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Feedback Stabilization via Center Manifold Reduction with Application to Tethered Satellites

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ABSTRACT

Title of Dissertation: Feedback Stabilization via Center Manifold Reduction with Application to Tethered Satellites

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Center manifold reduction has recently been introduced as a tool for design of stabilizing control laws for nonlinear systems in critical cases. In this dissertation, the center manifold approach is elaborated for general such nonlinear systems in several critical cases of interest, and the results are applied to the control of tethered satellite systems (TSS). In addition, to address stability questions for satellite deployment via TSS, we obtain new results in finite-time stability theory.

The critical cases considered in the general feedback stabilization studies include the cases in which the system linearization possesses a simple zero eigenvalue (of multiplicity one or two), a pair of simple pure imaginary eigenvalues, one zero eigenvalues along with a pair of simple pure imaginary eigenvalues, and two pairs of simple pure imaginary eigenvalues. The calculations involve center manifold reduction, normal form transformations, and Liapunov function construction for critical systems. These calculations are explicit.

The tethered satellite systems considered here consist of a satellite and subsatellite connected by a tether, in orbit around the Earth. The Lagrangian formulation of dynamics is used to obtain a nonlinear system of ordinary differential equations for TSS dynamics. For simplicity, a rigid, massless tether is assumed. Linear analysis reveals the presence of critical eigenvalues in the station-keeping mode of operation. This renders useful results on stabilization in critical cases to this application. The control variable assumed is tether

tension feedback. Besides the design of stabilizing station-keeping controllers, stability of deployment and instability of retrieval are also shown for a constant angle deployment/retrieval scheme.

FEEDBACK STABILIZATION VIA CENTER MANIFOLD REDUCTION WITH APPLICATION TO TETHERED SATELLITES

by

Der-Cherng Liaw

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DEDICATION

To my parents

and my brothers and sisters

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CHAPTER ONE

INTRODUCTION

Stability analysis and stabilization for nonlinear autonomous systems are subjects which have been studied extensively. Many publications have appeared: Some of them emphasize the development of new control theories for general systems [6], [15]-[19], [23], [25]-[26], [29]-[34], [44], [45], [53]-[55], [60]-[62], and some concern practical applications, for instance, [5], [8], [9], [12]-[14], [20]-[22], [24], [37]-[39], [46]-[50], [66]-[73]. But until now, not many papers have been published in the study of the stability and stabilization of critical systems, wherein the system's Jacobian matrix possesses eigenvalues lying on the imaginary axis. Several approaches have been used to study such systems. One involves an application of bifurcation theorems [1], [2], [25], [34], and another is geometric in nature and uses center manifold reduction [4], [10], [51], [55]. Other, often less constructive, techniques have also been used; see the survey papers of Bacciotti and Boieri [7], and Sontag [83] for details and further references. In this dissertation, we extend existing results in the geometric approach to the study of stability and stabilization of general critical nonlinear autonomous systems. These results are then employed to design stabilizing control laws for a Tethered Satellite System (TSS) during station-keeping. Additionally, we extend known results on the so-called finite time stability, and use these results to study stability of the TSS during constant in-plane angle deployment and retrieval.

1.1. Motivation

In general, the linearization approach is a very popular and powerful method being used to study the local stability properties as well as the locally stabilizing control design for smooth, nonlinear autonomous systems. It is known (e.g., [17], [36]) that if the linearized system at an equilibrium has all its eigenvalues in the open left-half of the complex plane, then the nonlinear system is asymptotically stable. If, on the other hand, one of these eigenvalues has positive real part the system is unstable. In the critical cases, where some of the eigenvalues have zero real parts while the rest lie in the open-left-half plane, it is known that stability is not determined by the linearization.

It is known that the local stability of smooth, nonlinear autonomous system is implied by the asymptotic stability (or the instability) of its linearized model. For the critical cases, the results will not be as direct. This might involve using the results of the bifurcation theorems [1], [2], [25], [34], especially, when a system has only simple critical eigenvalues, i.e., one zero eigenvalue or a pair of pure nonzero imaginary eigenvalues. An example of such a situation can be found in Section 2.4. Thus, by using the technique of linearization with linear stability criteria and bifurcation theorems, the local stability of smooth, nonlinear systems might be possible to determine. This approach is used in this dissertation to investigate the stabilization of a tethered satellite system during station-keeping mode.

The other approach studying critical systems is to use a geometric method for constructing the stability conditions of the overall system from an auxiliary system, namely the reduced model, by employing the center manifold theorem (e.g., [4], [10], [18], [19], [29], [32]). The application of this method has been extended to many areas, for instance, the stabilization of two time-scale nonlinear systems [60]-[61], [82].

Among those using center manifold reduction, Aeyels [4] investigated the existence of smooth stabilizing feedback control laws for a class of third order nonlinear systems for which the linearization possesses an uncontrollable pair of pure imaginary eigenvalues. Behtash and Sastry [10] considered the stabilization for critical nonlinear systems whose linearization possesses a scalar stable mode, along with a double zero eigenvalue, two distinct complex conjugate pairs of pure imaginary eigenvalues, or a zero eigenvalue along with a pair of imaginary eigenvalues. Unfortunately, there does not currently exist an analogous understanding for more general nonlinear critical systems. For instance, it is clearly important to allow any finite number of stable modes. Also, calculations given directly in terms of the original higher order model are clearly desirable.

A main goal of this dissertation is to derive such stabilizing control algorithms for general nonlinear systems in critical cases. Previous results for simple critical systems [4] and the double critical systems [10] are extended to more general high dimensional systems. Moreover, the stability conditions and stabilizing control laws obtained here are stated explicitly in terms of the original system dynamics.

A convenient assumption for using center manifold reduction is that there exist two groups of system states with linearly decoupled dynamics. To employ this reduction technique in constructing the stabilizing controllers for general nonlinear systems, we observe that linear feedbacks will change the structure of the linearized model of system dynamics. Obtaining a change of coordinates facilitating the use of the center manifold theorem for such problems is analyzed in this thesis. This is useful in the design of linear and linear-plus-nonlinear stabilizing control laws for critical systems.

In constructing Liapunov functions for critical nonlinear systems, Fu and Abed [26] have obtained results for the simple critical cases by using an asymptotic approach. Analogous results for general critical nonlinear systems do not currently exist. One goal of this thesis is to alleviate this deficiency.

In this thesis, we are strongly concerned with the applications of the existing stability criteria and stabilization techniques to the study of the behavior of the TSS, where analytical results are few. Based on a derived rigid-body model of the TSS, the bifurcation theorems and the new geometric results are applied to obtain criteria for stability and stabilization during station-keeping. In addition, Liapunov-like finite-time stability criteria are proposed and considered in the context of studying the behavior of the TSS during constant angle deployment and retrieval.

1.2. Introduction to Tethered Satellite Systems (TSS)

The topic of TSS has received considerable attention in recent years (e.g., [5], [8], [9], [12], [20]-[22], [24], [37]-[39], [66]-[73]). The basic TSS configuration consists of a satellite and a subsatellite connected by a tether, in orbit around the Earth (see Figure 1.1). Potential TSS applications include deployment and retrieval of satellites, aiding in space-assembly tasks, use of electrodynamic tethers for electric power generation [74, p. 4-259], and tethering platforms with an infrared telescope above the Space Station for observing stellar and planetary objects [74, p. 4-263]. Other potential applications [81] include low altitude scientific applications (such as gravity and magnetic field mapping, Earth surveillance, plasma physics and pollutant measurement), release of artificial meteors, study of Earth's magnetic field, cargo transfer and disturbance avoidance for payload deployment. Control problems associated with satellite tethering which are of particular concern in this thesis concern stabilization of the TSS during the deployment, retrieval and station-keeping modes of operation.

The basic structure of the Tethered Satellite System is as shown in Figure 1.2 [81]. The main body ("satellite") of this configuration can be a Shuttle or a large satellite and the tethered object ("subsatellite") at the far end of the tether might be an experimental laboratory or a small satellite. The TSS should be capable of deploying the subsatellite either upward away from the Earth or downward toward the Earth. In [81], a reel mechanism (see Figure 1.2) is proposed to provide control of the tension force along the tether. Other options for control, besides direct control of the tether tension, include momentum-type controllers [46] and thrusters [38], [89].

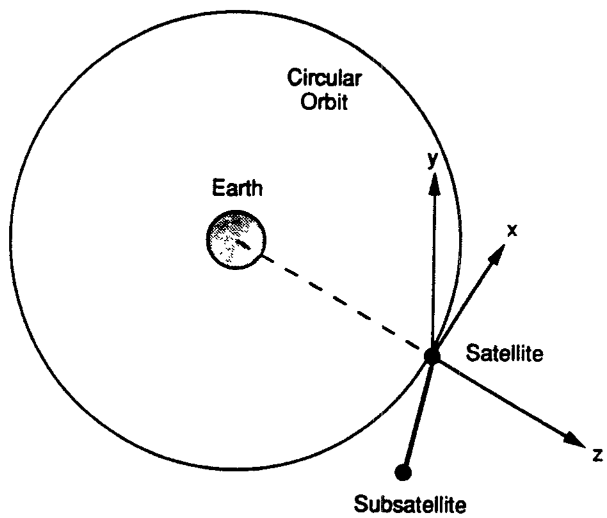

Figure 1.1. Tethered Satellite System in orbit

A variety of mathematical models for the TSS have been introduced and studied through analysis and simulation [5], [8], [24], [37], [57], [66]-[71], [85]. These models are based on assumptions on the mass and configuration of each element (satellite, subsatellite and tether), the flexibility and elasticity of the tether, orbit eccentricity, aero dynamic drag, electromagnetic forces, gravitational forces, thermal or solar radiation, and control techniques. In addition, as noted by Misra and Modi [66], several other factors should be considered in modeling TSS dynamics. These are longitudinal vibration of the tether, longitudinal strain variation along the tether, transverse vibration of the tether, torsional stiffness of the tether, rotational motion of the end masses, offset of the point of attachment at the satellite, and effects of a rotating atmosphere (in low altitude applications).

Several previous investigators consider simple cases and obtained lumpedparameter models [5], [47]-[50], while others propose more complicated models [66]-[71]. For instance, Eades and Wolf [24] obtain the relative motion equations and discuss the determination factors of the initial ejection velocity and the required techniques (constant tension force) to cause the trajectory to pass a desired spot. Arnold [5] proposes an approximate model and uses it to discuss the libration of the system through the gravity gradient method.

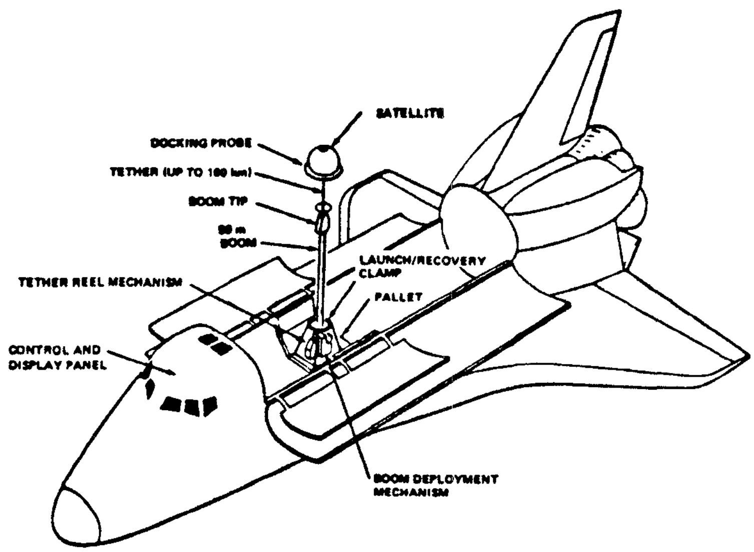

Figure 1.2. Basic structure of the Tethered Satellite System

As mentioned above, the major modes of operation of the TSS are payload deployment, payload retrieval and station-keeping. Among the possible control techniques for these three basic functions, a tension control method [80] and a constant in-plane angle method [5] were proposed for satellite deployment and retrieval. It was asserted in [5], [80] that constant angle retrieval of the TSS is inherently unstable. Several techniques have been proposed to overcome this unacceptable behavior. Kane [39] proposed that the tether be used as a guideline, wherein a dummy subsatellite serves as a pseudo-end object, with the true subsatellite "crawling" along the tether during retrieval. Kane and

Banerjee [38] proposed the use of a built-in thruster in the subsatellite, in addition to the tension control force. For station-keeping, Perrine [76] suggested to repeatedly let the tether be taut only at some discrete time instants and let it be slack for the remainder of the time.

Attitude control of tethered objects was studied by Lemke, Powell and He [46], who proposed using a moving attach point for attitude adjustment. Linearized controllability and observability of a tethered platform system during station-keeping motion were studied by Bainum, Woodard and Juang [8], where control input contains the tension control force and a momentum-type control device.

In [8], a minimum energy control law was also proposed for station-keeping motion by using a linear quadratic regulator design. Colombo and Arnold [20] discussed the anticipated orbit and speed of the subsatellite after release from the tether. In [21], Colombo considered use of a special reel mechanism to achieve a stable release motion of the system without significant loss of tension force. In [20], limitations on the tension force imposed by the configuration of the tether were studied and a tapered type tether was proposed for wider application.

A security problem might also arise during the operation of the system. If the tether breaks during any of the operating modes, then serious damage can result. Since the breakage of the tether will effectively change the payload, the tether itself might be forced back and hit the main satellite body. Moreover, the motion of the "lost" payload might also block the motion of the system and hit the main satellite body. Beramaschi [12] noted that parts of the tether would be able to reach the shuttle's altitude if this breakage occurred sufficiently close to the orbiter with the tether slackening after breakage. He also proposed either increasing the cross section of tether's terminal section or connecting the terminal section to the remaining part of the tether by a damper as ways of reducing the satellite safety problem. This study considered only the case of station-keeping and the safety of the satellite.

Many applications of the TSS have been proposed by Rupp and Laue [81]. In addition, Maunel and Gavit [63] considered an application in orbit modification by using forced tether length variations, and Lorenzini [58] considered a micro-g/variable-g application. Moreover, two-shuttle/two-tether systems or even multi-shuttle/multi-tether systems can be considered [67].

1.3. Outline

The development of this dissertation is as follows. In Chapter 2, we collect some basic results on nonlinear systems of ordinary differential equations, which will be referenced in the sequel. First, the definition of invariant and locally invariant manifold are given. This is followed by a discussion of the center manifold theorem. A convenient assumption in applying the center manifold theorem is that the system state variables separate into two groups, for which the linearized system dynamics are decoupled. A generally applicable linear transformation is introduced to facilitate systematic application of the center manifold theorem to linear feedback stabilization of critical nonlinear systems even when this assumption does not hold. This is followed by a summary of the definitions and some properties of multilinear functions, and the technique of normal form reduction. In Section 2.4, we review basic behaviors of one parameter families of nonlinear systems and simple bifurcation theorems, for cases in which the system Jacobian at a critical parameter value possesses one simple zero or a pair of simple pure imaginary eigenvalues. Definitions and results related to the so-called "finite time stability" are also given in the last section of Chapter 2.

Based on the existence theorems for the locally invariant manifold given in Section 2.1, composite-type linear and/or nonlinear controllers are proposed in Chapter 3 for stabilization of nonlinear systems in critical cases. Designs for feedback stabilizing controllers for the simple critical (SC) and the compound critical (CC) systems are proposed in Chapters 3 and 4, respectively. The simple critical cases (SC) occur when the linearized model has one simple zero or a pair of simple pure imaginary eigenvalues, while for the compound critical cases (CC) of interest here, the linearized model has two zero eigenvalues with geometric multiplicity one, one zero and a pair of simple pure imaginary eigenvalues, or two pairs of simple pure imaginary eigenvalues. In this design, the linear feedback is first constrained to ensure existence of a locally attracting invariant manifold. The remaining freedom in the controller is then employed to guarantee stability of the reduced model.

Families of Liapunov functions for critical systems are constructed in Chapter 5. The Center Manifold Theorem is employed in the development. Stability conditions for the simple critical systems (SC) and the compound critical systems (CC) are rederived by using these Liapunov functions.

In Chapter 6, tension control laws are designed guaranteeing asymptotic stability of the TSS during station-keeping. After deriving a set of dynamic equations governing the TSS dynamics, stabilizing tension control laws in feedback form are derived. The tether is assumed rigid and massless, and the equations of motion are derived using the system Lagrangian. It is observed that, to stabilize the system using tension control, tools from stability analysis of critical nonlinear systems must be applied. The results employs calculations related to the Hopf Bifurcation Theorem (recalled in Section 2.4). It is found that linear stabilizing feedback control laws exist. Simulations illustrate the nature of the conclusions, and demonstrate that nonlinear terms in the feedback can be used to significantly improve the transient response. The results given in Chapter 6 are found to be obtained by using the center manifold reduction technique proposed in Chapter 3.

In order to improve the transient responses of the TSS given in Chapter 6 without using high gains, a different technique is proposed in Chapter 7 for station-keeping control. In this approach, the linear feedback is first constrained to preserve the two distinct pairs of nonzero pure imaginary eigenvalues of the uncontrolled model of the TSS. The remaining freedom in the controller is then designed to provide the stability of the system by invoking the stability criterion as in Section 4.2.3 for a class of fourth-order critical systems. Simulations indicate that the nonlinear stabilizing feedback control law will improve the transient response significantly and the performance of the transient responses by using the new approach is better than those given in Chapter 6.

Based on the rigid model of TSS obtained in Chapter 6, a constant angle control method is hypothesized for subsatellite deployment and retrieval in Chapter 8. It is proved that this control law results in stable deployment but unstable retrieval. An enhanced control law for deployment is also proposed, which entails use of the constant angle method followed by a station-keeping control law once the tether length is sufficiently near the desired value. Simulations are given to illustrate the conclusions.

Finally, a summary of this dissertation and suggestions for the future study are given in Chapter 9.

Notation

$\sigma(\cdot)$ - Eigenvalue

$Re\{\cdot\}$ - Real part

$Im\{\cdot\}$ - Imaginary part

D, D_η, D_ξ - Differentiation operator, partial differentiation operator with respect to η and partial differentiation operator with respect to ξ

$\varphi_{ij}, \varphi_{ijk}$ - Coefficients of the quadratic terms ij and the cubic terms ijk of function φ, respectively, when $\varphi \in \{f, g, r, s, u, G\}$ and $i, j, k \in \{x, y, z, w\}$.

I, I_n, I_m - Identity matrices.

$O(\cdot)$ - High order terms of Taylor series expansion

prime denotes the transpose of vector and matrix

CHAPTER

TWO

MATHEMATICAL PRELIMINARIES

In this chapter, we collect some basic results on nonlinear systems of ordinary differential equations which will be employed in the remainder of this dissertation. The definitions of invariant and locally invariant manifold are recalled first, along with the Center Manifold Theorem. Next, definitions and properties associated with multilinear functions are recalled. This is followed by a description of the technique of normal form reduction. Results on generic simple bifurcations of equilibria of one-parameter families are given next. Finally, concepts and results on the so-called "finite time stability" are summarized.

2.1. Center Manifold Reduction

Consider the class of nonlinear autonomous systems

$$\dot{\eta} = A_{11}\eta + A_{12}\xi + F(\eta, \xi) \tag{2.1a}$$

$$\dot{\xi} = A_{21}\eta + A_{22}\xi + G(\eta, \xi), \tag{2.1b}$$

where $\eta \in I\!R^n$, $\xi \in I\!R^m$. In (2.1), A_{ij} for $i, j = 1, 2$ are constant matrices, and the functions F, G are sufficiently smooth, with their values and first derivatives vanishing at the origin. Let $\chi := (\eta', \xi')'$. Denote by $\chi(t, \chi_0)$ the solution to (2.1) at time t satisfying initial condition χ_0 at time t_0.

Definition 2.1. A manifold $\mathcal{D} \subset I\!R^{n+m}$, $0 \in \mathcal{D}$, is a *locally invariant manifold* for (2.1) if for each $\chi_0 \in \mathcal{D}$ with $||\chi_0||$ sufficiently small, there is a $\mathcal{T} > 0$ such that $\chi(t, \chi_0) \in \mathcal{D}$ for $|t| < \mathcal{T}$. Moreover, if this holds for any $\chi_0 \in \mathcal{D}$ with $\mathcal{T} = \infty$, then $\mathcal{D}$ is said to be an *invariant manifold* for (2.1).

Existence conditions for and some properties of a special locally invariant manifold, the so-called "center manifold," are given in the next theorem ([16], [18], [19], [32]).

Theorem 2.1. Let $A_{12} = 0$ and $A_{21} = 0$. If $Re\{\sigma(A_{22})\} < 0$ and $Re\{\sigma(A_{11})\} = 0$, then there exists a $\delta > 0$ and a locally invariant manifold for (2.1) given by the graph of a C^2 function $\xi = h(\eta)$, $||\eta|| < \delta$, where the function h satisfies

$$Dh(\eta) \cdot \{A_{11}\eta + F(\eta, h(\eta))\} = A_{22}h(\eta) + G(\eta, h(\eta)) \tag{2.2}$$

with $h(0) = 0$ and $Dh(0) = 0$. Moreover, the stability of the origin for (2.1) coincides with the stability of the origin for the reduced model (2.1a), with ξ replaced by $h(\eta)$.

Suppose $A_{12} = 0$, $A_{21} = 0$ and introduce the operator

$$\mathcal{N}(h(\eta)) = Dh(\eta) \cdot \{A_{11}\eta + F(\eta, h(\eta))\} - A_{22}h(\eta) - G(\eta, h(\eta)) \tag{2.3}$$

on the class of smooth functions h with $h(0) = 0$ and $Dh(0) = 0$. Clearly, $\mathcal{N}(h(\eta)) = 0$ precisely when h solves Eq. (2.2). In most cases, h cannot be solved for exactly. In this context, we note that although center manifolds are not necessarily unique, they are unique to finite order [16]. A well known result useful in constructing an approximate solution for h is recalled next.

Theorem 2.2. (Carr [16], Henry [32]). Let $\psi : I\!R^m \to I\!R^n$ be a C^1 mapping with $\psi(0) = 0$ and $D\psi(0) = 0$. If $\mathcal{N}(\psi(\eta)) = O(||\eta||^\gamma)$ for some $\gamma > 1$, then any h solving (2.2) satisfies

$$h(\eta) = \psi(\eta) + O(||\eta||^\gamma). \tag{2.4}$$

The following extension of Theorem 2.1 appears in [6].

Theorem 2.3. Let $A_{12} = 0$ and $A_{21} = 0$. If there is a $\beta \geq 0$ such that $Re\{\sigma(A_{22})\} < -\beta$ and $Re\{\sigma(A_{11})\} \geq -\beta$, then there exists $\delta > 0$ and a locally invariant manifold for (2.1) given by the graph of a C^2 function $\xi = h(\eta)$, $||\eta|| < \delta$.

Since (2.1) is an autonomous (i.e., time-invariant) system, reversing the sense of time yields the following result.

Corollary 2.1. Let $A_{12} = 0$ and $A_{21} = 0$. If there is a $\beta \geq 0$ such that $Re\{\sigma(A_{22})\} > \beta$ and $Re\{\sigma(A_{11})\} \leq \beta$, then there exists $\delta > 0$ and a locally invariant manifold for (2.1) given by the graph of a C^2 function $\xi = h(\eta)$, $||\eta|| < \delta$.

In Theorems 2.1 and 2.3, a convenient assumption in ascertaining existence of a locally invariant manifold is that the linearized dynamics in the variables η and ξ (as given in (2.1)) are decoupled. A linear transformation is introduced in the following discussions to facilitate application of the results above to general nonlinear systems, for which the linear decoupling does not apply.

First, recall the following matrix equality (e.g., [36]).

Equality 2.1. Let A and D be square matrices, with A nonsingular. Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B). \tag{2.5}$$

Next, we use this identity to show that the stability of a smooth nonlinear system is preserved under a specific linear transformation defined below, which facilitates application of the center manifold theorem to cases in which A_{12} and A_{21} do not vanish.

Consider a general nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2), \tag{2.6a}$$

$$\dot{x}_2 = f_2(x_1, x_2), \tag{2.6b}$$

where $x_1 \in I\!R^n$, $x_2 \in I\!R^m$, and f_1, f_2 are smooth vector functions.

Let $z_1 := x_1 - Px_2$ and $z_2 := x_2 - Ez_1$, where P and E are constant matrices. System (2.6) can then be rewritten in the form

$$\dot{z}_1 = \hat{f}_1(z_1, z_2), \tag{2.7a}$$

$$\dot{z}_2 = \hat{f}_2(z_1, z_2). \tag{2.7b}$$

It is implied by the inverse function theorem and Lemma 2.1 below that the local stability of the origin is preserved under the linear coordinate transformation defined above.

Theorem 2.4. (An Inverse Function Theorem) Let $\mathcal{D}$ be an open subset of $I\!R^n$ and $F : \mathcal{D} \to I\!R^n$ be C^1. Suppose the Jacobian matrix $DF(\eta_1)$ is invertible for some $\eta_1 \in \mathcal{D}$. Then there exists an $r > 0$ and an open subset $\mathcal{D}_1$ of $\mathcal{D}$ containing η_1 such that $F : \mathcal{D}_1 \to B_r(F(\eta_1))$ is invertible, where $B_r(F(\eta_1))$ denotes the open ball centered at $F(\eta_1)$ and of radius r. Moreover, the inverse mapping is also C^1.

■

By using Theorem 2.4, it is easy to have following result.

Lemma 2.1. The origin of (2.7) is asymptotically stable if and only if the origin of (2.6) is asymptotically stable.

Proof: We have

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} I_n & -P \\ -E & I_m + EP \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.8}$$

where I_n and I_m denote identity matrices of dimension of n and m, respectively. From Equality 2.1, we have

$$\det \begin{pmatrix} I_n & -P \\ -E & I_m + EP \end{pmatrix}$$

$$= \det(I_n) \cdot \det[I_m + EP - (-E) \cdot I \cdot (-P)] = 1. \tag{2.9}$$

The conclusion now follows from (2.8) and Theorem 2.4.

■

In our study of systems for which the linearized decoupling assumption does not hold, we shall encounter equations of the form (2.10) below. We now proceed to study this linear matrix equation. Consider

$$AM + MB = C, \tag{2.10}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $M, C \in \mathbb{C}^{m\times n}$. For $n = m$ and $B = A'$, Eq. (2.10) is a Liapunov matrix equation [17].

Let $\mathcal{F}$ denote the linear operator

$$\mathcal{F} : M \mapsto AM + MB \tag{2.11}$$

for $M \in \mathbb{C}^{m\times n}$.

For the case of $m = n$, we have the following results (see, e.g., [17]).

Lemma 2.2. Let $m = n$. Any eigenvalue of the linear operator $\mathcal{F}$ is the sum of an eigenvalue of A and an eigenvalue of B.

Lemma 2.3. Let $m = n$. Any matrix representation of the linear operator $\mathcal{F}$ is nonsingular if and only if the sum of any eigenvalue of A and any eigenvalue of B is nonzero.

The proofs of Lemmas 2.2 and 2.3 given in [17, p. 572-574] are easily extended to show validity of these lemmas for the case of $n \neq m$. We thus have the following result.

Theorem 2.5. Let n, m be arbitrary positive integers. If the sum of any eigenvalue of A and any eigenvalue of B is nonzero, then the linear matrix equation (2.10) has a unique solution.

We now study the application of Theorem 2.1 (or Theorem 2.3) to the stability analysis of (2.1) for the case in which the assumption that $A_{12} = 0$ and $A_{21} = 0$ does not apply. First, consider the case in which $A_{12} = 0$ and $A_{21} \neq 0$.

Letting $\nu := \xi - E\eta$, we have that

$$\dot{\eta} = A_{11}\eta + F(\eta, \nu + E\eta) \tag{2.12a}$$

$$\dot{\nu} = A_{22}\nu + G(\eta, \nu + E\eta) - E \cdot F(\eta, \nu + E\eta), \tag{2.12b}$$

for a matrix E solving the linear equation

$$A_{22}E - EA_{11} + A_{21} = 0. \tag{2.13}$$

A condition for the existence of a matrix E solving (2.13) has been given in Theorem 2.5. Applying Theorems 2.1 and 2.5 and Lemma 2.1 to (2.12), we have

Lemma 2.4. Assume $A_{12} = 0$, $Re\{\sigma(A_{22})\} < 0$ and $Re\{\sigma(A_{11})\} = 0$. Then the origin of (2.1) is asymptotically stable if the origin is asymptotically stable for the reduced model

$$\dot{\eta} = A_{11}\eta + F(\eta, h(\eta) + E\eta), \tag{2.14}$$

where h satisfies the partial differential equation

$$\begin{aligned} &Dh(\eta) \cdot \{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} \\ &\quad = A_{22}h(\eta) + G(\eta, h(\eta) + E\eta) - E \cdot F(\eta, h(\eta) + E\eta), \end{aligned} \tag{2.15}$$

with $h(0) = 0$ and $Dh(0) = 0$.

A similar result can be obtained for the case in which $A_{21} = 0$ but $A_{12} \neq 0$. Letting $\zeta := \eta - P\xi$, (2.1) gives

$$\dot{\zeta} = A_{11}\zeta + F(\zeta + P\xi, \xi) - P \cdot G(\zeta + P\xi, \xi) \tag{2.16a}$$

$$\dot{\xi} = A_{22}\xi + G(\zeta + P\xi, \xi), \tag{2.16b}$$

where P solves

$$A_{11}P - PA_{22} + A_{12} = 0. \tag{2.17}$$

Thus, we have

Lemma 2.5. Assume that $A_{21} = 0$, $Re\{\sigma(A_{22})\} < 0$ and $Re\{\sigma(A_{11})\} = 0$. Then the origin is asymptotically stable for (2.1) if the reduced model (2.16*a*) with $\xi = h(\zeta)$ is asymptotically stable, where h satisfies

$$\begin{aligned} &Dh(\zeta) \cdot \{A_{11}\zeta + F(\zeta + Ph(\zeta), h(\zeta)) - P \cdot G(\zeta + Ph(\zeta), h(\zeta))\} \\ &\quad = A_{22}h(\zeta) + G(\zeta + Ph(\zeta), h(\zeta)), \end{aligned} \tag{2.18}$$

with $h(0) = 0$ and $Dh(0) = 0$.

2.2. Multilinear Functions

To construct a Liapunov function for nonlinear systems, the technique of Taylor series expansion is a very important tool, which can be conveniently represented in terms of multilinear functions. In this section, we recall basic facts on multilinear functions.

Definition 2.2. (e.g., [26]) Let $V_1, V_2, \ldots, V_k$ and W be vector spaces over the same field. A map $\psi : V_1 \times V_2 \times \ldots \times V_k \to W$ is said to be multilinear (or k-linear) if it is linear in each of its arguments. That is, for any $v_i, \tilde{v}_i \in V_i$, $i = 1, \ldots, k$, and for any scalars $a, \tilde{a}$, we have

$$\begin{aligned}\psi(v_1, \ldots, av_i + \tilde{a}\tilde{v}_i, \ldots, v_k) = a\psi(v_1, \ldots, v_i, \ldots, v_k) \\ + \tilde{a}\psi(v_i, \ldots, \tilde{v}_i, \ldots, v_k).\end{aligned} \tag{2.19}$$

The integer k is the degree of the multilinear function ψ.

■

Next, we consider a special case in which $V_1 = V_2 = \ldots = V_k = V$.

Definition 2.3. [26] A k-linear function $\psi : V \times V \times \ldots \times V \to W$ is symmetric if the vector $\psi(v_1, v_2, \ldots, v_k)$ is invariant under arbitrary permutations of the argument vectors v_i. A function $\phi : I\!R^n \to I\!R^m$ is homogeneous of degree k (k an integer), if for each scalar α, $\phi(\alpha\eta) = \alpha^k \phi(\eta)$ for all $\eta \in I\!R^n$.

A representation of such maps can also be given in terms of multilinear functions. A very important property of homogeneous functions represented in terms of multilinear functions is given next.

Proposition 2.1. [26] Let $\psi : (I\!R^n)^k \to I\!R^m$ be a symmetric k-linear function. For any vector $v \in I\!R^n$,

$$D\psi(\eta, \eta, \ldots, \eta) \cdot v = k\psi(\eta, \eta, \ldots, \eta, v). \tag{2.20}$$

2.3. Normal Form Reduction

Normal form reduction consists of a nonlinear transformation usually used to study the local stability of time-invariant nonlinear systems, specifically, when all eigenvalues of system lie on the imaginary axis. This transformation results in a locally equivalent model of the system for which stability conditions are more easily obtained. Thus, the technique of normal form reduction provides a means to study local stability of critical nonlinear systems.

Consider a nonlinear system

$$\dot{\eta} = F(\eta) \tag{2.21}$$

where $\eta \in I\!R^n$ and F is a sufficiently smooth function with $F(0) = 0$. Let $\eta = \zeta + P(\zeta)$, where P is a purely nonlinear function with $P(0) = 0$. Local stability properties are preserved under such nonlinear transformations.

Lemma 2.6. Let $P(\zeta)$ be smooth mapping with $DP(0) = 0$. Then there exists an open subset $\mathcal{D}$ of $I\!R^n$ containing the origin for which the nonlinear mapping $\eta = \zeta + P(\zeta)$ is one-to-one and onto. Thus, local stability of the origin is preserved under the nonlinear transformation $\eta \mapsto \zeta$.

■

Under the nonlinear transformation $\eta = \zeta + P(\zeta)$, system (2.21) becomes

$$\dot{\zeta} = (I + DP(\zeta))^{-1} F(\zeta + P(\zeta)). \tag{2.22}$$

Write

$$F(\eta) = F_1\eta + F_2(\eta, \eta) + F_3(\eta, \eta, \eta) + O(||\eta||^4), \tag{2.23}$$

where F_1, F_2 and F_3 denote the linear, quadratic and cubic terms of the Taylor expansion of F at the origin, respectively. Here, we have presumed F is at least four times continuously differentiable. Analogously, write

$$P(\zeta) = P_2(\zeta, \zeta) + P_3(\zeta, \zeta, \zeta) + O(||\zeta||^4), \tag{2.24}$$

where P_2 and P_3 are the quadratic and cubic terms in the expansion of P. The transformed model (2.22) becomes

$$\dot{\zeta} = \mathcal{F}_1\zeta + \mathcal{F}_2(\zeta, \zeta) + \mathcal{F}_3(\zeta, \zeta, \zeta) + O(||\zeta||^4), \tag{2.25}$$

where $\mathcal{F}_1 = F_1$ and $\mathcal{F}_2$, $\mathcal{F}_3$ are given by

$$\mathcal{F}_2(\zeta,\zeta) = F_2(\zeta,\zeta) + F_1 \cdot P_2(\zeta,\zeta) - DP_2(\zeta,\zeta) \cdot F_1\zeta, \tag{2.26}$$

$$\begin{aligned}\mathcal{F}_3(\zeta,\zeta,\zeta) = & F_3(\zeta,\zeta,\zeta) - DP_2(\zeta,\zeta) \cdot \mathcal{F}_2(\zeta,\zeta) + DF_2(\zeta,\zeta) \cdot P_2(\zeta,\zeta) \\ & + F_1 \cdot P_3(\zeta,\zeta,\zeta) - DP_3(\zeta,\zeta,\zeta) \cdot F_1\zeta.\end{aligned} \tag{2.27}$$

Normal form reduction involves choosing a nonlinear function $P(\zeta)$ for which the nonlinear terms $\mathcal{F}_i$ up to a certain order in Eq. (2.25) either vanish or have as few nonzero components as possible.

2.4. Stability of Critical Nonlinear Systems

In this section, we recall several bifurcation-theoretic results on stability of one-parameter families of nonlinear systems

$$\dot{x} = f(x,\mu) \tag{2.28}$$

where $x \in I\!R^n$ and $\mu \in I\!R$. The vector field f is assumed to be sufficiently smooth in x and μ and $f(0,0) = 0$.

The equilibrium solutions of system (2.28) are the solutions of $f(x,\mu) = 0$, and thus clearly depend on the value of the parameter μ. For any given $\mu = \mu_0$ with $D_1 f(x_0(\mu),\mu)$ nonsingular, the Implicit Function Theorem guarantees the existence of a locally unique equilibrium solution $x_0(\mu)$ for μ near μ_0. For a parameter value μ_c at which the Jacobian $D_1 f(x_0(\mu_c),\mu_c)$ is singular, the possibility arises of (2.28) possessing several equilibrium paths emanating from $x_0(\mu_c)$ for μ near μ_c. If such a joining of equilibria occurs, the critical point $(x_0(\mu_c),\mu_c)$ is called a *stationary* (or *static*) *bifurcation point.* Another type of bifurcation from equilibrium is the so-called *Hopf* bifurcation, which may occur when the Jacobian $D_1 f(x_0(\mu_c),\mu_c)$ has a conjugate pair of simple pure imaginary eigenvalues. In the Hopf bifurcation, a family of periodic solutions merges with the equilibrium $x_0(\mu_c)$ at $\mu = \mu_c$.

Two main approaches are generally used in studying the stability of bifurcated solutions. One is to apply the center manifold theorem given in Theorem

2.1 to obtain stability criteria for the system from criteria derived for the reduced model. For details see, for instance, [16], [18], [29], [32] and [75]. In the other approach (as in [1], [2], [25], [34]), a Taylor series expansion of system (2.28) at the bifurcation point up to cubic terms can be used to determine stability criteria from the Jacobian of the bifurcated solution.

Here, we follow the notation of [25]. Write the Taylor series expansion of (2.28) as

$$\begin{aligned}
\dot{x} &= f(x, \mu) \\
&= L_0 x + Q_0(x, x) + C_0(x, x, x) + \cdots \\
&\qquad + \mu(L_1 x + Q_1(x, x) + \cdots) + \mu^2(L_2 x + \cdots) + \cdots
\end{aligned} \tag{2.29}$$

where $Q_k(x, x) := D_{\mu^k xx} f(x, \mu)$, $C_k(x, x, x) := D_{\mu^k xxx} f(x, \mu)$, etc., are the quadratic terms, cubic terms, etc., of $f(x, \mu)$. Here, the quadratic and high order terms in the expansion are chosen symmetric. For instance, $Q_k(x, y) = Q_k(y, x)$ for each $k \geq 0$. In (2.29), L_0 denotes the Jacobian $D_x f(0, 0)$ and $L_k := D_{\mu}^k x f(0, 0)$, $k \geq 1$.

Let l and r denote the left (row) and right (column) eigenvectors of the matrix L_0 corresponding to the simple zero eigenvalue or to the pair of pure imaginary eigenvalues $\pm j\omega_c$. Here, for definiteness, the first component of r is set to 1 and the left eigenvector l is chosen such that $lr = 1$. In some cases, a reordering of the components of the state vector is required in order for this normalization to be possible.

Suppose L_0 has only simple critical eigenvalues with the remaining eigenvalues stable. In stationary bifurcation, the stability conditions for the bifurcated solutions are found to be determined by the values of β_1 and β_2, the so-called *bifurcation stability coefficients*. If β_1 is zero and β_2 is negative, then the bifurcated solution will be asymptotically stable. In Hopf bifurcation, the stability of the periodic solution (and of the origin as well) may be derived from computing the Floquet exponent β_2 by applying Floquet theory or by considering the linearization of the so-called Poincaré return map. If $\beta_2 < 0$, the bifurcated periodic solution is asymptotically stable. For purposes of this thesis, we require only the stability criterion for Hopf bifurcation. A stability criterion for simple stationary bifurcation and further details can be found in [25].

Lemma 2.7. Suppose system (2.29) undergoes a Hopf bifurcation with a pair of pure imaginary eigenvalues $\pm j\omega_c$. Then the bifurcated periodic solution will be orbitally asymptotically stable with asymptotic phase in a neighborhood of the bifurcation point if

$$\beta_2 = 2Re\{l[2Q_0(r,a) + Q_0(\bar{r},b) + \frac{3}{4}C_0(r,r,\bar{r})]\} < 0,$$

where the vectors a and b solve

$$-L_0 a = \frac{1}{2}Q_0(r,\bar{r}), \tag{2.30}$$

$$(2j\omega_c - L_0)b = \frac{1}{2}Q_0(r,r), \tag{2.31}$$

respectively, and where $\bar{r}$ denotes the complex conjugate of the vector r.

Moreover, stability of the trivial solution $x = 0$ for system (2.28) is known (e.g., [25]) to coincide with the stability of the bifurcated periodic orbits if $\beta_2 \neq 0$.

2.5. Finite-Time Stability

In our study of the stability properties of satellite deployment and retrieval in Chapter 8, we shall find standard notions of stability inadequate from a physical point of view. The nonasymptotic notions arising in the theory of finite-time stability will, on the other hand, be of considerable value. Below, we summarize basic concepts of finite-time stability. Further extensions will be given in Chapter 8.

Consider the system

$$\dot{x} = f(t,x), \tag{2.32}$$

where $f : \Gamma \times I\!R^n \to I\!R^n$ and $\Gamma := [t_0, t_0 + \mathcal{T})$ with $t_0 \in R$ and $\mathcal{T} > 0$. Let x_0 denote the initial condition of (2.32) at t_0, and let $\phi(t; t_0, x_0)$ be the solution of (2.32) at time t satisfying this initial condition. We recall several relevant definitions [40].

Definition 2.4. System (2.32) is *finite-time stable* with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$, $\alpha \leq \beta$ (see Figure 2.1), if for each x_0 with $||x_0|| < \alpha$, we have $||\phi(t; t_0, x_0)|| < \beta$, $\forall\ t \in \Gamma$. System (2.32) is *finite-time unstable* with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$, $\alpha \leq \beta$, if there exist an x_0 and a t_1 with $||x_0|| < \alpha$ and $t_1 \in \Gamma$ such that $||\phi(t_1; t_0, x_0)|| = \beta$.

Definition 2.5. System (2.32) is *uniformly finite-time stable* with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$, $\alpha \leq \beta$ (see Figure 2.1), if for each $s \in \Gamma$ and x with $||x|| < \alpha$, we have $||\phi(t; s, x)|| < \beta, \forall\ t \in \Gamma$.

Definition 2.6. System (2.32) is *quasi-contractively stable* with respect to $(\alpha, \gamma, \Gamma, ||\cdot||)$, $\gamma < \alpha$ (see Figure 2.1), if for each x_0 with $||x_0|| < \alpha$, there is a $t_1 \in \Gamma$ for which $||\phi(t; t_0, x_0)|| < \gamma$, $\forall\ t \in [t_1, t_0 + \mathcal{T})$.

Definition 2.7. System (2.32) is *contractively stable* with respect to $(\alpha, \beta, \gamma, \Gamma, ||\cdot||)$, $\gamma < \alpha \leq \beta$, if it is finite-time stable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ and quasi-contractively stable with respect to $(\alpha, \gamma, \Gamma, ||\cdot||)$.

For given α, β, Γ, and $||\cdot||$, a necessary and sufficient condition for uniform finite-time stability is recalled next.

Lemma 2.8. ([40]) System (2.32) is uniformly finite-time stable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$, $\alpha < \beta$, if and only if there exists a continuous function $V(t, x)$ such that

$$\dot{V}(t, x) \leq 0, \quad \forall\ \ x \in \overline{B(\beta)}, \qquad t \in \Gamma, \tag{2.33}$$

$$V_M^{\delta}(t_1) < V_m^{\beta}(t_2), \quad \forall\ \ t_2 > t_1, \quad \forall\ \ \delta < \alpha, \text{ and } t_1, t_2 \in \Gamma, \tag{2.34}$$

where

$$B(\beta) := \{x : ||x|| < \beta\}, \tag{2.35}$$

$||\cdot||$ denotes a norm on $I\!R^n$, $\overline{B(\beta)}$ is the closure of $B(\beta)$, and

$$V_M^\alpha(t) := \sup_{||x||=\alpha} V(t,x), \tag{2.36}$$

$$V_m^\alpha(t) := \inf_{||x||=\alpha} V(t,x). \tag{2.37}$$

Here, $\dot{V}(t,x)$ is the time derivative of $V(t,x)$ along trajectories of system (2.32).

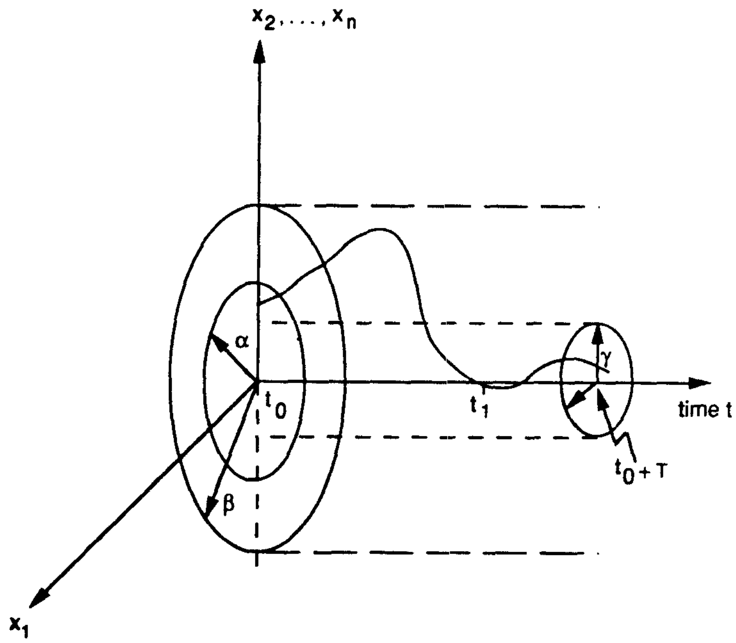

Figure 2.1. Illustrating finite-time stability

CHAPTER THREE

STABILIZATION OF NONLINEAR SYSTEMS IN SIMPLE CRITICAL CASES

The center manifold theorem has been applied to the local feedback stabilization of nonlinear systems in critical cases. In the present chapter, this approach is explicated for two particular critical cases in stability. The system linearization at the equilibrium point of interest is assumed to possess either a simple zero eigenvalue or a complex conjugate pair of simple, pure imaginary eigenvalues. In either case, the noncritical eigenvalues are taken to be stable. The results on stabilizability and stabilization are given explicitly in terms of the nonlinear model of interest in its original form, i.e., before reduction to the center manifold. Moreover, the formulation given in this chapter uncovers connections between results obtained using the center manifold reduction and those of an alternative approach.

3.1. Introduction

Recently, the center manifold reduction has been employed in nonlinear stabilization, resulting in stabilizing control laws for various classes of nonlinear systems in the so-called "critical cases." Critical cases occur when the linearized system at an equilibrium point has at least one eigenvalue on the imaginary axis, with the remaining eigenvalues in the open left half of the complex plane.

Aeyels [4], who initiated application of the center manifold reduction in nonlinear stabilization, investigated the existence of smooth stabilizing feedback control laws for a class of third-order nonlinear systems for which the linearized model possesses an uncontrollable pair of pure imaginary eigenvalues. Behtash and Sastry [10] used the same approach to study stabilization for nonlinear systems whose linearized model has two distinct pairs of complex conjugate pure imaginary eigenvalues, or a double pole at the origin, or a pole at the origin and a complex conjugate pair of pure imaginary eigenvalues. In [10], the design was undertaken for the reduced system on the center manifold using normal form calculations, and for simplicity, a scalar stable mode was assumed. Generally, there is a need for considering cases with any finite number of stable modes. Moreover, the control laws will be more convenient if they are given directly in terms of the original model rather than in terms of transformed versions.

A main goal of this chapter is to derive such stabilizing control algorithms for general nonlinear systems in critical cases. The development focuses on general nonlinear systems in two specific critical cases. In the first critical case of interest here, a simple zero eigenvalue occurs, while in the second case a pair of pure imaginary eigenvalues occurs. In either case, the critical eigenvalues of the linearized model need not be controllable.

Stabilizing control algorithms for such systems have been obtained [1], [2], [25] by using asymptotic expansions of critical eigenvalues and Floquet exponents of bifurcated solution branches of one-parameter embeddings of the nonlinear models. In this chapter, we use the center manifold reduction approach to obtain criteria for existence of stabilizing feedback control laws for critical nonlinear control systems. Moreover, explicit designs of these control laws are given when the existence criteria hold. The algorithms for controller design involve a preliminary stabilization of the noncritical modes, followed by a setting of control gains to stabilize the so-called reduced model whose eigenvalues lie on the imaginary axis. The feedback laws obtained include purely linear state feedback and feedback control laws containing both linear and nonlinear terms in the state.

3.2. General Design via Center Manifold Reduction

Referring to the results given in Theorem 2.1, a convenient assumption for the existence of a locally invariant manifold is that the linearized model of a nonlinear system must have two groups of states that are decoupled. In the application of the theorem, the decoupling may be destroyed as a result of the linear terms in the feedback controller. To overcome this difficulty, we introduced a similarity transformation in Section 2.1. Thus, we will be able to deal with systems whose states are linearly coupled. Stability criteria for such cases have been given in Lemmas 2.4 and 2.5, which can then be employed to design linear and/or nonlinear stabilizing feedback control laws for nonlinear control system.

Consider the class of nonlinear control systems

$$\dot{z}_1 = Z_{11}z_1 + Z_{12}z_2 + B_1u + f_1(z_1, z_2). \tag{3.1}$$

$$\dot{z}_2 = Z_{21}z_1 + Z_{22}z_2 + B_2u + f_2(z_1, z_2). \tag{3.2}$$

Using a block diagonalizing transformation for the uncontrolled model of the system, we can rewrite Eqs. (3.1)-(3.2) as

$$\dot{\eta} = A_{11}\eta + \tilde{B}_1u + F(\eta, \xi), \tag{3.3a}$$

$$\dot{\xi} = A_{22}\xi + \tilde{B}_2u + G(\eta, \xi). \tag{3.3b}$$

We assume that the linear transformation is chosen such that A_{22} is stable, while A_{11} is not stable. (It is always possible to achieve this by suitable choice of states η, ξ and of diagonalizing transformation.) Since the states η and ξ in (3.3) are linearly decoupled, we have the following result.

Lemma 3.1. If $\{A_{11}, \tilde{B}_1\}$ is a controllable pair or the subsystem (3.3a) with $\xi = 0$ is linearly stabilizable, then the original system (3.1)-(3.2) is stabilizable by linear state feedback.

Note that, in the sequel the nonlinear control system (3.1)-(3.2) is supposed to have been transformed into block diagonal form as given in (3.3). The stability analysis below focuses on the block diagonalized model. The implications for stability of the original system are then readily obtained. For simplicity, we assume a scalar control input u. It is not difficult to extend the analysis to general multi-input nonlinear control systems. Consequently, we assume that $\tilde{B}_i$ is a column vector denoted by b_i for $i = 1, 2$ and we can rewrite system (3.3) as

$$\dot{\eta} = A_{11}\eta + b_1 u + F(\eta, \xi), \tag{3.4a}$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(\eta, \xi). \tag{3.4b}$$

Next, we apply the center manifold reduction technique given in Lemma 2.4 to the design of stabilizing control laws for the class of nonlinear systems (3.4) in which all eigenvalues of the matrix A_{11} lie on the imaginary axis.

Let us first consider the case in which $b_1 = 0$ and assume the feedback control to be of the form

$$u(\eta, \xi) = K_1\eta + K_2\xi + U(\eta, \xi), \tag{3.5}$$

where $U(\cdot, \cdot)$ is a smooth, purely nonlinear function whose first derivatives vanish at the origin. Rewrite the system dynamics (3.4) as

$$\dot{\eta} = A_{11}\eta + F(\eta, \xi), \tag{3.6}$$

$$\dot{\xi} = b_2 K_1 \eta + (A_{22} + b_2 K_2)\xi + b_2 U(\eta, \xi) + G(\eta, \xi). \tag{3.7}$$

From Eq. (3.7), the linear decoupling property of the original uncontrolled system has been destroyed. Thus, the center manifold reduction technique given in Theorem 2.1 cannot be applied directly.

As introduced in Section 2.1, there is a constant matrix E such that, with $\nu := \xi - E\eta$, the transformed version of the control system (3.6)-(3.7) has a block diagonal form if E is the (unique) solution of the Liapunov-like equation

$$b_2K_1 + (A_{22} + b_2K_2)E - EA_{11} = 0. \tag{3.8}$$

We assume that $A_{22}+b_2K_2$ is stable. Moreover, since all the eigenvalues of A_{11} lie on the imaginary axis, then by Theorem 2.5 we can guarantee existence of a solution E of Eq. (3.8). The new system dynamics for states η and ξ can then be obtained from Eq. (3.6)-(3.7) as

$$\dot{\eta} = A_{11}\eta + F(\eta, \nu + E\eta), \tag{3.9a}$$

$$\dot{\nu} = (A_{22} + b_2K_2)\nu + b_2U(\eta, \nu + E\eta) + G(\eta, \nu + E\eta). \tag{3.9b}$$

Theorem 2.1 guarantees the existence of a C^2 locally invariant manifold, which is given by the graph of a function $\nu = h(\eta)$, for the transformed model (3.9). The function h satisfies

$$\begin{aligned} Dh(\eta) \cdot \{A_{11}\eta + F(\eta, h(\eta) + E\eta)\} = (A_{22} + b_2K_2)h(\eta) \\ + b_2U(\eta, h(\eta) + E\eta) + G(\eta, h(\eta) + E\eta) \end{aligned} \tag{3.10}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

As required by Lemma 2.4, the stability of system (3.9) can be guaranteed if the control gains K_1, K_2 and the nonlinear function U are chosen such that (i) $A_{22} + b_2K_2$ is stable, and (ii) the reduced model (3.9a) with $\nu = h(\eta)$ (the solution of Eq. (3.10)) is also stable.

