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A Theory of Adaptive Quasi Linear Representations

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A Theory of Adaptive Quasi Linear Representations

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Abstract

The analysis of the discrete multiscale edge representation is considered. A general signal description, called an inherently bounded Adaptive Quasi Linear Representation (AQLR), motivated by two important examples: the wavelet maxima representation and the wavelet zero-crossings representation, is introduced. This paper addresses the questions of uniqueness, stability, and reconstruction. It is shown, that the dyadic wavelet maxima (zero-crossings) representation is, in general, nonunique. Namely, for all maxima (zero-crossings) representation based on a dyadic wavelet transform, there exists a sequence having a nonunique representation. Nevertheless, these representations are always stable. Using the idea of the inherently bounded AQLR two stability results are proven. For a general perturbation, a global BIBO stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied. A reconstruction algorithm, based on the minimization of an appropriate cost function, is proposed. The convergence of the algorithm is guaranteed for all inherently bounded AQLR. In the case, where the representation is based on a wavelet transform, this method yields an efficient, parallel algorithm, especially promising in an analog-hardware implementation.

1 Introduction

An interesting and promising approach to a signal representation is to make explicit important features in the data. The first example, taught in elementary calculus, is a "sketch" of a function based on extrema of a signal and possibly of its first few derivatives. The second instance, widely used in computer vision, is an edge representation of an image. If the size of an expected feature is apriori unknown, a need for a multiscale analysis is apparent. Therefore, it is not surprising that multiscale sharper variation points (edges) are meaningful features for many signals, and they have been applied, for example, in edge detection, signal compression, pattern classification, pattern matching, and speech analysis.

1.1 A Brief Review

Traditionally, multiscale edges are determined either as extrema of Gaussian-filtered signals [16] or as zero-crossings of signals convolved with the Laplacian of a Gaussian (see e.g. [5] for a comprehensive review). S. Mallat in series of papers [12, 10, 11] introduced zero-crossings and extrema of the wavelet transform as a multiscale edge representation. Two important advantages of this method are low algorithmic complexity and flexibility in choosing the basic filter. Moreover, [10] and [11] propose reconstruction procedures and show accurate numerical reconstruction results from zero-crossings and maxima representations. In [11, 10], as in many other works in this area, the basic algorithm was developed using continuous variables. The continuous approach gives an excellent background to motivate and justify the use of either local extrema or zero-crossings as important signal features. Unfortunately, in the continuous framework, analytic tools to investigate the information content of the representation are not yet available. The knowledge about properties of the representations is mainly based on empirical reconstruction results. From the theoretical point of view, there are still important open problems, e.g. stability, uniqueness, and structure of a reconstruction set (a family of signals having the same representation).

Our objective is to analyze these theoretical questions using a model of an actual implementation. The main assumption is that the data is discrete and finite. The discrete multiscale maxima and zero-crossings representations are defined in a general set-up of a linear filters bank, however, the main goal is to consider a particular case where the filters bank describes the wavelet transform. Since reconstruction sets of both maxima and zero-crossings representations have a similar structure, a general form called the Adaptive Quasi Linear Representation (AQLR) is introduced. Moreover, many generalizations of the basic maxima and zero-crossings representations fit into the framework of the AQLR. This paper uses the idea of the AQLR to investigate rigorously three fundamental questions: uniqueness, stability, and reconstruction.

Regarding the uniqueness question, first, conditions for uniqueness are presented. By applying these conditions to the wavelet transform-based representation, a conclusive result is obtained. It turns out, that neither the wavelet maxima representations nor the wavelet zero-crossings representations are , in general, unique. The proof is based on showing a sinusoidal sequence, whose maxima (zero-crossings) representations cannot be unique for any dyadic wavelet transform.

The next subject is stability of the representation. This issue is of great importance because there are many known examples of unstable zero-crossings representations. In order to improve the stability properties, Mallat has included additional sums in the standard zero-crossings representation and, together with Zhong, they have introduced the wavelet maxima representation, as a stable alternative to the zero-crossings representation. Indeed, very good numerical results have been reported, but stability analysis has not been pursued. Using the idea of the inherently bounded AQLR, we are able to prove stability results. For a general perturbation, a global BIBO (bounded input, bounded output) stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied.

One of the most important practical problems is a need for an effective reconstruction scheme. Mallat and Zhong in [11] and [10] have used an algorithm based on alternate projections. In this paper, an alternative reconstruction scheme is proposed. The procedure is valid for any inherently bounded AQLR and it is based on an appropriate cost function, whose minimum is achieved at the reconstruction set. Specifically, we focus on an algorithm which is based on the integration of the gradient of the cost function. It is shown that this algorithm approaches the reconstruction set. This method yields efficient, parallel algorithms, promising especially in the case of the wavelet transform. In particular, the analog-hardware implementation, which is similar to a neural network, may lead to a very efficient and fast scheme.

1.2 Previous works

The multiscale edge representation has mainly been investigated in the zero-crossings case. The best-known result concerning the reconstruction of a signal from zero-crossings is the Logan Theorem [9]. This theorem basically states that zero-crossings uniquely define the signal within the family of band-pass signals having the property that the width of the band is smaller than the lower frequency of the band. Proving this theorem, Logan made an analytic extension of the signal and used standard properties of zeros of analytic functions. These tools are known as unstable and Logan has noticed that the reconstruction from zero-crossings appears to be very difficult and impractical. Under certain restrictions on the class of signals, usually polynomial data have been assumed, several additional proofs that zero-crossings form a complete (unique) signal representation have been published. All known proofs do not provide any stability results since they are based on unstable characterizations of analytic functions. The reader is referred to [5] for more details and further references.

In addition, in the case of general initial data, the restriction to polynomial data or even to band-limited signals may provide a poor approximation of the original signal. The situation is similar to the fact that a polynomial is determined by its zeros, but any nonzero value of a continuous function cannot be determined from zero-crossings of the function.

In spite of the last remark, there have been a number of attempts to reconstruct signals from multiscale zero-crossings, especially in image processing, e.g. [3, 18, 14]. They have been based on the believe that the restriction of a given reconstruction scheme into "natural" image data will be sufficiently stable and precise. Although good reconstruction results have been shown, however, any stability results have not been proven.

R. Hummel and R. Moniot ([5]) have exhibited the stability problem by showing two significantly different signals having almost the same multiscale zero-crossings representations. In order to stabilize the reconstruction of a function from its zero-crossings, the authors have included the gradient along each zero-crossing. In fact, improved numerical results have been reported but stability has not been analyzed. The reconstruction algorithm in [5] is based on the solution of a Heat Equation, this approach is valid only for the Laplacian of a Gaussian filter and it is required to record the zero-crossings on the dense sequence of scales.

Aware of the above problems, S. Mallat [10] proposed to use the wavelet zero-crossings representation as a complete and stable signal description. In order to overcome the apparent instability of zero-crossings, he has included the values of the wavelet transform integral calculated between two consecutive zero-crossings. Using a reconstruction algorithm based on alternate projections, very accurate reconstruction results have been shown. In the next work, S. Mallat together with S. Zhong [11] introduced the wavelet maxima representation as an alternative to the wavelet zero-crossings representation. As in the zero-crossings case, they have demonstrated very accurate reconstruction results. But, in both papers, neither uniqueness nor stability has been proven.

From reading the related papers, we have concluded that using a continuous variable approach, several very promising representations and reconstruction algorithms have been developed. Especially algorithms based on the wavelets deserve particular attention, because of low complexity of the fast wavelet transform and because of possible flexibility in choosing the basic filter. These algorithms provide accurate numerical reconstruction results, but their basic properties have not been analyzed yet. The reason seems to be, that the analysis of a continuous multiscale representation is a very difficult mathematical problem. On the other hand, even if the continuous analysis is given, the conclusions about the discrete realization are not obvious.

The work is mainly motivated as an attempt to analyze rigorously the numerical reconstruction results from the wavelet maxima representation and from the wavelet zero-crossings representation. This objective leads to the discrete and finite data assumption. It turns out, that the discrete implementation of a continuous framework is a delicate procedure and many details should be worked out. Even, in the case where the discretization of the linear transform is straightforward [1], the maxima and zero-crossings representations should be redefined and the investigated problems should be restated. Surprisingly, in the discrete framework, uniqueness, stability, and reconstruction questions can be answered analytically. The following sections describe the obtained results. Before getting into details, let us point out that the analysis gives rise to generalizations of the basic wavelet maxima and zero-crossings representations. As long as one keeps the structure of an inherently bounded Adaptive Quasi Linear Representation (defined in the next section), stability and reconstruction are guaranteed.

[1]An additional advantage of the wavelet transform is a very clear correspondence between the continuous and the discrete transforms.

2 The Multiscale Maxima Representation

This section describes the definitions of a discrete multiscale extrema (maxima) representation, an Adaptive Quasi Linear Representation (AQLR), and an inherently bounded AQLR. The main subsequent result is to show that the multiscale maxima representation, based on a wavelet transform, is an inherently bounded AQLR.

