TECHNICAL RESEARCH REPORT

Dynamical Properties of TCP System with AQM Routers

by Huigang Chen, John S. Baras, Nelson X. Liu

TR 2003-45

The Center for Satellite and Hybrid Communication Networks is a NASA-sponsored Commercial Space Center also supported by the Department of Defense (DOD), industry, the State of Maryland, the University of Maryland and the Institute for Systems Research. This document is a technical report in the CSHCN series originating at the University of Maryland.

Web site http://www.isr.umd.edu/CSHCN/

Dynamical Properties of TCP System with AQM Routers

Huigang Chen, John S. Baras, and Nelson X. Liu

Abstract

In this report we discuss the dynamics of heterogeneous TCP systems with propagation delays. Instead of studying the local linearized TCP dynamics, we study the global stability conditions and obtain the stability regions. Also we provide proof of periodic behavior of a single TCP connection when stability conditions are not met.

I. Introduction

Today's Internet traffic is mostly consisted of flows based on TCP which provides reliable transmission and resilient performance in response to varying network conditions. TCP, as a window-based end-to-end control scheme, adapts the sender's transmission rate by the feedback information of the receiver. The sender exploits the available network network by linearly increasing the window size and when a network congestion is detected from either packet drop or ECN marking, the sender reduces the window by half. Using TCP fluid-flow model, with such additive-increase-multiplicative-decrease window control algorithm, coupled with the AQM scheme in the intermediate routers, we can formulate the network as a nonlinear delay feedback system. The dynamical behavior of such system has been studied in [1], [2], [3], [4], [5], [6], and etc. Due to the nonlinearity of the system and the existence of delay in the feedback, people either focus on the small signal linearized system in the neighborhood of equilibrium state [5], or use Lyapunov-Razumikhin type theorems to analyze single source and single bottleneck network [3], [4]. Recently people start to use powerful nonlinear stability analysis tools, such passivity theory and ISS Small-Gain theorem, to obtain robust stability conditions for network flow control problems [7] [8], but it is very difficult to apply these tools to the TCP systems since TCP is not an ISS system. This work tries to establish global nonlinear stability conditions for a general AQM scheme and study the dynamical behavior of a simple TCP system when the stability conditions are not met. The paper is organized as follows. Section II proves global stability theorems for heterogeneous TCP systems. Section III proves the existence of periodic dynamics for a single source and single bottleneck TCP system with delay. We conclude in Section IV.

II. Global Stability of TCP systems

We are considering here a network of TCP sources and links as in the Figure (1) with N sources and L links. Packets from each source i flow at the rate of $r_i \in \mathbb{R}^N$ through the link j with the aggregated arrival rate $y_j = [Rr]_j \in \mathbb{R}^L$. And accordingly each link generates a penalty value $p_j \in \mathbb{R}^L$ from the arrival rate and its queue size and sends the penalty value back to the sender. Then each sender adjusts its sending rate appropriately by the aggregated penalty value $q_i = [R^T p]_i \in \mathbb{R}^N$ it receives. Here matrix R is the routing matrix and we assume it is constant. Also another assumption here is that the aggregated penalty value is the summation of the penalty values from related links rather than multiplication, which is a fine approximation when the penalty values are small. Notice that the system is nonlinear, and we allow different delays in the routing matrix. In this study we wish to establish conditions for global stability.

The system dynamics takes the form of the following,

$$\begin{aligned}
\dot{r}_i(t) &= \frac{r_i(t-\tau_i)}{r_i(t)}\left(\frac{1-q_i(t-\tau_i)}{d_i^2} - \frac{1}{2}r_i(t)^2 q_i(t-\tau_i)\right), \quad i = 1,\ldots,N & (1)\\
\dot{s}_j(t) &= ((1-p_j)y_j - c_j)_s^+, \quad p_j = h_j(s_j), \quad j = 1,\ldots,L & (2)\\
y &= Rr, \quad q = R^T p
\end{aligned}$$

where c_j is the link capacity at the link j, and τ_i is the link delay seen by the source i. The above equations describe the scenario in which the router drops the incoming packets by some schemes instead of just marking them. First of all we transform Equation (2) into

$$\dot{p}_j = H_j(p_j)((1-p_j)y_j - c_j), \quad j = 1,\ldots,L. \qquad (3)$$

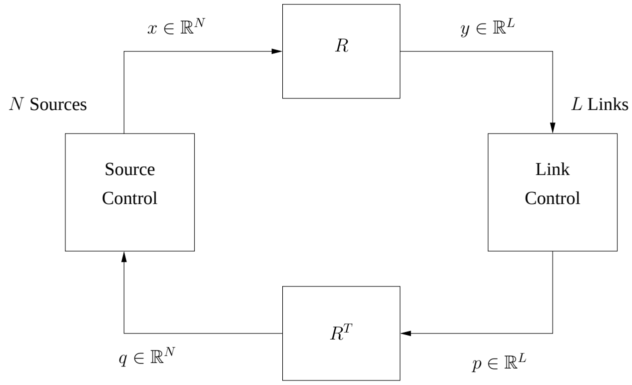

Fig. 1. Network of TCP system

Here $H_j(p_j) = h'_j(h_j^{-1}(p_j))$. Under some mild conditions, Equations (2) and (3) are equivalent. So we instead study the dynamical system with state equations (1) and (3). The delay-free case can be solved by Lyapunov's direct method. Here "global" means the region where $r_i \geq 0$, $i = 1, \dots, N$, and $p_j \in [0, 1]$, $j = 1, \dots, L$. We begin with a simple observation:

Lemma 1: The states of the dynamical system described by Equations (1) and (3) are uniformly bounded.

Proof: We only need to consider r_i's. We designate $c_M = \max_{j=1,\dots,L} c_j$. Without loss of generality suppose there is a t_0 such that $r_i(t) < r_i(t_0) = 2c_M$, for all $t < t_0$ and r_i is still increasing. Then

$$\dot{p}_j = H_j(p_j)((1 - p_j)y_j - c_j) \geq H_j(p_j)(1 - 2p_j)c_M, \quad j \in \mathcal{L}_i$$

for all t at which $r_i(t) \geq 2c_M$. So p_j peaks at least at $1/2$ and surpasses any η, $\eta \in (0, 1/2)$ in finite time t_η. Thus we have

$$r_i(t) \leq 2c_M + \max\{\frac{t_\eta}{d_i^2}, \frac{1}{d_i}\sqrt{\frac{1-\eta}{\eta}}\} + \frac{\tau_i}{d_i^2}, \quad \forall t$$

which completes the proof. ■

Proposition 1: If H_j is bounded away from 0 for all j, the delay-free ($\tau_i = 0$, $i = 1, \dots, N$) TCP system described by Equations (1) and (3) is globally asymptotically stable.

Proof: In the following proof, i is the index from 1 to N and j is the index from 1 to L. Let us define displaced state variables $\tilde{r}_i = r_i - r_i^*$ and $\tilde{p}_j = p_j - p_j^*$, where r_i^* and p_j^* are the rate of the ith source and the dropping probability at the jth link in equilibrium state respectively. Also we designate $\tilde{r} = [\tilde{r}_1, \dots, \tilde{r}_N]^T$ and $\tilde{p} = [\tilde{p}_1, \dots, \tilde{p}_L]^T$. So the dynamical equations becomes

$$\dot{\tilde{r}}_i = -\left(\frac{1}{2}\tilde{r}_i + \bar{r}_i\right) q_i \tilde{r}_i - \left(d_i^{-2} + \frac{1}{2}\bar{r}_i^2\right)\tilde{q}_i \tag{4}$$

$$\dot{\tilde{p}}_j = H_j(p_j)((1 - \bar{p}_j)\tilde{y}_j - y_j \tilde{p}_j) \tag{5}$$

We define $J_j(\cdot)$ as

$$J_j(u) \triangleq \begin{cases} \int_0^{u^2} 1/H_j(\bar{p}_j + \sqrt{x})dx, & u \geq 0 \\ \int_0^{u^2} 1/H_j(\bar{p}_j - \sqrt{x})dx, & u < 0 \end{cases}.$$

It is easy to see that since $H_j(u) > 0$ for all j and u is defined on a compact set, those integrals are well defined, continuously differentiable and positive definite.