Next, we consider the case in which b_1 is nonzero. In the simple critical cases where A_{11} has only one zero eigenvalue or a pair of pure imaginary eigenvalues, Lemma 3.1 will imply the existence of a linear stabilizing feedback control for system (3.4). Consider next the existence of a purely nonlinear smooth feedback.

Let the control input be as in Eq. (3.5). Since now we focus on purely nonlinear stabilizing controllers (i.e., $K_1 = 0$ and $K_2 = 0$ in Eq. (3.5)), then we have that system (3.4) is still linearly decoupled. Thus, if A_{22} is stable and all eigenvalues of A_{11} lie on the imaginary axis, then there exists a locally invariant manifold given by the graph of a function $\xi = h(\eta)$. Furthermore, the function h satisfies

$$\begin{aligned} Dh(\eta)\cdot\{A_{11}\eta + b_1 U(\eta, h(\eta)) + F(\eta, h(\eta))\} \\ = A_{22}h(\eta) + b_2 U(\eta, h(\eta)) + G(\eta, h(\eta)) \end{aligned} \tag{3.11}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Suppose that A_{22} is stable. Then, a purely nonlinear stabilizing feedback control law may be designed (by using Theorem 2.1 and Eq. (3.11)) from the stability conditions for the reduced model

$$\dot{\eta} = A_{11}\eta + b_1 U(\eta, h(\eta)) + F(\eta, h(\eta)). \tag{3.12}$$

Note that, for the case in which A_{22} is not stable, a linear state feedback $K_2\xi$ is needed to first stabilize $A_{22} + b_2 K_2$. Then the procedure discussed above can be employed to design stabilizing control laws for the system.

In the following sections, we consider two special cases in which the system has only simple critical modes (i.e., one zero eigenvalue or a pair of pure imaginary eigenvalues) and the rest of the eigenvalues are controllable or stabilizable.

3.3. One Zero Eigenvalue

In this section, we consider the case in which a simple zero eigenvalue occurs in the linearization. As discussed above, the stability of the overall system can be studied by a consideration of the reduced model only. Because of this, we first consider stability conditions for scalar systems with a zero eigenvalue. Then these conditions can be employed to design stabilizing control laws for general higher order systems.

Consider a scalar nonlinear system

$$\dot{x} = dx^2 + ex^3 + \cdots, \tag{3.13}$$

where d and e are real scalars.

Stability conditions for system (3.13) are given in the next lemma.

Lemma 3.2. System (3.13) is asymptotically stable if $d = 0$ and $e < 0$. Moreover, system (3.13) is unstable in case $d \neq 0$.

In the following, we apply the stability criteria of Lemma 3.2 to the design of stabilizing control laws for the general higher order system (3.4). We now make the following assumption, which applies throughout the remainder of this section. Suppose $A_{11} = 0$, a scalar, and let (A_{22}, b_2) be a controllable (or stabilizable) pair.

Let $x := \eta$ be a scalar and write

$$\begin{aligned}
f(x,\xi) :=& F(x,\xi) \\
=& f_{xx}x^2 + xf_{x\xi}\xi + \xi' f_{\xi\xi}\xi + f_{xxx}x^3 + x^2 f_{xx\xi}\xi \\
& + x \cdot \xi' f_{x\xi\xi}\xi + f_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,\xi)||^4), \qquad (3.14)\\
G(x,\xi) =& x^2 G_{xx} + xG_{x\xi}\xi + G_{\xi\xi}(\xi,\xi) + x^3 G_{xxx} \\
& + x^2 G_{xx\xi}\xi + xG_{x\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,\xi)||^4). \qquad (3.15)
\end{aligned}$$

The coefficients in the Taylor series expansion (3.14)-(3.15) are either constants or symmetric multilinear functions of their arguments. For instance, $f_{\xi\xi\xi}(\xi,\xi,\xi)$ and $G_{\xi\xi}(\xi,\xi)$ denote symmetric trilinear scalar function and bilinear vector function of ξ, respectively.

3.3.1. The case $b_1 = 0$

In this subsection, we consider the case in which $b_1 = 0$, and consider feedback controls

$$u(x,\xi) = k_1 x + K_2\xi + U(x,\xi), \qquad (3.16)$$

with k_1 a scalar control gain.

As observed in Section 3.2, the stability of control system (3.4) in this critical case coincides with the stability of the reduced model

$$\dot{x} = f(x, h(x) + Ex), \qquad (3.17)$$

where E and $h(\cdot)$ solve Eqs. (3.8) and (3.10), respectively, under the conditions: $A_{11} = 0$, $(A_{22} + b_2 K_2)$ is stable, with η substituted by x and K_1 substituted by k_1.

Referring to the boundary conditions (i.e., $h(0) = 0$ and $Dh(0) = 0$) of the solution h of Eq. (3.10), we can approximate h as

$$h(x) = x^2 h_{xx} + O(|x|^3). \tag{3.18}$$

We assume that $A_{22} + b_2 K_2$ is stable, and rewrite the control input (3.16) as

$$\begin{aligned} u(x,\xi) =& k_1 x + K_2 \xi + u_{xx} x^2 + x u_{x\xi} \xi + \xi' u_{\xi\xi} \xi \\ &+ u_{xxx} x^3 + x^2 u_{xx\xi} \xi + x \xi' u_{x\xi\xi} \xi + u_{\xi\xi\xi}(\xi,\xi,\xi) + \hat{U}(x,\xi), \end{aligned} \tag{3.19}$$

where $u_{\xi\xi\xi}$ is a symmetric trilinear function of ξ, and $\hat{U}$ is a higher order nonlinear function which vanishes along with its partial derivatives up to third order at $(x,\xi) = (0,0)$.

Solving Eqs. (3.8) and (3.10), we have

$$E = -(A_{22} + b_2 K_2)^{-1} b_2 k_1 \tag{3.20}$$

$$\begin{aligned} h_{xx} =& (A_{22} + b_2 K_2)^{-1} \{ [f_{xx} + f_{x\xi} E + E' f_{\xi\xi} E] E - [b_2 u_{xx} + G_{xx} \\ &+ (b_2 u_{x\xi} + G_{x\xi}) E + (b_2 E' u_{\xi\xi} E + G_{\xi\xi}(E,E))] \}. \end{aligned} \tag{3.21}$$

The reduced model (3.17) is then given by

$$\begin{aligned} \dot{x} =& \{ f_{xx} + f_{x\xi} E + E' f_{\xi\xi} E \} x^2 + \{ f_{x\xi} h_{xx} + 2E' f_{\xi\xi} h_{xx} \\ &+ f_{xxx} + f_{xx\xi} E + E' f_{x\xi\xi} E + f_{\xi\xi\xi}(E,E,E) \} x^3 + O(|x|^4). \end{aligned} \tag{3.22}$$

Using Lemma 3.2, we have the following stabilization result for control system (3.4).

Lemma 3.3. Suppose $b_1 = 0$ and let the control input be of the form (3.19). Then system (3.4) is asymptotically stable if $A_{22} + b_2 K_2$ is stable and the following conditions are satisfied:

$$f_{xx} + f_{x\xi} E + E' f_{\xi\xi} E = 0, \tag{3.23}$$

$$\begin{aligned} & f_{x\xi} h_{xx} + 2E' f_{\xi\xi} h_{xx} + f_{xxx} + f_{xx\xi} E \\ &\quad + E' f_{x\xi\xi} E + f_{\xi\xi\xi}(E,E,E) < 0, \end{aligned} \tag{3.24}$$

where E and h_{xx} are as given in Eqs. (3.20) and (3.21).

From Eqs. (3.20)-(3.21), and the fact that A_{22} is invertible, we have $E = 0$ and $h_{xx} = -A_{22}^{-1}G_{xx}$ for the uncontrolled system. The next stability criterion for the uncontrolled version of system (3.4) follows readily from Lemma 3.3.

Corollary 3.1. System (3.4) (with $u = 0$) is asymptotically stable if A_{22} is stable, $f_{xx} = 0$ and $f_{xxx} - f_{x\xi}A_{22}^{-1}G_{xx} < 0$.

In the rest of this section, we assume that the stability conditions given in Corollary 3.1 do not hold, and seek stabilizing control laws for system (3.4).

Linear stabilizing control laws follow from Lemma 3.3, and are as given next.

Proposition 3.1. Suppose $b_1 = 0$. Then there is a purely linear feedback which stabilizes (3.4) if there exist feedback gains k_1 and K_2 for which $(A_{22} + b_2K_2)$ is stable,

$$f_{xx} - k_1 f_{x\xi}Mb_2 + k_1^2 b_2' M' f_{\xi\xi} M b_2 = 0, \tag{3.25}$$

and

$$\begin{aligned} f_{xxx} - f_{x\xi}MG_{xx} &+ k_1\{f_{x\xi}MG_{x\xi} + 2G_{xx}'M'f_{\xi\xi} - f_{xx\xi}\}Mb_2 \\ &+ k_1^2\{b_2'M'f_{x\xi\xi}Mb_2 - f_{x\xi}MG_{\xi\xi}(Mb_2, Mb_2) - 2b_2'M'f_{\xi\xi}MG_{x\xi}Mb_2\} \\ &+ k_1^3\{2b_2'M'f_{\xi\xi}MG_{\xi\xi}(Mb_2, Mb_2) - f_{\xi\xi\xi}(Mb_2, Mb_2, Mb_2)\} < 0, \end{aligned} \tag{3.26}$$

where $M := (A_{22} + b_2K_2)^{-1}$.

Remark 3.1. The linear stabilizing control rule proposed in Proposition 3.1 is a composite-type controller design. First, the feedback gain K_2 is chosen to stabilize state ξ. Then the remaining feedback gain k_1 is selected to satisfy the conditions (3.25) and (3.26) based on the chosen gain K_2. Since the stability of state ξ will not be influenced by the feedback gain k_1, no extra constraints are required for the choice of k_1, such as the one given in [2].

Since k_1 is scalar, conditions (3.25) and (3.26) do not necessarily hold for any given K_2. Thus, a stabilizing linear feedback does not always follow from

Corollary 3.1. As observed from the stability conditions given in Lemma 3.3, the cubic terms of both the function G and the control input u do not contribute to the stability criteria of system (3.4). A general linear-plus-quadratic feedback control law can be abstracted as

$$u(x,\xi) = k_1 x + K_2\xi + u_{xx}x^2 + x u_{x\xi}\xi + \xi' u_{\xi\xi}\xi \tag{3.27}$$

if the control gains satisfy the conditions of Lemma 3.3.

Moreover, as observed from stability conditions (3.23) and (3.24) given in Lemma 3.3, when $k_1 = 0$ a possibility for smooth, a purely nonlinear feedback stabilization of the origin of (3.4) is to have an input of the form $u = u_{xx}x^2$. We have the following special result.

Corollary 3.2. Suppose A_{22} is stable and $b_1 = 0$. Then a *purely quadratic* stabilizing feedback $u = u_{xx}x^2$ for (3.4) exists if $f_{xx} = 0$ and $f_{xxx} - f_{x\xi}A_{22}^{-1}(G_{xx} + b_2 u_{xx}) < 0$.

3.3.2. The case $b_1 \neq 0$

Next, we consider the case in which $b_1 \neq 0$. To obtain a nontrivial stabilization problem in this case, we now restrict the control law to be purely nonlinear. We assume A_{22} is stable and the control input is given by Eq. (3.19) with $k_1 = 0$ and $K_2 = 0$. Then according to Section 3.2, the stability of system (3.4) is determined by the stability of the reduced model

$$\dot{x} = b_1 U(x, h(x)) + F(x, h(x)), \tag{3.28}$$

where h solves Eq. (3.11) with η substituted by x and with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Similarly, we can approximate h as in (3.18). Solving Eq. (3.11), we have

$$\begin{aligned} h_{xx} = A_{22}^{-1}\{&[b_1 u_{xx} + f_{xx} + (b_1 u_{x\xi} + f_{x\xi})E + E'(b_1 u_{\xi\xi} + f_{\xi\xi})E]E \\ &- [b_2 u_{xx} + G_{xx} + (b_2 u_{x\xi} + G_{x\xi})E + (b_2 E' u_{\xi\xi} E + G_{\xi\xi}(E,E))]\}. \end{aligned} \tag{3.29}$$

Applying Lemma 3.2 to the reduced model (3.28), we have

Lemma 3.4. Let A_{22} be stable and let $b_1 \neq 0$. Then system (3.4) is asymptotically stable if $f_{xx} + b_1 u_{xx} = 0$ and $f_{xxx} + b_1 u_{xxx} - (f_{x\xi} + b_1 u_{x\xi}) A_{22}^{-1} (G_{xx} + b_2 u_{xx}) < 0$.

Moreover, a purely cubic stabilizing controller exists when $f_{xx} = 0$.

Corollary 3.3. If $f_{xx} = 0$, then system (3.4) is stabilizable by a *purely cubic* feedback. A stabilizing cubic control of the form $u = u_{xxx} x^3$ exists.

For the case in which A_{22} is not stable, a linear feedback $K_2 \xi$ is needed to guarantee the existence of a locally invariant manifold. Then the design of stabilizing control laws proposed in Lemma 3.4 and Corollary 3.3 can be applied directly.

3.4. Pair of Pure Imaginary Eigenvalues

Next, we consider the case in which A_{11} has a pair of pure imaginary eigenvalues. Specifically, we take A_{11} to be of the form (3.31) below.

First, consider the stability of a planar system

$$\dot{\eta} = A_{11}\eta + Q(\eta, \eta) + C(\eta, \eta, \eta) + \cdots, \tag{3.30}$$

where $\eta = (x, y)'$, and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 \\ -\Omega_2 & 0 \end{pmatrix} \tag{3.31}$$

with $\Omega_1 \Omega_2 > 0$. Without loss of generality, we may express the quadratic and cubic terms in Eq. (3.30) as

$$Q(\eta, \eta) = \begin{pmatrix} q_{11}x^2 + q_{12}xy + q_{13}y^2 \\ q_{21}x^2 + q_{22}xy + q_{23}y^2 \end{pmatrix}, \tag{3.32}$$

$$C(\eta, \eta, \eta) = \begin{pmatrix} c_{11}x^3 + c_{12}x^2y + c_{13}xy^2 + c_{14}y^3 \\ c_{21}x^3 + c_{22}x^2y + c_{23}xy^2 + c_{24}y^3 \end{pmatrix}, \tag{3.33}$$

respectively. Note that system (3.30) has the pair of pure imaginary eigenvalues $\pm i\sqrt{\Omega_1 \Omega_2}$.

Applying a general stability criterion for planar systems undergoing Hopf bifurcation (see, e.g., [29]), we find that a sufficient condition for the stability of the origin for (3.30) is:

$$\frac{1}{8}\{q_{22}(\frac{1}{\Omega_2}q_{21}+\frac{1}{\Omega_1}q_{23})-q_{12}(\frac{1}{\Omega_1}q_{11}+\frac{\Omega_2}{\Omega_1^2}q_{13})+\frac{2}{\Omega_2}q_{11}q_{21}$$

$$-\frac{2\Omega_2}{\Omega_1^2}q_{13}q_{23}+3(c_{11}+\frac{\Omega_2}{3\Omega_1}c_{13}+\frac{1}{3}c_{22}+\frac{\Omega_2}{\Omega_1}c_{24})\}<0. \qquad (3.34)$$

In the following, we apply the stability criterion (3.34) to the design of a stabilizing control law for the more general (nonplanar) system (3.4). We now make the following assumption, which holds throughout the remainder of this section. Assume A_{11} (given in Eq. (3.4)) is of the form given by Eq. (3.31), and let $\eta = (x,y)'$ be a two-dimensional vector, $b_1 := (b_{11}, b_{12})'$ and $F(\eta,\xi) = (f(x,y,\xi), g(x,y,\xi))'$.

System (3.4) may be rewritten as

$$\dot{x} = \Omega_1 y + b_{11}u + f(x,y,\xi) \qquad (3.35a)$$

$$\dot{y} = -\Omega_2 x + b_{12}u + g(x,y,\xi) \qquad (3.35b)$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(x,y,\xi). \qquad (3.35c)$$

Here b_{11}, b_{12} are constant scalars, and f, g, G are given by

$$f(x,y,\xi) = f_{xx}x^2 + f_{xy}xy + f_{yy}y^2 + (xf_{x\xi} + yf_{y\xi})\xi$$
$$+\xi' f_{\xi\xi}\xi + f_{xxx}x^3 + f_{xxy}x^2y + f_{xyy}xy^2$$
$$+f_{yyy}y^3 + (x^2 f_{xx\xi} + xyf_{xy\xi} + y^2 f_{yy\xi})\xi$$
$$+\xi'(xf_{x\xi\xi} + yf_{y\xi\xi})\xi + f_{\xi\xi\xi}(\xi,\xi,\xi) \; + O(||(x,y,\xi)||^4), \qquad (3.36)$$

$$g(x,y,\xi) = g_{xx}x^2 + g_{xy}xy + g_{yy}y^2 + (xg_{x\xi} + yg_{y\xi})\xi$$
$$+\xi' g_{\xi\xi}\xi + g_{xxx}x^3 + g_{xxy}x^2y + g_{xyy}xy^2$$
$$+g_{yyy}y^3 + (x^2 g_{xx\xi} + xyg_{xy\xi} + y^2 g_{yy\xi})\xi$$
$$+\xi'(xg_{x\xi\xi} + yg_{y\xi\xi})\xi + g_{\xi\xi\xi}(\xi,\xi,\xi) \; + O(||(x,y,\xi)||^4), \qquad (3.37)$$

$$G(x,y,\xi) = x^2 G_{xx} + xyG_{xy} + y^2 G_{yy} + (xG_{x\xi} + yG_{y\xi})\xi$$

$$+ G_{\xi\xi}(\xi,\xi) + x^3 G_{xxx} + x^2 y G_{xxy} + xy^2 G_{xyy}$$
$$+ y^3 G_{yyy} + (x^2 G_{xx\xi} + xy G_{xy\xi} + y^2 G_{yy\xi})\xi$$
$$+ x G_{x\xi\xi}(\xi,\xi) + y G_{y\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) \ + O(||(x,y,\xi)||^4). \quad (3.38)$$

Similarly, the coefficients in (3.36)-(3.38) are either constants or symmetric multilinear functions of their arguments.

3.4.1. The case $b_1 = 0$

First, we consider the case in which $b_{11} = b_{12} = 0$. Let the control input be of the form

$$u = k_{11}x + k_{12}y + K_2\xi + U(x,y,\xi). \quad (3.39)$$

Assume that $A_{22} + b_2K_2$ is stable. According to Section 3.2, the stability of the origin of system (3.35) agrees with the stability of the reduced model

$$\dot{x} = \Omega_1 y + f(x,y,E_1x + E_2y + h(x,y)) \quad (3.40a)$$
$$\dot{y} = -\Omega_2 x + g(x,y,E_1x + E_2y + h(x,y)), \quad (3.40b)$$

where $E = (E_1, E_2)$ and $h(x,y)$ are the solutions of Eqs. (3.8) and (3.10), respectively, with $K_1 = (k_{11}, k_{12})$.

Similarly, referring to the boundary conditions of Eq. (3.10), we can write h in the form

$$h(x,y) = x^2 h_{xx} + xy h_{xy} + y^2 h_{yy} + O(||(x,y)||^3), \quad (3.41)$$

where h_{xx}, h_{xy}, h_{yy} are constant vectors.

Now, define the nonlinear control function U in Eq. (3.39) as

$$U(x,y,\xi) = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2 + (xu_{x\xi} + yu_{y\xi})\xi + \xi' u_{\xi\xi}\xi + u_{xxx}x^3$$
$$+ u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3 + (x^2 u_{xx\xi} + xy u_{xy\xi})\xi$$
$$+ y^2 u_{xy\xi}\xi + \xi'(xu_{x\xi\xi} + yu_{y\xi\xi})\xi + u_{\xi\xi\xi}(\xi,\xi,\xi) + \hat{U}(x,\xi), \quad (3.42)$$

where $u_{\xi\xi\xi}$ is a symmetric trilinear function of ξ, and $\hat{U}$ is a nonlinear function which vanishes along with its partial derivatives up to third order at $(x,\xi) = (0,0)$.

Since $(A_{22}+b_2K_2)$ is stable, by assumption matrices $(A_{22}+b_2K_2)^2+\Omega_1\Omega_2I$ and $(A_{22}+b_2K_2)^2+4\Omega_1\Omega_2I$ are both invertible, where I denotes the identity matrix.

Let

$$\begin{aligned} H(x,y) :=& b_2U(x,y,E_1x+E_2y)+G(x,y,E_1x+E_2y) \\ & -f(x,y,E_1x+E_2y)E_1-g(x,y,E_1x+E_2y)E_2 \\ =& x^2H_{xx}+xyH_{xy}+y^2H_{yy}+O(||(x,y)||^3), \end{aligned} \tag{3.43}$$

where U is as defined in Eq. (3.42). Solving Eqs. (3.8) and (3.10), we have

$$E_1=-\{(A_{22}+b_2K_2)^2+\Omega_1\Omega_2I\}^{-1}\{k_{11}(A_{22}+b_2K_2)-\Omega_2k_{12}I\}b_2 \tag{3.44}$$

$$E_2=-\{(A_{22}+b_2K_2)^2+\Omega_1\Omega_2I\}^{-1}\{k_{12}(A_{22}+b_2K_2)+\Omega_1k_{11}I\}b_2 \tag{3.45}$$

and

$$\begin{aligned} h_{xy} =& \{(A_{22}+b_2K_2)^2+4\Omega_1\Omega_2I\}^{-1}\{2\Omega_2H_{yy}-2\Omega_1H_{xx} \\ & -(A_{22}+b_2K_2)H_{xy}\}, \end{aligned} \tag{3.46}$$

$$h_{xx}=-(A_{22}+b_2K_2)^{-1}\cdot(H_{xx}+\Omega_2h_{xy}), \tag{3.47}$$

$$h_{yy}=-(A_{22}+b_2K_2)^{-1}\cdot(H_{yy}-\Omega_1h_{xy}). \tag{3.48}$$

The reduced model (3.40) is hence obtained as

$$\begin{aligned} \dot{x} =& \Omega_1y+\hat{f}_{xx}x^2+\hat{f}_{xy}xy+\hat{f}_{yy}y^2+\hat{f}_{xxx}x^3 \\ & +\hat{f}_{xxy}x^2y+\hat{f}_{xyy}xy^2+\hat{f}_{yyy}y^3+O(||(x,y)||^4) \end{aligned} \tag{3.49a}$$

$$\begin{aligned} \dot{y} =& -\Omega_2x+\hat{g}_{xx}x^2+\hat{g}_{xy}xy+\hat{g}_{yy}y^2+\hat{g}_{xxx}x^3 \\ & +\hat{g}_{xxy}x^2y+\hat{g}_{xyy}xy^2+\hat{g}_{yyy}y^3+O(||(x,y)||^4), \end{aligned} \tag{3.49b}$$

where $\hat{f},\hat{g}$ denote the new versions of the cubic terms, the values of which are given in Appendix 3.A.

Referring to the stability criterion (3.34) and the foregoing discussions, we obtain stability conditions for the control system (3.35) summarized in the following lemma.

Lemma 3.5. Suppose $b_{11} = b_{12} = 0$ and that the control input is of the form (3.39) with nonlinear function U as in (3.42). Then the origin of Eq. (3.35) is asymptotically stable if $A_{22} + b_2K_2$ is stable and

$$\hat{g}_{xy}(\frac{1}{\Omega_2}\hat{g}_{xx} + \frac{1}{\Omega_1}\hat{g}_{yy}) - \hat{f}_{xy}(\frac{1}{\Omega_1}\hat{f}_{xx} + \frac{\Omega_2}{\Omega_1^2}\hat{f}_{yy}) + \frac{2}{\Omega_2}\hat{f}_{xx}\cdot\hat{g}_{xx}$$
$$- \frac{2\Omega_2}{\Omega_1^2}\hat{f}_{yy}\cdot\hat{g}_{yy} + 3(\hat{f}_{xxx} + \frac{\Omega_2}{3\Omega_1}\hat{f}_{xyy} + \frac{1}{3}\hat{g}_{xxy} + \frac{\Omega_2}{\Omega_1}\hat{g}_{yyy}) < 0. \quad (3.50)$$

Remark 3.2. From (3.50) and Appendix 3.A, we observe that only quadratic terms of the function G, and the linear and quadratic terms of the control input u contribute to the stability conditions. A linear and/or quadratic feedback stabilizing control law follows from Lemma 3.5.

Although Lemma 3.5 provides a means for the design of a linear feedback stabilizing control law, such a linear stabilizing control law need not exist. Referring to Eqs. (3.43)-(3.45), we have $H(x,y) = b_2U(x,y,0) + G(x,y,0)$ when $k_{11} = k_{12} = 0$ and $K_2 = 0$. A purely quadratic stabilizing control law can then be proposed as follows.

Corollary 3.4. Assume that $b_{11} = b_{12} = 0$, A_{22} is stable and the origin of system (3.35) is unstable. Then a *purely quadratic* stabilizing feedback $u = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$ exists for system (3.35) if one of the values of $f_{x\xi}, f_{y\xi}, g_{x\xi}, g_{y\xi}$ is nonzero and

$$g_{xy}(\frac{1}{\Omega_2}g_{xx} + \frac{1}{\Omega_1}g_{yy}) - f_{xy}(\frac{1}{\Omega_1}f_{xx} + \frac{\Omega_2}{\Omega_1^2}f_{yy}) + \frac{2}{\Omega_2}f_{xx}g_{xx}$$
$$- \frac{2\Omega_2}{\Omega_1^2}f_{yy}g_{yy} + 3\{f_{xxx} + f_{x\xi}h_{xx} + \frac{\Omega_2}{3\Omega_1}(f_{xyy} + f_{x\xi}h_{yy} + f_{y\xi}h_{xy})$$
$$+ \frac{1}{3}(g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx}) + \frac{\Omega_2}{\Omega_1}(g_{yyy} + g_{y\xi}h_{yy})\} < 0, \quad (3.51)$$

where

$$h_{xy} = \{A_{22}^2 + 4\Omega_1\Omega_2 I\}^{-1}\{2\Omega_2(u_{yy}b_2 + G_{yy}) - 2\Omega_1(u_{xx}b_2 + G_{xx})$$

$$- A_{22}(u_{xy}b_2 + G_{xy})\}, \tag{3.52}$$

$$h_{xx} = - A_{22}^{-1}(u_{xx}b_2 + G_{xx} + \Omega_2 h_{xy}), \tag{3.53}$$

$$h_{yy} = - A_{22}^{-1}(u_{yy}b_2 + G_{yy} - \Omega_1 h_{xy}). \tag{3.54}$$

We note that Aeyels' stabilization conditions for a third-order system [4] are special cases of those given in Corollary 3.4. Corollary 3.4 can also be extended to the case in which A_{22} is not stable but the pair (A_{22}, b_2) is stabilizable. Under this condition, an additional linear feedback $K_2\xi$ is needed to ensure the existence of the locally invariant manifold and the stability of the noncritical states ξ (obtained by setting $x_1 = x_2 = 0$ and $u = 0$ in Eq. (3.35*c*)).

A stability criterion for the uncontrolled model of system (3.35) readily follows from Corollary 3.4.

Corollary 3.5. The origin of Eq. (3.35) with $u = 0$ is asymptotically stable if condition (3.51) holds with $u_{xx} = u_{xy} = u_{yy} = 0$.

3.4.2. The case $b_1 \neq 0$

Next, we consider the case in which either b_{11} or b_{12} is nonzero. Since this assumption guarantees the controllability of the subsystem (3.35*a*)-(3.35*b*), for a nontrivial stabilization problem in this case we only consider a purely nonlinear control law. Assume A_{22} is stable and the control input is as given in (3.39) with $k_{11} = k_{12} = 0$ and $K_2 = 0$. According to Section 3.2, the stability of system (3.35) is then determined by the stability of the reduced model

$$\dot{x} = \Omega_1 y + b_{11}U(x, y, h(x, y)) + f(x, y, h(x, y)) \tag{3.55a}$$

$$\dot{y} = -\Omega_2 x + b_{12}U(x, y, h(x, y)) + g(x, y, h(x, y)), \tag{3.55b}$$

where h solves Eq. (3.11) with η substituted by $(x, y)'$ and with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

As before, we take h to be of the form (3.41), and the nonlinear control function U to be a function of x and y only, as follows:

$$\begin{aligned} U(x, y, \xi) =& u_{xx}x^2 + u_{xy}xy + u_{yy}y^2 + u_{xxx}x^3 \\ &+ u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3. \end{aligned} \tag{3.56}$$

We have the following stability criterion for the control system (3.35) in this case.

Lemma 3.6. Suppose A_{22} is stable and that $b_1 \neq 0$. Then the origin of system (3.35) is asymptotically stable if

$$
\begin{aligned}
&(b_{12}u_{xy} + g_{xy}) \cdot \{\frac{1}{\Omega_2}(b_{12}u_{xx} + g_{xx}) + \frac{1}{\Omega_1}(b_{12}u_{yy} + g_{yy})\} \\
&\quad - (b_{12}u_{xy} + f_{xy}) \cdot \{(\frac{1}{\Omega_1}(b_{11}u_{xx} + f_{xx}) + \frac{\Omega_2}{\Omega_1^2}(b_{11}u_{yy} + f_{yy})\} \\
&\quad + \frac{2}{\Omega_2}(b_{11}u_{xx} + f_{xx}) \cdot (b_{12}u_{xx} + g_{xx}) - \frac{2\Omega_2}{\Omega_1^2}(b_{11}u_{yy} + f_{yy}) \cdot (b_{12}u_{yy} + g_{yy}) \\
&\quad + 3\{b_{11}u_{xxx} + f_{xxx} + f_{x\xi}h_{xx} + \frac{\Omega_2}{3\Omega_1}(b_{11}u_{xyy} + f_{xyy} + f_{x\xi}h_{yy} + f_{y\xi}h_{xy}) \\
&\quad + \frac{1}{3}(b_{12}u_{xxy} + g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx}) \\
&\quad + \frac{\Omega_2}{\Omega_1}(b_{12}u_{yyy} + g_{yyy} + g_{y\xi}h_{yy})\} < 0,
\end{aligned}
\tag{3.57}
$$

where h_{xx}, h_{xy}, h_{yy} are as given in Eqs. (3.52)-(3.54).

A purely cubic stabilizing control law is readily obtained from Lemma 3.6.

Corollary 3.6. Let A_{22} be stable and $b_1 \neq 0$. (We do not require (3.35) to be stable.) Then system (3.35) is stabilizable by a *purely cubic* state feedback $u = u_{xxx}x^3 + u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3$.

Note that the stabilizing control laws obtained in Corollaries 3.4 and 3.6 agree with those obtained by Abed and Fu [1], where an asymptotic expansion method based on bifurcation analysis is used for controller design.

3.5. Concluding Remarks

In this chapter, the center manifold reduction technique has been applied to study the smooth feedback stabilization problem for nonlinear systems in two critical cases. The stabilizing control law designs were composite-type designs. Stability is ensured first for the noncritical state ξ, and the remaining control gains are then chosen to stabilize the reduced model, all of whose eigenvalues lie on the imaginary axis. Stabilizing control laws for two simple critical cases, where the system has one zero eigenvalue or a pair of nonzero pure imaginary eigenvalues with the remaining eigenvalues either stable or linearly controllable, have been designed in both linear and/or nonlinear feedback forms. It was found that results given in this chapter agree with those obtained by Abed and Fu [1], [2], [25], where the stabilizing control laws are obtained by applying bifurcation stability analysis.

Appendix 3.A

The coefficients in the Taylor expansions of $\hat{f}, \hat{g}$ are given in terms of those of f, g, by the following. Here, ρ denotes either f or g, and $i \neq j$ for $i, j \in \{x, y\}$ with $E_{[x]} = E_1$, and $E_{[y]} = E_2$.

$$\hat{\rho}_{ii} = \rho_{ii} + \rho_{i\xi} E_{[i]} + E'_{[i]} \rho_{\xi\xi} E_{[i]}$$

$$\hat{\rho}_{ij} = \rho_{ij} + \rho_{i\xi} E_{[j]} + \rho_{j\xi} E_{[i]} + 2E'_{[i]} \rho_{\xi\xi} E_{[j]}$$

$$\begin{aligned}\hat{\rho}_{iii} = \rho_{iii} &+ \rho_{ii\xi} E_{[i]} + E'_{[i]} \rho_{i\xi\xi} E_{[i]} + \rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[i]}) \\ &+ \rho_{i\xi} h_{ii} + 2E'_{[i]} \rho_{\xi\xi} h_{ii}\end{aligned}$$

$$\begin{aligned}\hat{\rho}_{iij} = \rho_{j\xi} h_{ii} &+ \rho_{i\xi} h_{ij} + 2E'_{[j]} \rho_{\xi\xi} h_{ii} + 2E'_{[i]} \rho_{\xi\xi} h_{ij} + \rho_{iij} + \rho_{ij\xi} E_{[i]} \\ &+ \rho_{ii\xi} E_{[j]} + E'_{[i]} \rho_{j\xi\xi} E_{[i]} + 2E'_{[i]} \rho_{i\xi\xi} E_{[j]} + 3\rho_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[j]}).\end{aligned}$$

CHAPTER

FOUR

STABILIZATION OF NONLINEAR SYSTEMS IN COMPOUND CRITICAL CASES

In this chapter, we continue to study the stabilization of nonlinear systems in critical cases by using the center manifold reduction technique. Three degenerate cases are considered in this chapter, wherein the linearized model of the system has two zero eigenvalues, one zero eigenvalue and a pair of nonzero pure imaginary eigenvalues, or two distinct pairs of nonzero pure imaginary eigenvalues; while the remaining eigenvalues are stable. Using a local nonlinear mapping (normal form reduction) and Liapunov stability criteria, one can obtain the stability conditions for the degenerate reduced models in terms of the original system dynamics. The stabilizing control laws, in linear and/or nonlinear feedback forms, are then designed for both linearly controllable and linearly uncontrollable cases. The normal form transformations obtained in this chapter have been verified by using MACSYMA.

4.1. Introduction

Recently, the center manifold theorem has been applied to the stabilization of nonlinear systems. Aeyels [4] obtained a stabilizing control law for third order systems which possess a pair of pure imaginary eigenvalues and one stable eigenvalue. This result has been extended in Chapter 3 to more general high dimensional, nonlinear systems, in which the linearized model has either a pair of pure imaginary eigenvalues or one zero eigenvalue; while the remaining eigenvalues are stable or stabilizable.

More degenerate cases have been considered by Behtash and Sastry [10]. They obtained results for nonlinear systems whose linear part has: two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two distinct pairs of pure imaginary eigenvalues. Unfortunately, they consider only the case in which the state vector dimension is one more than the number of critical modes. Most of their results are given in terms of the system dynamics after normal form reduction.

In this chapter we extend their results to more general high dimensional, nonlinear systems, where the noncritical modes are either stable or stabilizable and the number of these noncritical modes is not restricted. Moreover, the stabilizing control laws are given in terms of the original system dynamics before normal form reduction.

First, the normal form reduction technique discussed briefly in Section 2.3 is applied to derive stability conditions for low dimensional, critical nonlinear systems, specifically, where the linearized model of the system has exactly two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two pairs of pure imaginary eigenvalues. This is followed by a study of stabilization of general high dimensional, critical nonlinear systems. In Section 4.3, the stability condition derived in Section 4.2.1 for planar systems with two zero eigenvalues, along with the center manifold reduction technique reviewed in Section 3.2, are employed to design the stabilizing feedback control laws for high dimensional, nonlinear systems. A linear and/or nonlinear feedback stabilizing control law is proposed for linearly uncontrollable systems, while a purely nonlinear stabilizing control law is designed for linearly controllable systems. Similar results are obtained for the remaining two degenerate cases, in which the uncontrolled model has one zero eigenvalue and a pair of pure imaginary eigenvalues, or two distinct pairs of pure imaginary eigenvalues; while remaining eigenvalues are stable or stabilizable by linear feedback. These are given in Sections 4.4 and 4.5, respectively.

4.2. Stability Conditions for Critical Reduced Models

In the following discussion, we continue to study the stabilization of critical nonlinear systems

$$\dot{\eta} = A_{11}\eta + b_1 u + F(\eta, \xi), \tag{4.1a}$$

$$\dot{\xi} = A_{22}\xi + b_2 u + G(\eta, \xi), \tag{4.1b}$$

where functions F, G are sufficiently smooth with $F(0,0) = 0$, $DF(0,0) = 0$, $G(0,0) = 0$ and $DG(0,0) = 0$. Specifically, we consider three degenerate cases in which A_{11} has exactly two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and a pair of pure imaginary eigenvalues; or two distinct pairs of pure imaginary eigenvalues. Similar to Chapter 3, the control input u in (4.1) is taken to be a scalar. So, b_1, b_2 are both vectors. It is not difficult to extend the results to the case of multiple inputs. Details are omitted.

First, the stability conditions for the low dimensional critical system (4.1a) with $u = 0$, $\xi = 0$ are derived in this section by employing the technique of normal form reduction as in Section 2.3 and Liapunov stability criteria. These stability conditions and the center manifold reduction technique given in Section 3.2 are applied to study the stabilization of the system (4.1) in the next three sections.

In the rest of this section, we focus on the derivation of stability conditions for the low dimensional critical system (4.1a) with $u = 0$ and $\xi = 0$ as given by

$$\begin{aligned}\dot{\eta} &= A_{11}\eta + F(\eta) \\ &= A_{11}\eta + F_2(\eta, \eta) + F_3(\eta, \eta, \eta) + O(\|\eta\|^4),\end{aligned} \tag{4.2}$$

where $F(\eta) := F(\eta, 0)$ and F_2, F_3 denote quadratic and cubic terms of the Taylor expansion of F, respectively. Here, we have presumed that F is at least four times continuously differentiable.

As mentioned in Section 2.3, a nonlinear transformation $\eta = \zeta + P(\zeta)$ can be applied to simplify the expressions of the critical nonlinear systems, where P is a purely nonlinear vector function

$$P(\zeta) = P_2(\zeta,\zeta) + P_3(\zeta,\zeta,\zeta) + O(||\zeta||^4), \tag{4.3}$$

where P_2 and P_3 are the quadratic and cubic terms in P, respectively.

Applying this method to Eq. (4.2), we obtain

$$\begin{aligned}\dot{\zeta} &= (I + DP(\zeta))^{-1} F(\zeta + P(\zeta)) \\ &= \mathcal{F}_1\zeta + \mathcal{F}_2(\zeta,\zeta) + \mathcal{F}_3(\zeta,\zeta,\zeta) + O(||\zeta||^4),\end{aligned} \tag{4.4}$$

where $\mathcal{F}_1 = A_{11}$ and $\mathcal{F}_2, \mathcal{F}_3$ are as given in Eqs. (2.26)-(2.27).

The main goal of this section is to obtain the homogeneous functions P_i for which the nonlinear terms $\mathcal{F}_i$ of the transformed model (4.4) allow a simple analysis of the local stability of the origin.

4.2.1. Stability of the Second-Order Model

First, consider the case in which $\eta = (x, y)'$ is a two dimensional vector, and Eq. (4.2) is a planar system

$$\begin{aligned}\dot{x} = y &+ f_{xx}x^2 + f_{xy}xy + f_{yy}y^2 + f_{xxx}x^3 + f_{xxy}x^2y \\ &+ f_{xyy}xy^2 + f_{yyy}y^3 + O(||(x,y)||^4),\end{aligned} \tag{4.5a}$$

$$\begin{aligned}\dot{y} = g_{xx}x^2 &+ g_{xy}xy + g_{yy}y^2 + g_{xxx}x^3 + g_{xxy}x^2y \\ &+ g_{xyy}xy^2 + g_{yyy}y^3 + O(||(x,y)||^4).\end{aligned} \tag{4.5b}$$

By using the technique given in Section 2.3, it is not difficult to obtain a normal form expression for (4.5). For instance, a general form has been obtained by Takens [84]. A result of [84] for the normal form of (4.5) up to sixth order can be written as

$$\dot{x_1} = x_2 + O(||(x_1,x_2)||^6), \tag{4.6a}$$

$$\begin{aligned}\dot{x_2} = \delta_1 x_1^2 &+ \delta_2 x_1 x_2 + \delta_3 x_1^3 + \delta_4 x_1^2 x_2 + \delta_5 x_1^4 + \delta_6 x_1^3 x_2 \\ &+ \delta_7 x_1^5 + \delta_8 x_1^4 x_2 + O(||(x_1,x_2)||^6),\end{aligned} \tag{4.6b}$$

where x_1, x_2 are the transformed states after normal form reduction and δ_i are constants.

To study the local stability of (4.6) by Liapunov's direct method, we invoke a special locally positive definite function. A class of such functions has been introduced by Fu and Abed [26] for constructing families of Liapunov functions for critical nonlinear systems the linear part of which has exactly one zero eigenvalue or a pair of nonzero pure imaginary eigenvalues with the remaining eigenvalues stable. This result is extended below to a more general case, which will provide a means to obtain the stability conditions for the model (4.6).

Lemma 4.1. The scalar function

$$\begin{aligned} V(x_1,x_2) =& v_1x_2^2 + v_2x_1x_2^2 + v_3x_2^3 + v_4x_1^4 + v_5x_1^3x_2 + v_6x_1^2x_2^2 \\ &+ v_7x_1x_2^3 + v_8x_2^4 + v_9x_1^5 + v_{10}x_1^6 \end{aligned} \tag{4.7}$$

is locally positive definite near the origin if $v_1, v_4 > 0$.

Lemma 4.1 follows directly from ([26], Lemma 1). Details are omitted. It is obvious to have the following result.

Corollary 4.1. The scalar function

$$\begin{aligned} V(x_1,x_2) =& x_2^4(\delta_1 + \rho_1(x_1,x_2)) + x_1^2x_2^2(\delta_2 + \rho_2(x_1,x_2)) \\ &+ \delta_3x_1^6 + O(||(x_1,x_2)||^7) \end{aligned} \tag{4.8}$$

is locally negative definite near the origin if $\delta_i < 0$, for $i = 1,2,3$ and the smooth scalar functions ρ_1, ρ_2 satisfy $\rho_i(0,0) = 0$ for $i = 1,2$.

Next, we employ Lemma 4.1 and Corollary 4.1 to study the local stability of Eq. (4.6). Choose as a Liapunov function candidate for (4.6) a function V as in (4.7) with $v_2 = v_6 = 0$. The time derivative of V along trajectories of Eq. (4.6) is given by

$$\begin{aligned} \dot{V} =& 2v_1(\delta_1x_1^2x_2 + \delta_2x_1x_2^2) + v_7x_2^4 + (2v_1\delta_3 + 4v_4)x_1^3x_2 \\ &+ (2v_1\delta_4 + 3v_5 + 3v_3\delta_1 + \rho(x_1,x_2))x_1^2x_2^2 + v_5\delta_1x_1^5 \\ &+ (5v_9 + v_5\delta_2 + 2v_1\delta_5)x_1^4x_2 + v_5\delta_3x_1^6 \\ &+ (v_5\delta_4 + 2v_1\delta_7 + 6v_{10})x_1^5x_2 + O(||(x_1,x_2)||^7), \end{aligned} \tag{4.9}$$

where ρ is a smooth, scalar function with $\rho(0,0)=0$.

According to Lemma 4.1, V is locally positive definite if $v_1, v_4 > 0$. By employing Corollary 4.1 to check the local negative definiteness of $\dot{V}$ (given in (4.9)) and applying Liapunov stability criteria, we have

Proposition 4.1. Let $\delta_1 = \delta_2 = 0$. Then the origin of (4.6) is asymptotically stable if the values of v_i in (4.7) can be chosen such that

(i) $v_1, v_4 > 0$, $v_2 = v_6 = 0$,

(ii) $v_7, v_5\delta_3, 2v_1\delta_4 + 3v_5 < 0$,

(iii) $5v_9 + 2v_1\delta_5 = 0$, $v_5\delta_4 + 2v_1\delta_7 + 6v_{10} = 0$ and $2v_1\delta_3 + 4v_4 = 0$.

Assume $\delta_1 = \delta_2 = 0$ and $\delta_3, \delta_4 < 0$. With these assumptions we can choose v_i such that the stability conditions in Proposition 4.1 hold. As implied by Lemma 2.6, the local stability of the origin is preserved under normal form reduction. Thus, we have

Lemma 4.2. The origin is asymptotically stable for (4.5) if $\delta_1 = \delta_2 = 0$, $\delta_3, \delta_4 < 0$.

By suitable choice of nonlinear functions P_2 and P_3 (in (4.3)), we obtain the values of the δ_i as: $\delta_1 = g_{xx}$, $\delta_2 = g_{xy} + 2f_{xx}$ and

$$\delta_3 = g_{xxx} + g_{xx}f_{xy} - g_{xy}f_{xx}, \tag{4.10}$$

$$\delta_4 = g_{xxy} + 3f_{xxx} + \frac{1}{2}\{f_{yy}g_{xx} + (g_{xy} - 2f_{xx})g_{yy} + f_{xy}g_{xy}\}. \tag{4.11}$$

In the next corollary, the stability conditions of Lemma 4.2 are stated in terms of the functions f and g.

Corollary 4.2. The origin of (4.5) is asymptotically stable if $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$, $g_{xxx} + 2f_{xx}^2 < 0$ and

$$g_{xxy} + 3f_{xxx} - f_{xx}(f_{xy} + 2g_{yy}) < 0. \tag{4.12}$$

Note that the stability conditions for (4.5) given in Corollary 4.2 agree with a result of Behtash and Sastry ([10], Lemma 4.1).

4.2.2. Stability of the Third-Order Reduced Model

Next, consider the case in which $\eta = (x, y, z)'$ and model (4.2) is the three dimensional system

$$\dot{x} = \Omega_1 y + f(x, y, z), \tag{4.13a}$$

$$\dot{y} = -\Omega_2 x + g(x, y, z), \tag{4.13b}$$

$$\dot{z} = r(x, y, z), \tag{4.13c}$$

where $\Omega_1 \Omega_2 > 0$ and functions f, g, r are sufficiently smooth and take the general form

$$\begin{aligned}\varphi(x, y, z) =& \varphi_{xx} x^2 + \varphi_{xy} xy + \varphi_{xz} xz + \varphi_{yy} y^2 + \varphi_{yz} yz + \varphi_{zz} z^2 \\ &+ \varphi_{xxx} x^3 + \varphi_{xxy} x^2 y + \varphi_{xxz} x^2 z + \varphi_{xyy} xy^2 + \varphi_{xyz} xyz + \varphi_{xzz} xz^2 \\ &+ \varphi_{yyy} y^3 + \varphi_{yyz} y^2 z + \varphi_{yzz} yz^2 + \varphi_{zzz} z^3 + O(||(x, y, z)||^4).\end{aligned} \tag{4.14}$$

As explained above, it is not difficult to derive the normal form for system (4.13). For instance, a normal form for the case of $\Omega_1 = \Omega_2 = -\omega$ up to the third order approximation has been obtained in cylindrical polar coordinates by Guckenheimer and Holmes [29]. A similar result is also obtained by Behtash and Sastry [10] for designing a purely nonlinear feedback stabilizing control law for the case in which ξ in (4.1) is a scalar. However, in both results mentioned above, the values of the coefficients in the normal form for (4.13) have not been expressed in terms of the original system dynamics (i.e., the functions f, g, r). In the following discussions, a normal form representation for a general system (4.13) up to third order will be given explicitly in terms of the original system dynamics. The result will be easy to apply to the stability analysis and stabilization of higher dimensional systems (4.1). Note that we do not assume $\Omega_1 = \Omega_2$ in the following discussions.

By employing the technique given in Section 2.3 with $P = P_2$ a quadratic function as given in Appendix 4.A, we can remove parts of quadratic terms of the dynamics in (4.13), and Eq. (4.13) becomes

$$\dot{z}_1 = \Omega_1 \{ z_2 + \frac{1}{\Omega_1 + \Omega_2} (g_{yz} + f_{xz}) z_1 z_3$$

$$+ \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{zz}) z_2 z_3\} + \tilde{f}(z_1, z_2, z_3) \tag{4.15a}$$

$$\begin{aligned}\dot{z}_2 =& \Omega_2\{-z_1 + \frac{1}{\Omega_1 + \Omega_2}(g_{yz} + f_{zz}) z_2 z_3 \\ & - \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{zz}) z_1 z_3\} + \tilde{g}(z_1, z_2, z_3)\end{aligned} \tag{4.15b}$$

$$\dot{z}_3 = \frac{1}{\Omega_1 + \Omega_2}(\Omega_1 r_{zz} + \Omega_2 r_{yy}) \cdot (z_1^2 + z_2^2) + r_{zz} z_3^2 + \tilde{r}(z_1, z_2, z_3). \tag{4.15c}$$

Assume that the nonlinear vector function P in the normal form transformation is chosen as $P(\eta) = P_2(\eta) + P_3(\eta)$ with P_2 and P_3 as given in Appendix 4.A. The new transformed version of (4.13) is then

$$\begin{aligned}\dot{x}_1 =& \Omega_1\{x_2 + \frac{1}{\Omega_1 + \Omega_2}(g_{yz} + f_{xz}) x_1 x_3 \\ & + \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{xz}) x_2 x_3 + \delta_1 x_1 (x_1^2 + x_2^2) \\ & + \epsilon_1 x_2 (x_1^2 + x_2^2) + x_3^2(\delta_2 x_1 + \epsilon_2 x_2)\} + O(||(x, y, z)||^4),\end{aligned} \tag{4.16a}$$

$$\begin{aligned}\dot{x}_2 =& \Omega_2\{-x_1 + \frac{1}{\Omega_1 + \Omega_2}(g_{yz} + f_{xz}) x_2 x_3 \\ & - \frac{1}{2\Omega_1\Omega_2}(\Omega_2 f_{yz} - \Omega_1 g_{xz}) x_1 x_3 + \delta_1 x_2 (x_1^2 + x_2^2) \\ & - \epsilon_1 x_1 (x_1^2 + x_2^2) + x_3^2(\delta_2 x_2 - \epsilon_2 x_1)\} + O(||(x, y, z)||^4),\end{aligned} \tag{4.16b}$$

$$\begin{aligned}\dot{x}_3 =& \frac{1}{\Omega_1 + \Omega_2}(\Omega_1 r_{xx} + \Omega_2 r_{yy}) \cdot (x_1^2 + x_2^2) + r_{zz} x_3^2 \\ & + \delta_3 x_3 (x_1^2 + x_2^2) + \tilde{r}_{zzz} x_3^3 + O(||(x, y, z)||^4),\end{aligned} \tag{4.16c}$$

where

$$\epsilon_1 = \frac{1}{4\Omega_1\Omega_2(\Omega_1 + \Omega_2)} \cdot \{3\Omega_2^2 \tilde{f}_{222} + \Omega_1\Omega_2(\tilde{f}_{112} - \tilde{g}_{122}) - 3\Omega_1^2 \tilde{g}_{111}\}, \tag{4.17}$$

$$\epsilon_2 = \frac{1}{2\Omega_1\Omega_2}(\Omega_2 \tilde{f}_{233} - \Omega_1 \tilde{g}_{133}), \tag{4.18}$$

$$\delta_1 = \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2} \cdot \{\Omega_1(3\tilde{f}_{111} + \tilde{g}_{112}) + \Omega_2(3\tilde{g}_{222} + \tilde{f}_{122})\}, \tag{4.19}$$

$$\delta_2 = \frac{1}{\Omega_1 + \Omega_2}(\tilde{f}_{133} + \tilde{g}_{233}), \tag{4.20}$$

$$\delta_3 = \frac{1}{\Omega_1 + \Omega_2}(\Omega_1 \tilde{r}_{113} + \Omega_2 \tilde{r}_{223}). \tag{4.21}$$

Here, φ_{ijk} denotes the coefficient of the cubic term $z_i z_j z_k$ of a function $\varphi \in \{\tilde{f}, \tilde{g}, \tilde{r}\}$ and $i, j, k = 1, 2, 3$.

Using Corollary 4.1 and Lemma 2.6, we obtain the following stability conditions for (4.13) based on the transformed model (4.16).

Lemma 4.3. The origin of (4.13) is asymptotically stable if $r_{zz} = 0$, $\Omega_1 \delta_1$, $\tilde{r}_{333} < 0$, and either of the following conditions hold:

(i) $g_{yz} + f_{xz} = 0$, $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$, and $\Omega_1 \delta_2, \delta_3 \leq 0$ or $\Omega_1 \delta_2$ and δ_3 are nonzero and of opposite sign,

(ii) $\Omega_1(g_{yz} + f_{xz})$ and $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ are nonzero and are of opposite sign, and $\Omega_1 \delta_2, \delta_3 \leq 0$,

where the values of δ_i, $i = 1, 2, 3$ are given in (4.19)-(4.21).

Proof: As discussed above, Eq. (4.13) can be transformed into (4.16) by normal form reduction. Choose

$$V = p_1(x_1^2 + \frac{\Omega_1}{\Omega_2} x_2^2) + p_2 x_3^2 \tag{4.22}$$

with $p_1, p_2 > 0$ as a Liapunov function candidate for the transformed model (4.16).

