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Vickrey Auctions with Reserve Pricing*

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Summary. We generalize the Vickrey auction to allow for reserve pricing in a multi-unit auction with interdependent values. In the Vickrey auction with reserve pricing, the seller determines the quantity to be made available as a function of the bidders' reports of private information, and then efficiently allocates this quantity among the bidders. Truthful bidding is a dominant strategy with private values and an *ex post* equilibrium with interdependent values. If the auction is followed by resale, then truthful bidding remains an equilibrium in the auction-plus-resale game. In settings with perfect resale, the Vickrey auction with reserve pricing maximizes seller revenues.

Keywords and Phrases: Auctions, Vickrey auctions, Multi-unit auctions, Reserve price, Resale

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1 Introduction

A Vickrey auction has the distinct advantage of assigning goods efficiently—putting the goods in the hands of those who value them most. However, one critique of a Vickrey auction is that it may yield low revenues for the seller. Indeed, Vickrey expressed this concern in his seminal article (Vickrey 1961). When competition is weak and the bidders are asymmetric, revenues from a Vickrey auction may be small. A vivid example was the 1990 New Zealand sale of spectrum licenses by second-price auction. In one case, the winner bid $100,000, but paid only $6; in another, the winner bid $7,000,000, but paid only $5,000 (McMillan 1994). Reserve pricing is a simple and effective device to avoid such disasters. The seller may charge the reserve price or reduce the quantity sold if the bids are too low. Reserve pricing is also an effective device for mitigating collusion, since it limits the maximum gain collusion can reap.

Reserve pricing is especially important in auctions, such as electricity auctions, spectrum auctions or Treasury auctions, where participants bid for multiple items. Then the largest market participant may be so large that removing this bidder may lead to no excess demand. In a Vickrey auction, prices are based on the opportunity cost of winning; that is, a winner pays the value that the goods would have in their best use without the winner. If a bidder's winnings are greater than the excess demand in the auction with the bidder removed, then some of the Vickrey prices are undefined (or zero). In auctions to supply electricity during peak periods, it is common for the capacity of the largest generator to be far greater than the excess capacity in the system. In such a setting, a Vickrey auction must involve reserve pricing.

We generalize the Vickrey auction to allow for reserve pricing in a multi-unit auction with interdependent values. In the Vickrey auction with reserve pricing, the seller determines the quantity to be made available as a function of the bidders' reports of their private information, and then efficiently allocates this quantity among the bidders. We prove in Theorem 1 that truthful bidding by all bidders is an *ex post* equilibrium in a model with interdependent values (a bidder's value also depends on the private information of other bidders) and that truthful bidding is a dominant strategy in a model with private values (a bidder's value depends only on its own private information). . Thus, reserve pricing does not interfere with many of the desirable features of a Vickrey auction.

An important motivation for this article is the possibility of resale after an auction. The "optimal auctions" literature requires the seller to misassign items, that is, to put the items in hands other than those who value them the most, with positive probability (except in symmetric models). However, the seller's ability to do this may be undermined when resale cannot be prevented; bidders will anticipate the resale market and adjust their bids accordingly. In Ausubel and Cramton (1999), we prove that whenever resale markets are *fully* efficient, a seller *cannot* increase revenues by misassigning the items among the bidders. The revenue-maximizing auction is simply for the seller to decide on an optimal quantity to sell based on the bidders' reports and to assign these units efficiently among the bidders. Thus, faced with a perfect resale market, the best that the seller can do is to withhold some of the supply, but then to sell the remaining supply efficiently.

This immediately raises the question of how to construct a mechanism that limits the quantity sold but allocates efficiently whatever quantity is sold. The Vickrey auction with reserve pricing defined in the current article performs precisely this job, and thus does exactly what is required for an optimizing seller facing a perfect resale market.[1] In particular, we prove in Theorem 2 that truthful bidding in the Vickrey auction with reserve pricing (and hence an efficient assignment of the goods that are sold) is an *ex post* equilibrium of the two-stage game consisting of the auction followed by resale. Indeed, for this result, we do not need the resale procedure to be perfect. Truthful bidding will be an *ex post* equilibrium whenever the resale game is such that no bidder expects to get more than 100% of the gains from trade that it brings to the table.

The current article is related to three strands of literature. First, a number of articles extend the Vickrey auction to settings where bidders have interdependent values. Crémer and McLean (1985) construct a mechanism through which the full surplus can be extracted from bidders and, as a step along the way, they construct a mechanism for discrete types that yields an efficient assignment as an *ex post* equilibrium. Maskin (1992) defined a modified second-price auction, which yields an efficient assignment in a single-good setting with interdependent values. Ausubel (1999) extends Maskin's approach by defining a "generalized Vickrey auction" for multiple identical items with interdependent values. Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001) and Perry and Reny (2002) also define auction mechanisms that, for the case of multiple identical objects, are outcome-equivalent to the generalized Vickrey auction. None of these papers explore reserve pricing or the implications of resale markets.

