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On An Overdetermined Neumann Problem

by

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<u>On an overdetermined Neumann problem</u>

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Several questions in harmonic analysis, partial differential equations and applied mathematics lead to the question of characterizing domains for which overdetermined boundary value problems have solutions. Given the existence of an excellent bibliography in [1] I will not attempt to trace the history of the problem (D) and (N) introduced below, except to say that their origins go back to the treatise [2] of Lord Rayleigh.

Let Ω be an open relatively compact subset of real analytic Riemannian manifold M. Assume further that $\partial\Omega$ is connected and

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of Lipshitz class. What can we say about Ω if any of the following problems (D) or (N) has an eigenvalue $\alpha > 0$?

$$(D)\quad \begin{cases} \Delta u + \alpha u = 0 \quad \text{in} \quad \Omega \\ u = 0 \text{ and } \dfrac{\partial u}{\partial n} = 1 \quad \text{on} \quad \partial\Omega \\ \text{(overdetermined Dirichlet problem)} \end{cases}$$

$$(D)\quad \begin{cases} \Delta u + \alpha u = 0 \quad \text{in} \quad \Omega \\ u = 1 \text{ and } \dfrac{\partial u}{\partial n} = 1 \quad \text{on} \quad \partial\Omega \\ \text{(overdetermined Neumann problem)} \end{cases}$$

It is remarkable that the existence of such an eigenvalue already implies that $\partial\Omega$ is a real analytic submanifold of M (at least if $\partial\Omega$ is of class $C^{2+\varepsilon}$; if $M = \mathbb{R}^n$ or $\mathbb{H}^n$ - real hyperbolic space - it is enough if $\partial\Omega$ is Lipschitz, this follows from work of Cafarelli [3]. It is quite possible that this last condition is always sufficient). In order to proceed any further we have to impose restrictions on both M and α. For instance, if $M = \mathbb{R}^n$ (resp. $\mathbb{H}^n$) and α in (D) is the first eigenvalue of Dirichlet problem, i.e. $u > 0$, then Ω = ball (resp. geodesic ball). This follows from a theorem of Serrin [4]. A few other cases are known [5], [6], [7] of the non-existence of eigenvalues if $M = \mathbb{R}^n$, $\Omega \neq$ ball. On the otherhand, if $M = \mathbb{R}^n$ (resp. $\mathbb{H}^n$) and Ω = ball (resp. geodesic ball) then one can see that, considering the radial eigenfunctions of the usual Dirichlet or Neumann problems in the role of u, there are infinitely many eigenvalues for (D) and for (N). Sometime ago I have proved in $\mathbb{R}^2$ that the converse was true [5], later I obtained the same result with P. Yang in $\mathbb{H}^2$[8]. We have now:

<u>Theorem</u> [9] Let $M = \mathbb{R}^n$(resp. $\mathbb{H}^n$), the existence of infinitely many eigenvalues for either of the problems (D) or (N) characterizes the balls (resp.geodesic balls) among all the

relatively compact domains Ω with connected Lipshitz boundary.

Note that in the Poincaré model $\mathbb{H}^n$ = unit ball of $\mathbb{R}^n$ and the geodesic balls are then euclidean balls.

It would seem to be natural to jump to the conclusion that this theorem should remain true in all M. The following example shows that one needs some caution.

Example Let $u = u(x) = x_1^2 - x_2^2 + \cdots + x_{2u-1}^2 - x_2^2$ defined in $\mathbb{R}^{2n}$, denote by v its restriction to $M = S^{2n-1}$. Denote by L the Laplace operator in $\mathbb{R}^{2n}$, Δ the Laplace Beltrami operator in M. It is well known they are related by

$$(1) \qquad L = \frac{\partial}{\partial r^2} + \frac{2n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta$$

We also have

$$\frac{\partial u}{\partial r} = \nabla u \cdot \frac{x}{r} = \frac{2u}{r}, \quad r = |x| ,$$

where the last identity holds by Euler's formula, u being homogeneous of degree 2. Therefore

$$(2) \qquad \frac{\partial^2 u}{\partial r^2} = \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u = \frac{2u}{r^2} .$$

It follows that

$$\frac{2u}{r^2} + \frac{(2n-1)}{r^2}2u + \frac{1}{r^2}\Delta u = Lu = 0,$$

and the function v satisfies $\Delta v + 4n\, v = 0$ on $\{r=1\} = M$. Consider now the set Ω = connected component containing $(1,0,\cdots,0)$ of $\{x \in M: u(x) > 0\}$. Then on $\partial\Omega$ we have $v(x) = 0$ and therefore $\alpha = 4n$ is the first eigenvalue of Dirichlet problem for Ω. We claim now that $\frac{\partial v}{\partial n}$ constant on $\partial\Omega$. Note that the (exterior) normal derivative is an operator <u>tangent</u> to the sphere M. First, observe that (2) implies