Consider $\mathcal{L}$, a linear space of real, finite sequences:

$$\mathcal{L} \triangleq \left\{ f : f = \{f(n)\}_{n=0}^{N-1} \quad f(n) \in \Re \right\}.$$

Let X and Y denote operators on $\mathcal{L}$ which provide the sets of local maxima and minima, respectively, of a sequence $f \in \mathcal{L}$. The formal definitions are :

$$Xf = \{k : f(k+1) \leq f(k) \text{ and } f(k-1) \leq f(k) \quad k = 0, 1, 2, \ldots, N-1\} \quad (1)$$

$$Yf = \{k : f(k+1) \geq f(k) \text{ and } f(k-1) \geq f(k) \quad k = 0, 1, 2, \ldots, N-1\}. \quad (2)$$

In this work, in order to avoid boundary problems, an N-periodic extension of finite sequences is assumed.

Let $W_1, W_2, \ldots, W_J, S_J$ be linear operators on $\mathcal{L}$. The sets XW_jf, YW_jf are local maxima and minima points of the sequence W_jf. The values of W_jf at extreme points are denoted by $\{W_jf(k)\}_{k \in XW_jf \cup YW_jf}$. The multiscale local extrema representation, R_mf is defined as:

$$R_mf \triangleq \left\{ \left\{ XW_jf, YW_jf, \{W_jf(k)\}_{k \in XW_jf \cup YW_jf} \right\}_{j=1}^{J}, S_Jf \right\}. \quad (3)$$

Mallat and Zhong ([11]) have further modified this transformation to include only local maxima of absolute values. They have used the term "maxima representation" for this signal description. Following [11], R_mf, even in the version (3), will be called the multiscale maxima representation as well. In the particular case, when $W_1, W_2, \ldots, W_J, S_J$ correspond to a wavelet transform, R_mf will be called the wavelet maxima representation.

Generally speaking, R_m is a nonlinear operator and its analysis is not easy. Our approach is to separate linear and nonlinear components. The determination of the extrema points sets is highly nonlinear. However, for the given extrema sets, XW_jf and YW_jf, the remaining data are obtained by a linear operation of sampling an image of a linear operator at fixed points. This observation is the motivation to consider R_mf as consists of two parts: the sampling information and the maxima information. The sampling information is the sequence $S_J f$ and the values of $W_j f$ at the points $XW_j f \cup YW_j f$ (j=1,2,...,J). The maxima information consists of the sets $XW_j f, YW_j f$ and the of fact that the elements of $XW_j f$ and $YW_j f$ are local maxima and minima of $W_j f$.

Let T_{mf} denote the linear operator associated with the sampling information. The following is its precise definition.

$$T_{mf} : \mathcal{L} \to \mathcal{L}^e$$

such that for all $h \in \mathcal{L}$

$$T_{mf} h = \{S_J h, \{W_1 h(k)\}_{k \in XW_1 f \cup YW_1 f}, \ldots, \{W_J h(k)\}_{k \in XW_J f \cup YW_J f}\}. \quad (4)$$

$\mathcal{L}^e$ is the linear space of finite, real sequences of length N^e, where

$$N^e = N + \sum_{j=1}^{J} (| XW_j f | + | YW_j f |).$$

Now, $R_m f$ is written in an alternative way as:

$$R_m f = \left\{ \{XW_j f, YW_j f\}_{j=1}^{J}, T_{mf} f \right\}. \quad (5)$$

This form will lead to a definition of a general family of signal descriptions having a common structure of a reconstruction set. For a given representation Rf, a reconstruction set $\Gamma(Rf)$ is defined as a set of all sequences satisfying this representation, i.e.

$$\Gamma(Rf) \triangleq \{\gamma \in \mathcal{L} : R\gamma = Rf\}. \quad (6)$$

At this point, the structure of the reconstruction set of the multiscale maxima representation is considered. It is clear that in order to satisfy a given maxima representation, a sequence $h \in \mathcal{L}$, in addition to obeying the sampling information $T_{mf} h = T_{mf} f$, needs to meet the requirement that $W_j h$ has local extrema at points of $XW_j f$ and $YW_j f$. Suppose that $T_{mf} h = T_{mf} f$ and for a moment let us dwell upon the latter condition. Roughly speaking, we have to assure that $W_j h$ is increasing after a minimum and before a maximum and it is increasing otherwise. In order to make it rigorous we need to introduce several definitions. For any $k \in XW_j f \cup YW_j f$, the segment of k with respect to extrema of f at level j, $P_j^{mf}(k)$ is defined as:

$$P_j^{mf}(k) \triangleq \{k, k+1, \ldots, k+r\} \quad (7)$$

such that:

$$r \geq 1$$
$$k + r \in XW_jf \cup YW_jf$$
$$k+1, \ldots, k+r-1 \in \overline{XW_jf \cup YW_jf}$$

where $\overline{A}$ denotes the complement of the set A with respect to $\{0, 1, \ldots, N-1\}$. Remark: due to the N-periodic extension, $k+i$ is defined modulo N.

For all $i \in \overline{XW_jf \cup YW_jf}$, its type $t(i)$ is introduced by:

$$t(i) \triangleq \begin{cases} -1 & \text{if } k \in XW_jf \text{ and } i \in P_j^{mf}(k) \\ 1 & \text{otherwise.} \end{cases}$$

This is a valid definition because it is easy to show that for a given j and for all $i \in \overline{XW_jf \cup YW_jf}$ there exists exactly one $k \in XW_jf \cup YW_jf$ such that $i \in P_j^{mf}(k)$.

The desired monotonic property can be achieved by enforcing an appropriate constraint on $W_jf(k+1) - W_jf(k)$ $(> 0, \geq 0, < 0, \leq 0)$. If one of the points $k+1, k$ is not an extremum, the kind of the constraint is a function of k and will be defined by the type of k, $t(k)$. If both k and $k+1$ are extreme points, the specific constraint cannot be defined solely either by k or by $k+1$. However, in the latter case, the sampling information assures the right relationship between $W_jf(k+1)$ and $W_jf(k)$. Consequently, the regular subset of $XW_jf \cup YW_jf$ is defined by:

$$(XW_jf \cup YW_jf)^r \triangleq \{k \in XW_jf \cup YW_jf : k+1 \in \overline{XW_jf \cup YW_jf}\}. \quad (8)$$

For all $k \in (XW_jf \cup YW_jf)^r$, the type of k, $t(k)$ is defined by:

$$t(k) \triangleq \begin{cases} -1 & \text{if } k \in XW_jf \\ 1 & \text{otherwise.} \end{cases}$$

In view of these considerations, the following theorem is easily verified.

Theorem 1 *R_mf is a given multiscale maxima representation. $h \in \Gamma(R_mf)$ if and only if*

$$T_{mf}h = T_{mf}f \quad (9)$$
$$t(k) \cdot (W_jh(k+1) - W_jh(k)) > 0 \quad (10)$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for all

$$k \in \overline{XW_jf \cup YW_jf} \cup (XW_jf \cup YW_jf)^r.$$

The maxima representations can be cast into the form $Rf = \{Vf, Tf\}$, where Vf is a set of points and T is a linear operator which may depend on Vf. However, the key feature of the maxima representation is the fact that the set Vf yields additional constraints in the form of linear inequalities, which do not appear directly in Rf. Stimulated by this observation, we define the following general family of signal representations.

Definition 1 *$Rf = \{Vf, Tf\}$ is called an Adaptive Quasi Linear Representation (AQLR) if there exists a linear operator A and a sequence a such that:*

$$x \in \Gamma(Rf) \Leftrightarrow Tx = Tf \ \ \textit{and} \ \ Ax > a. \tag{11}$$

A, a may depend on Vf, but they must be independent of Tf.

The reasoning behind the name "Adaptive Quasi Linear Representation" (AQLR) is the following. This representation is adaptive since T, A, a depend on the sequence f (via the set Vf). It is quasi linear because it is based on a set of linear equalities and inequalities.

Clearly, the following is true.

Proposition 1 *The multiscale maxima representations is an AQLR.*

The next definition is a generalization of an essential boundness property of the wavelet maxima representation.

Definition 2 *An AQLR is called inherently bounded if there exists a real $K > 0$ such that*

$$x \in \Gamma(Rf) \Rightarrow ||x|| \leq K||Tf||.$$

In this work, $||\cdot||$ denotes the Euclidean norm. The coefficient K can depend on the parameters of the representation e.g. $N, J, W_1, \ldots, W_J, S_J$ but it must be independent of Vf and Tf.

Proposition 2 *The wavelet maxima representation is inherently bounded AQLR.*

Proof: Let $h \in \Gamma(R_m f)$. We need to find $K > 0$ such that $||h|| \leq K||T_{mf} f||$. First recall (or see [11]) that the discrete wavelet transform satisfies Parseval's equality, namely:

$$||h||^2 = ||S_J h||^2 + \sum_{j=1}^{J} ||W_j h||^2. \tag{12}$$

Therefore, it suffices to bound $||W_j h||$, $||S_J h||$. $S_J h$ is included in $T_{mf} f$, hence:

$$||S_J h|| \leq ||T_{mf} f||. \tag{13}$$

Consider:

$$| W_j h(n) | \leq \max_n | W_j h(n) | = \max_n | W_j f(n) | \leq ||T_{mf} f||. \tag{14}$$

The middle equality holds because $W_j h$ has the same local extrema as $W_j f$, in particular it has the same global extrema as $W_j f$. The right inequality is valid since $\max_n | W_j h(n) |$ appears (with its original sign) as a component of $T_{mf} f$. Therefore we conclude

$$||W_j h|| \leq \sqrt{N} ||T_{mf} f||. \tag{15}$$

Substituting (15) and (13) to (12) yields:

$$||h|| \leq \sqrt{(NJ+1)} ||T_{mf} f||. \tag{16}$$

□

Remark: The above bound is not the best possible, for example the factor $\sqrt{J}$ can easily be removed. However, we conjecture that the best bound has to of order $\sqrt{N} ||T_{mf} f||$.

