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Bin Packing and Dynamic File Storage: An "Any-Fit" Algorithm Can Stabilize

by

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Bin Packing and Dynamic File Storage: An "Any-Fit" Algorithm Can Stabilize *

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Abstract

In this paper we first consider the one-dimensional bin-packing problem and show that a class of "any-fit" algorithms can bound the expected wasted space in the system under certain conditions.

In the second part of the paper we consider a dynamic file-storage problem and show, under certain conditions, that a class of "any-fit" algorithms can bound the expected wasted space left by deleted files.

BIN PACKING; DYNAMIC FILE STORAGE

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1 Introduction

In this paper we first consider the one-dimensional bin-packing problem and show that a class of "any-fit" algorithms can bound the expected wasted space in the system under certain conditions. This result is a simple application of the theory developed in Section 2 of Gubner, Gopinath, and Varadhan (1988), summarized here as Lemma A1 in Appendix A.

For comparison, we note that Courcoubetis and Weber (1986) call a bin-packing system "stabilizable," if there exists a bin-packing algorithm which will bound the expected wasted space for all time. They state and prove necessary and sufficient conditions for a stabilizing algorithm to exist. In particular, they give conditions under which a (complicated) stabilizing algorithm is guaranteed to exist. Using a completely different approach, we give a simple set of sufficient conditions under which it is guaranted that a class of simple algorithms, which includes the so called "first-fit" and "best-fit" algorithms, will stabilize a bin-packing system. We point out that the model of Courcoubetis and Weber (1986) is a continuous-time one, while our model is the discrete-time analog.

In the second part of the paper we consider a dynamic file-storage problem and show, under certain conditions, that a class of "any-fit" algorithms can bound the expected wasted space left by deleted files. This result relies on the extended theory in Section 5 of Gubner, Gopinath, and Varadhan (1988), summarized here as Lemma A2 in Appendix A. The reader may wish to contrast our discrete-time model with the continuous-time $M/M/\infty$ queuing models in Coffman, Kadota, and Shepp (1985) and Coffman et al. (1986).

2 Bin Packing

Consider a "packing station" in a warehouse in which objects arrive on a conveyor belt. At the end of the conveyor belt there is a packer who stands ready with an infinite stack of empty bins, each of height N. The heights of the objects may range between 1 and $N-1$, though not all heights need to be represented. As objects arrive, empty bins are removed from the stack and placed on the warehouse floor while they are filled. When a bin is full it is removed from the warehouse. Bins which are not yet full remain on the floor. The sum of all unused portions of all bins on the warehouse floor is called the *wasted space*. Observe that the wasted space is a time-varying quantity which changes as new objects are packed. The goal of the packer is to ensure that the expected wasted space is a uniformly bounded function of time. We assume that repacking of bins is *not* permitted.

To illustrate some of the problems involved, consider the following. When the first object arrives, the packer takes an empty bin from the stack and puts the bin on the warehouse floor. The object is then placed in the bin. This bin is not full; when the second object arrives, it may fit in the bin with the first object. If the second object would fit in the bin with the first object, the packer must make a decision. Should the packer put the second object with the first or should he take an empty bin from the stack, put the bin on the warehouse floor, and place the second object there? To show that this question is not entirely trivial, consider the following situations.

Suppose that the bins are of height 5 and that the objects are of heights 2 and 3. Suppose that the first two objects to arrive are of height 2. Clearly, the packer should put the second object into a new bin. Otherwise, the wasted space will be greater than or equal to 1 for all time onward.

Now consider the situation in which there are objects of height 1 together with objects of various other heights. Intuitively, if the "percentage" of objects of height

1 is "high enough," we should expect that an "any-fit" algorithm, which always packs a new object in a partially full bin already on the floor whenever possible, can control the wasted space. More precisely, we will show that under certain conditions, an "any-fit" algorithm can stabilize a bin-packing system in the sense that the expected wasted space will be bounded for all time, regardless of the inital condition of any bins on the warehouse floor.

3 A Mathematical Model for Bin Packing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space on which a time-homogeneous Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ is defined as follows. The "state" of the warehouse at time n will be denoted by X_n. The state, X_n, consists of the number of bins on the warehouse floor together with the wasted space in each bin on the floor. Given that $X_n = x$, for $k = 1, \ldots, N-1$, let λ_k denote the probability (independent of n) that at time $n+1$, an object of height k arrives on the conveyor belt. If $\sum_{k=1}^{N-1} \lambda_k < 1$, the remaining probability mass is assigned to the event

$$\{\text{no object arrives at time } n+1\}.$$

Let $w(X_n)$ denote the sum of the wasted space in the individual bins on the floor at time n. Clearly, w is a nonnegative function defined on the state space of the Markov process $\{X_n\}$.