(3) $\qquad \nabla u(x) \perp x \quad \text{if} \quad x \in \partial\Omega .$

Since x is the normal vector to M in $\mathbb{R}^{2n}$ we have $\nabla u(x) \in T_x M$ for $x \in \partial\Omega$. Therefore we have

(4) $\qquad \nabla u(x) = \frac{\partial v}{\partial n}(x) \cdot \vec{n}(x) \qquad \text{if } x \in \partial\Omega ,$

where $\vec{n}$ represents the unit exterior normal to Ω in M. We only need to verify that $|\nabla u(x)| = \text{constant}$ when $x \in \partial\Omega$. But

$$\nabla u(x) = 2(x_1, -x_2, \cdots, x_{2n-1}, -x_{2n})$$

and

(5) $\qquad |\nabla u|^2 = 4|x|^2 = 4 \qquad \text{if} \quad x \in M,$

which says that v satisfies:

(6) $$\begin{cases} \Delta v + 4n \cdot v = 0 & \text{in} \quad \Omega \\ v > 0 & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial v}{\partial n} = -2 & \text{on} \quad \partial\Omega . \end{cases}$$

Note that topologically $\partial\Omega = S^{n-1} \times S^{n-1}$ which shows that Serrin's theorem fails on M. We want to show now that there are infinitely many solutions for both (D) and (N) in Ω.

Let f be a twice differentiable function of a single real variable and define

$$\varphi(x) := f(u(x)).$$

Using again identity (2) we have

$$L\varphi = f''(u)|\nabla u|^2 + f'(u)Lu = f''(u)4|x|^2 = 4f''(u) \quad \text{on} \quad M.$$

$$\frac{\partial \varphi}{\partial r} = f'(u)\frac{\partial u}{\partial r} = 2f'(u)\ \frac{u}{r} = 2f'(u)u \qquad \text{on} \quad M.$$

$$\frac{\partial^2 \varphi}{\partial r^2} = 4f''(u)\left[\frac{u}{r}\right]^2 + \frac{2}{r}\ f'(u)\ \frac{\partial u}{\partial r} - \frac{2}{r^2}\ f'(u)u =$$

$$= \frac{4}{r^2}f''(u)u^2 + \frac{2}{r}\ f'(u)u = 4f''(u)u^2 + 2f'(u)u \qquad \text{on} \quad M.$$

Hence, on M,

$$L\varphi = 4f''(u) = 4f''(u)u^2 + 4nf'(u)u + \Delta\varphi,$$

and

(7) $$\Delta\varphi + \alpha\varphi = 4f''(u)(1-u^2) - 4nf'(u)u + \alpha f(u).$$

Therefore the equation $\Delta\varphi + \alpha\varphi = 0$ in Ω becomes the ordinary differential equation

(8) $$4(1-t^2)f''(t) - 4ntf''(t) + \alpha f(t) = 0, \quad 0 \le t \le 1$$

This equation has a regular singular point at end point $t = 1$. Each eigenvalue α and eigenfunction of (8) satisfying

(9) $$f(1) \text{ bounded}, \quad f(0) = 0$$

provides an eigenvalue for (D) since

$$\nabla\varphi = f'(u)\nabla u$$

hence again $\nabla\varphi\cdot x = 0$ on $\partial\Omega$ and to check whether $\frac{\partial\varphi}{\partial n} = \text{constant}$ we only need to compute $|\nabla\varphi|^2$ on $\partial\Omega$. But

$$|\nabla\varphi|^2 = (f'(u))^2|\nabla u|^2 = 4(f'(u))^2,$$

which shows

(10) $$\frac{\partial\varphi}{\partial n} = \pm\ 2f'(0) \qquad \text{on } \partial\Omega.$$

(This is different from zero since the eigenfunctions of (8) - (9) satisfy $f'(0) \neq 0$). This same computation shows that the eigenfunctions of (8) satisfying

(11) $$f(1) \text{ bounded}, \ f'(0) = 0,$$

will provide eigenfunctions $\varphi(x)$ for (N). This time $\frac{\partial\varphi}{\partial n} = 0$ on $\partial\Omega$ and $\varphi = f(0) \neq 0$ on $\partial\Omega$. Since it is a well known theorem of the theory of ordinary differential equations that (8)-(9) and (8)-(10) have infinitely many eigenvalues, the domain Ω has infinitely many eigenvalues for (D) and (N). Ω is not even topologically a geodesic ball in M.

On this note we leave the reader to reflect on these beautiful questions.

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