The next section describes a very similar treatment for the multiscale zero-crossings representation. The main observation is that the wavelet zero-crossings representation is an inherently bounded AQLR.

3 The Multiscale Zero-Crossings Representation

In defining the multiscale zero-crossings representation, we essentially follow [10], but minor changes are necessary due to our basic assumption that only a discrete signal version is available. Let Z be an operator which provide a set of zero-crossings points of a given sequence $f \in \mathcal{L}$, i.e.

$$Zf \triangleq \{k : f(k-1) \cdot f(k) \leq 0\}. \tag{17}$$

Mallat in [10] has stabilized the zero-crossings representation by including the values of the wavelet transform integral calculated between consecutive zero-crossings points. For the purpose of the precise discrete definition of these values, the segment of k, with respect to zero-crossings of f at level

j, is introduced. It is denoted by $P_j^{zf}(k)$ and defined, for all $k \in ZW_jf$, as follows:

$$P_j^{zf}(k) \triangleq \{k, k+1, \ldots, k+r\} \tag{18}$$

such that

$$r \geq 0$$

$$k+r+1 \in ZW_jf$$

$$k+1, k+2, \ldots, k+r \in \overline{ZW_jf}.$$

The sequence of sums of $h(n)$ between consecutive zero-crossings points of f at level j, $U_j^{zf}h$ is given by:

$$U_j^{zf}h \triangleq \left\{ \sum_{i \in P_j^{zf}(k)} W_jh(i) \right\}_{k \in ZW_jf} . \tag{19}$$

The multiscale zero-crossings representation, R_zf, is defined as:

$$R_zf \triangleq \left\{ \{ZW_jf, U_j^{zf}f\}_{j=1}^{J}, S_Jf \right\}. \tag{20}$$

As in the maxima representation case, for fixed sets ZW_jf, the remaining data $U_j^{zf}f$ and S_Jf are obtained by a linear operator, denoted by T_{zf}.

$$T_{zf} : \mathcal{L} \rightarrow \mathcal{L}^o$$

such that:

$$T_{zf}h = \{S_Jh, U_1^{zf}h, \ldots, U_J^{zf}h\}. \tag{21}$$

$\mathcal{L}^o$ is a linear space of finite, real sequences of length $N + \sum_{j=1}^{J} \mid ZW_jf \mid$. The zero crossings representation becomes:

$$R_zf = \left\{ \{ZW_jf\}_{j=1}^{J}, T_{zf}f \right\}. \tag{22}$$

The above form is helpful in the study of the structure of the reconstruction set. Note, that in order to have $h \in \Gamma(R_zf)$, in addition to obeying

$$T_{zf}h = T_{zf}f,$$

W_jh has to satisfy sign constraints yielding zero-crossings exactly at ZW_jf points. For the purpose of stating precisely the latter constraint, the set $(ZW_jf)^r$ is defined.

$$(ZW_jf)^r \triangleq \{k \in ZW_jf : \left(U_j^{zf}f\right)(k) \neq 0\}. \tag{23}$$

Observe that $(ZW_jf)^r$ consists of "proper" zero-crossings points, namely only points k for which $W_jf(k) \neq 0$ are taken into account.

Theorem 2 *Let R_zf be a given multiscale zero-crossings representation. $h \in \Gamma(R_zf)$ if and only if*

$$T_{zf}h = T_{zf}f \tag{24}$$

$$sgn\left(\left(U_j^{zf}f\right)(k)\right) \cdot W_jh(i) > 0. \tag{25}$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for all $i \in \overline{ZW_jf} \cup (ZW_jf)^r$ where k satisfies $i \in P_j^{zf}(k)$.

Proof: By straightforward applications of the above definitions.

□

As an immediate consequence of the theorem we are given:

Proposition 3 *The multiscale zero-crossings representation is an AQLR.*

The following characteristic of the zero-crossings representation requires the perfect reconstruction property, which is one of the basic features of the wavelet transform.

Theorem 3 *The wavelet zero-crossings representation is an inherently bounded AQLR.*

Proof: Let $h \in \Gamma(R_zf)$. Tf will be an abbreviated notation for $T_{zf}f$. We need to find a constant $K > 0$ such that $\|h\| \leq K\|Tf\|$.

Let j and $k \in ZW_jf$ be arbitrary and fixed. It follows from the sampling information constraint (refeq:24) that:

$$\sum_{l \in P_j^{zf}(k)} W_jh(l) = \sum_{l \in P_j^{zf}(k)} W_jf(l) = \left(U_j^{zf}f\right)(k). \tag{26}$$

Since W_jh has the same zero-crossings points as W_jf, for all $l \in P_j^{zf}(k)$ the values of $W_jh(l)$ have the same, fixed sign. Therefore

$$\sum_{l \in P_j^{zf}(k)} | W_jh(l) | = \left|\left(U_j^{zf}f\right)(k)\right|. \tag{27}$$

Applying

$$\sum x_i^2 \leq \sum x_i^2 + 2 \cdot \sum x_i x_j = (\sum x_i)^2$$

for nonnegative x_i's, we obtain:

$$\sum_{l \in P_j^{zf}(k)} \mid W_j h(l) \mid^2 \leq \left| \left(U_j^{zf} f \right)(k) \right|^2 . \tag{28}$$

Now, using the Parseval's equality and the definition of the Euclidian norm:

$$||h||^2 = ||S_J h||^2 + \sum_{j=1}^{J} ||W_j h||^2 =$$

$$= \sum_{i=1}^{N} \mid S_J h(i) \mid^2 + \sum_{j=1}^{J} \sum_{k \in ZW_j f} \sum_{l \in P_j^{zf}(k)} \mid W_j h(l) \mid^2 \leq$$

$$\leq \sum_{i=1}^{N} \mid S_J h(i) \mid^2 + \sum_{j=1}^{J} \sum_{k \in ZW_j f} \left| \left(U_j^{zf} f \right)(k) \right|^2 = ||Tf||^2.$$

Thus, finally

$$||h|| \leq ||Tf||. \tag{29}$$

□

Notice, that for the wavelet zero-crossings representation, $K = 1$, regardless of the values of N and J.

4 Basic Properties of AQLR's

After two important examples of inherently bounded AQLR's, the wavelet maxima representation and the wavelet zero-crossings representation, have been described, several basic properties of AQLR's are presented. The introduced results are: uniqueness characterization, description of the reconstruction set by its vertices, and bounds on the reconstruction set. The first result is valid for any AQLR, while the remaining two are valid only for inherently bounded AQLR. They are based on convex analysis and parametric linear programming. There are many relevant sources for the subject, we have mostly used [15, 4]. The primary objective of this section is to establish foundations for the subsequent discussion about uniqueness, stability and reconstruction.

A representation $Rf = \{Vf, Tf\}$ is said to be unique, if the reconstruction set $\Gamma(Rf)$ consists of exactly one element. We have the following uniqueness characterization for AQLR's.

Lemma 1 *Let $Rf = \{Vf, Tf\}$ be an AQLR. Then Rf is unique if and only if the kernel of the operator T is trivial, i.e. $\mathcal{N}T = \{0\}$.*

Proof: The lemma becomes obvious by topological arguments. Nevertheless, an elementary but constructive proof will be given. Initially, let us assume that the representation is not unique. Then there exists $h \neq f$ such that $Rh = Rf$. In particular, $Th = Tf$, but then $0 \neq h - f \in \mathcal{N}T$.

Next, consider the case where the kernel of T, $\mathcal{N}T$ is not trivial. Let $h \neq 0$ be such that $Th = 0$. Suppose $\alpha > 0$ and consider $f_\alpha \triangleq \alpha h + f$, with $f \in \Gamma(Rf)$, as a candidate to belong to $\Gamma(Rf)$. Of course $Tf_\alpha = Tf$, therefore $f_\alpha \in \Gamma(Rf)$ if and only if $Af_\alpha > a$ (see Definition 1). The latter is equivalent to:

$$\alpha \cdot Ah > a - Af. \tag{30}$$

Let $(a - Af)_i$ be the i-th component on the vector $a - Af$. Observe that $(a - Af)_i$ is negative for all i. Define:

$$\alpha_0 \triangleq min\left\{\frac{(a - Af)_i}{(Ah)_i} : (Ah)_i < 0\right\}. \tag{31}$$

Note that $\alpha_0 > 0$. It is easy to show that for all α such that $0 \leq \alpha < \alpha_0$:

$$Af_\alpha > a \tag{32}$$

Consequently, the representation Rf is not unique.

□

This claim has some significant consequences. Using the above lemma, an algorithm which tests for uniqueness can be developed. One option is to derive it from a rank test of the operator T. Another, more ambitious, approach is to characterize, for a particular application, all sets Vf giving rise to a unique representation. Perhaps the most important consequence of Lemma 1 is the fact that uniqueness of the representation Rf is equivalent to uniqueness of the underlying irregular sampling Tf. In other words, in the unique case, all the information about the signal is already contained in Tf. Additional constraints $Af > a$ are redundant. On the other hand, from the signal compression, understanding and interpretation point of view, it seems to be desirable that a little information would be specified explicitly by Tf and as much as possible information about a signal should be described implicitly by $Af > a$. Therefore, in our opinion, the most important and interesting features of AQLR's appear in the nonunique case.