The time derivative of V along trajectories of (4.16) is

$$\begin{aligned} \dot{V} &= 2\Omega_1 p_1 \delta_1 (x_1^2 + x_2^2)^2 + 2(p_1 \Omega_1 \delta_2 + p_2 \delta_3) x_3^2 (x_1^2 + x_2^2) \\ &\quad + 2p_2 \tilde{r}_{333} x_3^4 + 2p_2 r_{zz} x_3^3 + \frac{2}{\Omega_1 + \Omega_2} \{\Omega_1 p_1 (g_{yz} + f_{xz}) \\ &\quad + p_2(\Omega_1 r_{xx} + \Omega_2 r_{yy})\} x_3 (x_1^2 + x_2^2) + O(||(x_1, x_2, x_3)||^5). \end{aligned} \tag{4.23}$$

Since $p_1, p_2 > 0$, the scalar function V given in (4.22) is positive definite. Suppose $r_{zz} = 0$ and $\Omega_1 \delta_1, \tilde{r}_{333} < 0$. From Corollary 4.1, it follows that $\dot{V}$

(given in (4.23)) is locally negative definite if either condition (i) or (ii) holds. The application of Liapunov stability criteria to (4.16) indicates that the origin is asymptotically stable. As implied by Lemma 2.6 the origin is also asymptotically stable for the model (4.13).

■

Note that the stability condition (i) of Lemma 4.3 above agrees with that obtained by Behtash and Sastry ([10], Theorem 4.2).

By expressing the values δ_i in terms of the original system dynamics, we have the following result for the case (i) of Lemma 4.3.

Corollary 4.3. The origin is asymptotically stable for (4.13) if $r_{zz} = 0$, $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$, $f_{xz} + g_{yz} = 0$, $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of opposite sign, where

$$
\begin{aligned}
S_1 :=& \tilde{r}_{333} \\
=& \frac{1}{\Omega_1 \Omega_2}\{\Omega_1 \Omega_2 r_{zzz} - \Omega_2 f_{zz} r_{yz} + \Omega_1 g_{zz} r_{xz}\}, && (4.24) \\
S_2 :=& \Omega_1 \delta_1 \\
=& \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\{(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy} - \Omega_1 g_{yz} r_{xy} + 3\Omega_1\Omega_2 g_{yyy} \\
&+ (\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx} g_{xy} + \Omega_1^2 g_{xxy} + \frac{2\Omega_1^2}{\Omega_2} f_{xx} g_{xx} \\
&- \Omega_2 f_{xy} f_{yy} + \Omega_1\Omega_2 f_{xyy} - \Omega_1 f_{xx} f_{xy} + 3\Omega_1^2 f_{xxx}\}, && (4.25) \\
S_3 :=& \Omega_1 \delta_2 \\
=& \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\{2\Omega_2 f_{zz} r_{yz} - 2\Omega_1 g_{zz} r_{xz} + \Omega_1 (g_{xy} + 2 f_{xx}) g_{zz} \\
&+ \Omega_1\Omega_2 g_{yzz} - 2\Omega_2 f_{zz} g_{yy} - \Omega_2 f_{xy} f_{zz} + \Omega_1\Omega_2 f_{xzz}\}, && (4.26) \\
S_4 :=& \delta_3 \\
=& \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\{\Omega_2^2 r_{yyz} - \frac{\Omega_2}{\Omega_1}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) r_{yz} - \frac{\Omega_2}{\Omega_1}(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy} \\
&+ (\Omega_2 g_{yy} + \Omega_1 g_{xx}) r_{xz} + \Omega_2 g_{yz} r_{xy} + \Omega_1\Omega_2 r_{xxz}\}. && (4.27)
\end{aligned}
$$

Similarly, the case (ii) of Lemma 4.3 is addressed in terms of the original dynamics as follows.

Corollary 4.4. The origin of (4.13) is asymptotically stable if (i) $r_{zz} = 0$, (ii) $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ and $f_{xz} + g_{yz}$ have nonzero values and of opposite sign, (iii) $S_1, \tilde{S}_2 < 0$ and $\tilde{S}_3, \tilde{S}_4 \leq 0$, where S_1 is given by (4.24) and

$$\begin{aligned}
\tilde{S}_2 =& \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\{-\frac{\Omega_1 g_{xz} + \Omega_2 f_{yz}}{(\Omega_1 + \Omega_2)\Omega_2}[(\Omega_2 + 2\Omega_1)\Omega_2 r_{yy} \\
&+ (2\Omega_2 + 3\Omega_1)\Omega_1 r_{xx}] + \frac{\Omega_1}{2}(g_{yz} + 3f_{xz})r_{xy} + 3\Omega_1\Omega_2 g_{yyy} \\
&+ (\Omega_1 g_{xy} - 2\Omega_2 f_{yy})g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx} g_{xy} + \Omega_1^2 g_{xxy} + \frac{2\Omega_1^2}{\Omega_2} f_{xx} g_{xx} \\
&- \Omega_2 f_{xy} f_{yy} + \Omega_1\Omega_2 f_{xyy} - \Omega_1 f_{xx} f_{xy} + 3\Omega_1^2 f_{xxx}\},
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\tilde{S}_3 =& \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\{2\Omega_2 f_{zz} r_{yz} - 2\Omega_1 g_{zz} r_{xz} + \Omega_1(g_{xy} + 2f_{xx})g_{zz} \\
&+ \Omega_1\Omega_2 g_{yzz} - 2\Omega_2 f_{zz} g_{yy} - \Omega_2 f_{xy} f_{zz} + \Omega_1\Omega_2 f_{xzz} \\
&- \frac{\Omega_1}{2(\Omega_1 + \Omega_2)}(f_{xz} + g_{yz}) \cdot (\Omega_1 g_{xz} + \Omega_2 f_{yz})\},
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\tilde{S}_4 =& \frac{1}{\Omega_2(\Omega_1 + \Omega_2)}\{\Omega_2^2 r_{yyz} - \frac{\Omega_2}{\Omega_1}(\Omega_2 f_{yy} + \Omega_1 f_{xx})r_{yz} \\
&+ (\Omega_1 g_{xz} + \Omega_2 f_{yz})r_{xx} + (\Omega_2 g_{yy} + \Omega_1 g_{xx})r_{xz} \\
&- \frac{\Omega_2}{\Omega_1 + \Omega_2}[\Omega_1 g_{yz} + (\Omega_2 + 2\Omega_1)f_{xz}]r_{xy} + \Omega_1\Omega_2 r_{xxz}\}.
\end{aligned} \tag{4.30}$$

4.2.3. Stability of Fourth Order Systems

In this section, we derive stability conditions for (4.2) in which $\eta := (x, y, z, w)'$, $F(\eta) = (f(\eta), g(\eta), r(\eta), s(\eta))'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix}. \tag{4.31}$$

Here, $\Omega_1\Omega_2, \Omega_3\Omega_4 > 0$ and f, g, r, s are smooth, purely nonlinear scalar functions with the form as

$$\begin{aligned}\varphi =&\varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{xw}xw + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{yw}yw \\ &+ \varphi_{zz}z^2 + \varphi_{zw}zw + \varphi_{ww}w^2 + \varphi_{xxx}x^3 + (\varphi_{xxy}y + \varphi_{xxz}z + \varphi_{xxw}w)x^2 \\ &+ (\varphi_{xyy}x + \varphi_{yyy}y + \varphi_{yyz}z + \varphi_{yyw}w)y^2 + \varphi_{xyz}xyz + \varphi_{xyw}xyw + \varphi_{xzw}zxw \\ &+ \varphi_{yzw}yzw + (\varphi_{xzz}x + \varphi_{yzz}y + \varphi_{zzz}z + \varphi_{zzw}w)z^2 + (\varphi_{xww}x + \varphi_{yww}y \\ &+ \varphi_{zww}z + \varphi_{www}w)w^2 + O(||(x,y,z,w)||^4). \end{aligned} \tag{4.32}$$

For the case in which $\Omega_1 = \Omega_2 = -1$ and $\Omega_3 = \Omega_4 = -\omega \notin \{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$, a normal form for the model (4.2) has been obtained by using the technique given in Section 2.3; see for instance, [10], [29]. In the following analysis, we do not assume that $\Omega_1 = \Omega_2$ nor that $\Omega_3 = \Omega_4$.

Assume that $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. By using the technique of normal form reduction as discussed in Section 2.3 to let $\eta = \zeta + P(\zeta)$ with P defined in Eq. (4.3), we can write model (4.2) as Eq. (4.4). First, consider the case in which the nonlinear function P is a purely quadratic function only (i.e., $P = P_2$) as given in Appendix 4.B, we can make $\mathcal{F}_2$ (given in Eq. (2.26)) zero and Eq. (4.4) then becomes

$$\dot{\zeta} = A\zeta + \tilde{F}(\zeta), \tag{4.33}$$

where $\tilde{F}(\zeta) = (\tilde{f}(\zeta), \tilde{g}(\zeta), \tilde{r}(\zeta), \tilde{s}(\zeta))'$. Now, let P be a nonlinear function as given in (4.3) with P_2 having being as discussed above such that $\mathcal{F}_2 = 0$. By a suitable choice of cubic function P_3, as detailed in Appendix 4.B, the transformed model (4.4) takes the form

$$\begin{aligned}\dot{x}_1 =&\Omega_1\{x_2 + (\delta_1 x_1 + \epsilon_1 x_2)(x_1^2 + x_2^2) + (\delta_2 x_1 + \epsilon_2 x_2)(x_3^2 + x_4^2)\} \\ &+ O(||(x_1, x_2, x_3, x_4)||^4), \end{aligned} \tag{4.34a}$$

$$\begin{aligned}\dot{x}_2 =&\Omega_2\{-x_1 + (\delta_1 x_2 - \epsilon_1 x_1)(x_1^2 + x_2^2) + (\delta_2 x_2 - \epsilon_2 x_1)(x_3^2 + x_4^2)\} \\ &+ O(||(x_1, x_2, x_3, x_4)||^4), \end{aligned} \tag{4.34b}$$

$$\begin{aligned}\dot{x}_3 =&\Omega_3\{x_4 + (\delta_3 x_3 + \epsilon_3 x_4)(x_1^2 + x_2^2) + (\delta_4 x_3 + \epsilon_4 x_4)(x_3^2 + x_4^2)\} \\ &+ O(||(x_1, x_2, x_3, x_4)||^4), \end{aligned} \tag{4.34c}$$

$$\dot{x}_4 = \Omega_4\{-x_3 + (\delta_3 x_4 - \epsilon_3 x_3)(x_1^2 + x_2^2) + (\delta_4 x_4 - \epsilon_4 x_3)(x_3^2 + x_4^2)\} + O(||(x_1, x_2, x_3, x_4)||^4), \tag{4.34d}$$

where

$$\delta_1 = \frac{\Omega_2(3\tilde{g}_{222} + \tilde{f}_{122}) + \Omega_1(\tilde{g}_{112} + 3\tilde{f}_{111})}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2} \tag{4.35}$$

$$\epsilon_1 = \frac{\Omega_1\Omega_2(\tilde{f}_{112} - \tilde{g}_{122}) + 3\Omega_2^2\tilde{f}_{222} - 3\Omega_1^2\tilde{g}_{111}}{4\Omega_1\Omega_2(\Omega_1 + \Omega_2)} \tag{4.36}$$

$$\delta_2 = \frac{\Omega_3(\tilde{f}_{133} + \tilde{g}_{233}) + \Omega_4(\tilde{f}_{144} + \tilde{g}_{244})}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)} \tag{4.37}$$

$$\epsilon_2 = \frac{\Omega_2(\Omega_3\tilde{f}_{233} + \Omega_4\tilde{f}_{244}) - \Omega_1(\Omega_3\tilde{g}_{133} + \Omega_4\tilde{g}_{144})}{2\Omega_1\Omega_2(\Omega_3 + \Omega_4)} \tag{4.38}$$

$$\delta_3 = \frac{\Omega_1(\tilde{r}_{113} + \tilde{s}_{114}) + \Omega_2(\tilde{r}_{223} + \tilde{s}_{224})}{(\Omega_1 + \Omega_2)\cdot(\Omega_3 + \Omega_4)} \tag{4.39}$$

$$\epsilon_3 = \frac{\Omega_4(\Omega_1\tilde{r}_{114} + \Omega_2\tilde{r}_{224}) - \Omega_3(\Omega_1\tilde{s}_{113} + \Omega_2\tilde{s}_{223})}{2\Omega_3\Omega_4(\Omega_1 + \Omega_2)} \tag{4.40}$$

$$\delta_4 = \frac{\Omega_4(3\tilde{s}_{444} + \tilde{r}_{344}) + \Omega_3(\tilde{s}_{334} + 3\tilde{r}_{333})}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2} \tag{4.41}$$

$$\epsilon_4 = \frac{\Omega_3\Omega_4(\tilde{r}_{334} - \tilde{s}_{344}) + 3\Omega_4^2\tilde{r}_{444} - 3\Omega_3^2\tilde{s}_{333}}{4\Omega_3\Omega_4(\Omega_3 + \Omega_4)}. \tag{4.42}$$

Here, let $\zeta := (z_1, z_2, z_3, z_4)'$ in (4.33) then φ_{ijk} denotes the coefficient of the cubic term $z_i z_j z_k$ of a function φ, for $\varphi = \tilde{f}, \tilde{g}, \tilde{r}, \tilde{s}$ and $i, j, k = 1, \cdots, 4$.

Referring to the transformed model (4.34), we readily obtain the following stability conditions for the original model (4.2).

Lemma 4.4. Let $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. The origin is asymptotically stable for system (4.2) if $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$ and either $\Omega_1\delta_2 \leq 0$ and $\Omega_3\delta_3 \leq 0$, or $\Omega_1\delta_2$ and $\Omega_3\delta_3$ are nonzero and of opposite sign.

Proof: As discussed above, system (4.2) can be transformed into Eq. (4.34) if $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. Let

$$V = \frac{1}{2}p_1(x_1^2 + \frac{\Omega_1}{\Omega_2}x_2^2) + \frac{1}{2}p_2(x_3^2 + \frac{\Omega_3}{\Omega_4}x_4^2) \tag{4.43}$$

be a Liapunov function candidate for model (4.34) with $p_1, p_2 > 0$. Taking the time derivative of V along trajectories of the model (4.34), we then have

$$\begin{aligned}\dot{V} =& p_1\Omega_1\delta_1(x_1^2+x_2^2)^2 + (p_1\Omega_1\delta_2 + p_2\Omega_3\delta_3)\cdot(x_1^2+x_2^2)\cdot(x_3^2+x_4^2) \\ &+ p_2\Omega_3\delta_4(x_3^2+x_4^2)^2 + O(||(x_1,x_2,x_3,x_4)||^5). \end{aligned} \tag{4.44}$$

Since $p_1, p_2 > 0$ and $\Omega_1\Omega_2, \Omega_3\Omega_4 > 0$, the scalar function V given in (4.43) is positive definite. First, consider the case in which $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$, $\Omega_1\delta_2 \leq 0$ and $\Omega_3\delta_3 \leq 0$. Since $p_1, p_2 > 0$, $\dot{V}$ given in (4.44) is locally negative definite. So the origin is asymptotically stable for the transformed model (4.34). By Lemma 2.6, the origin is also asymptotically stable for the original model (4.2).

Next, consider the case in which $\Omega_1\delta_1 < 0$, $\Omega_3\delta_4 < 0$, $\Omega_1\delta_2$ and $\Omega_3\delta_3$ are nonzero and of opposite sign. Similarly, we can show that $\dot{V}$ given in (4.44) is locally negative definite by choosing $p_1, p_2 > 0$ such that $p_1\Omega_1\delta_2 + p_2\Omega_3\delta_3 = 0$. The stability of the origin for model (4.2) is hence implied by the Liapunov stability criteria and Lemma 2.6.

■

Note that, for the case in which $\Omega_1 = \Omega_2 = -1$ and $\Omega_3 = \Omega_4 = -\omega \notin \{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$, Lemma 4.4 agrees with a result of Behtash and Sastry ([10], Theorem 4.3). The stability conditions of Lemma 4.4 are expressed in terms of the original nonlinear dynamics before normal form reduction are given in the next result.

Corollary 4.5. Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. The origin of (4.2) is asymptotically stable if $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of opposite sign, where

$$\begin{aligned} S_1 = & \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\{\Omega_1[3(\Omega_2 g_{yyy} + \Omega_1 f_{xxx}) + (\Omega_1 g_{xxy} + \Omega_2 f_{xyy})] \\ &+ g_{yy}(\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) - f_{xy}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) + \frac{\Omega_1^2}{\Omega_2} g_{xx}(g_{xy} + 2f_{xx}) \\ &+ \frac{\Omega_1}{\Omega_4}[(3\Omega_2 s_{yy} + \Omega_1 s_{xx})g_{yz} + (3\Omega_1 s_{xx} + \Omega_2 s_{yy})f_{xz}] \end{aligned}$$

$$
\begin{aligned}
&-\frac{\Omega_1}{\Omega_3}[(\Omega_1 r_{xx}+3\Omega_2 r_{yy})g_{yw}+(\Omega_2 r_{yy}+3\Omega_1 r_{xx})f_{xw}]\\
&+\frac{\Omega_1}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_4}[\Omega_1(\Omega_4 g_{xw}-2\Omega_2 g_{yz})+\Omega_2(\Omega_4 f_{yw}+2\Omega_1 f_{xz})]\cdot\\
&(\Omega_4 r_{xy}-2\Omega_1 s_{xx}+2\Omega_2 s_{yy})-\frac{\Omega_1}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_3}[\Omega_1(2\Omega_2 g_{yw}+\Omega_3 g_{xz})\\
&-\Omega_2(2\Omega_1 f_{xw}-\Omega_3 f_{yz})]\cdot(\Omega_3 s_{xy}-2\Omega_2 r_{yy}+2\Omega_1 r_{xx})\},
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
S_2=&\frac{1}{3\Omega_3^2+2\Omega_3\Omega_4+3\Omega_4^2}\{\Omega_3[3(\Omega_4 s_{www}+\Omega_3 r_{zzz})+(\Omega_3 s_{zzw}+\Omega_4 r_{zww})]\\
&+s_{ww}(\Omega_3 s_{zw}-2\Omega_4 r_{ww})-r_{zw}(\Omega_4 r_{ww}+\Omega_3 r_{zz})+\frac{\Omega_3^2}{\Omega_4}s_{zz}(s_{zw}+2r_{zz})\\
&+\frac{\Omega_3}{\Omega_2}[(3\Omega_4 g_{ww}+\Omega_3 g_{zz})s_{xw}+(3\Omega_3 g_{zz}+\Omega_4 g_{ww})r_{xz}]\\
&-\frac{\Omega_3}{\Omega_1}[(\Omega_3 f_{zz}+3\Omega_4 f_{ww})s_{yw}+(\Omega_4 f_{ww}+3\Omega_3 f_{zz})r_{yz}]\\
&+\frac{\Omega_3}{(4\Omega_3\Omega_4-\Omega_1\Omega_2)\Omega_2}[\Omega_3(\Omega_2 s_{yz}-2\Omega_4 s_{xw})\\
&+\Omega_4(\Omega_2 r_{yw}+2\Omega_3 r_{xz})]\cdot(\Omega_2 f_{zw}-2\Omega_3 g_{zz}+2\Omega_4 g_{ww})\\
&-\frac{\Omega_3}{(4\Omega_3\Omega_4-\Omega_1\Omega_2)\Omega_1}[\Omega_3(2\Omega_4 s_{yw}+\Omega_1 s_{xz})\\
&-\Omega_4(2\Omega_3 r_{yz}-\Omega_1 r_{xw})]\cdot(\Omega_1 g_{zw}-2\Omega_4 f_{ww}+2\Omega_3 f_{zz})\},
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
S_3=&\frac{\Omega_3}{(\Omega_1+\Omega_2)\cdot(\Omega_3+\Omega_4)}\{2f_{zz}r_{yz}+\frac{1}{\Omega_3}[2\Omega_4 f_{ww}s_{yw}\\
&+\Omega_1\Omega_4(f_{xww}+g_{yww})]+\Omega_1(f_{xzz}+g_{yzz})\\
&-\frac{2\Omega_1}{\Omega_2}g_{zz}r_{xz}+\frac{\Omega_1}{\Omega_3^2\Omega_4}[\Omega_3(\Omega_4 s_{ww}+\Omega_3 s_{zz})(g_{yz}+f_{xz})\\
&-\Omega_4(\Omega_4 r_{ww}+\Omega_3 r_{zz})(g_{yw}+f_{xw})]-\frac{2\Omega_1\Omega_4}{\Omega_3\Omega_2}g_{ww}s_{xw}\\
&+\frac{1}{\Omega_3\Omega_2}[\Omega_1(\Omega_3 g_{zz}+\Omega_4 g_{ww})(g_{xy}+2f_{xx})\\
&-\Omega_2(\Omega_3 f_{zz}+\Omega_4 f_{ww})(f_{xy}+2g_{yy})]
\end{aligned}
$$

$$+\frac{1}{(4\Omega_3\Omega_4-\Omega_1\Omega_2)\Omega_3}[\Omega_4(\Omega_1 r_{xw}-2\Omega_3 r_{yz})$$

$$+\Omega_3(\Omega_1 s_{xz}+2\Omega_4 s_{yw})]\cdot(\Omega_1 g_{zw}-2\Omega_4 f_{ww}+2\Omega_3 f_{zz})$$

$$-\frac{\Omega_1}{(4\Omega_3\Omega_4-\Omega_1\Omega_2)\Omega_3\Omega_2}[\Omega_4(\Omega_2 r_{yw}+2\Omega_3 r_{xz})$$

$$+\Omega_3(\Omega_2 s_{yz}-2\Omega_4 s_{xw})]\cdot(\Omega_2 f_{zw}-2\Omega_3 g_{zz}+2\Omega_4 g_{ww})\}, \tag{4.47}$$

$$S_4=\frac{\Omega_1}{(\Omega_1+\Omega_2)\cdot(\Omega_3+\Omega_4)}\{2r_{xx}f_{xw}+\frac{1}{\Omega_1}[2\Omega_2 r_{yy}g_{yw}+\Omega_3\Omega_2(r_{yyz}+s_{yyw})]$$

$$+\frac{\Omega_3}{\Omega_1^2\Omega_2}[\Omega_1(\Omega_2 g_{yy}+\Omega_1 g_{xx})(s_{xw}+r_{xz})-\Omega_2(\Omega_2 f_{yy}+\Omega_1 f_{xx})(s_{yw}+r_{yz})]$$

$$-\frac{2\Omega_3}{\Omega_4}s_{xx}f_{xz}+\frac{1}{\Omega_1\Omega_4}[\Omega_3(\Omega_1 s_{xx}+\Omega_2 s_{yy})(s_{zw}+2r_{zz})$$

$$-\Omega_4(\Omega_1 r_{xx}+\Omega_2 r_{yy})(r_{zw}+2s_{ww})]-\frac{2\Omega_3\Omega_2}{\Omega_1\Omega_4}s_{yy}g_{yz}$$

$$+\frac{1}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_1}[\Omega_2(\Omega_3 f_{yz}-2\Omega_1 f_{xw})$$

$$+\Omega_1(\Omega_3 g_{xz}+2\Omega_2 g_{yw})]\cdot(\Omega_3 s_{xy}-2\Omega_2 r_{yy}+2\Omega_1 r_{xx})$$

$$-\frac{\Omega_3}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_1\Omega_4}[\Omega_2(\Omega_4 f_{yw}+2\Omega_1 f_{xz})+\Omega_1(\Omega_4 g_{xw}$$

$$-2\Omega_2 g_{yz})]\cdot(\Omega_4 r_{xy}-2\Omega_1 s_{xx}+2\Omega_2 s_{yy})+\Omega_3(r_{xxz}+s_{xxw})\}. \tag{4.48}$$

4.3. Double Zero Eigenvalue

In this section, we consider the case in which η and b_1 are both two dimensional vectors. Thus, $\eta := (x,y)'$ and $b_1 = (b_{11}, b_{12})'$, $F := (f,g)'$ and

$$A_{11}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \tag{4.49}$$

$$\begin{aligned}G(x,y,\xi)=&x^2G_{xx}+xyG_{xy}+y^2G_{yy}+(xG_{x\xi}+yG_{y\xi})\xi\\&+G_{\xi\xi}(\xi,\xi)+x^3G_{xxx}+x^2yG_{xxy}+xy^2G_{xyy}\\&+y^3G_{yyy}+(x^2G_{xx\xi}+xyG_{xy\xi}+y^2G_{yy\xi})\xi+xG_{x\xi\xi}(\xi,\xi)\\&+yG_{y\xi\xi}(\xi,\xi)+G_{\xi\xi\xi}(\xi,\xi,\xi)\ +O(||(x,y,\xi)||^4).\end{aligned} \tag{4.50}$$

The scalar functions f, g are taken to be the form

$$\begin{aligned}\varphi(x,y,\xi) = &\varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{yy}y^2 + (x\varphi_{x\xi} + y\varphi_{y\xi})\xi \\ &+ \xi'\varphi_{\xi\xi}\xi + \varphi_{xxx}x^3 + \varphi_{xxy}x^2y + \varphi_{xyy}xy^2 \\ &+ \varphi_{yyy}y^3 + (x^2\varphi_{xx\xi} + xy\varphi_{xy\xi} + y^2\varphi_{yy\xi})\xi \\ &+ \xi'(x\varphi_{x\xi\xi} + y\varphi_{y\xi\xi})\xi + \varphi_{\xi\xi\xi}(\xi,\xi,\xi) \ + O(||(x,y,\xi)||^4). \quad (4.51)\end{aligned}$$

The coefficients in the expansions in (4.50) and (4.51) are either constants or symmetric multilinear functions of their arguments. For instance, $\varphi_{\xi\xi\xi}$ and $G_{\xi\xi}$ denote a symmetric trilinear function and a symmetric bilinear function, respectively.

4.3.1. The case $b_1 = 0$

In this subsection, we consider the case in which $b_1 = 0$ and let the feedback control u be given by

$$u(x,y,\xi) = k_{11}x + k_{12}y + K_2\xi + U(x,y,\xi), \quad (4.52)$$

where k_{11}, k_{12} are scalars and U is a smooth function with $U(0,0,0) = 0$ and $DU(0,0,0) = 0$.

Suppose $A_{22} + b_2K_2$ is stable. As discussed in Section 3.2, the stability of system (4.1) agrees with the stability of the reduced model

$$\dot{x} = y + f(x,y,E_1x + E_2y + h(x,y)) \quad (4.53a)$$

$$\dot{y} = g(x,y,E_1x + E_2y + h(x,y)), \quad (4.53b)$$

where $E = (E_1, E_2)$ and $h(x,y)$ solve Eqs. (4.54) and (4.55) below, respectively:

$$b_2K_1 + (A_{22} + b_2K_2)E - EA_{11} = 0, \quad (4.54)$$

$$\begin{aligned}Dh(\eta)\cdot\{A_{11}\eta + F(\eta,h(\eta)+E\eta)\} = &(A_{22}+b_2K_2)h(\eta) \\ &+ b_2U(\eta,h(\eta)+E\eta) + G(\eta,h(\eta)+E\eta), \quad (4.55)\end{aligned}$$

with boundary conditions $h(0,0) = 0$ and $Dh(0,0) = 0$.

The boundary conditions above dictate that h be of the form

$$h(x,y) = x^2 h_{xx} + xy h_{xy} + y^2 h_{yy} + O(||(x,y)||^3), \tag{4.56}$$

where h_{xx}, h_{xy}, h_{yy} are constant vectors.

Let the nonlinear control function U have the form (4.51) and

$$\begin{aligned} H(x,y) :=& b_2 U(x,y,E_1 x + E_2 y) + G(x,y,E_1 x + E_2 y) \\ &- f(x,y,E_1 x + E_2 y)E_1 - g(x,y,E_1 x + E_2 y)E_2 \\ =& x^2 H_{xx} + xy H_{xy} + y^2 H_{yy} + O(||(x,y)||^3). \end{aligned} \tag{4.57}$$

By solving Eqs. (4.54)-(4.55), we then have

$$E_1 = -k_{11}(A_{22} + b_2 K_2)^{-1} b_2 \tag{4.58}$$

$$E_2 = -\{(A_{22} + b_2 K_2)^2\}^{-1} \cdot \{k_{12}(A_{22} + b_2 K_2) + k_{11} I\} b_2 \tag{4.59}$$

and

$$h_{xx} = -(A_{22} + b_2 K_2)^{-1} H_{xx}, \tag{4.60}$$

$$h_{xy} = -2\{(A_{22} + b_2 K_2)^2\}^{-1} H_{xx} - (A_{22} + b_2 K_2)^{-1} H_{xy}, \tag{4.61}$$

$$h_{yy} = -(A_{22} + b_2 K_2)^{-1}(H_{yy} - h_{xy}). \tag{4.62}$$

The reduced model (4.53) is hence obtained as

$$\begin{aligned} \dot{x} =& y + \hat{f}_{xx} x^2 + \hat{f}_{xy} xy + \hat{f}_{yy} y^2 + \hat{f}_{xxx} x^3 \\ &+ \hat{f}_{xxy} x^2 y + \hat{f}_{xyy} xy^2 + \hat{f}_{yyy} y^3 + O(||(x,y)||^4) \end{aligned} \tag{4.63a}$$

$$\begin{aligned} \dot{y} =& \hat{g}_{xx} x^2 + \hat{g}_{xy} xy + \hat{g}_{yy} y^2 + \hat{g}_{xxx} x^3 \\ &+ \hat{g}_{xxy} x^2 y + \hat{g}_{xyy} xy^2 + \hat{g}_{yyy} y^3 + O(||(x,y)||^4), \end{aligned} \tag{4.63b}$$

where φ_{ij} and φ_{ijk} denote the coefficients of quadratic terms ij and cubic terms ijk of function φ, for $\varphi = \hat{f}, \hat{g}$ and $i,j,k \in \{x,y\}$, respectively, and are given in Appendix 4.C.

Now, referring to the stability criterion given in Corollary 4.2 and the foregoing discussions, we have

Proposition 4.2. Assume that $b_{11} = b_{12} = 0$, the control input is given by (4.52) and the nonlinear function U has the form as the one given in (4.51). Then the origin of (4.1) is asymptotically stable if (i) $A_{22} + b_2 K_2$ is stable, (ii) $\hat{g}_{xx} = 0$, $\hat{g}_{xy} + 2\hat{f}_{xx} = 0$, (iii) $\hat{g}_{xxx} + 2\hat{f}_{xx}^2 < 0$ and (iv) $\hat{g}_{xxy} + 3\hat{f}_{xxx} - \hat{f}_{xx}(\hat{f}_{xy} + 2\hat{g}_{yy}) < 0$.

It can be seen from Proposition 4.2 and Appendix 4.C that only the quadratic terms of the function G, and the linear and quadratic terms of control input u contribute to the stability conditions. Thus, a linear and/or quadratic feedback stabilizing control law is implied by Proposition 4.2. Although a purely linear feedback stabilizing control law might conceivably be obtained by using Proposition 4.2, in general construction of such a control law is not feasible.

Consider a special case of system (4.1) in which ξ is a scalar. So, b_2 is a scalar. Referring to Eqs. (4.57)-(4.62), we can determine the values of E_1, E_2, h_{xx}, h_{xy} and h_{yy} from the linear and quadratic gains of the control input. A linear-plus-quadratic stabilizing control law can hence be obtained as follows.

Lemma 4.5. Assume that ξ is a scalar, $b_{11} = b_{12} = 0$ and system (4.1) may or may not be stable. If $A_{22} + b_2 K_2$ is stable and $g_{x\xi} \neq 0$, then a *linear-plus-quadratic* feedback can be designed to guarantee the stability of the origin of (4.1). The proposed feedback control has the form as $u = k_{11}x + k_{12}y + K_2\xi + u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$.

Note that a purely quadratic feedback stabilizing control law, under the assumptions: $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$ and $g_{x\xi} \neq 0$, given by Behtash and Sastry ([10], Corollary 4.1) for a three dimensional version of (4.1) is a special case of Lemma 4.5.

Suppose the control input u is a purely nonlinear function. Then a purely quadratic stabilizing control law follows readily from Proposition 4.2.

Lemma 4.6. Assume that $b_{11} = b_{12} = 0$, A_{22} is stable and system (4.1) may not be stable. Then there exists a *purely quadratic* stabilizing feedback

$u = u_{xx}x^2 + u_{xy}xy + u_{yy}y^2$ for the origin of (4.1) if the following conditions hold:

(i) $g_{xx} = 0,\ g_{xy} + 2f_{xx} = 0,$

(ii) $g_{xxx} + g_{x\xi}h_{xx} + 2f_{xx}^2 < 0,$

(iii) $g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + f_{x\xi}h_{xx}) - f_{xx}(f_{xy} + 2g_{yy}) < 0,$

where

$$h_{xx} = -\,A_{22}^{-1}(u_{xx}b_2 + G_{xx}), \tag{4.64}$$

$$h_{xy} = -\,2(A_{22}^2)^{-1}(u_{xx}b_2 + G_{xx}) - A_{22}^{-1}(u_{xy}b_2 + G_{xy}). \tag{4.65}$$

A stability criterion for the uncontrolled version of (4.1) is obtained as follows.

Corollary 4.6. Assume that $u = 0$. The origin of (4.1) is asymptotically stable if (i) A_{22} is stable, (ii) $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$, (iii) $g_{xxx} + g_{x\xi}h_{xx} + 2f_{xx}^2 < 0$, (iv) $g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + f_{x\xi}h_{xx}) - f_{xx}(f_{xy} + 2g_{yy}) < 0$, where h_{xx} and h_{xy} are given in (4.64)-(4.65) by letting $u_{xx} = u_{xy} = 0$.

4.3.2. The case $b_1 \neq 0$

Next, we consider the case in which either b_{11} or b_{12} is nonzero. It is known that $b_{12} \neq 0$ implies the controllability of the subsystem (4.1a). For simplicity, the control law is restricted to be purely nonlinear such that the control input u has the form as given in (4.51).

Let A_{22} be stable. As discussed in Section 3.2, the stability of system (4.1) agrees with that of the reduced model

$$\dot{x} = y + b_{11}u(x, y, h(x, y)) + f(x, y, h(x, y)) \tag{4.66a}$$

$$\dot{y} = b_{12}u(x, y, h(x, y)) + g(x, y, h(x, y)), \tag{4.66b}$$

where h is the solution of

$$\begin{aligned} Dh(\eta)\cdot\{A_{11}\eta + b_1 u(\eta, h(\eta)) + F(\eta, h(\eta))\}& \\ = A_{22}h(\eta) + b_2 u(\eta, h(\eta)) + G(\eta, h(\eta))& \end{aligned} \tag{4.67}$$

with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Similarly, the function h is assumed to be given by Eq. (4.56). Choose the control input to be a function of only x and y as follows

$$\begin{aligned} u(x, y, \xi) = & u_{xx}x^2 + u_{xy}xy + u_{yy}y^2 + u_{xxx}x^3 \\ & + u_{xxy}x^2y + u_{xyy}xy^2 + u_{yyy}y^3. \end{aligned} \tag{4.68}$$

A stability criterion for control system (4.1) is obtained as follows

Proposition 4.3. Assume that $b_1 \neq 0$ and A_{22} is stable. Then the origin is asymptotically stable for (4.1) if

(i) $g_{xx} + b_{12}u_{xx} = 0$, $g_{xy} + b_{12}u_{xy} + 2(f_{xx} + b_{11}u_{xx}) = 0$,

(ii) $g_{xxx} + b_{12}u_{xxx} + g_{x\xi}h_{xx} + 2(f_{xx} + b_{11}u_{xx})^2 < 0$,

(iii) $g_{xxy} + b_{12}u_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} + 3(f_{xxx} + b_{11}u_{xxx} + f_{x\xi}h_{xx})$
$-(f_{xx} + b_{11}u_{xx}) \cdot \{f_{xy} + b_{11}u_{xy} + 2(g_{yy} + b_{12}u_{yy})\} < 0$,

where h_{xx} and h_{xy} are given in Eqs. (4.64)-(4.65).

According to Proposition 4.3 above, b_{12} plays a key role in all stability conditions (i)-(iii). So we have the following result.

Lemma 4.7. Let A_{22} be stable, but the full system need not be stable. If $b_{12} \neq 0$, then the stability of the origin of (4.1) can be guaranteed by a *purely quadratic-plus-cubic* state feedback of the form (4.68).

4.4. One Zero and a Pair of Pure Imaginary Eigenvalues

In this section, we apply Corollaries 4.3 and 4.4 to design stabilizing control laws for control system (4.1), where $\eta := (x, y, z)'$ and $b_1 = (b_{11}, b_{12}, b_{13})'$ are both three dimensional vectors, $F := (f, g, r)'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 \\ -\Omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.69}$$

Also, in the following analysis, φ_{ij} and φ_{ijk} denote the coefficients of the quadratic terms ij and the cubic terms ijk of function φ, respectively, for all

$i, j, k \in \{x, y, z, \xi\}$ and $\varphi \in \{f, g, r, G\}$. As usual, these coefficients are either constants or symmetric multilinear functions of their arguments.

4.4.1. The case $b_1 = 0$

Let the control input u be of the form

$$u(x, y, z, \xi) = k_{11}x + k_{12}y + k_{13}z + K_2\xi + U(x, y, z, \xi), \tag{4.70}$$

where $k_{1i}, i = 1, 2, 3$ are scalars and function U is smooth enough with $U(0, 0, 0, 0) = 0$ and $DU(0, 0, 0, 0) = 0$.

Let $A_{22} + b_2K_2$ be stable. As discussed in Section 3.2, the stability of (4.1) agrees with the stability of the reduced model

$$\dot{x} = \Omega_1 y + f(x, y, z, E\eta + h(x, y, z)) \tag{4.71a}$$

$$\dot{y} = -\Omega_2 x + g(x, y, z, E\eta + h(x, y, z)), \tag{4.71b}$$

$$\dot{z} = r(x, y, z, E\eta + h(x, y, z)), \tag{4.71c}$$

where $E = (E_1, E_2, E_3)$ and $h(x, y, z)$ solve Eqs. (4.54) and (4.55), respectively, with $\eta := (x, y, z)'$ and boundary conditions $h(0, 0, 0) = 0$ and $Dh(0, 0, 0) = 0$.

Referring to the boundary conditions above, we can write h as

$$\begin{aligned} h(x, y, z) =& x^2 h_{xx} + xy h_{xy} + xz h_{xz} + y^2 h_{yy} + yz h_{yz} \\ &+ z^2 h_{zz} + O(\|(x, y, z)\|^3), \end{aligned} \tag{4.72}$$

where h_{ij}, $i, j \in \{x, y, z\}$ are constant vectors.

Let

$$\begin{aligned} H(x, y, z) :=& b_2 U(x, y, z, E\eta) + G(x, y, z, E\eta) - f(x, y, z, E\eta)E_1 \\ &- g(x, y, z, E\eta)E_2 - r(x, y, z, E\eta)E_3 \\ =& x^2 H_{xx} + xy H_{xy} + xz H_{xz} + y^2 H_{yy} + yz H_{yz} \\ &+ z^2 H_{zz} + O(\|(x, y, z)\|^3). \end{aligned} \tag{4.73}$$

Solving Eqs. (4.54)-(4.55), we have

$$E_1 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{11}(A_{22} + b_2K_2) - \Omega_2 k_{12} I\}b_2, \tag{4.74}$$

$$E_2 = -\{(A_{22} + b_2K_2)^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{12}(A_{22} + b_2K_2) + \Omega_1 k_{11}\}b_2, \tag{4.75}$$

$$E_3 = -k_{13}(A_{22} + b_2K_2)^{-1}b_2, \tag{4.76}$$

and

$$
\begin{aligned}
h_{xy} =& -\{(A_{22}+b_2K_2)^2+4\Omega_1\Omega_2I\}^{-1}\{-2\Omega_2H_{yy} \\
&+2\Omega_1H_{xx}+(A_{22}+b_2K_2)H_{xy}\}, && (4.77)\\
h_{xx} =& -(A_{22}+b_2K_2)^{-1}(H_{xx}+\Omega_2h_{xy}), && (4.78)\\
h_{yy} =& -(A_{22}+b_2K_2)^{-1}(H_{yy}-\Omega_1h_{xy}), && (4.79)\\
h_{xz} =& -\{(A_{22}+b_2K_2)^2+\Omega_1\Omega_2I\}^{-1}\{(A_{22}+b_2K_2)H_{xz}-\Omega_2H_{yz}\} && (4.80)\\
h_{yz} =& -\{(A_{22}+b_2K_2)^2+\Omega_1\Omega_2I\}^{-1}\{(A_{22}+b_2K_2)H_{yz}+\Omega_1H_{xz}\} && (4.81)\\
h_{zz} =& -(A_{22}+b_2K_2)^{-1}H_{zz}. && (4.82)
\end{aligned}
$$

Let $\hat{\varphi}(x,y,z) := \varphi(x,y,z,E\eta+h(x,y,z))$, for $\varphi = f,g,r$, where the elements of E are given in (4.74)-(4.76) and function h is defined in (4.72) with h_{ij} given in (4.77)-(4.82). The coefficients of the quadratic terms and the cubic terms of functions $\hat{f},\hat{g},\hat{r}$ expressed in terms of E_i and h_{jk} are also given in Appendix 4.C.

The reduced model (4.71) can hence be rewritten as

$$
\begin{aligned}
\dot{x} =& \Omega_1 y+\hat{f}(x,y,z), && (4.83a)\\
\dot{y} =& -\Omega_2 x+\hat{g}(x,y,z), && (4.83b)\\
\dot{z} =& \hat{r}(x,y,z). && (4.83c)
\end{aligned}
$$

As discussed above, the stability of the overall system (4.1) agrees with that of the reduced model (4.83) if $A_{22}+b_2K_2$ is stable. In the following design, we will focus on the stabilization of (4.83) by assuming $A_{22}+b_2K_2$ is stable.

The next result follows readily from Corollaries 4.3 and 4.4 and the foregoing discussions.

Proposition 4.4. Let $b_{11}=b_{12}=0$ and the control input be given by (4.70). Then the origin of (4.1) is asymptotically stable if $A_{22}+b_2K_2$ is stable, $\hat{r}_{zz}=0$, and either of the following conditions holds:

(i) $\Omega_1 \hat{r}_{xx} + \Omega_2 \hat{r}_{yy} = 0$, $\hat{f}_{xz} + \hat{g}_{yz} = 0$, $S_1, S_2 < 0$ and $S_3, S_4 \le 0$ or S_3 and S_4 are nonzero and of opposite sign, where S_i, $i = 1, \cdots, 4$ are given in (4.24)-(4.27) with coefficients $\varphi_{ij}, \varphi_{ijk}$ replaced by $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$, respectively, for all $\varphi = f, g, r$.

(ii) $\Omega_1 \hat{r}_{xx} + \Omega_2 \hat{r}_{yy}$ and $\hat{f}_{xz} + \hat{g}_{yz}$ have nonzero values and of opposite sign, $S_1, \tilde{S}_2 < 0$ and $\tilde{S}_3, \tilde{S}_4 \le 0$, where S_1 is given in (4.24) and $\tilde{S}_i$, $i = 2, 3, 4$ are given in (4.28)-(4.30) with coefficients $\varphi_{ij}, \varphi_{ijk}$ replaced by $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$, respectively, for all $\varphi = f, g, r$.

Here, $\hat{\varphi}(x, y, z) := \varphi(x, y, z, E\eta + h(x, y, z))$ for $\varphi = f, g, r$, as defined above.

■

It is obvious from Proposition 4.4 and Appendix 4.C that only up to the quadratic terms of function G and the control input u contribute to the stability conditions of Proposition 4.4 in the case $b_1 = 0$. A linear and/or quadratic feedback stabilizing control law can hence be obtained from Proposition 4.4. Similar to the results given in Proposition 4.2, a purely linear feedback stabilizing control law might conceivably be obtained by using Proposition 4.4, however, in general construction of such a control law is not feasible. A stability criterion for the uncontrolled version of (4.1) can also be obtained from Proposition 4.4 by letting $u = 0$.

Consider a special case of system (4.1) in which ξ is a scalar. So, b_2 is a scalar. Suppose the nonlinear control function U in (4.70) is a function of x, y and z only and has the form given in (4.14). According to Eqs. (4.74)-(4.82), the values of E_i, and h_{ij} can be determined by the linear and quadratic gains of control input. A linear-plus-quadratic stabilizing control law can hence be obtained from Proposition 4.4 as follows.

Lemma 4.8. Let ξ be a scalar, $b_{1i} = 0$, $i = 1, 2, 3$ and system (4.1) need not be stable. Then a *linear-plus-quadratic* feedback can be designed to guarantee the stability of the origin for (4.1), if (i) $A_{22} + b_2 K_2$ is stable, (ii) $r_{\xi\xi} = 0$, (iii) $r_{z\xi} \neq 0$ (iv) $\Omega_1 r_{x\xi} g_{z\xi} - \Omega_2 r_{y\xi} f_{z\xi} \neq 0$, and (v) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$, or

$g_{y\xi} + \alpha f_{x\xi} \neq 0$ for $\alpha = 1$ and $\alpha = \frac{1}{3}$. This feedback control has the form

$$\begin{aligned} u(x,y,z,\xi) =& k_{11}x + k_{12}y + k_{13}z + K_2\xi + u_{xx}x^2 \\ &+ u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2. \end{aligned} \tag{4.84}$$

Proof: In the following, we check the stability conditions of Proposition 4.4 under the assumptions of Lemma 4.8. Suppose ξ is a scalar, $b_{1i} = 0$, for $i = 1, 2, 3$, and conditions (i)-(iii) hold. Then the values of $\hat{r}_{zz}$ and S_1 (given in (4.24)) can be made to be real numbers through $r_{z\xi}$ by the choice of E_3 and h_{zz}. Moreover, since condition (iv) holds, the values of $\Omega_1\hat{r}_{xx} + \Omega_2\hat{r}_{yy}$ and $\hat{f}_{xz} + \hat{g}_{yz}$ can be assigned arbitrarily by a proper choice of E_1 and E_2, while the values of S_3 and S_4 (given in (4.26)-(4.27)) or $\tilde{S}_3$ and $\tilde{S}_4$ in (given in (4.29)-(4.30)) can be assigned by proper choice of h_{xz} and h_{yz}.

Finally, condition (v) provides the opportunity for assigning the values of S_2 (gievn in (4.25)) and $\tilde{S}_2$ (given in (4.28)) by proper choice of h_{xx} or h_{yy}. According to Appendix 4.C and Eqs. (4.73)-(4.82), $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$ can be determined by the linear and quadratic control gains through the linear matrix E and the vector function h. The conclusions of the lemma follow.

■

A purely quadratic feedback stabilizing control law can also be obtained as given below. The proof is similar to that of Lemma 4.8. Details are omitted.

Lemma 4.9. Let A_{22} be stable, ξ be a scalar, $b_{1i} = 0$, $i = 1, 2, 3$ and system (4.1) need not be stable. Then a *purely quadratic* feedback

$$u(x,y,z) = u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2 \tag{4.85}$$

can be designed to guarantee the stability of the origin of (4.1), if the following conditions hold:

(i) $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$ and $f_{xz} + g_{yz} = 0$, or $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ and $f_{xz} + g_{yz}$ have nonzero values and of opposite sign,

(ii) $r_{zz} = 0$ and $r_{z\xi} \neq 0$, and

(iii) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$, and $g_{z\xi} \neq 0$ or $f_{z\xi} \neq 0$.

■

4.4.2. The case $b_1 \neq 0$

Next, we consider the case in which one of b_{1i}, $i = 1,2,3$ is nonzero. It is known that $b_{13} \neq 0$, and $b_{11} \neq 0$ or $b_{12} \neq 0$ implies the controllability of subsystem (4.1a). For simplicity, the control law is restricted here to be a purely nonlinear function of x, y and z only and to have the form (4.14).

Let A_{22} be stable. As discussed in Section 3.2, the stability of system (4.1) agrees with that of the reduced model (4.83). Here,

$$\hat{f}(x,y,z) = b_{11}u(x,y,z) + f(x,y,z,h(x,y,z)), \tag{4.86a}$$

$$\hat{g}(x,y,z) = b_{12}u(x,y,z) + g(x,y,z,h(x,y,z)), \tag{4.86b}$$

$$\hat{r}(x,y,z) = b_{13}u(x,y,z) + r(x,y,z,h(x,y,z)), \tag{4.86c}$$

and h is the solution for (4.67) with boundary conditions $h(0) = 0$ and $Dh(0) = 0$. Similarly, function h is assumed to be given by Eq. (4.72).

By letting

$$\begin{aligned} H(x,y,z) :=& b_2u(x,y,z) + G(x,y,z,0) \\ =& x^2H_{xx} + xyH_{xy} + xzH_{xz} + y^2H_{yy} + yzH_{yz} \\ &+ z^2H_{zz} + O(||(x,y,z)||^3), \end{aligned} \tag{4.87}$$

we can obtain h_{ij} as given in (4.77)-(4.82) with $K_2 = 0$ and H_{ij} given in (4.87).

A stability criterion for control system (4.1) in the case of $b_1 \neq 0$ is obtained as follows.

Proposition 4.5. Let $b_1 \neq 0$ and A_{22} be stable. Then the origin of (4.1) is asymptotically stable if $\hat{r}_{zz} = 0$, and either of conditions (i) and (ii) given in Proposition 4.4 hold. Here, $\hat{\varphi}_{ij}$ and $\hat{\varphi}_{ijk}$ denote the coefficients of quadratic terms and cubic terms of function $\hat{\varphi}$ $(= \hat{f}, \hat{g}, \hat{r}$ given in (4.86)), respectively.

It is obvious from Proposition 4.5 above that the vector b_1 plays a key role in all stability conditions (i)-(iii). The next two results follow readily from Proposition 4.5.

Lemma 4.10. Let A_{22} be stable, but the whole system may not be stable. If $b_{13} \neq 0$ and one of b_{11} and b_{12} is not zero, then the stability of the origin of (4.1) can be guaranteed by a *purely quadratic-plus-cubic* state feedback as follows

$$\begin{aligned} u(x,y,z) =& u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{yy}y^2 + u_{yz}yz + u_{zz}z^2 \\ & + u_{xxx}x^3 + u_{xxy}x^2y + u_{xxz}x^2z + u_{xyy}xy^2 + u_{xyz}xyz + u_{xzz}xz^2 \\ & + u_{yyy}y^3 + u_{yyz}y^2z + u_{yzz}yz^2 + u_{zzz}z^3. \end{aligned}$$

Lemma 4.11. Let A_{22} be stable, but the full system need not be stable. Then the stability of the origin for (4.1) can be guaranteed by a *purely cubic* state feedback

$$\begin{aligned} u(x,y,z) =& u_{xxx}x^3 + u_{xxy}x^2y + u_{xxz}x^2z + u_{xyy}xy^2 + u_{xzz}xz^2 \\ & + u_{yyy}y^3 + u_{yyz}y^2z + u_{yzz}yz^2 + u_{zzz}z^3, \end{aligned} \tag{4.88}$$

if $r_{zz} = 0$ and following conditions hold:

(i) $b_{13} \neq 0$ and one of b_{11} and b_{12} is not zero, and

(ii) $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$ and $f_{xz} + g_{yz} = 0$, or the expressions $\Omega_1 r_{xx} + \Omega_2 r_{yy}$ and $f_{xz} + g_{yz}$ have nonzero values and of opposite sign,

4.5. Two Distinct Pairs of Pure Imaginary Eigenvalues

In this section, we continue our study of the stability and stabilization of control system (4.1) in which $\eta := (x, y, z, w)'$ and $b_1 = (b_{11}, b_{12}, b_{13}, b_{14})'$ are both four dimensional vectors, $F := (f, g, r, s)'$ and

$$A_{11} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix}. \tag{4.89}$$

As in the previous two sections, in the following analysis, φ_{ij} and φ_{ijk} denote the coefficients of the quadratic terms ij and the cubic terms ijk of function φ, respectively, for all $i, j, k \in \{x, y, z, w, \xi\}$ and $\varphi \in \{f, g, r, s, G\}$. As usual, these coefficients are either constants or symmetric multilinear functions of their arguments.

4.5.1. The case $b_1 = 0$

First, we consider the case in which $b_1 = 0$, and

$$u(x, y, z, w, \xi) = k_{11}x + k_{12}y + k_{13}z + k_{14}w + K_2\xi + U(x, y, z, w, \xi), \quad (4.90)$$

where $k_{1i}, i = 1, \cdots, 4$ are scalars and U is sufficiently smooth with $U(0, 0, 0, 0, 0) = 0$ and $DU(0, 0, 0, 0, 0) = 0$.

Let $A_{22} + b_2K_2$ be stable. Similarly, the stability of (4.1) is known to agree with the stability of the reduced model

$$\dot{x} = \Omega_1 y + f(x, y, z, w, E\eta + h(x, y, z, w)), \quad (4.91a)$$

$$\dot{y} = -\Omega_2 x + g(x, y, z, w, E\eta + h(x, y, z, w)), \quad (4.91b)$$

$$\dot{z} = \Omega_3 w + r(x, y, z, w, E\eta + h(x, y, z, w)), \quad (4.91c)$$

$$\dot{w} = -\Omega_4 z + s(x, y, z, w, E\eta + h(x, y, z, w)), \quad (4.91d)$$

where $E = (E_1, E_2, E_3, E_4)$ and $h(x, y, z, w)$ solve Eqs. (4.54) and (4.55), respectively, with $\eta := (x, y, z, w)'$ and boundary conditions $h(0, 0, 0, 0) = 0$ and $Dh(0, 0, 0, 0) = 0$.