We assume that an "any-fit" algorithm is employed. With regard to the behavior of the process $\{w(X_n)\}$, this implies the following. Given that $X_n = x$, with conditional probability 1,

$$(1) \qquad |w(X_{n+1}) - w(x)| \ \leq \ N - 1.$$

This implies immediately that

$$(2) \qquad \mathsf{E}[\, |w(X_{n+1}) - w(x)|^2 \mid X_n = x \,] \ \leq \ (N-1)^2.$$

Further, we can easily compute that when $w(x) \geq 1$,

$$\mathsf{E}[\, w(X_{n+1}) - w(x) \mid X_n = x\,] \;\leq\; -[\lambda_1 - \sum_{k=2}^{N-1}(N-k)\lambda_k]. \tag{3}$$

Using these facts, we have

Theorem 1. *Let* $\nu \triangleq \lambda_1 - \sum_{k=2}^{N-1}(N-k)\lambda_k$. *Assume* $\nu > 0$. *Choose* $0 < C \leq \nu/36$. *Choose* $L = \max\{1, \frac{N-1}{1/2}, \frac{(N-1)^2}{C}\}$ *so that whenever* $w(x) > L$,

$$|w(X_{n+1}) - w(x)| \;\leq\; \tfrac{1}{2}w(x), \tag{4}$$

$$\mathsf{E}[\, w(X_{n+1}) - w(x) \mid X_n = x\,] \;\leq\; -\nu, \tag{5}$$

and

$$\mathsf{E}[\, |w(X_{n+1}) - w(x)|^2 \mid X_n = x\,] \;\leq\; C w(x). \tag{6}$$

Set $A = (N-1) + L$. *Then*

$$\mathsf{E}[\, w(X_n) \mid X_0 = x\,] \;\leq\; \frac{w(x)^3}{[w(x) + \frac{\nu}{2}n]^2} + A(1 + 4(\tfrac{A}{\nu})^2 \sum_{k=1}^{n-1}\frac{1}{k^2}). \tag{7}$$

Proof. Observe that when $w(x) \leq L$, (1) implies $w(X_{n+1}) \leq (N-1) + L = A$. Thus

$$\mathsf{E}[\, w(X_{n+1}) \mid X_n = x\,] \;\leq\; A, \qquad \text{if } w(x) \leq L, \tag{8}$$

and

$$\mathsf{E}[\, w(X_{n+1})^3 \mid X_n = x\,] \;\leq\; A^3, \qquad \text{if } w(x) \leq L. \tag{9}$$

Now, (1), (3), and (2) clearly imply (4), (5), and (6) when $w(x) > L$. The fact that (4), (5), (6), (8), and (9) imply (7) is simply Lemma A1 in Appendix A. ∎

Clearly, (7) implies both (see Papadimitriou (1973)).

$$\mathsf{E}[\, w(X_n) \mid X_0 = x\,] \;\leq\; w(x) + A(1 + \tfrac{2}{3}(\tfrac{\pi A}{\nu})^2), \tag{10}$$

and

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}[\, w(X_n) \mid X_0 = x\,] \;\leq\; A(1 + \tfrac{2}{3}(\tfrac{\pi A}{\nu})^2) \;<\; \infty, \tag{11}$$

which is the desired result. We should point out that the assumption that ν be positive is not vacuous. Take, for example, $N = 3$, $\lambda_1 = \frac{3}{4}$, and $\lambda_2 = \frac{1}{4}$. Then $\nu = \frac{1}{2}$.

4 Dynamic File Storage

Suppose we are given a file-storage unit, a floppy disk, for example. Such a unit will be called a "volume" for the sake of generality. In our model we shall assume that all volumes have infinite storage capacity. Suppose that initially the volume is unused. As files arrive, they are stored in order beginning at an "origin" or "starting point." After several files have been stored, some will be deleted. This will leave empty holes where new files could be stored, if they are small enough. In general, even if new files are put in the holes, gaps will remain. Let V_n denote the "state" of the volume at time $n = 0, 1, 2, \ldots$. Let $w(V_n)$ denote the sum of the sizes of all gaps or holes left by deleted files. We will show that under certain conditions, an "any-fit" algorithm, which always places an arriving file in a gap if a large enough gap exists, will stabilize the system in the sense that the expected wasted space is uniformly bounded for all time.