Next consider the candidate Lyapunov function

$$V(\tilde{r},\tilde{p}) = \frac{1}{2}\tilde{r}^T \begin{bmatrix} d_1^{-2}+\bar{r}_1^2/2 & & \\ & \ddots & \\ & & d_N^{-2}+\bar{r}_N^2/2 \end{bmatrix}^{-1} \tilde{r} + \frac{1}{2} \begin{bmatrix} 1-\bar{p}_1 & & \\ & \ddots & \\ & & 1-\bar{p}_L \end{bmatrix}^{-1} [J_1(\tilde{q}_1),\ldots,J_L(\tilde{q}_L)]^T . \tag{6}$$

Take the derivative along the trajectory of the dynamical system, we get

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}(d_i^{-2}+\bar{r}_i^2/2)^{-1}\tilde{r}_i\dot{\tilde{r}}_i + \sum_{j=1}^{L}(1-\bar{p}_j)^{-1}\dot{J}_j(\tilde{q}_j) \\
&= \sum_{i=1}^{N}(d_i^{-2}+\bar{r}_i^2/2)^{-1}\left(-(\tilde{r}_i/2+\bar{r}_i)q_i\tilde{r}_i^2-(d_i^{-2}+\bar{r}_i^2/2)\tilde{q}_i\tilde{r}_i\right) \\
&+ \sum_{j=1}^{N}(1-\bar{p}_j)^{-1}H_j(p_j)^{-1}\tilde{p}_jH_j(p_j)\left((1-\bar{p}_j)\tilde{y}_j-y_j\tilde{p}_j\right) \\
&= -\tilde{r}^T \begin{bmatrix} \frac{\tilde{r}_1/2+\bar{r}_1}{d_1^{-2}+\bar{r}_1^2/2}q_1 & & \\ & \ddots & \\ & & \frac{\tilde{r}_N/2+\bar{r}_N}{d_N^{-2}+\bar{r}_N^2/2}q_N \end{bmatrix} \tilde{r} \\
&- \tilde{p}^T \begin{bmatrix} \frac{y_1}{1-\bar{p}_1} & & \\ & \ddots & \\ & & \frac{y_L}{1-\bar{p}_L} \end{bmatrix} \tilde{p} - \tilde{q}^T\tilde{r} + \tilde{p}^T\tilde{y}.
\end{aligned}$$

But

$$\tilde{q}^T\tilde{r} = \tilde{p}^T R\tilde{r} = \tilde{p}^T\tilde{y}.$$

So

$$\dot{V} = -\tilde{r}^T \mathrm{diag}\left[\frac{\tilde{r}_i/2+\bar{r}_i}{d_i^{-2}+\bar{r}_i^2/2}q_i\right]\tilde{r} - \tilde{p}^T\mathrm{diag}\left[\frac{y_j}{1-\bar{p}_j}\right]\tilde{p}$$

From the fact that

$$\begin{aligned}
\tilde{r}_i/2+\bar{r}_i &= r_i/2+\bar{r}_i/2 \geq \bar{r}_i/2 \\
y_j &= \sum_{i\in\mathcal{L}_j} r_i \geq 0 \\
q_i &= \sum_{j\in\mathcal{S}_i} p_j \geq 0,
\end{aligned}$$

we know that the derivative of $V(\tilde{r},\tilde{q})$ is positive semidefinite and since the only invariant set is at the equilibrium, so global asymptotic stability follows from Invariance Theorem. ∎

Remark 1: The above Proposition considers the AIMD form of TCP. Actually by using similar Lyapunov technique, one can prove that for the fluid-flow model of general TCP window update algorithm like:

$$W_{k+1} = \begin{cases} W_k + aW_k^{-\alpha}, & \text{packet} \quad \text{received} \\ W_k - bW_k^{\beta}, & \text{packet} \quad \text{loss} \end{cases}$$

global asymptotic stability still holds.

Next we consider the case when there are no delays from the feedbacks of the penalty information, but there are state delays in the rate dynamics, which is,

$$\begin{aligned} \dot{r}_i(t) &= \frac{r_i(t-\tau_i)}{r_i(t)}\left(\frac{1-q_i(t)}{d_i^2}-\frac{1}{2}r_i(t)^2 q_i(t)\right), \quad i=1,\dots,N & (7)\\ \dot{s}_j(t) &= ((1-p_j)y_j-c_j)_s^+, \quad p_j=h_j(s_j), \quad j=1,\dots,L & (8)\\ y &= Rr, \quad q=R^T p \end{aligned}$$

Proposition 2: If H_j is bounded away from 0 for all j, the TCP system described by Equations (7) and (3) is globally asymptotically stable.

Proof: For any initial conditions $r_i(\theta_i)=\phi_i(\theta_i)$, $\phi_i(\theta_i)\geq 0$, $\theta_i\in[-\tau_i,0]$, and $p_j(0)=p_{j0}$, there is a solution $r_i(t)$ and $p_j(t)$ for $\forall t>0$.

Let us study a particular trajectory of such solution. Define functions $\xi_i:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\xi_i(x,t)\triangleq\begin{cases} r_i(T-\tau_i), & T=\max_{\tilde{r}_i(\theta)=x,0\leq\theta\leq t}\theta\\ \bar{r}_i, & \tilde{r}_i(\theta)\neq x, 0\leq\theta\leq t\end{cases},$$

and define functions $\zeta_i:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\zeta_i(x,t)\triangleq\begin{cases} \bar{r}_i+x, & \exists\theta,\mathrm{st.}0\leq\theta\leq t,\tilde{r}_i(\theta)=x\\ \bar{r}_i, & \tilde{r}_i(\theta)\neq x,0\leq\theta\leq t\end{cases}.$$

Observe that the discontinuity with regard to t in $\xi_i(x,t)$ takes place only at $x=\tilde{r}(t)$ and $\partial\xi_i(x,t)/\partial t=0$ for all other x's . So

$$\int_0^{\tilde{r}(t)}\frac{\partial\xi_i(x,t)}{\partial t}dx=0$$

Consider the functions $W_i(t)$'s

$$W_i(t)=\begin{cases} \frac{1}{2}\int_0^{\tilde{r}_i(t)^2}\frac{\zeta_i(\sqrt{x},t)}{\xi_i(\sqrt{x},t)}dx, & \tilde{r}>0\\ \frac{1}{2}\int_0^{\tilde{r}_i(t)^2}\frac{\zeta_i(-\sqrt{x},t)}{\xi_i(-\sqrt{x},t)}dx, & \tilde{r}<0\end{cases}.$$

First of all, the above integrals are well defined for all t, $t\in(0,\infty)$ since the integrands are nothing other than $r_i(t)/r_i(t-\tau_i)$ or 1 and the integral can be written as

$$\begin{aligned} &\frac{1}{2}\int_0^{\tilde{r}_i(t)^2}\frac{\zeta_i(\pm\sqrt{x},t)}{\xi_i(\pm(\sqrt{x},t)}\frac{dx}{dt}dt\\ \leq\;&\int_0^t\frac{r_i(t)}{r_i(t-\tau_i)}r_i(t-\tau_i)\left(\frac{1-q_i}{d_i^2}-\frac{1}{2}r_i^2q_i\right)dt\leq(1+d_i^{-2})r_Mt\end{aligned}$$

where r_M is the upperbound of the rate. Second, since $r_i(t)$ remains positive all the time so $W_i(t)$'s are positive definite functions. And the derivative of $W_i(t)$s are

$$\begin{aligned}\dot{W}_i(t) &= \tilde{r}_i(t)\dot{\tilde{r}}_i\frac{r_i(t)}{r_i(t-\tau_i)}+\frac{1}{2}\int_0^{\tilde{r}_i(t)^2}\frac{\partial}{\partial t}\frac{\zeta_i(\pm\sqrt{x},t)}{\xi_i(\pm\sqrt{x},t)}dx\\ &= -(\tilde{r}_i/2+\bar{r}_i)q_i\tilde{r}_i^2-(d_i^{-2}+\bar{r}_i^2/2)\tilde{q}_i\tilde{r}_i\end{aligned}$$

