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ABSTRACT

In this paper we provide an accurate analysis of the performance of a random-carrier (RC) code-division multiple-access (CDMA) scheme recently introduced for use in high-capacity optical networks. According to this scheme coherent optical techniques are employed to exploit the huge bandwidth of single-mode optical fibers and are coupled with spread-spectrum direct-sequence modulation in order to mitigate the interference from other signals due to the frequency overlap caused by the instability of the carrier frequency of the laser, or to the mistakes in the frequency coordination and assignment.

The average bit error probability of this multiple-access scheme is evaluated by using the characteristic function of the other-user interference at the output of the matched optical filter. Both phase noise and thermal noise are taken into account in the computation. Time-synchronous as well as asynchronous systems are analyzed in this context. Binary phase-shift-keying (BPSK) and on-off-keying (OOK) data modulation schemes are considered. The analysis is valid for arbitrary values of the spreading gain and the number of interfering users. The performance evaluation of RC CDMA establishes the potential advantage in employing hybrids of wavelength-division multiple-access (WDMA) and CDMA to combat inter-carrier interference in dense WDMA systems.

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1. Introduction

The random-carrier (RC) code-division multiple-access (CDMA) scheme was first introduced in [1] for use in high-capacity optical networks. According to this scheme, coherent optical techniques are employed to exploit the huge bandwidth (tens of thousands of GHz) of single-mode optical fibers. The inherent instabilities of present-day semiconductor lasers are circumvented by coupling the optical multiple-access system, which is assumed to place randomly the modulated carriers in the available optical band, with CDMA. In particular, spread-spectrum direct-sequence modulation is employed in order to mitigate the interference from other signals, that is due to the frequency overlap caused by the instability of the carrier frequency of the laser.

Analysis of RC CDMA provides the basis for exploiting the potential advantage of employing hybrids of wavelength-division multiple-access (WDMA) and CDMA to combat inter-carrier interference in dense WDMA schemes. That hybrid scheme used WDMA for providing multiple-access capability and CDMA for protection against laser-frequency instabilities and mistakes in the frequency coordination and assignment. Such a combination of the best features of both WDMA and CDMA was recently proposed in [2].

We complete, extend, and validate the work of [1] by providing a more accurate evaluation of the performance of the RC CDMA scheme, without making all the approximations and limiting assumptions made there. Here are the specific contributions of our work: (a) The performance measure considered is the average probability of a bit error at the output of the RC CDMA system's optical matched filter and is evaluated via an approach that can achieve any desirable accuracy. (b) Our analysis is valid for an arbitrary spreading gain (number of chips per bit) and an arbitrary number of interfering users. (c) In this evaluation, we also take into account the effects of phase noise of the lasers and shot noise; the former is modeled as a Brownian motion process and the latter as additive white Gaussian noise (AWGN). (d) Not only chip-synchronous, but also asynchronous multi-user systems are analyzed; thus the effect of time delays between the various users are taken into account. (e) Besides PSK, on-off-keying (OOK) data modulation is also considered.

For the evaluation of the bit error probability, we use the characteristic-function method introduced in [3] for radio-frequency (RF) CDMA communication systems. The evaluation of error probability in [3] was carried out for arbitrary deterministic signature sequences. In [4], the results of [3] were extended to CDMA systems employing random signature sequences (i.e., i.i.d. sequences that assume the values $+1$ and -1 with equal probability and are mutually independent for different users). In this work, we analyze optical RC CDMA systems with random signature sequences. In our analysis, BPSK modulation, as well as OOK modulation is used to modulate the data bit stream, while M-ary PSK modulation is employed for the signature sequence stream. Electro-optical phase modulators [6] make these features feasible.

We evaluate the average bit error probability by averaging over the data streams, signature sequences, carrier frequencies, phases of the interfering users, and time delays (for asynchronous systems). This is accomplished by computing the characteristic function of the interference due to the other users at the output of the optical filter that is matched to a particular signal. The accuracy of this computational technique can be completely controlled by the user and is determined by the accuracy of the integration routines invoked: as the the number of points in the integration rule increases, the required computer CPU time increases. Any desirable accuracy can be attained with the help of this technique.

In the calculation of the characteristic function of the multiuser interference the Central Limit Theorem (CLT) is used once; however, its use is properly justified and the results are compared to those obtained via another approach that makes no use of the CLT.

In our analysis, the characteristic function method is actually applied to more general situations than those considered in [3]-[5], which deal with other-user interference and multipath interference and have a symmetric probability density function (pdf). In our work, the characteristic function technique used for the evaluation of the average error probability of systems with additive interference is extended to the case of interference with non-symmetric pdfs (the case when the interference does not assume positive and negative values with equal probability).

Another useful product of our analysis of RC CDMA systems is an exact expression for the error probability of a single-user coherent optical system disturbed by phase noise and AWGN.

We provide numerical results describing the performance of the RC CDMA systems.

The tradeoffs between the various system parameters are illustrated and interpreted. Besides the average error probability as a function of system parameters, the maximum number of users that can be supported with this scheme at a given error probability is also obtained. Moreover, the performance of a single user coherent optical system disturbed by phase noise and AWGN is analyzed in detail.

Our analysis of the RC CDMA system paves the way for accurately evaluating the performance of dense WDMA systems and of hybrid WDMA/CDMA systems, and for comparing the performance of such systems in terms of average error probability and multiple-access capability (i.e., the maximum number of simultaneous users that can be supported at a prespecified error probability) for realistic environments characterized by asynchronous users, laser phase noise, shot (AWGN) noise, and frequency instability of lasers. In Chapter 3, we provide the analysis of pure WDMA and hybrid WDMA/CDMA systems and a comparison of their performance.

This paper is organized as follows. In Section 2, the model and the receiver structure are described. In Section 3, the single user performance for the BPSK and OOK modulation is obtained. Section 4 extends the analysis to the multiuser case, in which K active users share a common optical channel. The average bit error probability for the intended user is obtained by using the characteristic functions of the interference and AWGN. Section 5 contains computation of the characteristic function of other-user interference. Based on two different sets of assumptions, this function is obtained for both BPSK and OOK modulation. In Section 6, the pdf of a useful random variable, which plays an important role in the analysis of both single user and multiuser systems, is estimated. Section 7 extends the analysis to the asynchronous system, in which random delays are introduced to the signals of the interfering users. Section 8 presents the numerical results and comparisons of different systems. Finally, Section 9 contains the summary.

2. Model

K high data rate users share a common optical channel in a multi-access fashion. These users are scattered in an optical bandwidth as big as $W = 10THz$. Due to the frequency instabilities of lasers, each carrier frequency wanders around its designated frequency. We assume that each carrier frequency is randomly distributed in the total optical bandwidth W. The transmitted optical signal is denoted by $S(t)$, which is a complex signal as

$$S(t) = \sum_{m=1}^{K} \sqrt{P}\, b_m(t-\tau_m) a_m(t-\tau_m) e^{i[\omega_m(t-\tau_m)+\theta_m(t)]} \tag{2.1}$$

where associated parameters are as follows:

- P is the transmitted signal power of each user
- $b_m(t)$ is the data stream of the m-th user given by

  $$b_m(t) = \sum_{n=-\infty}^{\infty} b_n^{(m)} p(t-nT)$$

  where $b_n^{(m)}$ denotes the n-th bit of the m-th user; $b_n^{(m)} \epsilon \{-1, 1\}$ for BPSK modulation and $b_n^{(m)} \epsilon \{0, 1\}$ for OOK modulation; $p(t)$ is a pulse of unit amplitude in [0, T].
- $a_m(t)$ is the addressing function or signature sequence stream used by user m. That is

  $$a_m(t) = e^{i\phi_m(t)} = \sum_{n=-\infty}^{\infty} e^{i\phi_{mn}} h(t-nT_c)$$

  where $h(t)$ is a pulse of unit amplitude in $[0, T_c]$ and $T_c = \frac{T}{N}$ is the chip duration, where N is the number of chips per bit. ϕ_{mn} is a phase taking values in $[0, 2\pi]$.
- ω_m is the carrier on which the m-th signal is sent. This value is randomly chosen by the transmitter laser for (RC) CDMA, or is preassigned for hybrid WDMA/CDMA.
- $\theta_m(t)$ is the phase noise associated with the m^{th} transmitter laser, which is a Brownian motion process with Lorentzian bandwidth β. The mean of this process is zero and the variance is $2\pi\beta t$.
- τ_m is the m-th time delay which is a uniform random variable distributed in [0, T]. For the synchronous system, this time delay is zero.

At the k-th receiver, the optical signal $S(t)$ is first despread by $a_k^*(t)$, which is the complex conjugate of $a_k(t)$, and then homodyne-detected for the transmitted signal from user k (see Fig. 1). The output of the photodetector is

$$r(t) = \sqrt{P}\ b_k(t)e^{i\Delta\theta_k(t)} + n(t) +$$
$$\sum_m{}' \sqrt{P}\ b_m(t-\tau_m)e^{i[\omega'_m(t-\tau_m)+\phi_m(t-\tau_m)-\phi_k(t-\tau_m)+\Delta\theta_m(t)]} \tag{2.2}$$

where

$$\sum_m{}' \triangleq \sum_{\substack{m=1 \\ m\neq k}}^{K}$$

$$\omega'_m \triangleq \omega_m - \omega_k$$

$$\Delta\theta_k(t) \triangleq \theta_k(t) - \theta_L(t)$$

$$\Delta\theta_m(t) \triangleq \theta_m(t) - \theta_L(t)$$

where $\theta_L(t)$ is the phase noise of the local laser and $n(t)$ is the complex AWGN process with double-sided spectral density $\frac{N_0}{2}$. The receiver used is a correlation receiver. which is optimum for the single user case with no phase noise (see Fig. 2). The performance of this suboptimum receiver in the presense of phase noise and AWGN is obtained for the single user and multiuser situations in the subsequent sections.

3. Single User Analysis

Here the performance of the system described in Section 2, is evaluated in terms of BER for the single-user case. The real part of the output of the integrator is denoted by Y, which is

$$Y = b_0^{(k)} X \sqrt{P} + \eta \sqrt{P} \tag{3.1}$$

where

$$X = \frac{1}{T} \int_0^T \cos\left[\Delta\theta_k(t)\right] dt \tag{3.2}$$

and η is a zero mean Gaussian random variable of variance $N_0/2PT$. According to Appendix A, the probability of error P_e for BPSK modulation is obtained as

$$P_e = \overline{\frac{1}{2} Q\left((\rho + X)\sqrt{\frac{2PT}{N_0}}\right)} + \overline{\frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)} \tag{3.3}$$

where $Q(\cdot)$ and $\Phi(\cdot)$ are related to the standard normal distribution as follows:

$$Q(\alpha) = \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx$$

$$\Phi(\alpha) = \int_{-\infty}^\alpha \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx$$

The overlines in (3.3) indicate expectation with respect to the random variable X. Similarly, for OOK modulation, P_e is obtained as

$$P_e = \frac{1}{2} Q\left(\rho\sqrt{\frac{2PT}{N_0}}\right) + \overline{\frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)}. \tag{3.4}$$

It is easy to show that the optimum value of ρ for (3.3) and (3.4) is 0 and 1/2, respectively.

4. Multiuser Analysis

Here the performance of the system of Section 2.1 is evaluated in terms of BER for the multiuser case. The synchronous case ($\tau_m = 0$) is considered first; the asynchronous case is studied in a separate section. The output of the integrator V is

$$V = b_0^{(k)}\sqrt{P}x + \sqrt{P}\sum_m{}' i_m + \tilde{\eta}\sqrt{P}, \tag{4.1}$$

where

$$x = \frac{1}{T}\int_0^T e^{i\Delta\theta_k(t)}dt \tag{4.2}$$

$$i_m = b_0^{(m)} \cdot \frac{1}{T}\int_0^T e^{i\left[\omega'_m t+\phi_m(t)-\phi_k(t)+\Delta\theta_m(t)\right]}dt \tag{4.3}$$

and

$$\tilde{\eta} = \frac{1}{T\sqrt{P}}\int_o^T n(t)dt. \tag{4.4}$$

In order to calculate the integral in (4.3) we make some assumptions. First, we assume that the phase addressing functions $\phi_m(t)$ and $\phi_k(t)$ in the chip interval $((n-1)\,T_c, nT_c]$ take the values ϕ_{mn} and ϕ_{kn} in $[0, 2\pi]$, respectively. Second, for large values of N, we assume that

$$\Delta\theta_m(t) = \Delta\theta_m(nT_c) \triangleq \theta_{mn}, \qquad for \quad (n-1)T_c < t \leq nT_c \tag{4.5}$$

where θ_{mn} is a zero-mean Gaussian random variable of variance $4\pi\beta nT_c$. Under these assumptions,

$$i_m = \frac{b_0^{(m)}}{N} sinc\left(\frac{\omega'_m T_c}{2}\right)\sum_{n=1}^{N} e^{i\left[\phi_{mn}-\phi_{kn}+\theta_{mn}+\omega'_m(n-1/2)T_c\right]}. \tag{4.6}$$

By taking the real part of V in (4.1), we have

$$Y = \left(b_0^{(k)} \cdot X + \sum_m{}' I_m + \eta\right)\sqrt{P} \tag{4.7}$$

where X and η are defined in (3.1) and (3.2) and I_m is

$$I_m = \frac{b_0^{(m)}}{N} sinc\left(\frac{\omega'_m T_c}{2}\right) \sum_{n=1}^{N} \cos(X_{mn}) \tag{4.8}$$

where

$$X_{mn} = < \phi_{mn} - \phi_{kn} + \theta_{mn} + \omega'_m (n - 1/2) T_c > . \tag{4.9}$$

In (4.9), $< \cdot >$ represents $[\cdot]\ mod.2\pi$. To evaluate the bit error probability let us rewrite (4.7) as

$$Y = \left(b_0^{(k)} \cdot X + I + \eta\right) \sqrt{P} \tag{4.10}$$

where

$$I = \sum_{m}^{\prime} I_m . \tag{4.11}$$

According to Appendix A, the average probability of error for BPSK modulation is obtained as

$$\begin{aligned} P_e = {} & \overline{\frac{1}{2} Q\left((\rho + X)\sqrt{\frac{2PT}{N_0}}\right)} + \overline{\frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)} \\ & + \frac{1}{\pi}\int_0^{\infty} \left(1 - \Phi_I(u)\right) \Phi_\eta(u) \cos(\rho u) \frac{\overline{\sin(uX)}}{u} du \end{aligned} \tag{4.12}$$

provided that I and X are independent. However, this assumption is discussed in detail in Section 2.4, where it is justified by the evaluation of the characteristic function of I via two different methods. Similarly, for OOK modulation, P_e is

$$\begin{aligned} P_e = {} & \frac{1}{2} Q\left(\rho\sqrt{\frac{2PT}{N_0}}\right) + \overline{\frac{1}{2}\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)} \\ & + \frac{1}{2\pi}\int_0^{\infty} \left(1 - \Phi_I(u)\right) \Phi_\eta(u) \frac{\sin(\rho u) - \overline{\sin(\rho - X)u}}{u} \cdot du \end{aligned} \tag{4.13}$$

where

$$\Phi_\eta(u) = \exp\left(-\frac{N_0}{4PT} u^2\right). \tag{4.14}$$

As shown in [3]-[5] where the characteristic-function method was applied to other systems with additive interference, expressions similar to those in (4.12) and (4.13) can be evaluated with any desirable accuracy. Therefore, for all practical purposes results obtained via this method are considered exact and have been used in the literature for checking the accuracy of other techniques. In particular, the infinite integrals in (4.12) and (4.13) are first truncated (for the specific acceptable truncation error) and then the integration is carried out using standard methods (for the specific acceptable integration error).