At this point, the structure of the reconstruction set is described. Let $Rf = \{Vf, Tf\}$ be an AQLR, its reconstruction set is given as: [2]

$$\Gamma = \{x : Tx = Tf, \quad Ax > a\}. \tag{33}$$

The closure of the reconstruction set, Γ^c is the following convex polyhedron.

$$\Gamma^c = \{x : Tx = Tf, \quad Ax \geq a\}. \tag{34}$$

Since every equality of the form $x_i = t_i$ can be replaced by two inequalities $x_i \geq t_i$, $-x_i \geq -t_i$, without loss of generality, we can assume that

$$\Gamma^c = \{x : Bx \geq b\}$$

for a given $p \times N$ matrix B and a p-dimensional vector b.

For an inherently bounded AQLR's, the associated set Γ^c is bounded. Therefore as a special case of the theorem of Krein and Milman [8], the following holds.

Theorem 4 *For an inherently bounded AQLR, the closure of the reconstruction set is the convex hull of its finitely many vertices.*

In the sequel, the following property of a polyhedron vertex will be used. Let $\{x : Bx \geq b\}$ be a polyhedron and v^i its vertex. Then, there exist N rows of B, which constitute a regular matrix $[B]^i$ such that:

$$v^i = \left([B]^i\right)^{-1} \cdot [b]^i \tag{35}$$

where $[b]^i$ is a subvector of b corresponding to these N rows. By inserting zero columns to the matrix $([B]^i)^{-1}$, the matrix D^i is obtained, such that:

$$v^i = D^i b. \tag{36}$$

Since the closure of the reconstruction set is the convex hull of its vertices, the above equation can characterize the changes in the reconstruction set

[2]The abbreviated notation Γ is used instead of $\Gamma(Rf)$.

due to perturbations in either the matrix B or the vector b. Accordingly, it will be used to prove the stability results.

The last part of this section, addresses the problem of finding bounds for the set $\Gamma^c = \{x : Bx \geq b\}$. Especially, we will focus on the bound in which the dependency on the matrix B and on the vector b will appear in different factors.

Consider the following characterization of a bounded polyhedron.

Theorem 5 ([15] pp 65)
The polyhedron $\Gamma^c = \{x : Bx \geq b\}$ is bounded if and only if it contains no halfline. If $\Gamma^c \neq \emptyset$ the latter statement holds if and only if the associated homogeneous system of inequalities:

$$Bx \geq 0 \tag{37}$$

admits no nonzero solution.

Notice, that the homogeneous system of inequalities is b independent. Therefore, if exists one b_0 yielding a bounded polyhedron, then $\{x : Bx \geq b\}$ is bounded for all b. Let us assume that the matrix B is fixed and arbitrary, but there exists b_0 such that the set $\{x : Bx \geq b_0\}$ is nonempty and bounded. Then, from the second statement of the theorem, for all $x \neq 0$ there exits at least one index i such that $(Bx)_i < 0$. Let us define

$$\mathcal{I}(x) \triangleq \{i : (Bx)_i < 0\}. \tag{38}$$

Observe, that the following function is well defined for all $x \neq 0$.

$$\lambda_m(x) \triangleq \min\left\{\frac{b_i}{(Bx)_i} : \quad i \in \mathcal{I}(x)\right\}. \tag{39}$$

Next, consider Γ_U, the projection of Γ^c on the unit ball:

$$\Gamma_U \triangleq \{\hat{x} : \|\hat{x}\| = 1, \quad \text{and} \quad \exists\lambda(\hat{x}) > 0 \quad \text{such that} \quad \lambda(\hat{x}) \cdot \hat{x} \in \Gamma^c\}. \tag{40}$$

Proposition 4 *$\lambda_m(\hat{x})$ is a positive, continuous function for all $\hat{x} \in \Gamma_U$.*

Proof: If $\hat{x} \in \Gamma_U$ then $\lambda(\hat{x})(Bx)_i \geq b_i$ $(i = 1, 2, \ldots, p)$. Since for $i \in \mathcal{I}(\hat{x})$, $(B\hat{x})_i$ is negative, therefore, in this case, b_i has to be negative and

$$\frac{b_i}{(B\hat{x})_i} > 0$$

for all $i \in \mathcal{I}(\hat{x})$. Thus, in particular, $\lambda_m(\hat{x}) > 0$ for all $\hat{x} \in \Gamma_U$. Consequently:

$$\beta_m(x) \triangleq \frac{1}{\lambda_m(x)} = \max\left\{\frac{(Bx)_i}{b_i} : i \in \mathcal{I}(x)\right\}. \tag{41}$$

To proceed we need to get rid of the set $\mathcal{I}(x)$ inside the above maximum. Let us define :

$$\mathcal{I}_b \triangleq \{i : b_i < 0\}. \tag{42}$$

Notice, that $\mathcal{I}(\hat{x}) \subseteq \mathcal{I}_b$ for all $\hat{x} \in \Gamma_U$. Thus, $\beta_m(\hat{x})$ can be written as:

$$\beta_m(\hat{x}) = max\left\{max\left\{\frac{(Bx)_i}{b_i}, 0\right\} : i \in \mathcal{I}_b\right\}. \tag{43}$$

From the above formula, it is clear that $\beta_m(\hat{x})$ is a continuous function for all $\hat{x} \in \Gamma_U$. Therefore, since $\beta_m(\hat{x}) > 0$ for all $\hat{x} \in \Gamma_U$,

$$\lambda_m(\hat{x}) = \frac{1}{\beta_m(\hat{x})}$$

is a continuous function for all $\hat{x} \in \Gamma_U$ as well.

□

Because Γ_U is a compact set, the following maximum is well defined.

$$\lambda_m \triangleq \max\{\lambda_m(\hat{x}) : \hat{x} \in \Gamma_U\}. \tag{44}$$

In view of the above considerations, it is easy to show that λ_m is a tight bound on Γ^c, namely:

$$\forall x \in \Gamma^c \quad \|x\| \leq \lambda_m \tag{45}$$

$$\exists x \in \Gamma^c \text{ such that } \|x\| = \lambda_m. \tag{46}$$

The bound λ_m is clearly the best possible. However, it has two important disadvantages. The first is the need to know Γ_U which, although, can be determined independently of calculating Γ^c, may involve complex computations. The second disadvantage is that the effects on the bound of B and b are not separated. In what follows, a less accurate bound, but without the above drawbacks, will be calculated. Consider:

$$\lambda_m(x) = \min\left\{\frac{b_i}{(Bx)_i} : i \in \mathcal{I}(x)\right\} \leq$$

$$\leq \max\{| b_i |: i = 1,2,\ldots,p\} \cdot \min\Big\{\Big\{\frac{-1}{(Bx)_i} : i \in \mathcal{I}(x)\Big\} \leq$$

$$\leq \|b\| \cdot \min\Big\{\Big\{\frac{-1}{(Bx)_i} : i \in \mathcal{I}(x)\Big\}.$$

Let us define

$$\lambda_o(x) \triangleq \min\Big\{\Big\{\frac{-1}{(Bx)_i} : i \in \mathcal{I}(x)\Big\}. \tag{47}$$

Proposition 5 *$\lambda_o(x)$ is a positive and continuous function for all $x \neq 0$.*

Proof: As a consequence of the definition of $\mathcal{I}(x)$, $\lambda_o(x)$ is positive for all $x \neq 0$. To show continuity, let us consider:

$$\beta_o(x) \triangleq \frac{1}{\lambda_o(x)} = \max\{-(Bx)_i : i \in \mathcal{I}(x)\}. \tag{48}$$

Which can also be written as:

$$\beta_o(x) = \max\{\max\{-(Bx)_i, 0\} : i = 1,2,\ldots,p\}. \tag{49}$$

It is apparent from (49) that $\beta_o(x)$ is a continuous function of x, and thus $\lambda_o(x)$ is continuous as well.

□

From the latter form one can see that $\beta_o(x)$ depends on B and x but it is b-independent. Consider any compact set U^c containing nonzero elements such that $\Gamma_U \subseteq U^c$ Then for all $\widehat{x} \in \Gamma_U$:

$$\lambda_m(\widehat{x}) \leq \|b\| \cdot \frac{1}{\beta_o(\widehat{x})} \leq$$

$$\leq \|b\| \cdot \max\Big\{\frac{1}{\beta_o(x)} : x \in U^c\Big\}. \tag{50}$$

For example, the unit ball $U = \{x : \|x\| = 1\}$ is used as a set U^c. Then, using only the matrix B, the coefficient β_U is calculated as:

$$\beta_U \triangleq \min\{\max\{\max\{-(Bx)_i, 0\} : i = 1,2,\ldots,p\} : x \in U\}. \tag{51}$$

Combining together (50), (45), (44) the following result is obtained.

$$\|x\| \leq \frac{\|b\|}{\beta_U} \quad \forall x \in \Gamma^c. \tag{52}$$

The above bound will be used to prove the convergence of the reconstruction algorithm.