The boundary conditions above require h to have the form

$$\begin{aligned} h(x, y, z, w) &= x^2 h_{xx} + xy h_{xy} + xz h_{xz} + xw h_{xw} + y^2 h_{yy} + yz h_{yz} \\ &+ yw h_{yw} + z^2 h_{zz} + zw h_{zw} + w^2 h_{ww} + O(||(x, y, z, w)||^3), \quad (4.92) \end{aligned}$$

where h_{ij}, $i, j \in \{x, y, z, w\}$ are constant vectors.

Similarly, let

$$\begin{aligned} H(x, y, z, w) &:= b_2 U(x, y, z, w, E\eta) + G(x, y, z, w, E\eta) - f(x, y, z, w, E\eta)E_1 \\ &- g(x, y, z, w, E\eta)E_2 - r(x, y, z, w, E\eta)E_3 - s(x, y, z, w, E\eta)E_4 \\ &= x^2 H_{xx} + xy H_{xy} + xz H_{xz} + xw H_{xw} + y^2 H_{yy} + yz H_{yz} \\ &+ yw H_{yw} + z^2 H_{zz} + zw H_{zw} + w^2 H_{ww} + O(||(x, y, z, w)||^3), \quad (4.93) \end{aligned}$$

By solving Eqs. (4.54)-(4.55), we have

$$E_1 = -\{M_1^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{11}M_1 - \Omega_2 k_{12} I\}b_2, \quad (4.94)$$

$$E_2 = -\{M_1^2 + \Omega_1\Omega_2 I\}^{-1}\{k_{12}M_1 + \Omega_1 k_{11}\}b_2, \tag{4.95}$$

$$E_3 = -\{M_1^2 + \Omega_3\Omega_4 I\}^{-1}\{k_{13}M_1 - \Omega_4 k_{14} I\}b_2, \tag{4.96}$$

$$E_4 = -\{M_1^2 + \Omega_3\Omega_4 I\}^{-1}\{k_{14}M_1 + \Omega_3 k_{13}\}b_2, \tag{4.97}$$

$$h_{zw} = -(M_1^2 + 4\Omega_3\Omega_4 I)^{-1}(-2\Omega_4 H_{ww} + 2\Omega_3 H_{zz} + M_1 H_{zw}), \tag{4.98}$$

$$h_{zz} = -M_1^{-1}(H_{zz} + \Omega_4 h_{zw}), \tag{4.99}$$

$$h_{ww} = -M_1^{-1}(H_{ww} - \Omega_3 h_{zw}), \tag{4.100}$$

$$\begin{pmatrix} h_{xz} \\ h_{xw} \end{pmatrix} = (M_2^2 + \Omega_1\Omega_2 I)^{-1}\{M_2 \begin{pmatrix} H_{xz} \\ H_{xw} \end{pmatrix} - \Omega_2 \begin{pmatrix} H_{yz} \\ H_{yw} \end{pmatrix}\}, \tag{4.101}$$

$$\begin{pmatrix} h_{yz} \\ h_{yw} \end{pmatrix} = (M_2^2 + \Omega_1\Omega_2 I)^{-1}\{\Omega_1 \begin{pmatrix} H_{xz} \\ H_{xw} \end{pmatrix} + M_2 \begin{pmatrix} H_{yz} \\ H_{yw} \end{pmatrix}\}, \tag{4.102}$$

where the expressions of h_{xx}, h_{xy}, h_{yy} are given in Eqs. (4.77)-(4.79) with H_{ij} defined in (4.93), $M_1 := A_{22} + b_2 K_2$ and

$$M_2 := \begin{pmatrix} M_1 & \Omega_4 I \\ -\Omega_3 I & M_1 \end{pmatrix}. \tag{4.103}$$

The reduced model (4.91) can hence be obtained as

$$\dot{x} = \Omega_1 y + \hat{f}(x, y, z, w), \tag{4.104a}$$

$$\dot{y} = -\Omega_2 x + \hat{g}(x, y, z, w), \tag{4.104b}$$

$$\dot{z} = \Omega_3 w + \hat{r}(x, y, z, w), \tag{4.104c}$$

$$\dot{w} = -\Omega_4 z + \hat{s}(x, y, z, w). \tag{4.104d}$$

Here, $\hat{\varphi}(x, y, z, w) := \varphi(x, y, z, w, E\eta + h(x, y, z, w))$, for $\varphi = f, g, r, s$ with E_i given in (4.94)-(4.97) and h defined in (4.92). The values of h_{ij} are given in (4.77)-(4.79) and (4.98)-(4.102), and the coefficients of the quadratic terms and cubic terms of the functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ expressed in terms of E_i and h_{jk} are given in Appendix 4.C.

A linear and/or quadratic feedback stabilizing control law readily follows from Corollary 4.5 and the foregoing discussions.

Proposition 4.6. Let $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and $b_{1i} = 0$ for $i = 1, \cdots, 4$. The origin is asymptotically stable for control system (4.1) if

$S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of opposite sign, where S_i are given in (4.45)-(4.48) with coefficients $\varphi_{ij}, \varphi_{ijk}$ replaced by $\hat{\varphi}_{ij}, \hat{\varphi}_{ijk}$, respectively, for all $\varphi = f, g, r, s$. Here, $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are defined above and the control input is given by (4.90).

Note that a stability criterion for the uncontrolled model of (4.1) can also be obtained from Proposition 4.6 by letting $u = 0$.

Next, consider a special case in which ξ is a scalar. Referring to Eqs. (4.93), (4.77)-(4.79) and (4.98)-(4.102), we can determine h_{ij} from the quadratic gains of the control input. A purely quadratic stabilizing control law is hence obtained as follows.

Lemma 4.12. Let ξ be a scalar, $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$, $b_{1i} = 0$ for $i = 1, \cdots, 4$ and system (4.1) may not be stable. A *purely quadratic* feedback

$$u(x,y,z,w) = u_{xx}x^2 + x(u_{xy}y + u_{xz}z + u_{xw}w) + u_{yy}y^2 \\ + y(u_{yz}z + u_{yw}w) + u_{zz}z^2 + u_{zw}zw + u_{ww}w^2 \quad (4.105)$$

exists guaranteeing the asymptotic stability of the origin for (4.1), if $f_{x\xi} + g_{y\xi} \neq 0$, $r_{z\xi} + s_{w\xi} \neq 0$ and either of the following two conditions hold:

(i) $f_{x\xi} \neq g_{y\xi}$ and $r_{z\xi} \neq s_{w\xi}$,

(ii) $\Omega_1 g_{x\xi} + \Omega_2 f_{y\xi} \neq 0$ and $\Omega_3 s_{z\xi} + \Omega_4 r_{w\xi} \neq 0$.

Proof: In the following, we check the stability conditions of Proposition 4.6 under the hypotheses of Lemma 4.12. Suppose ξ is a scalar, $b_{1i} = 0$, $i = 1, \cdots, 4$, $f_{x\xi} + g_{y\xi} \neq 0$ and $r_{z\xi} + s_{w\xi} \neq 0$. Then the values of S_3 and S_4 (given in (4.47)-(4.48)) can be made equal to any real numbers by a proper choice of $\Omega_1 h_{xx} + \Omega_2 h_{yy}$ and $\Omega_3 h_{zz} + \Omega_4 h_{ww}$.

If condition (i) holds, then the value of S_1 (given in (4.45)) will be determined by h_{xx} and h_{yy}, independent of the value of S_4. Similarly, the value of S_2 is determined by h_{zz} and h_{ww}, irrespective of the value of S_3. The values of S_1 and S_2 can also be adjusted by the choice of h_{xy} and h_{zw} when condition

(ii) holds.

According to Eqs. (4.77)-(4.79), (4.93) and (4.98)-(4.102), the values of h_{ij} can be directly determined by the quadratic feedback gains when ξ is scalar. The conclusion is hence implied.

■

A similar stabilizing control law can also be designed as follows.

Lemma 4.13. Suppose ξ is a scalar, $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$, $b_{1i} = 0$ for $i = 1, \cdots, 4$ and system (4.1) may not be stable. A *purely quadratic* feedback as given in (4.105) can be designed to guarantee the stability of the origin for (4.1) if $f_{x\xi} \neq \alpha g_{y\xi}$ and $r_{z\xi} \neq \alpha s_{w\xi}$ for $\alpha = -3$ and $\alpha = -\frac{1}{3}$ and either of the following conditions holds:

(i) $\Omega_2 f_{w\xi}s_{y\xi} - \Omega_1 g_{w\xi}s_{x\xi} \neq 0$ or $\Omega_2 f_{z\xi}r_{y\xi} - \Omega_1 g_{z\xi}r_{x\xi} \neq 0$,

(ii) $\Omega_4 f_{w\xi}r_{x\xi} - \Omega_3 f_{z\xi}s_{x\xi} \neq 0$ or $\Omega_2\Omega_4 f_{w\xi}r_{y\xi} - \Omega_1\Omega_3 g_{z\xi}s_{x\xi} \neq 0$, or

(iii) $\Omega_1\Omega_4 g_{w\xi}r_{x\xi} - \Omega_2\Omega_3 f_{z\xi}s_{y\xi} \neq 0$ or $\Omega_4 g_{w\xi}r_{y\xi} - \Omega_3 g_{z\xi}s_{y\xi} \neq 0$.

Proof: The proof is very similar to that of Lemma 4.12. Suppose $f_{x\xi} \neq \alpha g_{y\xi}$ and $r_{z\xi} \neq \alpha s_{w\xi}$ for $\alpha = -3$ and $\alpha = -\frac{1}{3}$. The values of S_1 and S_2 (given in (4.45)-(4.46)) can then be adjusted by h_{xx} (or h_{yy}) and h_{zz} (or h_{ww}). Moreover, the values of S_3 and S_4 (given in (4.47)-(4.48)) can be any real numbers by a proper choice of h_{xw}, h_{yw}, h_{xz} or h_{yz}, when either of conditions (i) to (iii) holds. Since the values of h_{ij} can be directly determined from the quadratic control gains when ξ is a scalar, the conclusion is hence implied.

■

4.5.2. The case $b_1 \neq 0$

In this subsection, we consider the case in which one of b_{1i}, $i = 1, \cdots, 4$ is nonzero. It is known that $b_{11} \neq 0$ or $b_{12} \neq 0$, and $b_{13} \neq 0$ or $b_{14} \neq 0$ imply the controllability of subsystem (4.1a). Similar to Section 4.4.2, the control law, here, is also restricted to be a purely nonlinear function of x, y, z, w and has the form as given in (4.32).

Let A_{22} be stable. Then according to the discussions in Section 3.2, the stability of (4.1) is determined from the reduced model (4.104), where

$$\hat{f}(x,y,z) = b_{11}u(x,y,z) + f(x,y,z,h(x,y,z)), \tag{4.106a}$$

$$\hat{g}(x,y,z) = b_{12}u(x,y,z) + g(x,y,z,h(x,y,z)), \tag{4.106b}$$

$$\hat{r}(x,y,z) = b_{13}u(x,y,z) + r(x,y,z,h(x,y,z)), \tag{4.106c}$$

$$\hat{s}(x,y,z) = b_{14}u(x,y,z) + s(x,y,z,h(x,y,z)), \tag{4.106d}$$

and h is the solution for (4.67) with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Suppose h is given by Eq. (4.92) and let

$$\begin{aligned} H(x,y,z,w) &:= b_2u(x,y,z,w) + G(x,y,z,w,0) \\ &= x^2H_{xx} + xyH_{xy} + xzH_{xz} + xwH_{xw} + y^2H_{yy} + yzH_{yz} \\ &\quad + ywH_{yw} + z^2H_{zz} + zwH_{zw} + w^2H_{ww} + O(||(x,y,z,w)||^3). \end{aligned} \tag{4.107}$$

h_{ij} are hence obtained as given in (4.77)-(4.79) and (4.98)-(4.102) with $K_2 = 0$ and H_{ij} given in (4.107). A stability criterion for control system (4.1) in the case $b_1 \neq 0$ readily follows from Corollary 4.5.

Proposition 4.6. Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and $b_{1i} = 0$ for $i = 1, \cdots, 4$. The origin is asymptotically stable for control system (4.1) if $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of opposite sign, where S_i are given in (4.45)-(4.48) with coefficients $\varphi_{ij}, \varphi_{ijk}$ replaced by $\hat{\varphi}_{ij}, \hat{\varphi}_{ijk}$, respectively, for all $\varphi = f, g, r, s$. Here, $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are defined in (4.106) and the control input u is a purely nonlinear function and has the form as given in (4.32).

A purely cubic stabilizing control law is obtained as follows.

Lemma 4.14. Let A_{22} be stable, but the full system need not be stable. If $b_{11} \neq 0$ or $b_{12} \neq 0$, and $b_{13} \neq 0$ or $b_{14} \neq 0$, then the stability of the origin of (4.1) can be guaranteed by a *purely cubic* state feedback

$$u(x,y,z,w) = u_{xxx}x^3 + (u_{xxy}y + u_{xxz}z + u_{xxw}w)x^2 + (u_{xyy}x + u_{yyy}y$$

$$+ u_{yyz}z + u_{yyw}w)y^2 + (u_{xzz}x + u_{yzz}y + u_{zzz}z + u_{zzw}w)z^2$$
$$+ (u_{xww}x + u_{yww}y + u_{zww}z + u_{www}w)w^2. \tag{4.108}$$

4.6. Concluding Remarks

The center manifold reduction technique discussed in Section 3.2, along with the normal form reduction recalled in Section 2.3, are applied in this chapter to study the stability and stabilization of smooth, nonlinear autonomus systems in doubly critical cases. Specifically, the linearized model of the system has two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and a pair of nonzero pure imaginary eigenvlaues; or two distinct pairs of nonzero pure imaginary eigenvalues. The feedback stabilizing control laws are proposed for both linearly controllable and linearly uncontrollable cases, while a purely nonlinear feedback design is considered in the former case and linear and/or nonlinear control designs are obtained for the latter case.

Some of the results given in this chapter agree with those obtained by Behtash and Sastry [10]. However, the results obtained in this chapter cover more detailed design for general high dimensional systems. For instance, the stability criteria and stabilizing control laws are given in terms of the original system dynamics before normal form reduction. Moreover, there is no restriction on the number of the noncritical modes and the stabilizing control algorithms proposed in this chapter can be coded easily.

Appendix 4.A

The polynomial functions P_2 and P_3 for deriving the normal form for the case in which A_{11} has exactly one zero eigenvalue and a pair of nonzero, pure imaginary eigenvalues are given below.

Let $P_2(x,y,z) = (P_2^1, P_2^2, P_2^3)'$, where $P_2^i(x,y,z)$ has the form as

$$\varphi = \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{yy}y^2 + \varphi_{yz}yz + \varphi_{zz}z^2,$$

for all $\varphi = P_2^i$, $i = 1, \cdots, 3$.

The coefficients of polynomial functions P_2^i are

$$P^1_{2,xx} = \frac{(2g_{yy}+f_{xy})\Omega_2 + g_{xx}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^1_{2,xy} = -\frac{(g_{xy}+2f_{xx})\Omega_1 - 2f_{yy}\Omega_2}{3\Omega_1\Omega_2}$$

$$P^1_{2,xz} = \frac{f_{yz}\Omega_2 + g_{xz}\Omega_1}{4\Omega_1\Omega_2}$$

$$P^1_{2,yy} = -\frac{(f_{xy}-g_{yy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_2^2}$$

$$P^1_{2,yz} = -\frac{f_{xz}\Omega_2 - g_{yz}\Omega_1}{2(\Omega_2^2+\Omega_1\Omega_2)}$$

$$P^1_{2,zz} = \frac{g_{zz}}{\Omega_2}$$

$$P^2_{2,xx} = -\frac{2f_{yy}\Omega_2 + (f_{xx}-g_{xy})\Omega_1}{3\Omega_1^2}$$

$$P^2_{2,xy} = \frac{(2g_{yy}+f_{xy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{2,xz} = -\frac{f_{xz}\Omega_2 - g_{yz}\Omega_1}{2(\Omega_1^2+\Omega_1\Omega_2)}$$

$$P^2_{2,yy} = -\frac{f_{yy}\Omega_2 + (g_{xy}+2f_{xx})\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{2,yz} = -\frac{f_{yz}\Omega_2 + g_{xz}\Omega_1}{4\Omega_1\Omega_2}$$

$$P^2_{2,zz} = -\frac{f_{zz}}{\Omega_1}$$

$$P^3_{2,xx} = \frac{r_{xy}}{4\Omega_1}$$

$$P^3_{2,xy} = \frac{r_{yy}-r_{xx}}{\Omega_2+\Omega_1}$$

$$P^3_{2,xz} = \frac{r_{yz}}{\Omega_1}$$

$$P^3_{2,yy} = -\frac{r_{xy}}{4\Omega_2}$$

$$P^3_{2,yz} = -\frac{r_{xz}}{\Omega_2}$$

$$P^3_{2,zz} = 0$$

Next, let $P_3(z_1, z_2, z_3) = (P^1_3, P^2_3, P^3_3)'$, where $P^i_3(z_1, z_2, z_3)$ has the form as

$$\begin{aligned}\varphi =& \varphi_{111}z_1^3 + (\varphi_{112}z_2 + \varphi_{113}z_3)z_1^2 + (\varphi_{122}z_1 + \varphi_{222}z_2)z_2^2 \\ &+ \varphi_{223}z_2^2 z_3 + \varphi_{123}z_1 z_2 z_3 + (\varphi_{133}z_1 + \varphi_{233}z_2 + \varphi_{333}z_3)z_3^2,\end{aligned}$$

for all $\varphi = P^i_3$, $i = 1, \cdots, 3$.

The coefficients are given as follows.

$$P^1_{3,111} = \frac{(-3\tilde{f}_{222} + 2\tilde{f}_{112})\Omega_2^2 + (\tilde{g}_{122} - 2\tilde{g}_{111} + \tilde{f}_{112})\Omega_1\Omega_2 + \tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2}$$

$$\begin{aligned}P^1_{3,112} =& -\{(-3\tilde{g}_{222} + 3\tilde{g}_{112} - \tilde{f}_{122} + 9\tilde{f}_{111})\Omega_2 + (-9\tilde{g}_{222} + \tilde{g}_{112} \\ &- 3\tilde{f}_{122} + 3\tilde{f}_{111})\Omega_1\}/\{6\Omega_2^2 + 4\Omega_1\Omega_2 + 6\Omega_1^2\}\end{aligned}$$

$$P^1_{3,113} = \frac{(2\tilde{g}_{223} + \tilde{f}_{123})\Omega_2 + \tilde{g}_{113}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^1_{3,122} = 0$$

$$P^1_{3,123} = \frac{2\tilde{f}_{223}\Omega_2 + (-\tilde{g}_{123} - 2\tilde{f}_{113})\Omega_1}{3\Omega_1\Omega_2}$$

$$P^1_{3,133} = 0$$

$$P^1_{3,222} = -\frac{\tilde{f}_{122}\Omega_2^2 + (-2\tilde{g}_{222} + \tilde{g}_{112} + 3\tilde{f}_{111})\Omega_1\Omega_2 - 3\tilde{g}_{222}\Omega_1^2}{3\Omega_2^3 + 2\Omega_1\Omega_2^2 + 3\Omega_1^2\Omega_2}$$

$$P^1_{3,223} = \frac{(\tilde{g}_{223} - \tilde{f}_{123})\Omega_2 + 2\tilde{g}_{113}\Omega_1}{3\Omega_2^2}$$

$$P^1_{3,233} = -\frac{\tilde{f}_{133}\Omega_2 - \tilde{g}_{233}\Omega_1}{\Omega_2^2 + \Omega_1\Omega_2}$$

$$P^1_{3,333} = \frac{\tilde{g}_{333}}{\Omega_2}$$

$$P^2_{3,111} = \{(-3\tilde{g}_{222} + 3\tilde{g}_{112} - \tilde{f}_{122} + 3\tilde{f}_{111})\Omega_2^2 + (-3\tilde{g}_{222} + \tilde{g}_{112}$$

$$- \tilde{f}_{122} - \tilde{f}_{111})\Omega_1\Omega_2 + 2\tilde{g}_{112}\Omega_1^2\}/\{6\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2 + 6\Omega_1^3\}$$

$$P^2_{3,112} = \frac{(-3\tilde{f}_{222} + \tilde{f}_{112})\Omega_2 + (\tilde{g}_{122} - 3\tilde{g}_{111})\Omega_1}{2\Omega_1\Omega_2 + 2\Omega_1^2}$$

$$P^2_{3,113} = -\frac{2\tilde{f}_{223}\Omega_2 + (\tilde{f}_{113} - \tilde{g}_{123})\Omega_1}{3\Omega_1^2}$$

$$P^2_{3,122} = 0$$

$$P^2_{3,123} = \frac{(2\tilde{g}_{223} + \tilde{f}_{123})\Omega_2 - 2\tilde{g}_{113}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{3,133} = 0$$

$$P^2_{3,222} = \frac{-\tilde{f}_{222}\Omega_2^2 + (-\tilde{g}_{122} - 4\tilde{f}_{222} + \tilde{f}_{112})\Omega_1\Omega_2 - 3\tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2}$$

$$P^2_{3,223} = -\frac{\tilde{f}_{223}\Omega_2 + (\tilde{g}_{123} + 2\tilde{f}_{113})\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{3,233} = \frac{-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1}{2\Omega_1\Omega_2}$$

$$P^2_{3,333} = -\frac{\tilde{f}_{333}}{\Omega_1}$$

$$P^3_{3,111} = \frac{2\Omega_2\tilde{r}_{222} + \Omega_1\tilde{r}_{112}}{3\Omega_1^2}$$

$$P^3_{3,112} = -\frac{\tilde{r}_{111}}{\Omega_2}$$

$$P^3_{3,113} = \frac{\tilde{r}_{123}}{2\Omega_1}$$

$$P^3_{3,122} = \frac{\tilde{r}_{222}}{\Omega_1}$$

$$P^3_{3,123} = \frac{\tilde{r}_{223} - \tilde{r}_{113}}{\Omega_2 + \Omega_1}$$

$$P^3_{3,133} = \frac{\tilde{r}_{233}}{\Omega_1}$$

$$P^3_{3,222} = -\frac{\Omega_2\tilde{r}_{122} + 2\Omega_1\tilde{r}_{111}}{3\Omega_2^2}$$

$$P^3_{3,223} = 0$$

$$P^3_{3,233} = -\frac{\tilde{r}_{133}}{\Omega_2}$$

$$P^3_{3,333} = 0$$

Appendix 4.B

The polynomial functions P_2 and P_3 for deriving the normal form for the case in which A_{11} has exactly two distinct pairs of nonzero, pure imaginary eigenvalues are given below.

Let $P_2(x,y,z,w) = (P_2^1, P_2^2, P_2^3, P_2^4)'$, where $P_2^i(x,y,z,w)$ has the form as

$$\begin{aligned}\varphi =& \varphi_{xx}x^2 + \varphi_{xy}xy + \varphi_{xz}xz + \varphi_{xw}xw + \varphi_{yy}y^2 \\ &+ \varphi_{yz}yz + \varphi_{yw}yw + \varphi_{zz}z^2 + \varphi_{zw}zw + \varphi_{ww}w^2,\end{aligned}$$

for all $\varphi = P_2^i$, $i = 1, \cdots, 4$. The coefficients are given as follows

$$P^1_{2,xx} = \frac{(2g_{yy} + f_{xy})\Omega_2 + g_{xx}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^1_{2,xy} = -\frac{(g_{xy} + 2f_{xx})\Omega_1 - 2f_{yy}\Omega_2}{3\Omega_1\Omega_2}$$

$$P^1_{2,xz} = \frac{f_{xw}\Omega_3\Omega_4 + \Omega_1((-2g_{yw} - 2f_{xw})\Omega_2 - g_{xz}\Omega_3) - f_{yz}\Omega_2\Omega_3}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^1_{2,xw} = -\frac{\Omega_1(g_{xw}\Omega_4 + (-2g_{yz} - 2f_{xz})\Omega_2) + f_{xz}\Omega_3\Omega_4 + f_{yw}\Omega_2\Omega_4}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^1_{2,yy} = -\frac{(f_{xy} - g_{yy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_2^2}$$

$$P^1_{2,yz} = \frac{\Omega_3(f_{yw}\Omega_4 + (f_{xz} - g_{yz})\Omega_1) - 2f_{yw}\Omega_1\Omega_2 + 2g_{xw}\Omega_1^2}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^1_{2,yw} = \frac{((f_{xw} - g_{yw})\Omega_1 - f_{yz}\Omega_3)\Omega_4 + 2f_{yz}\Omega_1\Omega_2 - 2g_{xz}\Omega_1^2}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^1_{2,zz} = \frac{2g_{ww}\Omega_4^2 + (2g_{zz}\Omega_3 + f_{zw}\Omega_2)\Omega_4 - g_{zz}\Omega_1\Omega_2}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^1_{2,zw} = \frac{2f_{ww}\Omega_4 - 2f_{zz}\Omega_3 - g_{zw}\Omega_1}{4\Omega_3\Omega_4 - \Omega_1\Omega_2}$$

$$P^1_{2,ww} = \frac{2g_{ww}\Omega_3\Omega_4 + 2g_{zz}\Omega_3^2 - f_{zw}\Omega_2\Omega_3 - g_{ww}\Omega_1\Omega_2}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^2_{2,xx} = -\frac{2f_{yy}\Omega_2 + (f_{xx} - g_{xy})\Omega_1}{3\Omega_1^2}$$

$$P^2_{2,xy} = \frac{(2g_{yy} + f_{xy})\Omega_2 - 2g_{xx}\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{2,xz} = \frac{\Omega_3(g_{xw}\Omega_4 + (f_{xz} - g_{yz})\Omega_2) + 2f_{yw}\Omega_2^2 - 2g_{xw}\Omega_1\Omega_2}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^2_{2,xw} = -\frac{(g_{xz}\Omega_3 + (g_{yw} - f_{xw})\Omega_2)\Omega_4 + 2f_{yz}\Omega_2^2 - 2g_{xz}\Omega_1\Omega_2}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^2_{2,yy} = -\frac{f_{yy}\Omega_2 + (g_{xy} + 2f_{xx})\Omega_1}{3\Omega_1\Omega_2}$$

$$P^2_{2,yz} = \frac{g_{yw}\Omega_3\Omega_4 + \Omega_1(g_{xz}\Omega_3 + (-2g_{yw} - 2f_{xw})\Omega_2) + f_{yz}\Omega_2\Omega_3}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^2_{2,yw} = -\frac{\Omega_1((-2g_{yz} - 2f_{xz})\Omega_2 - g_{xw}\Omega_4) + g_{yz}\Omega_3\Omega_4 - f_{yw}\Omega_2\Omega_4}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^2_{2,zz} = -\frac{2f_{ww}\Omega_4^2 + (2f_{zz}\Omega_3 - g_{zw}\Omega_1)\Omega_4 - f_{zz}\Omega_1\Omega_2}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^2_{2,zw} = \frac{2g_{ww}\Omega_4 - 2g_{zz}\Omega_3 + f_{zw}\Omega_2}{4\Omega_3\Omega_4 - \Omega_1\Omega_2}$$

$$P^2_{2,ww} = -\frac{2f_{ww}\Omega_3\Omega_4 + 2f_{zz}\Omega_3^2 + \Omega_1(g_{zw}\Omega_3 - f_{ww}\Omega_2)}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^3_{2,xx} = -\frac{2\Omega_2^2 s_{yy} - \Omega_3\Omega_4 s_{xx} + 2\Omega_1\Omega_2 s_{xx} + \Omega_2\Omega_4 r_{xy}}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^3_{2,xy} = -\frac{-\Omega_3 s_{xy} + 2\Omega_2 r_{yy} - 2\Omega_1 r_{xx}}{\Omega_3\Omega_4 - 4\Omega_1\Omega_2}$$

$$P^3_{2,xz} = \frac{\Omega_4(2\Omega_3 s_{yw} + \Omega_1 r_{xw}) + \Omega_1\Omega_3 s_{xz} + (2\Omega_3\Omega_4 - \Omega_1\Omega_2)r_{yz}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^3_{2,xw} = \frac{-2\Omega_3^2 s_{yz} + \Omega_1\Omega_3(s_{xw} - r_{xz}) + (2\Omega_3\Omega_4 - \Omega_1\Omega_2)r_{yw}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^3_{2,yy} = \frac{\Omega_3\Omega_4 s_{yy} - 2\Omega_1\Omega_2 s_{yy} - 2\Omega_1^2 s_{xx} + \Omega_1\Omega_4 r_{xy}}{\Omega_3\Omega_4^2 - 4\Omega_1\Omega_2\Omega_4}$$

$$P^3_{2,yz} = \frac{\Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw} - 2\Omega_4 r_{xz}) + \Omega_2\Omega_4 r_{yw} + \Omega_1\Omega_2 r_{xz}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^3_{2,yw} = -\frac{-\Omega_2\Omega_3 s_{yw} - 2\Omega_3^2 s_{xz} + \Omega_2\Omega_3 r_{yz} + 2\Omega_3\Omega_4 r_{xw} - \Omega_1\Omega_2 r_{xw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^3_{2,zz} = \frac{2\Omega_4 s_{ww} + \Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_3\Omega_4}$$

$$P^3_{2,zw} = -\frac{\Omega_3 s_{zw} - 2\Omega_4 r_{ww} + 2\Omega_3 r_{zz}}{3\Omega_3\Omega_4}$$

$$P^3_{2,ww} = -\frac{-\Omega_4 s_{ww} - 2\Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_4^2}$$

$$P^4_{2,xx} = \frac{\Omega_3(-\Omega_2 s_{xy} - \Omega_4 r_{xx}) + 2\Omega_2^2 r_{yy} + 2\Omega_1\Omega_2 r_{xx}}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^4_{2,xy} = -\frac{2\Omega_2 s_{yy} - 2\Omega_1 s_{xx} + \Omega_4 r_{xy}}{\Omega_3\Omega_4 - 4\Omega_1\Omega_2}$$

$$P^4_{2,xz} = -\frac{\Omega_1(\Omega_2 s_{yz} - \Omega_4 s_{xw} + \Omega_4 r_{xz}) - 2\Omega_3\Omega_4 s_{yz} + 2\Omega_4^2 r_{yw}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^4_{2,xw} = \frac{\Omega_4(2\Omega_3 s_{yw} - \Omega_1 r_{xw}) - \Omega_1\Omega_2 s_{yw} - \Omega_1\Omega_3 s_{xz} + 2\Omega_3\Omega_4 r_{yz}}{4\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2}$$

$$P^4_{2,yy} = -\frac{-\Omega_1\Omega_3 s_{xy} + (\Omega_3\Omega_4 - 2\Omega_1\Omega_2) r_{yy} - 2\Omega_1^2 r_{xx}}{\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3}$$

$$P^4_{2,yz} = -\frac{\Omega_4(2\Omega_3 s_{xz} - \Omega_2 s_{yw}) - \Omega_1\Omega_2 s_{xz} + \Omega_2\Omega_4 r_{yz} - 2\Omega_4^2 r_{xw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^4_{2,yw} = -\frac{\Omega_3(\Omega_2 s_{yz} + 2\Omega_4 s_{xw} + 2\Omega_4 r_{xz}) - \Omega_1\Omega_2 s_{xw} + \Omega_2\Omega_4 r_{yw}}{4\Omega_2\Omega_3\Omega_4 - \Omega_1\Omega_2^2}$$

$$P^4_{2,zz} = -\frac{-\Omega_3 s_{zw} + 2\Omega_4 r_{ww} + \Omega_3 r_{zz}}{3\Omega_3^2}$$

$$P^4_{2,zw} = \frac{2\Omega_4 s_{ww} - 2\Omega_3 s_{zz} + \Omega_4 r_{zw}}{3\Omega_3\Omega_4}$$

$$P^4_{2,ww} = -\frac{\Omega_3 s_{zw} + \Omega_4 r_{ww} + 2\Omega_3 r_{zz}}{3\Omega_3\Omega_4}$$

Let $\zeta := (z_1, z_2, z_3, z_4)'$ and $P_3(z_1, z_2, z_3, z_4) = (P_3^1, P_3^2, P_3^3, P_3^4)'$, where $P_3^i(z_1, z_2, z_3, z_4)$ has the form as

$$\begin{aligned}\varphi =&\varphi_{111}z_1^3 + (\varphi_{112}z_2 + \varphi_{113}z_3 + \varphi_{114}z_4)z_1^2 \\ &+ (\varphi_{122}z_1 + \varphi_{222}z_2 + \varphi_{223}z_3 + \varphi_{224}z_4)z_2^2 + \varphi_{123}z_1z_2z_3 + \varphi_{124}z_1z_2z_4 \\ &+ \varphi_{134}z_3z_1z_4 + \varphi_{234}z_2z_3z_4 + (\varphi_{133}z_1 + \varphi_{233}z_2 + \varphi_{333}z_3 + \varphi_{334}z_4)z_3^2 \\ &+ (\varphi_{144}z_1 + \varphi_{244}z_2 + \varphi_{344}z_3 + \varphi_{444}z_4)z_4^2,\end{aligned}$$

for all $\varphi = P_3^i$, $i = 1, \cdots, 4$. The coefficients of P_3^i are given as follows.

$$P_{3,111}^1 = \frac{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} + \tilde{f}_{112})\Omega_1\Omega_2 + \tilde{g}_{111}\Omega_1^2}{4\Omega_1^2\Omega_2}$$

$$\begin{aligned}P_{3,112}^1 =& \{2\tilde{f}_{122}\Omega_2^2 + (-\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122} - 3\tilde{f}_{111})\Omega_1\Omega_2 \\ &+ (3\tilde{g}_{222} - \tilde{g}_{112} + 3\tilde{f}_{122} - 3\tilde{f}_{111})\Omega_1^2\}/\{6\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2 + 6\Omega_1^3\}\end{aligned}$$

$$\begin{aligned}P_{3,113}^1 =& \{\tilde{f}_{114}\Omega_3\Omega_4^2 + (\Omega_2((-2\tilde{g}_{124} - 7\tilde{f}_{114})\Omega_1 - \tilde{f}_{123}\Omega_3) - \tilde{g}_{113}\Omega_1\Omega_3 \\ &- 2\tilde{f}_{224}\Omega_2^2)\Omega_4 + (6\tilde{g}_{223} + 3\tilde{f}_{123})\Omega_1\Omega_2^2 + 3\tilde{g}_{113}\Omega_1^2\Omega_2\} \\ &/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}\end{aligned}$$

$$\begin{aligned}P_{3,114}^1 =& -\{(\tilde{f}_{113}\Omega_3^2 + \tilde{f}_{124}\Omega_2\Omega_3 + \tilde{g}_{114}\Omega_1\Omega_3)\Omega_4 + \Omega_2(\Omega_1(-2\tilde{g}_{123}\Omega_3 \\ &- 7\tilde{f}_{113}\Omega_3) - 3\tilde{g}_{114}\Omega_1^2) + \Omega_2^2((-6\tilde{g}_{224} - 3\tilde{f}_{124})\Omega_1 \\ &- 2\tilde{f}_{223}\Omega_3)\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}\end{aligned}$$

$$P_{3,122}^1 = \frac{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} + 4\tilde{f}_{222} - \tilde{f}_{112})\Omega_1\Omega_2 + 3\tilde{g}_{111}\Omega_1^2}{4\Omega_1\Omega_2^2 + 4\Omega_1^2\Omega_2}$$

$$\begin{aligned}P_{3,123}^1 =& \{\tilde{f}_{124}\Omega_3\Omega_4^2 + (\Omega_1(2\tilde{f}_{113}\Omega_3 - \tilde{g}_{123}\Omega_3) + \Omega_2((-4\tilde{g}_{224} \\ &- 5\tilde{f}_{124})\Omega_1 - 2\tilde{f}_{223}\Omega_3) + 4\tilde{g}_{114}\Omega_1^2)\Omega_4 + 6\tilde{f}_{223}\Omega_1\Omega_2^2 \\ &+ (-3\tilde{g}_{123} - 6\tilde{f}_{113})\Omega_1^2\Omega_2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}\end{aligned}$$

$$\begin{aligned}P_{3,124}^1 =& -\{(\tilde{f}_{123}\Omega_3^2 + 2\tilde{f}_{224}\Omega_2\Omega_3 + (\tilde{g}_{124} - 2\tilde{f}_{114})\Omega_1\Omega_3)\Omega_4 \\ &+ \Omega_2(\Omega_1(-4\tilde{g}_{223}\Omega_3 - 5\tilde{f}_{123}\Omega_3) + (3\tilde{g}_{124} + 6\tilde{f}_{114})\Omega_1^2) \\ &+ 4\tilde{g}_{113}\Omega_1^2\Omega_3 - 6\tilde{f}_{224}\Omega_1\Omega_2^2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}\end{aligned}$$

$$P_{3,133}^1 = \frac{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4 + (\tilde{f}_{233}\Omega_2 + \tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{g}_{234} + \tilde{f}_{134})\Omega_1\Omega_2}{4\Omega_1\Omega_2\Omega_3}$$

$$
\begin{aligned}
P^1_{3,134} = {} & \{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4^2 + (((2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_2 \\
& + (2\tilde{g}_{244} - 2\tilde{g}_{233} + 4\tilde{f}_{144} - 4\tilde{f}_{133})\Omega_1)\Omega_3 \\
& - \tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_4 \\
& + (2\tilde{g}_{233}\Omega_1 - 2\tilde{f}_{133}\Omega_2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} \\
& - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_3 + (-2\tilde{g}_{244} + 2\tilde{g}_{233} \\
& - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (-2\tilde{g}_{244} + 2\tilde{g}_{233} \\
& - 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1^2\Omega_2\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 \\
& - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\} \\
P^1_{3,144} = {} & \{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_3\Omega_4 + (\tilde{f}_{233}\Omega_2 + \tilde{g}_{133}\Omega_1)\Omega_3^2 \\
& + (\tilde{g}_{234} - \tilde{f}_{134})\Omega_1\Omega_2\Omega_3 - 2\tilde{f}_{244}\Omega_1\Omega_2^2 - 2\tilde{g}_{144}\Omega_1^2\Omega_2\} \\
& /\{4\Omega_1\Omega_2\Omega_3\Omega_4 - 4\Omega_1^2\Omega_2^2\} \\
P^1_{3,222} = {} & 0 \\
P^1_{3,223} = {} & \{\tilde{f}_{224}\Omega_3\Omega_4^2 + (\Omega_1(\tilde{f}_{123}\Omega_3 - \tilde{g}_{223}\Omega_3) - 7\tilde{f}_{224}\Omega_1\Omega_2 \\
& + (2\tilde{g}_{124} - 2\tilde{f}_{114})\Omega_1^2)\Omega_4 + (3\tilde{g}_{223} - 3\tilde{f}_{123})\Omega_1^2\Omega_2 \\
& + 6\tilde{g}_{113}\Omega_1^3\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\} \\
P^1_{3,224} = {} & -\{(\tilde{f}_{223}\Omega_3^2 + (\tilde{g}_{224} - \tilde{f}_{124})\Omega_1\Omega_3)\Omega_4 + \Omega_2((3\tilde{f}_{124} \\
& - 3\tilde{g}_{224})\Omega_1^2 - 7\tilde{f}_{223}\Omega_1\Omega_3) + \Omega_1^2(2\tilde{g}_{123}\Omega_3 - 2\tilde{f}_{113}\Omega_3) \\
& - 6\tilde{g}_{114}\Omega_1^3\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\} \\
P^1_{3,233} = {} & \{(2\tilde{f}_{234}(\Omega_2 + \Omega_1)\Omega_3 - 2\tilde{f}_{144}\Omega_1\Omega_2 + 2\tilde{g}_{244}\Omega_1^2)\Omega_4^2 + (2\tilde{f}_{234}(\Omega_2 + \Omega_1)\Omega_3^2 \\
& + ((2\tilde{g}_{244} - 2\tilde{g}_{233})\Omega_1\Omega_2 + (2\tilde{f}_{133} - 2\tilde{f}_{144})\Omega_1^2)\Omega_3 \\
& - \tilde{f}_{234}\Omega_1\Omega_2^2 + (\tilde{g}_{134} - \tilde{f}_{234})\Omega_1^2\Omega_2 + \tilde{g}_{134}\Omega_1^3)\Omega_4 \\
& + (2\tilde{f}_{133}\Omega_1\Omega_2 - 2\tilde{g}_{233}\Omega_1^2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_1\Omega_2^2 + (\tilde{g}_{134} \\
& - \tilde{f}_{234})\Omega_1^2\Omega_2 + \tilde{g}_{134}\Omega_1^3)\Omega_3\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3^2\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^3 \\
& + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3^2\} \\
P^1_{3,234} = {} & \{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4^2 + (((3\tilde{f}_{244} - 3\tilde{f}_{233})\Omega_2
\end{aligned}
$$

$$- \tilde{g}_{144}\Omega_1 + \tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2)\Omega_4$$

$$+ (-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1)\Omega_3^2 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2\Omega_3$$

$$+ (2\tilde{f}_{233} - 2\tilde{f}_{244})\Omega_1\Omega_2^2 + (2\tilde{g}_{144}\Omega_1^2 - 2\tilde{g}_{133}\Omega_1^2)\Omega_2\}$$

$$/\{4\Omega_2\Omega_3\Omega_4^2 + (4\Omega_2\Omega_3^2 - 4\Omega_1\Omega_2^2)\Omega_4 - 4\Omega_1\Omega_2^2\Omega_3\}$$

$$P^1_{3,244} = 0$$

$$P^1_{3,333} = \{6\tilde{f}_{444}\Omega_4^3 + (3\tilde{f}_{334}\Omega_3 - 2\tilde{g}_{344}\Omega_1)\Omega_4^2 + (-7\tilde{g}_{333}\Omega_1\Omega_3$$

$$- \tilde{f}_{334}\Omega_1\Omega_2)\Omega_4 + \tilde{g}_{333}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^1_{3,334} = -\{6\tilde{g}_{444}\Omega_1\Omega_4^2 + (9\tilde{f}_{333}\Omega_3^2 + \Omega_1(3\tilde{g}_{334}\Omega_3 + 2\tilde{f}_{344}\Omega_2))\Omega_4$$

$$- 3\tilde{f}_{333}\Omega_1\Omega_2\Omega_3 - \tilde{g}_{334}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^1_{3,344} = \{9\tilde{f}_{444}\Omega_3\Omega_4^2 + (-3\tilde{g}_{344}\Omega_1\Omega_3 - 3\tilde{f}_{444}\Omega_1\Omega_2)\Omega_4 - 6\tilde{g}_{333}\Omega_1\Omega_3^2$$

$$+ \Omega_2(2\tilde{f}_{334}\Omega_1\Omega_3 + \tilde{g}_{344}\Omega_1^2))\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^1_{3,444} = -\{(3\tilde{f}_{344}\Omega_3^2 + 7\tilde{g}_{444}\Omega_1\Omega_3)\Omega_4 + 6\tilde{f}_{333}\Omega_3^3 + \Omega_1(2\tilde{g}_{334}\Omega_3^2$$

$$- \tilde{f}_{344}\Omega_2\Omega_3) - \tilde{g}_{444}\Omega_1^2\Omega_2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^2_{3,111} = -\{2\tilde{f}_{122}\Omega_2^3 + ((-\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122}$$

$$+ 3\tilde{f}_{111})\Omega_1)\Omega_2^2 + (-3\tilde{g}_{222} - \tilde{g}_{112} + \tilde{f}_{122}$$

$$+ \tilde{f}_{111})\Omega_1^2\Omega_2 - 2\tilde{g}_{112}\Omega_1^3\}/\{6\Omega_1^2\Omega_2^2 + 4\Omega_1^3\Omega_2 + 6\Omega_1^4\}$$

$$P^2_{3,112} = \{\tilde{f}_{222}\Omega_2^2 + (\tilde{g}_{122} - 2\tilde{f}_{222} + \tilde{f}_{112})\Omega_1\Omega_2$$

$$+ (2\tilde{g}_{122} - 3\tilde{g}_{111})\Omega_1^2\}/\{4\Omega_1^2\Omega_2 + 4\Omega_1^3\}$$

$$P^2_{3,113} = \{\tilde{g}_{114}\Omega_3\Omega_4^2 + (\Omega_2(-\tilde{g}_{123}\Omega_3 + \tilde{f}_{113}\Omega_3 - 7\tilde{g}_{114}\Omega_1)$$

$$+ (2\tilde{f}_{124} - 2\tilde{g}_{224})\Omega_2^2)\Omega_4 - 6\tilde{f}_{223}\Omega_2^3 + (3\tilde{g}_{123}$$

$$- 3\tilde{f}_{113})\Omega_1\Omega_2^2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,114} = \{((\tilde{f}_{114} - \tilde{g}_{124})\Omega_2\Omega_3 - \tilde{g}_{113}\Omega_3^2)\Omega_4 + \Omega_2^2(2\tilde{g}_{223}\Omega_3$$

$$- 2\tilde{f}_{123}\Omega_3 + (3\tilde{g}_{124} - 3\tilde{f}_{114})\Omega_1) + 7\tilde{g}_{113}\Omega_1\Omega_2\Omega_3$$

$$- 6\tilde{f}_{224}\Omega_2^3\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,122} = -\frac{\tilde{f}_{122}\Omega_2^2 + (-2\tilde{g}_{222} + \tilde{g}_{112} + 3\tilde{f}_{111})\Omega_1\Omega_2 - 3\tilde{g}_{222}\Omega_1^2}{3\Omega_1\Omega_2^2 + 2\Omega_1^2\Omega_2 + 3\Omega_1^3}$$

$$\begin{aligned}
P^2_{3,123} = &\{\tilde{g}_{124}\Omega_3\Omega_4^2 + (\Omega_2(-2\tilde{g}_{223}\Omega_3 + \tilde{f}_{123}\Omega_3 + (-5\tilde{g}_{124} \\
&- 4\tilde{f}_{114})\Omega_1) + 2\tilde{g}_{113}\Omega_1\Omega_3 + 4\tilde{f}_{224}\Omega_2^2)\Omega_4 + (6\tilde{g}_{223} \\
&+ 3\tilde{f}_{123})\Omega_1\Omega_2^2 - 6\tilde{g}_{113}\Omega_1^2\Omega_2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}
\end{aligned}$$

$$\begin{aligned}
P^2_{3,124} = &-\{(\tilde{g}_{123}\Omega_3^2 + (2\tilde{g}_{224} - \tilde{f}_{124})\Omega_2\Omega_3 - 2\tilde{g}_{114}\Omega_1\Omega_3)\Omega_4 \\
&+ \Omega_2(\Omega_1(-5\tilde{g}_{123}\Omega_3 - 4\tilde{f}_{113}\Omega_3) + 6\tilde{g}_{114}\Omega_1^2) + \Omega_2^2(4\tilde{f}_{223}\Omega_3 \\
&+ (-6\tilde{g}_{224} - 3\tilde{f}_{124})\Omega_1)\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}
\end{aligned}$$

$$\begin{aligned}
P^2_{3,133} = &-\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4 + (2\tilde{f}_{133}\Omega_2 - 2\tilde{g}_{233}\Omega_1)\Omega_3 \\
&+ \tilde{f}_{234}\Omega_2^2 + (\tilde{f}_{234} - \tilde{g}_{134})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2\}/\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\}
\end{aligned}$$

$$\begin{aligned}
P^2_{3,134} = &\{(\tilde{f}_{244}\Omega_2 + \tilde{g}_{144}\Omega_1)\Omega_4^2 + (((\tilde{f}_{233} - \tilde{f}_{244})\Omega_2 \\
&+ 3\tilde{g}_{144}\Omega_1 - 3\tilde{g}_{133}\Omega_1)\Omega_3 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2)\Omega_4 \\
&+ (-\tilde{f}_{233}\Omega_2 - \tilde{g}_{133}\Omega_1)\Omega_3^2 + (\tilde{f}_{134} - \tilde{g}_{234})\Omega_1\Omega_2\Omega_3 \\
&+ (2\tilde{f}_{244} - 2\tilde{f}_{233})\Omega_1\Omega_2^2 + (2\tilde{g}_{133}\Omega_1^2 - 2\tilde{g}_{144}\Omega_1^2)\Omega_2\} \\
&/\{4\Omega_1\Omega_3\Omega_4^2 + (4\Omega_1\Omega_3^2 - 4\Omega_1^2\Omega_2)\Omega_4 - 4\Omega_1^2\Omega_2\Omega_3\}
\end{aligned}$$

$$\begin{aligned}
P^2_{3,144} = &-\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_3\Omega_4^2 + +(((2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_2 \\
&+ (-2\tilde{g}_{244} - 2\tilde{g}_{233})\Omega_1)\Omega_3^2 + (\tilde{f}_{234}\Omega_2^2 + (\tilde{g}_{134} \\
&+ \tilde{f}_{234})\Omega_1\Omega_2 + \tilde{g}_{134}\Omega_1^2)\Omega_3 - 4\tilde{f}_{144}\Omega_1\Omega_2^2 + 4\tilde{g}_{244}\Omega_1^2\Omega_2)\Omega_4 \\
&+ (2\tilde{f}_{133}\Omega_2 - 2\tilde{g}_{233}\Omega_1)\Omega_3^3 + (\tilde{f}_{234}\Omega_2^2 + (\tilde{g}_{134} \\
&+ \tilde{f}_{234})\Omega_1\Omega_2 + \tilde{g}_{134}\Omega_1^2)\Omega_3^2 + ((2\tilde{g}_{244} - 2\tilde{g}_{233} \\
&- 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (2\tilde{g}_{244} + 2\tilde{g}_{233} \\
&- 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_1^2\Omega_2)\Omega_3\}/\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\Omega_4^2 + ((4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3^2 \\
&- 4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_4 + (-4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_3\}
\end{aligned}$$

$$P^2_{3,222} = 0$$

$$P^2_{3,223} = \{\tilde{g}_{224}\Omega_3\Omega_4^2 + (\Omega_2(\tilde{f}_{223}\Omega_3 + (-7\tilde{g}_{224} - 2\tilde{f}_{124})\Omega_1)$$

$$+ \tilde{g}_{123}\Omega_1\Omega_3 - 2\tilde{g}_{114}\Omega_1^2)\Omega_4 - 3\tilde{f}_{223}\Omega_1\Omega_2^2 + (-3\tilde{g}_{123}$$

$$- 6\tilde{f}_{113})\Omega_1^2\Omega_2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,224} = \{(-\tilde{g}_{223}\Omega_3^2 + \tilde{f}_{224}\Omega_2\Omega_3 + \tilde{g}_{124}\Omega_1\Omega_3)\Omega_4 + \Omega_2(\Omega_1(7\tilde{g}_{223}\Omega_3$$

$$+ 2\tilde{f}_{123}\Omega_3) + (-3\tilde{g}_{124} - 6\tilde{f}_{114})\Omega_1^2) + 2\tilde{g}_{113}\Omega_1^2\Omega_3$$

$$- 3\tilde{f}_{224}\Omega_1\Omega_2^2\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^2_{3,233} = \{(2\tilde{g}_{234}\Omega_3 - \tilde{f}_{244}\Omega_2 - \tilde{g}_{144}\Omega_1)\Omega_4 + (\tilde{f}_{233}\Omega_2$$

$$+ \tilde{g}_{133}\Omega_1)\Omega_3 + (-\tilde{g}_{234} - \tilde{f}_{134})\Omega_1\Omega_2\}/\{4\Omega_3^2\Omega_4 - 4\Omega_1\Omega_2\Omega_3\}$$

$$P^2_{3,234} = -\{(2\tilde{f}_{144}\Omega_2 - 2\tilde{g}_{244}\Omega_1)\Omega_4^2 + (((-4\tilde{g}_{244} + 4\tilde{g}_{233}$$

$$- 2\tilde{f}_{144} + 2\tilde{f}_{133})\Omega_2 + (2\tilde{g}_{233} - 2\tilde{g}_{244})\Omega_1)\Omega_3$$

$$- \tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134} - \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_4$$

$$+ (2\tilde{g}_{233}\Omega_1 - 2\tilde{f}_{133}\Omega_2)\Omega_3^2 + (-\tilde{f}_{234}\Omega_2^2 + (-\tilde{g}_{134}$$

$$- \tilde{f}_{234})\Omega_1\Omega_2 - \tilde{g}_{134}\Omega_1^2)\Omega_3 + (2\tilde{g}_{244} - 2\tilde{g}_{233}$$

$$+ 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1\Omega_2^2 + (2\tilde{g}_{244} - 2\tilde{g}_{233}$$

$$+ 2\tilde{f}_{144} - 2\tilde{f}_{133})\Omega_1^2\Omega_2\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2$$

$$- 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}$$

$$P^2_{3,244} = 0$$

$$P^2_{3,333} = \{6\tilde{g}_{444}\Omega_4^3 + (3\tilde{g}_{334}\Omega_3 + 2\tilde{f}_{344}\Omega_2)\Omega_4^2 + (7\tilde{f}_{333}\Omega_2\Omega_3$$

$$- \tilde{g}_{334}\Omega_1\Omega_2)\Omega_4 - \tilde{f}_{333}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^2_{3,334} = \{6\tilde{f}_{444}\Omega_2\Omega_4^2 + (\Omega_2(3\tilde{f}_{334}\Omega_3 - 2\tilde{g}_{344}\Omega_1) - 9\tilde{g}_{333}\Omega_3^2)\Omega_4$$

$$+ 3\tilde{g}_{333}\Omega_1\Omega_2\Omega_3 - \tilde{f}_{334}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^2_{3,344} = \{9\tilde{g}_{444}\Omega_3\Omega_4^2 + (3\tilde{f}_{344}\Omega_2\Omega_3 - 3\tilde{g}_{444}\Omega_1\Omega_2)\Omega_4 + 6\tilde{f}_{333}\Omega_2\Omega_3^2$$