5. Computation of the Characteristic Function

The computation of P_e in (4.12) and (4.13) requires the characteristic function of multiuser interference $\Phi_I(u)$. This is defined as

$$\Phi_I(u) = E[e^{iuI}] = E\left[e^{iu \sum_m' \frac{b_0^{(m)}}{N} sinc(\frac{\omega_m' T_c}{2}) \sum_{n=1}^{N} \cos(X_{mn})}\right] \tag{5.1}$$

where X_{mn} is defined in (4.9). We pursue two approaches for obtaining the characteristic function.

According to the first approach, it is assumed that the carrier frequencies of the users are *uniformly* distributed in a bandwidth of W. The phase signature sequences are uniformly distributed in the set of equally spaced phases, $\{0, 2\pi/M, \ldots, (M-1)2\pi/M\}$, where M is the number of points in this set. This means that the phase signature sequence is modulated by an M-ary phase shift keying scheme. M=2 corresponds to the commonly used BPSK case.

According to the second approach, it is assumed that the phase signature sequences are *continuous* and *uniformly*-distributed in $[0, 2\pi]$. This approximates the case in which the number of levels M is large. The choice of carrier frequencies distributions is *arbitrary*.

As it will become clear in the following, the two approaches not only correspond to two different sets of assumptions that can be useful under a variety of system conditions, but are also necessary to validate the use of characteristic-function method and the evaluation of the characteristic function of other-user interference.

First we prove the following two lemmas. follows.

- **Lemma 1**

  Let X be a random variable uniformly distributed in the bandwidth $[0, W]$. Let $Y = < X >$. Then the distribution of Y approaches to a uniform distribution in $[0, 2\pi]$, as $W \to \infty$.

- **Proof:** See Appendix B.

- **Lemma 2**

  Let the sequence $\{\phi_n\}_n$ be i.i.d. and uniform in $[0, 2\pi]$ and let the sequence $\{\lambda_n\}_n$ be arbitrary. Assume that these two sequences are independent for all n. Then the sequence $\{X_n\}_n$ defined as

  $$X_n = < \phi_n + \lambda_n >$$

  is i.i.d. and uniform in $[0, 2\pi]$.

- **Proof:** See Appendix C.

1. Assumption of Uniform Carriers

Let X_{mn} in (4.9) be

$$X_{mn} = < \gamma_{mn} + \beta_{mn} > \tag{5.2}$$

where

$$\gamma_{mn} = \phi_{mn} - \phi_{kn} + \theta_{mn} \tag{5.3}$$

$$\beta_{mn} = \omega'_m (n - 1/2) T_c. \tag{5.4}$$

$\{\omega'_m\}$ is assumed to be i.i.d. and uniformly distributed in a bandwidth W as big as 10 THz. Therefore, $\{\beta_{mn}\}$ for fixed n are i.i.d. and uniformly distributed in a bandwidth of $(n - 1/2)W/(RN)$, where R is the data rate. For typical values of R and N, this value is still very large and according to Lemma 1, it is reasonable to consider $\{< \beta_{mn} >\}$ as i.i.d. random variables uniformly distributed in $[0, 2\pi]$. Consequently, Lemma 2 asserts that $\{X_{mn}\}$ with respect to m are i.i.d. and uniformly distributed in $[0, 2\pi]$. Therefore, (5.1) is expressed as

$$\Phi_I(u) = \prod_m^{I} E\left[e^{iu \frac{b_0^{(m)}}{N} sinc(\frac{\omega'_m T_c}{2}) \sum_{n=1}^{N} \cos(X_{mn})} \right]. \tag{5.5}$$

This is independent of the index m . Hence

$$\Phi_I(u) = \left\{ E\left[e^{iu\frac{b_0}{N} sinc(\frac{\omega' T_c}{2}) \sum_{n=1}^{N} \cos(X_n)} \right] \right\}^{(K-1)} . \tag{5.6}$$

Conditioned on ω in (5.6), it is easy to show that the sequence $\{\cos X_n\}$ has zero-mean and zero correlation. Therefore, this sequence is a "$\rho - mixing$" one with $\rho(n) = 0$ (See [7] and [8]). We want to apply the CLT to the sequence $\{\cos X_n\}$. To achieve that we report a theorem from [8].

- **Theorem**

Let $\{X_n\}$ be a second-order stationary, centered, and ρ-mixing sequence; let $\sigma_n^2 \to \infty$ and $\sum_i \rho(2^i) < \infty$. Then $\{s_n/\sigma_n\}$ satisfies the CLT, where $s_n = \sum_{i=1}^{n} X_i$ and $\sigma_n^2 = Var(s_n)$.

The variance of the sequence $\{\cos X_n\}$ turns out to be

$$\sigma_n^2 = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos[\omega'(2n-1)T_c]E\{\cos(2\theta_n)\}, & \text{for} \quad M = 2 \\ \frac{1}{2}, & \text{for} \quad M > 2 \end{cases} \tag{5.7}$$

Since $2\theta_n$ is a zero-mean Gaussian random variable of variance $16\pi\beta n T_C$, it is easy to show that the expectation in (5.7) is $\exp(-8\pi\beta n T_c)$. Hence,

$$\sigma_n^2 = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos[\omega'(2n-1)T_c]\exp(-8\pi\beta n T_c), & \text{for} \quad M = 2 \\ \frac{1}{2}, & \text{for} \quad M > 2 \end{cases} \tag{5.8}$$

Apparently, all conditions of the theorem are satisfied for $M > 2$. But, for $M = 2$, the sequence is not second-order stationary. The variance of the term $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \cos(X_n)$ is

$$\sigma^2 = \begin{cases} \frac{1}{2} + \frac{1}{2N}\Sigma_{n=1}^{N}\cos[\omega'(2n-1)T_c]\exp(-8\pi\beta n T_c), & \text{for} \quad M = 2 \\ \frac{1}{2}, & \text{for} \quad M > 2 \end{cases} \tag{5.9}$$

It is easy to show that

$$\lim_{N\to\infty} \frac{1}{N}\Sigma_{n=1}^{N}\cos[\omega'(2n-1)T_c]\exp(-8\pi\beta n T_c) = \int_0^1 \cos(2\omega' T)\exp(-8\pi\beta T t)dt$$

which is easy to compute. Therefore, as $N \to \infty$, (5.9) converges to

$$\sigma^2 = \begin{cases} \frac{1}{2} + \frac{1}{2(a^2+b^2)}\left[(b\sin(b) - a\cos(b))\exp(-a) + a\right], & \text{for} \quad M = 2 \\ \frac{1}{2}, & \text{for} \quad M > 2 \end{cases} \tag{5.10}$$

where $a = 8\pi\beta T$ and $b = 2\omega'_m T$. This limit is positive (non zero) for almost all values of ω'. Despite the fact that for $M = 2$ the condition of stationarity required by the theorem is not satisfied, we have shown that for almost any given ω' the sequence $\{\frac{1}{N}\sum_{n=1}^{N}\sigma_n^2\}$ converges to a positive limit. This also leads to the satisfaction of the condition $\sum_{n=1}^{N}\sigma_n^2 \to \infty$, as $N \to \infty$, stated in the theorem. We may thus proceed under the assumption that the CLT holds. Then, for large N, the term $\frac{1}{\sqrt{N}}\sum\limits_{n=1}^{N}\cos(X_n)$ in (5.6) can be replaced by ζ , where ζ is a Gaussian random variable of zero-mean and variance σ^2. Conditioned on b_0 and ω', (5.6) becomes

$$\Phi_I(u) = \left\{E_{b_0,\omega'}\left\{E\left[e^{iu\frac{b_o}{\sqrt{N}}sinc(\frac{\omega T_c}{2})\zeta}\right]\right\}\right\}^{(K-1)}. \tag{5.11}$$

The inner expectation in (5.11) is the characteristic function of a Gaussian random variable. Therefore,

$$\Phi_I(u) = \left\{E\left[e^{-u^2\frac{b_o^2}{2N}\left(sinc(\frac{\omega' T_c}{2})\right)^2\sigma^2}\right]\right\}^{(K-1)} \tag{5.12}$$

where the expectation is with respect to b_0 and ω'. For BPSK, $b_0\epsilon\{-1,1\}$ and consequently we obtain

$$\Phi_I(u) = \left\{\overline{\exp\left[-\frac{u^2}{2N}\left(sinc(\frac{\omega' T_c}{2})\right)^2\sigma^2\right]}\right\}^{(K-1)}. \tag{5.13}$$

For OOK, $b_0\epsilon\{0,1\}$ and thus we obtain

$$\Phi_I(u) = \left\{\frac{1}{2} + \frac{1}{2}\overline{\exp\left[-\frac{u^2}{2N}\left(sinc(\frac{\omega' T_c}{2})\right)^2\sigma^2\right]}\right\}^{(K-1)}. \tag{5.14}$$

Notice that this characteristic function does not depend on the phase noise for $M > 2$. Thus, when conditioned on the phase noise random process, the pdf of the other-user interference term I defined by (4.11) and (4.8) turns out to be independent of X. This provides partial justification for the independence of I and X in (4.10) and thus for the validity of (4.12) and (4.13) under this approach. For $M = 2$, σ^2 in (5.10) depends on this phase noise. However, the numerical calculation of the characteristic functions for both cases ($M = 2$ and $M > 2$) shows a negligible difference between these two (see Figure 2.6). Therefore, for M=2, the independence assumption is also well made and consequently, the validity of (4.12) and (4.13) is justified. The issue of the independence of I and X [necessary for the validity of (4.12)-(4.13)] is raised by the fact that, although I_m and X are mutually independent for all m (and so are I_m and $I_{m'}$ for $m \neq m'$), X and $(I_1, I_2, \cdots, I_K)$ may not be mutually independent. Final justification for the validity of this approach is provided by its close agreement (refer to the section on numerical results, in particular to Figures 2.9 and 2.10) with the results obtained via the second approach that follows. Under the second approach, I and X are guaranteed to be independent.

2. Assumption of Uniform Phase Signature Sequence

For the case in which the signature sequence phases are uniformly distributed in a set of equally spaced discrete levels, if the number of levels in this set is reasonably large, this discrete uniform distribution is approximated with a continuous uniform phase in $[0, 2\pi]$. Let us express the characteristic function of the interference I as

$$\Phi_I(u) = E_{\overline{\omega'}}\ E_{\overline{b_0}}\ E_{\overline{X}}\left[e^{iuI}\right] \tag{5.15}$$

where I is given by (4.11) and (4.8). $E_{\overline{X}}$ is the expectation with respect to the $N(K-1)$ dimensional vector in $\{X_{mn}\}$. $E_{\overline{b_0}}$ is the expectation with respect to the $K-1$ dimensional vector in $\{b_0^{(m)}\}$. Finally, $E_{\overline{\omega'}}$ is the expectation with respect to the $K-1$ dimensional vector $\overline{\omega'} = (\omega'_1, \ldots, \omega'_K)$. Since $\{\phi_{mn}\}$ in (4.9) are i.i.d. and uniformly distributed in $[0, 2\pi]$ for all m and n , $\{X_{mn}\}$ are also i.i.d. and uniform in $[0, 2\pi]$, for all m and n (based on Lemma 2). Therefore,

$$\Phi_I(u) = \prod_m' E_{\omega'_m}\ \left\{E_{b_0^{(m)}}\left\{\prod_{n=1}^{N} E_{X_{mn}}\left[e^{i\frac{u}{N}b_0^{(m)}\cdot \sin c\left(\frac{\omega'_m T_c}{2}\right)\cos(X_{mn})}\right]\right\}\right\}. \tag{5.16}$$

We use the identity

$$E_{X_{mn}}\left[e^{iu\cos(X_{mn})}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iu\cos x}\cdot dx =$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \cos\left(u\cos x\right) dx = J_0(u) \tag{5.17}$$

where $J_0(\cdot)$ is the Bessel function of the first kind. Using (5.17) the inner expectation in (5.16) becomes $J_0\left(b_0^{(m)}\cdot\frac{u}{N}\cdot\sin c\left(\frac{\omega_m' T_c}{2}\right)\right)$, which is independent of n. Also using the fact that $J_0(\alpha) = J_0(-\alpha)$ we obtain the following expression for BPSK:

$$E_{b_0^{(m)}}\left\{\prod_{n=1}^{N} E_{X_{mn}}[\cdot]\right\} = \left[J_0\left(\frac{u}{N}sinc\left(\frac{\omega_m' T_c}{2}\right)\right)\right]^N . \tag{5.18}$$

Moreover, if we assume that the sequence $\{\omega_m\}$ is i.i.d., (5.18) is independent of the index m and (5.16) becomes

$$\Phi_I(u) = \left\{\overline{\left[J_0\left(\frac{u}{N}sinc\left(\frac{\omega' T_c}{2}\right)\right)\right]^N}\right\}^{K-1} \tag{5.19}$$

where the overline is the expectation with respect to the generic ω' scattered in the bandwidth of the channel. Similarly , for OOK modulation. $\Phi_I(u)$ is obtained as

$$\Phi_I(u) = \left\{\frac{1}{2} + \frac{1}{2}\overline{\left[J_0\left(\frac{u}{N}sinc\left(\frac{\omega' T_c}{2}\right)\right)\right]^N}\right\}^{K-1} . \tag{5.20}$$

Under the conditions of this approach, $(I_1, I_2, \cdots, I_K)$ and X are mutually independent and thus X and I are mutually independent; this justifies the use of (4.12) and (4.13).