The second strand of literature considers multi-unit auctions with variable supply. Back and Zender (2001) show that in a uniform-price auction the seller can eliminate low-price equilibria (Back and Zender 1993) by restricting supply after the bids are in. Lengwiler (1999), in a model allowing two possible price levels, considers the effects of variable supply on seller revenues in both uniform-price and pay-your-bid auctions. McAdams (2002) also examines variations on the uniform-price auction in which the auctioneer is

[1] The current article does not concern itself with the determination of the optimal quantity based on bidders' reports, and instead takes as given the quantity as a function of bidders' reports. For a treatment of the problem of determining the optimal quantity, see Ausubel and Cramton (1999).

able to increase or decrease quantity after receiving the bids. None of these papers consider Vickrey pricing or resale.

The third strand of literature considers auctions with resale. Haile (1999, 2003) demonstrates that, in auctions followed by resale, bidders will anticipate the resale market and adjust their bids accordingly. Furthermore, Haile (1999) considers Vickrey auctions with resale in the case of auctioning a single item. Ausubel and Cramton (1999) consider optimal multi-unit auctions with efficient resale. Here we consider how to implement these optimal auctions. Zheng (2002) and Calzolari and Pavan (2002) consider single-item optimal auctions with alternative resale games.[2]

Section 2 presents a general model for the auction of a divisible good. Bidders' demands for the items may be interdependent. Section 3 defines the Vickrey auction with reserve pricing, and demonstrates that truthful bidding is an *ex post* equilibrium, despite the fact that the bidding affects the quantity sold. Section 4 analyzes an auction followed by resale. It is shown that the possibility of resale does not distort the Vickrey auction with reserve pricing. Truthful bidding remains an *ex post* equilibrium, despite the presence of a resale market following the auction. Section 5 concludes.

2 The General Divisible Good Model

A seller has a quantity 1 of a divisible good to sell to n bidders, $N \equiv \{1,\ldots,n\}$. The seller's valuation for the good equals zero. Each bidder i can consume any quantity $q_i \in [0,1]$. We can interpret q_i as bidder i's share of the total quantity. Let $q \equiv (q_1,\ldots,q_n)$, and let $Q \equiv \{q \mid \sum_i q_i \leq 1\}$ be the set of all feasible assignments. Each bidder's value for the good depends on the private information of all the bidders. Let $t_i \in T_i \equiv [0,t_i^{\max}]$ be bidder i's type (i's private information), $t \equiv (t_1,\ldots,t_n) \in T \equiv T_1 \times \cdots \times T_n$, and $t_{-i} \equiv t \setminus t_i = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$. A bidder's value $V_i(t,q_i)$ for the quantity q_i depends on its own type t_i and the other bidders' types t_{-i}. A bidder's utility is its value less the amount it pays: $V_i(t,q_i) - X_i$. Let $v_i(t,q_i)$ denote the marginal value for bidder i, given the vector t of types and quantity q_i. Then $V_i(t,q_i) = \int_0^{q_i} v_i(t,y)\,dy$.

We assume that $v_i(t,q_i)$ satisfies the following assumptions:

Continuity. For all i, t, and q_i, $v_i(t,q_i)$ is jointly continuous in (t,q_i).

Value monotonicity. For all i, t and q_i, $v_i(t,q_i)$ is nonnegative, strictly increasing in t_i, and weakly decreasing in q_i.

[2] The principal difference between the approaches of Ausubel and Cramton (1999) and Zheng (2002) is that we assume that the resale market is fully efficient, whereas Zheng assumes that the winner of the auction has full monopoly power in the resale game. Calzolari and Pavan (2002) assume that the resale market comprises a single take-it-or-leave-it offer by a seller (with probability λ) or a buyer (with probability $1-\lambda$).

Single-crossing property. For all i, $j \neq i$, q_i, q_j, t_{-i}, and $t_i' > t_i$,

$$v_i(t,q_i) > v_j(t,q_j) \Rightarrow v_i(t_i',t_{-i},q_i) > v_j(t_i',t_{-i},q_j) \text{ and } v_i(t_i',t_{-i},q_i) < v_j(t_i',t_{-i},q_j) \Rightarrow v_i(t,q_i) < v_j(t,q_j).$$

Value monotonicity implies that types are naturally ordered, and that the bidders have weakly downward-sloping demand curves. The single-crossing property implies that, if a fixed quantity is assigned efficiently among the bidders, then bidder i's quantity $q_i(t)$ may be chosen to be weakly increasing in t_i. The single-crossing property holds if an increase in bidder i's type raises bidder i's marginal value at least as much as any other bidder's.