$$+ \Omega_1(2\tilde{g}_{334}\Omega_2\Omega_3 - \tilde{f}_{344}\Omega_2^2)\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^2_{3,444} = \{(7\tilde{f}_{444}\Omega_2\Omega_3 - 3\tilde{g}_{344}\Omega_3^2)\Omega_4 - 6\tilde{g}_{333}\Omega_3^3 + \Omega_2(2\tilde{f}_{334}\Omega_3^2$$

$$+ \tilde{g}_{344}\Omega_1\Omega_3) - \tilde{f}_{444}\Omega_1\Omega_2^2\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^3_{3,111} = \{\Omega_2^2(3\Omega_1\tilde{r}_{112} - 2\Omega_3\tilde{s}_{122}) + (\Omega_3^2\Omega_4 - 7\Omega_1\Omega_2\Omega_3)\tilde{s}_{111}$$

$$+ 6\Omega_2^3\tilde{r}_{222} - \Omega_2\Omega_3\Omega_4\tilde{r}_{112}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^3_{3,112} = \{\Omega_3(\Omega_2(-3\Omega_1\tilde{s}_{112} - 2\Omega_4\tilde{r}_{122}) - 6\Omega_2^2\tilde{s}_{222})$$

$$+ \Omega_3^2\Omega_4\tilde{s}_{112} + (3\Omega_1\Omega_3\Omega_4 - 9\Omega_1^2\Omega_2)\tilde{r}_{111}\}$$

$$/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^3_{3,113} = -\{\Omega_3(\Omega_2\tilde{s}_{223} + \Omega_4(-\tilde{s}_{124} - \tilde{r}_{123}) - \Omega_1\tilde{s}_{113})$$

$$+ \Omega_2\Omega_4\tilde{r}_{224} + 2\Omega_1\Omega_2\tilde{r}_{123} - \Omega_1\Omega_4\tilde{r}_{114}\}/\{4\Omega_1\Omega_3\Omega_4 - 4\Omega_1^2\Omega_2\}$$

$$P^3_{3,114} = -\{\Omega_3(-2\Omega_2\tilde{s}_{224} - 2\Omega_1\tilde{s}_{114}) + \Omega_3\Omega_4(\tilde{s}_{123} - \tilde{r}_{124})$$

$$+ \Omega_3^2\tilde{s}_{123} + 2\Omega_2\Omega_4\tilde{r}_{223} - \Omega_4^2\tilde{r}_{124} + 2\Omega_1\Omega_4\tilde{r}_{113}\}$$

$$/\{4\Omega_1\Omega_4^2 + 4\Omega_1\Omega_3\Omega_4\}$$

$$P^3_{3,122} = -\{\Omega_4(-\Omega_3^2\tilde{s}_{122} + 3\Omega_2\Omega_3\tilde{r}_{222} - 2\Omega_1\Omega_3\tilde{r}_{112}) + 3\Omega_1\Omega_2\Omega_3\tilde{s}_{122}$$

$$+ 6\Omega_1^2\Omega_3\tilde{s}_{111} - 9\Omega_1\Omega_2^2\tilde{r}_{222}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^3_{3,123} = \{\Omega_4(\Omega_3^2(2\tilde{s}_{224} - 2\tilde{s}_{114}) + \Omega_3(\Omega_2(\tilde{s}_{123} + \tilde{r}_{124})$$

$$+ \Omega_1(\tilde{s}_{123} + \tilde{r}_{124}))) + \Omega_4^2(\Omega_3(2\tilde{s}_{224} - 2\tilde{s}_{114})$$

$$+ \Omega_2\tilde{r}_{124} + \Omega_1\tilde{r}_{124}) + \Omega_3(2\Omega_2^2\tilde{s}_{224} + \Omega_1\Omega_2(2\tilde{s}_{114}$$

$$- 2\tilde{s}_{224}) - 2\Omega_1^2\tilde{s}_{114}) + \Omega_3^2(\Omega_2\tilde{s}_{123} + \Omega_1\tilde{s}_{123})$$

$$+ (2\Omega_3\Omega_4^2 + (2\Omega_3^2 - 2\Omega_2^2 - 2\Omega_1\Omega_2)\Omega_4 - 4\Omega_1\Omega_2\Omega_3)\tilde{r}_{223}$$

$$+ (-2\Omega_3\Omega_4^2 + (-2\Omega_3^2 + 2\Omega_1\Omega_2 + 2\Omega_1^2)\Omega_4 + 4\Omega_1\Omega_2\Omega_3)\tilde{r}_{113}\}$$

$$/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4$$

$$+ (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}$$

$$P^3_{3,124} = \{\Omega_3(\Omega_2(\Omega_1(\tilde{s}_{223} - \tilde{s}_{113}) + \Omega_4(\tilde{s}_{124} - \tilde{r}_{123}))$$

$$- \Omega_2^2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{s}_{124} - \tilde{r}_{123}) + \Omega_1^2\tilde{s}_{113})$$

$$+ \Omega_3^2(2\Omega_4\tilde{s}_{113} - 2\Omega_4\tilde{s}_{223}) + (2\Omega_3\Omega_4^2 - \Omega_2^2\Omega_4 - 3\Omega_1\Omega_2\Omega_4)\tilde{r}_{224}$$

$$+ (-2\Omega_3\Omega_4^2 + 3\Omega_1\Omega_2\Omega_4 + \Omega_1^2\Omega_4)\tilde{r}_{114}\}/\{\Omega_3(4\Omega_2\Omega_4^2$$

$$+ 4\Omega_1\Omega_4^2) - 4\Omega_1\Omega_2^2\Omega_4 - 4\Omega_1^2\Omega_2\Omega_4\}$$

$$P^3_{3,133} = \{\Omega_2\Omega_4(\Omega_3(-2\tilde{s}_{234} - 7\tilde{r}_{233}) - \Omega_1\tilde{r}_{134}) + \Omega_4^2(6\Omega_3\tilde{s}_{144}$$

$$
\begin{aligned}
&- 2\Omega_2\tilde{r}_{244} + 3\Omega_3\tilde{r}_{134}) + (3\Omega_3^2\Omega_4 - \Omega_1\Omega_2\Omega_3)\tilde{s}_{133} + \Omega_1\Omega_2^2\tilde{r}_{233}\} \\
&/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^3_{3,134} = {}&-\{\Omega_4(\Omega_2\Omega_3(4\tilde{s}_{244} + 5\tilde{r}_{234}) + 3\Omega_3^2\tilde{s}_{134} + 2\Omega_1\Omega_2\tilde{r}_{144}) \\
&- 4\Omega_2\Omega_3^2\tilde{s}_{233} + \Omega_1\Omega_2\Omega_3\tilde{s}_{134} - \Omega_1\Omega_2^2\tilde{r}_{234} - 6\Omega_3\Omega_4^2\tilde{r}_{144} \\
&+ (6\Omega_3^2\Omega_4 - 2\Omega_1\Omega_2\Omega_3)\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^3_{3,144} = {}&\{\Omega_2(\Omega_3^2(2\tilde{s}_{234} - 2\tilde{r}_{233}) - \Omega_1\Omega_3\tilde{s}_{144} + \Omega_1\Omega_3\tilde{r}_{134}) \\
&+ \Omega_4(3\Omega_3^2\tilde{s}_{144} - 7\Omega_2\Omega_3\tilde{r}_{244} - 3\Omega_3^2\tilde{r}_{134}) + 6\Omega_3^3\tilde{s}_{133} \\
&+ \Omega_1\Omega_2^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^3_{3,222} = {}&-\{\Omega_3(7\Omega_1\Omega_2\tilde{s}_{222} + 2\Omega_1^2\tilde{s}_{112} - \Omega_1\Omega_4\tilde{r}_{122}) - \Omega_3^2\Omega_4\tilde{s}_{222} \\
&+ 3\Omega_1^2\Omega_2\tilde{r}_{122} + 6\Omega_1^3\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\} \\
P^3_{3,223} = {}&0 \\
P^3_{3,224} = {}&-\{\Omega_4(\Omega_3^2(\Omega_1(-2\tilde{s}_{224} - 2\tilde{s}_{114}) - 4\Omega_2\tilde{s}_{224}) - \Omega_3\Omega_1(\Omega_1 + \Omega_2)(\tilde{s}_{123} \\
&+ \tilde{r}_{124})) + \Omega_3(\Omega_1^2\Omega_2(2\tilde{s}_{224} + 2\tilde{s}_{114}) + 2\Omega_1\Omega_2^2\tilde{s}_{224} \\
&+ 2\Omega_1^3\tilde{s}_{114}) + \Omega_4^2(\Omega_1\Omega_3(2\tilde{s}_{114} - 2\tilde{s}_{224}) - \Omega_1\Omega_2\tilde{r}_{124} \\
&- \Omega_1^2\tilde{r}_{124}) - \Omega_3^2(\Omega_2 + \Omega_1)\Omega_1\tilde{s}_{123} + ((4\Omega_2 + 2\Omega_1)\Omega_3\Omega_4^2 \\
&+ (2\Omega_1\Omega_3^2 - 2\Omega_1\Omega_2^2 - 2\Omega_1^2\Omega_2)\Omega_4)\tilde{r}_{223} + (2\Omega_1\Omega_3\Omega_4^2 \\
&+ (-2\Omega_1\Omega_3^2 - 2\Omega_1^2\Omega_2 - 2\Omega_1^3)\Omega_4)\tilde{r}_{113}\}/\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^3 \\
&+ ((4\Omega_2 + 4\Omega_1)\Omega_3^2 - 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4^2 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\Omega_4\} \\
P^3_{3,233} = {}&\{\Omega_4^2(\Omega_3(6\tilde{s}_{244} + 3\tilde{r}_{234}) + 2\Omega_1\tilde{r}_{144}) + \Omega_4(3\Omega_3^2\tilde{s}_{233} \\
&+ 2\Omega_1\Omega_3\tilde{s}_{134} - \Omega_1\Omega_2\tilde{r}_{234}) - \Omega_1\Omega_2\Omega_3\tilde{s}_{233} + (7\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2)\tilde{r}_{133}\} \\
&/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^3_{3,234} = {}&-\{\Omega_2(\Omega_3(\Omega_1\tilde{s}_{234} - 2\Omega_1\tilde{r}_{233}) + \Omega_1^2\tilde{r}_{134}) + \Omega_4(\Omega_3^2(3\tilde{s}_{234} \\
&+ 6\tilde{r}_{233}) - 4\Omega_1\Omega_3\tilde{s}_{144} + 2\Omega_1\Omega_2\tilde{r}_{244} - 5\Omega_1\Omega_3\tilde{r}_{134}) \\
&+ 4\Omega_1\Omega_3^2\tilde{s}_{133} - 6\Omega_3\Omega_4^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^3_{3,244} = {}&\{\Omega_4(\Omega_3^2(3\tilde{s}_{244} - 3\tilde{r}_{234}) + 7\Omega_1\Omega_3\tilde{r}_{144}) + \Omega_1\Omega_2\Omega_3(\tilde{r}_{234}
\end{aligned}
$$

$$- \tilde{s}_{244}) + 6\Omega_3^3\tilde{s}_{233} - 2\Omega_1\Omega_3^2\tilde{s}_{134} - \Omega_1^2\Omega_2\tilde{r}_{144}$$
$$+ 2\Omega_1\Omega_3^2\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$
$$P^3_{3,333} = -\{\Omega_3\Omega_4(2\tilde{s}_{333} - \tilde{s}_{344}) - \Omega_3^2\tilde{s}_{333} + 3\Omega_4^2\tilde{r}_{444}$$
$$+ (-2\Omega_4^2 - \Omega_3\Omega_4)\tilde{r}_{334}\}/\{4\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4\}$$
$$P^3_{3,334} = \{\Omega_3(9\tilde{s}_{444} - \tilde{s}_{334}) + \Omega_4(3\tilde{s}_{444} - 3\tilde{s}_{334})$$
$$+ (\Omega_4 + 3\Omega_3)\tilde{r}_{344} + (-9\Omega_4 - 3\Omega_3)\tilde{r}_{333}\}/\{6\Omega_4^2 + 4\Omega_3\Omega_4 + 6\Omega_3^2\}$$
$$P^3_{3,344} = 0$$
$$P^3_{3,444} = -\frac{-3\Omega_3^2\tilde{s}_{444} + \Omega_4(\Omega_3(\tilde{s}_{334} - 2\tilde{s}_{444}) + \Omega_4^2\tilde{r}_{344} + 3\Omega_3\Omega_4\tilde{r}_{333}}{3\Omega_4^3 + 2\Omega_3\Omega_4^2 + 3\Omega_3^2\Omega_4}$$
$$P^4_{3,111} = -\{-6\Omega_2^3\tilde{s}_{222} + \Omega_2^2(-3\Omega_1\tilde{s}_{112} - 2\Omega_4\tilde{r}_{122}) + \Omega_2\Omega_3\Omega_4\tilde{s}_{112}$$
$$+ (\Omega_3\Omega_4^2 - 7\Omega_1\Omega_2\Omega_4)\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$
$$P^4_{3,112} = \{\Omega_4(\Omega_2(3\Omega_1\tilde{r}_{112} - 2\Omega_3\tilde{s}_{122}) + 6\Omega_2^2\tilde{r}_{222}) + (3\Omega_1\Omega_3\Omega_4 - 9\Omega_1^2\Omega_2)\tilde{s}_{111}$$
$$- \Omega_3\Omega_4^2\tilde{r}_{112}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$
$$P^4_{3,113} = \{\Omega_4(\Omega_3(2\Omega_1^2\tilde{s}_{114} - 2\Omega_2^2\tilde{s}_{224}) + (\Omega_3^2 - 2\Omega_1\Omega_2)(\Omega_1 + \Omega_2)\tilde{s}_{123})$$
$$+ \Omega_4^2(\Omega_1\Omega_2(2\tilde{s}_{114} - 2\tilde{s}_{224}) + \Omega_3(\Omega_1 + \Omega_2)(\tilde{s}_{123} - \tilde{r}_{124}))$$
$$+ \Omega_3(\Omega_2^2(-2\Omega_1\tilde{s}_{123}) - 2\Omega_1^2\Omega_2\tilde{s}_{123}) + (2\Omega_2^2\Omega_4^2 + 2\Omega_1\Omega_2\Omega_3\Omega_4)\tilde{r}_{223}$$
$$+ \Omega_4^3(-\Omega_2\tilde{r}_{124} - \Omega_1\tilde{r}_{124}) + (-2\Omega_1^2\Omega_4^2 - 2\Omega_1\Omega_2\Omega_3\Omega_4)\tilde{r}_{113}\}$$
$$/\{(4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3\Omega_4^2 + ((4\Omega_1\Omega_2 + 4\Omega_1^2)\Omega_3^2$$
$$- 4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_4 + (-4\Omega_1^2\Omega_2^2 - 4\Omega_1^3\Omega_2)\Omega_3\}$$
$$P^4_{3,114} = -\{\Omega_3(\Omega_2\tilde{s}_{223} + \Omega_4(-\tilde{s}_{124} - \tilde{r}_{123}) + \Omega_1\tilde{s}_{113})$$
$$+ \Omega_2\Omega_4\tilde{r}_{224} + \Omega_1\Omega_4\tilde{r}_{114}\}/\{4\Omega_1\Omega_3\Omega_4\}$$
$$P^4_{3,122} = \{\Omega_3(-3\Omega_2\Omega_4\tilde{s}_{222} + 2\Omega_1\Omega_4\tilde{s}_{112} - \Omega_4^2\tilde{r}_{122}) + 9\Omega_1\Omega_2^2\tilde{s}_{222}$$
$$+ 3\Omega_1\Omega_2\Omega_4\tilde{r}_{122} + 6\Omega_1^2\Omega_4\tilde{r}_{111}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$
$$P^4_{3,123} = -\{\Omega_3(\Omega_2(\Omega_1(3\tilde{s}_{223} - 3\tilde{s}_{113}) + \Omega_4(\tilde{r}_{123} - \tilde{s}_{124}))$$
$$+ \Omega_2^2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{r}_{123} - \tilde{s}_{124}) - \Omega_1^2\tilde{s}_{113})$$

$$+ \Omega_3^2(2\Omega_4\tilde{s}_{113} - 2\Omega_4\tilde{s}_{223}) + (2\Omega_3\Omega_4^2 + \Omega_2^2\Omega_4 - \Omega_1\Omega_2\Omega_4)\tilde{r}_{224}$$
$$+ (-2\Omega_3\Omega_4^2 + \Omega_1\Omega_2\Omega_4 - \Omega_1^2\Omega_4)\tilde{r}_{114}\}/\{\Omega_3^2(4\Omega_2\Omega_4$$
$$+ 4\Omega_1\Omega_4) + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}$$

$$P^4_{3,124} = \{\Omega_4(\Omega_3^2(2\tilde{s}_{224} - 2\tilde{s}_{114}) + \Omega_1\Omega_2(4\tilde{s}_{114} - 4\tilde{s}_{224})$$
$$+ \Omega_3(\Omega_2(-\tilde{s}_{123} - \tilde{r}_{124}) + \Omega_1(-\tilde{s}_{123} - \tilde{r}_{124})))$$
$$+ \Omega_4^2(\Omega_3(2\tilde{s}_{224} - 2\tilde{s}_{114}) - \Omega_2\tilde{r}_{124} - \Omega_1\tilde{r}_{124})$$
$$+ \Omega_3(-2\Omega_2^2\tilde{s}_{224} + \Omega_1\Omega_2(2\tilde{s}_{114} - 2\tilde{s}_{224}) + 2\Omega_1^2\tilde{s}_{114})$$
$$+ \Omega_3^2(-\Omega_2\tilde{s}_{123} - \Omega_1\tilde{s}_{123}) + (2\Omega_3\Omega_4^2 + (2\Omega_3^2 + 2\Omega_2^2 - 2\Omega_1\Omega_2)\Omega_4)\tilde{r}_{223}$$
$$+ ((-2\Omega_3^2 + 2\Omega_1\Omega_2 - 2\Omega_1^2)\Omega_4 - 2\Omega_3\Omega_4^2)\tilde{r}_{113}\}/$$
$$\{(4\Omega_2 + 4\Omega_1)\Omega_3\Omega_4^2 + ((4\Omega_2 + 4\Omega_1)\Omega_3^2$$
$$- 4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_4 + (-4\Omega_1\Omega_2^2 - 4\Omega_1^2\Omega_2)\Omega_3\}$$

$$P^4_{3,133} = -\{\Omega_4^2(\Omega_2(2\tilde{s}_{244} - 2\tilde{r}_{234}) - 3\Omega_3\tilde{s}_{134}) + \Omega_4(7\Omega_2\Omega_3\tilde{s}_{233}$$
$$+ \Omega_1\Omega_2\tilde{s}_{134}) - \Omega_1\Omega_2^2\tilde{s}_{233} + 6\Omega_4^3\tilde{r}_{144} + (3\Omega_3\Omega_4^2$$
$$- \Omega_1\Omega_2\Omega_4)\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^4_{3,134} = -\{\Omega_2\Omega_4(\Omega_3(5\tilde{s}_{234} + 4\tilde{r}_{233}) + 2\Omega_1\tilde{s}_{144} - \Omega_1\tilde{r}_{134})$$
$$- \Omega_1\Omega_2^2\tilde{s}_{234} + \Omega_4^2(-6\Omega_3\tilde{s}_{144} - 4\Omega_2\tilde{r}_{244} - 3\Omega_3\tilde{r}_{134})$$
$$+ (6\Omega_3^2\Omega_4 - 2\Omega_1\Omega_2\Omega_3)\tilde{s}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^4_{3,144} = -\{\Omega_4(\Omega_2\Omega_3(7\tilde{s}_{244} + 2\tilde{r}_{234}) + 3\Omega_3^2\tilde{s}_{134} - \Omega_1\Omega_2\tilde{r}_{144})$$
$$- \Omega_1\Omega_2^2\tilde{s}_{244} + 2\Omega_2\Omega_3^2\tilde{s}_{233} - \Omega_1\Omega_2\Omega_3\tilde{s}_{134} + 3\Omega_3\Omega_4^2\tilde{r}_{144}$$
$$+ 6\Omega_3^2\Omega_4\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\}$$

$$P^4_{3,222} = -\{\Omega_4(-\Omega_1\Omega_3\tilde{s}_{122} - 7\Omega_1\Omega_2\tilde{r}_{222} - 2\Omega_1^2\tilde{r}_{112}) + 3\Omega_1^2\Omega_2\tilde{s}_{122}$$
$$+ 6\Omega_1^3\tilde{s}_{111} + \Omega_3\Omega_4^2\tilde{r}_{222}\}/\{\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + 9\Omega_1^2\Omega_2^2\}$$

$$P^4_{3,223} = 0$$

$$P^4_{3,224} = -\{2\Omega_3^2\Omega_4\tilde{s}_{223} + \Omega_3(-\Omega_1\Omega_2\tilde{s}_{223} + \Omega_1\Omega_4(\tilde{r}_{123} - \tilde{s}_{124})$$
$$- \Omega_1^2\tilde{s}_{113}) + (2\Omega_3\Omega_4^2 - \Omega_1\Omega_2\Omega_4)\tilde{r}_{224} - \Omega_1^2\Omega_4\tilde{r}_{114}\}$$

$$
\begin{aligned}
&/\{4\Omega_3^2\Omega_4^2 - 4\Omega_1\Omega_2\Omega_3\Omega_4\} \\
P^4_{3,233} =& \{\Omega_4^2(\Omega_3(3\tilde{s}_{234} - 3\tilde{r}_{233}) + 2\Omega_1\tilde{s}_{144} - 2\Omega_1\tilde{r}_{134}) \\
&+ \Omega_2\Omega_4(\Omega_1\tilde{r}_{233} - \Omega_1\tilde{s}_{234}) + (7\Omega_1\Omega_3\Omega_4 - \Omega_1^2\Omega_2)\tilde{s}_{133} \\
&- 6\Omega_4^3\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^4_{3,234} =& \{\Omega_4^2(\Omega_3(6\tilde{s}_{244} + 3\tilde{r}_{234}) - 4\Omega_1\tilde{r}_{144}) + \Omega_4(\Omega_1\Omega_2(\tilde{r}_{234} \\
&- 2\tilde{s}_{244}) - 6\Omega_3^2\tilde{s}_{233} + 5\Omega_1\Omega_3\tilde{s}_{134}) + 2\Omega_1\Omega_2\Omega_3\tilde{s}_{233} \\
&- \Omega_1^2\Omega_2\tilde{s}_{134} + 4\Omega_1\Omega_3\Omega_4\tilde{r}_{133}\}/\{9\Omega_3^2\Omega_4^2 \\
&- 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^4_{3,244} =& \{\Omega_2(\Omega_1\Omega_3\tilde{s}_{234} - \Omega_1^2\tilde{s}_{144}) + \Omega_4(\Omega_3^2(-3\tilde{s}_{234} - 6\tilde{r}_{233}) \\
&+ 7\Omega_1\Omega_3\tilde{s}_{144} + \Omega_1\Omega_2\tilde{r}_{244} + 2\Omega_1\Omega_3\tilde{r}_{134}) + 2\Omega_1\Omega_3^2\tilde{s}_{133} \\
&- 3\Omega_3\Omega_4^2\tilde{r}_{244}\}/\{9\Omega_3^2\Omega_4^2 - 10\Omega_1\Omega_2\Omega_3\Omega_4 + \Omega_1^2\Omega_2^2\} \\
P^4_{3,333} =& -\{\Omega_4(\Omega_3(3\tilde{s}_{444} - \tilde{s}_{334})) + \Omega_4^2(3\tilde{s}_{444} - 3\tilde{s}_{334}) \\
&- 2\Omega_3^2\tilde{s}_{334} + (\Omega_4^2 + \Omega_3\Omega_4)\tilde{r}_{344} + (\Omega_3\Omega_4 - 3\Omega_4^2)\tilde{r}_{333}\} \\
&/\{6\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4 + 6\Omega_3^3\} \\
P^4_{3,334} =& -\frac{\Omega_3(3\tilde{s}_{333} - \tilde{s}_{344}) + 3\Omega_4\tilde{r}_{444} - \Omega_4\tilde{r}_{334}}{2\Omega_3\Omega_4 + 2\Omega_3^2} \\
P^4_{3,344} =& 0 \\
P^4_{3,444} =& -\frac{\Omega_4\Omega_3\tilde{s}_{344} + 3\Omega_3^2\tilde{s}_{333} + (\Omega_4^2 + 4\Omega_3\Omega_4)\tilde{r}_{444} - \Omega_3\Omega_4\tilde{r}_{334}}{4\Omega_3\Omega_4^2 + 4\Omega_3^2\Omega_4}
\end{aligned}
$$

Appendix 4.C

The coefficients of the quadratic terms and cubic terms of functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are as follows:

$$
\begin{aligned}
\hat{\varphi}_{ii} =& \varphi_{ii} + \varphi_{i\xi}E_{[i]} + E'_{[i]}\varphi_{\xi\xi}E_{[i]} \\
\hat{\varphi}_{ij} =& \varphi_{ij} + \varphi_{i\xi}E_{[j]} + \varphi_{j\xi}E_{[i]} + 2E'_{[i]}\varphi_{\xi\xi}E_{[j]} \\
\hat{\varphi}_{iii} =& \varphi_{iii} + \varphi_{ii\xi}E_{[i]} + E'_{[i]}\varphi_{i\xi\xi}E_{[i]} + \varphi_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[i]}) \\
&+ \varphi_{i\xi}h_{ii} + 2E'_{[i]}\varphi_{\xi\xi}h_{ii}
\end{aligned}
$$

$$\begin{aligned}\hat{\varphi}_{iij} =&\varphi_{j\xi}h_{ii} + \varphi_{i\xi}h_{ij} + 2E'_{[j]}\varphi_{\xi\xi}h_{ii} + 2E'_{[i]}\varphi_{\xi\xi}h_{ij} + \varphi_{iij} + \varphi_{ij\xi}E_{[i]}\\ &+ \varphi_{ii\xi}E_{[j]} + E'_{[i]}\varphi_{j\xi\xi}E_{[i]} + 2E'_{[i]}\varphi_{i\xi\xi}E_{[j]} + 3\varphi_{\xi\xi\xi}(E_{[i]}, E_{[i]}, E_{[j]})\\ \hat{\varphi}_{ijk} =&\varphi_{ijk} + \varphi_{i\xi}h_{jk} + \varphi_{j\xi}h_{ik} + \varphi_{k\xi}h_{ij}\\ &+ 2E'_{[i]}\varphi_{\xi\xi}h_{jk} + 2E'_{[j]}\varphi_{\xi\xi}h_{ik} + 2E'_{[k]}\varphi_{\xi\xi}h_{ik}.\end{aligned}$$

Here, i, j, k are distinct, $\varphi \in \{f, g, r, s\}$, and $i, j, k \in \{x, y, z, w\}$ with $E_{[x]} = E_1$, $E_{[y]} = E_2$, $E_{[z]} = E_3$, $E_{[w]} = E_4$.

CHAPTER FIVE

LIAPUNOV FUNCTIONS FOR NONLINEAR SYSTEMS VIA CENTER MANIFOLD REDUCTION

In this chapter, we construct families of "composite" Liapunov function candidates for general nonlinear critical systems using center manifold reduction technique. One part of the composite Liapunov function is based on the reduced subsystem on the center manifold. The other part is based on the Jacobian matrix of the noncritical subsystem. Detailed constructions of Liapunov functions are given for the simple critical cases and the compound critical cases discussed in Chapters 3 and 4. The stability conditions for these critical cases obtained in the previous two chapters are also reconstructed in this chapter by using the Liapunov function approach.

5.1. Introduction

Behtash and Sastry [10] employed Liapunov functions for reduced order models of nonlinear critical systems in normal form to obtain stability criteria. However, Liapunov functions have not been constructed directly for the original system without the need for the reduction. Motivated by Fu and Abed's results [26] on the construction of Liapunov functions for nonlinear systems in simple critical cases (i.e, cases in which the linearized system has either one zero eigenvalue or a pair of pure imaginary eigenvalues), we construct families of Liapunov functions for the nonlinear systems within the framework of center manifold reduction. The result relies on the stability of the linearization of the noncritical subsystem and the identification of Liapunov functions for the reduced model on the center manifold. In the following, composite-type Liapunov functions are constructed, in a sense to be explicated below.

Two categories of critical systems are considered in this chapter, which include the simple critical case and the compound critical case. The simple critical case considered here is that of the linearized model of the system has one zero eigenvalue or a pair of pure imaginary eigenvalues with remaining eigenvalues stable; while the compound critical case is that when the linearized model possesses two zero eigenvalues with geometric multiplicity one, one zero eigenvalue and a pair of pure imaginary eigenvalues or two distinct pairs of pure imaginary eigenvalues with remaining eigenvalues stable. The main difference between this result and Fu and Abed's results [26] is that the technique of center manifold reduction is used in this chapter instead of the approach using eigenvector decomposition of the vector space in [26]. In addition, [26] is concerned only with the simple critical cases, whereas the compound critical cases are also considered here.

The results in this chapter are obtained as follows. First, results on locally positive definite function are obtained. It is followed by the construction of Liapunov function candidates for general critical nonlinear systems using center manifold reduction. The detailed designs of families of Liapunov functions for the simple critical cases and the compound critical cases are given in Section 5.3 and 5.4 to demonstrate the main results.

5.2. Locally Definite Functions

The technique of Taylor series expansion is a very important tool to construct Liapunov functions for nonlinear systems, which can be conveniently represented as multilinear function. By using the notations of multilinear functions given in Section 2.2, some locally positive definite functions are introduced in this section, which will be employed in the next three sections for the construction of Liapunov function candidates.

First, recall the next two definitions.

Definition 5.1. (e.g., [86]) A continuous function $\psi : I\!R \to I\!R$ is said to be of class K if (i) $\psi(\cdot)$ is strictly increasing, and (ii) $\psi(0) = 0$.

Definition 5.2. (e.g., [86]) A continuous function $\psi : I\!R^n \to I\!R$ is an l.p.d.f. if and only if $\psi(0) = 0$, and $\psi(x) > 0$ for all $x \neq 0$ and $||x|| < \delta$ for some $\delta > 0$. ψ is a p.d.f. if and only if $\psi(0) = 0$, $\psi(x) > 0$ and $\psi(x) \to \infty$ as $||x|| \to \infty$ uniformly in x. Moreover, a continuous function ψ is said to be (locally) negative definite if $-\psi$ is an (l.p.d.f.) p.d.f.

Now, we introduce some results on the existence of locally positive definite functions.

Consider a scalar function as given by

$$v(\eta, \xi) = \xi' \mathcal{P} \xi + \xi' \rho_{\eta\xi}(\eta) + \rho_{\eta\eta}(\eta) + \rho_{\eta\xi\xi}(\eta, \xi) + \rho_{\xi\xi\xi}(\xi), \tag{5.1}$$

where $\eta \in I\!R^n, \xi \in I\!R^m, \rho_{\eta\xi}(\eta)$ is a vector polynomial function of η of which each component has order in η no less than j_1; $\rho_{\eta\eta}(\eta)$ is a scalar $2j_2$-linear function; $\rho_{\eta\xi\xi}(\eta, \xi)$ is a scalar polynomial function of which each component has order in (η, ξ) no less than one and two, respectively; and $\rho_{\xi\xi\xi}(\xi)$ is a scalar polynomial function of ξ of which each component has order in ξ no less than three. Here, j_1 and j_2 denote positive integers with $j_1 \geq j_2$.

We have the following result.

Lemma 5.1. (Locally Positive Definite Function for Two Sets of Variables) Suppose there exist $\alpha_1, \alpha_2 > 0$ and $\beta_1 > 0$ such that $\xi' \mathcal{P} \xi \geq \alpha_1 ||\xi||^2$, $\rho_{\eta\eta}(\eta) \geq \alpha_2 ||\eta||^{2j_2}$ and $||\xi' \rho_{\eta\xi}(\eta)|| \leq \beta_1 ||\xi|| \cdot ||\eta||^{j_2}$. If $4\alpha_1\alpha_2 > \beta_1^2$, then $v(\eta, \xi)$ given by (5.1) is an l.p.d.f.

Proof: It is known that there exists $\delta_1, \beta_2 > 0$ such that

$$||\rho_{\eta\xi\xi}(\eta,\xi) + \rho_{\xi\xi\xi}(\xi)|| \leq \beta_2(||\eta|| + ||\xi||) \cdot ||\xi||^2, \tag{5.2}$$

for all $||\eta||, ||\xi|| < \delta_1$.

Suppose the assumptions of Lemma 5.1 hold, i.e., there exist $\alpha_1, \alpha_2 > 0$ and $\beta_1 > 0$ such that $\xi'\mathcal{P}\xi \geq \alpha_1||\xi||^2$, $\rho_{\eta\eta}(\eta) \geq \alpha_2||\eta||^{2j_2}$ and $||\xi'\rho_{\eta\xi}(\eta)|| \leq \beta_1||\xi|| \cdot ||\eta||^{j_2}$.

Then we have

$$\begin{aligned} v(\eta,\xi) &\geq \Delta \cdot ||\xi||^2 - \beta_1||\xi|| \cdot ||\eta||^{j_2} + \alpha_2||\eta||^{2j_2} \\ &= \Delta \cdot (||\xi|| - \frac{\beta_1}{2\Delta} \cdot ||\eta||^{j_2})^2 + \frac{1}{4\Delta}(4\alpha_2\Delta - \beta_1^2) \cdot ||\eta||^{2j_2}, \end{aligned} \tag{5.3}$$

where $\Delta := \alpha_1 - \beta_2(||\eta|| + ||\xi||)$.

The conclusion of Lemma 5.1 is hence implied when $4\alpha_1\alpha_2 > \beta_1^2$.

■

It is known (e.g., [86]) that there exist $\alpha_1, \alpha_2 > 0$ such that $\xi'\mathcal{P}\xi \geq \alpha_1||\xi||^2$ and $\rho_{\eta\eta}(\eta) \geq \alpha_2||\xi||^{2j_2}$ when matrix $\mathcal{P}$ is positive definite and function $\rho_{\eta\eta}(\eta)$ is a p.d.f. Moreover, when $j_1 > j_2$, where j_i are defined above, we will have $||\xi'\rho_{\eta\xi}(\eta)|| \leq \beta_1(||\eta||) \cdot ||\xi|| \cdot ||\eta||^{j_2}$, where β_1 is a function of class K instead of a positive constant. Thus, we have the following result from Lemma 5.1.

Corollary 5.1. If matrix $\mathcal{P}$ is positive definite and the scalar function $\rho_{\eta\eta}(\eta)$ is a p.d.f., and the integer $j_1 > j_2$, then $v(\eta,\xi)$ given by (5.1) is an l.p.d.f.

The next result follows readily from Corollary 5.1.

Corollary 5.2. The scalar function $v(\eta,\xi)$ given in (5.1) is locally negative definite when matrix $\mathcal{P}$ is negative definite matrix and function $\rho_{\eta\eta}(\eta)$ is negative definite, with $j_1 > j_2$.

A special extension of the locally definite function given in (5.1) is introduced as follows.

Consider a scalar function

$$v(\xi,\eta,\zeta) = \xi'\mathcal{P}_1\xi + \eta'\mathcal{P}_2\eta + \mathcal{C}(\xi,\eta,\zeta) + \mathcal{D}(\eta,\zeta) + \mathcal{E}(\xi,\eta) + \mathcal{F}(\xi,\zeta), \tag{5.4}$$

where $\eta \in \mathbb{R}^n, \xi \in \mathbb{R}^m$, and $\zeta \in \mathbb{R}^r$. Here, $\mathcal{C}$ is a scalar polynomial function of (ξ, η, ζ) of which each component has order in each argument no less than one; $\mathcal{D}$ is a smooth function; $\mathcal{E}$ is a scalar polynomial functions of (ξ, η) of which each component has order in (ξ, η) no less than (j_1, j_2), respectively; and $\mathcal{F}$ is a scalar polynomial functions of (ξ, ζ) of which each component has order in (ξ, ζ) no less than (j_3, j_4), respectively. Here, j_i denote positive integers, for $i = 1, \cdots, 4$ with $j_1, j_3 \geq 2$, $j_1 + j_2 \geq 3$, and $j_3 + j_4 \geq 3$.

According to the proofs of Lemma 5.1, it is not difficult to prove the following result.

Lemma 5.2. (Locally Positive Definite Function for Three Sets of Variables) If $\mathcal{P}_1$, $\mathcal{P}_2$ are positive definite matrices and $\mathcal{D}(\eta, \zeta)$ is an l.p.d.f., then the scalar function $v(\xi, \eta, \zeta)$ given in (5.4) is an l.p.d.f.

The next result follows readily from Lemma 5.2.

Corollary 5.3. If $\mathcal{P}_1$, $\mathcal{P}_2$ are negative definite and $\mathcal{D}(\eta, \zeta)$ is locally negative definite, then the scalar function $v(\xi, \eta, \zeta)$ given in (5.4) is locally negative definite.

5.3. Liapunov Function Candidates for Critical Systems

In the following, we construct families of Liapunov functions for nonlinear critical systems. First, a general set-up of candidates is proposed for critical systems. It is observed that such construction can be simplified by invoking a result on the solution of a class of scalar multilinear equations. Using the solvability of scalar multilinear equations, along with center manifold reduction (discussed in Section 3.2), we propose a class of "composite" Liapunov function candidates for general critical systems. Detailed construction of families of Liapunov function for the simple critical cases and the compound critical cases are given in Sections 5.4 and 5.5, respectively, to demonstrate the main results.

In this section, we construct Liapunov function candidates for a class of nonlinear autonomous system as given by

$$\dot{\eta} = A_{11}\eta + F(\eta, \xi) \tag{5.5}$$

$$\dot{\xi} = A_{22}\xi + G(\eta, \xi), \tag{5.6}$$

where $\eta \in I\!R^n$, $\xi \in I\!R^m$. Here, A_{11} and A_{22} are constant matrices, and the functions F, G are sufficiently smooth, with their values and first derivatives vanishing at the origin.

Taylor series expansion of system (5.5)-(5.6) at the origin gives

$$\begin{aligned}\dot{\eta} =& A_{11}\eta + F_{\eta\eta}(\eta,\eta) + F_{\eta\xi}(\eta,\xi) + F_{\xi\xi}(\xi,\xi) + F_{\eta\eta\eta}(\eta,\eta,\eta)\\ &+ F_{\eta\eta\xi}(\eta,\eta,\xi) + F_{\eta\xi\xi}(\eta,\xi,\xi) + F_{\xi\xi\xi}(\xi,\xi,\xi) + \ldots \end{aligned}\tag{5.7a}$$

$$\begin{aligned}\dot{\xi} =& A_{22}\xi + G_{\eta\eta}(\eta,\eta) + G_{\eta\xi}(\eta,\xi) + G_{\xi\xi}(\xi,\xi) + G_{\eta\eta\eta}(\eta,\eta,\eta)\\ &+ G_{\eta\eta\xi}(\eta,\eta,\xi) + G_{\eta\xi\xi}(\eta,\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) + \ldots \end{aligned}\tag{5.7b}$$

where components of the approximation of functions F and G on the right side of Eq. (5.7) are multilinear (but not necessarily symmetric) functions of their arguments.

Without loss of generality, we choose

$$\begin{aligned}\mathcal{V} =& \xi'\mathcal{P}_1\xi + \eta'\mathcal{P}_2\eta + \mathcal{V}_{\eta\eta\eta}(\eta,\eta,\eta) + \mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi) + \mathcal{V}_{\eta\xi\xi}(\eta,\xi,\xi)\\ &+ \mathcal{V}_{\xi\xi\xi}(\xi,\xi,\xi) + \mathcal{V}_{\eta\eta\eta\eta}(\eta,\eta,\eta,\eta) + \mathcal{V}_{\eta\eta\eta\xi}(\eta,\eta,\eta,\xi)\\ &+ \mathcal{V}_{\eta\eta\xi\xi}(\eta,\eta,\xi,\xi) + \mathcal{V}_{\eta\xi\xi\xi}(\eta,\xi,\xi,\xi) + \mathcal{V}_{\xi\xi\xi\xi}(\xi,\xi,\xi,\xi) + \ldots \end{aligned}\tag{5.8}$$

as a Liapunov function candidate for (5.7), where the dots denote the high order terms and $\mathcal{P}_1, \mathcal{P}_2$ are two symmetric square matrices with the remaining terms on the right side of Eq. (5.8) being multilinear functions.

Differentiating $\mathcal{V}$ along trajectories of (5.7) gives

$$\dot{\mathcal{V}} = \dot{\mathcal{V}}^{(2)} + \dot{\mathcal{V}}^{(3)} + \dot{\mathcal{V}}^{(4)} + \ldots + \dot{\mathcal{V}}^{(i)} + \ldots, \tag{5.9}$$

where

$$\dot{\mathcal{V}}^{(2)} = \xi'(\mathcal{P}_1 A_{22} + A_{22}'\mathcal{P}_1)\xi + \eta'(\mathcal{P}_2 A_{11} + A_{11}'\mathcal{P}_2)\eta, \tag{5.10}$$

$$\dot{\mathcal{V}}^{(3)} = \dot{\mathcal{V}}^{(3)}_{\eta\eta\eta}(\eta,\eta,\eta) + \dot{\mathcal{V}}^{(3)}_{\eta\eta\xi}(\eta,\eta,\xi) + \dot{\mathcal{V}}^{(3)}_{\eta\xi\xi}(\eta,\xi,\xi) + \dot{\mathcal{V}}^{(3)}_{\xi\xi\xi}(\xi,\xi,\xi), \tag{5.11}$$

$$\begin{aligned}\dot{\mathcal{V}}^{(4)} =& \dot{\mathcal{V}}^{(4)}_{\eta\eta\eta\eta}(\eta,\eta,\eta,\eta) + \dot{\mathcal{V}}^{(4)}_{\eta\eta\eta\xi}(\eta,\eta,\eta,\xi) + \dot{\mathcal{V}}^{(4)}_{\eta\eta\xi\xi}(\eta,\eta,\xi,\xi)\\ &+ \dot{\mathcal{V}}^{(4)}_{\eta\xi\xi\xi}(\eta,\xi,\xi,\xi) + \dot{\mathcal{V}}^{(4)}_{\xi\xi\xi\xi}(\xi,\xi,\xi,\xi), \end{aligned}\tag{5.12}$$

and $\dot{\mathcal{V}}^{(i)}$ are the quadratic, cubic, quartic, i-th order terms, respectively, of $\dot{\mathcal{V}}$.

In the above (using Proposition 2.1),

$$\dot{\mathcal{V}}^{(3)}_{\eta\eta\eta}(\eta,\eta,\eta) = 2F'_{\eta\eta}(\eta,\eta)\mathcal{P}_2\eta + 3\mathcal{V}_{\eta\eta\eta}(\eta,\eta,A_{11}\eta), \tag{5.13}$$

$$\begin{aligned}\dot{\mathcal{V}}^{(3)}_{\eta\eta\xi}(\eta,\eta,\xi) &= 2(\xi'\mathcal{P}_1 G_{\eta\eta}(\eta,\eta) + F'_{\eta\xi}(\eta,\xi)\mathcal{P}_2\eta) \\ &\quad + D_\eta\mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi)A_{11}\eta + D_\xi\mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi)A_{22}\xi,\end{aligned} \tag{5.14}$$

$$\begin{aligned}\dot{\mathcal{V}}^{(4)}_{\eta\eta\eta\eta}(\eta,\eta,\eta,\eta) &= 2F'_{\eta\eta\eta}(\eta,\eta,\eta)\mathcal{P}_2\eta + 3\mathcal{V}_{\eta\eta\eta}(\eta,\eta,F_{\eta\eta}(\eta,\eta)) \\ &\quad + D_\xi\mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi)G_{\eta\eta}(\eta,\eta) + 4\mathcal{V}_{\eta\eta\eta\eta}(\eta,\eta,\eta,A_{11}\eta),\end{aligned} \tag{5.15}$$

$$\begin{aligned}\dot{\mathcal{V}}^{(4)}_{\eta\eta\eta\xi}(\eta,\eta,\eta,\xi) &= 2(\xi'\mathcal{P}_1 G_{\eta\eta\eta}(\eta,\eta,\eta) + F'_{\eta\eta\xi}(\eta,\eta,\xi)\mathcal{P}_2\eta) \\ &\quad + 3\mathcal{V}_{\eta\eta\eta}(\eta,\eta,F_{\eta\xi}(\eta,\xi)) + D_\eta\mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi)F_{\eta\eta}(\eta,\eta) \\ &\quad + D_\xi\mathcal{V}_{\eta\eta\xi}(\eta,\eta,\xi)G_{\eta\xi}(\eta,\xi) + D_\xi\mathcal{V}_{\eta\xi\xi}(\eta,\xi,\xi)G_{\eta\eta}(\eta,\eta) \\ &\quad + D_\eta\mathcal{V}_{\eta\eta\eta\xi}(\eta,\eta,\eta,\xi)A_{11}\eta + D_\xi\mathcal{V}_{\eta\eta\eta\xi}(\eta,\eta,\eta,\xi)A_{22}\xi.\end{aligned} \tag{5.16}$$

The remaining terms are obviously implied and are omitted. Note that the components of $\dot{\mathcal{V}}(\cdot)$ on the right side of Eqs. (5.11)-(5.12) are multilinear functions.

Now, we can check the suitability of $\mathcal{V}$ given in (5.8) as a Liapunov function for (5.7). In the trivial case in which both A_{11} and A_{22} are stable, it is known (e.g., [17], [86]) that there exist positive definite matrices $\mathcal{P}_1$ and $\mathcal{P}_2$ such that both $\mathcal{P}_1 A_{22} + A'_{22}\mathcal{P}_1$ and $\mathcal{P}_2 A_{11} + A'_{11}\mathcal{P}_2$ are negative definite, which provide the local negative definiteness of $\dot{\mathcal{V}}$. Thus, in this case, we can choose

$$\mathcal{V} = \xi'\mathcal{P}_1\xi + \eta'\mathcal{P}_2\eta \tag{5.17}$$

as a Liapunov function to prove the asymptotic stability of the origin for (5.7).

Throughout the rest of this section, we consider the nontrivial case in which A_{22} is stable but all eigenvalues of A_{11} lie on the imaginary axis. Motivated by the results on the existence of locally positive definite function given in Section 5.2, the possibility of $\mathcal{V}$ in (5.8) being an l.p.d.f. is considered as follows.

Let $\mathcal{V}_1$ denote the scalar function containing all the components, which are functions of η only, of $\mathcal{V}$ (given in (5.8)). That is,

$$\mathcal{V}_1(\eta) := \eta'\mathcal{P}_2\eta + \mathcal{V}_{\eta\eta\eta}(\eta,\eta,\eta) + \mathcal{V}_{\eta\eta\eta\eta}(\eta,\eta,\eta,\eta) + \ldots, \tag{5.18}$$

where the dots denote high order terms.

The next result readily follows from Corollary 5.1 and Lemma 5.2.

Lemma 5.3. The scalar function $\mathcal{V}$ given by (5.8) is an l.p.d.f., if either of the following two conditions hold:

(i) both $\mathcal{P}_1$ and $\mathcal{P}_2$ are positive definite,

(ii) $\mathcal{P}_1$ is positive definite, $\mathcal{V}_1(\eta)$ defined in (5.18) above is an l.p.d.f., where either $\mathcal{P}_{21} = 0$ and $\mathcal{P}_{22}$ is positive definite with all k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \xi) = \mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \zeta_2, \xi)$, or $\mathcal{P}_{22} = 0$ and $\mathcal{P}_{21}$ is positive definite with all k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \xi) = \mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \zeta_1, \xi)$. Here, we assume that $\eta'\mathcal{P}_2\eta = \zeta_1'\mathcal{P}_{21}\zeta_1 + \zeta_2'\mathcal{P}_{22}\zeta_2$ and $\eta := (\zeta_1, \zeta_2)'$.

Next, we consider the possibility of $\dot{\mathcal{V}}$ given in (5.9) being a locally negative definite function. Since A_{22} is stable, as discussed above, there exists a symmetric positive definite matrix $\mathcal{P}_1$ such that $\mathcal{P}_1 A_{22} + A_{22}'\mathcal{P}_1$ is negative definite. Motivated by the results of Corollaries 5.2 and 5.3, we observe that the local negative definiteness of $\dot{\mathcal{V}}$ (given in (5.9)) can be easily proven if the components $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\xi}$ of $\dot{\mathcal{V}}$ can be set to zero, for all $k = 3, 4, \ldots$.

It is observed that the expressions of k-linear function $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\xi}$, for instance, Eqs. (5.13) and (5.15), have a general form as given in (5.19) below. To simplify the expressions of $\dot{\mathcal{V}}$, we might need to obtain the solutions of k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}$ for the scalar multilinear equation $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\xi} = 0$, for each $k = 3, 4, \ldots$. The solvability of such equations is first discussed in Section 5.3.1 below, where a general result is obtained by employing the result on linear matrix equation given in Theorem 2.5. Then this result is applied to construct Liapunov function candidates for system (5.7) in Section 5.3.2.

5.3.1. Solvability of a Class of Scalar Multilinear Equations

Let $\mathcal{T}, \mathcal{M} : (I\!R^n)^k \times I\!R^m \to I\!R$ denote $(k+1)$-linear functions, for $k \geq 2$ an integer and consider a class of scalar multilinear equations

$$D_\eta\mathcal{T}(\eta, \cdots, \eta, \xi)A\eta + D_\xi\mathcal{T}(\eta, \cdots, \eta, \xi)B\xi - \mathcal{M}(\eta, \cdots, \eta, \xi) = 0, \quad (5.19)$$

for all $\eta \in I\!R^n, \xi \in I\!R^m$, where $A \in I\!R^{n\times n}$ and $B \in I\!R^{m\times m}$ are two constant matrices.

From Definition 2.2, we have two k-linear vector functions $\mathcal{T}^*$ and $\mathcal{M}^*$ such that $\mathcal{T}(\eta, \cdots, \eta, \xi) = \xi'\mathcal{T}^*(\eta, \cdots, \eta)$ and $\mathcal{M}(\eta, \cdots, \eta, \xi) = \xi'\mathcal{M}^*(\eta, \cdots, \eta)$.

Rewriting Eq. (5.19), we have

$$\xi'\{D\mathcal{T}^*(\eta, \cdots, \eta)A\eta + B'\mathcal{T}^*(\eta, \cdots, \eta) - \mathcal{M}^*(\eta, \cdots, \eta)\} = 0 \qquad (5.20)$$

for all $\eta \in I\!R^n, \xi \in I\!R^m$. The existence of a solution $\mathcal{T}$ for the partial differential equation (5.19) can then be provided by the existence of a solution $\mathcal{T}^*$ for the matrix equation (5.21) below:

$$D\mathcal{T}^*(\eta, \cdots, \eta)A\eta + B'\mathcal{T}^*(\eta, \cdots, \eta) - \mathcal{M}^*(\eta, \cdots, \eta) = 0 \qquad (5.21)$$

for all $\eta \in I\!R^n$.

By using the principle of induction, we can have the following result.

Lemma 5.4. Let $A \in I\!R^{n\times n}$ and $B \in I\!R^{m\times m}$. Then for given positive integer k and k-linear function $\mathcal{M}^*(\eta, \cdots, \eta)$, there exists a k-linear function $\mathcal{T}^*(\eta, \cdots, \eta)$ such that Eq. (5.21) holds for all $\eta \in I\!R^n$ if $\sum_{i=1}^{k} \sigma_i(A) + \sigma(B) \neq 0$. Here, $\sigma_i(A)$ and $\sigma(B)$ denote eigenvalues of matrices A and B, respectively.

Proof: The principle of induction is employed here to prove the existence of each k-linear solution T^* for Eq. (5.21) under the assumptions of Lemma 5.4. Details are given as follows.

First, consider the case of $k = 1$, and let $\mathcal{T}^*(\eta) := \mathcal{T}^*\eta$ and $\mathcal{M}^*(\eta) := \mathcal{M}^*\eta$. Eq. (5.21) can hence be written as

$$(\mathcal{T}^*A + B'\mathcal{T}^* - \mathcal{M}^*)\eta = 0. \qquad (5.22)$$

It is implied by Theorem 2.5 that there exists a unique solution $\mathcal{T}^*$ for Eq. (5.22) if $\sigma(A) + \sigma(B') \neq 0$ (i.e., $\sigma(A) + \sigma(B) \neq 0$). So, the conclusion of Lemma 5.4 holds for the case of $k = 1$.

Next, we suppose there exists a k-linear function $\mathcal{T}^*$ such that Eq. (5.21) holds for given k-linear function $\mathcal{M}^*$ and k is some positive integer. Here, we do not have any restriction on the dimensions of the row of $\mathcal{T}^*$.

Now, consider the case in which $\mathcal{T}^*$ is a $(k+1)$-linear vector function in η. According to Definition 2.2, we can let $\mathcal{T}^* = \sum_{i=1}^{n} \eta_i \mathcal{T}_i(\eta, \cdots, \eta)$ and $\mathcal{M}^* = \sum_{i=1}^{n} \eta_i \mathcal{M}_i(\eta, \cdots, \eta)$, where $\eta := (\eta_1, \eta_2 \cdots, \eta_n)'$ and each $\mathcal{T}_i$ and $\mathcal{M}_i$ are k-linear functions in η.