Therefore the positive definite function

$$\begin{aligned} V(t) &= \begin{bmatrix} d_1^{-2}+\bar{r}_1^2/2 & & \\ & \ddots & \\ & & d_N^{-2}+\bar{r}_N^2/2\end{bmatrix}^{-1}[W_1(t),\dots,W_N(t)]^T\\ &+\frac{1}{2}\begin{bmatrix}1-\bar{p}_1 & & \\ & \ddots & \\ & & 1-\bar{p}_L\end{bmatrix}^{-1}[J_1(\tilde{q}_1),\dots,J_L(\tilde{q}_L)]^T \qquad (9)\end{aligned}$$

has the derivative

$$\dot{V}(t) = -\tilde{r}^T \mathrm{diag}\left[\frac{\tilde{r}_i/2+\bar{r}_i}{d_i^{-2}+\bar{r}_i^2/2} q_i\right]\tilde{r} - \tilde{p}^T \mathrm{diag}\left[\frac{y_j}{1-\bar{p}_j}\right]\tilde{p}.$$

Since $V(t)$ is always non-negative and decreasing with time, so $\lim_{t\to\infty} V(t)$ exists. From boundedness of the trajectory, $V(t)$ and $\dot{V}(t)$ is bounded. Thus by applying Barbalat lemma, $\lim_{t\to\infty}\dot{V}(t)=0$. Because $\dot{V}(t)$ is a positive definite function of $\tilde{r}_i$'s and $\tilde{p}_j$'s, so it follows that the dynamical system converges to its equilibrium state. ■

The previous theorem establishes that in the source dynamics, the delay term of the sending rate is not the cause of instability. So we should study the effect of delayed feedback to the global stability. It is a very difficult problem even in a single flow scenario. By using Lyapunov-Razumikhin type theorems, previous studies ([4], [3]) gave very narrow stability regions. This is because in the source dynamics the gain of the delayed term can be large and the gain of the delay-free term can be small. Also Razumikhin type methods often give rather poor stability region estimates for delay-dependent stability. Here we try to utilize a more sophiticated Lyapunov functional to obtain a much better stability region estimates.

Due to complexity of the problem we face, we only present a single-flow scenario with a linear queue marking function. Consider a TCP/AQM dynamics described as below,

$$\begin{aligned}\dot{r} &= \frac{1-kq(t-\tau)}{\tau^2} - \frac{1}{2}r(t)^2 kq(t-\tau),\\ \dot{q} &= (r-C)_q^+,\end{aligned}$$

r and q are TCP rate and bottleneck queue size as usual and k is the slope of the marking function. By re-scaling the time $t=\tau s$ to normalize the delay, we get

$$\begin{aligned}\dot{\tilde{r}} &= -\frac{\tilde{q}(s-1)}{\tau q^*} - \tau(\tilde{r}/2+C)kq(s-1)\tilde{r}(s)\\ \dot{\tilde{q}} &= \tau\tilde{r}.\end{aligned} \tag{10}$$

Here we let $r=r^*+\tilde{r}$ and $q=q^*+\tilde{q}$, where $r^*=C$ and $q^*=k^{-1}\left(\left(1+C^2\tau^2/2\right)^{-1}\right)$ are equilibrium rate and queue length respectively. We have the following result.

Theorem 1: Suppose $\tau C \gg 1$, we can choose k to make the system (10) globally stable.

Proof: Local linear stability result in Lemma 2 tells us when $\tau C\gg 1$, we can choose $k=4C^{-3}\tau^{-3}$ to obtain local stability. For simplicity, let us denote $\psi(t)=[\tilde{r}(t),\tilde{q}(t)]^T$. Also denote η as a 2×2 matrix-valued function with bounded variation on $[-1,0]$:

$$\eta(s)=\begin{cases}[0 \quad 0; 0 \quad 0], & s=-1\\ [0 - \tau^{-1}/q^*; 0 \quad 0], & -1<s<0\\ [-\tau C k q^* - \tau^{-1}/q^*; \tau \quad 0], & s=0\end{cases}$$

It is easy to see that the linearized version of (10) can be written as

$$\dot{\psi}(t)=\int_{-1}^{0} d\eta(\theta)\psi_t(\theta).$$

From [9], we can choose the following Lyapunov functional for (10),

$$\begin{aligned}V(\psi) &= \psi^T(0)Y(0)\psi(0)\\ &+ 2\psi^T(0)\int_{-1}^{0}\int_{u}^{0} Y(-u+\theta)d\eta(u)\psi(\theta)d\theta\\ &+ \int_{-1}^{0}\int_{-h}^{0} ds\psi^T(s)d\eta^T(h)\\ &\times \int_{-1}^{0}\int_{-1}^{0} Y(-s+h-u+\theta)d\eta(u)\psi(\theta)d\theta\end{aligned} \tag{11}$$

It is known that if (10) is locally stable, we can find $Y\in C([-1,1],\mathrm{R}^{2\times 2})$ satisfies $\dot{Y}(0)+\dot{Y^T}(0)=-W$ where W is a positive definite matrix. Here $\dot{Y}(0)$ is defined as $d^+Y(0)/dt$. Then $V(\psi)$ is positive definite if Y satisfies additionally:

(i) $Y(t)$ is continuously differentiable for $t \neq 0$.
(ii) $Y(0)$ is symmetric and $Y(t) = Y^T(-t)$.
(iii) $Y(t) = \int_{-1}^{0} d\eta^T(s) Y(s+t)$.

Take the derivative of $V(\psi)$ we can show

$$\begin{aligned} \dot{V}\psi &= -\psi^T(0)W\psi(0) - 2k\tau(\tilde{r}(0)q(-1)/2 + \tilde{q}(-1)C) \\ &\times \ (\tilde{r}(0)\tilde{q}(0)Y_{12}(0) + \tilde{r}(0)^2 Y_{11}(0)) \end{aligned}$$

Denote $Y(t) = [y_1 \quad y_2; y_3 \quad y_4]$. From the conditions at which $Y(t)$ has to satisfy we have

$$\begin{bmatrix} \dot{y}_1(t) & \dot{y}_2(t) \\ \dot{y}_3(t) & \dot{y}_4(t) \end{bmatrix} = \begin{bmatrix} -kCq^*\tau & \tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{1}{\tau q^*} & 0 \end{bmatrix} \begin{bmatrix} y_1(1-t) & y_2(1-t) \\ y_3(1-t) & y_4(1-t) \end{bmatrix}.$$

We can solve the above equation explicitly. By some calculation, $y_1(t)$ is the solution of the following fourth-degree differential equation,

$$y_1^{(4)} - k^2C^2\tau^2q^{*2}y_1^{(2)} - \frac{1}{q^{*2}}y_1 = 0.$$

Given initial condition $\dot{y}_1(0) = -1/2$, a solution of the above equation is

$$y_1(t) = \frac{1}{2u}\frac{(u^2 + kC\tau q^* u)e^{-ut} - q^{*-1}e^{u(t-1)}}{u^2 + kC\tau q^* u + q^{*-1}e^{-u}}$$

where

$$u = \sqrt{\frac{k^2C^2\tau^2q^{*2}}{2} + \sqrt{\frac{k^4C^4\tau^4q^{*4}}{4} + \frac{1}{q^{*2}}}}.$$

It is easy to see that when $\tau C \gg 1$, $u = \sqrt{2/(\tau C)}$ and $y_1(t) = (1-t)/2$. We can also deduct the following relations,

$$y_3(1) + \frac{1}{\tau q^*} = y_3(0),$$

and for $\dot{Y}(0) + \dot{Y}^T(0) = -W = -[W_1 \quad W_2; W_2 \quad W_4]$,

$$\begin{cases} 2\dot{y}(0) = -1, \\ -kC\tau q^* y_2(0) + \tau y_4(0) - (\tau q^*)^{-1}y_1(1) = W_2, \\ -2(\tau q^*)^{-1}y_3(1) = W_4. \end{cases}$$

So we can set $y_2(0) = y_3(0) = \tau^{-1}q^{*-1}$, and choose $y_4(0)$ properly so that

$$W = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{2}{\tau^2q^{*2}} \end{bmatrix} < 0.$$

Now $\dot{V}(\psi)$ becomes

$$\begin{aligned} \dot{V}(\psi) &= -\tilde{r}(0)^2 - 2\tau^{-2}/q^*\tilde{q}(0)^2 - 2k\tau(\tilde{r}(0)q(-1)/2 + \tilde{q}(-1)C) \\ &\times \ (\frac{2}{\tau q^*}\tilde{r}(0)\tilde{q}(0) + \frac{1}{2}\tilde{r}(0)^2). \end{aligned}$$