6. PDF of X

The pdf of X as defined in (3.2) depends on βT, where 2β is the Lorentzian bandwidth of the Brownian motion process $\Delta\theta_k(t)$.

a) Monte Carlo

In order to obtain this pdf through Monte Carlo simulation, we write (3.2) as

$$X = \frac{1}{T} \sum_{n=1}^{F} \int_{(n-1)\delta}^{n\delta} \cos(\Delta\theta_k(t))\, dt \tag{6.1}$$

where F is the number of divisions of T into smaller portions of size δ, where $\delta = T/F$. For F large enough, let

$$\Delta\theta_k(t) = \Delta\theta_k(n\delta) \triangleq \Theta_n \qquad for \quad (n-1)\delta < t \leq n\delta. \tag{6.2}$$

By substitution of (6.2), (6.1) becomes

$$X = \frac{1}{F} \sum_{n=1}^{F} \cos(\Theta_n) \triangleq G(\underline{\Theta}) \tag{6.3}$$

where $\{\Theta_n\}_{n=1}^{F}$ is a sequence of zero-mean Gaussian random variables of variance $4\pi\beta n\delta$ and correlation $4\pi\beta\delta \min(m, n)$. The statistics of X are estimated through the generation of the random sequence $\{\Theta_n\}_{n=1}^{F}$ in a computer by using (6.3). The probability that X lies in a set $A = [a_1, a_2]$, where $a_1, a_2 \epsilon [0, 1]$, is obtained as

$$P_A = \int_{R^F} 1_A[G(\underline{\theta})].P_{\underline{\Theta}}(\underline{\theta})d\underline{\theta} \tag{6.4}$$

where $P_{\underline{\Theta}}$ is the joint distribution of $\underline{\Theta} = \{\Theta_1, \Theta_2, ..., \Theta_F\}$ and $1_A[.]$ the indicator function. The Monte Carlo estimate of this probability is

$$\hat{P}_A = \frac{1}{M} \sum_{i=1}^{M} 1_A[G(\underline{\Theta}_i)] \tag{6.5}$$

where M is the number of independent simulation runs in the computer and $\underline{\Theta}_i$ is generated according to the pdf $P_{\underline{\Theta}}$. The sequence $\{\Theta_n\}_{n=1}^{F}$ generated in (6.5) is obtained via the iteration

$$\begin{cases} \Theta_1 = \nu_1 \\ \Theta_n = \Theta_{n-1} + \nu_n \quad 2 \leq n \leq F \end{cases} \tag{6.6}$$

where $\{\nu_n\}$ is a sequence of i.i.d. zero-mean Gaussian random variables of the variance $4\pi\beta\delta$. It is easy to show that the sequence $\{\Theta_n\}$ has the required properties.

The estimate in (6.5) is an unbiased estimate whose variance is given by

$$var(\hat{P}_A) = \frac{P_A - P_A^2}{M}. \tag{6.7}$$

The accuracy of this estimate is limited by the statistical fluctuations in the simulation. More specifically the accuracy of the estimate probability of a quantile q obtained after M independent simulation trials is approximately $\sqrt{1/Mq}$. At low probabilities, the number of simulation trials increases drastically, which prohibits the utilization of the simple Monte Carlo method. To circumvent this problem a modified Monte Carlo simulation, namely the Importance Sampling method, is implemented.

b) Importance Sampling

The idea is that the input distribution is biased so that more samples lie in the region of interest at the output. Let us write (6.4) as

$$P_A = \int_{R^F} 1_A[G(\underline{\theta})].W(\underline{\theta}).P_{\underline{\Theta}^*}(\underline{\theta})d\underline{\theta} \tag{6.8}$$

where $W(\underline{\theta}) = \frac{P_{\underline{\Theta}}(\underline{\theta})}{P_{\underline{\Theta}^*}(\underline{\theta})}$ and $P_{\underline{\Theta}^*}$ is the biased distribution of $\underline{\Theta}$. The Importance Sampling estimate of P_A in (6.8) is given by

$$P_A^* = \frac{1}{M}\sum_{i=1}^{M} 1_A[G(\underline{\Theta}_i^*)].W(\underline{\Theta}_i^*) \tag{6.9}$$

where $\underline{\Theta}_i^*$ is generated from the distribution $P_{\underline{\Theta}_i^*}$. The variance of this estimate is given by (see [10])

$$var(P_A^*) = \frac{\overline{W} - P_A^2}{M} \tag{6.10}$$

where the average weight $\overline{W}$ is

$$\overline{W} = \int_{R^F} 1_A[G(\underline{\theta})]W(\underline{\theta})P_{\underline{\Theta}}(\underline{\theta})d\underline{\theta}. \tag{6.11}$$

The fundamental issue in Importance Sampling is the determination of the biasing distribution $P_{\underline{\Theta}^*}$, so that the variance of the Importance Sampling estimator is minimized. The proposed biasing distribution is obtained by shifting the original input samples Θ_n by $\pm\alpha n\delta$, i.e.,

$$\Theta_n^* = \Theta_n + Z\alpha n\delta \tag{6.12}$$

where Z takes values in $\{-1, 1\}$ with probability $1/2$. The parameter $\alpha > 0$ is the identity which should be optimized accordingly. Given $P_{\underline{\Theta}}$ and using (6.12) we can easily show that

$$W(\underline{\theta}) = \frac{P_{\underline{\Theta}}(\underline{\theta})}{P_{\underline{\Theta}^*}(\underline{\theta})} = \frac{\exp(\frac{(\alpha T)^2}{8\pi\beta T})}{\cosh(\frac{\alpha T\theta_F}{4\pi\beta T})}. \tag{6.13}$$

This is the same expression which was given in [11] in a similar context. However, it was not proposed in [11] how to optimize (6.13) in terms of α.

In order to reduce the variance in (6.10) with respect to the one in (6.7) we need to make sure that $\overline{W} < P_A$. In what follows, although we do not provide the optimum value for α such that this condition holds, but we argue for an approximation in the hope that the numerical results will provide encouraging results.

First let rewrite expressions (6.4) and (6.11) as

$$P_A = \int_{\mathcal{B}} P_{\underline{\Theta}}(\underline{\theta})d\underline{\theta} \tag{6.14}$$

$$\overline{W} = \int_{\mathcal{B}} W(\underline{\theta}).P_{\underline{\Theta}}(\underline{\theta})d\underline{\theta} \tag{6.15}$$

respectively, where $\mathcal{B}$ is the inverse image of the set A, i.e., $\mathcal{B} = G^{-1}(A)$. Also let the set A be $A = \{a\}, a\epsilon[-1, 1]$, which is a reasonable assumption due to the probabilities of bins of small sizes. $P_{\underline{\Theta}}(\underline{\theta})$ is the joint pdf of a multivariate Gaussian with zero mean and covariance Σ, where

$$\Sigma = 4\pi\beta\delta \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & F \end{pmatrix}. \tag{6.16}$$

It is easy to show that

$$P_{\Theta}(\underline{\theta}) = (2\pi)^{\frac{-n}{2}}.|\Sigma|^{\frac{-1}{2}}.e^{-2\pi\beta\delta[\theta_1^2+(\theta_2-\theta_1)^2+\ldots+(\theta_F-\theta_{F-1})^2]}. \tag{6.17}$$

At this point, we attempt to minimize $W(\underline{\theta})$ in (6.15) with respect to α for the values of $\underline{\theta}$ in $\mathcal{B}$ which maximize $P_{\Theta}(\underline{\theta})$. In other words, the effect of the integrand in (6.15) is reduced when it is maximum. As we proceed, we find the optimum value of α in (6.13), which minimizes $W(\underline{\theta})$ when $P_{\Theta}(\underline{\theta})$ is maximum, and we also express the conditions under which this minimum value is less than one.

First, (6.17) is maximum when $\underline{\theta}$ is in the set $S = \{\underline{\theta}\colon \theta_1 = \theta_2 = \ldots = \theta_F\} \subset \mathcal{R}^F$. This reduces (6.17) into

$$P_{\Theta}(\underline{\theta}) = (2\pi)^{\frac{-n}{2}}.|\Sigma|^{\frac{-1}{2}}.e^{-2\pi\beta\delta\theta_F^2}, \qquad for \quad \underline{\theta}\epsilon S. \tag{6.18}$$

Second, we need to maximize (6.18) with the constraint $\underline{\theta}\epsilon\mathcal{B}$. In other words, we should maximize (6.18) for $\underline{\theta}\epsilon S \cap \mathcal{B}$. But it is obvious that

$$S \cap \mathcal{B} = \{\underline{\theta}\colon \theta_1 = \theta_2 = \ldots = \theta_F = 2K\pi \pm \cos^{-1}(a), \quad \text{for K integer}\} \tag{6.19}$$

and the values which maximize (6.18), and consequently (6.17), are in the set

$$L = \{\underline{\theta}\colon \theta_1 = \theta_2 = \ldots = \theta_F = \pm\cos^{-1}(a)\} \subset (S \cap \mathcal{B}) \subset \mathcal{B}. \tag{6.20}$$

Next, we minimize $W(\underline{\theta})$ in (6.13) with the constraint $\underline{\theta}\epsilon L$, i.e.,

$$\min_{\alpha}\{W(\underline{\theta}) = \frac{\exp(\frac{(\alpha T)^2}{8\pi\beta T})}{\cosh(\frac{\alpha T\theta_F}{4\pi\beta T})} \mid \underline{\theta}\epsilon L\} \tag{6.21}$$

or

$$\min_{\alpha}\{W(\underline{\theta}) = \frac{\exp(\frac{(\alpha T)^2}{8\pi\beta T})}{\cosh(\frac{\alpha T o}{4\pi\beta T})}\} \tag{6.22}$$

where $\phi = \cos^{-1}(a)$. A simple derivative of $W(\underline{\theta})$ in (6.22) with respect to α renders the necessary condition for the minimum points. This is

$$y = \tanh(\gamma y) \tag{6.23}$$

where $y = \frac{\alpha T}{\phi}$ and $\gamma = \frac{\phi^2}{4\pi\beta T}$. For $\gamma \leq 1$, (6.23) has only one solution at $y = 0$. This means that $W(\underline{\theta})$ is minimized at $\alpha = 0$ and $W(\underline{\theta}) = 1$. This is the useless trivial reduction of the problem to the simple case of Monte Carlo. For $\gamma > 1$, (6.23) has three roots. At $\alpha = 0$ $W(\underline{\theta})$ is maximized; at the other two symmetric roots ($\pm\alpha_{min}$), $W(\underline{\theta})$ is minimized and this minimum value is less than one. In other words, the minimum of $W(\underline{\theta})$ in (6.22) for nonzero α is achieved when $\frac{\phi^2}{4\pi\beta T} > 1$ and this minimum is less than one. However, for practical values of ϕ and βT, this condition is always satisfied.

7. Asynchronous System

In this section, the analysis developed for the synchronous system is extended to the asynchronous case. The time delay τ_m in (2.1) is considered to be a uniformly distributed random variable in $[0, T]$, which represents the m-th user's time delay. At the k-th receiver, the matched filter is synchronized with the k-th signal, i.e. $\tau_k = 0$. Therefore, (4.10) is still valid and the average probability of error is given in (4.12) and (4.13) for BPSK and OOK modulation, respectively. Evaluation of the characteristic function of multiuser interference for the asynchronous case follows next. Let us write τ_m as

$$\tau_m = \ell_m T_c + \tau'_m \tag{7.1}$$

where $\ell_m = \lfloor \frac{\tau_m}{T_c} \rfloor$ is a discrete random variable taking values in $\{0, 1, \ldots, N-1\}$ with equal probability. τ'_m is uniformly distributed in $[0, T_c]$. The counterpart of (4.3) becomes

$$i_m = \frac{1}{T} \int_0^T b_m(t - \tau_m) e^{i\left[\omega'_m t - \omega'_m \tau_m + \phi_m(t-\tau_m) - \phi_k(t) + \Delta\theta_m(t)\right]}. \tag{7.2}$$

The integral in (7.2) can be written as

$$\int_0^T = \int_0^{\tau_m} + \int_{\tau_m}^T \tag{7.3}$$

where

$$\int_0^{\tau_m} = \sum_{n=1}^{\ell_m+1} \int_{(n-1)T_c}^{(n-1)T_c+\tau'_m} + \sum_{n=1}^{\ell_m} \int_{(n-1)T_c+\tau'_m}^{nT_c} \tag{7.4}$$

$$\int_{\tau_m}^{T} = \sum_{n=\ell_m+2}^{N} \int_{(n-1)T_c}^{(n-1)T_c+\tau'_m} + \sum_{n=\ell_m+1}^{N} \int_{(n-1)T_c+\tau'_m}^{nT_c} . \tag{7.5}$$