Rewriting Eq. (5.21), we have

$$
\begin{aligned}
&\{[\mathcal{T}_1(\eta, \cdots, \eta), \cdots, \mathcal{T}_n(\eta, \cdots, \eta)] + \sum_{i=1}^{n} \eta_i D\mathcal{T}_i(\eta, \cdots, \eta)\} A\eta \\
&\quad + B' \sum_{i=1}^{n} \eta_i \mathcal{T}_i(\eta, \cdots, \eta) - \sum_{i=1}^{n} \eta_i \mathcal{M}_i(\eta, \cdots, \eta) \\
&= \sum_{i=1}^{n} \eta_i \{D\mathcal{T}_i(\eta, \cdots, \eta) A\eta + \sum_{j=1}^{n} a_{ji} \mathcal{T}_j(\eta, \cdots, \eta) \\
&\quad + B'\mathcal{T}_i(\eta, \cdots, \eta) - \mathcal{M}_i(\eta, \cdots, \eta)\} = 0, \qquad (5.23)
\end{aligned}
$$

where $A := [a_{ij}]$. From Eq. (5.23), we can say that Eq. (5.21) holds for all $\eta \in I\!R^n$ if for all $\eta \in I\!R^n$ the following relationship holds:

$$D\hat{\mathcal{T}}(\eta, \cdots, \eta) A\eta + (A' \oplus B')\hat{\mathcal{T}}(\eta, \cdots, \eta) - \hat{\mathcal{M}}(\eta, \cdots, \eta) = 0. \qquad (5.24)$$

Here, $\oplus$ denotes the Kronecker sum and two k-linear vector functions $\hat{\mathcal{T}}$ and $\hat{\mathcal{M}}$ are defined as: $\hat{\mathcal{T}}(\eta, \cdots, \eta) := (\mathcal{T}_1(\eta, \cdots, \eta), \cdots, \mathcal{T}_n(\eta, \cdots, \eta))'$ and $\hat{\mathcal{M}}(\eta, \cdots, \eta) := (\mathcal{M}_1(\eta, \cdots, \eta), \cdots, \mathcal{M}_n(\eta, \cdots, \eta))'$.

Now, we can iteratively solve for the existing conditions for the solution T^* for Eq. (5.21) for each positive integer $k > 1$. For instance, for the case of $k = 2$. It is not difficult to find out that there exists a solution T^* for Eq. (5.21) if $\sigma(A) + \sigma(A' \oplus B') \neq 0$ by comparing Eq. (5.24) with Eq. (5.22). Iteratively, we have a solution, the trilinear function T^*, for Eq. (5.21) for the case of $k = 3$, if $\sigma(A) + \sigma(A' \oplus (A' \oplus B')') \neq 0$. It is known (e.g., [36]) that any eigenvalue of matrix $A' \oplus B'$ is the sum of one eigenvalue of A' and one eigenvalue of B'. So, from the foregoing discussion the existence conditions for the solution T^* for Eq. (5.21) will be $\sigma_1(A) + \sigma_2(A) + \sigma(B) \neq 0$ and $\sum_{i=1}^{3} \sigma_i(A) + \sigma(B) \neq 0$ for the case of $k = 2$ and $k = 3$, respectively. Here, $\sigma_i(A)$ denotes one of eigenvalues of A.

By the principle of induction, we then have a solution T^* for Eq. (5.21) for each positive integer k if $\sum_{i=1}^{k} \sigma_i(A) + \sigma(B) \neq 0$.

■

The next result follows readily from Lemma 5.4 and the foregoing discussion.

Corollary 5.4. Let $A \in I\!R^{n\times n}$ and $B \in I\!R^{m\times m}$. If $Re\{\sigma(A)\} = 0$ and $Re\{\sigma(B)\} < 0$, then for any given positive integer k and given $(k+1)$-linear function $\mathcal{M}(\eta, \cdots, \eta, \xi)$ there exists a $(k+1)$-linear function $\mathcal{T}(\eta, \cdots, \eta, \xi)$ such that Eq. (5.19) holds for all $\eta \in I\!R^n$ and $\xi \in I\!R^m$.

5.3.2. Construction of the Liapunov Function Candidates

Now, we can apply Corollary 5.4 to simplify the expressions of $\dot{\mathcal{V}}$ given in (5.9) such that we can construct Liapunov function candidates for system (5.7) easily. Since A_{22} is stable and all eigenvalues of A_{11} lie on the imaginary axis, then according to Corollary 5.4 the solution (k-linear function) $\mathcal{V}_{\eta\cdots\eta\xi}$ will always exist for the scalar multilinear equation $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\xi} = 0$ for all $\eta \in I\!R^n$ and $\xi \in I\!R^m$.

From Corollaries 5.2 and 5.3, Lemma 5.3 and the foregoing discussions, we then have the following two criteria for constructing Liapunov function candidates for nonlinear critical system (5.7).

Lemma 5.5. Suppose $\mathcal{P}_1, \mathcal{P}_2$ are two symmetric positive definite matrices with $\mathcal{P}_1 A_{22} + A'_{22}\mathcal{P}_1$ being negative definite and $\mathcal{P}_2 A_{11} + A'_{11}\mathcal{P}_2 = 0$. Then the function $\mathcal{V}$ given in (5.8) is a Liapunov function for ascertaining the asymptotic stability of the origin of (5.7) if there exists a positive integer j^* such that

(i) the $(2j^*)$-linear function $\dot{\mathcal{V}}_{\eta\cdots\eta}(\eta, \cdots, \eta)$ is negative definite,

(ii) $\dot{\mathcal{V}}^{(i)}_{\eta\cdots\eta\eta}(\eta, \cdots, \eta, \eta) = 0$, for each $i = 3, \ldots, 2j^* - 1$,

(iii) $\mathcal{V}^{(i)}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \xi) = 0$ for each $i = 3, \ldots, j^* + 1$.

Lemma 5.6. Suppose condition (ii) of Lemma 5.3 holds with $\mathcal{P}_1 A_{22} + A'_{22}\mathcal{P}_1$ being negative definite and $\mathcal{P}_2 A_{11} + A'_{11}\mathcal{P}_2 = 0$. Then function $\mathcal{V}$ given in (5.8) is a Liapunov function for ascertaining asymptotic stability of the origin of (5.7) if there exists a positive integer j^* such that conditions (i)-(iii) of Lemma 5.5 hold.

As implied by Corollary 5.4, for each $k \geq 3$, there exists a k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \xi)$ of $\mathcal{V}$ (given in (5.8)), which is $(k-1)$-linear in η and linear in ξ, such that condition (iii) of Lemma 5.3 holds. For instance, there exist $\mathcal{V}_{\eta\eta\xi}(\eta, \eta, \xi)$ and $\mathcal{V}_{\eta\eta\eta\xi}(\eta, \eta, \eta, \xi)$ to make $\dot{\mathcal{V}}^{(3)}_{\eta\eta\xi}(\eta, \eta, \xi)$ and $\dot{\mathcal{V}}^{(4)}_{\eta\eta\eta\xi}(\eta, \eta, \eta, \xi)$ given in Eqs. (5.14) and (5.16) equal to zero for all $\eta \in I\!R^n, \xi \in I\!R^m$. To prove that the scalar function $\mathcal{V}$ (given in (5.8)) is a Liapunov function for (5.7) by employing Lemma 5.5, we need to have a scalar function $\mathcal{V}_1(\eta)$ given in (5.18) such that conditions (i) and (ii) of Lemma 5.5 hold. Moreover, in the application of Lemma 5.6, there is one more restriction on the k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}$ as defined in Lemma 5.3.

The problem stated above for finding $\mathcal{V}_1$ such that conditions (i) and (ii) of Lemma 5.5 hold is in general hard to solve. In the rest of this section, we employ the technique of center manifold reduction to delete the contributed terms in each k-linear function $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta}(\eta, \cdots, \eta)$ from the nonlinear function G so that the problem can be simplified. According to Theorem 2.1, in the case of which A_{22} is stable and all eigenvalues of A_{11} lie on the imaginary axis, system (5.7) has a locally invariant manifold given by the graph of a function $\xi = h(\eta)$. Moreover, this h satisfies the partial differential equation (2.2) with boundary conditions $h(0) = 0$ and $Dh(0) = 0$.

Let $\phi(\eta)$ be an approximate polynomial function of h such that

$$h(\eta) - \phi(\eta) = O(||\eta||^{\gamma+1}), \tag{5.25}$$

and let $\nu = \xi - \phi(\eta)$.

Then we can rewrite (5.7) as

$$\dot{\eta} = A_{11}\eta + F(\eta, \nu + \phi(\eta))$$

$$
\begin{aligned}
&= A_{11}\eta + \hat{F}_{\eta\eta}(\eta,\eta) + F_{\eta\nu}(\eta,\nu) + \ldots + O(||(\eta,\nu)||^{\gamma+2}), && (5.26a)\\
\dot{\nu} &= \dot{\xi} - D\phi(\eta)\cdot\dot{\eta} \\
&= A_{22}\nu + [G(\eta,\nu+\phi(\eta)) - G(\eta,\phi(\eta))] \\
&\quad - D\phi(\eta)\cdot[F(\eta,\nu+\phi(\eta)) - F(\eta,\phi(\eta))] + O(||(\eta,\nu)||^{\gamma+2}) \\
&= A_{22}\nu + G_{\eta\nu}(\eta,\nu) + G_{\nu\nu}(\nu,\nu) + G_{\eta\eta\nu}(\eta,\eta,\nu) \\
&\quad + G_{\eta\nu\nu}(\eta,\nu,\nu) + G_{\nu\nu\nu}(\nu,\nu,\nu) + \ldots + O(||(\eta,\nu)||^{\gamma+2}), && (5.26b)
\end{aligned}
$$

where $\hat{F}_{\eta\eta}$ is the new version of bilinear function $F_{\eta\eta}$ and the remaining terms on the right side of Eqs. (5.26a) and (5.26b) are obviously implied by the approximations of functions F and G, which are supposed to be represented in multilinear forms.

It is observed from Eq. (5.26b) that there are no terms in the Taylor series expansion of the dynamics $\dot{\nu}$, which are function of η only. Thus, we have the following result by modifying the conditions of Lemma 5.5 and referring to the discussion above.

Theorem 5.1. Suppose A_{22} is stable. If there is a scalar function $\mathcal{V}_1$ given in (5.18) with $\mathcal{P}_2$ being symmetric positive definite to show the asymptotic stability of the origin for the reduced model (5.26a) with $\nu = 0$, then the origin is asymptotically stable for the whole system (5.26).

Next, we implement the result of Lemma 5.6. Suppose there exists no square matrix $\mathcal{P}_2$ such that conditions of Theorem 5.1 hold. However, there exists a square matrix

$$
\mathcal{P}_2 = \begin{pmatrix} \mathcal{P}_{21} & 0 \\ 0 & \mathcal{P}_{22} \end{pmatrix} \tag{5.27}
$$

such that $\mathcal{V}_1$ (given in (5.18)) is an l.p.d.f., with $\mathcal{P}_{21} = 0$ and $\mathcal{P}_{22}$ being positive definite, or $\mathcal{P}_{22} = 0$ and $\mathcal{P}_{21}$ being positive definite. Let $\eta := (\zeta_1, \zeta_2)'$ such that $\eta'\mathcal{P}_2\eta = \zeta_1'\mathcal{P}_{21}\zeta_1 + \zeta_2'\mathcal{P}_{22}\zeta_2$.

Consider the case in which $\mathcal{P}_{21} = 0$ and the scalar function $\mathcal{V}$ (given in (5.8)) is applied to the new model (5.26), i.e., ξ is replaced by $\nu + \phi(\eta)$. It is observed that the k-linear function $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta,\nu}(\eta,\cdots,\eta,\nu)$, for instance, see Eqs. (5.14) and (5.16), has the form as given by

$$\begin{aligned}\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\nu}(\eta,\cdots,\eta,\nu) =& D_\eta \mathcal{V}_{\eta\cdots\eta\nu}(\eta,\cdots,\eta,\nu)A_{11}\eta + D_\nu \mathcal{V}_{\eta\cdots\eta\nu}(\eta,\cdots,\eta,\nu)A_{22}\nu \\ &+ \mathcal{M}_{\eta\cdots\eta\zeta_2\nu}(\eta,\cdots,\eta,\zeta_2,\nu),\end{aligned} \tag{5.28}$$

where k-linear function $\mathcal{M}_{\eta\cdots\eta\zeta_2\nu}(\eta,\cdots,\eta,\zeta_2,\nu)$ is linear in ν, $(k-1)$-linear in ζ_2 and $(k-2)$-linear in ζ_1. Under this condition, we only need to solve the k-linear functions $\mathcal{V}_{\eta\cdots\eta\zeta_2\xi}$ from the scalar multilinear equation $\dot{\mathcal{V}}^{(k)}_{\eta\cdots\eta\xi} = 0$, for each $k = 3, 4, \ldots$, in which the existence of such solution is guaranteed by Corollary 5.4. That is, the k-linear functions $\mathcal{V}_{\eta\cdots\eta\xi}$ of $\mathcal{V}$ in (5.8) only need to contain the components $\mathcal{V}_{\eta\cdots\eta\zeta_2\xi}$.

For the case in which $\mathcal{P}_{22} = 0$ and $\mathcal{P}_{21}$ is positive definite, similarly, we can show that the k-linear functions $\mathcal{V}_{\eta\cdots\eta\xi}$ of $\mathcal{V}$ in (5.8) only need to contain the components $\mathcal{V}_{\eta\cdots\eta\zeta_1\xi}$.

From the discussion above and Lemma 5.6, we then have the following stability criterion for (5.26).

Theorem 5.2. Suppose A_{22} is stable. If there is a scalar function $\mathcal{V}_1$ given in (5.18) to show the asymptotic stability of the origin for the reduced model (5.26a) with $\nu = 0$, then the origin is asymptotically stable for the whole system (5.26). Here, the square matrix P_2 is defined in (5.27) with $\mathcal{P}_{21} = 0$ and $\mathcal{P}_{22}$ being positive definite, or $\mathcal{P}_{22} = 0$ and $\mathcal{P}_{21}$ being positive definite.

Remark 5.1. The result given in Theorems 5.1 and 5.2 provide special versions of the stability conditions given in Theorem 2.1.

To conclude the discussion above, we have the following algorithm for constructing families of Liapunov functions for (5.26) in the case when all the eigenvalues of A_{11} lie on the imaginary axis and A_{22} is stable. We denote $\mathcal{V}_{\eta\eta}(\eta,\eta) := \eta'\mathcal{P}_2\eta$.

Algorithm 5.1. (Algorithm for Constructing Liapunov Function)

Step 1. Choose $\mathcal{P}_1$ to be a symmetric and positive definite matrix with $\mathcal{P}_1 A_{22} + A'_{22}\mathcal{P}_1$ negative definite.

Step 2. Apply Theorem 5.1 (or Theorem 5.2) to find a scalar function $\mathcal{V}_1$ as given in (5.18) such that conditions (i) and (ii) of Lemma 5.5 hold for (5.26a) with $\nu = 0$ (i.e., to construct a Liapunov function for the reduced model (5.26a) with $\nu = 0$).

Step 3. For each integer $k \geq 3$, solve for k-linear function $\mathcal{V}_{\eta\cdots\eta\xi}(\eta, \cdots, \eta, \xi)$ (or $\mathcal{V}_{\eta\cdots\eta\zeta_i\xi}(\eta, \cdots, \eta, \zeta_i, \xi)$) such that condition (iii) of Lemma 5.5 holds, the solution for which is guaranteed by Corollary 5.4.

In the next two sections, we construct families of Liapunov function for two categories of critical systems (SC) and (CC) defined below to demonstrate the main results of this chapter. For simplicity, we only focus on the construction of Liapunov function for the reduced model (5.26a) with $\nu = 0$. It is easy to construct families of Liapunov function for the whole system (5.26) by employing Algorithm 5.1. Details are omitted. The two critical cases (SC) and (CC) considered next are defined as follows:

(SC) The matrix A_{22} is stable and A_{11} has exactly one zero eigenvalue or a pair of pure imaginary eigenvalues.

(CC) The matrix A_{22} is stable and A_{11} has exactly two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and a pair of nonzero pure imaginary eigenvalues or two distinct pairs of nonzero pure imaginary eigenvalues.

Rewrite the reduced model (5.26a) as (by setting $\nu = 0$)

$$\dot{\eta} = A_{11}\eta + \hat{F}_{\eta\eta}(\eta, \eta) + \hat{F}_{\eta\eta\eta}(\eta, \eta, \eta) + \ldots, \tag{5.29}$$

where $\hat{F}_{\eta\eta}$ and $\hat{F}_{\eta\eta\eta}$ denote the quadratic terms and cubic terms of the approximation of dynamics F after substituting ξ with the approximate solution $\phi(\eta)$ for $h(\eta)$, respectively.

According to Theorem 2.1, a function describing the locally invariant manifold, $h(\cdot)$, should satisfy the partial differential equation as given in (2.2) with boundary conditions: $h(0) = 0$ and $Dh(0) = 0$. This leads us to take the linear term of the approximate solution $\phi(\eta)$ to be zero and

$$\phi(\eta) \equiv \phi_{\eta\eta}(\eta, \eta) + \phi_{\eta\eta\eta}(\eta, \eta, \eta) + \cdots, \tag{5.30}$$

where $\phi_{\eta\eta}(\cdot)$ and $\phi_{\eta\eta\eta}(\cdot)$ denote the quadratic terms and cubic terms of ϕ, respectively.

Thus, we have $\hat{F}_{\eta\eta}(\eta, \eta) = F_{\eta\eta}(\eta, \eta)$ and

$$\hat{F}_{\eta\eta\eta}(\eta, \eta, \eta) = F_{\eta\eta\eta}(\eta, \eta, \eta) + F_{\eta\xi}(\eta, \phi(\eta, \eta)). \tag{5.31}$$

5.4. Liapunov Functions for Simple Critical Cases

First, we consider the critical case (SC). For simplicity, we have the following hypotheses, stated in terms of matrices A_{11} and A_{22} are defined in (5.7) (or (5.26)).

Case (S): The matrix A_{22} is stable and $A_{11} = 0$ a scalar.

Case (H): The matrix A_{22} is stable and A_{11} is a 2×2 matrix possessing a pair of nonzero, pure imaginary eigenvalues.

5.4.1. Case (S)

Consider the critical case (S). Let $x := \eta$, which is a scalar and

$$\begin{aligned} f(x,\xi) :=& F(x,\xi) \\ =& f_{xx}x^2 + x f_{x\xi}\xi + \xi' f_{\xi\xi}\xi + f_{xxx}x^3 + x^2 f_{xx\xi}\xi \\ &+ x \cdot \xi' f_{x\xi\xi}\xi + f_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,\xi)||^4), \end{aligned} \tag{5.32}$$

$$\begin{aligned} G(x,\xi) =& x^2 G_{xx} + x G_{x\xi}\xi + G_{\xi\xi}(\xi,\xi) + x^3 G_{xxx} \\ &+ x^2 G_{xx\xi}\xi + x G_{x\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,\xi)||^4). \end{aligned} \tag{5.33}$$

Here, the coefficients of the approximation in Taylor series expansion (5.32)-(5.33) are either constants or symmetric multilinear functions of their arguments as usual.

Choose the approximate solution $\phi(\eta) = \phi(x) = h_{xx}x^2$, where $h_{xx} = -A_{22}^{-1}G_{xx}$ as obtained in Section 3.3. The reduced model given in (5.29) can then be rewritten as

$$\dot{x} = f_{xx}x^2 + (f_{xxx} - f_{x\xi}A_{22}^{-1}G_{xx})x^3 + O(|x|^4). \tag{5.34}$$

Choose

$$\mathcal{V}_r = p_{xx}x^2 + \mathcal{V}_{r,xxx}x^3, \tag{5.35}$$

as a Liapunov function candidate for the reduced model (5.34) with $p_{xx} > 0$. Taking the derivative of $\mathcal{V}_r$ along the trajectory of (5.34), we have

$$\begin{aligned}\dot{\mathcal{V}}_r =& 2p_{xx}f_{xx}x^3 + \{2p_{xx}(f_{xxx} - f_{x\xi}A_{22}^{-1}G_{xx})\\ &+ 3\mathcal{V}_{r,xxx}f_{xx}\}x^4 + O(|x|^5).\end{aligned} \tag{5.36}$$

Since $p_{xx} > 0$, the scalar function $\mathcal{V}_r$ (given in (5.35)) is locally positive definite. By checking the locally negative definiteness of $\dot{\mathcal{V}}_r$ given in (5.36) and employing Liapunov stability criteria, we have the following obvious result.

Theorem 5.3. Under hypothesis (S), the origin of (5.34) is asymptotically stable if $f_{xx} = 0$ and $f_{xxx} - f_{x\xi}A_{22}^{-1}G_{xx} < 0$.

Remark 5.2. The result of Theorem 5.3 coincides with the one given in Corollary 3.1. Moreover, families of Liapunov function (5.35) for the reduced model (5.34) have only one restriction, i.e., $p_{xx} > 0$, There is no restriction on the value of $\mathcal{V}_{r,xxx}$. In this case, families of Liapunov functions for the full model (5.26) can be constructed by using Theorem 5.3 and Algorithm 5.1.

5.4.2. Case (H)

Next, we consider the critical case (H) in which A_{11} has eigenvalues $\omega_c = \pm i\sqrt{\Omega_1\Omega_2}$, with $\Omega_1\Omega_2 > 0$ and $i = \sqrt{-1}$. By letting $\eta = (x,y)'$ be a two dimensional vector and $F(\eta,\xi) = (f(x,y,\xi), g(x,y,\xi))'$, we can describe system (5.7) as

$$\dot{x} = \Omega_1 y + f(x,y,\xi) \tag{5.37a}$$

$$\dot{y} = -\Omega_2 x + g(x,y,\xi) \tag{5.37b}$$

$$\dot{\xi} = A_{22}\xi + G(x,y,\xi), \tag{5.37c}$$

with

$$
\begin{aligned}
f(x,y,\xi) = & f_{xx}x^2 + f_{xy}xy + f_{yy}y^2 + (xf_{x\xi} + yf_{y\xi})\xi + \xi' f_{\xi\xi}\xi + f_{xxx}x^3 \\
& + f_{xxy}x^2y + f_{xyy}xy^2 + f_{yyy}y^3 + (x^2 f_{xx\xi} + xyf_{xy\xi} + y^2 f_{yy\xi})\xi \\
& + \xi'(xf_{x\xi\xi} + yf_{y\xi\xi})\xi + f_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,y,\xi)||^4), \qquad (5.38) \\
g(x,y,\xi) = & g_{xx}x^2 + g_{xy}xy + g_{yy}y^2 + (xg_{x\xi} + yg_{y\xi})\xi + \xi' g_{\xi\xi}\xi + g_{xxx}x^3 \\
& + g_{xxy}x^2y + g_{xyy}xy^2 + g_{yyy}y^3 + (x^2 g_{xx\xi} + xyg_{xy\xi} + y^2 g_{yy\xi})\xi \\
& + \xi'(xg_{x\xi\xi} + yg_{y\xi\xi})\xi + g_{\xi\xi\xi}(\xi,\xi,\xi) + O(||(x,y,\xi)||^4), \qquad (5.39) \\
G(x,y,\xi) = & x^2G_{xx} + xyG_{xy} + y^2G_{yy} + (xG_{x\xi} + yG_{y\xi})\xi \\
& + G_{\xi\xi}(\xi,\xi) + x^3G_{xxx} + x^2yG_{xxy} + xy^2G_{xyy} \\
& + y^3G_{yyy} + (x^2G_{xx\xi} + xyG_{xy\xi} + y^2G_{yy\xi})\xi \\
& + xG_{x\xi\xi}(\xi,\xi) + yG_{y\xi\xi}(\xi,\xi) + G_{\xi\xi\xi}(\xi,\xi,\xi) \ + O(||(x,y,\xi)||^4). \ (5.40)
\end{aligned}
$$

The coefficients of the approximations (5.38)-(5.40) are either constants or symmetric multilinear functions of their arguments as usual. Similarly, we choose the approximate solution of h as

$$\phi(\eta) = \phi(x,y) = x^2h_{xx} + xyh_{xy} + y^2h_{yy}, \qquad (5.41)$$

where h_{xx}, h_{xy}, h_{yy} have been obtained in Section 3.4 as

$$
\begin{aligned}
h_{xy} =& (A_{22}^2 + 4\Omega_1\Omega_2 I)^{-1}(2\Omega_2 G_{yy} - 2\Omega_1 G_{xx} - A_{22}G_{xy}) \qquad (5.42) \\
h_{xx} =& -A_{22}^{-1}(G_{xx} + \Omega_2 h_{xy}) \qquad (5.43) \\
h_{yy} =& -A_{22}^{-1}(G_{yy} - \Omega_1 h_{xy}). \qquad (5.44)
\end{aligned}
$$

The reduced model (5.26a) can hence be obtained as

$$
\begin{aligned}
\dot{x} =& \Omega_1 y + f_{xx}x^2 + f_{xy}xy + f_{yy}y^2 + (f_{xxx} + f_{x\xi}h_{xx})x^3 \\
& + (f_{y\xi}h_{xx} + f_{x\xi}h_{xy} + f_{xxy})x^2y + (f_{x\xi}h_{yy} + f_{y\xi}h_{xy} + f_{xyy})xy^2 \\
& + (f_{yyy} + f_{y\xi}h_{yy})y^3 + O(||(x,y)||^4), \qquad (5.45a) \\
\dot{y} =& -\Omega_2 x + g_{xx}x^2 + g_{xy}xy + g_{yy}y^2 + (g_{xxx} + g_{x\xi}h_{xx})x^3 \\
& + (g_{y\xi}h_{xx} + g_{x\xi}h_{xy} + g_{xxy})x^2y + (g_{x\xi}h_{yy} + g_{y\xi}h_{xy} + g_{xyy})xy^2 \\
& + (g_{yyy} + g_{y\xi}h_{yy})y^3 + O(||(x,y)||^4). \qquad (5.45b)
\end{aligned}
$$

Now, choose

$$\mathcal{V}_r = p_{xx}x^2 + p_{yy}y^2 + \mathcal{V}_{r,xxx}x^3 + \mathcal{V}_{r,xxy}x^2y + \mathcal{V}_{r,xyy}xy^2 + \mathcal{V}_{r,yyy}y^3$$
$$+ \mathcal{V}_{r,xxxx}x^4 + \mathcal{V}_{r,xxxy}x^3y + \mathcal{V}_{r,xxyy}x^2y^2 + \mathcal{V}_{r,xyyy}xy^3 + \mathcal{V}_{r,yyyy}y^4 \quad (5.46)$$

as a Liapunov function candidate for the reduced model (5.45) with $p_{xx}, p_{yy} > 0$. The derivative of $\mathcal{V}_r$ along the trajectory of (5.45) is

$$\dot{\mathcal{V}}_r = 2(\Omega_1 p_{xx} - \Omega_2 p_{yy})xy + v_1 xy^2 + v_2 x^2 y + v_3 y^3 + v_4 x^3 + v_5 xy^3$$
$$+ v_6 x^3 y + v_7 x^2 y^2 + v_8 x^4 + v_9 y^4 + O(||(x, y)||^5), \quad (5.47)$$

where

$$v_1 = 2g_{xy}p_{yy} + 2f_{yy}p_{xx} - 3\Omega_2\mathcal{V}_{r,yyy} + 2\Omega_1\mathcal{V}_{r,xxy}, \quad (5.48)$$

$$v_2 = 2g_{xx}p_{yy} + 2f_{xy}p_{xx} - 2\Omega_2\mathcal{V}_{r,xyy} + 3\Omega_1\mathcal{V}_{r,xxx}, \quad (5.49)$$

$$v_3 = 2g_{yy}p_{yy} + \Omega_1\mathcal{V}_{r,xyy}, \quad (5.50)$$

$$v_4 = 2f_{xx}p_{xx} - \Omega_2\mathcal{V}_{r,xxy}, \quad (5.51)$$

$$v_5 = -4\Omega_2\mathcal{V}_{r,yyyy} + 2\Omega_1\mathcal{V}_{r,xxyy} + 2(g_{xyy} + g_{x\xi}h_{yy} + g_{y\xi}h_{xy})p_{yy} + 2f_{yy}\mathcal{V}_{r,xxy}$$
$$+ 2(f_{yyy} + f_{y\xi}h_{yy})p_{xx} + 3g_{xy}\mathcal{V}_{r,yyy} + (2g_{yy} + f_{xy})\mathcal{V}_{r,xyy}, \quad (5.52)$$

$$v_6 = -2\Omega_2\mathcal{V}_{r,xxyy} + 4\Omega_1\mathcal{V}_{r,xxxx} + 2(f_{xxy} + f_{x\xi}h_{xy} + f_{y\xi}h_{xx})p_{xx} + 3f_{xy}\mathcal{V}_{r,xxx}$$
$$+ 2(g_{xxx} + g_{x\xi}h_{xx})p_{yy} + 2g_{xx}\mathcal{V}_{r,xyy} + (g_{xy} + 2f_{xx})\mathcal{V}_{r,xxy}, \quad (5.53)$$

$$v_7 = -3\Omega_2\mathcal{V}_{r,xyyy} + 3\Omega_1\mathcal{V}_{r,xxxy} + 2(g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx})p_{yy}$$
$$+ 2(f_{xyy} + f_{x\xi}h_{yy} + f_{y\xi}h_{xy})p_{xx} + 3g_{xx}\mathcal{V}_{r,yyy} + 2g_{xy}\mathcal{V}_{r,xyy}$$
$$+ f_{xx}\mathcal{V}_{r,xyy} + (g_{yy} + 2f_{xy})\mathcal{V}_{r,xxy} + 3f_{yy}\mathcal{V}_{r,xxx}, \quad (5.54)$$

$$v_8 = -\Omega_2\mathcal{V}_{r,xxxy} + 2(f_{xxx} + f_{x\xi}h_{xx})p_{xx} + g_{xx}\mathcal{V}_{r,xxy} + 3f_{xx}\mathcal{V}_{r,xxx}, \quad (5.55)$$

$$v_9 = \Omega_1\mathcal{V}_{r,xyyy} + 2(g_{yyy} + g_{y\xi}h_{yy})p_{yy} + 3g_{yy}\mathcal{V}_{r,yyy} + f_{yy}\mathcal{V}_{r,xyy}. \quad (5.56)$$

Since $p_{xx}, p_{yy} > 0$, the scalar function $\mathcal{V}_r$ given in (5.46) is hence an l.p.d.f. Similarly, by checking the locally negative definiteness of $\dot{\mathcal{V}}_r$ (given in (5.47)) and employing Liapunov stability criteria, we have

Proposition 5.1. Under hypothesis (H), the origin is asymptotically stable for (5.45) if there exists a function as given in (5.46) with $p_{xx}, p_{yy} > 0$ such that $\Omega_1 p_{xx} - \Omega_2 p_{yy} = 0$ and $v_i = 0$, $i = 1, \ldots, 6$, $v_7 \le 0$ and $v_8, v_9 < 0$, where the values of v_i are defined in (5.48)-(5.56) above.

From Proposition 5.1, there exist solutions $p_{xx}, p_{yy} > 0$ and $V_{r,ijk}$ for $\Omega_1 p_{xx} = \Omega_2 p_{yy}$ and $v_i = 0$, $i = 1, \ldots, 6$, where v_i are given in (5.48)-(5.53). The stability conditions given in Proposition 5.1 for the reduced model (5.45) can be obtained from the solutions of $v_7 \le 0$ and $v_8, v_9 < 0$. Consider a special case, by letting $v_8 = v_9$ and solve for solutions to $v_7 \le 0$ and $v_8 < 0$. A stability criterion for the reduced model (5.45) is obtained as follows.

Theorem 5.4. Under hypothesis (H), the origin is asymptotically stable for (5.45) if

$$\begin{aligned} &3\Omega_1\Omega_2(g_{yyy} + g_{y\xi}h_{yy}) + (\Omega_1 g_{xy} - 2\Omega_2 f_{yy})g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx}g_{xy} \\ &\quad + \Omega_1^2(g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx}) + \frac{2\Omega_1^2}{\Omega_2} f_{xx}g_{xx} - \Omega_2 f_{xy}f_{yy} - \Omega_1 f_{xx}f_{xy} \\ &\quad + \Omega_1\Omega_2(f_{xyy} + f_{x\xi}h_{yy} + f_{y\xi}h_{xy}) + 3\Omega_1^2(f_{xxx} + f_{x\xi}h_{xx}) < 0. \end{aligned} \tag{5.57}$$

Remark 5.3. The result of Theorem 5.4 agrees with the one given in Corollary 3.5. Moreover, families of Liapunov function for the whole system (5.7) can be obtained by using Proposition 5.1, Theorem 5.4 and Algorithm 5.1. Details are omitted.

5.5. Liapunov Functions for Compound Critical Cases

Next, we consider constructing families of Liapunov functions for the compound critical cases (CC) of a nonlinear system (5.7). To simplify the notations, we define

Case (SS): The matrix A_{22} is stable, and A_{11} is a 2×2 matrix possessing two zero eigenvalues with geometric multiplicity one and has the form as given in (4.49).

Case (HS): The matrix A_{22} is stable and A_{11} is a 3×3 matrix possessing one zero eigenvalue and a pair of nonzero, pure imaginary eigenvalues with the form as given in (4.69).

Case (HH): The matrix A_{22} is stable and A_{11} is a 4×4 matrix possessing two distinct pairs of nonzero, pure imaginary eigenvalues with the form as given in (4.89).

where matrices A_{11} and A_{22} are defined in (5.7) (or (5.26)).

5.5.1. Case (SS)

In this subsection, we consider the critical case (SS) in which $\eta := (x, y)'$ and the nonlinear system (5.7) is assumed given in the form

$$\dot{x} = y + f(x, y, \xi) \tag{5.58a}$$

$$\dot{y} = g(x, y, \xi) \tag{5.58b}$$

$$\dot{\xi} = A_{22}\xi + G(x, y, \xi), \tag{5.58c}$$

where functions f, g, G have the forms as given in (5.38)-(5.40). Similarly, we choose the approximate solution $\phi(\eta)$ of the manifold h as the one given in (5.41), where h_{xx}, h_{xy} and h_{yy} have been obtained in Section 4.3.1 and are given as

$$h_{xx} = -A_{22}^{-1} G_{xx} \tag{5.59}$$

$$h_{xy} = -A_{22}^{-1}(G_{xy} + 2A_{22}^{-1} G_{xx}) \tag{5.60}$$

$$h_{yy} = -A_{22}^{-1}(G_{yy} - h_{xy}). \tag{5.61}$$

Let $\hat{\varphi}(x, y) = \varphi(x, y, h(x, y))$, for $\varphi = f, g$, which implies that $\hat{\varphi}_{ij} = \varphi_{ij}$ for $\varphi = f, g$ and $i, j \in \{x, y\}$. The reduced model (5.26a) can then be written as

$$\dot{x} = \Omega_1 y + \hat{f}(x, y), \tag{5.62a}$$

$$\dot{y} = -\Omega_2 x + \hat{g}(x, y). \tag{5.62b}$$

Motivated by the scalar function (5.1) and Lemma 5.1, we choose

$$\begin{aligned} \mathcal{V}_r = & p_{yy} y^2 + \mathcal{V}_{r,xyy} x^2 y + \mathcal{V}_{r,xxy} x^2 y + \mathcal{V}_{r,yyy} y^3 \\ & + \mathcal{V}_{r4}(x, y) + \mathcal{V}_{r5}(x, y) + \mathcal{V}_{r6}(x, y) \end{aligned} \tag{5.63}$$

as a Liapunov function candidate for the reduced model (5.62) above. Here, $\mathcal{V}_{r4}, \mathcal{V}_{r5}$ and $\mathcal{V}_{r6}$ denote the fourth order, fifth order and sixth order homogeneous polynomial functions of x and y, respectively.

By assuming $p_{yy}, \mathcal{V}_{r4,xxxx} > 0$ and $4p_{yy}\mathcal{V}_{r4,xxxx} > \mathcal{V}_{r,xxy}^2$, the locally positive definiteness of $\mathcal{V}_r$ (given in (5.63)) is implied by Lemma 5.1. Moreover, the derivative of $\mathcal{V}_r$ along the trajectory of the reduced model (5.62) is obtained as

$$\begin{aligned}\dot{\mathcal{V}}_r = {} & 2p_{yy}g_{xx}x^2y + 2(\mathcal{V}_{r,xxy} + p_{yy}g_{xy})xy^2 + (2p_{yy}g_{yy} + \mathcal{V}_{r,xyy})y^3 \\ & + (\mathcal{V}_{r,xxy}g_{xx})x^4 + v_1x^3y + v_2x^2y^2 + v_3xy^3 + v_4y^4 + v_5x^5 \\ & + v_6x^4y + v_7x^6 + \mathcal{R}_1(x,y) + O(||(x,y)||^7), \end{aligned} \tag{5.64}$$

where $\mathcal{R}_1(x,y)$ denotes the remaining terms of fifth order and sixth order homogeneous functions of $\dot{\mathcal{V}}_r$. The expressions of $\mathcal{R}_1$ are very lengthy and it can be made to zero for all $x, y \in I\!R$, by a suitable choice of the functions $\mathcal{V}_{r5}$ and $\mathcal{V}_{r6}$, which are independent of the nonlinear dynamics f, g and the values of v_i. The expressions of $\mathcal{R}_1(x,y)$ are omitted.

It is observed from (5.64) that to provide the locally negative definiteness of $\dot{\mathcal{V}}_r$, following conditions must hold: $g_{xx} = 0$, $\mathcal{V}_{r,xyy} = -2p_{yy}g_{yy}$ and $\mathcal{V}_{r,xxy} = -p_{yy}g_{xy}$. Suppose $g_{xx} = 0$, the expressions of v_i are obtained as

$$v_1 = 2p_{yy}\hat{g}_{xxx} + 4\mathcal{V}_{r4,xxxx} + \mathcal{V}_{r,xxy}(2f_{xx} + g_{xy}), \tag{5.65}$$

$$v_2 = 2p_{yy}\hat{g}_{xxy} + 3\mathcal{V}_{r4,xxxy} + \mathcal{V}_{r,xyy}(f_{xx} + 2g_{xy}) + \mathcal{V}_{r,xxy}(2f_{xy} + g_{yy}), \tag{5.66}$$

$$\begin{aligned} v_3 = {} & 2p_{yy}\hat{g}_{xyy} + 2\mathcal{V}_{r4,xxyy} + \mathcal{V}_{r,xyy}(f_{xy} + 2g_{yy}) \\ & + 2\mathcal{V}_{r,xxy}f_{yy} + 3\mathcal{V}_{r,yyy}g_{xy}, \end{aligned} \tag{5.67}$$

$$v_4 = 2p_{yy}\hat{g}_{yyy} + \mathcal{V}_{r4,xyyy} + \mathcal{V}_{r,xyy}f_{yy} + 3\mathcal{V}_{r,yyy}g_{yy}, \tag{5.68}$$

$$v_5 = 4\mathcal{V}_{r4,xxxx}f_{xx} + \mathcal{V}_{r,xxy}\hat{g}_{xxx}, \tag{5.69}$$

$$\begin{aligned} v_6 = {} & 2p_{yy}\hat{g}_{xxxx} + 4\mathcal{V}_{r4,xxxx}f_{xy} + 3\mathcal{V}_{r4,xxxy}f_{xx} + \mathcal{V}_{r4,xxxy}g_{xy} \\ & + \mathcal{V}_{r,xxy}(2\hat{f}_{xxx} + \hat{g}_{xxy}) + 2\mathcal{V}_{r,xyy}\hat{g}_{xxx} + 5\mathcal{V}_{r5,xxxxx}, \end{aligned} \tag{5.70}$$

$$v_7 = 4\mathcal{V}_{r4,xxxx}\hat{f}_{xxx} + \mathcal{V}_{r4,xxxy}\hat{g}_{xxx} + 5\mathcal{V}_{r5,xxxxx}f_{xx} + \mathcal{V}_{r,xxy}\hat{g}_{xxxx}. \tag{5.71}$$

By using Corollary 5.2 to check the locally negative definiteness of $\dot{\mathcal{V}}_r$ and employing Liapunov stability criteria, we then have the next result.

Proposition 5.2. Under hypothesis (SS), the origin is asymptotically stable for (5.62) if there exists a function given in (5.63) such that (i) $p_{yy}, \mathcal{V}_{r4,xxxx} > 0$, (ii) $4p_{yy}\mathcal{V}_{r4,xxxx} > \mathcal{V}^2_{r,xxy}$, (iii) $g_{xx} = 0$, (iv) $\mathcal{V}_{r,xyy} = -2p_{yy}g_{yy}$, (v) $\mathcal{V}_{r,xxy} = -p_{yy}g_{xy}$, (vi) $\mathcal{R}_1(x,y) = 0$ for all $x, y \in I\!R$, (vii) $v_1 = v_3 = v_5 = v_6 = 0$, (viii) $v_2 \le 0$ and (ix) $v_4, v_7 < 0$, where v_i are defined in (5.65)-(5.71) above.

Next, we implement the stability conditions of Proposition 5.2 in terms of dynamics f and g for the reduced model (5.62). From the foregoing discussion, it is not difficult to have function the $\mathcal{V}_r$ such that conditions (i)-(vi) of Proposition 5.2 hold. Thus, in the following discussion, we assume conditions (i)-(vi) hold. Then the rest of job of implementing Proposition 5.2 is to check for conditions (vii)-(ix) of Proposition 5.2.

Solving for $v_1 = v_5 = 0$ from Eqs. (5.65) and (5.69), we have

$$p_{yy}(2f_{xx} + g_{xy}) \cdot (\hat{g}_{xxx} - g_{xy}f_{xx}) = 0, \tag{5.72}$$

$$\begin{aligned} \mathcal{V}_{r4,xxxx} &= \frac{g_{xy}}{4f_{xx}} \cdot p_{yy}\hat{g}_{xxx} \quad \text{for} \quad f_{xx} \neq 0, \\ &= -\frac{1}{4}p_{yy}(2\hat{g}_{xxx} - g^2_{xy}) \quad \text{for} \quad f_{xx} = 0. \end{aligned} \tag{5.73}$$

By checking the conditions (i)-(ii) and (v) of Proposition 5.2, we have $\hat{g}_{xxx} \neq g_{xy}f_{xx}$. Thus, the only possible solution to Eq. (5.72) is that $2f_{xx} + g_{xy} = 0$. From conditions (i) and (v) and Eq. (5.73), the condition (ii) now becomes $\hat{g}_{xxx} + 2f^2_{xx} < 0$.

Moreover, it is observed from Eqs. (5.67) and (5.68) that there exist $\mathcal{V}_{r4,xxyy}$ and $\mathcal{V}_{r4,xyyy}$ such that $v_3 = 0$ and $v_4 < 0$. Next, we solve for conditions $v_2 \le 0$, $v_6 = 0$ and v_7 by using the results obtained above. It is found that there exist $\mathcal{V}_{r4,xxxy}$ and $\mathcal{V}_{r5,xxxxx}$ such that $v_2 \le 0$, $v_6 = 0$ and $v_7 < 0$ if

$$\begin{aligned} \hat{g}_{xxy} + 3\hat{f}_{xxx} - f_{xx}(f_{xy} + 2g_{yy}) &= g_{xxy} + g_{x\xi}h_{xy} + g_{y\xi}h_{xx} \\ &\quad + 3(f_{xxx} + f_{x\xi}h_{xx}) - f_{xx}(f_{xy} + 2g_{yy}) < 0, \end{aligned} \tag{5.74}$$

where h_{xx}, h_{xy} are given in (5.59)-(5.60).

To conclude the discussion above, we have the following result.

Theorem 5.5. Under hypothesis (SS), the origin is asymptotically stable for (5.62), if $g_{xx} = 0$, $g_{xy} + 2f_{xx} = 0$, $\hat{g}_{xxx} + 2f_{xx}^2 = g_{xxx} + g_{x\xi}h_{xx} + 2f_{xx}^2 < 0$ and Eq. (5.74) holds.

Remark 5.4. The result given in Theorem 5.5 agrees with the one given in Corollary 4.6. A stability criterion for the critical subsystem (5.5) with $\xi = 0$ can also be obtained from Theorem 5.5, which agrees with the one given by Behtash and Sastry ([10], Theorem 4.1). However, the proofs given in [10] were not stated clearly enough. Families of Liapunov functions for the whole system (5.7) can also be obtained by using Theorem 5.5, Proposition 5.2 and Algorithm 5.1. Details are omitted.

5.5.2. Case (HS)

Next, we consider the critical case (HS), where $\eta := (x, y, z)'$, $F(\eta, \xi) = (f(x, y, z, \xi), g(x, y, z, \xi), r(x, y, z, \xi))'$ are two three dimensional vectors. Let the approximate solution of h be given as

$$\phi(x, y, z) = h_{xx}x^2 + h_{xy}xy + h_{xz}xz + h_{yy}y^2 + h_{yz}yz + h_{zz}z^2, \qquad (5.75)$$

where the values of $h_{xx}, h_{xy}, h_{xz}, h_{yy}, h_{yz}$ and h_{zz} have been obtained in Section 4.4 and are as given in Appendix 5.A.

Let $\hat{\varphi}(x, y, z) = \varphi(x, y, z, \phi(x, y, z))$ for $\varphi = f, g, r$. Thus, we have $\hat{\varphi}_{ij} = \varphi_{ij}$, for $i, j \in \{x, y, z\}$. The reduced model (5.26a) can then be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \Omega_1 & 0 \\ -\Omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \hat{f}(x, y, z) \\ \hat{g}(x, y, z) \\ \hat{r}(x, y, z) \end{pmatrix}. \qquad (5.76)$$

We now choose

$$\mathcal{V}_r = p_{xx}x^2 + p_{yy}y^2 + p_{zz}z^2 + \mathcal{V}_{r3}(x, y, z) + \mathcal{V}_{r4}(x, y, z), \qquad (5.77)$$

as a Liapunov function candidate for the reduced model (5.76), where p_{xx}, p_{yy}, $p_{zz} > 0$ and $\mathcal{V}_{r3}, \mathcal{V}_{r4}$ denote a cubic and a quartic functions of x, y, z, respectively.

Let $\mathcal{V}_{ri,\psi}$ be the coefficient of the term ψ in scalar function $\mathcal{V}_{ri}$ for $i = 3, 4$, we have the derivative of $\mathcal{V}_r$ along the trajectory of (5.76) as

$$\begin{aligned}\dot{\mathcal{V}}_r =& 2(\Omega_1 p_{xx} - \Omega_2 p_{yy})xy + 2p_{zz}r_{zz}z^3 + v_1 x^2 z + v_2 y^2 z \\ &+ v_3 x^2 z^2 + v_4 x^2 y^2 + v_5 y^2 z^2 + v_6 x^4 + v_7 y^4 \\ &+ v_8 z^4 + \mathcal{R}_2(x,y,z) + O(||(x,y,z)||^5)\end{aligned} \tag{5.78}$$

with

$$v_1 = 2r_{xx}p_{zz} + 2f_{xz}p_{xx} - \Omega_2 \mathcal{V}_{r3,xyz}, \tag{5.79}$$

$$v_2 = 2r_{yy}p_{zz} + 2g_{yz}p_{yy} + \Omega_1 \mathcal{V}_{r3,xyz}, \tag{5.80}$$

$$\begin{aligned}v_3 =& -\Omega_2 \mathcal{V}_{r4,xyzz} + 2\hat{r}_{xxz}p_{zz} + 2\hat{f}_{xzz}p_{xx} + \mathcal{V}_{r3,xxz}r_{zz} + 2\mathcal{V}_{r3,xzz}r_{xz} \\ &+ 3\mathcal{V}_{r3,zzz}r_{xx} + g_{xx}\mathcal{V}_{r3,yzz} + f_{xx}\mathcal{V}_{r3,xzz} + g_{xz}\mathcal{V}_{r3,xyz} \\ &+ 2f_{xz}\mathcal{V}_{r3,xxz} + g_{zz}\mathcal{V}_{r3,xxy} + 3f_{zz}\mathcal{V}_{r3,xxx},\end{aligned} \tag{5.81}$$

$$\begin{aligned}v_4 =& -3\Omega_2 \mathcal{V}_{r4,xyyy} + 3\Omega_1 \mathcal{V}_{r4,xxxy} + 2\hat{g}_{xxy}p_{yy} + 2\hat{f}_{xyy}p_{xx} + \mathcal{V}_{r3,xxz}r_{yy} \\ &+ \mathcal{V}_{r3,xyz}r_{xy} + \mathcal{V}_{r3,yyz}r_{xx} + 3g_{xx}\mathcal{V}_{r3,yyy} + 2g_{xy}\mathcal{V}_{r3,xyy} \\ &+ f_{xx}\mathcal{V}_{r3,xyy} + (g_{yy} + 2f_{xy})\mathcal{V}_{r3,xxy} + 3f_{yy}\mathcal{V}_{r3,xxx},\end{aligned} \tag{5.82}$$

$$\begin{aligned}v_5 =& \Omega_1 \mathcal{V}_{r4,xyzz} + 2\hat{r}_{yyz}p_{zz} + 2\hat{g}_{yzz}p_{yy} + \mathcal{V}_{r3,yyz}(r_{zz} + 2g_{yz}) \\ &+ \mathcal{V}_{r3,yzz}(2r_{yz} + g_{yy}) + 3\mathcal{V}_{r3,zzz}r_{yy} + 3g_{zz}\mathcal{V}_{r3,yyy} \\ &+ f_{yy}\mathcal{V}_{r3,xzz} + f_{yz}\mathcal{V}_{r3,xyz} + f_{zz}\mathcal{V}_{r3,xyy},\end{aligned} \tag{5.83}$$

$$v_6 = 2\hat{f}_{xxx}p_{xx} - \Omega_2 \mathcal{V}_{r4,xxxy} + \mathcal{V}_{r3,xxz}r_{xx} + g_{xx}\mathcal{V}_{r3,xxy} + 3f_{xx}\mathcal{V}_{r3,xxx}, \tag{5.84}$$

$$v_7 = \Omega_1 \mathcal{V}_{r4,xyyy} + 2\hat{g}_{yyy}p_{yy} + \mathcal{V}_{r3,yyz}r_{yy} + 3g_{yy}\mathcal{V}_{r3,yyy} + f_{yy}\mathcal{V}_{r3,xyy}, \tag{5.85}$$

$$v_8 = 2\hat{r}_{zzz}p_{zz} + 3\mathcal{V}_{r3,zzz}r_{zz} + g_{zz}\mathcal{V}_{r3,yzz} + f_{zz}\mathcal{V}_{r3,xzz}, \tag{5.86}$$

$\mathcal{R}_2(x,y,z)$ denotes the remaining terms of cubic and quartic terms and the coefficients of cubic terms of functions $\hat{f}, \hat{g}$ and $\hat{r}$ are given in Appendix 5.A. It is found that $\mathcal{R}_2(x,y,z)$ can be zero for all $x, y, z \in I\!R$ independently, by suitable choice of the functions $\mathcal{V}_{r3}$ and $\mathcal{V}_{r4}$, while there is no assumption on nonlinear functions f, g and r for the existence of such choice. The expressions for $\mathcal{R}_2$ are very lengthy and hence are not given.

Since $p_{xx}, p_{yy}, p_{zz} > 0$, the scalar function $\mathcal{V}_r$ given in (5.77) is an l.p.d.f. To check the locally negative definiteness of $\dot{\mathcal{V}}_r$ and employ Liapunov stability criteria, we then have

Proposition 5.3. Under hypothesis (HS), the origin is asymptotically stable for (5.76) if there exists a function given in (5.77) such that (i) $p_{xx}, p_{yy}, p_{zz} > 0$, (ii) $r_{zz} = 0$, (iii) $\Omega_1 p_{xx} = \Omega_2 p_{yy}$, (iv) $\mathcal{R}_2(x, y, z) = 0$ for all $x, y, z \in I\!R$, (v) $v_1 = v_2 = 0$, (vi) $v_i \leq 0$, $i = 3, 4, 5$ and (vii) $v_i < 0$, $i =$ 6,7,8, where v_i is defined in (5.79)-(5.86) above.

To implement Proposition 5.3, we assume conditions (i)-(iv) hold in the following discussions. Solving for $v_1 = v_2 = 0$ from Eqs. (5.79)-(5.80), we have

$$2p_{zz}(\Omega_1 r_{xx} + \Omega_2 r_{yy}) + 2\Omega_1 p_{xx}(f_{xz} + g_{yz}) = 0. \tag{5.87}$$

From the previous discussion, there is no assumption on the sign of Ω_1 (we only assume $\Omega_1\Omega_2 > 0$). For simplicity, we choose $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$ and $f_{xz} + g_{yz} = 0$, which will guarantee that $v_1 = v_2 = 0$. Moreover, by letting $v_6 = v_7$ and solving for conditions (vi) and (vii) of Proposition 5.3 we obtain the following result.