Three special cases of the general model are particularly useful.

PRIVATE VALUES. A bidder's value $V_i(t_i,q_i)$ only depends on its own type.

COMMON VALUE. The bidders' values are the same: $V_i(t,q_i) = V_j(t,q_i)$.

INDEPENDENT TYPES. The bidders' types are drawn independently from the distribution functions F_i with positive and finite density f_i on T_i.

The private values assumption enables us to strengthen many of the results. In particular, truthful bidding becomes a dominant strategy, rather than simply a best response. Also, value monotonicity automatically implies the single-crossing property in the private value setting.

The common value assumption often is made in models of oil lease auctions and in models of Treasury and other financial auctions.

Independent types is needed in the optimal auction analysis (our final result). Expected revenues depend on the probability distribution of types, and independence is needed for a general revenue equivalence theorem. However, most of our analysis is based on "*ex post*" arguments, which do not require any assumptions about the distribution of types.

Our starting point for describing a Vickrey auction with reserve pricing is to specify the aggregate quantity $\bar{q}(t) \equiv \Sigma_i \, q_i(t)$ that the seller assigns to the bidders, as a function of the vector of reported types. The description of the Vickrey auction is only guaranteed to make sense if the aggregate quantity $\bar{q}(t)$ is weakly increasing. We therefore require

Monotonic aggregate quantity. The aggregate quantity rule $\bar{q}(t)$ is weakly increasing in each bidder's type.

This assumption, together with the single-crossing property, guarantees that the quantity $\bar{q}(t)$ can be assigned efficiently among the bidders in such a way that each bidder i's quantity $q_i(t)$ is weakly increasing in t_i.

3 Vickrey Auction with Reserve Pricing

The Vickrey auction with reserve pricing can be thought of as a three-step procedure. First, the bidders simultaneously and independently report their types t to the seller, and the seller determines the aggregate quantity $\bar{q}(t)$ that it wishes to assign to bidders. Second, the seller determines an efficient assignment of this aggregate quantity; that is, the seller solves for $q^*(t) \equiv (q_1^*(t), \cdots, q_n^*(t))$ that maximizes $\Sigma_i V_i(t, q_i^*(t))$ subject to $\Sigma_i q_i^*(t) = \bar{q}(t)$. When the efficient assignment is not unique due to flat regions in the aggregate demand curve, $q_i^*(t)$ is chosen so that it is weakly increasing in t_i. Third, the seller determines a payment $X_i^*(t)$ for each bidder i associated with the assignment of $q_i^*(t)$, where $q_i^*(t)$ and $X_i^*(t)$ must be specified so that truthful bidding is incentive compatible and individually rational for every type of every bidder.

The determination of the payment rule is most easily understood in an environment with discrete units. Hold the reports t_{-i} of bidders other than bidder i fixed, and consider the quantity $q_i^*(t_i, t_{-i})$ assigned to i as a function of t_i. Let $t_i^1(t_{-i})$ denote the minimum type such that i is awarded at least one unit, let $t_i^2(t_{-i})$ denote the minimum type such that i is awarded at least two units, etc. More precisely, for every $k \geq 1$, let $t_i^k(t_{-i}) = \inf\left\{t_i : q_i^*(t_i, t_{-i}) \geq k\right\}$, the minimum type such that bidder i is awarded at least k units. By hypothesis, $q_i^*(t)$ is weakly increasing in t_i. Therefore, by value monotonicity and the single-crossing property, $t_i^k \leq t_i^{k+1}$ for all $k \geq 1$.

> *Discrete payment rule*. If bidder i is assigned K units, then for every k ($1 \leq k \leq K$), bidder i is charged a price of $v_i(t_i^k(t_{-i}), t_{-i}, k)$ for the k^{th} unit.

Vickrey pricing is best thought of in terms of opportunity costs. The winner pays the opportunity cost of its winnings. In a standard Vickrey auction, the opportunity cost is always the value to the other bidder that would receive the good if the winner did not participate. In a Vickrey auction with reserve pricing, the opportunity cost can come instead from the seller. This occurs for a good that the seller would withhold were it not for the winner's bids. Critical to the analysis, observe that bidder i's value is evaluated at the *minimal* type at which i receives the k^{th} unit. This specification has the effect of subsuming the proper pricing rule both for the case where the k^{th} unit of bidder i comes from another bidder as well as for the case where the k^{th} unit of bidder i comes from the seller's reserve. If the k^{th} unit for bidder i is assigned to bidder i from another bidder j, then bidder i is charged the other bidder's value $v_j(t_i^k(t_{-i}), t_{-i}, q_j)$, assuming i's type is just high enough to receive k units, as by definition, $t_i^k(t_{-i})$ is the minimal type of bidder i such that bidder i receives this unit, so $v_i(t_i^k(t_{-i}), t_{-i}, k) = v_j(t_i^k(t_{-i}), t_{-i}, q_j)$. Meanwhile, if the k^{th} unit for bidder i is assigned to bidder i out of the seller's reserve, then the seller's implicit "reserve price" for this unit also equals $v_i(t_i^k(t_{-i}), t_{-i}, k)$, since all types of bidder i greater than $t_i^k(t_{-i})$ are receiving this unit while all types of bidder i less than $t_i^k(t_{-i})$ are not.