By using (7.3), (7.4) and (7.5) in (7.2) i_m becomes

$$i_m = \frac{1}{T} \sum_{n=1}^{N} \left[e^{+}_{mn} \int_{(n-1)T_c}^{(n-1)T_c+\tau'_m} e^{ig^{+}_{mn}(t)} dt + e^{-}_{mn} \int_{(n-1)T_c+\tau'_m}^{nT_c} e^{ig^{-}_{mn}(t)} dt \right] \tag{7.6}$$

where

$$g^{+}_{mn}(t) \triangleq \omega'_m t - \omega'_m \tau_m + \phi_{m(n-1)} - \phi_{kn} + \theta_{mn} \tag{7.7}$$

$$g^{-}_{mn}(t) \triangleq \omega'_m t - \omega'_m \tau_m + \phi_{mn} - \phi_{kn} + \theta_{mn} \tag{7.8}$$

$$e^{+}_{mn} \triangleq b^{(m)}_{-1} \cdot 1_{[1,\ell_m+1]}(n) + b^{(m)}_{0} \cdot 1_{[\ell_m+2,N]}(n) \tag{7.9}$$

$$e^{-}_{mn} \triangleq b^{(m)}_{-1} \cdot 1_{[1,\ell_m]}(n) + b^{(m)}_{0} \cdot 1_{[\ell_m+1,N]}(n) \tag{7.10}$$

$$1_{[i,j]}(n) \triangleq \begin{cases} 1 & i \leq n \leq j \\ 0 & \text{other} \end{cases} . \tag{7.11}$$

The counterpart of (4.8) is obtained as

$$I_m = R_e\{i_m\} = \frac{1}{N} \sum_{n=1}^{N} \left[e^{+}_{mn} \alpha^{+}_{m} \cos X^{+}_{mn} + e^{-}_{mn}\ \alpha^{-}_{m} \cos X^{-}_{mn} \right] \tag{7.12}$$

where

$$\alpha^{+}_{m} \triangleq \frac{\tau'_m}{T_c} sinc\left(\frac{\omega'_m \tau'_m}{2}\right) \tag{7.13}$$

$$\alpha^{-}_{m} \triangleq (1 - \frac{\tau'_m}{T_c}) sinc\left[\frac{\omega'_m (T_c - \tau'_m)}{2}\right] \tag{7.14}$$

$$X^{+}_{mn} \triangleq < \omega'_m (n-1) T_c + \frac{\omega'_m \tau'_m}{2} - \omega'_m \tau_m + \phi_{m(n-1)} - \phi_{kn} + \theta_{mn} > \tag{7.15}$$

$$X^{-}_{mn} \triangleq < \omega'_m (n-1/2) T_c + \frac{\omega'_m \tau'_m}{2} - \omega'_m \tau_m + \phi_{mn} - \phi_{kn} + \theta_{mn} > . \tag{7.16}$$

At this point, we make two assumptions similar to those of the synchronous case. two assumptions as follows.

1. Assumption of Uniform Carriers

As before, we observe that, conditioned on $\overline{\tau} = (\tau_1, \tau_2, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_K)$, $\{X^+_{mn}\}$ and $\{X^-_{mn}\}$ are i.i.d. with respect to m and uniformly distributed in $[0, 2\pi]$. Therefore,

$$\Phi_I(u) = E_{\overline{\tau}} \left\{ \prod_m^I E \left[e^{j \frac{u}{N} \left(\alpha^+_m \sum_{n=1}^N e^+_{mn} \cos X^+_{mn} + \alpha^-_m \sum_{n=1}^N e^-_{mn} \cos X^-_{mn} \right)} \right] \right\}. \tag{7.17}$$

The sequences $\{e^+_{mn} \cos X^+_{mn}\}_n$ and $\{e^-_{mn} \cos X^-_{mn}\}_n$ are zero-mean and have zero correlations. Therefore, they are ρ-mixing sequences. Also notice that these two sequences are uncorrelated with respect to each other. By conditioning on ω'_m and by using the CLT (7.17) yields

$$\Phi_I(u) = E_{\overline{\tau}} \left\{ \prod_m^I E_{\omega'_m} E \left[e^{j \frac{u}{\sqrt{N}} \left(\alpha^+_m \eta^+_m + \alpha^-_m \eta^-_m \right)} \right] \right\} \tag{7.18}$$

where η^+_m and η^-_m are zero-mean Gaussian random variables of variances $\sigma^2_{\eta^+_m}$ and $\sigma^2_{\eta^-_m}$, respectively. It is easy to show that, for BPSK modulation,

$$\sigma^2_{\eta^+_m} = \begin{cases} \frac{1}{2} + \frac{1}{2N} \sum_{j=1}^N \cos(2\omega'_m (j - \ell_m - 1) T_c - \omega'_m \tau'_m) e^{-8\pi\beta j T_c}, & M = 2 \\ \frac{1}{2}, & M > 2 \end{cases} \tag{7.19}$$

$$\sigma^2_{\eta^-_m} = \begin{cases} \frac{1}{2} + \frac{1}{2N} \sum_{j=1}^N \cos(\omega'_m (2j - 2\ell_m - 1) T_c - \omega'_m \tau'_m) e^{-8\pi\beta j T_c}, & M = 2 \\ \frac{1}{2}, & M > 2 \end{cases}. \tag{7.20}$$

The two summations in (7.19) and (7.20), as $N \to \infty$, are equal to the integral

$$\int_0^1 \cos \left[b_m t - b_m \frac{\ell_m}{N} - \omega'_m \tau'_m \right] \exp(-at) dt \tag{7.21}$$

where $b_m = 2\omega'_m T$ and $a = 8\pi\beta T$. Let define η_m as

$$\eta_m \triangleq \alpha^+_m \eta^+_m + \alpha^-_m \eta^-_m. \tag{7.22}$$

The variance of this zero-mean Gaussian random variable is

$$\sigma^2_{\eta_m} = (\alpha^+_m)^2 \sigma^2_{\eta^+_m} + (\alpha^-_m)^2 \sigma^2_{\eta^-_m} = \left[(\alpha^+_m)^2 + (\alpha^-_m)^2 \right] \sigma^2_m(\ell_m) \tag{7.23}$$

where $\sigma_m^2(n) = \frac{1}{2}$ for $M > 2$, and for $M = 2$,

$$\sigma_m^2(n) = \frac{1}{2} + \frac{1}{2(a^2 + b_m^2)} \Big[a\cos(\frac{n}{N}b_m + \omega'_m \tau'_m) + b_m \sin(\frac{n}{N}b_m + \omega'_m \tau'_m) + \Big(b_m \sin(b_m(1 - \frac{n}{N}) - \omega'_m \tau'_m) - a\cos(b_m(1 - \frac{n}{N}) - \omega'_m \tau'_m) \Big) \exp(-a) \Big] \tag{7.24}$$

If we substitute (7.22) in (7.18) and perform the inner expectation, we obtain

$$\Phi_I(u) = E_{\overline{\tau}} \left\{ \prod_m' E_{\omega'_m} \left[e^{\frac{-u^2}{2N}\sigma^2_{\eta m}} \right] \right\}. \tag{7.25}$$

$\overline{\tau}$ is an i.i.d. and uniform random vector. Hence, the outer expectation is interchanged with the product and the result is independent of the index m.

$$\Phi_I(u) = \left\{ E_\tau E_{\omega'} \left[e^{\frac{-u^2}{2N}\sigma^2_\eta} \right] \right\}^{(K-1)}. \tag{7.26}$$

The expectation with respect to τ is reduced to τ' and ℓ. Hence, finally

$$\Phi_I(u) = \left\{ \frac{1}{N} \sum_{n=1}^{N} \Phi_n(u) \right\}^{(K-1)} \tag{7.27}$$

where

$$\Phi_n(u) = E_{\tau'} E_{\omega'} \left[e^{\frac{-u^2}{2N}\sigma^2_\eta} \right] \tag{7.28}$$

$$\sigma^2_\eta = \left[(\alpha^+)^2 + (\alpha^-)^2 \right] \sigma^2(n). \tag{7.29}$$

For OOK modulation, (7.27), (7.28) and (7.29) still hold and only $\sigma^2(n)$ is divided by a factor of 2.

2. Assumption of Uniform Phase Signature Sequence

Since the two sequences $\{\phi_{m(n-1)}\}$ in (7.15) and $\{\phi_{mn}\}$ in (7.16) are i.i.d. and uniformly distributed in $[0, 2\pi]$, for all m and n, the two sequences $\{X^+_{mn}\}$ and $\{X^-_{mn}\}$ are also i.i.d. and uniformly distributed in $[0, 2\pi]$, for all m and n (Lemma 2). Therefore,

$$\Phi_I(u) = \prod_m' E \left[e^{j\frac{u}{T}\left(\alpha^+_m \sum_{N=1}^{N} e^+_{mn} \cos X^+_{mn} + \alpha^-_m \sum_{n=1}^{N} e^-_{mn} \cos X^-_{mn}\right)} \right]. \tag{7.30}$$

Moreover, the two sequences $\{X_{mn}^{+}\}$ and $\{X_{mn}^{-}\}$ are independent of each other. To prove this we need to show that the two random variables X_{mn}^{+} and X_{ij}^{-} are independent for all m, n, i and j. For all cases, except the case in which $i = m$ and $j = n-1$, the i.i.d. and uniform phase sequences $\{\phi_{m(n-1)}\}$ in (7.15) and $\{\phi_{ij}\}$ in (7.16) provide the proof in a straightforward manner. For the case $i = m$ and $j = n-1, X_{mn}^{+}$ and X_{ij}^{-} are still independent, because ϕ_{kn} and $\phi_{k(n-1)}$ are independent and uniform for this case. Using these facts

$$\Phi_I(u) = \prod_{m}^{I} E_m \left\{ \prod_{n=1}^{N} E_{X_{mn}^{+}} \left[e^{\frac{iu}{T}\alpha_m^{+} e_{mn}^{+} \cos X_{mn}^{+}} \right] \cdot E_{X_{mn}^{-}} \left[e^{\frac{iu}{T}\alpha_m^{-} e_{mn}^{-} \cos X_{mn}^{-}} \right] \right\} \qquad (7.31)$$

where the expectation E_m is with respect to the all random variables with index m. Upon substitution from (5.17)

$$\Phi_I(u) = \prod_{m}^{I} E_m \left\{ \prod_{n=1}^{N} J_o\left(\frac{u}{T}\alpha_m^{+} e_{mn}^{+}\right) J_0\left(\frac{u}{T}\alpha_m^{-} e_{mn}^{-}\right) \right\}. \qquad (7.32)$$

For BPSK modulation, e_{mn}^{+} and e_{mn}^{-} belong to $\{-1, 1\}$. Therefore, the inner product is independent of n and (7.32) becomes

$$\Phi_I(u) = \prod_{m}^{I} E_m \left[J_0\left(\frac{u}{T}\alpha_m^{+}\right) J_o\left(\frac{u}{T}\alpha_m^{-}\right) \right]^N. \qquad (7.33)$$

Moreover, by assuming that the sequences $\{\omega'_m\}$ and $\{\tau'_m\}$ are i.i.d., the expectation above becomes independent of the index m and, finally, the characteristic function of the multiuser interference for BPSK modulation is obtained as

$$\Phi_I(u) =$$

$$\left\{ \overline{\left[J_0\left(\frac{u}{N} \cdot \frac{\tau'}{T_c} sinc\left[\frac{\omega'\tau'}{2}\right]\right) J_0\left(\frac{u}{N}\left(1 - \frac{\tau'}{T_c}\right) sinc\left[\frac{\omega'(T_c - \tau')}{2}\right]\right)\right]^N} \right\}^{(K-1)} \qquad (7.34)$$

where the expectation in (7.34) is with respect to ω' and τ', which are distributed in $[-W/2, W/2]$ and $[0, T_c]$, respectively.

The evaluation of the characteristic function of the multiuser interference for OOK in this case, is more tedious than for $BPSK$. We know that ℓ_m is a uniform discrete random variable which takes values in $\{0, 1, \ldots, N-1\}$. By assuming

$$c_m \triangleq J_0(\frac{u}{T}\alpha_m^+ b_{-1}^{(m)}) J_0(\frac{u}{T}\alpha_m^- b_{-1}^{(m)}) \tag{7.35}$$

$$d_m \triangleq J_0(\frac{u}{T}\alpha_m^+ b_0^{(m)}) J_0(\frac{u}{T}\alpha_m^- b_0^{(m)}) \tag{7.36}$$

$$v_m \triangleq J_0(\frac{u}{T}\alpha_m^+ b_{-1}^{(m)}) J_0(\frac{u}{T}\alpha_m^- b_0^{(m)}) \tag{7.37}$$

(7.32) becomes

$$\Phi_I(u) = \prod_m^I E_m \left\{ \left[\prod_{n=1}^{\ell_m} c_m \right] \left[\prod_{n=\ell_m+2}^{N} d_m \right] v_m \right\}. \tag{7.38}$$

Since c_m , d_m and v_m are independent of n, then

$$\Phi_I(u) = \prod_m^I E_m \left\{ v_m c_m^{\ell_m} d_m^{N-\ell_m-1} \right\}. \tag{7.39}$$

Performing the expectation above with respect to ℓ_m, yields

$$\Phi_I(u) = \prod_m^I E_m \left\{ \frac{v_m}{N} \sum_{i=0}^{N-1} c_m^i d_m^{N-1-i} \right\}. \tag{7.40}$$

By taking expectation with respect to $(b_{-1}^{(m)}, b_0^{(m)})$ and rearranging the terms the final result becomes

$$\Phi_I(u) = \left\{ \frac{1}{4} + \frac{1}{4}\overline{[R_u(\omega', \tau')R_u(\omega', T_c - \tau')]^N} \right.$$
$$\left. + \frac{1}{4N}\overline{\left[[R_u(\omega', \tau') + R_u(\omega', T_c - \tau')] \frac{1 - [R_u(\omega', \tau')R_u(\omega', T_c - \tau')]^N}{1 - [R_u(\omega', \tau')R_u(\omega', T_c - \tau')]} \right]} \right\}^{(K-1)}$$

(7.41)

where

$$R_u(\omega', \tau') = J_0\left(\frac{u}{T}\tau' sinc\left(\frac{\omega'\tau'}{2}\right)\right). \tag{7.42}$$

8. Numerical Results

Figures 3a and 3b illustrate the pdf for the random variable X, which was defined in (3.2). According to Section 2.5, by using the importance sampling technique this pdf is obtained for different values of the parameter βT. As expected from (3.2), for lower values of phase noise (low β), this pdf tends to accumulate around the point X=1 which corresponds to the noiseless case. By using the importance sampling method we were able to find the tail of this pdf down to 10^{-9}.