5 A Mathematical Model for Dynamic File Storage

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space on which the time-homogeneous Markov chain $\{V_n, n = 0, 1, 2, \ldots\}$ is defined as follows. Let the "state" of the volume at time n be denoted by V_n. The state, V_n, consists of the total number of active files together with their starting addresses and lengths. We let $w(V_n)$ denote the total wasted space on the volume at time n. For example, suppose that at time n there are N

files of lengths $\ell_1, \ldots, \ell_N$. If we let the address of the origin be 0, and if the starting addresses are $a_1 < \cdots < a_N$, where $a_i + \ell_i \leq a_{i+1}$, then the wasted space is

$$a_1 + \sum_{i=1}^{N-1} [a_{i+1} - (a_i + \ell_i)], \tag{12}$$

which is equal to

$$a_N - \sum_{i=1}^{N-1} \ell_i. \tag{13}$$

In our model we shall assume that the maximum file size is M records. Let $\mu, \lambda_1, \ldots, \lambda_M$ be nonnegative numbers such that $\mu + \sum_{m=1}^{M} \lambda_m \leq 1$. Given that $V_n = v$, λ_m will be the conditional probability that at time $n+1$ a file of m records arrives and is stored (somewhere, to be specified later) on the volume. If at time n there are $N \geq 1$ files already stored, $\frac{\mu}{N}$ will denote the conditional probability that one of the stored files is deleted at time $n+1$. This leaves $1 - \sum_{m=1}^{M} \lambda_m - \mu$ as the conditional probability that nothing happens at time $n+1$.

Given that $V_n = v$, let B_v denote the event that the last file (the one starting at a_N) is deleted. We want to ignore the following situation. Suppose $a_N - (a_{N-1} + \ell_{N-1})$ is very large, say t. Then with conditional probability $\frac{\mu}{N}$, the last file is deleted, and

$$w(V_{n+1}) - w(v) \;=\; -t. \tag{14}$$

Let B_v^c denote the complement of the event B_v. Let $I_{B_v^c}$ denote the indicator function of the event B_v^c. Then, whenever $w(v) \geq 1$,

$$I_{B_v^c}(V_{n+1})|w(V_{n+1}) - w(v)| \;\leq\; M. \tag{15}$$

This implies

$$\mathsf{E}[\, I_{B_v^c}(V_{n+1})|w(V_{n+1}) - w(v)|^2 \mid V_n = v\,] \;\leq\; M^2. \tag{16}$$

A little reflection yields, when $w(v) \geq 1$,

$$\mathsf{E}[\, I_{B_v^c}(V_{n+1})(w(V_{n+1}) - w(v)) \mid V_n = v\,] \;\leq\; M\frac{N-1}{N}\mu - \lambda_1 \;\leq\; M\mu - \lambda_1. \tag{17}$$

We note that if $w(v) \geq 1$, then $N \geq 1$. We conclude with

Theorem 2. Let $\nu \triangleq \lambda_1 - M\mu$. *Assume* $\nu > 0$. *Choose* $0 < C \leq \nu/36$. *Choose* $L = \max\{1, \frac{M}{1/2}, \frac{M^2}{C}\}$ *so that whenever* $w(x) > L$,

$$I_{B_v^c}(V_{n+1})|w(V_{n+1}) - w(v)| \;\leq\; \tfrac{1}{2}w(v), \tag{18}$$

$$\mathsf{E}[\, I_{B_v^c}(V_{n+1})(w(V_{n+1}) - w(v)) \mid V_n = v\,] \;\leq\; -\nu, \tag{19}$$

and

$$\mathsf{E}[\, I_{B_v^c}(V_{n+1})|w(V_{n+1}) - w(v)|^2 \mid V_n = v\,] \;\leq\; Cw(v). \tag{20}$$

Let $A \triangleq 2L$. *Then*

$$\mathsf{E}[\, w(V_n) \mid V_0 = v\,] \;\leq\; \frac{w(v)^3}{[w(v) + \frac{\nu}{2}n]^2} + A(1 + 4(\tfrac{A}{\nu})^2 \sum_{k=1}^{n-1} \frac{1}{k^2}). \tag{21}$$