Suppose q is confined in the region $[0, mC\tau]$, for any positive integer m. We can upperbound the above equation by

$$\dot{V}(\psi) \leq \begin{cases} -\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 + \frac{4\tilde{r}(0)}{C\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}), & \tilde{r}(0)(\tilde{r}(0)/2 + 2\tilde{q}(0)/(\tau q^*)) \geq 0 \\ -\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 - (2mC + \tilde{r}(0)(m+1/2))\frac{4\tilde{r}(0)}{C^2\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}), & \tilde{r}(0)(\tilde{r}(0)/2 + 2\tilde{q}(0)/(\tau q^*)) \leq 0. \end{cases} \tag{12}$$

We can observe from Figure 2 in Region I and Region III the upper condition of (12) is satisfied and $\dot{V}(\psi)$

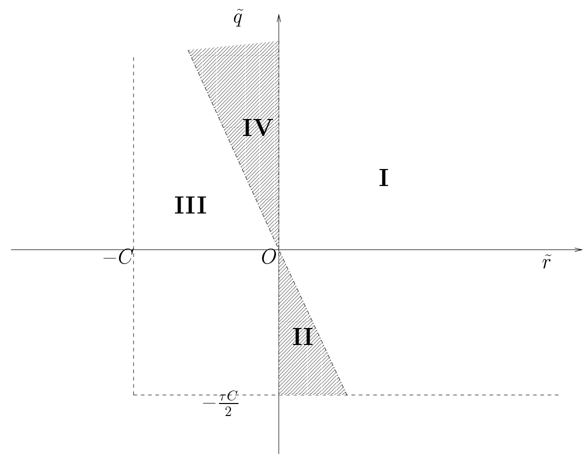

Fig. 2. Region of $\dot{V}(\psi)$ in (12)

becomes

$$\begin{aligned}\dot{V}(\psi) &\leq -\tilde{r}(0)^2 - \frac{2}{\tau^2 q^{*2}}\tilde{q}(0)^2 + \frac{4\tilde{r}(0)}{C\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}) \\ &= -(1 - 2(C\tau)^{-1})\tilde{r}(0)^2 - \frac{2}{\tau^2 q^{*2}}\tilde{q}(0)^2 + \frac{4}{\tau q^{*2}}\tilde{r}(0)\tilde{q}(0)\end{aligned}$$

The last part of the above equation is strictly below zero when ψ is other than 0 if the condition

$$4\left(1 - \frac{2}{C\tau}\right)\frac{2}{q^{*2}} > \frac{16}{q^{*4}}$$

holds. This is true from our assumption $C\tau \gg 1$.

In Region II and Region IV of Figure 2, the maximum value of $\dot{V}(\psi)$ should be reached for $\tilde{r} \geq 0$ (Region II). This is because for every $\tilde{r}$ and $\tilde{q}$ in Region IV, we can choose $-\tilde{r}$ and $-\tilde{q}$ in Region II so that the right handside of (12) in the second situation is larger. This argument can be immediately verified from the existence of the $(2mC + \tilde{r}(0)(m + 1/2))$ term. So now let us focus on Region II. The part which involves $\tilde{q}$ is

$$-\frac{2}{\tau^2 q^{*2}}\tilde{q}(0)^2 - (2mC + \tilde{r}(0)(m + 1/2)))\frac{8\tilde{r}(0)}{C^2\tau^2 q^*)}\tilde{q}(0).$$

This reaches maximum in Region II at $\tilde{q}(0) = 0$. So we showed in Region II and IV we also have

$$\dot{V}(\psi) \leq 0.$$

Therefore we conclude that in all regions of the state space, the derivative of the Lyapunov functional (11) is strictly below 0 except when the trajectory is at the equilibrium state. Consequently global stability holds. ∎

III. Periodic Solution of a Single TCP Connection

Now we study a single TCP source with the round-trip time R and go through a bottleneck link with fixed bandwidth C. The bottleneck router is implemented with some ECN marking scheme. We have the following dynamics,

$$\begin{aligned}\frac{dr}{dt} &= \frac{1 - f(q(t-\tau))}{\tau^2}\frac{r(t-\tau)}{r(t)} \\ &- \frac{1}{2}r(t)r(t-\tau)f(q(t-\tau)) \\ \frac{dq}{dt} &= (r - C)_q^+ \end{aligned} \tag{13}$$

The function $f(q) \in C^1(\mathbb{R}, [0,1])$ is a nondecreasing marking function satisfying the following constraints

$$\begin{aligned} f(q) = 0 \quad &, \quad \text{for} \quad q \leq 0 \\ f(q) = 1 \quad &, \quad \text{for} \quad q \geq B. \end{aligned} \tag{14}$$

From observations of simulations, we notice that dynamics (13) has periodic solutions when the delay τ is greater than a certain value. Low [5] proved that the condition of stability of linearized version of (13). In this section, we try to prove some properties of nonlinear dynamics, including boundedness and periodic behavior of the solution.

First of all, we want to derive the local behavior of the system (13) when the delay τ is near the boundary of the stability region established from the linearized system. To begin with, we need to obtain the local stability conditions with regard to τ. The linearized version of Equations (13) is

$$\begin{aligned} \frac{d\tilde{r}}{dt} &= -f'(q^*)\left(\frac{1}{\tau^2} + \frac{1}{2}r^{*2}\right)\tilde{q}(t-\tau) - r^* f(q^*)\tilde{r}(t) \\ \frac{d\tilde{q}}{dt} &= \tilde{r}. \end{aligned}$$

Here we let $r = r^* + \tilde{r}$ and $q = q^* + \tilde{q}$, where $r^* = C$ and $q^* = f^{-1}(\left(1 + C^2R^2/2\right)^{-1})$ are equilibrium rate and queue length respectively. Re-scale the time $t = \tau s$ to normalize the delay, we get

$$\begin{aligned} \frac{d\tilde{r}}{ds} &= -\frac{f'(q^*)}{\tau f(q^*)}\tilde{q}(s-1) - \tau f(q^*) C\tilde{r}(s) \\ \frac{d\tilde{q}}{ds} &= \tau\tilde{r}. \end{aligned} \tag{15}$$

The following lemma holds.

Lemma 2: Suppose $f'(q^*) < 0.5006$ and let t_k, $k = \{0, 1\}$ be the solutions of the following equation,

$$\frac{\omega_k^2}{\cos\omega_k} = f'(q^*)(1 + t_k^2 C^2/2), \tag{16}$$

and ω_0, ω_1 be respectively the first and the second smallest positive $\omega \in [0, \pi/2]$ satisfying

$$\frac{\cos\omega}{\omega^2} f'(q^*) + \frac{\omega^4 \tan^2\omega}{2f'(q^*)\cos\omega} = 1. \tag{17}$$

If $t_0 < \tau < t_1$, then all the characteristic roots $\lambda(\tau)$ of Equations (15) have negative real parts. There is an infinite series $\tau_k, k = 0, 1, \ldots$, such that there are exactly 2 pure imaginary roots when $\tau = \tau_k$. $\lambda(\tau)$ is differentiable at $\tau = t_1$, and $\mathrm{Re}\lambda'(t_1) > 0$. If $f'(q^*) > 0.5006$, the system (15) is unstable for all τ. For $\tau > t_1$, there are precisely two characteristic roots λ of Equations (15) in the region $\mathrm{Re}\lambda > 0$ and $-\pi < \mathrm{Im}\lambda < \pi$.