Figure 4 shows the performance of the single-user system for the BPSK case. Degradation of this performance is due to two factors, additive white Gaussian noise and phase noise. For values of βT higher than 0.05, the average probability of bit error is higher than 10^{-3}, even for values of E_b/N_0 up to 20 dB, where E_b is the average bit energy. Reduction of the phase noise by 50% ($\beta T = 0.025$) reduces P_e to the range of 10^{-5}. Even higher performance is obtained for the lower values of βT. The lower bound is achieved when $\beta T = 0$ (no phase noise).

In Figure 5, the performance of the two schemes, BPSK and OOK, for the single-user case are compared. For the no-phase-noise case, OOK is worse than BPSK for 3 dB. This value increases adversely due to phase noise. As shown, for $\beta T = 0.015$, the two curves for BPSK and OOK diverge even further for higher values of SNR. This asserts the inferior performance for OOK due to phase noise.

Numerical calculation of the characteristic function, for typical values of the system parameters, reveals that the binary and M-ary phase signature sequence in the first method yields values that are very close to each other. This not only validates the average probability of bit error formulation for the case of binary phase signature sequence (M=2), but also suggests considering the M-ary case ($M > 2$) (which is computationally easier to obtain) as the representative case for all numerical results. Figure 6 shows how close these two cases are.

Multiuser performance of the asynchronous system for BPSK is illustrated in Figure 7. For 5000 high data rate users of data rate 10 Mbs, the average probability of error is obtained for different values of βT and SNR. For this high number of high data rate users and for values of βT such as 0.015, the average probability of bit error in the range of 10^{-6} becomes feasible by introducing more signal power (higher SNR). The total optical bandwidth W and the spreading gain N are set to be 10 THz and 1000, respectively. The multiuser capability of the same system is obtained in Figure 8. For a P_e in the range of 10^{-4}, this system can support almost 10000 high data rate users by tolerating a phase noise as high as $\beta T = 0.015$, while keeping E_b/N_0 higher than 12 dB.

Performance of the synchronous and asynchronous systems using the two proposed methods of evaluation of the characteristic functions is obtained for BPSK and OOK in Figure 9 and Figure 10, respectively. For the proposed values of parameters indicated, the performance of the synchronous system using either method and the asynchronous system using the first method are exactly the same. In fact, we obtained computationally exactly the same values for the corresponding characteristic functions. The asynchronous system using second method is slightly different from the others. Also notice that, in the OOK case, the degradation of the performance is mainly due to the single-user contribution of error in the total average probability of error.

The close agreement of the results provided by the two methods observed in Figures 9 and 10 justifies the use of the first method, which corresponds to more realistic system modeling assumptions, whereas the second method, which is analytically very accurate (actually exact within the framework of the characteristic-function method), relies on modeling assumptions that are less realistic. Therefore, depending on the modeling assumptions we can use the most suitable method with full knowledge of the accuracy of the method.

9. Conclusions

In this paper we provided the modeling and the tool for analyzing the performance of an RC CDMA network with coherent optical detection. The average bit error probability of this multiple-access scheme was evaluated using the characteristic function of multiuser interference at the output of the matched optical filter. Based on two set of assumptions, we proposed two approaches for obtaining the characteristic function. These two methods were numerically evaluated and the results proved to be very close to each other. Both phase noise and thermal noise were taken into account in the analysis. The effects of AWGN and phase noise were studied numerically for single-user systems with spreading and despreading and for multiuser systems. Importance-sampling techniques were developed to simulate a given function of phase noise. It was shown that thousands of high data rate users can communicate reliably in a multiaccess optical channel corrupted by AWGN and phase noise. Time-synchronous as well as asynchronous systems were analyzed in this context. The nearly equal performance of synchronous and asynchronous schemes indicated that synchronization does not enhance performance significantly. Binary phase-shift-keying (BPSK) and on-off-keying (OOK) data modulation schemes were considered. Considerable performance degradation of OOK due to phase noise left the BPSK scheme as the dominant viable scheme.

The performance evaluation of RC CDMA established the potential advantage in employing hybrids of wavelength-division multiple-access (WDMA) and CDMA to combat inter-carrier interference in dense WDMA systems.

Appendix A

In this appendix, we provide the average probability of bit error for single and multiuser systems. BPSK and OOK modulation are both considered.

1. Single User Analysis

The probability of error P_e for BPSK modulation is

$$P_e = \frac{1}{2} Pr\left[Y > \rho\sqrt{P} | b_0^{(k)} = -1\right] + \frac{1}{2} Pr\left[Y < \rho\sqrt{P} | b_0^{(k)} = 1\right] \tag{A.1}$$

where $\rho\sqrt{P}$ is the threshold. Upon substitution for Y from (3.1)

$$P_e = \frac{1}{2} Pr\left[\eta > \rho + X\right] + \frac{1}{2} Pr\left[\eta < \rho - X\right]. \tag{A.2}$$

The random variable X takes values in $[-1, 1]$. P_e in (A.2) takes two possible forms, depending on the values of X,

$$P_e = \begin{cases} \frac{1}{2} - \frac{1}{2} Pr\left[\rho - X < \eta < \rho + X / X > 0\right] & w.p. \quad p^* \triangleq Pr\left[X > 0\right] \\ \frac{1}{2} + \frac{1}{2} Pr\left[\rho + X < \eta < \rho - X / X < 0\right] & w.p. \quad q^* \triangleq Pr\left[X < 0\right] \end{cases} \tag{A.3}$$

where "$w.p.$" means "with probability". By taking the average of (A.3), we obtain

$$\begin{aligned} P_e = & \frac{1}{2} + \frac{q^*}{2} Pr\left[\rho + X < \eta < \rho - X / X < 0\right] \\ & - \frac{p^*}{2} Pr\left[\rho - X < \eta < \rho + X / X > 0\right]. \end{aligned} \tag{A.4}$$

Next we find the two probabilities in (A.4). The first one is

$$\begin{aligned} & \frac{1}{q^*} Pr\left[\rho + X < \eta < \rho - X, X < 0\right] \\ & = \frac{1}{q^*} \int_{-1}^{0} Pr\left[\rho + x < \eta < \rho - x\right] f_X(x) dx \end{aligned} \tag{A.5}$$

where $f_X(\cdot)$ is the *pdf* of the random variable X. By using the functions $\Phi(.)$ and $Q(.)$ (A.5) takes the form

$$\frac{1}{q^*} \int_{-1}^{0} \left[\Phi\left((\rho - x) \sqrt{\frac{2PT}{N_0}} \right) + Q\left((\rho + x) \sqrt{\frac{2PT}{N_0}} \right) - 1 \right] f_X(x) dx \tag{A.6}$$

where $Q(\cdot)$ and $\Phi(\cdot)$ are related to the standard normal distribution as follows:

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx$$

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx$$

Similarly, the second integral in (A.4) is

$$\frac{1}{p^*} \int_0^1 \left[1 - \Phi\left((\rho - x)\sqrt{\frac{2PT}{N_0}} \right) - Q\left((\rho + x)\sqrt{\frac{2PT}{N_0}} \right) \right] f_X(x)dx. \tag{A.7}$$

Upon substitution of (A.6) and (A.7) in (A.4) the final result is obtained as

$$P_e = \frac{1}{2}\overline{Q\left((\rho + X)\sqrt{\frac{2PT}{N_0}} \right)} + \frac{1}{2}\overline{\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}} \right)}. \tag{A.8}$$

The overlines in (A.8) indicate expectation with respect to the random variable X. Similarly, for OOK modulation P_e is obtained as

$$P_e = \frac{1}{2}Q\left(\rho\sqrt{\frac{2PT}{N_0}} \right) + \frac{1}{2}\overline{\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}} \right)}. \tag{A.9}$$

It is easy to show that the optimum value of ρ for (A.8) and (A.9) is 0 and 1/2, respectively.

2. Multiuser Analysis

The probability of error P_e for BPSK modulation is obtained from (A.1) as follows. If we replace "η" with "$I + \eta$", (A.2),(A.3), and (A.4) are still valid in the multiuser case. The first probability in (A.4) is

$$\frac{1}{q^*} Pr\left[\rho + X < I + \eta < \rho - X, X < 0\right]$$

$$= \frac{1}{q^*} \int_{-1}^{0} Pr\left[\rho + x < I + \eta < \rho - x\right] f_X(x)dx. \tag{A.10}$$

Here we made the assumption that I and X are independent. The validity of this assumption is justified in Section 2.4 by evaluating the characteristic function of multiuser interference. Upon substitution of $Pr[\cdot]$ in (A.10) by an integral, we obtain

$$\frac{1}{q^*}\int_{-1}^{0} f_X(x)\int_{\rho+x}^{\rho-x} f_{I+\eta}(y)\; dydx \tag{A.11}$$

where $f_{I+\eta}(\cdot)$ is the *pdf* of $I+\eta$. Since the *pdf* $f_{I+\eta}(\cdot)$ is a real function, it is easy to show that

$$f_{I+\eta}(y) = \frac{1}{\pi}\int_{0}^{\infty} \left[Re\left\{\Phi_{I+\eta}(u)\right\}\cdot\cos(yu) + Im\left\{\phi_{I+\eta}(u)\right\}\sin(yu)\right] du \tag{A.12}$$

where $Re\{\cdot\}$ and $Im\{\cdot\}$ denote the real and imaginary parts, respectively, of their complex arguments. Considering that I and η are independent and that $\Phi_I(u)$ turned out to be real (see Section 2.4),

$$f_{I+\eta}(y) = \frac{1}{\pi}\int_{0}^{\infty} \Phi_I(u)\;\Phi_\eta(u)\;\cos(yu)du. \tag{A.13}$$

Substituting (A.13) in (A.11) and performing the integration $\int_{\rho+x}^{\rho-x}$, the first probability in (A.4) is

$$-\frac{2}{\pi q^*}\int_{0}^{\infty} \Phi_I(u)\;\Phi_\eta(u)\;\cos(\rho u)\left[\int_{-1}^{0} f_X(x)\;\sin(ux)\;dx\right]\frac{du}{u}. \tag{A.14}$$

Using the same procedure, the second probability in (A.4) is

$$\frac{2}{\pi p^*}\int_{0}^{\infty} \Phi_I(u)\;\Phi_\eta(u)\;\cos(\rho u)\left[\int_{0}^{1} f_X(x)\;\sin(ux)\;dx\right]\frac{du}{u}. \tag{A.15}$$

Combining (A.14) and (A.15) in (A.4) we obtain

$$P_e = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} \Phi_I(u)\;\Phi_\eta(u)\;\cos(\rho u)\frac{\overline{\sin(uX)}}{u}\;du, \tag{A.16}$$

where

$$\overline{\sin(uX)} = \int_{-1}^{1} f_X(x)\;\sin(ux)\;dx.$$

To evaluate P_e for OOK modulation we follow the same method as before and obtain

$$P_e = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \Phi_I(u)\ \Phi_\eta(u) \cdot \frac{\sin(\rho u) - \overline{\sin(\rho - X)u}}{u} \cdot du. \tag{A.17}$$

In order to put (A.16) in a more meaningful format and also to facilitate the computations, we rewrite (A.16) as

$$P_e = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Phi_\eta(u)\ \cos(\rho u)\frac{\overline{\sin(uX)}}{u}du$$
$$+ \frac{1}{\pi}\int_0^\infty (1 - \Phi_I(u))\ \Phi_\eta(u)\ \cos(\rho u)\frac{\overline{\sin(uX)}}{u}du. \tag{A.18}$$

The third term in (A.18) includes the contribution of multiuser interference in BER. The first two terms are single user contributions. Therefore, by using (A.9) in (A.18) we obtain

$$P_e = \frac{1}{2}\overline{Q\left((\rho + X)\sqrt{\frac{2PT}{N_0}}\right)} + \frac{1}{2}\overline{\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)}$$
$$+ \frac{1}{\pi}\int_0^\infty (1 - \Phi_I(u))\,\Phi_\eta(u)\cos(\rho u)\frac{\overline{\sin(uX)}}{u}du. \tag{A.19}$$

Similarly, for OOK modulation, P_e is

$$P_e = \frac{1}{2}Q\left(\rho\sqrt{\frac{2PT}{N_0}}\right) + \frac{1}{2}\overline{\Phi\left((\rho - X)\sqrt{\frac{2PT}{N_0}}\right)}$$
$$+ \frac{1}{2\pi}\int_0^\infty (1 - \Phi_I(u))\,\Phi_\eta(u)\frac{\sin(\rho u) - \overline{\sin(\rho - X)u}}{u}du \tag{A.20}$$

where

$$\Phi_\eta(u) = \exp(-\frac{N_0}{4PT}u^2). \tag{A.21}$$

Appendix B

In this appendix, the distribution of a random variable which is the $mod.2\pi$ of a uniform random variable is derived. Let X be a uniform random variable distributed in the bandwidth $[0, W]$. We are interested in knowing the distribution of the random variable Y, which is defined as

$$Y = < X > . \tag{B.1}$$

The random variable Y is expressed in terms of some disjoint intervals as follows:

$$Y = \begin{cases} X - 2n\pi & if \quad 2n\pi < X < 2(n+1)\pi; \quad 0 \leq n < M \\ X - 2M\pi & if \quad 2M\pi < X < W \end{cases} \tag{B.2}$$

where

$$M = \lfloor W/2\pi \rfloor .$$

For $\alpha \epsilon [0, 2\pi]$, the event $(Y \leq \alpha)$ is expressed as

$$\bigcup_{n=0}^{M-1} (X - 2n\pi \leq \alpha, \quad 2n\pi < X < 2(n+1)\pi) \bigcup (X - 2M\pi \leq \alpha, \quad 2n\pi < X < W) \tag{B.3}$$

The probability of (B.3) is the summation of the probabilities of the individual events in (B.3). By using the conditional probability and after some simplifications, $F_Y(\alpha)$ (the cumulative distribution function of Y) becomes

$$F_Y(\alpha) = \frac{\alpha \lfloor W/2\pi \rfloor + min(\alpha \ , \ W - 2\pi \lfloor W/2\pi \rfloor)}{W} . \tag{B.4}$$

As $W \to \infty$, this distribution approaches to the uniform one.