Returning to the case of continuous quantity, let $q_{-i}^*(t) \equiv \bar{q}(t) - q_i^*(t)$ denote the aggregate quantity allocated to bidders other than i (bidders $N \setminus i$) following reports t. Furthermore, for any quantity y, let $v_{-i}(t, y)$ denote the marginal value to bidders $N \setminus i$ if the quantity y is allocated *efficiently* among bidders $N \setminus i$. Observe that, for any aggregate quantity rule $\bar{q}(t)$ and for any reports t, an efficient assignment rule $q^*(t)$ satisfies

$$v_i(t, q_i^*(t)) \begin{cases} \leq v_{-i}(t, q_{-i}^*(t)), & \text{for } i \text{ such that } q_i^*(t) = 0 \\ = v_{-i}(t, q_{-i}^*(t)), & \text{for } i \text{ such that } 0 < q_i^*(t) < \bar{q}(t) \\ \geq v_{-i}(t, q_{-i}^*(t)), & \text{for } i \text{ such that } q_i^*(t) = \bar{q}(t). \end{cases} \tag{1}$$

Otherwise, from continuity and value monotonicity, if $0 < q_i^*(t) < \bar{q}(t)$ and $v_i(t, q_i^*(t)) > v_{-i}(t, q_{-i}^*(t))$, then there exists $\varepsilon > 0$ such that allocating $q_i^*(t) + \varepsilon$ to bidder i and $q_{-i}^*(t) - \varepsilon$ to bidders $-i$ would generate social improvement, and similarly if $v_i(t, q_i^*(t)) < v_{-i}(t, q_{-i}^*(t))$.

From Eq. (1) and the single-crossing property, for any monotonic aggregate quantity rule $\bar{q}(t)$, there exists an associated efficient assignment rule $q_i^*(t)$ that is weakly increasing in t_i. To see this, note that the single-crossing property implies that, in an efficient assignment, any quantity that must go to i when t_i is reported must still go to i when $t_i' > t_i$ is reported, and any quantity that cannot go to i when $t_i' > t_i$ is reported still cannot go to i when t_i is reported. This would guarantee that if aggregate demand were strictly downward sloping, then $q_i^*(t)$ would be uniquely defined, and it would be weakly increasing in t_i. However, when the aggregate demand curve has a flat region and the flat portion includes more than one bidder, then $q_i^*(t)$ is no longer unique, and indeed some efficient assignment rules may not be monotonic. In this case, the seller must choose a tie-breaking rule that is consistent with a monotonic efficient assignment. For example, in the flat portion of aggregate demand, award the good first to the bidder with the higher type, and split the quantity equally among bidders with the same type.

Also observe that, although $\bar{q}(t)$ is monotonic, $\bar{q}(t)$ need not be continuous in t_i, so it is useful to define limits of $\bar{q}(t)$ from above and below in t_i:

$$\overline{q}_i^+(\hat{t}_i, t_{-i}) = \lim_{t_i \downarrow \hat{t}_i} \overline{q}(t_i, t_{-i}) \text{ and } \overline{q}_i^-(\hat{t}_i, t_{-i}) = \lim_{t_i \uparrow \hat{t}_i} \overline{q}(t_i, t_{-i}).$$

We can now define the generalized Vickrey auction with reserve pricing.

DEFINITION. *Vickrey auction with reserve pricing.* Given any monotonic efficient assignment rule $q^*(t)$, and for reports t_{-i} of bidders other than bidder i and for any quantity z such that $0 \le z \le q_i^*(t_i^{\max}, t_{-i})$, define:

$$\hat{t}_i(t_{-i}, z) = \inf_{t_i} \left\{ t_i \mid q_i^*(t_i, t_{-i}) \ge z \right\}. \tag{2}$$

Following reports t, bidder i is assigned $q_i^*(t)$ units and is charged a payment $X_i^*(t)$ computed by:

$$X_i^*(t) = \int_0^{q_i^*(t)} v_i(\hat{t}_i(t_{-i}, z), t_{-i}, z)\, dz. \tag{3}$$

Note that the payment formula of Eq. (3) is well defined, since the value monotonicity assumption assures that, for any reports t and for any quantity $z \in [0, q_i^*(t)]$, we have $0 \le v_i(\hat{t}_i(t_{-i}, z), t_{-i}, z) \le v_i(t, 0)$.

