networking* **1** (1993), pp. 424–428.

[13] W. Leland, M. Taqqu, W. Willinger, and D. Wilson, "On the self-similar nature of ethernet traffic (extended version)," *IEEE/ACM Transactions on Networking* **2** (1994), pp. 1–15.

[14] M. Livny, B. Melamed and A.K. Tsiolis, "The impact of autocorrelation on queueing systems," *Management Science* **39** (1993), pp. 322–339.

[15] N. Likhanov, B. Tsybakov and N.D. Georganas, "Analysis of an ATM buffer with self-similar ("fractal") input traffic," in *Proceedings of Infocom '95*, Boston (MA), April 1995, pp. 985–992.

[16] R.M. Loynes, "The stability of a queue with non–independent inter–arrival and service times," *Proceedings of the Cambridge Philosophical Society* **58** (1962), pp. 497–520.

[17] I. Norros, "A storage model with self-similar input," *Queueing Systems – Theory & Applications* **16** (1994), pp. 387–396.

[18] M. Parulekar and A.M. Makowski, "Tail probabilities for a multiplexer with self-similar traffic," in *Proceedings of Infocom' 96*, April 1996, San Francisco (CA), pp. 1452–1459.

[19] M. Parulekar and A.M. Makowski, *Buffer Overflow Probabilities for a Multiplexer With Self-Similar Traffic*, Technical Report **TR–95-67**, Institute for Systems Research, University of Maryland, College Park (MD).

[20] M. Parulekar and A.M. Makowski, "Tail probabilities for $M|G|\infty$ processes (II): Buffer asymptotics," in preparation.

[21] M. Parulekar, *Buffer Engineering for Self-Similar Traffic*, Ph.D. Thesis, Electrical Engineering Department, University of Maryland, College Park (MD). Expected December 1996.

[22] V. Paxson and S. Floyd, "Wide area traffic: The failure of Poisson modeling," *IEEE/ACM Transactions on Networking* **3** (1993), pp. 226–244.

[23] W. Willinger, M. S. Taqqu, W. E. Leland and D. V. Wilson, "Self–similarity in high–speed packet traffic: Analysis and modeling of ethernet traffic measurements," *Statistical Science* **10** (1995), pp. 67–85.