Appendix C

Let $\{X_n\}$ be

$$X_n = < \phi_n + \lambda_n > \tag{C.1}$$

where $\{\phi_n\}$ is assumed to be i.i.d. and uniform in $[0, 2\pi]$, for all n, and $\{\lambda_n\}$ is a sequence of random variables with arbitrary distribution. These two sequences are independent of each other for all n. Here we establish two claims:

- Claim 1: $\{X_n\}$ are uniform for all n.

- Proof : It is a known fact that

$$X_n|\text{given } \lambda_{\mathrm{n}} \stackrel{\mathrm{d}}{=} \phi_{\mathrm{n}}. \tag{C.2}$$

The pdf of X_n is obtained as

$$f_{X_n}(x) = \int_\lambda f_{X_n|\lambda_n}(x|\lambda) f_{\lambda_n}(\lambda) d\lambda. \tag{C.3}$$

By using (C.2) (C.3) becomes

$$f_{X_n}(x) = \int_\lambda f_{\phi_n}(x) f_{\lambda_n}(\lambda) d\lambda = f_{\phi_n}(x). \tag{C.4}$$

- Claim 2: $\{X_n\}$ are independent for all n.
- Proof : We show that any two random variables X_n and X_l $(n \neq l)$ are independent. To show this, it suffices to show that, for any function g$(\cdot)$,

$$E[g(X_n)g(X_l)] = E[g(X_n)]E[g(X_l)]. \tag{C.5}$$

Given λ_n and λ_l in (C.1), the first side of (C.5) conditioned on these values is

$$E[g(<\phi_n + \lambda_n>).g(<\phi_l + \lambda_l>)|\lambda_n, \lambda_l]. \tag{C.6}$$

Since ϕ_n and ϕ_l are independent, (C.6) becomes

$$E[g(<\phi_n + \lambda_n>)|\lambda_n] \cdot E[g(<\phi_l + \lambda_l>)|\lambda_l] \tag{C.7}$$

or

$$E[g(X_n)|\lambda_n] \cdot E[g(X_l)|\lambda_l]. \tag{C.8}$$

From Claim 1:

$$X_n \stackrel{d}{=} X_n|\text{given } \lambda_{\mathrm{n}}$$

$$X_l \stackrel{d}{=} X_l|\text{given } \lambda_{\mathrm{l}}. \tag{C.9}$$

Therefore, (C.8) is

$$E[g(X_n)] \cdot E[g(X_l)]. \tag{C.10}$$

This means that

$$E[g(X_n)g(X_l)|\lambda_n, \lambda_l] = E[g(X_n)] \cdot E[g(X_l)]. \tag{C.11}$$

Taking expectation with respect to λ_n and λ_l gives (C.5) and this completes the proof.

REFERENCES

[1] G. J. Foschini and G. Vannucci. "Using Spread Spectrum in a High-Capacity Fiber-Optic Local Network." *Journal of Lightwave Technology*, Vol. 6, March 1988, pp. 370-379.

[2] G. Vannucci. "Combining Frequency-Division and Code-Division Multiplexing in a High-Capacity Optical Network." *IEEE Network*, March 1989, 21-30.

[3] E. A. Geraniotis and M. B. Pursley. "Error probability for Direct-Sequence Spread-Spectrum Multiple-Access Communications–Part II: Approximations." Special Issue on Spread-Spectrum Communications of the *IEEE Transactions on Communications*, Vol. COM-30. pp. 985-995. May 1982.

[4] E. Geraniotis and B. Ghaffari. "Performance of Binary and Quaternary Direct-Sequence Spread-Spectrum Multiple-Access Communications with Random Signature Sequences." To appear in the *IEEE Transactions on Communications*, 1991.

[5] E. A. Geraniotis and M. B. Pursley. "Performance of Coherent Direct-Sequence Spread-Spectrum Communications over Specular Multipath Fading Channels." *IEEE Transactions on Communications*, Vol. COM-33, pp. 502-508, June 1985.

[6] R. C. Alferness. "Waveguide Electrooptic Modulators." *IEEE Transactions on Microwave Theory and Techniques*, Vol. MTT-30, No. 8, August 1982.

[7] R. C. Bradley, "The Basic Properties of Strong Mixing Conditions," in *Dependence in Probability and Statistics*, Birkhauser, 1985.

[8] M. Peligrad, "Recent Advances in the Central Limit Theorem and Its Weak Invariance Principle for Mixing Sequences of Random Variables," in *Dependence in Probability and Statistics*, Birkhauser, 1985.

[9] G. Orsak and B. Aazhang. "On the Theory of Importance Sampling Applied to the Analysis of Detection Systems." *IEEE Transactions on Communications*. Vol. 37, No. 4, April 1989.

[10] G. J. Foschini and G. Vannucci. "Characterizing Filtered Light Waves Corrupted by Phase Noise." *IEEE Transactions on Information Theory*, Vol. 34, No. 6, November 1988.