In the Vickrey auction, a bidder pays the opportunity cost of its winning for each incremental quantity won. Hence, the marginal payment made at each quantity z is determined by the bidder's marginal value assuming the bidder makes the *lowest possible* report consistent with winning a quantity z. This marginal value may be determined either by the opportunity to sell to another bidder or by the opportunity to withhold the good. In this way, the bidder receives 100 percent of the gains from trade that it brings to the table. The fact that the bidder receives 100 percent of its incremental contribution is what gives the bidder the incentive for truthful bidding.

THEOREM 1. *For any monotonic aggregate quantity rule $\overline{q}(t)$ and associated monotonic efficient assignment rule $q_i^*(t)$, and for any valuation functions $v_i(t, q_i)$ satisfying continuity, value monotonicity and the single-crossing property, the Vickrey auction with reserve pricing has truthful bidding as an ex post equilibrium.*

PROOF. By continuity, value monotonicity and the single-crossing property, we can choose $q_i^*(t)$ to be weakly increasing in t_i. Then $\hat{t}_i(t_{-i}, z)$ defined by Eq. (2) is weakly increasing in z. Substituting Eq. (3) into the expression, $V_i(t,q_i) - X_i$, for bidder i's utility yields the following integral for bidder i's utility from reporting its type as t_i' when its true type is t_i and the other bidders' true and reported types are t_{-i}:

$$U_i(t_i' \mid t) = \int_0^{q_i^*(t_i', t_{-i})} \left[v_i(t, z) - v_i(\hat{t}_i(t_{-i}, z), t_{-i}, z) \right] dz. \tag{4}$$

Observe that the integrand of Eq. (4) is independent of t_i', bidder i's reported type; t_i' enters into Eq. (4) *only* through the upper limit on the integral. Moreover, by value monotonicity, the integrand of Eq. (4) is nonnegative for all $z \le q_i^*(t)$ and is nonpositive for all $z \ge q_i^*(t)$. Hence, $U_i(t_i' \mid t)$ is maximized for every t when the upper limit on the integral equals $q_i^*(t)$, which is attained by truthful bidding. ■

For the special case of private values, truthful bidding is a dominant strategy. Then truthful bidding is a best response for *any* reports by the other bidders. Without private values, the dominant strategy result is lost, since a bidder's value depends on the types of the other bidders, and so the bidder cares whether the reports of the others are truthful. Truthful bidding is only a best response if the other bidders are truthful; but it remains a best response after the bidder learns the opposing bidders' (truthful) reports. Hence, the *ex post* equilibrium property of truthful bidding always holds.

4 Auction followed by Resale

A main motivation for assigning goods efficiently is the possibility of resale (Ausubel and Cramton 1999). Resale undermines the seller's incentive to misassign the goods, since the misassignment may be undone in the resale market. The bidders anticipate the possibility of resale, which alters their incentives and distorts the bidding in the initial auction. Hence, an equilibrium in the auction game typically is not an equilibrium in the auction-plus-resale game.

Here we wish to show that a Vickrey auction with reserve pricing is not distorted by the possibility of resale. To prove this, we need to show that a bidder i with type t_i does not wish to misreport type t_i' in a Vickrey auction with reserve pricing followed by resale. Let $\Delta_i(t_i' \mid t)$ denote the optimal quantity of resale between bidder i and the coalition $N \setminus i$ if bidder i misreports its type as t_i' when its true type is t_i and the other bidders' true and reported types are t_{-i}, and let $\mathrm{GFT}_i(t_i' \mid t)$ denote the gains from trade available via resale between bidder i and the coalition $N \setminus i$ if bidder i misreports its type as t_i' when its true type is t_i and the other bidders' true and reported types are t_{-i}.

LEMMA 1. *If bidder i misreports its type as t_i' when its true type is t_i and the other bidders' true and reported types are t_{-i}, the (minimum) optimal quantity of resale between bidder i and the coalition $N \setminus i$ is given by*

$$\Delta_i(t_i' \mid t) = \begin{cases} \min\{z \ge 0 \mid v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) \le v_i(t, q_i^*(t_i', t_{-i}) - z)\}, & \text{if } t_i' > t_i, \\ \min\{z \ge 0 \mid v_i(t, q_i^*(t_i', t_{-i}) + z) \le v_{-i}(t, q_{-i}^*(t_i', t_{-i}) - z)\}, & \text{if } t_i' < t_i, \end{cases} \tag{5}$$

and the gains from trade available via resale between bidder i and the coalition $N \setminus i$ are given by

$$\text{GFT}_i(t_i' \mid t) = \int_0^{\Delta_i(t_i'|t)} \Big[v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) - v_i(t, q_i^*(t_i', t_{-i}) - z) \Big] dz \,. \tag{6}$$