5 The theory of nonuniqueness

The section aim is to show that, in general, the discrete dyadic wavelet maxima (zero-crossings) representation is not unique. The results are consequences of Lemma 1, which relates uniqueness of the representation to the set $\mathcal{N}T$, the kernel of the sampling information. The main idea is to show a sequence f such that the set $\mathcal{N}T$ corresponding to the representation Rf cannot be $\{0\}$. The precise statement of the theorem is as follows.

Theorem 6 *A discrete dyadic wavelet maxima (zero-crossings) representation based on a discrete low pass filter $H(w)$ is given. If $H(\pi) = 0$, $J \geq 3$, and N is a multiple of 2^J then there exists a sequence f which has a nonunique maxima (zero-crossings) representation.*

Let us point out that, although, the hypothesis of the theorem may seem to be demanding, it is just a technical condition. Usually the number of levels, J, satisfies $J \geq 3$. In order to benefit from the fast wavelet transform N has to be a multiple of 2^J. Since $H(w)$ is a low pass filter, it is natural to assume that $\mid H(w) \mid$ reaches its minimum at π. If this minimum is nonzero, then essentially $S_J f$ contains all information about f and the maxima (zero-crossings) information is redundant. Indeed, all filters used by Mallat, Zhong and many others fulfill the conditions the theorem.

The most of the section describes the proof the theorem, which will be divided to proofs of several propositions.

The construction of the counter example is based on the set $\mathcal{B}$, defined as follows:

$$\mathcal{B} = \left\{ \{c_r\}_{r=1}^{2^{J-1}}, \{s_r\}_{r=1}^{2^{J-1}-1} \right\} \tag{53}$$

where

$$c_r(n) = \cos(\frac{2\pi rn}{2^J}) \quad n = 0, 1, \ldots, N-1 \tag{54}$$

$$s_r(n) = \sin(\frac{2\pi rn}{2^J}) \quad n = 0, 1, \ldots, N-1. \tag{55}$$

Proposition 6 *The set $\mathcal{B}$ is included in $\mathcal{N}S_J$, the kernel of the operator S_J.*

Proof: This proof applies some specific properties of the discrete dyadic wavelet transform. For through description see [11]. In the sequel, the same notation as in [11] will be used, but only the aspects, necessary for the proof, will be described.

In the discrete dyadic wavelet transform case, the sequence $S_J f$ is defined by the following recursion formula:

$$S_{j+1}f = S_j f * h_j \quad j = 0, 1, 2, \ldots, J-1 \tag{56}$$

with $S_0 f = f$. The symbol $*$ denotes the discrete (N periodic) convolution operator, and the sequence h_j is obtained, from the given 2π-periodic transfer function $H(w)$, in the way described below. Let $\widehat{h}_j$ denote the Discrete Fourier Transform (DFT) of the sequence h_j, i.e.

$$\widehat{h}_j(k) = \sum_{n=0}^{N-1} h_j(n) exp(-2\pi i \frac{nk}{N}) \quad k = 0, 1, \ldots, N-1. \tag{57}$$

The DFT of $h_0 = h$ is obtained from the continuous function $H(w)$ by sampling at $w_k = 2\pi \frac{k}{N}$, namely

$$\widehat{h}(k) = H(2\pi \frac{k}{N}) \quad k = 0, 1, \ldots, N-1.$$

Similarly $\widehat{h}_j(k)$ is defined as the value of $H(2^j w)$ at $w = 2\pi \frac{k}{N}$.

$$\widehat{h}_j(k) = H(2^j 2\pi \frac{k}{N}) \;\; k = 0, 1, \ldots, N-1.$$

Since the periodic convolution corresponds to the multiplication of DFT's, the DFT of $S_J f$ can be written as:

$$\widehat{S_J f}(k) = \widehat{S_J}(k) \widehat{f}(k) \quad k = 0, 1, \ldots, N-1$$

where

$$\widehat{S_J}(k) = \prod_{l=0}^{J-1} \widehat{h}_l(k) \tag{58}$$

is the discrete transfer function of the operator S_J. At this point, let us consider:

$$m(r) = \frac{rN}{2^J} \quad r = \pm 1, \pm 2, \ldots, \pm 2^{J-1}. \tag{59}$$

Recall that N is a multiple of 2^J, thus $m(r)$ is an integer. The next step is to show that $\widehat{S_J}(m(r)) = 0$. Notice that r can be written as $r = 2^l r_1$ where $0 \le l \le J-1$ and r_1 is an odd number. See that:

$$\widehat{h}_{J-1-l} = H\left(2^{J-1-l} 2\pi \frac{2^l r_1 N}{N 2^J}\right) = H(\pi r_1) = 0. \tag{60}$$

Therefore, using (58), indeed we obtain:

$$\widehat{S_J}(m(r)) = 0. \tag{61}$$

The integers $m(r)$'s, as zeros of the transfer function $\widehat{S_J}$, will be used to define sequences belonging to the null space of S_J. Let e_r be the following exponential sequence

$$e_r(n) = exp\left(2\pi i \frac{rn}{N}\right) \qquad n = 0, 1, \ldots, N-1. \tag{62}$$

Its Discrete Fourier Transform, $\widehat{e_r}$ is given by:

$$\widehat{e_r}(k) = N\delta_r(k) \tag{63}$$

where

$$\delta_r(k) = \begin{cases} 1 & \text{if } k = r \\ 0 & \text{otherwise.} \end{cases}$$

Combining together (61) and (63), one can conclude that $\widehat{S_J} e_{m(r)} = 0$, thus

$$S_J e_{m(r)} = 0. \tag{64}$$

The sequences c_r, s_r are expressed by $e_{m(r)}$'s in the subsequent way:

$$c_r(n) = \cos(\frac{2\pi rn}{2^J}) = \frac{1}{2}(e_{m(r)} + e_{m(-r)})$$

$$s_r(n) = \sin(\frac{2\pi rn}{2^J}) = \frac{1}{2i}(e_{m(r)} - e_{m(-r)}).$$

Therefore $S_J c_r = 0$ and $S_J s_r = 0$ for $r = 1, 2, \ldots, 2^{J-1}$.

□

Notice, that $s_{2^{J-1}}$ does not appear in the set $\mathcal{B}$. The reason is that $s_{2^{J-1}} \equiv 0$ and in the next proposition the independence of the set $\mathcal{B}$ is asserted.

Proposition 7 *The set $\mathcal{B}$ is linearly independent.*

Proof: It is a well known fact, which can easily be proven by showing that the set $\mathcal{B}$ is orthogonal and does not contain zero.

□

As an universal counter example of nonuniqueness the following sequence is proposed.

$$f(n) = \cos(2\pi \frac{n}{2^J}) \qquad n = 0, 1, \ldots, N-1. \tag{65}$$

Observe that the same sequence is proposed for all dyadic wavelet transforms and for both the maxima representation and the zero-crossings representation.

The representation $R_m f$ $(R_z f)$ is unique if and only if $\mathcal{N}T_{mf} = \{0\}$ $(\mathcal{N}T_{zf} = \{0\})$. Consequently, the nonuniqueness of $R_m f$ $(R_m f)$ is easily deduced from the following proposition.

Proposition 8 *The equation*

$$T_{mf}h = 0 \;\; (T_{zf}h = 0) \qquad h \in span(\mathcal{B}) \tag{66}$$

has a nontrivial solution.

Proof: Consider an arbitrary $h \in span(\mathcal{B})$.

$$h = \sum_{i=1}^{2^{J-1}} \alpha_i c_i + \sum_{i=1}^{2^{J-1}-1} \alpha'_i s_i \tag{67}$$

The dimension of $span(\mathcal{B})$ is $2^J - 1$. The idea is to show that the set of equations $T_{mf}h = 0$ $(T_{zf}h = 0)$ yields less than $2^J - 1$ independent equations with unknowns $(\{\alpha_i\}, \{\alpha'_i\})$. Recall that:

$$T_{mf}h = \{S_J h, \{W_1 h(k)\}_{k \in XW_1 f \cup YW_1 f}, \ldots, \{W_J h(k)\}_{k \in XW_J f \cup YW_J f}\}.$$

and

$$T_{zf}h = \{S_J h, U_1^{zf} h, \ldots, U_J^{zf} h\}.$$

From Proposition 6 we see that $S_J h = 0$ for all $(\{\alpha_i\}, \{\alpha'_i\})$. Let j be fixed. Consider the equations:

$$W_j h(k) = 0 \qquad k \in XW_j f \cup YW_j f \tag{68}$$

or

$$U_j^{zf} h(k) = 0 \qquad k \in ZW_j f. \tag{69}$$

Recognize that f is a 2^J-periodic sinusoid, therefore $W_j f$ is a 2^J-periodic sinusoid as well. Moreover $W_j c_i, W_j s_i$ are 2^J-periodic too. Therefore solving the equation (68), it suffices to consider

$$k \in \{0, 1, \ldots, 2^J - 1\} \bigcap (XW_j f \cup YW_j f).$$

But $W_j f$ has only two local extreme points in a 2^J period, consequently (68) contains only two different equations with unknowns $(\{\alpha_i\}, \{\alpha_i'\})$! Similarly for zero-crossings, (69) has only two different equations. There are J levels ($j = 1, 2, \ldots, J$) so the set of equations $T_{mf} h = 0$ ($T_{zf} h = 0$) consists, at most, of $2J$ independent equations, but

$$2^J - 1 > 2J \quad \forall J \geq 3. \tag{70}$$

Accordingly, the equation (66) has a nontrivial solution and the representation is not unique.