Proof: The characteristic equation of Equation (15) is

$$\Delta(\lambda) = \lambda^2 + \tau f(q^*) C\lambda + \frac{f'(q^*)}{f(q^*)} e^{-\lambda} = 0. \tag{18}$$

Suppose Equation (18) has pure imaginary roots $j\omega$, then

$$\tau f(q^*) C = \omega \tan\omega \tag{19}$$

$$\left(\frac{f'(q^*)}{f(q^*)}\right)^2 = \omega^4 + \tau^2 f(q^*)^2 C^2 \omega^2. \tag{20}$$

Substitute equality (19) into (20), we immediately get

$$\frac{f'(q^*)}{f(q^*)} = \frac{\omega^2}{\cos\omega}. \tag{21}$$

Multiply this with (19) and after some calculations we obtain Equation (17). This indicates that the pure imaginary roots of Equation (18) do not depend on $f(q^*)$ whose value is decided by the delay τ. Further examination of the signs of real and imaginary terms of the equation tells us that all possible positive pure imaginary roots of Equation (18) can only lie in the intervals $[2k\pi, 2k\pi + \pi/2]$, $k = 0, 1, \ldots$. For sufficiently large k, the first term in the left hand side of (17) can be small at $2k\pi$ and the second term is zero. So there is a smallest positive solution to the equation (17) and we denote it ω_0. There is a sequence of positive pure imaginary roots $\{\omega_n\}$, $\omega_n < \omega_{n+1}$.

For each ω_k, the value of τ such that (19) and (20) hold can be easily deducted. For t_0 and t_1 the result is in Equation (16) and the rest is the same. We will show that at each τ there are at most 2 pure imaginary roots possible. Assume the contrary holds. Then there exist ω_1' and ω_2' such that (20) and (21) both hold for some τ. Therefore

$$\begin{aligned}\frac{\cos\omega_1'}{\sin^2\omega_1'} &= \frac{1}{\tau^2 f(q^*)^2 C^2} \times \frac{f'(q^*)}{f(q^*)} \\ &= \frac{\cos\omega_2'}{\sin^2\omega_2'}.\end{aligned}$$

Since $\cos\theta/\sin^2\theta$ is monotone for $\theta \in [2k\pi + \pi/2]$, so $\omega_1' = 2k\pi + \omega_2'$. It is obviously not possible for any $k \neq 0$ by checking (20). Therefore at each τ there are at most 2 conjugated pure imaginary roots.

By applying Theorem 13.9 of [10] we know that the necessary and sufficient condition that all roots of $\Delta(\lambda)$ reside to the left of the imaginary axis is

$$\frac{C\tau f(q^*)^2}{f'(q^*)} > \frac{\sin a}{a}, \tag{22}$$

where

$$\cot a = \frac{a}{C\tau f(q^*)}. \tag{23}$$

It can be observed that for all a's that satisfy (23), we only need to check the restriction of (22) for $a \in [0,\pi]$, since $\sin a/a$ is a monotonically decreasing function. For convenience denote $b \triangleq C\tau$, we have from (23)

$$b = a^{-1}\cot a \pm \sqrt{a^{-2}\cot^2 a - 2}$$

for $a \in [0,\theta] \subset [0,\pi/2]$ where $\cot\theta/\theta = 2$. Thus after some calculations (22) is the same as,

$$f'(q^*)^{-1} > \frac{\cos^2 a}{a^3 \sin a}\left(\frac{\cot a}{a} - \sqrt{\frac{\cot^2 a}{a^2} - 2}\right) \tag{24}$$

for $\tau \leq \sqrt{2}/C$ and

$$f'(q^*)^{-1} > \frac{\cos^2 a}{a^3 \sin a}\left(\frac{\cot a}{a} + \sqrt{\frac{\cot^2 a}{a^2} - 2}\right) \tag{25}$$

for $\tau \geq \sqrt{2}/C$.

The right hand side is lower bounded away from 0 for $a \in [0,\theta]$. We calculated numerically its minimum as 1.9976 and we denote the delay τ at this moment as τ_1 whose value is $\tau_1 = 0.8355/C$. Consequently there is an upper bound equal to $\beta_0 \triangleq 0.5006$ for $f'(q*)$ so that there exists stability region for τ. Therefore by properties of a with regard to τ and Theorem 13.9 in [10] we obtain the stability region described in Figure 3. So we see that if $f'(q^*) < \beta_0$, the system is stable at $\forall\tau$, $\tau \in [t_0, t_1]$ for some t_0, t_1.

It is easy to observe that there are no roots of (18) having the form $\lambda = u + i\pi$. Otherwise the imaginary part of (18) implies

$$u = -C\tau f(q^*)/2,$$

and the real part of (18) becomes

$$\begin{aligned} 0 &= u^2 - \pi^2 + C\tau f(q^*)u - \frac{f'(q^*)}{f(q^*)}e^{-u} \\ &= -\frac{1}{4}C^2\tau^2 f(q^*)^2 - \pi^2 - \frac{f'(q^*)}{f(q^*)}e^{-u} < 0.\end{aligned}$$

This is a contradiction. Since $\omega_1 < \pi$, applying Rouche's theorem, it can be shown that there are two roots in the region of $\mathrm{Re}\lambda > 0$, $-\pi < \mathrm{Im}\lambda < \pi$ for $\forall\tau > t_1$.

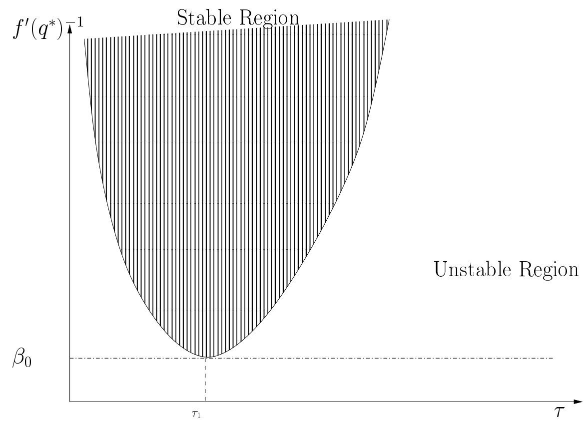

Fig. 3. Stability region of (15) with regard to τ and $f'(q^*)$

Suppose $\lambda = u + iv$ is the solution and denote $w = 1/f(q^*)$. It follows,

$$u^2 - v^2 + C\tau w^{-1}u + f'(q^*)we^{-u}\cos v = 0$$
$$2uv + C\tau w^{-1}v - f'(q^*)we^{-u}\sin v = 0.$$

Differentiating the above equations with regard to w we get

$$\begin{aligned} & (2u + C\tau w^{-1} - f'(q^*)we^{-u}\cos v)\frac{du}{dw} \\ - \quad & (2v + f'(q^*)we^{-u}\sin v)\frac{dv}{dw} \\ = \quad & \frac{C\tau u}{w^2} - f'(q^*)e^{-u}\cos v, \end{aligned}$$

and

$$\begin{aligned} & (2v + f'(q^*)we^{-u}\sin v)\frac{du}{dw} \\ + \quad & (2u + C\tau w^{-1} - f'(q^*)we^{-u}\cos v)\frac{dv}{dw} \\ = \quad & \frac{C\tau v}{w^2} + f'(q^*)e^{-u}\sin v, \end{aligned}$$

From the Implicit Function Theorem, $\lambda(\tau_0)$ is continuously differentiable with regard to w (equivalently with regard to τ) and from some calculations the sign of $\mathrm{Re}\lambda'(t_0)$ is the same as the sign of

$$\begin{aligned} & wf'(q^*)^2 + 2C\tau v^2 w^{-2} + C\tau f'(q^*)vw^{-1}\sin v \\ + \quad & 2vf'(q^*)\sin v - f'(q^*)\tau C w^{-1}\cos v. \end{aligned} \tag{26}$$

But at $\tau = t_1$, $v = \omega_1$, using the relation (16) it follows

$$f'(q^*)C\tau w^{-1}\cos v = vf'(q^*)\sin v.$$

Substituting this back to (26) we obtain $\mathrm{Re}\lambda'(t_1) > 0$ ■

Theorem 2: Equations (13) has a Hopf bifurcation at t_1, where t_1 is defined in Lemma 2.

Proof: According to [11] in order to prove Hopf bifurcation point it is sufficient to check the following two conditions,

1) The linear equations (15) has a simple purely imaginary characteristic root $\lambda_0 = i\nu_0 \neq 0$ and all characteristic roots $\lambda_j \neq \lambda_0, \bar{\lambda}_0$, satisfy $\lambda_j \neq m\lambda_0$ for any integer m.
2) $\mathrm{Re}\lambda'(\tau_0) \neq 0$.

And from Lemma 2 both conditions are automatically satisfied. ■

Theorem 2 can be interpreted as the local solution to Equations (13) when τ is closely above t_1 is periodic with period close to $2\pi/\omega_1$. See [11] Theorem 11.1.1 for exact statement.