PROOF. Observe that the integrand of Eq. (6) gives the marginal gains of the z^{th} unit transferred from coalition $N \setminus i$ to bidder i. By value monotonicity, if $z' < z$, then $v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) > v_i(t, q_i^*(t_i', t_{-i}) - z)$ implies $v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z') > v_i(t, q_i^*(t_i', t_{-i}) - z')$ and $v_i(t, q_{-i}^*(t_i', t_{-i}) + z) > v_{-i}(t, q_i^*(t_i', t_{-i}) - z)$ implies $v_i(t, q_{-i}^*(t_i', t_{-i}) + z') > v_{-i}(t, q_i^*(t_i', t_{-i}) - z')$. Thus, $\Delta_i(t_i' \mid t)$ defined by Eq. (5) provides the (minimal) upper limit for the integral in Eq. (6) which maximizes the value of the integral. ■

The following calculation will be helpful in what follows:

LEMMA 2. *For any monotonic aggregate quantity rule* $\bar{q}(t)$ *and associated monotonic efficient assignment rule* $q_i^*(t)$, *for any valuation functions* $v_i(t,q_i)$ *satisfying continuity, value monotonicity and the single-crossing property, for any bidder i, for any true type* t_i, *for any overreport* $t_i' > t_i$, *for any vector* t_{-i} *of other bidders' reported and true types, and for any z such that* $0 \le z \le \Delta_i(t_i' \mid t)$,

$$v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) \le v_i(\hat{t}_i(t_{-i}, q_i^*(t_i', t_{-i}) - z), t_{-i}, q_i^*(t_i', t_{-i}) - z). \tag{7}$$

PROOF. Consider any z such that $0 \le z \le \Delta_i(t_i' \mid t)$, and define $\hat{t}_i^z \equiv \hat{t}_i(t_{-i}, q_i^*(t_i', t_{-i}) - z) \ge t_i$. By the definition of $\hat{t}_i^z$, for every $\tilde{t}_i > \hat{t}_i^z$, it is the case that $q_i^*(\tilde{t}_i, t_{-i}) \ge q_i^*(t_i', t_{-i}) - z$; therefore, $v_{-i}(\tilde{t}_i, t_{-i}, \bar{q}(\tilde{t}_i, t_{-i}) - q_i^*(t_i', t_{-i}) + z) \le v_i(\tilde{t}_i, t_{-i}, q_i^*(t_i', t_{-i}) - z)$, for every $\tilde{t}_i > \hat{t}_i^z$, and so taking the limit as $\tilde{t}_i \downarrow \hat{t}_i^z$ implies that $v_{-i}(\hat{t}_i^z, t_{-i}, \bar{q}^+(\hat{t}_i^z, t_{-i}) - q_i^*(t_i', t_{-i}) + z) \le v_i(\hat{t}_i^z, t_{-i}, q_i^*(t_i', t_{-i}) - z)$. Note that $v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) \equiv v_{-i}(t, \bar{q}(t_i', t_{-i}) - q_i^*(t_i', t_{-i}) + z) \le v_{-i}(\hat{t}_i^z, t_{-i}, \bar{q}^+(\hat{t}_i^z, t_{-i}) - q_i^*(t_i', t_{-i}) + z)$, since $\hat{t}_i^z \le t_i'$ implies $\bar{q}(t_i', t_{-i}) \ge \bar{q}^+(\hat{t}_i^z, t_{-i})$, and since $\hat{t}_i^z \ge t_i$. Combining inequalities, we conclude that $v_{-i}(t, q_{-i}^*(t_i', t_{-i}) + z) \le v_i(\hat{t}_i^z, t_{-i}, q_i^*(t_i', t_{-i}) - z)$, as desired. ■

To prove Theorem 2, we need some structure on the resale game. In particular, we need a constraint on how much a misreporting bidder can gain in the resale game. With two bidders, individual rationality is all that is required. In the resale game, a bidder cannot get a surplus that is greater than the available gains from trade, for to do so the other bidder would have to strictly lose from resale. In this case, the other bidder would simply refuse to participate in resale. With more than two bidders and interdependent values, we must extend the definition of individual rationality. This is because one bidder's misreport in the auction may create gains from trade among the other bidders. These other bidders, then, should consider the gains from trade they can secure among themselves in deciding whether to participate in resale with the misreporting bidder.