□

Some remarks need to be made at this point. From the proof, it turns out, that it is relatively easy to produce more examples of nonunique dyadic wavelet maxima (zero-crossings) representations using 2^p-periodic signals, where p is an integer. For example, consider $J = 5$ and let f be a 2^5-periodic signal. Then $W_j f$ ($j = 1, 2, \ldots, 5$) are 2^5-periodic as well. In this case, if $2^J - 1 = 31$ is greater than the total number of local extrema (zero-crossings) of $W_1 f, W_2 f, \ldots, W_5 f$ per one 2^J period, then the representation is not unique. In other words, if $W_j f$'s have, in the mean, less than $\frac{31}{5} = 6.2$ local extrema (zero-crossings) in one period, than $R_m f$ ($R_z f$) cannot be unique.

Hummel with Moniot [5], Mallat [10], and Mallat with Zhong [11] have reported that high frequency errors may occur in the discrete maxima (zero-crossings) representation. For these 2^J-periodic signals, components of the reconstruction error can appear as 2^p-periodic signals for $p = 1, 2, \ldots, J$. Most of them cannot be related as high frequency errors. For more details and for the specific example, the reader is referred to [1, 2].

From our simulations and from Mallat's results it turns out that for the vast majority of signals, the representation is unique. We even conjecture that the wavelet maxima (zero-crossings) representation is unique for a generic family of signals, but we are not able to prove it.

6 Stability

Addressing the stability issue, the standard approach is to introduce the notion of perturbations: of the representation and of the reconstruction set. In addition, measures for a distance between distinct representations and for a distance between different reconstruction sets should be defined. In general, it is not an easy task. Remember that Vf, Tf may have different sizes for different representations. Fortunately, for inherently bounded representations, the following characterization of BIBO (bounded input, bounded output) stability is easily verified.

Proposition 9 *Let $Rf_i = \{Vf_i, T_if_i\}$ $i = 1,2$ be inherently bounded AQLR's. Then for all $K_I > 0$ there exists K_O such that:*

$$\|T_if_i\| \leq K_I \quad (i = 1,2) \Rightarrow \|x_1 - x_2\| \leq K_O \quad \forall x_i \in \Gamma(Rf_i)$$

Proof: This claim is an immediate consequence of the definition of an inherently bounded AQLR.

$$x_i \in \Gamma(Rf_i) \Rightarrow \|x_i\| \leq K \cdot \|T_if_i\| \leq K \cdot K_I$$

$$\|x_1 - x_i\| \leq \|x_1\| + \|x_2\| \leq 2K \cdot K_I$$

□

The above result is strong in the sense that it is valid regardless of the sets Vf_1, Vf_2. It is weak in view of the fact that the bound on $\|x_1 - x_2\|$ is achieved by the bounds on absolute values of x_1, x_2. In this case, a small perturbation in the representation does not necessary yield a small bound of $\|x_1 - x_2\|$. The next result is complimentary in the sense that a certain structure of the perturbation is assumed, but a bound, proportional to the size of the perturbation, is given.

In many applications, the reasons for perturbations in a representation are arithmetic or quantization errors in a reconstruction algorithm. This kind of perturbations may change the continuous values of Tf but it preserves the discrete values of Vf. Therefore the perturbed representation, $(Rf)_p$, can be written as:

$$(Rf)_p = \{Vf, Tf + \Delta(Tf)\}. \tag{71}$$

Let Γ_p be the corresponding reconstruction set. In general, the distance between two reconstruction sets, Γ and Γ_p, is defined by:

$$d(\Gamma, \Gamma_p) \triangleq \sup\{\|\gamma - \gamma_p\| : \gamma \in \Gamma, \gamma_p \in \Gamma_p\}.$$

Observe, that for inherently bounded AQLR's, $d(\Gamma, \Gamma_p)$ is always finite. The measure of the perturbation in the reconstruction set is the difference between $d(\Gamma, \Gamma_p)$ and the size of Γ which is defined as follows:

$$s(\Gamma) \triangleq d(\Gamma, \Gamma) = \sup\{\|\gamma_1 - \gamma_2\| : \gamma_1, \gamma_2 \in \Gamma\}. \tag{72}$$

$s(\Gamma)$ and $d(\Gamma, \Gamma_p)$ describe the largest possible Euclidian norm of a reconstruction error, from the original representation and from a perturbed one, respectively.

One remark is in order. In general, for an arbitrary $\Delta(Tf)$, the associated reconstruction set may be empty and then $d(\Gamma, \Gamma_p)$ would not be defined. In the sequel, it is assumed that this problem is treated by a reconstruction algorithm and hence $\Delta(Tf)$ yields a nonempty Γ_p. In this case, the following Lipschitz condition is satisfied.

Theorem 7 *For all inherently bounded AQLR, there exists $K > 0$ such that:*

$$d(\Gamma, \Gamma_p) \leq K \cdot \|\Delta(Tf)\| + s(\Gamma). \tag{73}$$

Proof: Let Γ^c and Γ_p^c be the closures of the sets Γ and Γ_p, respectively. Since $\|v - \bar{v}\|$ is a continuous function on $\Gamma^c \times \Gamma_p^c$, which is a compact set, then there exist $v \in \Gamma^c$ and $\bar{v} \in \Gamma_p^c$ such that:

$$\|v - \bar{v}\| = d(\Gamma^c, \Gamma_p^c) = d(\Gamma, \Gamma_p). \tag{74}$$

Moreover, v and $\bar{v}$ have to be vertices of Γ^c and Γ_p^c. Indeed, if, for example, v is not a vertex, then there exists $\epsilon > 0$ such that $v + \epsilon(v - \bar{v}) \in \Gamma^c$, and:

$$\|v + \epsilon(v - \bar{v}) - \bar{v}\| = (1 + \epsilon) \cdot \|v - \bar{v}\| > \|v - \bar{v}\|. \tag{75}$$

It contradicts the fact that $\|v - \bar{v}\| = d(\Gamma^c, \Gamma_p^c)$.

Let $\Delta(Tf)$ be fixed and arbitrary, such that Γ_p is nonempty. We define

$$(Rf)_p^\lambda \triangleq \{Vf, Tf + \lambda \cdot \Delta(Tf)\} \quad 0 \leq \lambda \leq 1 \tag{76}$$

with the underlying reconstruction set denoted by Γ_p^λ. From the definition of an Adaptive Quasi Linear Representation (AQLR):

$$\Gamma_p^\lambda = \{x : Tx = Tf + \lambda \cdot \Delta(Tf) \text{ and } Ax > a\}. \tag{77}$$

The above formula yields the following observation: if $x_0 \in \Gamma = \Gamma_p^0$ and $x_1 \in \Gamma_p = \Gamma_p^1$ then

$$x_0 + \lambda \cdot (x_1 - x_0) \in \Gamma_p^\lambda \quad 0 \leq \lambda \leq 1. \tag{78}$$

Therefore Γ_p^λ is nonempty for $0 \leq \lambda \leq 1$ and $d(\Gamma, \Gamma_p^\lambda)$ is well defined.

Next, notice that the closure of Γ_p^λ is given by:

$$\left(\Gamma_p^\lambda\right)^c = \{x : Tx = Tf + \lambda \cdot \Delta(Tf) \;\; Ax \geq a\} = \{x : Bx \geq b + \lambda \Delta b\}. \tag{79}$$

where B is a $p \times N$ matrix and $b, \Delta b$ are p-dimensional vectors. Since every equality of $Tx = Tf + \lambda \cdot \Delta(Tf)$ appears in two rows in $Bx \geq b + \lambda \Delta b$:

$$||\Delta b|| = 2||\Delta(Tf)||. \tag{80}$$

We know that

$$d(\Gamma, \Gamma_p^\lambda) = ||v^\lambda - \bar{v}^\lambda|| \tag{81}$$

where v^λ is a vertex of Γ^c and $\bar{v}^\lambda$ is a vertex of $\left(\Gamma_p^\lambda\right)^c$. Using (36) we can write:

$$v^\lambda = D^i b \quad \text{and} \quad \bar{v}^\lambda = \bar{D}^i(b + \lambda \Delta b). \tag{82}$$

Both matrices D^i and $\bar{D}^i$ are obtained from an inverse of a regular submatrix of B. Note that $||Db - \bar{D}(b + \lambda \Delta b)||$ is a continuous function of λ for any two matrices $D, \bar{D}$. Therefore, if

$$||v^\lambda - \bar{v}^\lambda|| < ||v_o - \bar{v}_o|| \tag{83}$$

for all pairs $v_o, \bar{v}_o$ of vertices of Γ^c, $\left(\Gamma_p^\lambda\right)^c$, respectively, which are different from $v^\lambda, \bar{v}^\lambda$, then there exists a segment $[\lambda_i, \lambda_{i+1}]$ such that:

$$d(\Gamma, \Gamma_p^\lambda) = ||D^i b - \bar{D}^i(b + \lambda \Delta b)|| \quad \forall \lambda \in [\lambda_i, \lambda_{i+1}]. \tag{84}$$