To prove the existence of global periodic solution is equivalent to prove the invariant solution under an infinite dimensional operator. The basic idea is to use Schauder Fixed Point Theorem. To avoid trivial constant solution, certain criterion called ejectivity ([11], [12]) has to be checked.

Definition 1: Suppose X is a Banach space, U is a subset of X, and x is a given point in U. Given a map $A: \quad U \setminus \{x\} \to X$, the point $x \in U$ is said to be an *e*jective point of A if there is an open neighborhood $G \subseteq X$ of x such that for every $y \in G \cap U$, $y \neq x$, there is an integer $m = m(y)$ such that $A^m y \notin G \cap U$.

Periodic solution for infinite dimensional problems is studied by examining the mapping of a closed convex set, usually a cone, its boundedness and ejectivity under such mapping. So we list some useful tools (Theorem 3 and Theorem 4 for convenience [11].

For any $M > 0$, denote $S_M = \{x \in X : |x| = M\}$, and $B_M = \{x \in X : |x| < M\}$.

Theorem 3: If K is a closed convex set in X, $A : K \setminus \{0\} \to K$ is completely continuous, $0 \in K$ is an ejective point of A, and there is an $M > 0$ such that $Ax = \lambda x$, $x \in K \cap S_M$ implies $\lambda < 1$, then A has a fixed point in $K \cap B_M \setminus \{0\}$ if either K is infinite dimensional or 0 is an extreme point of K.

Suppose $L : C \to \mathbb{R}^n$ is linear and continuous, $f : C \to \mathbb{R}^n$ is completely continuous together with a continuous derivative f' and $f(0) = 0$, $f'(0) = 0$. Consider two equations

$$\begin{aligned} \dot{x}(t) &= Lx_t + f(x_t) \\ \dot{y}(t) &= Ly_t \end{aligned} \tag{27}$$

For any characteristic root λ of the above equation, there is a decomposition of C as $C = P_\lambda \oplus Q_\lambda$, where P_λ and Q_λ are invariant under the solution operator $T_L(t)$ of the above equation, $T_L(t)\phi = y_t(\phi)$, $\phi \in C$. Let the projection operators defined by the decomposition of C be π_λ, $I - \pi_\lambda$ with the range of π_λ equal to P_λ. Now we have the following conditions [12] to check the ejective point.

Theorem 4: Suppose the following conditions are fulfilled:

(i) There is a characteristic root λ of Equation (27) satisfying $\mathrm{Re}\lambda > 0$.

(ii) There is a closed convex set $K \subseteq C$, $0 \in K$, and $\delta > 0$, such that

$$v = v(\delta) \triangleq \inf\{|\pi_\lambda \phi| : \phi \in K, |\phi| = \delta\} > 0$$

(iii) There is a continuous function $\tau : K \setminus \{0\} \to [\alpha, \infty]$, $0 \leq \alpha$ such that the map defined by

$$A\phi = x_{\tau(\phi)}(\phi), \quad \phi \in K \setminus \{0\}$$

takes $K \setminus \{0\}$ into K and is completely continuous.

(iv) Given $G \subset C$ open, $0 \in G$, there is a neighborhood V of 0 such that $x_t(\cdot;\phi) \in G$, if $\phi \in V \cap K$, and $0 \leq t \leq \tau(\phi)$.

Then 0 is an ejective point of A.

In next two lemmas we prove the boundedness of rate $r(t)$ and queue size $q(t)$. Note that the function $f(q)$ is only a marking function, so it is not trivial to show $q(t)$ is bounded.

Lemma 3: Denote $C_1 > C$ as the solution to

$$\frac{\tau^2 C \left(C_1(1 - (C\tau)^{-1}) - C\right)^2}{2C_1} = B,$$

and C_2 as the solution to

$$\frac{C_2^2 - C^2}{2C_2}\tau = 2.$$

Let $C_M = \max\{C_1, C_2, C/(1 - (C\tau)^{-1})\}$. Then for any initial conditions, there exists a finite time t_M such that $\forall t > t_M$, $r(t)$ of Equation (13) is less than C_M.

Proof: Suppose at some time t_1, $r(t_1) = C_M$. Due to the continuity of $r(t)$, we can define

$$t_0 \triangleq \sup\{t_0 | r(t_0) \leq C, \quad t_0 < t_1\}$$

and $r(t_0) = C$. We note that from (13),

$$\frac{dr}{dt} \leq \frac{r(t-\tau)}{\tau^2 r(t)}$$

which is equivalent to

$$\frac{dr^2}{dt} \leq 2\frac{r(t-\tau)}{\tau^2}.$$

So it follows,

$$\begin{aligned} r(t)^2 - C^2 &\leq \frac{2}{\tau^2}\int_{t_0}^{t} r(s-\tau)ds \\ &\leq \frac{2}{\tau^2} C_M (t-t_0). \end{aligned}$$

Thus the time the trajectory takes from t_0 to t_1 is lower bounded by

$$t_1 - t_0 \geq \frac{C_M^2 - C^2}{2C_M}\tau^2$$

It can be easily shown that since $C_M \geq C_2$, $t - t_0 \geq 2\tau$.

In addition, we know that $\forall t \in [t_0 + \tau, t_1]$,

$$\frac{dr(t)}{dt} \leq \frac{C_M}{\tau^2 C}. \tag{28}$$

It follows that $r(t_1 - \tau) \geq C_M \left(1 - (C\tau)^{-1}\right)$. In real applications, since there are always more than one packet flowing in the network, so $C\tau >> 1$. Therefore the lower bound of $r(t_1 - \tau)$ is always positive. We want to obtain the following lower bound for the bottleneck queue size at time $t_1 - \tau$:

$$q(t_1 - \tau) \geq \int_{t_0+\tau}^{t_1-\tau} (r(s) - C)ds.$$

Since $r(t)$ is a continuous function with upper bounded first derivative (28), it turns out that the lower bound for $q(t_1 - \tau)$ is

$$\frac{\left(C_M\left(1 - \frac{1}{C\tau}\right) - C\right)^2}{2C_M/(\tau^2 C)}.$$

From our assumption we know that the above expression is no less than B and the marking function $f(B) = 1$. So we have

$$\begin{aligned} \frac{r(t_1)}{t} &= \frac{r(t_1-\tau)}{\tau^2 r(t_1)}\left(1 - (1 + \frac{\tau^2 r(t_1)^2}{2}) f(q(t_1 - \tau))\right) \\ &= -\frac{r(t_1-\tau)}{\tau^2 r(t_1)}\frac{\tau^2 r(t_1)^2}{2} \\ &< 0. \end{aligned}$$

Therefore the rate $r(t)$ can not increase beyond C_M.

The previous proof applies to the situations where $r(t) < C_M$, for $t \in [t_0 - \tau, t_0]$. In general case, from previous deduction we know that the queue size will reach B if the rate $r(t)$ exceeds C_M after rising from C. It will remain above B until the rate $r(t)$ falls below C. But there is a delay τ from the queue to the sender. Consequently the rate $r(t)$ will stay below C for the period of at least tau to return back to C. Therefore there always exists a time t_M such that $r(t_M) = C$, $r(t) < C_M, \forall t \in [t_M - \tau, t_M]$ and the upper bound applies after time t_M. ■

Lemma 4: We define q_M as

$$B + (C_M - C)\tau + 2\left(\frac{C_M}{C} - \ln\frac{C_M}{C} - 1\right)$$

where C_M is the same as in Lemma 3. Then $\forall t$, $t > t_M$, where t_M is also defined in Lemma 3, $q(t) < q_M$.