Proof. Observe that if $w(v) \leq L$, then $|w(V_{n+1}) - w(v)| \leq \max\{M, L\} = L$, since we are choosing $L \geq 2M$. This implies $w(V_{n+1}) \leq 2L = A$. So,

$$\mathsf{E}[\, w(V_{n+1}) \mid V_n = v\,] \;\leq\; A, \qquad \text{if } w(v) \leq L, \tag{22}$$

and

$$\mathsf{E}[\, w(V_{n+1})^3 \mid V_n = v\,] \;\leq\; A^3, \qquad \text{if } w(v) \leq L. \tag{23}$$

Clearly, (15), (17), and (16) imply (18), (19), and (20) when $w(v) > L$. The fact that (18), (19), (20), (22), and (23) imply (21) is simply Lemma A2 in Appendix A. Hence, (21) holds. ∎

Clearly, (21) implies both

$$\mathsf{E}[\, w(V_n) \mid V_0 = v\,] \;\leq\; w(v) + A(1 + \tfrac{2}{3}(\tfrac{\pi A}{\nu})^2), \tag{24}$$

and

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}[\, w(X_n) \mid X_0 = x\,] \;\leq\; A(1 + \tfrac{2}{3}(\tfrac{\pi A}{\nu})^2) \;<\; \infty, \tag{25}$$

which is the desired result.

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A Appendix

We state below two simple consequences of the so-called "Key Lemma" and its extension proved in Gubner, Gopinath, and Varadhan (1988).

Lemma A1. *Let* $(\Omega, \mathcal{F}, \mathsf{P})$ *be a probability space on which* $\{Y_n, n = 0, 1, 2, \ldots\}$ *is a time-homogeneous Markov process. Let* u *be a nonnegative function defined on the state space of* $\{Y_n\}$. *Suppose that there exist positive constants* C, ν, *and* L *with* $36C \leq \nu$ *such that whenever* $Y_n = y$ *and* $u(y) > L$,

$$|u(Y_{n+1}) - u(y)| \ \leq\ \tfrac{1}{2}u(y), \quad \mathsf{P}(\,\cdot \mid Y_n = y) - \text{a.s.}, \tag{26}$$

$$\mathsf{E}[\, u(Y_{n+1}) - u(y) \mid Y_n = y\,] \ \leq\ -\nu, \tag{27}$$

and

$$\mathsf{E}[\, |u(Y_{n+1}) - u(y)|^2 \mid Y_n = y\,] \ \leq\ C\, u(y). \tag{28}$$

If there exist finite, positive *constants,*[1] A_0 *and* B_0, *such that*

$$\mathsf{E}[\, u(Y_{n+1}) \mid Y_n = y\,] \ \leq\ A_0, \qquad \text{whenever } u(y) \leq L, \tag{29}$$

and

$$\mathsf{E}[\, u(Y_{n+1})^3 \mid Y_n = y\,] \ \leq\ B_0, \qquad \text{whenever } u(y) \leq L, \tag{30}$$

then

$$\mathsf{E}[\, u(Y_n) \mid Y_0 = y\,] \ \leq\ \frac{u(y)^3}{[u(y) + \frac{\nu}{2}n]^2} + A_0 + \frac{4B_0}{\nu^2}\sum_{k=1}^{n-1}\frac{1}{k^2}. \tag{31}$$

[1] In Gubner, Gopinath, and Varadhan (1988), constants A and B were used. They are related by $A_0 = A$ and $B_0 = B\nu^2/4$.

Lemma A2. *Lemma* A1 *holds if* (26)–(28) *are replaced by*

$$I_{B_y^c}(Y_{n+1})|u(Y_{n+1}) - u(y)| \;\le\; \tfrac{1}{2}u(y), \quad \mathsf{P}(\,\cdot \mid Y_n = y) - \text{a.s.}, \tag{32}$$

$$\mathsf{E}[\, I_{B_y^c}(Y_{n+1})(u(Y_{n+1}) - u(y)) \mid Y_n = y\,] \;\le\; \nu, \tag{33}$$

and

$$\mathsf{E}[\, I_{B_y^c}(Y_{n+1})|u(Y_{n+1}) - u(y)|^2 \mid Y_n = y\,] \;\le\; C\, u(y), \tag{34}$$

where B_y *is a subset of the state space such that*

$$y' \in B_y \implies u(y') \le u(y). \tag{35}$$