Furthermore, there exists another pair of vertices, with associated matrices D^{i-1} and $\bar{D}^{i-1}$, such that:

$$d\left(\Gamma, \Gamma_p^{\lambda_i}\right) = ||D^i b - \bar{D}^i(b + \lambda_i \Delta b)|| = ||D^{i-1} b - \bar{D}^{i-1}(b + \lambda_i \Delta b)||. \tag{85}$$

Next observe that since the number of regular submatrices of B is finite, the number of possible pairs $D, \bar{D}$ is finite as well. Consider $||Db - \bar{D}(b + \lambda \Delta b)||$ and $||D_o b - \bar{D}_o(b + \lambda \Delta b)||$ as two functions of λ. As square roots of quadratic forms, these expressions may coincide or be equal for at most two values of λ. Therefore all possible pairs of these functions intersect at finitely many points. Consequently, there exist L points:

$$0 = \lambda_0 < \lambda_1 \ldots < \lambda_{L-1} = 1 \tag{86}$$

and L pairs of matrices $(D^i, \bar{D}^i)$ $i = 0, 1, \ldots, L-1$ such that $D^i b$ is a vertex of Γ^c and $\bar{D}^i(b + \lambda\Delta b)$ is a vertex of $\left(\Gamma_p^\lambda\right)^c$ for all $\lambda \in [\lambda_i, \lambda_{i+1}]$. Moreover

$$d(\Gamma, \Gamma_p^\lambda) = \|D^i b - \bar{D}^i(b + \lambda\Delta b)\| \quad \forall\lambda \in [\lambda_i, \lambda_{i+1}]. \tag{87}$$

$$d\left(\Gamma, \Gamma_p^{\lambda_i}\right) = \|D^i b - \bar{D}^i(b + \lambda_i\Delta b)\| = \|D^{i-1} b - \bar{D}^{i-1}(b + \lambda_i\Delta b)\|. \tag{88}$$

Proposition 10

$$d\left(\Gamma, \Gamma_p^{\lambda_i}\right) \le d(\Gamma, \Gamma) + \sum_{k=0}^{i-1}(\lambda_{k+1} - \lambda_k) \cdot \|\bar{D}^k \Delta b\|. \tag{89}$$

Proof: By induction on i. Let $i = 1$.

$$\begin{aligned}
d(\Gamma, \Gamma_p^{\lambda_1}) &= \|D^1 b - \bar{D}^1(b + \lambda_1\Delta b)\| = \\
&= \|D^0 b - \bar{D}^0(b + \lambda_1\Delta b)\| = \\
&= \|D^0 b - \bar{D}^0 b + \bar{D}^0 b - \bar{D}^0(b + \lambda_1\Delta b)\| \le \\
&\le \|D^0 b - \bar{D}^0 b\| + \|\bar{D}^0 b - \bar{D}^0(b + \lambda_1\Delta b)\| = \\
&= d(\Gamma, \Gamma) + \lambda_1 \cdot \|\bar{D}\Delta b\|.
\end{aligned}$$

Since $\lambda_0 = 0$, the above is exactly the claim for $i = 1$. By induction, let us assume that the proposition holds for $i - 1$. Consider:

$$\begin{aligned}
d(\Gamma, \Gamma_p^{\lambda_i}) &= \|D^{i-1} b - \bar{D}^{i-1}(b + \lambda_i\Delta b)\| = \\
&= \|D^{i-1} b - \bar{D}^{i-1}(b + \lambda_{i-1}\Delta b) + \bar{D}^{i-1}(b + \lambda_{i-1}\Delta b) - \bar{D}^{i-1}(b + \lambda_i\Delta b)\| \le \\
&\le \|D^{i-1} b - \bar{D}^{i-1}(b + \lambda_{i-1}\Delta b)\| + \|\bar{D}^{i-1}(b + \lambda_{i-1}\Delta b) - \bar{D}^{i-1}(b + \lambda_{i-1}\Delta b)\| = \\
&= d(\Gamma, \Gamma_p^{\lambda_{i-1}}) + (\lambda_i - \lambda_{i-1})\|\bar{D}^{i-1}\Delta b\| \le
\end{aligned}$$

(using the induction assumption)

$$\le d(\Gamma, \Gamma) + \sum_{k=0}^{i-2}(\lambda_{k+1} - \lambda_k)\|\bar{D}^k \Delta b\| + (\lambda_i - \lambda_{i-1})\|\bar{D}^{i-1}\Delta b\| =$$

$$= d(\Gamma, \Gamma) + \sum_{k=0}^{i-1} (\lambda_{k+1} - \lambda_k) \|\bar{D}^k \Delta b\|.$$

This concludes the proof of the proposition.

□

Using the proposition we deduce that the distance between Γ and Γ_p satisfies:

$$d(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + \sum_{k=0}^{L-1} (\lambda_{k+1} - \lambda_k) \|\bar{D}^k \Delta b\|. \tag{90}$$

Let $\|\bar{D}^i\|$ be the induced matrix norm of $\bar{D}^i$. Then

$$\|\bar{D}^i \Delta B\| \leq \|\bar{D}^i\| \cdot \|\Delta b\|. \tag{91}$$

Since the number of possible matrix $\bar{D}^i$ is finite, there exists $K_D > 0$ such that:

$$\|\bar{D}^i\| \leq K_D \tag{92}$$

for all valid $\bar{D}^i$. Combining together (92),(91), (90) we show that

$$d(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + K_D \|\Delta b\|. \tag{93}$$

By taking $K = 2K_D$ and using (80), the desired relation is obtained:

$$(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + K \cdot \|\Delta(Tf)\|. \tag{94}$$

□

Observe that the above result is global in the sense that as long as $\Delta(Tf)$ gives rise to a nonempty reconstruction set, the theorem holds regardless of the size of $\Delta(Tf)$.

One can ask whether the shown kind of stability is indeed the property that has been desired to achieve. The answer has several different aspects and let us dwell a while upon this subject. First consider the following citation from Hummel and Moniot [5] " stability of the representation concerns continuity of the inverse map ". Theorem 7 is exactly of this type. Another citation, from [10] is as follows: " a representation is said to be unstable if a small perturbation of the representation may correspond to an arbitrary large perturbation of the original function." This definition refers to BIBO stability, which was given by Proposition 9. In view of these considerations, stability, as presented in this work, is indeed a necessary property of multiscale edge representation. But it does not mean that inherently bounded AQLR will always provide accurate reconstruction results. Somehow, perhaps because of partial uniqueness results obtained by unstable tools, poor reconstruction results are often regarded as evidences of instability. For instance, such was the case in the example given in [5], mentioned earlier. A careful discrete analysis may point out different possible reasons for inadequate reconstruction results, e.g. nonuniqueness of the discrete representation, instability or high sensitivity of the reconstruction algorithm. Therefore, stability should not be viewed as a sufficient condition of a signal description, and every practical signal representation has to be tested quantitatively with respect to the size and the structure of reconstruction sets and with respect to sensitivity of the reconstruction algorithm.

7 A Reconstruction Scheme

In a nonunique case, there are several ways to define a reconstruction algorithm. One can require to find all elements from the reconstruction set, sometimes it is desired to determine a smallest element satisfying a given representation. In this work, the reconstruction is defined as a procedure to find any element x belonging to the closure of the reconstruction set, Γ^c. As mentioned earlier, we propose a reconstruction algorithm based on an appropriate potential function $v(x)$. This function should satisfy:

$$v(x) = 0 \quad \forall x \in \Gamma^c \tag{95}$$

$$v(x) > 0 \quad \forall x \in \overline{\Gamma^c}. \tag{96}$$

where $\overline{\Gamma^c}$ denotes the complement of Γ^c in $\mathcal{L}$. Furthermore, it will be shown that the proposed $v(x)$ does not have any local extremum outside Γ^c, i.e.

$$||\nabla v(x)|| > 0 \quad \forall x \in \overline{\Gamma}. \tag{97}$$

$\nabla v(x)$ denotes the gradient of $v(x)$ with respect to x, namely it is a column vector of derivatives of v with respect to components of x. With such a potential function, the reconstruction is achieved by any minimization algorithm operating on $v(x)$. We will focus on the reconstruction algorithm based on the differential equation:

$$\dot{x}(t) = -\nabla\left(v\left(x(t)\right)\right) \tag{98}$$

whose analog hardware implementation give rise to a very fast algorithm.

In this section, a general inherently bounded Adaptive Quasi Linear Representation (AQLR) is considered. As mentioned in Section 4, the closure of the reconstruction set, Γ^c, can be written as:

$$\Gamma^c = \{x : Bx \geq b\} \tag{99}$$

for a given $p \times N$ matrix B and a p-dimensional vector b. The function $v(x)$ is derived from this representation in the subsequent way.

$$v(x) \triangleq \sum_{i=1}^{p} f(Bx - b)_i \tag{100}$$

where $(Bx - b)_i$ denotes the i-th component of the vector $Bx - b$. The function $f(\cdot)$ is defined by:

$$f(\xi) \triangleq \begin{cases} \xi^2 & \text{if } \xi < 0 \\ 0 & \text{otherwise} \end{cases} .$$

Using the above definitions, it is easy to verify that indeed (95) and (96) hold.

