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The Berry-Hannay Phase of the Equal-Sided Spring-Jointed
Four-Bar Mechanism

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Abstract

In this work we apply the moving systems approach developed by Marsden, Montgomery, and Ratiu to a free-floating, equal-sided, spring-jointed, four-bar mechanism that is being slowly rotated about its central axis and derive a formula for the induced geometric phase.

1 Introduction

Due to the immense number of applications, research on gyroscopes has been active for many years. With the absence of rotating parts, low power requirements, and inherent scalability, vibratory gyroscopes have become particularly popular [1]. These devices all take advantage of the same physical phenomenon, the Coriolis force [2]. Marsden, Montgomery, and Ratiu have developed a modern geometrical approach to moving systems through which the effect of the Coriolis force can be understood as the holonomy of a particular connection known as the Cartan-Hannay-Berry connection [5]. This holonomy is termed the Berry-Hannay phase (see [3] for a detailed discussion of Berry's phase).

In the moving systems approach one starts with a Riemannian manifold S, referred to as the ambient space, and a submanifold $Q \subset S$, the configuration space. Let M be the space of embeddings of Q into S. If a particle in Q follows a path $q(t)$ and Q follows the curve m_t then the particle in S follows the path $m_t(q(t))$. The velocity in S is then $T_{q(t)} m_t \cdot v + \mathcal{Z}_t(m_t(q(t)))$ where $\mathcal{Z}_t(m_t(q(t)))$ is the velocity vector $\frac{dm_t}{dt}$. The Hamiltonian on T^*Q is given by

$$H = \frac{1}{2}\|p\|^2 - \mathcal{P}(Z_t) - \frac{1}{2}\|\mathcal{Z}^\perp\|^2 + V(q) + U(m_t(q)) \quad (1)$$

where $V(\cdot)$ is a potential on Q, $U(\cdot)$ is a potential on

S, and $\mathcal{P}(Z_t)$ is defined to be

$$\mathcal{P}(Z_t) = p \cdot (T_{q(t)} m_t)^{-1} \left[\mathcal{Z}_t m_t(q)\right]^T \quad (2)$$

where $\left[\mathcal{Z}_t m_t(q)\right]^T$ is the orthogonal projection of $\mathcal{Z}_t(m_t(q(t)))$ onto $T_{m_t(q)} m_t(Q)$ relative to the metric on S.

We assume we have a Lie Group G acting on T^*Q relative to which we can average and replace the above Hamiltonian with its G-average. The Hamiltonian vector field of the averaged Hamiltonian term $< \mathcal{P}(Z_t) >$ has a natural interpretation as the horizontal lift of $\mathcal{Z}_t$ relative to the Cartan-Hannay-Berry connection on $T^*Q \times M$. The holonomy of this connection is the Berry-Hannay phase for the slowly moving system. For the details of the moving systems approach see [4] and [5].

The precession of the nodal points in a rotating vibrating ring was first analyzed in the late 1800's by G.H. Bryan [6]. This phenomenon has provided the basis for gyroscopes (e.g. [7] and [8]). While the linear analyses of these works is effective, it is to be expected that a deeper understanding will emerge by appealing to a nonlinear, geometric approach directed at more accurate constitutive models. The present paper takes a first such step by applying the geometric techniques of moving systems to compute the Berry-Hannay phase (=precession) for a specific class of linkages.

2 Results

The equal-sided four-bar mechanism is diagrammed in figure 1. We follow the setup and analysis of free-floating four-bar mechanisms by Yang and Krishnaprasad [9]. A frame is placed at the system center of mass. Define θ_i as the angle of the ith bar with respect to the center of mass frame, r_i as the vector from the system center of mass to the center of mass of the ith bar, and $d_\pm$ as the vector from the center of mass of the ith bar to the joint with the $i \pm 1$ bar. Under appropriate conditions the configuration space of the free-floating four-bar mechanism is $S^1 \times S^1$. We take
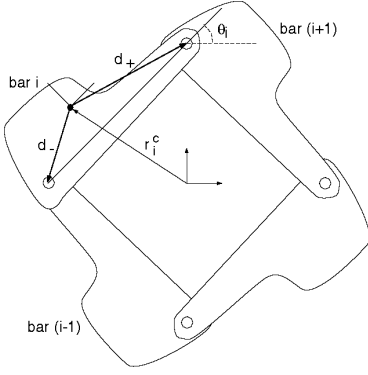
Figure 1: Equal-sided four-bar linkage