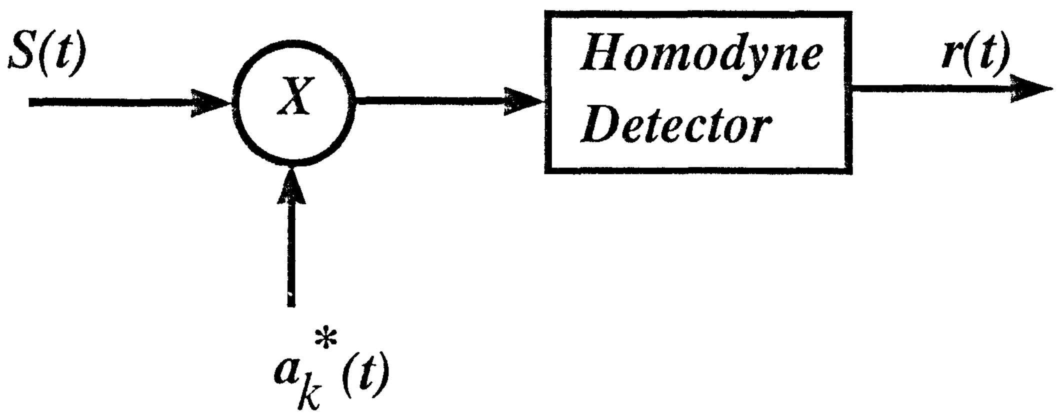

Fig.1 **Coherent detection of the optical signal $S(t)$**

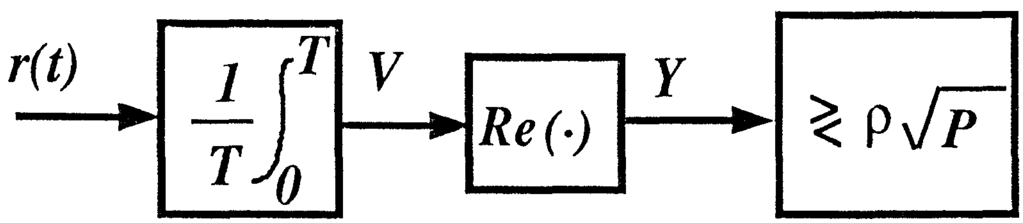

Fig.2 **Detection of the electrical signal $r(t)$**

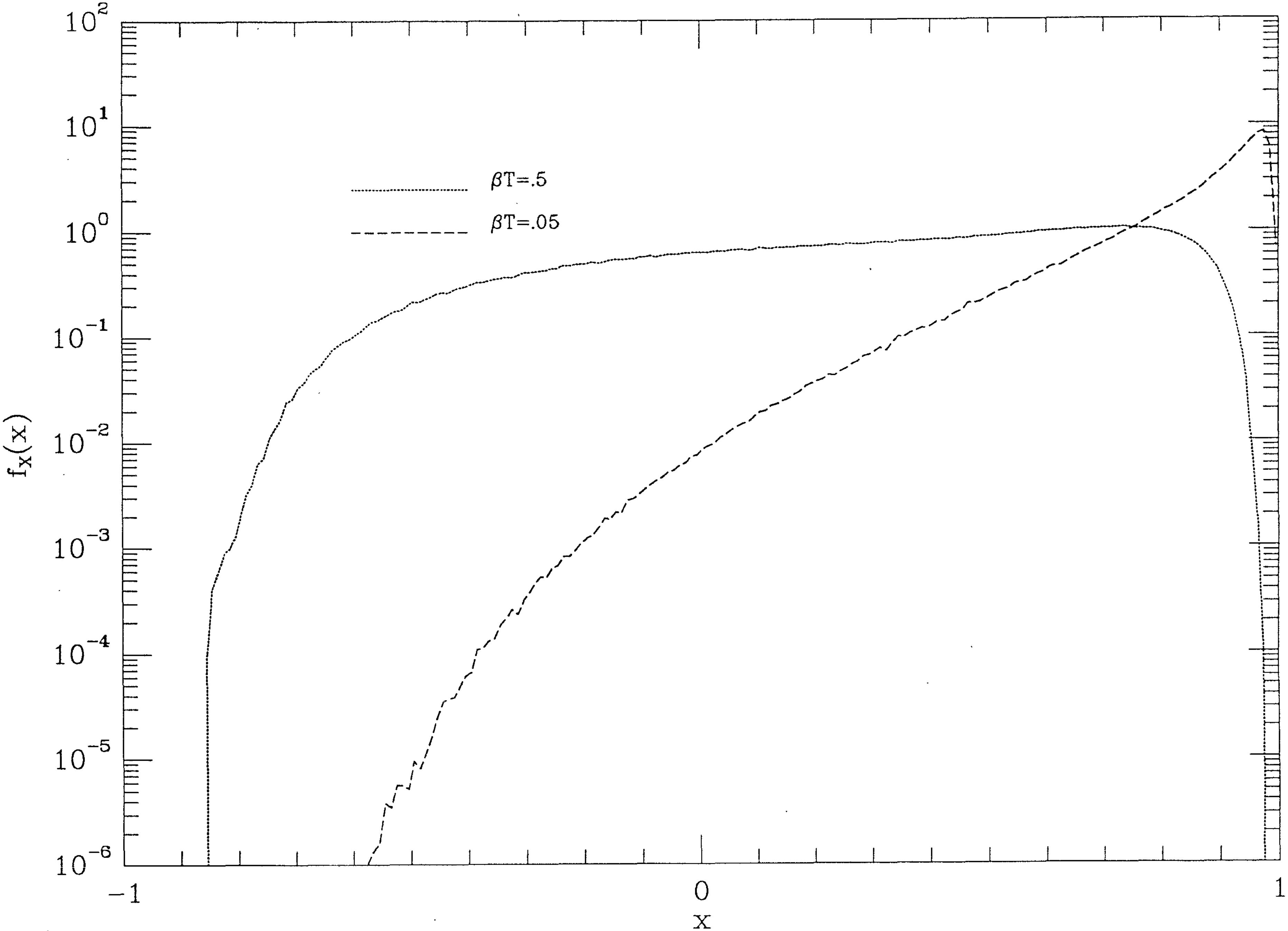

Fig. 3a. PDF of the random variable X

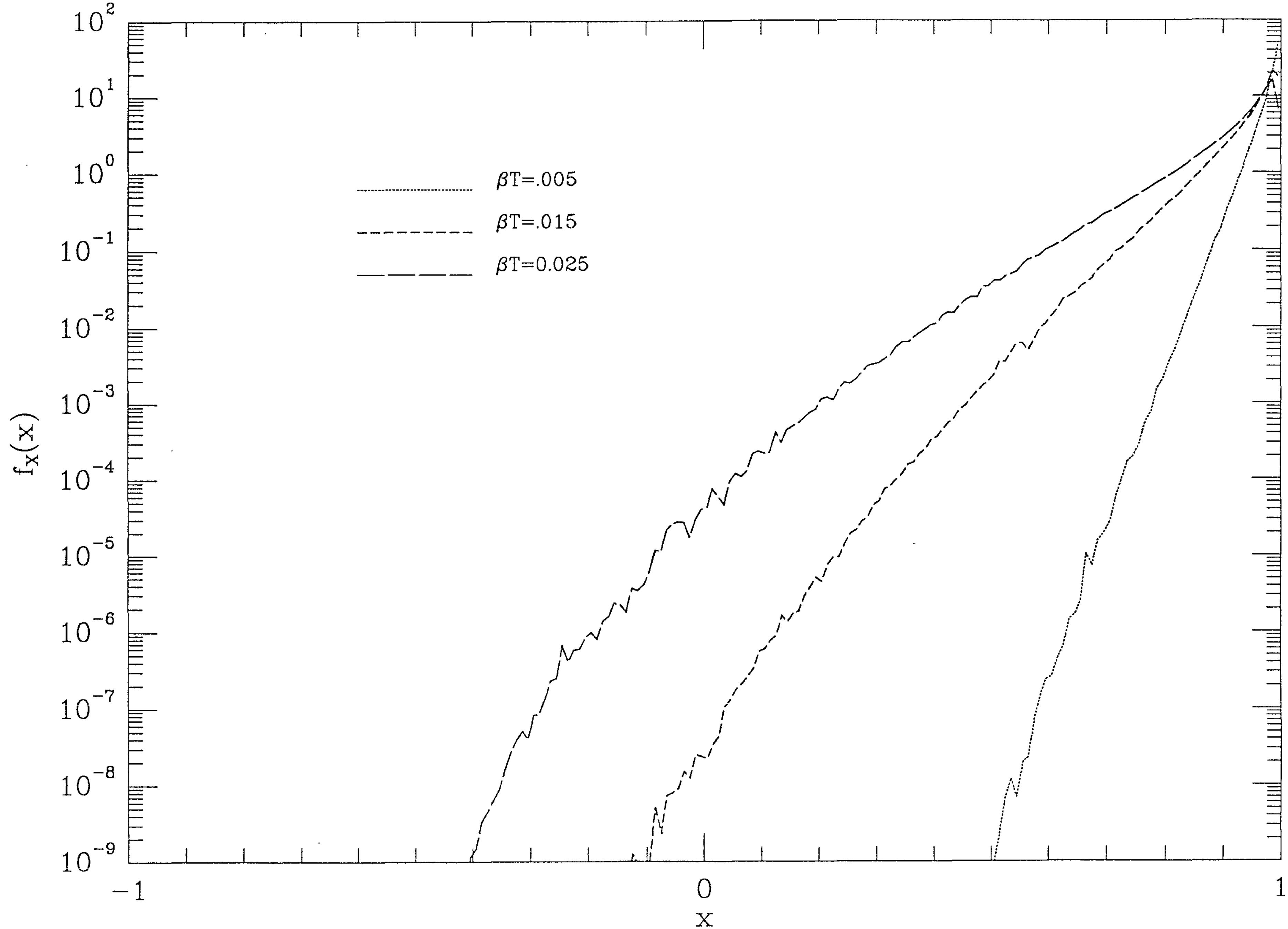

Fig. 3b. PDF of the random variable X

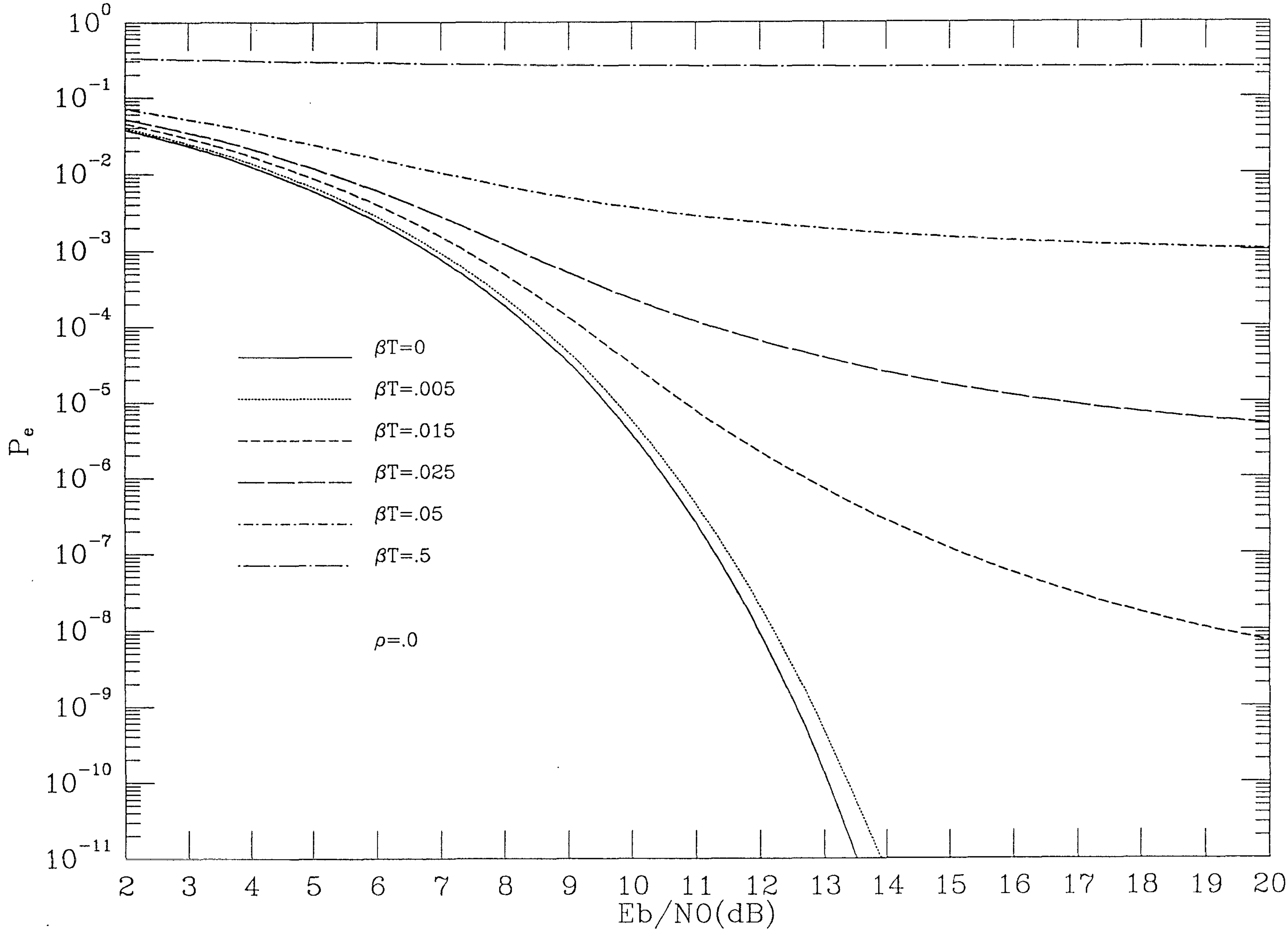

Fig. 4. Performance of the single user system (BPSK)

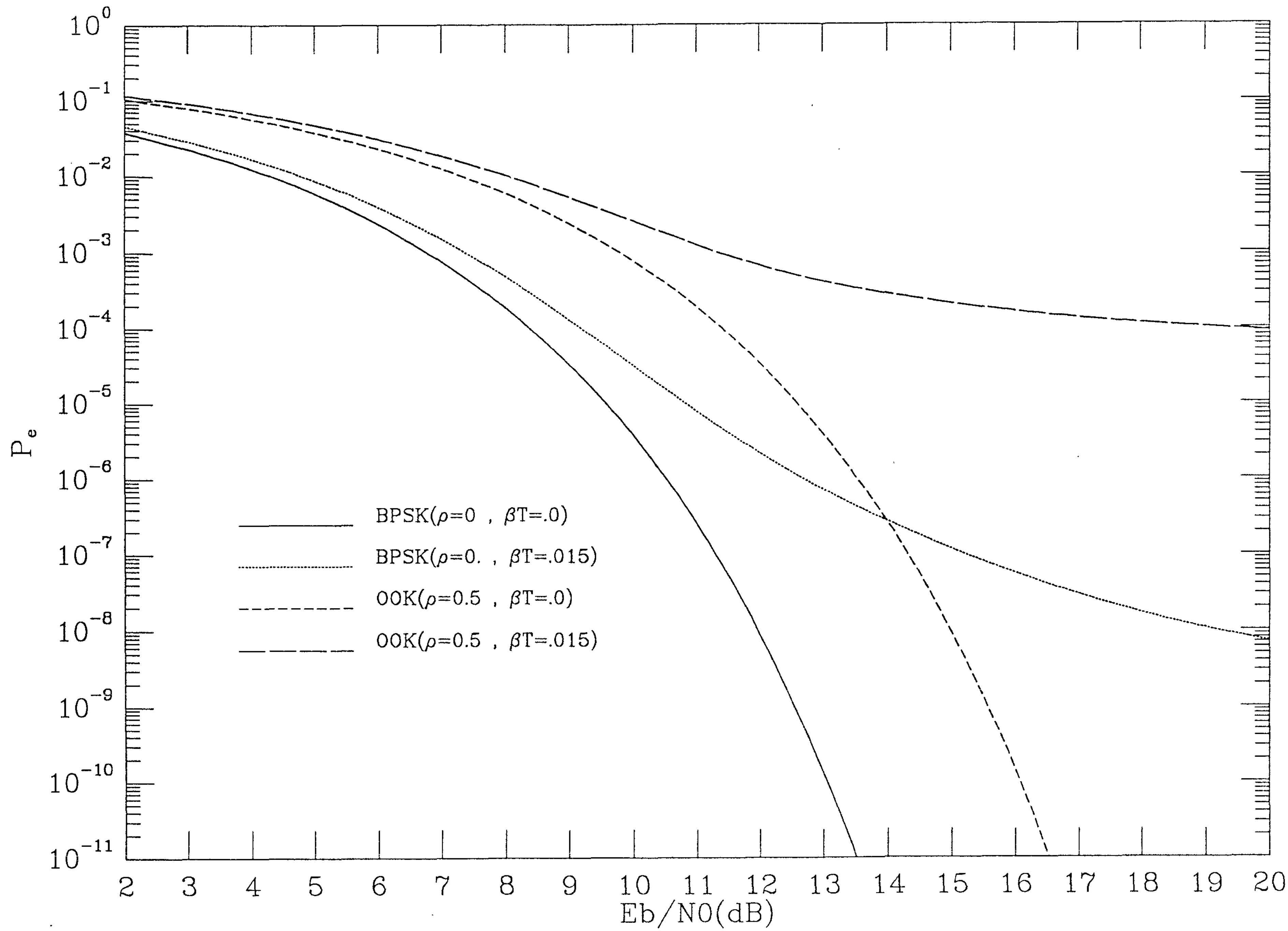

Fig. 5. Performance comparison of the single user system for BPSK and OOK

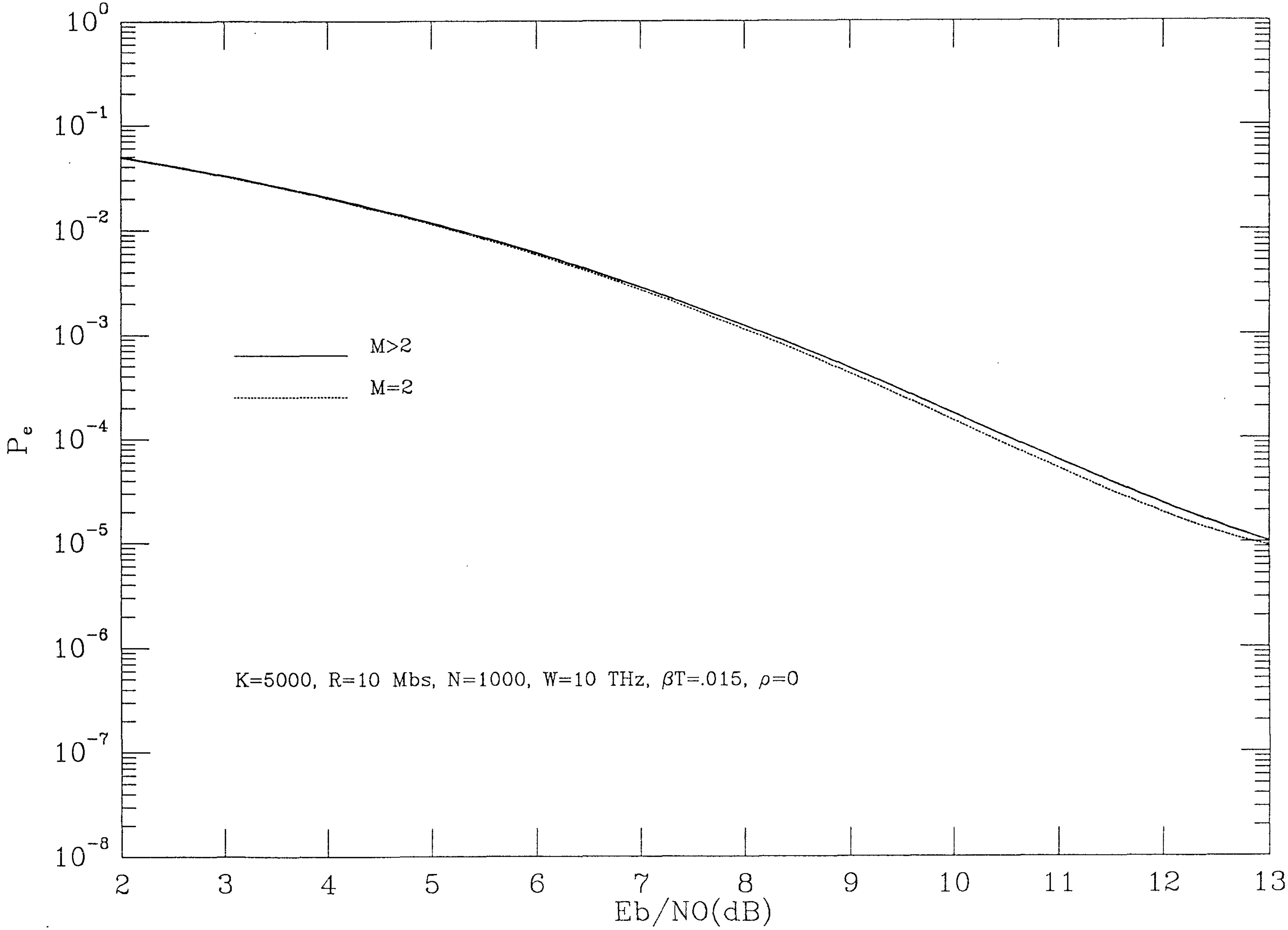

Fig. 6. Comparison of binary and M-ary phase signature sequence for first method (synchronous, BPSK)