Coalitional Rationality. For any initial allocation a of the good among bidders, for any vector t of types and for any subset S of the set N of bidders, let $v(S \mid a,t)$ denote the available gains from trade if the bidders in subset S trade only amongst themselves (starting at allocation a and evaluated at types t). Further, let s_i denote the surplus from the resale process realized by bidder i. The resale process is *coalitionally rational* if, for every subset S of the set N of bidders, the bidders in subset S obtain no more surplus s_i than they bring to the table:

$$\sum_{i \in S} s_i \leq v(N \mid a,t) - v(N \setminus S \mid a,t). \tag{8}$$

The resale process is *coalitionally-rational against individual bidders* if, for every element i of the set N of bidders, bidder i obtains no more surplus s_i than it brings to the table:

$$s_i \leq v(N \mid a,t) - v(N \setminus i \mid a,t). \tag{9}$$

The intuition behind this assumption is that, in the bargaining process underlying resale, the bidders in coalition $N \setminus S$ always have the outside option of excluding the bidders in the complementary set, S, from the bargaining and only trading amongst themselves. Hence, the bidders in S cannot deprive the bidders in $N \setminus S$ of the gains from trade that they could still obtain by trading amongst themselves.

We should remark that the assumption of coalitional rationality is quite natural and quite weak. It is implied, for example, by the requirement in the definition of the core that no coalition can improve upon an allocation. All we will need for our resale theorem is the still-weaker assumption of coalitional rationality against individual bidders. This is the requirement that any individual bidder i not receive any higher payoff than its marginal contribution to the set $N \setminus i$ of bidders. Observe that this is trivially implied by coalitional rationality. With superadditive values (which is always the case when value reflects potential gains from trade), it is also satisfied by standard solution concepts such as the Shapley value, which has every bidder i receiving its expected marginal contribution to the set S of bidders (the expectation taken over all subsets $S \subseteq N \setminus i$).

In the private values case, the definition of coalitional rationality reduces to individual rationality. With private values, if all bidders except bidder i report truthfully in the auction, then observe that in the resale round, $v(N \setminus i \mid a,t) = 0$, since the objects distributed to the coalition $N \setminus i$ are already assigned efficiently. Thus, individual rationality, $s_j \geq 0$, and feasibility, $\sum_{j \in N} s_j \leq v(N \mid a,t)$, imply that $s_i \leq v(N \mid a,t) - v(N \setminus i \mid a,t)$, which is coalitional rationality.

We now can prove our second theorem, which concerns the game with resale.

THEOREM 2. *For any monotonic aggregate quantity rule* $\bar{q}(t)$ *and associated monotonic efficient assignment rule* $q_i^*(t)$, *and for any valuation functions* $v_i(t,q_i)$ *satisfying continuity, value monotonicity and the single-crossing property, truthful bidding followed by no resale is an ex post equilibrium of the two-stage game consisting of the Vickrey auction with reserve pricing followed by any resale process that is coalitionally-rational against individual bidders.*

PROOF. Let $\pi_i(t_i'|t)$ denote the combined payoff to bidder i in the Vickrey auction and the resale market from misreporting t_i', when its true type is t_i and the other bidders' reported and true types are t_{-i}. By coalitional rationality against individual bidders, $\pi_i(t_i' \mid t) \le U_i(t_i' \mid t) + \mathrm{GFT}_i(t_i' \mid t)$, since $\mathrm{GFT}_i(t_i' \mid t)$ is defined to be the gains from trade available via resale between bidder i and the coalition $N \setminus i$. By Eqs. (4) and (6),

$$\begin{aligned}\pi_i(t_i' \mid t) \le &\int_0^{q_i^*(t)}\left[v_i(t,z) - v_i(\hat{t}_i(t_{-i},z),t_{-i},z)\right]dz \\ &+\int_0^{q_i^*(t_i',t_{-i})-q_i^*(t)}\left[v_i(t,q_i^*(t_i',t_{-i})-z) - v_i(\hat{t}_i(t_{-i},q_i^*(t_i',t_{-i})-z),t_{-i},q_i^*(t_i',t_{-i})-z)\right]dz \\ &+\int_0^{\Delta_i(t_i'|t)}\left[v_{-i}(t,q_{-i}^*(t_i',t_{-i})+z) - v_i(t,q_i^*(t_i',t_{-i})-z)\right]dz.\end{aligned} \tag{10}$$

Since $t_i \le \hat{t}_i(t_{-i},q_i^*(t_i',t_{-i})-z),$ for all z between 0 and $q_i^*(t_i',t_{-i})-q_i^*(t)$, the second integrand of Eq. (10) is weakly negative. Since $0 \le \Delta_i(t_i' \mid t) \le q_i^*(t_i',t_{-i}) - q_i^*(t),$ we further have

$$\begin{aligned}\pi_i(t_i' \mid t) \le &\int_0^{q_i^*(t)}\left[v_i(t,z) - v_i(\hat{t}_i(t_{-i},z),t_{-i},z)\right]dz \\ &+\int_0^{\Delta_i(t_i'|t)}\left[v_i(t,q_i^*(t_i',t_{-i})-z) - v_i(\hat{t}_i(t_{-i},q_i^*(t_i',t_{-i})-z),t_{-i},q_i^*(t_i',t_{-i})-z)\right]dz \\ &+\int_0^{\Delta_i(t_i'|t)}\left[v_{-i}(t,q_{-i}^*(t_i',t_{-i})+z) - v_i(t,q_i^*(t_i',t_{-i})-z)\right]dz.\end{aligned} \tag{11}$$