Theorem 5.6. Under hypothesis (HS), the origin is asymptotically stable for (5.76) if $r_{zz} = 0$, $dr_{xx} + er_{yy} = 0$, $f_{xz} + g_{yz} = 0$, $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of different sign, where

$$S_1 = \frac{24}{\Omega_1\Omega_2}\{\Omega_1\Omega_2 \hat{r}_{zzz} - \Omega_2 f_{zz} r_{yz} + \Omega_1 g_{zz} r_{xz}\}, \tag{5.88}$$

$$\begin{aligned} S_2 =& \frac{8}{\Omega_1^2 + \Omega_2^2}\{(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy} - \Omega_1 g_{yz} r_{xy} + 3\Omega_1\Omega_2 \hat{g}_{yyy} \\ & + (\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) g_{yy} + \frac{\Omega_1^2}{\Omega_2} g_{xx} g_{xy} + \Omega_1^2 \hat{g}_{xxy} + \frac{2\Omega_1^2}{\Omega_2} f_{xx} g_{xx} \\ & - \Omega_2 f_{xy} f_{yy} + \Omega_1\Omega_2 \hat{f}_{xyy} - \Omega_1 f_{xx} f_{xy} + 3\Omega_1^2 \hat{f}_{xxx}\}, \end{aligned} \tag{5.89}$$

$$\begin{aligned} S_3 =& 24\{2\Omega_2 f_{zz} r_{yz} - 2\Omega_1 g_{zz} r_{xz} + \Omega_1(g_{xy} + 2f_{xx}) g_{zz} \\ & + \Omega_1\Omega_2 \hat{g}_{yzz} - 2\Omega_2 f_{zz} g_{yy} - \Omega_2 f_{xy} f_{zz} + \Omega_1\Omega_2 \hat{f}_{xzz}\}, \end{aligned} \tag{5.90}$$

$$S_4 = 24\{\Omega_2^2 \hat{r}_{yyz} - \frac{\Omega_2}{\Omega_1}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) r_{yz} - \frac{\Omega_2}{\Omega_1}(\Omega_1 g_{xz} + \Omega_2 f_{yz}) r_{yy}$$
$$+ (\Omega_2 g_{yy} + \Omega_1 g_{xx}) r_{xz} + \Omega_2 g_{yz} r_{xy} + \Omega_1 \Omega_2 \hat{r}_{xxz}\}. \quad (5.91)$$

Sketch of proof: Suppose $r_{zz} = 0$, $\Omega_1 r_{xx} + \Omega_2 r_{yy} = 0$, $f_{xz} + g_{yz} = 0$ and hypothesis (HS) holds. Also, choose $\mathcal{V}_r$ (given in (5.77)) such that conditions (i), (iii)-(iv) of Proposition 5.3 hold. It is found that $v_8 < 0$ if $S_1 < 0$ and there exist $\mathcal{V}_{r4,xxxy}$ and $\mathcal{V}_{r4,xyyy}$ such that $v_6 = v_7 < 0$ and $v_4 \leq 0$ when $S_2 < 0$, under the pre-choice of $\mathcal{V}_{r3}$ and $\mathcal{V}_{r4}$ such that $\mathcal{R}_2(x, y, z) = 0$ for all $x, y, z \in I\!R$. The values of v_i are given in (5.79)-(5.86). Finally, we have $\mathcal{V}_{r4,xyzz}$ such that $v_3, v_5 \leq 0$ if $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of different sign. The conclusion is directly implied by Proposition 5.3.

■

Remark 5.5. The result of Theorem 5.6. agrees with condition (i) of Proposition 4.4 with $u = 0$. Similarly, families of Liapunov function for the whole system (5.7) can be implemented by using Proposition 5.3, Theorem 5.6 and Algorithm 5.1. Details are also omitted.

5.5.3. Case (HH)

Next, we consider the critical case (HH) for system (5.7), where $\eta := (x, y, z, w)'$ and $F(\eta, \xi) = (f(\eta, \xi), g(\eta, \xi), r(\eta, \xi), s(\eta, \xi))'$ are four dimensional vectors. Choose the approximate solution of center manifold h as

$$\phi(\eta) = \phi(x, y, z) = h_{xx}x^2 + h_{xy}xy + h_{xz}xz + h_{xw}xw + h_{yy}y^2 + h_{yz}yz$$
$$+ h_{yw}yw + h_{zz}z^2 + h_{zw}zw + h_{ww}w^2, \quad (5.92)$$

where the values of constant vectors h_{ij}, $i, j = x, y, z, w$ have been obtained in Section 4.5 and are as given in Appendix 5.B.

Let $\hat{\varphi}(x, y, z, w) = \varphi(x, y, z, w, \phi(x, y, z, w))$ for $\varphi = f, g, r, s$. Thus, we have $\hat{\varphi}_{ij} = \varphi_{ij}$, for $i, j \in \{x, y, z, w\}$. The reduced model (5.26a) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} \hat{f}(x,y,z,w) \\ \hat{g}(x,y,z,w) \\ \hat{r}(x,y,z,w) \\ \hat{s}(x,y,z,w) \end{pmatrix}. \tag{5.93}$$

Similarly, let us choose

$$\mathcal{V}_r = p_{xx}x^2 + p_{yy}y^2 + p_{zz}z^2 + p_{ww}w^2 + \mathcal{V}_{r3}(x,y,z) + \mathcal{V}_{r4}(x,y,z), \tag{5.94}$$

as a Liapunov function candidate for the reduced model (5.93), where p_{xx}, p_{yy}, p_{zz}, $p_{ww} > 0$ and $\mathcal{V}_{r3}, \mathcal{V}_{r4}$ denote cubic and quartic functions of x, y, z, w, respectively. Then the derivative of $\mathcal{V}_r$ along the trajectory of (5.93) is

$$\begin{aligned} \dot{\mathcal{V}}_r =& 2(\Omega_1 p_{xx} - \Omega_2 p_{yy})xy + 2(\Omega_3 p_{zz} - \Omega_4 p_{ww})zw + v_1 y^2 z^2 + v_2 y^2 w^2 \\ &+ v_3 x^2 z^2 + v_4 x^2 w^2 + v_5 x^2 y^2 + v_6 z^2 w^2 + v_7 x^4 \\ &+ v_8 y^4 + v_9 z^4 + v_{10} w^4 + \mathcal{R}_3(x,y,z,w) + O(||(x,y,z,w)||^5) \end{aligned} \tag{5.95}$$

with

$$\begin{aligned} v_1 =& -\Omega_4 \mathcal{V}_{r4,yyzw} + \Omega_1 \mathcal{V}_{r4,xyzz} + \mathcal{V}_{r3,yyw} s_{zz} + \mathcal{V}_{r3,yzw} s_{yz} + \mathcal{V}_{r3,zzw} s_{yy} \\ &+ \mathcal{V}_{r3,yyz}(r_{zz} + 2g_{yz}) + \mathcal{V}_{r3,yzz}(2r_{yz} + g_{yy}) + 2p_{zz}\hat{r}_{yyz} + 3\mathcal{V}_{r3,zzz} r_{yy} \\ &+ 2\hat{g}_{yzz} p_{yy} + 3g_{zz}\mathcal{V}_{r3,yyy} + f_{yy}\mathcal{V}_{r3,xzz} + f_{yz}\mathcal{V}_{r3,xyz} + f_{zz}\mathcal{V}_{r3,xyy}, \end{aligned} \tag{5.96}$$

$$\begin{aligned} v_2 =& \Omega_3 \mathcal{V}_{r4,yyzw} + \Omega_1 \mathcal{V}_{r4,xyww} + \mathcal{V}_{r3,yyw}(s_{ww} + 2g_{yw}) \\ &+ \mathcal{V}_{r3,yww}(2s_{yw} + g_{yy}) + 2p_{ww}\hat{s}_{yyw} + 3\mathcal{V}_{r3,www} s_{yy} \\ &+ \mathcal{V}_{r3,yyz} r_{ww} + \mathcal{V}_{r3,yzw} r_{yw} + \mathcal{V}_{r3,zww} r_{yy} + 2\hat{g}_{yww} p_{yy} \\ &+ 3g_{ww}\mathcal{V}_{r3,yyy} + f_{yy}\mathcal{V}_{r3,xww} + f_{yw}\mathcal{V}_{r3,xyw} + f_{ww}\mathcal{V}_{r3,xyy}, \end{aligned} \tag{5.97}$$

$$\begin{aligned} v_3 =& -\Omega_2 \mathcal{V}_{r4,xyzz} - \Omega_4 \mathcal{V}_{r4,xxzw} + \mathcal{V}_{r3,xxw} s_{zz} + \mathcal{V}_{r3,xzw} s_{xz} \\ &+ \mathcal{V}_{r3,zzw} s_{xx} + \mathcal{V}_{r3,xxz} r_{zz} + 2\mathcal{V}_{r3,xzz} r_{xz} + 2p_{zz}\hat{r}_{xxz} \\ &+ 3\mathcal{V}_{r3,zzz} r_{xx} + 2\hat{f}_{xzz} p_{xx} + g_{xx}\mathcal{V}_{r3,yzz} + f_{xx}\mathcal{V}_{r3,xzz} \\ &+ g_{xz}\mathcal{V}_{r3,xyz} + 2f_{xz}\mathcal{V}_{r3,xxz} + g_{zz}\mathcal{V}_{r3,xxy} + 3f_{zz}\mathcal{V}_{r3,xxx}, \end{aligned} \tag{5.98}$$

$$v_4 = -\Omega_2 \mathcal{V}_{r4,xyww} + \Omega_3 \mathcal{V}_{r4,xxzw} + \mathcal{V}_{r3,xxw}(s_{ww} + 2f_{xw})$$

$$
\begin{aligned}
&+ 2\mathcal{V}_{r3,xww}s_{xw} + 2p_{ww}\hat{s}_{xxw} + 3\mathcal{V}_{r3,www}s_{xx} + \mathcal{V}_{r3,xxz}r_{ww} \\
&+ \mathcal{V}_{r3,xzw}r_{xw} + \mathcal{V}_{r3,zww}r_{xx} + 2\hat{f}_{xww}p_{xx} + g_{xx}\mathcal{V}_{r3,yww} \\
&+ f_{xx}\mathcal{V}_{r3,xww} + g_{xw}\mathcal{V}_{r3,xyw} + g_{ww}\mathcal{V}_{r3,xxy} + 3f_{ww}\mathcal{V}_{r3,xxx}, \qquad (5.99)\\
v_5 =& - 3\Omega_2\mathcal{V}_{r4,xyyy} + 3\Omega_1\mathcal{V}_{r4,xxxy} + \mathcal{V}_{r3,xxw}s_{yy} + \mathcal{V}_{r3,xyw}s_{xy} + \mathcal{V}_{r3,yyw}s_{xx} \\
&+ \mathcal{V}_{r3,xxz}r_{yy} + \mathcal{V}_{r3,xyz}r_{xy} + \mathcal{V}_{r3,yyz}r_{xx} + 2\hat{g}_{xxy}p_{yy} \\
&+ 2\hat{f}_{xyy}p_{xx} + 3g_{xx}\mathcal{V}_{r3,yyy} + 2g_{xy}\mathcal{V}_{r3,xyy} + f_{xx}\mathcal{V}_{r3,xyy} \\
&+ (g_{yy} + 2f_{xy})\mathcal{V}_{r3,xxy} + 3f_{yy}\mathcal{V}_{r3,xxx}, \qquad (5.100)\\
v_6 =& - 3\Omega_4\mathcal{V}_{r4,zwww} + 3\Omega_3\mathcal{V}_{r4,zzzw} + \mathcal{V}_{r3,zzw}(s_{ww} + 2r_{zw}) + 2\mathcal{V}_{r3,zww}s_{zw} \\
&+ 2p_{ww}\hat{s}_{zzw} + 3\mathcal{V}_{r3,www}s_{zz} + 3\mathcal{V}_{r3,zzz}r_{ww} + 2p_{zz}\hat{r}_{zww} \\
&+ \mathcal{V}_{r3,zww}r_{zz} + g_{zz}\mathcal{V}_{r3,yww} + g_{zw}\mathcal{V}_{r3,yzw} + g_{ww}\mathcal{V}_{r3,yzz} \\
&+ f_{zz}\mathcal{V}_{r3,xww} + f_{zw}\mathcal{V}_{r3,xzw} + f_{ww}\mathcal{V}_{r3,xzz}, \qquad (5.101)\\
v_7 =& - \Omega_2\mathcal{V}_{r4,xxxy} + \mathcal{V}_{r3,xxw}s_{xx} + \mathcal{V}_{r3,xxz}r_{xx} + 2\hat{f}_{xxx}p_{xx} \\
&+ g_{xx}\mathcal{V}_{r3,xxy} + 3f_{xx}\mathcal{V}_{r3,xxx}, \qquad (5.102)\\
v_8 =& \Omega_1\mathcal{V}_{r4,xyyy} + \mathcal{V}_{r3,yyw}s_{yy} + \mathcal{V}_{r3,yyz}r_{yy} + 2\hat{g}_{yyy}p_{yy} \\
&+ 3g_{yy}\mathcal{V}_{r3,yyy} + f_{yy}\mathcal{V}_{r3,xyy}, \qquad (5.103)\\
v_9 =& - \Omega_4\mathcal{V}_{r4,zzzw} + \mathcal{V}_{r3,zzw}s_{zz} + 2p_{zz}\hat{r}_{zzz} + 3\mathcal{V}_{r3,zzz}r_{zz} \\
&+ g_{zz}\mathcal{V}_{r3,yzz} + f_{zz}\mathcal{V}_{r3,xzz}, \qquad (5.104)\\
v_{10} =& \Omega_3\mathcal{V}_{r4,zwww} + 2p_{ww}\hat{s}_{www} + 3\mathcal{V}_{r3,www}s_{ww} + \mathcal{V}_{r3,zww}r_{ww} \\
&+ g_{ww}\mathcal{V}_{r3,yww} + f_{ww}\mathcal{V}_{r3,xww}, \qquad (5.105)
\end{aligned}
$$

$\mathcal{R}_3(x, y, z, w)$ denotes the cubic and remaining quartic terms of $\dot{\mathcal{V}}_r$ and the coefficients of cubic terms of functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are given in Appendix 5.B. The expressions for $\mathcal{R}_3$ are very lengthy and it can be set to zero for all $x, y, z, w \in I\!R$ by suitable choice of $\mathcal{V}_{r3}$ and $\mathcal{V}_{r4}$, independent of nonlinear dynamics f, g, r, s and values of v_i. However, it is found that to guarantee the existence of solutions for $\mathcal{R}_3 = 0$, we need to have an assumption of $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. The expressions of $\mathcal{R}_3$ are not given.

Since $p_{xx}, p_{yy}, p_{zz}, p_{ww} > 0$, the scalar function $\mathcal{V}_r$ given in (5.94) is an l.p.d.f. To check the locally negative definiteness of $\dot{\mathcal{V}}_r$ (given in (5.95)) and employ Liapunov stability criteria, we then have

Proposition 5.4. Under hypothesis (HH), the origin is asymptotically stable for the reduced model (5.93) if $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$ and there exists a function as given in (5.94) such that (i) $p_{xx}, p_{yy}, p_{zz}, p_{ww} > 0$, (ii) $\Omega_1 p_{xx} = \Omega_2 p_{yy}$, (iii) $\Omega_3 p_{zz} = \Omega_4 p_{ww}$, (iv) $\mathcal{R}_3(x, y, z, w) = 0$ for all $x, y, z, w \in I\!R$, (v) $v_i \leq 0$, $i = 1, \ldots, 6$ and (vi) $v_i < 0$ for $i = 7, \ldots, 10$, where v_i are given in (5.96)-(5.105).

To implement the stability criterion given in Proposition 5.4 in terms of system dynamics, we assume conditions (i)-(iv) hold and let $v_{2i} = v_{2i-1}$, $i = 1, 2, 4, 5$. Solving for conditions (v) and (vi) of Proposition 5.4, we obtain a stability criterion for the reduced model (5.93) in the next theorem.

Theorem 5.7. Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. Under hypothesis (HH), the origin is asymptotically stable for the reduced model (5.93) if $S_1, S_2 < 0$ and $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of different sign, where

$$\begin{aligned}
S_1 = \frac{8}{\Omega_1^2 + \Omega_2^2}\{&\Omega_1[3(\Omega_2\hat{g}_{yyy} + \Omega_1\hat{f}_{xxx}) + (\Omega_1\hat{g}_{xxy} + \Omega_2\hat{f}_{xyy})] \\
&+ g_{yy}(\Omega_1 g_{xy} - 2\Omega_2 f_{yy}) - f_{xy}(\Omega_2 f_{yy} + \Omega_1 f_{xx}) + \frac{\Omega_1^2}{\Omega_2} g_{xx}(g_{xy} + 2f_{xx}) \\
&+ \frac{\Omega_1}{\Omega_4}[(3\Omega_2 s_{yy} + \Omega_1 s_{xx})g_{yz} + (3\Omega_1 s_{xx} + \Omega_2 s_{yy})f_{xz}] \\
&- \frac{\Omega_1}{\Omega_3}[(\Omega_1 r_{xx} + 3\Omega_2 r_{yy})g_{yw} + (\Omega_2 r_{yy} + 3\Omega_1 r_{xx})f_{xw}] \\
&+ \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_4}[\Omega_1(\Omega_4 g_{xw} - 2\Omega_2 g_{yz}) + \Omega_2(\Omega_4 f_{yw} + 2\Omega_1 f_{xz})] \cdot (\Omega_4 r_{xy} \\
&- 2\Omega_1 s_{xx} + 2\Omega_2 s_{yy}) - \frac{\Omega_1}{(4\Omega_1\Omega_2 - \Omega_3\Omega_4)\Omega_3}[\Omega_1(2\Omega_2 g_{yw} + \Omega_3 g_{xz}) \\
&- \Omega_2(2\Omega_1 f_{xw} - \Omega_3 f_{yz})] \cdot (\Omega_3 s_{xy} - 2\Omega_2 r_{yy} + 2\Omega_1 r_{xx})\},
\end{aligned} \tag{5.106}$$

$$S_2 = \frac{8}{\Omega_3^2 + \Omega_4^2}\{\Omega_3[3(\Omega_4 \hat{s}_{www} + \Omega_3 \hat{r}_{zzz}) + (\Omega_3 \hat{s}_{zzw} + \Omega_4 \hat{r}_{zww})]$$

$$+ s_{ww}(\Omega_3 s_{zw} - 2\Omega_4 r_{ww}) - r_{zw}(\Omega_4 r_{ww} + \Omega_3 r_{zz}) + \frac{\Omega_3^2}{\Omega_4} s_{zz}(s_{zw} + 2r_{zz})$$

$$+ \frac{\Omega_3}{\Omega_2}[(3\Omega_4 g_{ww} + \Omega_3 g_{zz})s_{xw} + (3\Omega_3 g_{zz} + \Omega_4 g_{ww})r_{xz}]$$

$$- \frac{\Omega_3}{\Omega_1}[(\Omega_3 f_{zz} + 3\Omega_4 f_{ww})s_{yw} + (\Omega_4 f_{ww} + 3\Omega_3 f_{zz})r_{yz}]$$

$$+ \frac{\Omega_3}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_2}[\Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw}) + \Omega_4(\Omega_2 r_{yw} + 2\Omega_3 r_{xz})] \cdot (\Omega_2 f_{zw}$$

$$- 2\Omega_3 g_{zz} + 2\Omega_4 g_{ww}) - \frac{\Omega_3}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_1}[\Omega_3(2\Omega_4 s_{yw} + \Omega_1 s_{xz})$$

$$- \Omega_4(2\Omega_3 r_{yz} - \Omega_1 r_{xw})] \cdot (\Omega_1 g_{zw} - 2\Omega_4 f_{ww} + 2\Omega_3 f_{zz})\}, \tag{5.107}$$

$$S_3 = \frac{24}{\Omega_1 + \Omega_2}\{2 f_{zz} r_{yz} + \frac{1}{\Omega_3}[2\Omega_4 f_{ww} s_{yw} + \Omega_1\Omega_4(\hat{f}_{xww} + \hat{g}_{yww})]$$

$$+ \Omega_1(\hat{f}_{xzz} + \hat{g}_{yzz}) - \frac{2\Omega_1}{\Omega_2} g_{zz} r_{xz} - \frac{2\Omega_1\Omega_4}{\Omega_3\Omega_2} g_{ww} s_{xw}$$

$$+ \frac{\Omega_1}{\Omega_3^2\Omega_4}[\Omega_3(\Omega_4 s_{ww} + \Omega_3 s_{zz})(g_{yz} + f_{xz}) - \Omega_4(\Omega_4 r_{ww} + \Omega_3 r_{zz})(g_{yw} + f_{xw})]$$

$$+ \frac{1}{\Omega_3\Omega_2}[\Omega_1(\Omega_3 g_{zz} + \Omega_4 g_{ww})(g_{xy} + 2f_{xx}) - \Omega_2(\Omega_3 f_{zz} + \Omega_4 f_{ww})(f_{xy} + 2g_{yy})]$$

$$+ \frac{1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3}[\Omega_4(\Omega_1 r_{xw} - 2\Omega_3 r_{yz}) + \Omega_3(\Omega_1 s_{xz} + 2\Omega_4 s_{yw})] \cdot (\Omega_1 g_{zw}$$

$$- 2\Omega_4 f_{ww} + 2\Omega_3 f_{zz}) - \frac{\Omega_1}{(4\Omega_3\Omega_4 - \Omega_1\Omega_2)\Omega_3\Omega_2}[\Omega_4(\Omega_2 r_{yw} + 2\Omega_3 r_{xz})$$

$$+ \Omega_3(\Omega_2 s_{yz} - 2\Omega_4 s_{xw})] \cdot (\Omega_2 f_{zw} - 2\Omega_3 g_{zz} + 2\Omega_4 g_{ww})\}, \tag{5.108}$$

$$S_4 = \frac{24}{\Omega_3 + \Omega_4}\{2 r_{xx} f_{xw} + \frac{1}{\Omega_1}[2\Omega_2 r_{yy} g_{yw} + \Omega_3\Omega_2(\hat{r}_{yyz} + \hat{s}_{yyw})]$$

$$+ \Omega_3(\hat{r}_{xxz} + \hat{s}_{xxw}) - \frac{2\Omega_3}{\Omega_4} s_{xx} f_{xz} - \frac{2\Omega_3\Omega_2}{\Omega_1\Omega_4} s_{yy} g_{yz}$$

$$+ \frac{\Omega_3}{\Omega_1^2\Omega_2}[\Omega_1(\Omega_2 g_{yy} + \Omega_1 g_{xx})(s_{xw} + r_{xz}) - \Omega_2(\Omega_2 f_{yy} + \Omega_1 f_{xx})(s_{yw} + r_{yz})]$$

$$+\frac{1}{\Omega_1\Omega_4}[\Omega_3(\Omega_1 s_{xx}+\Omega_2 s_{yy})(s_{zw}+2r_{zz})-\Omega_4(\Omega_1 r_{xx}+\Omega_2 r_{yy})(r_{zw}+2s_{ww})]$$

$$+\frac{1}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_1}[\Omega_2(\Omega_3 f_{yz}-2\Omega_1 f_{xw})+\Omega_1(\Omega_3 g_{xz}+2\Omega_2 g_{yw})]\cdot(\Omega_3 s_{xy}$$

$$-2\Omega_2 r_{yy}+2\Omega_1 r_{xx})-\frac{\Omega_3}{(4\Omega_1\Omega_2-\Omega_3\Omega_4)\Omega_1\Omega_4}[\Omega_2(\Omega_4 f_{yw}+2\Omega_1 f_{xz})+\Omega_1(\Omega_4 g_{xw}$$

$$-2\Omega_2 g_{yz})]\cdot(\Omega_4 r_{xy}-2\Omega_1 s_{xx}+2\Omega_2 s_{yy})\}. \tag{5.109}$$

Sketch of proof: Suppose $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$, for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$, and the hypothesis (HH) holds. Also, choose $\mathcal{V}_r$ (given in (5.94)) such that conditions (i)-(iv) of Proposition 5.4 hold. It is found that there exist $\mathcal{V}_{r4,xxxy}$ and $\mathcal{V}_{r4,xyyy}$ such that $v_7 = v_8 < 0$ and $v_5 \leq 0$ if $S_1 < 0$, and we have $\mathcal{V}_{r4,zwww}$ and $\mathcal{V}_{r4,zzzw}$ such that $v_9 = v_{10} < 0$ and $v_6 \leq 0$ when $S_2 < 0$. The values of v_i are defined in (5.96)-(5.105). Moreover, there exist $\mathcal{V}_{r4,xyzz}$, $\mathcal{V}_{r4,xyww}$, $\mathcal{V}_{r4,xxzw}$ and $\mathcal{V}_{r4,yyzw}$ such that $v_1 = v_2 \leq 0$ and $v_3 = v_4 \leq 0$ if $S_3, S_4 \leq 0$ or S_3 and S_4 are nonzero and of different sign. The conclusion readily follows from Proposition 5.4.

■

Remark 5.6. The result of Theorem 5.7 agrees with the one given in Proposition 4.6 with $u = 0$. Families of Liapunov function for the full model can be constructed by applying Proposition 5.4, Theorem 5.7 and Algorithm 5.1.

5.6. Concluding Remarks

In this chapter,we have proposed a method for constructing families of Liapunov functions for nonlinear systems, specifically, when the Jacobian matrix of the system has eigenvalues lying on the imaginary axis and the remaining eigenvalues are stable. The Center manifold reduction technique is employed here to simplify the complexity of the proposed Liapunov functions. Finally, families of Liapunov functions for the simple critical case (SC) and the compound critical case (CC) are obtained to demonstrate the proposed method. It is found that the stability criteria obtained in this chapter for the critical cases (SC) and (CC) agree with those derived in Chapters 3 and 4 by using normal form reduction.

Appendix 5.A

The values of $h_{xx}, h_{xy}, h_{xz}, h_{yy}, h_{yz}$ and h_{zz} are given below.

$$
\begin{aligned}
h_{xy} &= -(A_{22}^2 + 4\Omega_1\Omega_2 I)^{-1}(-2\Omega_2 G_{yy} + 2\Omega_1 G_{xx} + A_{22}G_{xy}) \\
h_{xx} &= -A_{22}^{-1}(G_{xx} + \Omega_2 h_{xy}) \\
h_{xz} &= -(A_{22}^2 + \Omega_1\Omega_2 I)^{-1}(A_{22}G_{xz} - \Omega_2 G_{yz}) \\
h_{yy} &= -A_{22}^{-1}(G_{yy} - \Omega_1 h_{xy}) \\
h_{yz} &= -(A_{22}^2 + \Omega_1\Omega_2 I)^{-1}(A_{22}G_{yz} + \Omega_1 G_{xz}) \\
h_{zz} &= -A_{22}^{-1}G_{zz}.
\end{aligned}
$$

Moreover, the coefficients of cubic terms of functions $\hat{f}, \hat{g}, \hat{r}$ are defined in the form as given below.

$$
\begin{aligned}
\hat{\varphi}_{iii} &= \varphi_{iii} + \varphi_{i\xi}h_{ii} \\
\hat{\varphi}_{iij} &= \varphi_{iij} + \varphi_{i\xi}h_{ij} + \varphi_{j\xi}h_{ii} \\
\hat{\varphi}_{ijk} &= \varphi_{ijk} + \varphi_{i\xi}h_{jk} + \varphi_{j\xi}h_{ik} + \varphi_{k\xi}h_{ij}
\end{aligned}
$$

where $\varphi \in \{f, g, r\}$ and all i, j, k are distinct with $i, j, k \in \{x, y, z\}$.

Appendix 5.B

The values of h_{xx}, h_{xy}, h_{yy} are the same as ones given in Appendix 5.A. The values of remaining terms of h_{ij}, $i, j \in \{x, y, z, w\}$ are given as below.

$$
\begin{aligned}
h_{zw} &= -(A_{22}^2 + 4\Omega_3\Omega_4 I)^{-1}(-2\Omega_4 G_{ww} + 2\Omega_3 G_{zz} + A_{22}G_{zw}) \\
h_{zz} &= -A_{22}^{-1}(G_{zz} + \Omega_4 h_{zw}) \\
h_{ww} &= -A_{22}^{-1}(G_{ww} - \Omega_3 h_{zw}) \\
\begin{pmatrix} h_{xz} \\ h_{xw} \end{pmatrix} &= (M^2 + \Omega_1\Omega_2 I)^{-1}\{M \begin{pmatrix} G_{xz} \\ G_{xw} \end{pmatrix} - \Omega_2 \begin{pmatrix} G_{yz} \\ G_{yw} \end{pmatrix}\} \\
\begin{pmatrix} h_{yz} \\ h_{yw} \end{pmatrix} &= (M^2 + \Omega_1\Omega_2 I)^{-1}\{\Omega_1 \begin{pmatrix} G_{xz} \\ G_{xw} \end{pmatrix} + M \begin{pmatrix} G_{yz} \\ G_{yw} \end{pmatrix}\}
\end{aligned}
$$

where

$$M := \begin{pmatrix} A_{22} & \Omega_4 I \\ -\Omega_3 I & A_{22} \end{pmatrix}.$$

Moreover, the coefficients of cubic terms of functions $\hat{f}, \hat{g}, \hat{r}, \hat{s}$ are of the form

$$\begin{aligned} \hat{\varphi}_{iii} &= \varphi_{iii} + \varphi_{i\xi} h_{ii} \\ \hat{\varphi}_{iij} &= \varphi_{iij} + \varphi_{i\xi} h_{ij} + \varphi_{j\xi} h_{ii} \\ \hat{\varphi}_{ijk} &= \varphi_{ijk} + \varphi_{i\xi} h_{jk} + \varphi_{j\xi} h_{ik} + \varphi_{k\xi} h_{ij} \end{aligned}$$

where $\varphi \in \{f, g, r, s\}$ and all i, j, k are distinct with $i, j, k \in \{x, y, z, w\}$.

CHAPTER SIX

STATION-KEEPING CONTROL OF TSS: ONE CRITICAL MODE

In the following three chapters, we apply the existing control theory to the stabilization and control of a tethered satellite system which is introduced in Section 1.2. First, in this chapter we employ a bifurcation stability result to design stabilizing control laws for the TSS during station-keeping. Another approach for station-keeping control is studied in Chapter 7 by using the center manifold reduction technique proposed in Section 4.6. Finally, a constant in-plane angle control technique is proposed in Chapter 8 for the deployment and the retrieval of the subsatellite of a TSS.

After deriving a set of dynamic equations governing the dynamics of a Tethered Satellite System (TSS), stabilizing tension control laws for the TSS during station-keeping in feedback form are derived in this chapter. The tether is assumed rigid and massless, and the equations of motion are derived using the system Lagrangian. It is observed that, to stabilize the system, tools from stability analysis of critical nonlinear systems must be applied. This result employs tools related to the Hopf Bifurcation Theorem in the construction of the stabilizing control laws, which may be taken purely linear. Simulations illustrate the nature of the conclusions, and show that nonlinear terms in the feedback can be used to significantly improve the transient response.

6.1. Introduction

The topic of Tethered Satellite Systems (TSS) has received considerable attention in recent years (e.g., [5], [8], [9], [12], [20]-[22], [24], [37]-[39], [66]-[73]). Potential applications of these systems include deployment and retrieval of satellites, aiding in space-assembly tasks, use of electrodynamic tethers for electric power generation [74, p. 4-259], and tethering platforms with an infrared telescope above the Space Station for observing stellar and planetary objects [74, p. 4-263]. For other potential applications, and a discussion of early research on tethered satellites, the reader is referred to Rupp and Laue [81].

In this chapter, we focus on the issue of stabilization of a tethered satellite system during the station-keeping mode. Specifically, consider the TSS depicted in Figure 1.1. Here, a large satellite is tethered to a smaller subsatellite and the configuration is in a circular orbit around the Earth. During station-keeping, a subsatellite is controlled so as to follow a prescribed orbit to within a set tolerance [59, p. 220]. By assuming the satellite to be much more massive than the subsatellite, and that the satellite follows a perfect circular orbit, we are able to focus attention on the station-keeping control of the subsatellite. This is accomplished through the design of tether tension control laws in feedback form which result in regulating the position of the subsatellite relative to the satellite, while simultaneously regulating the tether length at a prescribed nominal value. The proposed tension control law is implemented, say, using a reel-type mechanism.

The result makes use of several simplifying assumptions. For instance, the tether is assumed rigid and massless. With these assumptions, the TSS can be described by a set of ordinary differential equations. The system Lagrangian is used to obtain these equations. Next, we observe that linear feedback-type tension control laws can place all but two poles of the system. These two poles are a complex conjugate pair of pure imaginary eigenvalues of the system linearization. To stabilize the system, therefore, tools from stability analysis of critical nonlinear systems must be applied. Our approach is to use the technique of [1] as summarized in Section 2.4, in which Hopf bifurcation calculations are employed to construct stabilizing control laws. First, a class of purely linear stabilizing feedback control laws are given. Next, nonlinear stabilizing control laws are developed. Simulations are presented which allow one to compare the transient response of the system with the two types of feedback. The additional freedom afforded by the inclusion of nonlinear terms can be used to obtain a significant improvement in the speed of the transient response.

Notation

E - Earth

S - Satellite

m - Subsatellite and subsatellite mass

G - Gravitational constant

M, m_s - Mass of the Earth, mass of the satellite

(x_m, y_m, z_m) - Earth-based rotating Cartesian coordinates of subsatellite, with z_m in the local outgoing vertical direction, and x_m in the direction of motion of the satellite in its orbit (see Figure 6.1)

(x_s, y_s, z_s) - Earth-based rotating coordinates of the satellite

$(\hat{x}_m, \hat{y}_m, \hat{z}_m)$ - Inertial coordinates of subsatellite

$(\hat{x}_s, \hat{y}_s, \hat{z}_s)$ - Inertial coordinates of the satellite

Ω - Constant angular velocity of the satellite in circular orbit

θ, ϕ - In-plane angle and out-of-plane angle of subsatellite relative to the local vertical

$\omega_\phi := \dot{\phi}$, $\omega_\theta := \dot{\theta}$, ℓ - Tether length, $v := \dot{\ell}$

r_0, r_m - Radius of the satellite orbit, radius of subsatellite orbit

τ_θ, τ_ϕ - Generalized torques in directions θ, ϕ

F_ℓ - Generalized force along tether

$\mathbf{F} := \frac{\tau_\theta}{\ell\cos\phi}\hat{\theta} + \frac{\tau_\phi}{\ell}\hat{\phi} + F_\ell\hat{\ell}$, where a hat indicates a unit vector in the given direction

6.2. System Model

Referring to the depictions in Figures 1.1 and 6.1, a mathematical model of the TSS may be derived. Assume the satellite and the subsatellite are point masses and the tether is massless and rigid. Moreover, take the mass of the satellite to be much larger than that of the subsatellite (i.e., $m_s \gg m$). This allows us to take the center of mass of the TSS to be the satellite, and to consider the satellite as being in a circular orbit around the Earth. In addition, the gravitational attraction between the subsatellite and the satellite is neglected.

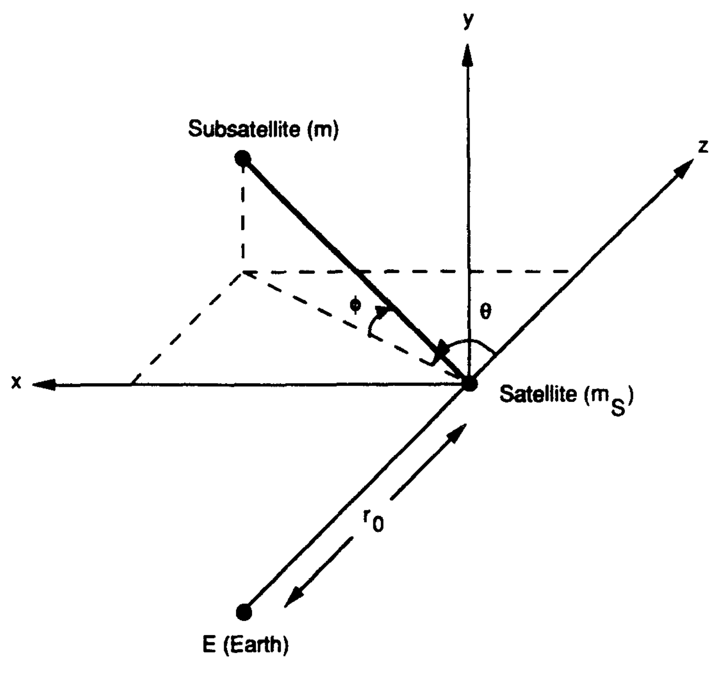

Figure 6.1. Rotating coordinate system

It is evident from Figure 6.1 that we have the relationships

$$x_m = \ell\cos\phi\sin\theta \tag{6.1}$$

$$y_m = \ell\sin\phi \tag{6.2}$$

$$z_m = r_0 + \ell \cos\phi \cos\theta \tag{6.3}$$

$$r_m^2 = r_0^2 + \ell^2 + 2r_0\ell\cos\phi\cos\theta. \tag{6.4}$$

Also,

$$\begin{pmatrix} \hat{x}_m \\ \hat{y}_m \\ \hat{z}_m \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & \sin\Omega t \\ 0 & 1 & 0 \\ -\sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} x_m \\ y_m \\ z_m \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_s \\ \hat{y}_s \\ \hat{z}_s \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 & \sin\Omega t \\ 0 & 1 & 0 \\ -\sin\Omega t & 0 & \cos\Omega t \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix}$$

where the equations above fix a particular choice of time reference.

Since the tether is assumed to be massless, the total kinetic energy of the system is given by

$$\begin{aligned} KE &= \frac{1}{2}m_s(\dot{\hat{x}}_s^2 + \dot{\hat{y}}_s^2 + \dot{\hat{z}}_s^2) + \frac{1}{2}m(\dot{\hat{x}}_m^2 + \dot{\hat{y}}_m^2 + \dot{\hat{z}}_m^2) \\ &= \frac{1}{2}m_s\Omega^2 r_0^2 + \frac{1}{2}m\{\dot{\ell}^2 + \ell^2\dot{\phi}^2 + \ell^2\cos^2\phi(\dot{\theta}+\Omega)^2 + \Omega^2 r_0^2 \\ &+ 2\Omega r_0\dot{\ell}\cos\phi\sin\theta - 2\Omega r_0\ell\sin\phi\sin\theta\dot{\phi} + 2\Omega r_0\ell\cos\phi\cos\theta(\dot{\theta}+\Omega)\} \end{aligned} \tag{6.5}$$

The potential energy of the system arises solely from gravity and is given by

$$PE = -\frac{GMm_s}{r_0} - \frac{GMm}{r_m}.$$

Moreover, since the satellite is assumed to be in a circular orbit, it is in a zero-g orbit. Thus

$$\frac{GMm_s}{r_0^2} = m_s\Omega^2 r_0,$$

or more succinctly $GM = \Omega^2 r_0^3$. Writing the expression for the system Lagrangian $L = KE - PE$ and invoking the Lagrangian formulation of dynamics, the dynamic equations of the system are found to be

$$\tau_\theta = m\ell^2\cos^2\phi\{\ddot{\theta} + 2\frac{\dot{\ell}}{\ell}(\dot{\theta}+\Omega) - 2\tan\phi(\dot{\theta}+\Omega)\dot{\phi}$$

$$+ \frac{\Omega^2 r_0 \sin\theta}{\ell\cos\phi}(1 - \frac{r_0^3}{r_m^3})\} \tag{6.6}$$

$$\tau_\phi = m\ell^2\{\ddot{\phi} + 2\frac{\dot{\ell}}{\ell}\dot{\phi} + \cos\phi\sin\phi(\dot{\theta}+\Omega)^2$$

$$+ \frac{\Omega^2 r_0}{\ell}\cos\theta\sin\phi(1 - \frac{r_0^3}{r_m^3})\} \tag{6.7}$$

$$F_\ell = m\{\ddot{\ell} - \ell(\dot{\phi})^2 - \ell\cos^2\phi(\dot{\theta}+\Omega)^2$$

$$+ \frac{\Omega^2 r_0^3 \ell}{r_m^3} - \Omega^2 r_0\cos\phi\cos\theta(1 - \frac{r_0^3}{r_m^3})\}. \tag{6.8}$$

For the limiting case $r_0 \gg \ell$, we have $r_m \simeq r_0$, and Eq. (6.4) implies

$$1 - \frac{r_0^3}{r_m^3} \simeq 3\cos\phi\cos\theta\frac{\ell}{r_0}. \tag{6.9}$$

According to Eq. (6.9), the approximate motion equation of the system for the case $r_0 \gg \ell$ is found to be

$$\mathbf{F} = m\hat{\ell}\{\ddot{\ell} - \ell\dot{\phi}^2 - \ell\cos^2\phi(\dot{\theta}+\Omega)^2 + \ell\Omega^2 - 3\Omega^2\ell\cos^2\phi\cos^2\theta\}$$

$$+ m\hat{\theta}\{\ddot{\theta}\ell\cos\phi + 2(\dot{\theta}+\Omega)(\dot{\ell}\cos\phi - \ell\dot{\phi}\sin\phi) + 3\ell\Omega^2\cos\theta\cos\phi\sin\theta\}$$

$$+ m\hat{\phi}\{\ell\ddot{\phi} + 2\dot{\ell}\dot{\phi} + \ell\cos\phi\sin\phi(\dot{\theta}+\Omega)^2 + 3\ell\Omega^2\cos^2\theta\cos\phi\sin\phi\},$$

which agrees with the model derived by Arnold [5] using the gravity gradient method. Note that in the analysis of the following sections, we do not assume $r_0 \gg \ell$.

6.3. Analysis and Control in the Station-Keeping Mode

6.3.1. Model in State-Space Form

Suppose $\cos\phi \neq 0$ (i.e., $\phi \neq \pm\frac{\pi}{2}$) and let the applied tension force be the only external force acting on the system. Thus, for instance, we neglect effects of a rotating atmosphere, the Earth's magnetic force, solar radiation, and the oblateness of the Earth. We do not take into account the mechanism for commanding the desired tether tension, although one can imagine it to be controlled by a reel mechanism.

Since (i) we have modeled satellite and subsatellite as point masses, (ii) the tether is assumed rigid, and (iii) there are no external forces besides the commanded tether tension, we conclude that the generalized forces acting on the subsatellite are $F_\ell = T$, $\tau_\theta = 0$, $\tau_\phi = 0$. Eqs. (6.6)-(6.8) can now be rewritten in state space form as follows:

$$\dot{\phi} = \omega_\phi \tag{6.10}$$

$$\dot{\omega}_\phi = -\frac{2v}{\ell}\omega_\phi - \frac{1}{2}\sin(2\phi)(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0}{\ell}\cos\theta\sin\phi(1 - \frac{r_0^3}{r_m^3}) \tag{6.11}$$

$$\dot{\theta} = \omega_\theta \tag{6.12}$$

$$\dot{\omega}_\theta = -\frac{2v}{\ell}(\omega_\theta + \Omega) + 2\tan\phi(\omega_\theta + \Omega)\omega_\phi - \frac{\Omega^2 r_0 \sin\theta}{\ell\cos\phi}(1 - \frac{r_0^3}{r_m^3}) \tag{6.13}$$

$$\dot{\ell} = v \tag{6.14}$$

$$\begin{aligned}\dot{v} = {} & \ell\omega_\phi^2 + \ell\cos^2\phi(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0^3 \ell}{r_m^3} \\ & + \Omega^2 r_0 \cos\theta\cos\phi(1 - \frac{r_0^3}{r_m^3}) + \frac{T}{m}.\end{aligned} \tag{6.15}$$

For the case in which the tether length is held constant (i.e., $\dot{\ell} = v = 0$, $\ell = \ell^* =$ a constant), the conditions for an equilibrium point are $\theta = n\pi$ and $\phi = m\pi$, where n, m are integers. (We disregard another apparent possibility for ϕ, since the corresponding equation has no solution for ϕ when $\theta = n\pi$.) At the equilibrium point $(0,0,0,0)$ when only Eqs. (6.10)-(6.13) are considered with $v = 0$, the linearized system of Eqs. (6.10)-(6.13) has the two conjugate pairs of pure imaginary eigenvalues

$$\begin{aligned}\lambda_{1,2} &= \pm i\Omega\sqrt{1 + \frac{r_0}{\ell^*}(1 - \frac{r_0^3}{r_{m,0}^3})}, \quad \text{and} \\ &\simeq \pm i2\Omega, \qquad \text{for } r_0 \gg \ell^*\end{aligned} \tag{6.16}$$

$$\begin{aligned}\lambda_{3,4} &= \pm i\Omega\sqrt{\frac{r_0}{\ell^*}(1 - \frac{r_0^3}{r_{m,o}^3})}, \\ &\simeq \pm i\sqrt{3}\Omega, \qquad \text{for } r_0 \gg \ell^*\end{aligned} \tag{6.17}$$

where

$$r_{m,0} := r_0 + \ell^*. \tag{6.18}$$

The eigenvalues $\lambda_{1,2}$ are associated with Eqs. (6.10) and (6.11), i.e., with the out-of-plane dynamics, and $\lambda_{3,4}$ are associated with Eqs. (6.12) and (6.13), i.e., with the in-plane dynamics.

The appearance of pure imaginary eigenvalues suggests the possibility of oscillations near the equilibrium point $(0,0,0,0)$. Specifically, if the reel mechanism acts like a latch, resulting in a fixed tether length, the system may have librations with two distinct frequencies along with the orbital motion.

In the sequel, we consider the problem of designing tension control laws rendering the TSS asymptotically stable in the station-keeping mode. The main difficulty will be the presence of the two pairs of pure imaginary eigenvalues, and the uncontrollability of one of these pairs.

The conditions for an equilibrium point of system (6.10)-(6.15) are

$$0 = \Omega^2 \sin\phi\{\,\cos\phi + \frac{r_0}{\ell}\cos\theta(1 - \frac{r_0^3}{r_m^3})\,\} \tag{6.19}$$

$$0 = \frac{\Omega^2 r_0 \sin\theta}{\ell\cos\phi}(1 - \frac{r_0^3}{r_m^3}) \tag{6.20}$$

$$0 = \ell\Omega^2\cos^2\phi - \frac{\Omega^2 r_0^3 \ell}{r_m^3} + \Omega^2 r_0\cos\theta\cos\phi(1 - \frac{r_0^3}{r_m^3}) + \frac{T}{m} \tag{6.21}$$

where the applied tension force T (applied through a reel mechanism) may be constrained to satisfy (6.21). From the definition of out-of-plane angle ϕ, we have $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. There are hence only two equilibrium points: (0, 0, 0, 0, ℓ^*, 0) and (0, 0, π, 0, ℓ^*, 0) if the tether length is fixed at, say, $\ell = \ell^*$. In our paper [3], it is observed that the set $\phi = 0$, $\omega_\phi = 0$ is an invariant manifold for Eqs. (6.10)-(6.15), regardless of the form of the tension control law T. Although this implies that the system (6.10)-(6.15) is uncontrollable, we find below that there does exist a control strategy stabilizing the system. With an assumed rigid and massless tether, there may appear to be no constraint on the value of the applied tension force T. In reality, however, the tether is not rigid. Thus the subsatellite cannot be pushed away from the satellite by the applied tension force through the tether. Note that, in this chapter, the sign convention implies that a positive value of tension would correspond to a slack tether. Hence, in the sequel we restrict the applied tension force T to assume only nonpositive values. Although the conclusions we will reach will also apply to the *model* (6.10)-(6.15) in the absence of this restriction, they would no longer relate to the physical problem.

6.3.2. Stabilization for In-Plane Angle Near $\theta = 0$

Let $x_0 = (0, 0, 0, 0, \ell^*, 0)'$ and $X = x - x_0$, where $x = (\phi,\ \omega_\phi,\ \theta,\ \omega_\theta,\ \ell,\ v)'$. Then the Taylor expansion of the right side of Eqs. (6.10)-(6.15) is, to third order in X,

$$\frac{d}{dt}X = L_0 X + Q_0(X, X) + C_0(X, X, X) + eU + e\frac{T}{m}, \tag{6.22}$$

where the matrix L_0, the quadratic form Q_0, the cubic form C_0, the vector e and the scalar U are given, in terms of parameters a_i defined in Appendix 6.A, by

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_2^2 & 0 & 0 & -\frac{2\Omega}{\ell^*} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2\ell^*\Omega & a_3 & 0 \end{pmatrix},$$

$$Q_0(X, X) = \begin{pmatrix} 0 \\ -2\Omega\phi\omega_\theta + a_{12}\phi\tilde{\ell} - \frac{2}{\ell^*}\omega_\phi v \\ 0 \\ a_{12}\theta\tilde{\ell} - \frac{2}{\ell^*}\omega_\theta v + 2\Omega\phi\omega_\phi + \frac{2\Omega}{\ell^{*2}}\tilde{\ell}v \\ 0 \\ a_4\theta^2 + \ell^*\omega_\theta^2 + 2\Omega\omega_\theta\tilde{\ell} + a_5\phi^2 + \ell^*\omega_\phi^2 + a_{13}\tilde{\ell}^2 \end{pmatrix},$$

$$C_0(X,X,X) = \begin{pmatrix} 0 \\ a_6\theta^2\phi - \omega_\theta^2\phi + a_7\phi^3 + a_8\phi\tilde{\ell}^2 + \frac{2}{\ell^{*2}}\omega_\phi\tilde{\ell}v \\ 0 \\ a_9\theta^3 + a_{10}\theta\phi^2 + a_8\theta\tilde{\ell}^2 + 2\phi\omega_\phi\omega_\theta + \frac{2}{\ell^{*2}}\omega_\theta\tilde{\ell}v - \frac{2\Omega}{\ell^{*3}}\tilde{\ell}^2 v \\ 0 \\ a_{14}\theta^2\tilde{\ell} + \omega_\theta^2\tilde{\ell} - 2\ell^*\Omega\omega_\theta\phi^2 + a_{11}\phi^2\tilde{\ell} + \omega_\phi^2\tilde{\ell} + a_{15}\tilde{\ell}^3 \end{pmatrix},$$

$$e = (0,0,0,0,0,1)'$$

$$U = \frac{(3r_0^2\ell^* + 3r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0+\ell^*)^2}$$

where $\tilde{\ell} := \ell - \ell^*$. The expressions above have been verified using the code MACSYMA.[1]

Case 6.1: Linear State Feedback

Our first design is that of a tension control law in linear state feedback form which stabilizes the system. The design is carried out in two steps. The first, addressed in Lemma 6.1 and Corollary 6.1 below, is to give conditions on the linear state feedback ensuring that four of the eigenvalues of system (6.10)-(6.15) are moved to the left half of the complex plane. These eigenvalues correspond to the "in-plane variables" (θ, ω_θ, $\tilde{\ell}$, v).

Lemma 6.1. If the following conditions hold, then the tension control force $T = m(-U - k_1\theta - k_2\omega_\theta - k_3\tilde{\ell} - k_4 v)$ stabilizes the "in-plane Jacobian matrix" at the equilibrium point x_0, i.e., the Jacobian matrix of Eqs. (6.12)-(6.15) with respect to the vector $(\theta, \omega_\theta, \tilde{\ell}, v)$:

(i) $k_4 > 0$

(ii) $b_1, b_2, b_3 > 0$

(iii) $k_4 b_1 b_2 - b_2^2 - k_4^2 b_3 > 0$

where

$$b_1 = k_3 - \frac{(2r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0+\ell^*)^3} - \frac{2\Omega k_2}{\ell^*} + 4\Omega^2, \tag{6.23}$$

[1] MACSYMA is a trademark of Symbolics, Inc., Cambridge, MA.

$$b_2 = \frac{k_4(3r_0^3 + 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^3} - \frac{2\Omega k_1}{\ell^*}, \tag{6.24}$$

$$b_3 = \frac{(3r_0^3 + 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^3}(k_3 - \frac{(3r_0^3 + 3r_0^2\ell^* + 3r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0 + \ell^*)^3}). \tag{6.25}$$

Sketch of proof: The linearization of (6.22) upon application of the linear state feedback T above is given by

$$\frac{d}{dt}X = \tilde{L}_0 X,$$

where

$$\tilde{L}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_2^2 & 0 & 0 & -\frac{2\Omega}{\ell^*} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -k_1 & -k_2 + 2\ell^*\Omega & -k_3 + a_3 & -k_4 \end{pmatrix}.$$

Hence the characteristic equation of the closed-loop system is

$$(\lambda^2 + a_1^2)\,(\lambda^4 + k_4\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3) = 0, \tag{6.26}$$

where b_i, $i = 1, \ldots, 3$ are as defined in Eqs. (6.23)-(6.25). The lemma follows readily by applying the Routh-Hurwitz test to the second polynomial factor in the left side of Eq. (6.26).

The next result follows readily from Lemma 6.1, and demonstrates that the set of feedbacks of Lemma 6.1 is not vacuous, at least in the practically interesting case $r_0 \gg \ell^*$. The result also holds for larger ℓ^*, but the corresponding conditions on the feedback gains become very complicated.

Corollary 6.1. If $r_0 \gg \ell^*$, the conclusion of Lemma 6.1 holds when any of the following three conditions is satisfied:

(i) $0 \geq k_1 > \ell^*\Omega k_4(1 - \frac{k_3}{2\Omega^2})$, $k_2 = 0$, $k_3 > 3\Omega^2$, and $k_4 > 0$;

(ii) $0 > k_1 > k_2 k_4$, $k_2 < 0$, $k_3 > 3\Omega^2$, and $k_4 > 0$;

(iii) $k_1 = 0$, $k_2 < \min\{2\Omega\ell^*, \frac{\ell^*}{2\Omega}k_3 + 3\Omega^2\}$, $k_3 > 3\Omega^2$, and $k_4 > 0$.

From the closed-loop characteristic equation (6.26), we infer that the system has an uncontrollable pair of pure imaginary eigenvalues. Moreover, it is easy to see that even when the states ϕ and ω_ϕ are used in the tension control law, these two pure imaginary eigenvalues remain fixed. Hence the stability of the closed-loop system cannot be identified from the linearized model alone.