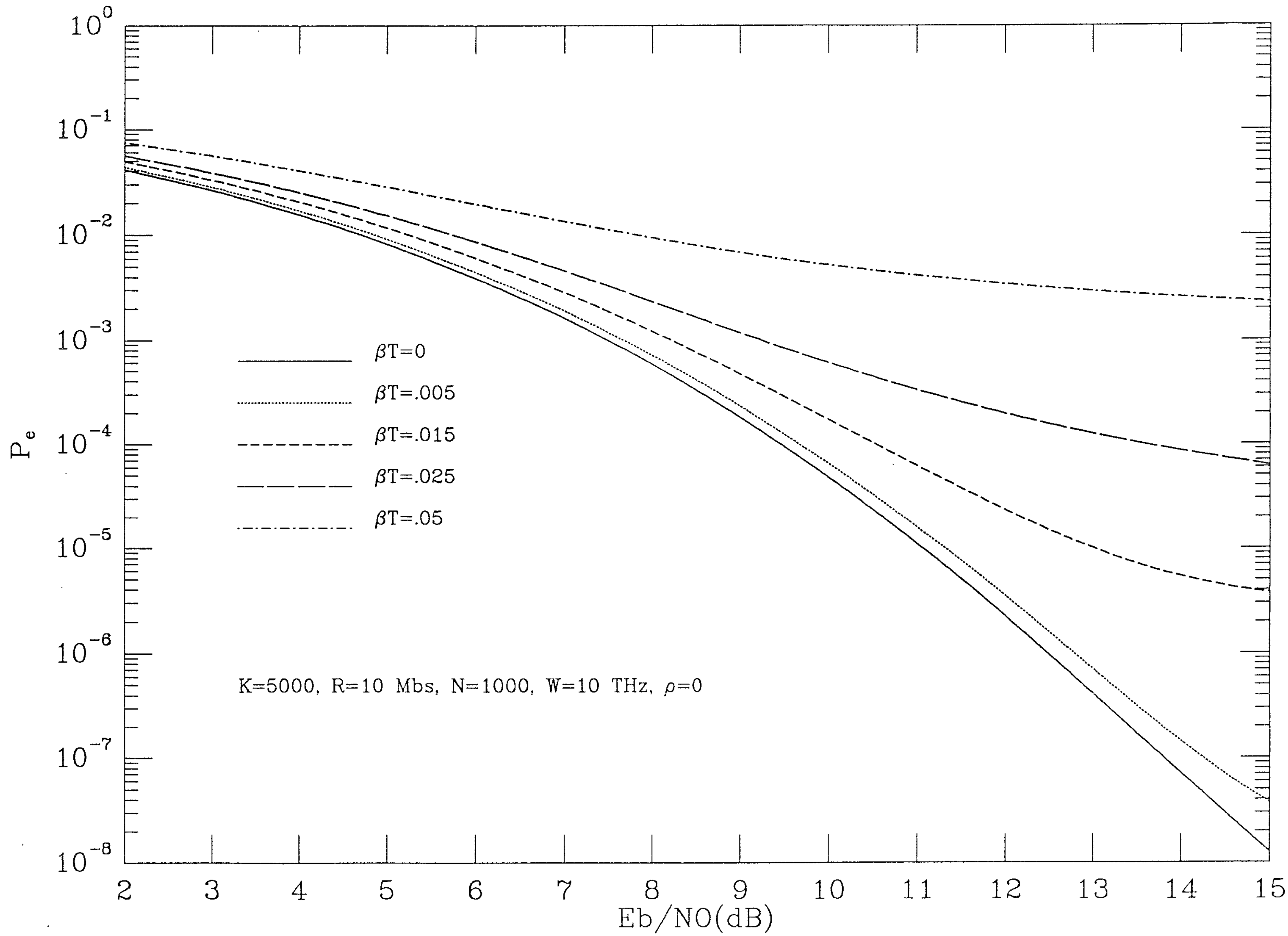

Fig. 7. Performance of the asynchronous multiuser system for BPSK with method 1

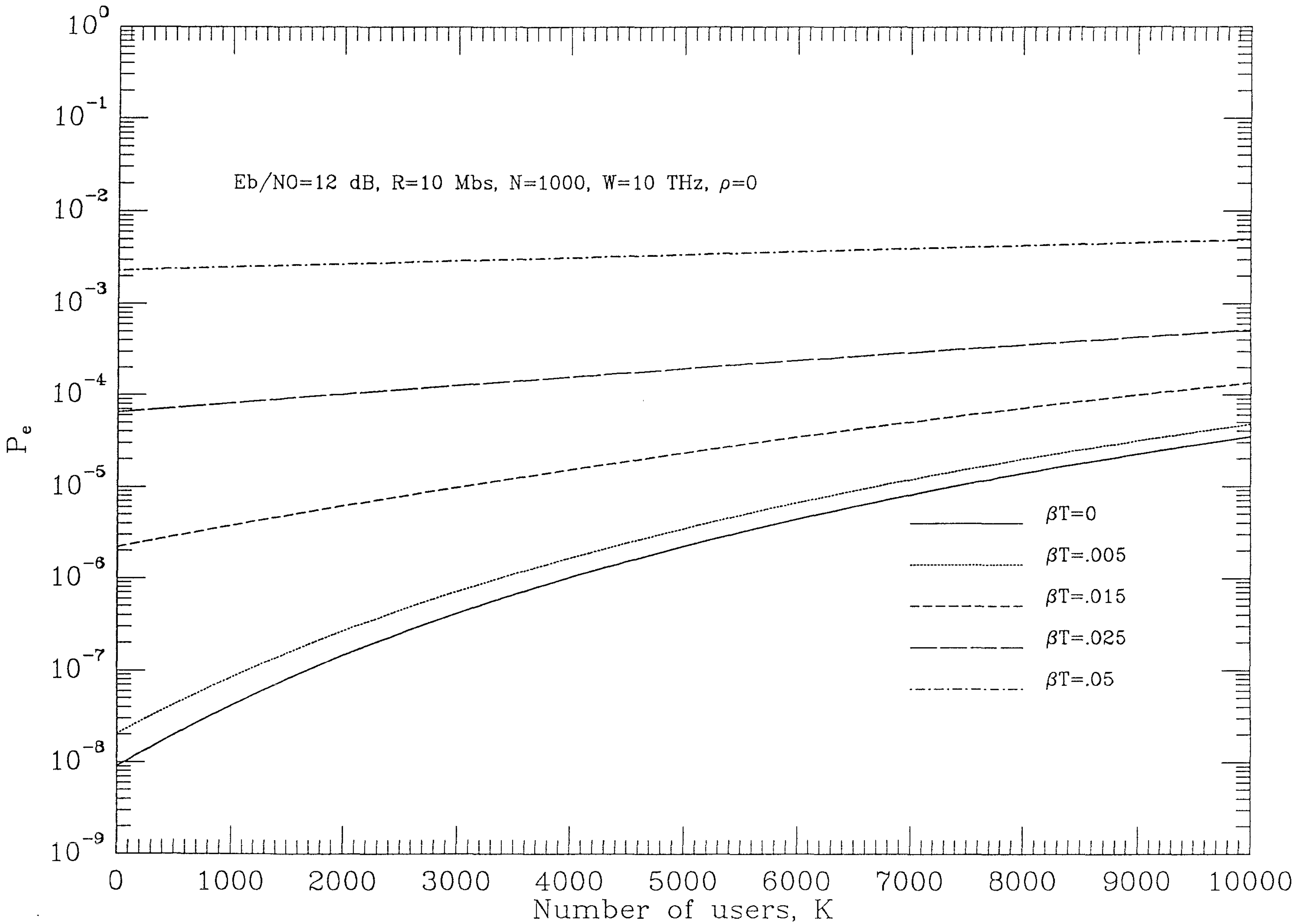

Fig. 8. Multiuser capability of the asynchronous system for BPSK with method 1

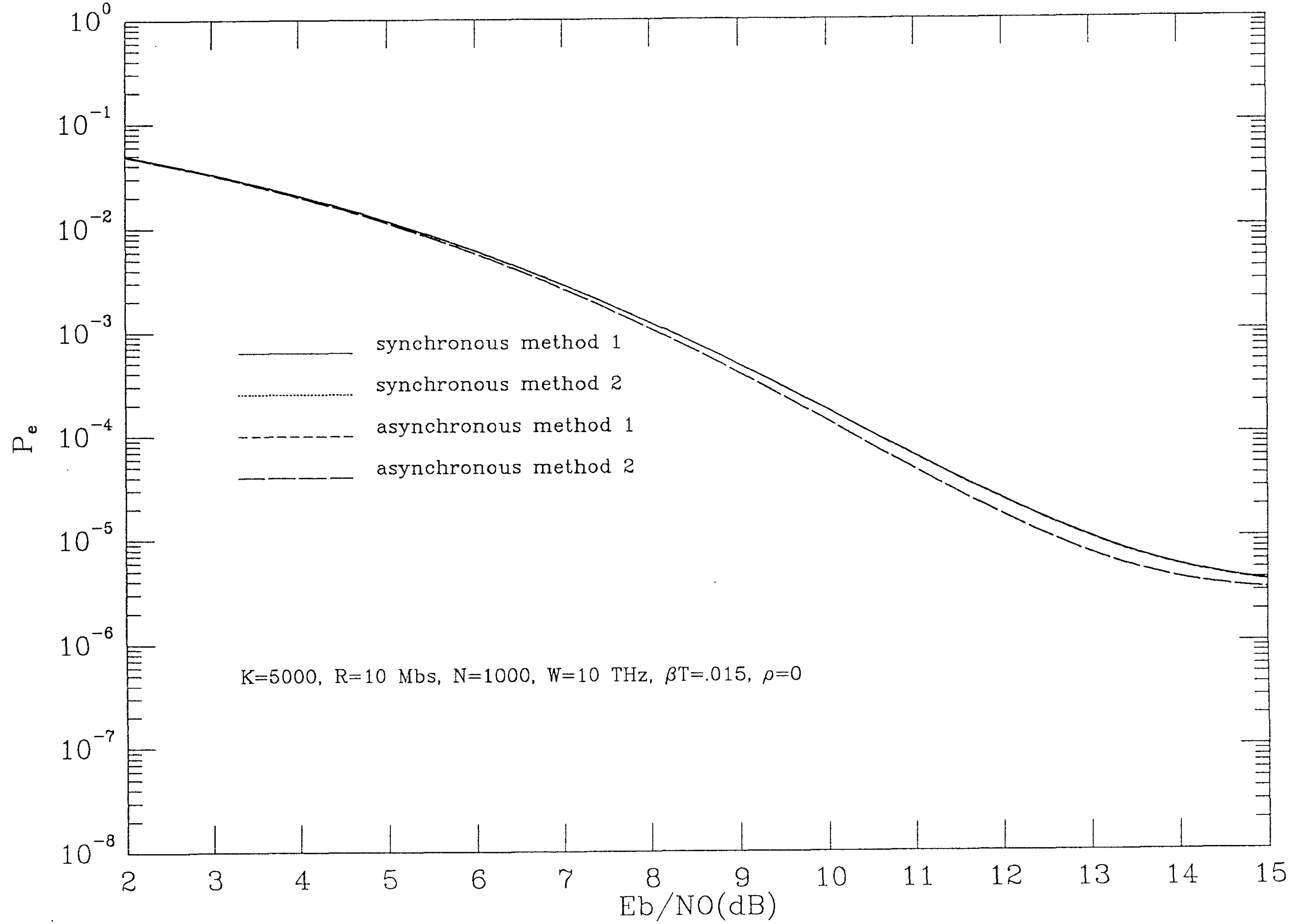

Fig. 9. Performance comparison of the synchronous and asynchronous systems for two methods (BPSK)

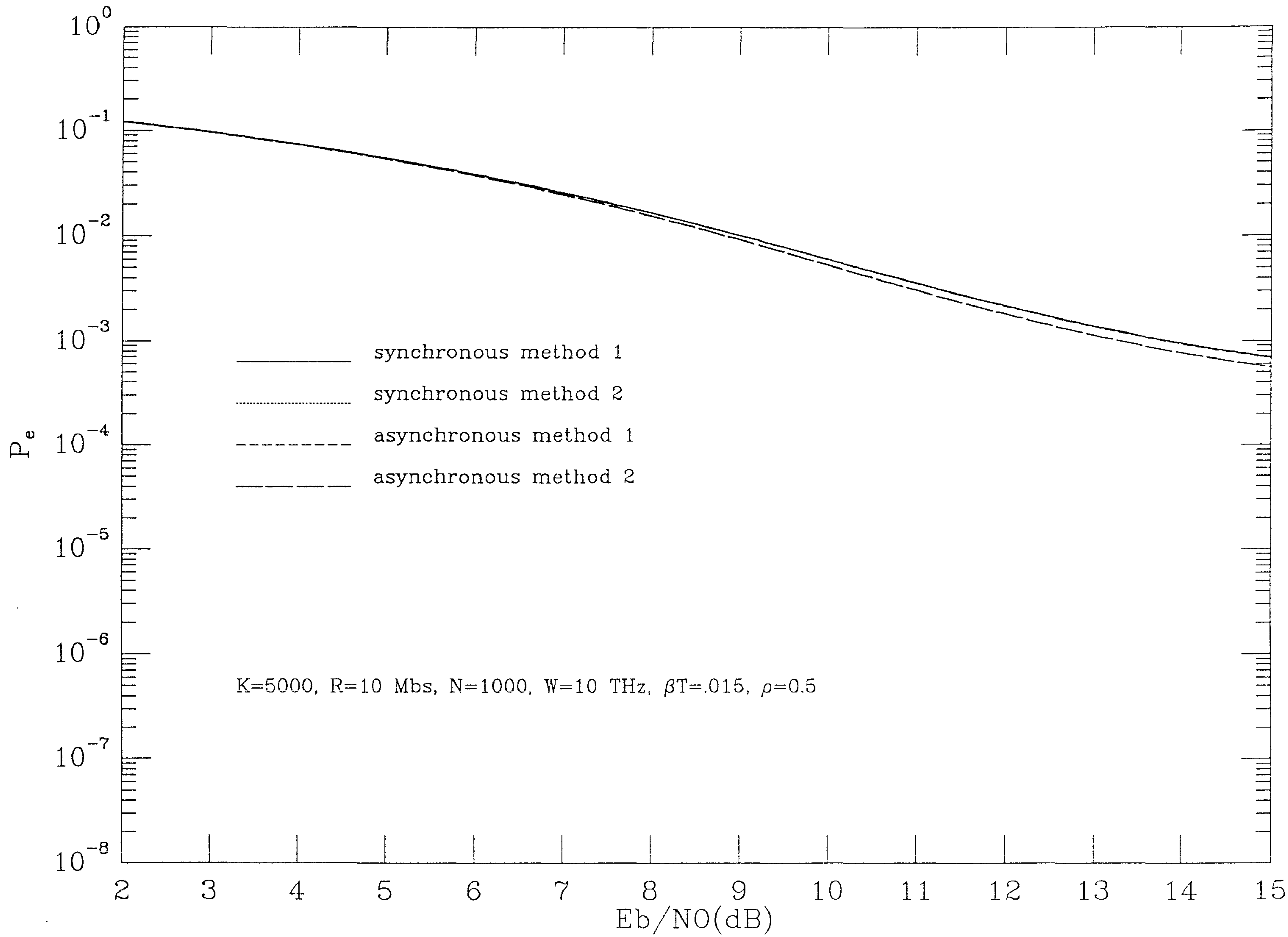

Fig. 10. Performance comparison of the OOK synchronous and asynchronous systems for two methods (OOK)