Observe that $f(\xi)$ is continuously differentiable. Therefore $v(x)$ is continuous and continuous differentiable. The gradient of $v(x)$ is given by:

$$\nabla v(x) = 2B'Z(Bx - b)$$

where Z is a $p \times p$ diagonal matrix satisfying:

$$Z(i,i) = \begin{cases} 1 & \text{if } (Bx - b)_i < 0 \\ 0 & \text{otherwise} \end{cases} .$$

Naturally, B' denotes the transpose of the matrix B.

The following theorem states that $v(x)$ does not have local extrema outside the set Γ^c.

Theorem 8 *Let Γ be nonempty. Then $\nabla v(x) = 0$ if and only if $x \in \Gamma^c$.*

Proof: If $x \in \Gamma^c$ then clearly $\nabla v(x) = 0$. Let us assume $\nabla v(x) = 2B'Z(Bx - b) = 0$. Since $x \in \Gamma^c$ if and only if $Z(Bx - b) = 0$, we need to show that $Z(Bx - b) = 0$. Consider the following decomposition of Zb:

$$Zb = ZBy + b_o \tag{101}$$

such that $b_o \perp \mathcal{R}(ZB)$, namely $b_o' ZBx = 0 \ \ \forall x$, or equivalently:

$$(ZB)' b_o = 0. \tag{102}$$

Substituting this decomposition into the hypothesis yields:

$$0 = 2B'Z(Bx - b) = 2B'(ZBx - ZBy - b_o) = 2B'(ZBx - ZBy).$$

Using $Z = Z'Z$ we see that

$$2(ZB)'(ZB)(x - y) = 0$$

which implies

$$(x - y)'(ZB)'(ZB)(x - y) = \|ZB(x - y)\| = 0.$$

Therefore

$$ZBx = ZBy. \tag{103}$$

Consequently, in this case:

$$Z(Bx - b) = ZBx - ZBy - b_o = -b_o. \tag{104}$$

Hence, it suffices to prove that $b_o = 0$. This will be based on the following statement of the Farkas theorem of the alternative ([13],p 472-474).

Theorem 9 *Exactly one of the two alternatives holds:*

1. $\exists x$ *s.t.* $ZBx \geq Zb$.
2. $\exists b_o$ *such that* $(ZB)'b_o = 0 \ \ b_o \geq 0 \ \ (Zb)'b_o > 0$.

We are already given $(ZB)'b_o = 0$. Observe that from the definition of Z, $ZBx - Zb \leq 0$, therefore $b_o \geq 0$. At this point, assume by contradiction that $\|b_o\| > 0$. Consider

$$(Zb)'b_o = (ZBy + b_o)'b_o = \|b_o\|^2 > 0.$$

Therefore the second alternative holds. For the first alternative take any $x \in \Gamma^c$. Then $Bx \geq b$, and for any matrix Z with nonnegative entries: $ZBx \geq Zb$. Hence the first alternative holds as well. This is the contradiction we were after, and eventually we have $b_o = 0$.

□

In view of these considerations, a reconstruction scheme can be implemented as:

$$\arg\min \{v(x) : x \in \mathcal{L}\}. \tag{105}$$

The minimization is significantly facilitated by the property that local extrema of $v(x)$ appear only in Γ^c. We are going to focus on the algorithm based on the differential equation (98). The desired property is that for all $x(0)$, $x(t)$ will approach the set Γ^c as $t \to \infty$. In other words, $x(t)$ should approximate an element from Γ^c for t large enough. The convergence result is based on La Salle's Theorem.

Theorem 10 *(La Salle)*
Let Ω be a compact set with the property that every solution of $\dot{x}(t) = f(x)$ which starts in Ω remains for all future time in Ω. Let $v : \Omega \to \Re$ be a continuously differentiable function such that $\dot{v}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{v}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Invariant set, M is defined by:

$$x(0) \in M \Rightarrow x(t) \in M \quad \forall t \geq 0.$$

The proof can be found, for example, in [7]. At this point we are able to prove the following convergence result.

Theorem 11 *Let Γ^c be the closure of the reconstruction set for of the inherently bounded AQLR. Then for all $x(0)$, the solution of*

$$\dot{x}(t) = -\nabla \left(v\left(x(t)\right)\right). \tag{106}$$

will approach Γ^c as $t \to \infty$.

Proof: Let $x(0)$ be arbitrary and fixed. Define:

$$\Omega = \{x : v(x) \leq v(x(0))\}. \tag{107}$$

Since $\dot{v}(x) = -||\nabla(v(x))||^2 \leq 0$, every solution of (106) which starts in Ω remains there. $\Gamma^c \subseteq \Omega$ because for all $x \in \Gamma^c$ $v(x) = 0$. As a consequence of Theorem 8 $E = \Gamma^c$. But Γ^c is an invariant set, therefore $M = \Gamma^c$. By showing that Ω is compact we will get the desired convergence result. Ω is closed because $v(x)$ is continuous. Boundness of Ω is based on the fact that the representation is inherently bounded. Let $x \in \Omega$ be arbitrary. Define the vector b_x by:

$$(b_x)_i \triangleq \begin{cases} (Bx)_i & \text{if } (Bx)_i < b_i \\ b_i & \text{otherwise.} \end{cases}$$

Proposition 11 *The norm of b_x is bounded in the following way*

$$||b_x||^2 \leq ||b||^2 + v(x). \tag{108}$$

Proof: For any vector y we define:

$$||y||_s^2 \triangleq \sum_{\{i:(Bx)_i \geq b_i\}} y_i^2$$

$$||y||_{ns}^2 \triangleq \sum_{\{i:(Bx)_i < b_i\}} y_i^2.$$

Note that $||y||_s, ||y||_{ns}$ are norms of appropriate projections of y. Therefore, we can write:

$$\begin{aligned} ||b_x||^2 &= ||b_x||_s^2 + ||b_x||_{ns}^2 = \\ &= ||b_x||_s^2 + ||b_x - b + b||_{ns}^2 \leq \\ &\leq ||b_x||_s^2 + ||b_x - b||_{ns}^2 + ||b||_{ns}^2 \end{aligned}$$

Using $v(x) = ||b_x - b||_{ns}^2$ and $||b_x||_s^2 = ||b||_s^2$, the claim of the proposition is shown.

□

To conclude the proof of boundness of Ω observe that

$$Bx \geq b_x.$$

Using the result (52) we see see that:

$$||x|| \leq K_1 ||b_x||. \tag{109}$$

From the definition of Ω and from the proposition it can be shown that for all $x \in \Omega$

$$||x|| \leq K_1 \sqrt{||b||^2 + v(x(0))} \tag{110}$$

namely, Ω is bounded and the proof is completed.

□

The idea to minimize a cost function in order to reconstruct a signal from the multiscale edge representation has appeared in many works, e.g. [5, 18, 14]. The comparison reveals the following advantages of the proposed algorithm.

- This algorithm is based on continuously differentiable cost function.
- It does not apply approximations.
- It is adapted for both unique and nonunique cases.
- Its validity and convergence are guaranteed

Although this algorithm has been developed independently, it is straightforward, therefore, it is possible that it has appeared elsewhere in the literature. The authors would be grateful to obtain any information about applying this kind of algorithms.

8 Conclusions

Perhaps the most important outcome of this work is to show feasibility and capability of discrete analysis. In general, the discrete approach described here may be applied for a variety of representations and reconstruction algorithms, providing new insights into their properties. We believe that, even for complex algorithms, testing for uniqueness and computing a precise reconstruction set, even for a few examples, is worth the effort.

As mentioned earlier, many important and interesting features of the multiscale edge representation appear in the nonunique case. However, the most of theoretical works has been developed in the framework of unique representations. In our opinion,the need to develop more analytical tools and applications for nonunique representations is apparent.

In addition, from the theoretical point of view, there are still many interesting open questions concerning the discrete analysis of the multiscale edge representation. Consider the following, partial list of problems desiring further research.

- What is the family of signal for which the wavelet maxima (zero-crossings) representation is unique ?
- What kind of information should be added to the dyadic wavelet maxima (zero-crossings) representation in order to assure general uniqueness ?
- Is multiscale zero-crossings points [3] representation indeed unstable ?

[3] without any additional information

- If it is, what is the minimal additional information which stabilizes it ?

As a first step in the undergoing research, this work dealt only with one dimensional signals. The reason is twofold: firstly, we thought that in the simpler case the basic properties would be better recognizable, secondly, the one dimensional multiscale edge representation has its own variety of applications. One of the most promising application areas is speech analysis, for example, pitch detection [6] or modeling signal transformations in auditory nervous system [17]. On the other hand, up to this point, the vast majority of multiscale edge representations has been implemented in computer vision. Therefore, it is advisable to extend these results for two dimensional signals. Surprisingly, there is an essential difference between maxima and zero-crossings representations. Two dimensional multiscale zero-crossings representation can easily be cast into the structure of inherently bounded AQLR, thus the related results are valid in this case. However, a two dimensional maxima representation appears to have a different structure. In order to proceed with a similar analysis one has to choose between:

- to extend the framework of the AQLR
- to change the definition of the two dimensional maxima representation to match the structure of the AQLR.

We are in the process of deciding which choice is more suitable for analysis and applications.

Summarizing, the described results about uniqueness and stability are new theoretical results. In our opinion, the most significant contribution of this work is to create a framework to define, analyze, and reconstruct a wide family of representations. Important examples are generalizations of a basic maxima representation obtained by using only a subset of local extreme points. Their properties are the subject of the undergoing research.

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