But, then, using Eq. (4), we can simplify this as

$$\pi_i(t_i' \mid t) \le U_i(t_i \mid t) + \int_0^{\Delta_i(t_i'|t)}\left[v_{-i}(t,q_{-i}^*(t_i',t_{-i})+z) - v_i(\hat{t}_i(t_{-i},q_i^*(t_i',t_{-i})-z),t_{-i},q_i^*(t_i',t_{-i})-z)\right]dz. \tag{12}$$

Finally, observe by Lemma 2 that the integrand of Eq. (12) is nonpositive for all z such that $0 \le z \le \Delta_i(t_i' \mid t)$; consequently the integral is nonpositive whenever $\Delta_i(t_i' \mid t) \ge 0$. By the single-crossing property and the monotonicity of $\bar{q}(t)$, $t_i' > t_i$ implies $\Delta_i(t_i' \mid t) \ge 0$. This allows us to conclude that $\pi_i(t_i' \mid t) \le U_i(t_i \mid t)$, for all $t_i' > t_i$, and for all t_{-i}. Analogous reasoning applies for all underreports $t_i' < t_i$. ■

Finally, consider the problem of a seller that seeks to maximize revenues, but cannot prevent resale. Ausubel and Cramton (1999) show that a seller faced with a perfect resale market cannot gain by misassigning goods. The best the seller can hope to do is to assign the goods efficiently, perhaps withholding quantity. This result requires independent types, so that the optimal auction program is well specified and a general revenue equivalence theorem holds.

Theorem 2 states that any monotonic aggregate quantity rule and associated monotonic efficient assignment rule can be implemented with a Vickrey auction with reserve pricing. This suggests that a revenue-maximizing seller then can optimize over all monotonic aggregate quantity rules to attain the upper bound on revenues given by the resale-constrained auction program in Ausubel and Cramton (1999). Indeed, this is the case provided the Vickrey auction with reserve pricing holds the lowest type ($t_i = 0$) of every bidder to a payoff of zero. To see this, note that $\hat{t}_i(t_{-i}, y) = 0$ for all t_{-i} and $y \in [0, q_i^*(0, t_{-i})]$, so that the lowest type's payment $X_i^*(0, t_{-i})$ is exactly equal to the value it gets from $q_i^*(0, t_{-i})$. Hence, we have:

COROLLARY. *With independent types, the Vickrey auction with reserve pricing attains the upper bound on revenues in the resale-constrained auction program.*

5 Conclusion

A Vickrey auction with reserve pricing has two main advantages. First, it assigns goods efficiently. Efficiency is especially important in auction markets with resale, since the apparent revenue benefits from misassignment are undermined by resale. Second, it allows the seller to withhold supply and set reserve prices to improve revenues. The use of reserve prices is especially important when competition is weak and the bidders are asymmetric. It is also important in auctions of multiple identical items, where one or more of the bidders purchases a significant share of the goods.

We have extended the Vickrey auction to include reserve pricing in a multi-unit setting with interdependent values. Truthful bidding remains an *ex post* equilibrium despite the fact that the seller varies the quantity based on the bids. This efficient outcome is robust to the possibility of resale. So long as the resale game satisfies a natural extension of individual rationality, truthful bidding followed by no resale is an equilibrium in the auction-plus-resale game. Moreover, if resale is efficient, then the Vickrey auction with appropriate reserve pricing is the optimal auction. No alternative auction can yield higher revenues.

A practical difficulty of using Vickrey pricing when auctioning multiple items is that identical items sell for different prices. Worse, large winners tend to pay lower average prices than small winners. This fact is an unavoidable implication of achieving efficiency. Large bidders have a greater incentive to reduce demands than small bidders. Hence, efficient pricing must reward large bidders for bidding their true demands by letting large bidders win the efficient quantity at lower average prices. In contrast, uniform pricing necessarily leads to an inefficient assignment (Ausubel and Cramton 2002), and hence to suboptimal revenues when resale is efficient.

Participants in many actual markets voice a strong preference for uniform pricing (Wilson 2002). Often the case for uniform pricing is made on efficiency grounds, and the case against Vickrey pricing is based on examples of lost revenue. With diminishing marginal valuations, these arguments have little merit. On either efficiency or revenue grounds, a Vickrey auction with reserve pricing should be preferred.

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