this as the ambient space S and choose coordinates (θ_0, θ_{10}) where $\theta_{10} = \theta_1 - \theta_0$. The shape space of this system is S^1. We take this as the configuration space Q and choose θ_{10} as a coordinate. At each joint is a spring and we assume that the total potential is such that the nominal (unrotated) system admits a periodic solution about some equilibrium shape. The entire system is adiabatically (slowly) rotated (moved) at rate Ω. The embedding $Q \hookrightarrow S$ is $m_t(\theta_{10}(t)) = (\Omega t + \theta_0(0), \theta_{10}(t))'$ with $\theta_0(0)$ some arbitrary initial angle. The Lagrangian is

$$L = \frac{1}{2}\begin{pmatrix} \Omega \\ \omega_{10} \end{pmatrix}' \widehat{M} \begin{pmatrix} \Omega \\ \omega_{10} \end{pmatrix} - V(\theta_{10}) \qquad (3)$$

where $\widehat{M}$ is a symmetric 2x2 matrix whose elements depend on θ_{10} and the mechanism parameters. From this we can see $\mathcal{Z}_t(m_t(q)) = (\Omega, 0)'$. Projecting $\mathcal{Z}_t(m_t(q))$ to $T_{m_t(q)}m_t(Q)$ relative to $\widehat{M}$ yields

$$[\mathcal{Z}_t m_t(q)]^T = \widehat{M}_{10}(\theta_{10})\widehat{M}_{11}^{-1}\Omega \qquad (4)$$

The Hamiltonian can be shown to be

$$H = \frac{1}{2\widehat{M}_{11}}p_{10}^2 + V(\theta_{10}) - \frac{\widehat{M}_{10}(\theta_{10})}{\widehat{M}_{11}}\Omega p_{10} \qquad (5)$$

where terms in Ω^2 have been dropped due to the adiabatic assumption. Since the configuration space is one-dimensional the system is integrable and thus there exist action-angle coordinates (I, ϕ) on T^*Q [10]. Let the coordinate transformations be given by

$$\theta_{10} = f_1(I, \phi) \qquad p_{10} = f_2(I, \phi) \qquad (6)$$

Using these formulas we then have

$$\mathcal{P}(Z) = \frac{\widehat{M}_{10}(f_1(I, \phi))}{\widehat{M}_{11}}\Omega f_2(I, \phi) \qquad (7)$$

Averaging over one cycle of ϕ yields $< \mathcal{P}(Z) > = \Omega g(I)$ for an appropriate function of the action. The Hamiltonian vector field given by the horizontal lift of $\mathcal{Z}(t)$ to T^*Q relative to the Cartan-Hannay-Berry connection is then

$$-X_{<\mathcal{P}(Z)>} = \left(-\Omega\frac{\partial g(I)}{\partial I}\frac{\partial}{\partial \phi}, 0\right) \qquad (8)$$

and the geometric phase is given by

$$\Delta\phi = -\oint \Omega\frac{\partial g}{\partial I}dt = -\int_0^{2\pi}\frac{\partial g}{\partial I}d\theta_0 = -2\pi\frac{\partial g}{\partial I} \qquad (9)$$

3 Conclusions

In this work we have found a formula for the Berry-Hannay phase for a generic equal-sided, spring-jointed, four-bar mechanism. Applying these results to a system with a quadratic potential (or a generic potential with a small-angle approximation) yields a geometric phase of 0.

The example here has given us insight into the use of the moving systems approach and we are now investigating the equal-sided n-bar mechanism. As this system is similar to a rotated vibrating ring we expect to find a non-zero Berry-Hannay phase even in the small angle approximation. Similar to the vibrating ring we expect the geometric phase to manifest itself as a rotating wave solution.

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