Proof: From Lemma 3 we know that $\forall t$, $t > t_M$, $r(t) < C_M$. We only consider this range of t in this proof. Suppose $q(t_0) = B$. We know that for $t > t_0 + \tau$,

$$\frac{dr(t)}{dt} = -\frac{r(t)r(t-\tau)}{2}$$

until for some t_1 such that $r(t_1) = C$. Since $r(t_0) > C$ as we showed in the previous proof, it follows,

$$\frac{dr(t)}{dt} \leq -\frac{r(t)C}{2}.$$

So,

$$r(t) \leq C_M e^{-C(t-t_0-\tau)/2},$$

and

$$t_1 - t_0 - \tau \leq \frac{2}{C} \ln \frac{C_M}{C}$$

Therefore the traffic accumulation at the bottleneck queue from $t_0 + \tau$ to t_1 is upper-bounded by

$$\begin{aligned} & \int_{t_0+\tau}^{t_1} (C_M e^{-C(t-t_0-\tau)/2} - C) dt \\ \leq \; & 2\left(\frac{C_M}{C} - \ln \frac{C_M}{C} - 1\right) \end{aligned} \tag{29}$$

The traffic accumulation from t_0 to $t_0 + \tau$ is upper-bounded by $(C_M - C)\tau$. Combine this and (29) we obtain the claim of this lemma. ∎

Consider Equations (13), let $C_0 = C([-\tau, 0], \mathbb{R}) \times \mathbb{R}$ and denote elements in C_0 by $\psi = (\phi, a)$, $\phi \in C([-\tau, 0], \mathbb{R})$, $a \in \mathbb{R}$. For any $\psi \in C_0$, Equations (13) has a unique solution $x_t(\psi)$, $x_t = (\tilde{r}_t, \tilde{q}(t - R))$, through ψ at zero. Let $K = \{\psi = (\phi, a) \in C_0 : 0 \leq a < \infty, 0 = \phi(-\tau) \geq \phi(\theta), -R \leq \theta \leq 0\}$.

Lemma 5: Suppose marking function $f(q)$ satisfies conditions (14), then there exists a continuous function $T_1 : K \setminus 0 \to (\tau, \infty)$, such that

$$x_{T_1(\psi)}(\psi) \in -K \triangleq \{-\psi : \psi \in K\}.$$

And there exists a continuous function $T_2 : -K \setminus 0 \to (2\tau, \infty)$, such that

$$x_{T_2(-\psi)}(-\psi) \in K$$

Proof: If the function T_1 exists, and from Equations (13) we see that the solution $x_t(\psi)$ is transversal to the $\tilde{q}$-axis at $t = T_1(\psi)$, so it immediately follows that $T_1(\psi)$ is continuous from the continuity of the solution with respect to the initial conditions. So we will prove the existence of the function T_1. The function T_2 can be proved in a similar way. To study the behavior of the solution of Equations (13) $x(\psi)$ for $\psi \in K$, we analyze its curve in the $(\tilde{r}, \tilde{q})$-plane. Denote

$$\Gamma = \{(r, q) \in \mathbb{R}^2 : \frac{1 - f(q)}{\tau^2} = \frac{1}{2} r^2 f(q)\}.$$

Because of the properties of $f(q)$, Γ is a curve that starts at $(-C, B - q^*)$, passes $(0, 0)$, and converges to $-q^*$ when $\tilde{r} \to \infty$ in the $(\tilde{r}, \tilde{q})$-plane.

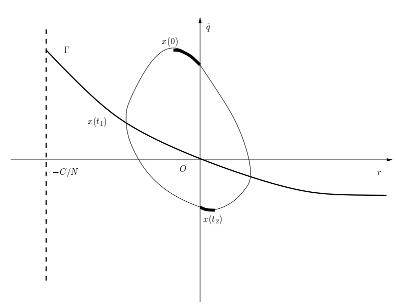

Fig. 4. Trajectory of Periodic Solution of Equations (13)

Suppose $x(0)=\psi(0)$ is above Γ. Denote $t_1 \triangleq \min\{t|x(t)\in\Gamma, t>0\}$. We will show t_1 is finite. As long as $x(t)$ remains above Γ, $\dot{\tilde{r}}(t)=r(t-\tau)r(t)^{-1}\left((1-f(q(t-\tau)))\tau^{-2}-r(t)^2 f(q(t-\tau))/2\right)<0$, and $\dot{\tilde{q}}(t-\tau)=\tilde{r}(t-\tau)<0$. If $x(t)$ never intersects Γ, we have $\dot{\tilde{q}}(t-\tau)=\tilde{r}(t-\tau)<\tilde{r}(0)<0, \forall t>\tau$. This is obviously not possible. So it follows that t_1 is finite. Also since $\dot{\tilde{r}}(t_1)=0$, and $\dot{\tilde{q}}(t_1-\tau)=\tilde{r}(t_1-\tau)<0$, so $t_1>\tau$.

For $t>t_1$, and $\tilde{r}(t)<0$, we have $\dot{\tilde{r}}(t)>0$ and $\dot{\tilde{q}}(t-\tau)<0$. If the trajectory of x_t intersects Γ before it intersects the $\tilde{q}$-axis, we know that at the intersection the trajectory must be vertical to the $\tilde{r}$-axis, so at that instance $\dot{\tilde{q}}>0$, which is contradictory. x_t cannot passes $(0,0)$ since local stability condition forbids this. So x_t does not cross the $\tilde{q}$-axis in the second quadrant of $(\tilde{r},\tilde{q})$-plane.

Next we show x_t will hit $\tilde{r}$-axis in finite time. If not so, thanks to the monotonicity of $\tilde{r}(t)$ and $\tilde{q}(t)$ as well as the boundedness of the third quadrant, x_t will converge to a certain $(\hat{r},\hat{q})$. That means the system (13) has another equilibrium state other than $(0,0)$ in the $(\tilde{r},\tilde{q})$-plane, which is impossible. So $\tilde{r}(t)$ will become positive in finite time. We denote $t_2 \triangleq \min\{t|\tilde{r}(t)=0, t>t_1\}$ and $T_1(\psi)=t_2+\tau$, then $T_1(\psi)$ is continuous and $x_{T_1(\psi)}(\psi)\in -K$.

Similarly denote $t_3 \triangleq \min\{t|x(t)\in\Gamma, t>t_2\}$ and we can show that t_3 is finite. This is because $d\tilde{q}/dt$ is strictly positive for $t\in[t_2+\tau, t_3]$. Use the same reasoning as before it can be shown that x_t can not first cross the $\tilde{q}$-axis in the fourth quadrant for $t>t_3$. It is left to be shown that x_t reaches $\tilde{q}$-axis in the first quadrant in finite time. Here we can utilize the actual boundedness of $\tilde{q}$ shown in Lemma 4 and monotonicities of $\tilde{r}$ and $\tilde{q}$ in the first quadrant to prove this. Therefore there exits a finite t_4 such that $x(t_4)$ belongs to $\tilde{q}$-axis and $T_2(\psi)\triangleq t_4+\tau$ is continuous and $x_{T_2(\psi)}(\psi)\in K$. ■

For any $\psi\in K\setminus\{0\}$, define $A: K\setminus\{0\}\to K$ by

$$A\psi = x_{T_1(\psi)+T_2(x_{T_1(\psi)(\psi)})}(\psi).$$

We need to prove the Condition (iv) of Theorem 4. We only show the mapping T_1 here, and due to symmetry, the proof of T_2 is the same.

Lemma 6: If G is a given open subset of $\mathbb{R}^2$, $0\in G$, there exists a neighborhood V of 0 in C such that $x_t(\cdot;\psi)\in G$, for any $\psi\in V\cap K$, $\psi\neq 0$, and any t, $0\le t\le T_1(\psi)$.

Proof: Because the vector field of (13) is continuous, and $(0,0)$ is the equilibrium point of the equations, we only need to prove that $x_t(\cdot;\psi)\in G$, for $0\le t\le t_2(\psi)$ as $t_2(\psi)$ defined in Lemma 5 ($\tilde{r}(t_2)=0$). Suppose the claim of this lemma is not true, we will later show a contradiction. Assume there exists a sequence $\psi_n\in K\setminus\{0\}$, $n=1,2,\ldots$, $\psi\to 0$, as $n\to\infty$, such that there is a τ_n, $0<\tau_n<t_2(\psi)+\tau$, $|\tilde{r}(\tau_n)|=M$, or $|\tilde{q}(\tau_n)|=N$, for some given positive M,N.

From smoothness of Γ, we know for any small ϵ, there is a constant α, such that $\max\{|\tilde{r}(t)|, 0\le t\le t_2(\psi)\} = |\tilde{r}(t_1)|\le\alpha\tilde{q}(0)$, for any $\tilde{q}(0)<\epsilon$. Then consider the period from the trajectory crossing the $\tilde{r}$-axis to t_2, we have

$$\begin{aligned}
\frac{d\tilde{r}}{dt} &= \frac{r(t-\tau)}{\tau^2 r(t)}\left(1-(1+\tau^2 r(t)^2/2)f(q(t-\tau))\right)\\
&\geq \frac{1-\delta}{\tau^2}\left(-\tau^2 Cf(q^*)\tilde{r}-\frac{\beta}{f(q^*)}\tilde{q}(t-\tau)\right)\\
&\geq -Cf(q^*)(1-\delta)\tilde{r}
\end{aligned}$$

for some positive constant $\delta<1$, and β. The last inequality is due to the negativity of $\tilde{q}(t-\tau)$ in the region we consider. Therefore, we get the upper bound of $|\tilde{q}(t_2-\tau)|$ as

$$\int_0^\infty \alpha\tilde{q}(0)e^{-Cf(q^*)t}dt = \frac{\alpha\tilde{q}(0)}{Cf(q^*)(1-\delta)}.$$

So the upper bound of $|q(t_2)|$ is just $\alpha(\tau+(Cf(q^*)(1-\delta))^{-1})\tilde{q}(0)$ for small enough $\tilde{q}(0)$. Using this we get after some calculations for $t_2\le t\le t_2+\tau$

$$\frac{d\tilde{r}}{dt}\le\frac{\gamma\alpha(\tau+(Cf(q^*)(1-\delta))^{-1})}{\tau^2 f(q^*)}\tilde{q}(0)$$

for some positive constant γ. So we can give an upper bound of $|\tilde{r}(t_2+\tau)|$ as

$$\frac{\gamma\alpha(\tau+(Cf(q^*)(1-\delta))^{-1})}{\tau f(q^*)}\tilde{q}(0).$$

Therefore, we proved for sufficiently small $\tilde{q}(0)$, for any $t \in [0, T_1(\psi)]$, $|x_t(\cdot;\psi)|$ never hits the boundary of $\{(x,y) : x \in [0,M], \quad y \in [0,N]\}$, for any positive M, N. That is equivalent to the argument of this lemma. ■

Lemma 7: For any $\tau > t_1$ and positive roots λ defined in Lemma 2, $\inf\{|\pi_\lambda \phi| : \phi \in K, |\phi| = \delta\} > 0$ for any $\delta > 0$.

Proof: It follows from Lemma 2 that there is a characteristic root λ with positive real part when $\tau > \tau_0$. Then the transposed equations of Equations (15) are

$$\begin{aligned}\frac{du(t)}{dt} &= \tau C f(q^*) u(t) - \tau v(t) \\ \frac{dv(t)}{dt} &= \frac{f'(q^*)}{\tau f(q^*)} u(t+1),\end{aligned}$$

and the bilinear form is

$$< \zeta, \psi >= b\phi(0) + \xi(0)a - \frac{f'(q^*)}{\tau f(q^*)} \int_{-1}^{0} \xi(\theta+1)\phi(\theta) d\theta$$

where $\zeta = (\xi, b)$, $b \in \mathbb{R}$, $\xi \in C([0,1], \mathbb{R})$ and $\psi = (a, \phi)$, $\phi \in C([-1,0], \mathbb{R})$, $a \in \mathbb{R}$. Since for any $\phi \in C$, $\pi_\lambda \phi = \Psi_\lambda < \Xi_\lambda, \phi >$ where Ψ_λ and Ξ_λ are the bases of generalized eigenspace of Equations (15) and its adjoint respectively. So it is sufficient to check $< \Xi_\lambda, \phi >$. It can be shown that the basis of the solutions to the transposed equations for projection π is

$$\zeta_i(s) = (-e^{-\lambda_i s}, -(Cf(q^*) + \lambda_i \tau^{-1}) e^{-\lambda_i s}), \quad i = 1, 2.$$

Consider $\lambda = \mu + i\nu$, where as proved in Lemma 2, $\mu > 0$ and $0 \leq \nu < \pi$. The real and imaginary parts of $< \zeta, \psi >$ are

$$\begin{aligned}\mathrm{Re} < \zeta, \psi > &= -(Cf(q^*) + \mu\tau^{-1})\phi(0) - a \\ &+ \frac{f'(q^*)}{\tau f(q^*)} \int_{-1}^{0} e^{-\mu(\theta+1)} \cos\nu(\theta+1)\phi(\theta) d\theta \\ \mathrm{Im} < \zeta, \psi > &= -\nu\tau^{-1}\phi(0) \\ &+ \frac{f'(q^*)}{\tau f(q^*)} \int_{-1}^{0} e^{-\mu(\theta+1)} \sin\nu(\theta+1)\phi(\theta) d\theta.\end{aligned}$$

If there is a sequence $\psi_n = (\phi_n, a_n) \in \partial B(1) \cap K$ such that $\pi_\lambda \psi_n \to \infty$ as $n \to \infty$, then $\mathrm{Im} < \zeta, \psi_n > \to 0$. But from the form of $\mathrm{Im} < \zeta, \psi_n >$ we know this is true only when $|\phi_n(\dot{})| \to 0$. This together with $\mathrm{Re} < \zeta, \psi_n > \to 0$ indicates $a_n \to 0$ as $n \to \infty$. So $\psi_n \to 0$ as $n \to \infty$ which contradicts the assumption. Therefore the second claim also holds. ■

From Lemma 3 and 4, and based on the fact that the solution continuously depends on initial conditions, we get the map $A : K \setminus \{0\} \to K$ defined above is completely continuous. Also it is obvious from boundedness of the solution that there is a constant $M > 0$, such that if $A\psi = \nu\psi$, $\psi \in K \setminus \{0\}$, $|\psi| = M$, then $\nu < 1$. Together with the results in Lemma 5, 6, and 7, applying Theorems 3 and 4, we conclude

Theorem 5: If marking function $f(q)$ satisfies Conditions (14) and $\tau > t_1$ where t_1 is given in Lemma 2, Equation (13) has a non-constant periodic solution.

ACKNOWLEDGEMENT

Effort sponsored by the Defense Advanced Research Projects Agency (DARPA) under contract number N66001-00-C-8063. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright annotation theron.

DISCLAIMER

References

[1] A. Misra, J. Baras, and T. Ott, "Generalized TCP congestion avoidance and its effect on bandwidth sharing and variability," in *IEEE GLOBCOM 2000, Communications: The Global Bridge*, San Francisco, CA, Nov. 27-Dec. 1 2000, pp. 329–337.

[2] ——, "Predicting bottleneck bandwidth sharing by generilized tcp flows," *Computer Networks Journal*, vol. 40, no. 4, pp. 557–576, Nov. 2002.

[3] S. Deb and R. Srikant, "Global stability of congestion controllers for the internet," University of Illinois, Urbana, IL, Tech. Rep., Feb. 2002.

[4] C. Hollot and Y. Chait, "Nonlinear stability analysis for a class of tcp/aqm networks," in *Proceedings of the IEEE Conference on Decision and Control*, Orlando, FL, Dec. 2001.

[5] S. Low, F. Paganini, and J. Doyle, "Internet congestion control," *IEEE Control Systems Magazine*, vol. 22, no. 1, pp. 28–43, 2002.

[6] Z. Wang and F. Paganini, "Global stability with time-delay in network congestion control," in *Proceedings of the IEEE Conference on Decision and Control*, Les Vegas, NV, Dec. 2002.

[7] J. Wen and M. Arcak, "A unifying passivity framework for network flow control," in *Proceedings of IEEE INFOCOM 2003*, San Francisco, CA, Apr. 2003.

[8] X. Fan, M. Arcak, and J. Wen, "Robustness of network flow control against disturbances and time-delay," *accepted for System and Control Letters*, 2003.

[9] W. Huang, "Generalization of Liapunov's theorem in a linear delay system," *Journal of Mathematical Analysis and Applications*, vol. 142, pp. 83–94, 1989.

[10] R. Bellman and K. Cooke, *Differential Difference Equations*. Academic Press, 1963.

[11] J. Hale and S. V. Lunel, *Introduction to Functional Differential Equations*. Springer-Verlag New York, 1993.

[12] P. Táboas, "Periodic solutions of a planar delay equation," *Proceedings of the Royal Society of Edinburgh*, vol. 116A, pp. 85–101, 1990.