The closed-loop system (upon application of a feedback law as in Lemma 6.1 or Corollary 6.1) is approximated, to third order in X, by

$$\frac{d}{dt}X = \tilde{L}_0 X + Q_0(X, X) + C_0(X, X, X) \tag{6.27}$$

in which $\tilde{L}_0$ has a complex conjugate pair of pure imaginary eigenvalues, with the remaining eigenvalues in the left half of the complex plane. This situation is an example of a *critical case* in nonlinear stability analysis. Its resolution may be approached via results on Hopf bifurcation for one-parameter families of nonlinear systems (see, e.g., [34]). The local asymptotic stability of the origin of Eq. (6.27) can be concluded from the negativity of an associated "stability coefficient," often denoted by β_2. The value of this coefficient can be obtained in several ways. One can, for instance, study normal forms of Eq. (6.27). Alternatively, one observes that, in the situation at-hand, smooth parametrizations of (6.27) will typically exhibit a Hopf bifurcation. The stability of the bifurcated periodic solutions, as well as that of the origin of (6.27), follows from the negativity of the Floquet exponents of these periodic solutions. The stability coefficient β_2 can be obtained as the leading coefficient in an asymptotic expansion of the critical Floquet exponent. The coefficient β_2 may be computed systematically. When $\beta_2 < 0$, the equilibrium point is locally asymptotically stable, while $\beta_2 > 0$ implies instability of the equilibrium. The case $\beta_2 = 0$ is inconclusive regarding stability. We now proceed to use an algorithm for the computation of β_2 as given in Lemma 2.9 (see for instance [1], [34]) to determine the dependence of β_2 on the gains k_i, $i = 1, \ldots, 4$.

Denote by r and l the right (column) and left (row) eigenvectors, respectively, of $\tilde{L}_0$ corresponding to the imaginary eigenvalue ia_1. Requiring $lr = 1$, we have

$$r = (1, ia_1, 0, 0, 0, 0)' \tag{6.28}$$

$$l = (\frac{1}{2}, -i\frac{1}{2a_1}, 0, 0, 0, 0). \tag{6.29}$$

Next, solve the equations

$$-\tilde{L}_0 a = \frac{1}{2} Q_0(r, \bar{r}) \tag{6.30}$$

$$(2ia_1 I - \tilde{L}_0)b = \frac{1}{2} Q_0(r, r) \tag{6.31}$$

for the vectors a and b. Here, an overbar denotes complex conjugation and I denotes the identity matrix. We find

$$a = (0, 0, 0, 0, -\frac{a_5 - \ell^* a_1^2}{2(a_3 - k_3)}, 0)' \tag{6.32}$$

$$b = (0, 0, c_1, c_2, c_3, c_4)', \tag{6.33}$$

where the a_i are as in Appendix 6.A, and the c_i and d_i are given by

$$c_1 = \frac{1}{d_1 + id_2}\{\frac{a_5}{2} + \frac{\ell^* a_1^2}{2} - \frac{\ell^*}{4}(k_3 - a_3) - \frac{i}{2} k_4 a_1 \ell^*\} \tag{6.34}$$

$$c_2 = 2ia_1 c_1 \tag{6.35}$$

$$c_3 = \frac{\ell^*}{4} - \frac{i\ell^*(4a_1^2 - a_2^2)}{4a_1 \Omega} c_1 \tag{6.36}$$

$$c_4 = \frac{i\ell^* a_1}{2} + \frac{\ell^*(4a_1^2 - a_2^2)}{2\Omega} c_1 \tag{6.37}$$

and

$$d_1 = k_1 + \frac{\ell^* k_4}{2\Omega}(4a_1^2 - a_2^2) \tag{6.38}$$

$$d_2 = 2a_1(k_2 - 2\ell^*\Omega) + \frac{\ell^*(4a_1^2 - a_2^2)}{4a_1\Omega}(a_3 + 4a_1^2 - k_3). \tag{6.39}$$

In general, the value of the bifurcation stability coefficient β_2 is given by the formula

$$\beta_2 = 2Re\{2lQ_0(r, a) + lQ_0(\bar{r}, b) + \frac{3}{4} lC_0(r, r, \bar{r})\}. \tag{6.40}$$

In our case, we then have

$$\beta_2 = \frac{1}{a_1}\{-\Omega Im(c_2) + \frac{(6r_0^3 + 4r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{2(r_0+\ell^*)^4} Im(c_3) + \frac{a_1}{\ell^*} Re(c_4)\}$$
$$= \frac{d_3}{a_1}\cdot Re(c_1), \tag{6.41}$$

where (assuming $\ell^* < r_0$)

$$d_3 = -2a_1\Omega - \frac{\ell^*(4a_1^2 - a_2^2)}{8a_1\Omega}\frac{(6r_0^3 + 4r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0+\ell^*)^4}$$
$$+ \frac{a_1(4a_1^2 - a_2^2)}{2\Omega} > 0 \tag{6.42}$$

$$Re(c_1) = \frac{1}{d_1^2 + d_2^2}\{(\frac{a_5}{2} + \frac{\ell^* a_1^2}{2} - \frac{\ell^*}{4}(k_3 - a_3))d_1 - \frac{1}{2}k_4 a_1 \ell^* d_2\}. \tag{6.43}$$

Since $a_1 > 0$, $d_3 > 0$, and since stability is implied by $\beta_2 < 0$, we have the following result.

Theorem 6.1. If a linear state feedback controller T as defined in Lemma 6.1 is applied, with

$$(\frac{a_5}{2} + \frac{\ell^* a_1^2}{2} - \frac{\ell^*}{4}(k_3 - a_3))d_1 - \frac{1}{2}k_4 a_1 \ell^* d_2 < 0,$$

then the equilibrium point x_0 will be rendered asymptotically stable for the system (6.10)-(6.15).

The next result readily follows from Theorem 6.1 and Corollary 6.1.

Corollary 6.2. If $r_0 \gg \ell^*$ and a linear state feedback control T as in Lemma 6.1 is applied, with either of the following three conditions satisfied, then the conclusion of Theorem 6.1 holds:

(i) $0 \geq k_1 > \ell^*\Omega k_4(1 - \frac{k_3}{2\Omega^2})$, $k_2 = 0$, $3\Omega^2 < k_3 < 14\Omega^2$, and $k_4 > 0$;

(ii) $0 > k_1 > k_2 k_4$, $0 > k_2 > 2\ell^*\Omega - \frac{13\ell^*\Omega}{32}(19 - \frac{k_3}{\Omega^2})$, $3\Omega^2 < k_3 < 14\Omega^2$, and $k_4 > 0$;

(iii) $k_1 = 0$, $\min\{2\Omega\ell^*, 3\Omega^2\} > k_2 > 2\ell^*\Omega - \frac{13\ell^*\Omega}{32}(19 - \frac{k_3}{\Omega^2})$, $3\Omega^2 < k_3 < 14\Omega^2$, and $k_4 > 0$.

Case 6.2: Nonlinear State Feedback

Next we present a result on stabilization with a tension control law including both linear and nonlinear terms. The nonlinear terms introduce more flexibility in the design, and, as will be seen in Section 6.4, can lead to superior transient response.

By computing the eigenvectors l and r and using the formula (6.40) for β_2, we find that *any cubic term in the feedback has no effect on the value of* β_2. We are therefore led to hypothesize a *feedback containing only linear and quadratic terms*. The component of the closed-loop quadratic term $Q_0(X, X)$ depending on the states θ, ω_θ, $\tilde{\ell}$ or v is also found to have no effect on β_2. The next theorem gives conditions for a nonlinear feedback, of the form motivated by these observations, to be stabilizing.

Theorem 6.2. If condition (6.44) below holds, then the applied tension control force $T = m(-U - k_1\theta - k_2\omega_\theta - k_3\tilde{\ell} - k_4 v - q_1\phi^2 - q_2\phi\omega_\phi - q_3\omega_\phi^2)$ stabilizes the system (6.10)-(6.15), where the k_i, $i = 1, \ldots, 4$ satisfy the conditions of Lemma 6.1.

$$(\frac{-q_1 + a_5}{2} + (\frac{\ell^* + q_3}{2})a_1^2 - (k_3 - a_3)\frac{\ell^*}{4})d_1 + \frac{a_1}{2}(-q_2 - \ell^* k_4)d_2 < 0. \quad (6.44)$$

Here, d_1, d_2 are given in Eqs. (6.38) and (6.39), and the a_i are as specified in Appendix 6.A.

The proof entails checking the effect on the value of β_2 of adding the extra quadratic term $-(q_1\phi^2 + q_2\phi\omega_\phi + q_3\omega_\phi^2)$ in the last row of $Q_0(X, X)$.

As mentioned above, inclusion of nonlinear terms in the feedback control may be used to improve the transient response of the stabilized system. In particular, the rate at which system trajectories decay toward the equilibrium point may be significantly increased. Simulation evidence for this is given in Section 6.4.

It is not difficult to give analytical reasoning to support this conclusion, and to guide in the tuning of the linear and quadratic feedback gains. Assume $r_0 \gg \ell^*$, and use Eq. (6.40) with a feedback of the form given in Theorem 6.2 to ascertain the approximate formula

$$\beta_2 \simeq \frac{9\Omega}{2(d_1^2 + d_2^2)}\{(\frac{-q_1 + a_5}{2} + (\frac{\ell^* + q_3}{2})a_1^2 - (k_3 - a_3)\frac{\ell^*}{4})d_1 + \frac{a_1}{2}(-q_2 - \ell^* k_4)d_2\}. \tag{6.45}$$

Eq. (6.45) can be used to show that, if $r_0 \gg \ell^*$ and the linear gains k_i, $i = 1$, ..., 4, are chosen according to condition (i) of Corollary 6.2, with $k_1 = 0$, then β_2 may be rendered as negative as desired simply by setting the quadratic gains $q_2 = q_3 = 0$ and taking $q_1 > 0$ and sufficiently large. Thus the gains k_i may be used to place four of the eigenvalues of (6.10)-(6.15) in the left half of the complex plane, while, *independently*, the gains q_i, i = 1,2,3, are used to make β_2 negative and of large magnitude.

6.3.3. Stabilization for In-Plane Angle Near $\theta = \pi$

Similarly, now let $x_\pi = (0, 0, \pi, 0, \ell^*, 0)'$ denote the equilibrium point of interest, and $X = x - x_\pi$ be the differential state variation. Then the system (6.10)-(6.15), to third order near the equilibrium point x_π, may be written as follows

$$\frac{d}{dt}X = L_\pi X + Q_\pi(X, X) + C_\pi(X, X, X) + eU_\pi + e\frac{T}{m}.$$

Here, L_π, Q_π, C_π are as identified in Appendix 6.B. The next lemma is analogous to Lemma 6.1, and so is stated without proof.

Lemma 6.2. Let the applied tension force be of the form $T = m(-U_\pi - k_1\tilde{\theta} - k_2\omega_\theta - k_3\tilde{\ell} - k_4 v)$, where $\tilde{\theta} := \theta - \pi$ and $\omega_\theta := \dot{\tilde{\theta}} = \dot{\theta}$. Then the "in-plane" Jacobian matrix of Eqs. (6.12)-(6.15), i.e., the Jacobian of the right side of (6.12)-(6.15) with respect to $(\tilde{\theta}, \omega_\theta, \tilde{\ell}, v)$, will be stable at the equilibrium point x_π, if k_i, $i = 1, \ldots, 4$, satisfy the following conditions:

(i) $k_4 > 0$,

(ii) $h_1, h_2, h_3 > 0$, and

(iii) $k_4 h_1 h_2 - h_2^2 - k_4^2 h_3 > 0$.

Here, the auxiliary parameters h_1, h_2, h_3 are given by

$$h_1 = k_3 - \frac{(2r_0\ell^{*2} - \ell^{*3})\Omega^2}{(r_0 - \ell^*)^3} - \frac{2\Omega k_2}{\ell^*} + 4\Omega^2, \tag{6.46}$$

$$h_2 = \frac{k_4(3r_0^3 - 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 - \ell^*)^3} - \frac{2\Omega k_1}{\ell^*}, \tag{6.47}$$

$$h_3 = \frac{(3r_0^3 - 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 - \ell^*)^3}(k_3 - \frac{(3r_0^3 - 3r_0^2\ell^* + 3r_0\ell^{*2} - \ell^{*3})\Omega^2}{(r_0 - \ell^*)^3}) \tag{6.48}$$

Corollaries 1 and 2 remain valid. The detailed statements need not be given.

6.4. Simulation Results

A TSS with the following characteristics is considered:

- Nominal tether length $\ell^* = 100$ km,
- Orbital radius $r_0 = 6598$ km,
- Satellite mass $m = 170$ kg,
- Orbital angular velocity $\Omega = 0.0011781$ radians/second.

Let the equilibrium point of interest of (6.10)-(6.15) be $x_0 = (0, 0, 0, 0, \ell^*, 0)'$. Simulation results will now be presented which illustrate the system dynamics for the various types of control studied in this chapter.

Let the initial conditions of the system be $\phi = 0.01$ radians, $\theta = -0.01$ radians, and $\omega_\theta = \omega_\phi = 0$.

Example 6.1. (No tension control: latch mechanism)

Suppose the reel mechanism acts like a latch fixing ℓ at ℓ^*. The system response for the in-plane angle θ and the out-of-plane angle ϕ are shown in Figures 6.2(a) and 6.2(b), respectively. We observe an apparent undamped oscillation near the equilibrium point x_0.

Example 6.2. (Linear stabilizing feedback)

The tension controller is taken as $T = -m(U + k_3\tilde{\ell} + k_4 v)$, with $k_3 = 3.1\Omega^2$, $k_4 = 0.0034$, and $U = 0.41019$. The control law is stabilizing, as can be checked using Theorem 6.1. Indeed, $\beta_2 \simeq -0.0004$ for the closed-loop system. The response of the variables of ϕ, θ, and the deviation $\tilde{\ell}$ of the tether length are shown in Figures 6.3(a), 6.3(b) and 6.3(c), respectively. However, it is not easy to see in Figure 6.3(a) any decay of the oscillation in the out-of-plane angle ϕ. This may be attributed to the fact that $|\beta_2|$ is small. The applied tension force is shown in Figure 6.3(d).

Example 6.3. (Linear-plus-quadratic stabilizing feedback)

Let the tension control law be of the form

$$T = -m(U + k_3\tilde{\ell} + k_4 v + q_1\phi^2 + q_2\phi\omega_\phi + q_3\omega_\phi^2), \tag{6.49}$$

where $U = 0.41019$. The out-of-plane angle ϕ decays when $k_3 = 3.1\Omega^2$, $k_4 = 0.0034$, $q_1 = 1500$, and $q_2 = q_3 = 0$ as, shown in Figure 6.4(a). However, this is at the expense of large variations in θ and $\tilde{\ell}$, as depicted in Figures 6.4(b) and 6.4(c). The applied tension force is shown in Figure 6.4(d).

Example 6.4. (Linear-plus-quadratic stabilizing feedback)

A further example is depicted in Figure 6.5. In this example, k_3, k_4 and q_3 are unchanged from their previous values (given in Example 6.3), but now $q_1 = 0$, and $q_2 = 10^6$.

Example 6.5. (Switching-type stabilizing feedback)

Figure 6.6 relates to an example invoking a switching control strategy. The nonlinear feedback control law of Example 6.3 is used for the first 5 hours of the simulation. Then the control law is switched to a purely linear feedback with the parameters values specified in Example 6.2.

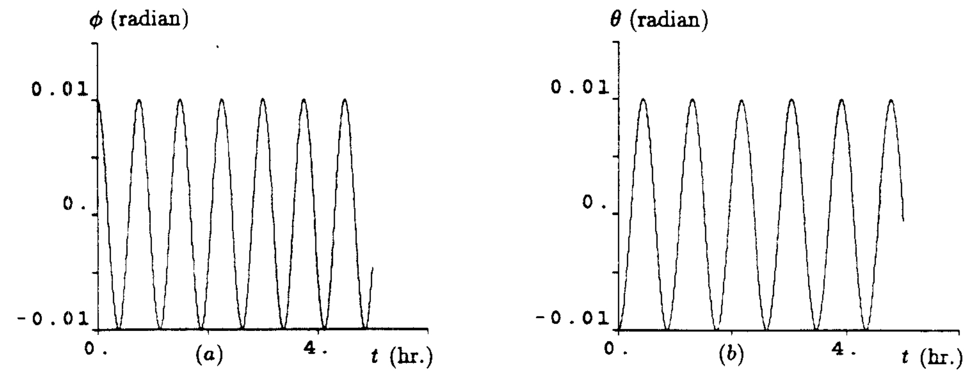

Figure 6.2. Simulation results for uncontrolled system

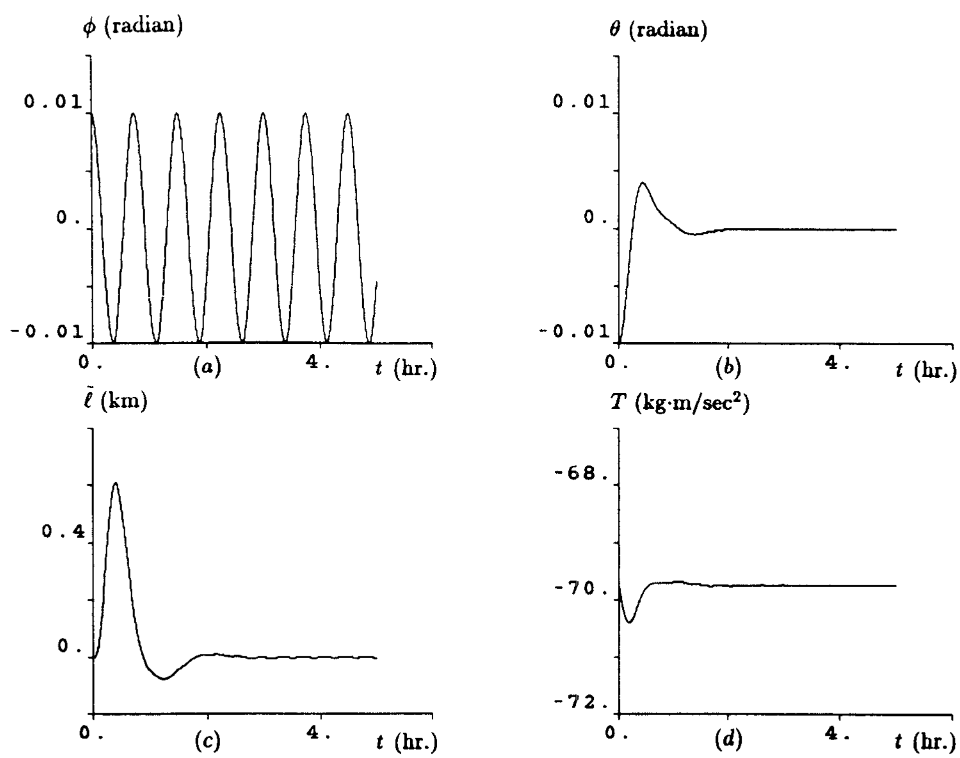

Figure 6.3. Simulation results for linear feedback system

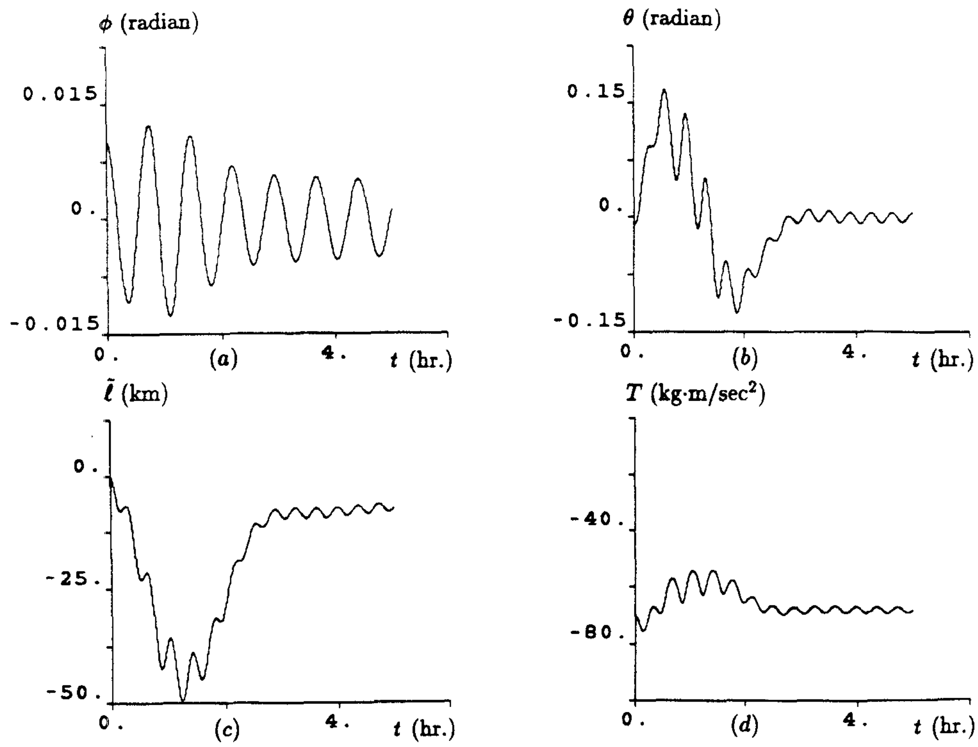

Figure 6.4. Simulation results for nonlinear feedback system ($q_1 = 1500$)

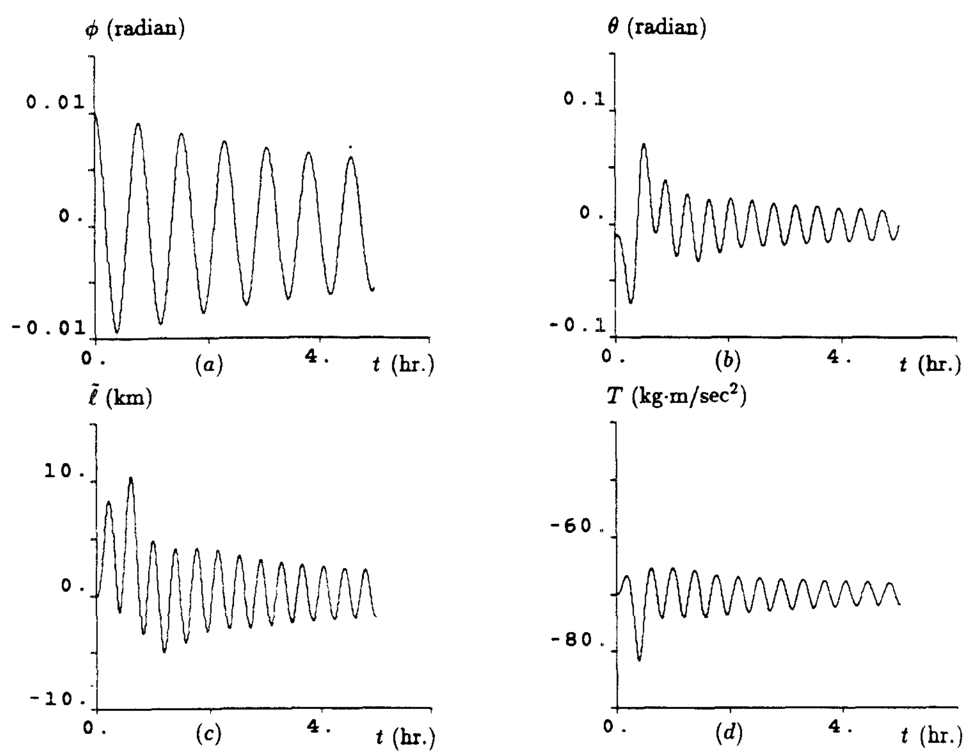

Figure 6.5. Simulation results for nonlinear feedback system ($q_2 = 10^6$)

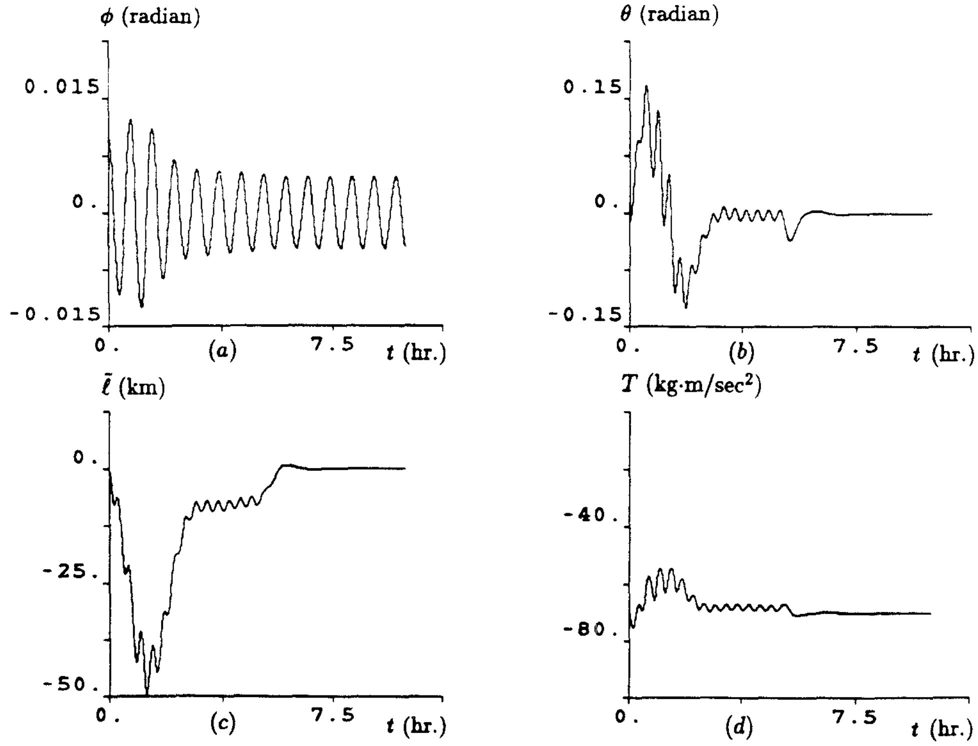

Figure 6.6. Simulation results for switching-controlled system

6.5. Concluding Remarks

In this chapter, we have presented analytical designs of tension feedback control laws for the stabilization of the tethered satellite system during station-keeping. These designs are based on calculations related to Hopf bifurcation stability. The calculations have been performed for a model of the tethered satellite system derived under several simplifying assumptions. This model is characterized by its nonlinearity and the existence of two critical modes, one of which cannot be removed by (linear) feedback. Notwithstanding this fact, we have been able to construct stabilizing controllers using linear and/or quadratic feedback. Cubic terms were not included in the feedback laws since the nonlinear stability calculations indicated that their effect might be of only secondary significance. Moreover, simulation was used to demonstrate the validity of the analytical designs. The simulations also indicated the importance of quadratic feedback of the out-of-plane angle ϕ and/or the out-of-plane angular rate ω_ϕ in improving the transient response of the out-of-plane variables, i.e., in dampening the roll oscillations.

Regarding the issue of how to achieve further improvements in the transient response, several possibilities arise. Optimization-based computer-aided design tools can be applied to systematically search for linear and/or nonlinear control gains resulting in a suboptimal transient response. If other actuators, such as subsatellite thrusters or tether base movement, are available in addition to tether tension control, then one expects improved transient performance.

Appendix 6.A

The values of the coefficients a_i, $i = 1, \ldots, 15$ are as listed below.

$$a_1 = \left(\frac{(4r_0^3 + 6r_0^2\ell^* + 4r_0\ell^{*2} + \ell^{*3})}{(r_0 + \ell^*)^3}\right)^{1/2}\Omega$$

$$a_2 = \left(\frac{(3r_0^3 + 3r_0^2\ell^* + r_0\ell^{*2})}{(r_0 + \ell^*)^3}\right)^{1/2}\Omega$$

$$a_3 = \frac{(3r_0^3 + 3r_0^2\ell^* + 3r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0 + \ell^*)^3}$$

$$a_4 = -\frac{(6r_0^4\ell^* + 6r_0^3\ell^{*2} + 4r_0^2\ell^{*3} + r_0\ell^{*4})\Omega^2}{2(r_0 + \ell^*)^4}$$

$$a_5 = -\frac{(8r_0^4\ell^* + 14r_0^3\ell^{*2} + 16r_0^2\ell^{*3} + 9r_0\ell^{*4} + 2\ell^{*5})\Omega^2}{2(r_0 + \ell^*)^4}$$

$$a_6 = \frac{(6r_0^5 + 9r_0^4\ell^* + 10r_0^3\ell^{*2} + 5r_0^2\ell^{*3} + r_0\ell^{*4})\Omega^2}{2(r_0 + \ell^*)^5}$$

$$a_7 = \frac{(16r_0^5 + 29r_0^4\ell^* + 50r_0^3\ell^{*2} + 45r_0^2\ell^{*3} + 21r_0\ell^{*4} + 4\ell^{*5})\Omega^2}{6(r_0 + \ell^*)^5}$$

$$a_8 = -\frac{(10r_0^3 + 5r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^5}$$

$$a_9 = \frac{(12r_0^5 + 9r_0^4\ell^* + 10r_0^3\ell^{*2} + 5r_0^2\ell^{*3} + r_0\ell^{*4})\Omega^2}{6(r_0 + \ell^*)^5}$$

$$a_{10} = -\frac{(27r_0^4\ell^* + 30r_0^3\ell^{*2} + 15r_0^2\ell^{*3} + 3r_0\ell^{*4})\Omega^2}{6(r_0 + \ell^*)^5}$$

$$a_{11} = -\frac{(4r_0^5 + 2r_0^4\ell^* + 10r_0^3\ell^{*2} + 10r_0^2\ell^{*3} + 5r_0\ell^{*4} + \ell^{*5})\Omega^2}{(r_0 + \ell^*)^5}$$

$$a_{12} = \frac{(6r_0^3 + 4r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^4}$$

$$a_{13} = -\frac{3r_0^3\Omega^2}{(r_0 + \ell^*)^4}$$

$$a_{14} = -\frac{(3r_0^5 - 3r_0^4\ell^*)\Omega^2}{(r_0 + \ell^*)^5}$$

$$a_{15} = \frac{4r_0^3\Omega^2}{(r_0 + \ell^*)^5}$$

Appendix 6.B

The system model (6.10)-(6.15) is approximated, to third order in the states, near the equilibrium point x_π by

$$\dot{X} = L_\pi X + Q_\pi(X, X) + C_\pi(X, X, X) + eU_\pi + e\frac{T}{m}$$

where

$$L_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -f_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -f_2^2 & 0 & 0 & -\frac{2\Omega}{\ell^*} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2\ell^*\Omega & f_3 & 0 \end{pmatrix}$$

$$Q_\pi = \begin{pmatrix} 0 \\ -2\Omega\phi\omega_\theta + f_{12}\phi\tilde{\ell} - \frac{2}{\ell^*}\omega_\phi v \\ 0 \\ f_{12}\tilde{\theta}\tilde{\ell} - \frac{2}{\ell^*}\omega_\theta v + 2\Omega\phi\omega_\phi + \frac{2\Omega}{\ell^{*2}}\tilde{\ell}v \\ 0 \\ f_4\tilde{\theta}^2 + \ell^*\omega_\theta^2 + 2\Omega\omega_\theta\tilde{\ell} + f_5\phi^2 + \ell^*\omega_\phi^2 + f_{13}\tilde{\ell}^2 \end{pmatrix}$$

$$C_\pi = \begin{pmatrix} 0 \\ f_6\tilde{\theta}^2\phi - \omega_\theta^2\phi + f_7\phi^3 + f_8\phi\tilde{\ell}^2 + \frac{2}{\ell^{*2}}\omega_\phi\tilde{\ell}v \\ 0 \\ f_9\tilde{\theta}^3 + f_{10}\tilde{\theta}\phi^2 + f_8\tilde{\theta}\tilde{\ell}^2 + 2\phi\omega_\phi\omega_\theta + \frac{2}{\ell^{*2}}\omega_\theta\tilde{\ell}v - \frac{2\Omega}{\ell^{*3}}\tilde{\ell}^2 v \\ 0 \\ f_{14}\tilde{\theta}^2\tilde{\ell} + \omega_\theta^2\tilde{\ell} - 2\ell^*\Omega\omega_\theta\phi^2 + f_{11}\phi^2\tilde{\ell} + \omega_\phi^2\tilde{\ell} + f_{15}\tilde{\ell}^3 \end{pmatrix}$$

$$U_\pi = \frac{(3r_0^2\ell^* - 3r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0 - \ell^*)^2}$$

e is given in Section 6.3.2 and the values of f_i, $i = 1, \ldots, 15$ are

$$f_1 = \left(\frac{(4r_0^3 - 6r_0^2\ell^* + 4r_0\ell^{*2} - \ell^{*3})}{(r_0 - \ell^*)^3}\right)^{1/2}\Omega$$

$$f_2 = \left(\frac{(3r_0^3 - 3r_0^2\ell^* + r_0\ell^{*2})}{(r_0 - \ell^*)^3}\right)^{1/2}\Omega$$

$$f_3 = \frac{(3r_0^3 - 3r_0^2\ell^* + 3r_0\ell^{*2} - \ell^{*3})\Omega^2}{(r_0 - \ell^*)^3}$$

$$f_4 = -\frac{(6r_0^4\ell^* - 6r_0^3\ell^{*2} + 4r_0^2\ell^{*3} - r_0\ell^{*4})\Omega^2}{2(r_0 - \ell^*)^4}$$

$$f_5 = -\frac{(8r_0^4\ell^* - 14r_0^3\ell^{*2} + 16r_0^2\ell^{*3} - 9r_0\ell^{*4} + 2\ell^{*5})\Omega^2}{2(r_0 - \ell^*)^4}$$

$$f_6 = \frac{(6r_0^5 - 9r_0^4\ell^* + 10r_0^3\ell^{*2} - 5r_0^2\ell^{*3} + r_0\ell^{*4})\Omega^2}{2(r_0 - \ell^*)^5}$$

$$f_7 = \frac{(16r_0^5 - 29r_0^4\ell^* + 50r_0^3\ell^{*2} - 45r_0^2\ell^{*3} + 21r_0\ell^{*4} - 4\ell^{*5})\Omega^2}{6(r_0 - \ell^*)^5}$$

$$f_8 = -\frac{(10r_0^3 - 5r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 - \ell^*)^5}$$

$$f_9 = \frac{(12r_0^5 - 9r_0^4\ell^* + 10r_0^3\ell^{*2} - 5r_0^2\ell^{*3} + r_0\ell^{*4})\Omega^2}{6(r_0 - \ell^*)^5}$$

$$f_{10} = \frac{(27r_0^4\ell^* - 30r_0^3\ell^{*2} + 15r_0^2\ell^{*3} - 3r_0\ell^{*4})\Omega^2}{6(r_0 - \ell^*)^5}$$

$$f_{11} = -\frac{(4r_0^5 - 2r_0^4\ell^* + 10r_0^3\ell^{*2} - 10r_0^2\ell^{*3} + 5r_0\ell^{*4} - \ell^{*5})\Omega^2}{(r_0 - \ell^*)^5}$$

$$f_{12} = -\frac{(6r_0^3 - 4r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 - \ell^*)^4}$$

$$f_{13} = \frac{3r_0^3\Omega^2}{(r_0 - \ell^*)^4}$$

$$f_{14} = -\frac{(3r_0^5 + 3r_0^4\ell^*)\Omega^2}{(r_0 - \ell^*)^5}$$

$$f_{15} = \frac{4r_0^3\Omega^2}{(r_0 - \ell^*)^5}.$$

CHAPTER SEVEN

STATION-KEEPING CONTROL OF TSS: TWO CRITICAL MODES

In this chapter, we continue to study the stabilization of the TSS during station-keeping. It has been found in Chapter 6 that the out-of-plane angle of tethered satellite systems (TSS) is uncontrollable and difficult to stabilize during the station-keeping mode. (This was in the setting where only tension control is allowed.) A new method is proposed in this chapter to improve the time response of the system. As opposed to the design involving the Hurwitz stability criterion plus Hopf bifurcation theorem in Chapter 6, the new approach relies upon controlling both the in-plane angle and the out-of-plane angle by invoking a nonlinear stability criterion for a fourth-order nonlinear critical system whose linearized model has two distinct pairs of nonzero pure imaginary eigenvalues. Both linear and nonlinear feedback control laws are obtained to guarantee the stability of the system. However, simulations show that the nonlinear feedbacks are superior for having better time responses. Moreover, compared with the results given in the previous chapter, simulation results also indicate that a better performance can be achieved by the new technique with smaller nonlinear control gains and small linear feedback gains.

7.1. Introduction

In recent years, lots of issues have been published for the study of the Tethered Satellite Systems (TSS) (e.g., [5], [49], [66]-[71], [80], [81]). During station-keeping, a satellite is controlled so as to follow a prescribed orbit to within a set tolerance. It is observed in Chapter 6 that the system linearization has two distinct pairs of nonzero pure imaginary eigenvalues while the tether length is fixed. At the expense of large nonlinear control gains, a linear-plus-quadratic feedback tension control law is obtained in the previous chapter to provide the stability and the significant deçaying of the time response of the out-of-plane angle, but having less effect on the large variations of the in-plane angle and the tether length.

Although a linear feedback can be designed to stabilize all but a pair of nonzero pure imaginary eigenvalues, it is shown in the following that large linear feedback gains are needed to place these stabilizable eigenvalues far from the imaginary axis on the complex plane, which will induce the large quadratic feedback gains in the design of Chapter 6; while significant transient response is required. This observation might explain why the transient responses of the system given in the previous chapter, where we used small linear gains, are not practically acceptable. To improve the transient response by using such a design proposed in Chapter 6, one might expect to have large linear and quadratic feedback gains.

In this chapter, we propose a different technique such that we can improve the transient responses of the TSS during station-keeping without using large linear and nonlinear feedbacks. Our approach is based on a stability criterion for a class of fourth-order nonlinear systems whose Jacobian matrix has two distinct pairs of nonzero pure imaginary eigenvalues. In this new design, a linear feedback is first selected to preserve the two pairs of nonzero pure imaginary eigenvalues of the uncontrolled model of the system linearization and to make the rest of eigenvalues of the system stable. Then the same linear feedback and/or an extra nonlinear feedback will provide the stability of the full model of the system by employing the center manifold reduction technique and the stability criterion for the fourth-order nonlinear systems. It is shown that the quadratic feedback gains corresponding to the out-of-plane angle and its derivative play a very important role in deciding the stability of the out-of-plane angle. Moreover, the dynamics of the in-plane angle are found to be stabilizable by a cubic feedback instead of a linear feedback as in Chapter 6.

The development of this chapter is as follows. First, a stability criterion for a special class of fourth-order nonlinear critical systems whose linearized model has two distinct pairs of nonzero pure imaginary is abstracted from Corollary 4.5. It is followed by the recall of the equations of motion for the TSS during station-keeping obtained in Section 6.3. The possible constraints on the poles placement and the nonlinear stability coefficient by using the stabilizing control laws proposed in Chapter 6 are also discussed in Section 7.3.2. The stability criterion for the fourth-order nonlinear critical systems and the Center Manifold Theorem are then applied to design the new stabilizing control laws in Section 7.4. Compared with the results of Chapter 6, simulations presented in Section 7.5 demonstrate that a better performance can be achieved by smaller nonlinear gains and small linear gains; while using the current approach. Moreover, the variations of the in-plane angle and the tether length are found to be less than the ones shown in Chapter 6. Finally, concluding remarks pinpoint the main conclusions of this chapter.

7.2. Stability Criterion for a Class of Fourth-Order Nonlinear Critical Systems

In Section 4.2.3, we have derived a stability criterion for the fourth-order nonlinear critical systems given in (7.1) below by employing normal form formulation.

$$\dot{\eta} = A\eta + F(\eta), \tag{7.1}$$

where $\eta := (x, y, z, w)'$, $F(\eta) = (f(\eta), g(\eta), r(\eta), s(\eta))'$ and

$$A = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ -\Omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_3 \\ 0 & 0 & -\Omega_4 & 0 \end{pmatrix}. \tag{7.2}$$

Here, $\Omega_1\Omega_2, \Omega_3\Omega_4 > 0$ and f, g, r, s are smooth, purely nonlinear scalar functions. For the interest of this chapter, we consider a special case of system (7.1) by letting $f(\eta) = 0$ and $g_{xw}xw$ being the only quadratic term for nonlinear function $g(\eta)$.

The next result readily follows from Corollary 4.5.

Corollary 7.1. Let $f(\eta) = 0$, $g_{xw}xw$ be the only quadratic term for nonlinear function $g(\eta)$, and $\Omega_1\Omega_2 \neq \alpha\Omega_3\Omega_4$ for each $\alpha \in \{\frac{1}{9}, \frac{1}{4}, 1, 4, 9\}$. Then the origin is asymptotically stable for system (7.1) if $S_1, S_2 < 0$, and $S_3, S_4 \leq 0$ or S_3 and S_4 have nonzero values and of different sign, where

$$\begin{aligned} S_1 = {} & \frac{1}{3\Omega_1^2 + 2\Omega_1\Omega_2 + 3\Omega_2^2}\{\Omega_1(3\Omega_2 g_{yyy} + \Omega_1 g_{xxy}) \\ & + \frac{\Omega_1^2 g_{xw}}{4\Omega_1\Omega_2 - \Omega_3\Omega_4} \cdot (\Omega_4 r_{xy} - 2\Omega_1 s_{xx} + 2\Omega_2 s_{yy})\}, \end{aligned} \tag{7.3}$$

$$\begin{aligned} S_2 = {} & \frac{1}{3\Omega_3^2 + 2\Omega_3\Omega_4 + 3\Omega_4^2}\{\Omega_3[\Omega_4(3s_{www} + r_{zww}) + \Omega_3(3r_{zzz} + s_{zzw})] \\ & + s_{ww}(\Omega_3 s_{zw} - 2\Omega_4 r_{ww}) - r_{zw}(\Omega_4 r_{ww} + \Omega_3 r_{zz}) + \frac{\Omega_3^2}{\Omega_4} s_{zz}(s_{zw} + 2r_{zz})\} \end{aligned} \tag{7.4}$$

$$S_3 = \frac{\Omega_1}{(\Omega_1 + \Omega_2) \cdot (\Omega_3 + \Omega_4)}\{\Omega_4 g_{yww} + \Omega_3 g_{yzz}\}, \tag{7.5}$$

$$\begin{aligned} S_4 = {} & \frac{\Omega_1}{(\Omega_1 + \Omega_2) \cdot (\Omega_3 + \Omega_4)}\{\frac{\Omega_3\Omega_2}{\Omega_1} \cdot (r_{yyz} + s_{yyw}) + \Omega_3(r_{xxz} + s_{xxw}) \\ & + \frac{1}{\Omega_1\Omega_4}[\Omega_3(\Omega_1 s_{xx} + \Omega_2 s_{yy})(s_{zw} + 2r_{zz}) - \Omega_4(\Omega_1 r_{xx} + \Omega_2 r_{yy})(r_{zw} \\ & + 2s_{ww})] - \frac{2\Omega_3 g_{xw}}{4\Omega_1\Omega_2 - \Omega_3\Omega_4} \cdot (\Omega_4 r_{xy} - 2\Omega_1 s_{xx} + 2\Omega_2 s_{yy})\}. \end{aligned} \tag{7.6}$$

7.3. TSS Dynamics During Station-Keeping

A point-mass model for the TSS during station-keeping has been obtained in Chapter 6 by invoking Lagrangian formulation. As shown in Chapter 6, the out-of-plane angle is hard to stabilize and system is uncontrollable. Although stabilizing feedback tension control laws can be obtained by using Hopf bifurcation stability criteria, the performance of the controlled system, that is, the transient responses, are not good enough. It is considered in this section to discuss the possible reasons and seek for another alternative to improve the system performance.

First, the equations of motion for the TSS obtained in the previous chapter are recalled. The stability criterion given in Corollary 7.1 is then applied to check the stability of the uncontrolled model of the system. It is followed by the discussions of the constraints on the poles placement and nonlinear stability coefficient by using Hopf bifurcation theorem, which might contribute to the reasons of ill performance of transient responses in Chapter 6.

7.3.1. System Dynamics for the TSS During Station-Keeping

By using several simplifying assumptions, we have derived a mathematical model for the TSS in Chapter 6. The state space model given in Section 6.3 is recalled in Eq. (7.7) below, where we assume the applied tension force T is the only external force acting on the system and $\cos\phi \neq 0$.

$$\dot{\phi} = \omega_\phi \tag{7.7a}$$

$$\dot{\omega}_\phi = -\frac{2v}{\ell}\omega_\phi - \frac{1}{2}\sin(2\phi)(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0}{\ell}\cos\theta\sin\phi(1 - \frac{r_0^3}{r_m^3}) \tag{7.7b}$$

$$\dot{\theta} = \omega_\theta \tag{7.7c}$$

$$\dot{\omega}_\theta = -\frac{2v}{\ell}(\omega_\theta + \Omega) + 2\tan\phi(\omega_\theta + \Omega)\omega_\phi - \frac{\Omega^2 r_0 \sin\theta}{\ell\cos\phi}(1 - \frac{r_0^3}{r_m^3}) \tag{7.7d}$$

$$\dot{\ell} = v \tag{7.7e}$$

$$\dot{v} = \ell\omega_\phi^2 + \ell\cos^2\phi(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0^3 \ell}{r_m^3}$$

$$+\Omega^2 r_0 \cos\theta\cos\phi(1-\frac{r_0^3}{r_m^3})+\frac{T}{m} \tag{7.7f}$$

where

$$r_m = (r_0^2+\ell^2+2r_0\ell\cos\theta\cos\phi)^{1/2}. \tag{7.8}$$

By assuming the tether length is held constant (i.e., $\dot{\ell}=v=0$, $\ell=\ell^*=$ a constant), we have only two equilibrium points of system (7.7): $(0,0,0,0,\ell^*,0)$ and $(0,0,\pi,0,\ell^*)$. For simplicity, only the stability and stabilization of system (7.7) at the equilibrium point $(0,0,0,0,\ell^*)$ is considered in the following. Similar results for the stability and stabilization at the other system equilibrium can easily be obtained by using the same approach.

Denote by $x_0 = (0,0,0,0,\ell^*,0)'$ the equilibrium point of (7.7) and let $X =: x - x_0 = (\phi,\ \omega_\phi,\ \theta,\ \omega_\theta,\ \tilde{\ell},\ v)'$, where $\tilde{\ell} := \ell - \ell^*$. The Taylor series expansion of the right hand side of Eqs. (7.7) is, to third order in X,

$$\frac{d}{dt}X = L_0X + Q_0(X,X) + C_0(X,X,X) + eU + e\frac{T}{m} \tag{7.9}$$

where the matrix L_0, the quadratic form Q_0, the cubic form C_0, the vector e and the scalar U are given in Section 6.3.

As shown in Section 6.3, the linearization at the equilibrium point x_0 (with $\dot{v}=v=0$) has two pairs of pure imaginary eigenvalues:

$$\lambda_{1,2} = \pm i\Omega\sqrt{1+\frac{r_0}{\ell^*}(1-\frac{r_0^3}{r_{m,0}^3})}$$

$$\simeq \pm i2\Omega \qquad \text{for} \quad r_0 \gg \ell^*, \tag{7.10}$$

$$\lambda_{3,4} = \pm i\Omega\sqrt{\frac{r_0}{\ell^*}(1-\frac{r_0^3}{r_{m,0}^3})}$$

$$\simeq \pm i\sqrt{3}\Omega \qquad \text{for} \quad r_0 \gg \ell^*, \tag{7.11}$$

where

$$r_{m,0} = r_0 + \ell^*. \tag{7.12}$$

The appearance of these two pairs of pure imaginary eigenvalues suggests the possibility of oscillations near the equilibrium point $(0,0,0,0)$. Application of Corollary 7.1 to the uncontrolled model (7.7a)-(7.7d) gives $S_i = 0$ for $i = 1, \cdots, 4$. Thus, no conclusion regarding stability is reached for the open-loop system.

7.3.2. Constraints on Poles Placement and Nonlinear Stability Coefficient

In the previous chapter, we have shown that system (7.7) is uncontrollable and obtained stabilizing feedback tension control laws by using Hopf bifurcation stability criteria. In which, a linear state feedback control is first employed for poles placement of those eigenvalues corresponding to the in-plane angle and the tether length, while same linear feedback or another nonlinear feedback will then drive the origin for the full model of the system to stable if the control gains satisfy the Hopf bifurcation stability conditions. Similar results can also be obtained by using center manifold reduction technique. Details of technique can be referred to, for instance, Chapter 3.

At the expense of large nonlinear feedback gains and large variations of the θ and $\tilde{\ell}$, simulations given in Section 6.4 demonstrate the stability of the system with significant decaying of the time response for ϕ. Nevertheless, the transient responses of the system are not good enough. This might be attributed to the fact that the stabilized eigenvalues are still close to the imaginary axis in the complex plane, and linear and nonlinear feedback gains used in the simulations are not large enough. Indeed, large linear feedback gains are shown below to be necessary to place all stabilizable eigenvalues far from the imaginary axis in the complex plane, which will propel a large increase of the quadratic feedback gains for having a significant decaying of the transient response. Details are given as follows.

Let the tension control law be governed by

$$T = -\,m\{U + k_1\phi + k_2\omega_\phi + k_3\theta + k_4\omega_\theta$$

$$+ k_5\tilde{\ell} + k_6 v + u(\phi, \omega_\phi, \theta, \omega_\theta, \tilde{\ell}, v)\}. \quad (7.13)$$

So, the linearized model of (7.7) at equilibrium point x_0 becomes

$$\frac{d}{dt}X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_2^2 & 0 & 0 & -\frac{2\Omega}{\ell^*} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 + 2\ell^*\Omega & -k_5 + a_3 & -k_6 \end{pmatrix} X. \quad (7.14)$$

The characteristic equation of the closed-loop system is hence as follows

$$(\lambda^2 + a_1^2)\,(\lambda^4 + k_6\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3) = 0, \quad (7.15)$$

where

$$b_1 = k_5 - \frac{(2r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0 + \ell^*)^3} - \frac{2\Omega k_4}{\ell^*} + 4\Omega^2, \quad (7.16)$$

$$b_2 = \frac{k_6(3r_0^3 + 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^3} - \frac{2\Omega k_3}{\ell^*}, \quad (7.17)$$

$$b_3 = \frac{(3r_0^3 + 3r_0^2\ell^* + r_0\ell^{*2})\Omega^2}{(r_0 + \ell^*)^3}\{k_5 - \frac{(3r_0^3 + 3r_0^2\ell^* + 3r_0\ell^{*2} + \ell^{*3})\Omega^2}{(r_0 + \ell^*)^3}\} \quad (7.18)$$

It has been shown in Lemma 6.1 that the tension control force T given in (7.13) with $u = 0$ stabilizes the Jacobian matrix of Eqs. (7.7c)-(7.7f) if (i) $k_6 > 0$, (ii) $b_i > 0$, for $i = 1, 2, 3$ and (iii) $k_6 b_1 b_2 - b_2^2 - k_6^2 b_3 > 0$, where the values of b_i are given in (7.16)-(7.18). It is observed from Eq. (7.15) that the linear control gains k_1, k_2 do not influence the eigenvalues of the system. For simplicity, we choose $k_1 = k_2 = 0$ in the following discussions.

Consider the linearization of Eqs. (7.7c)-(7.7f) with $\phi = \omega_\phi = 0$, we have

$$\begin{pmatrix} \dot\theta \\ \dot\omega_\theta \\ \dot{\tilde{\ell}} \\ \dot v \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a_2^2 & 0 & 0 & -\frac{2\Omega}{\ell^*} \\ 0 & 0 & 0 & 1 \\ 0 & 2\ell^*\Omega & a_3 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \omega_\theta \\ \tilde{\ell} \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tilde{u}, \quad (7.19)$$

where $\tilde{u} = \frac{T}{m} + U$.

The controllability matrix of (7.19) is calculated as

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & -\frac{2\Omega}{\ell^*} & 0 \\ 0 & -\frac{2\Omega}{\ell^*} & 0 & -\frac{2\Omega}{\ell^*}(a_3 - a_2^2 - 4\Omega^2) \\ 0 & 1 & 0 & a_3 - 4\Omega^2 \\ 1 & 0 & a_3 - 4\Omega^2 & 0 \end{pmatrix}, \tag{7.20}$$

where $\det(\mathcal{C}) = \frac{4\Omega^2 a_2^2}{\ell^{*2}} \neq 0$. Thus, the Jacobian matrix of Eqs. (7.7c)-(7.7f) is linearly controllable, which implies that there always exist linear feedback gains k_i to place all the eigenvalues of Jacobian matrix (7.7c)-(7.7f) to any positions in the complex plane. However, as calculated, $\det(\mathcal{C}) \simeq \frac{12\Omega^4}{\ell^{*2}}$ when $r_0 \gg \ell^*$. In general, $\Omega \simeq 0.001$ and $\ell^* > 1$, which implies that the determinant of the controllability matrix $\mathcal{C}$ is very close to zero. Thus, the linear controllability of the Jacobian matrix of Eqs. (7.7c)-(7.7f) is nearly uncontrollable. In fact, as shown in Lemma 7.1 below, large linear feedback gains are needed to move those stabilizable eigenvalues of (7.7c)-(7.7f) far from the imaginary axis in the complex plane.

From the foregoing discussions, it is obvious to have following result.

Lemma 7.1. Suppose $r_0 \gg \ell^*$ and let $\sigma^* := \min_i |Re\{\sigma_i\}|$, where σ_i denote the eigenvalues of the Jacobian matrix (7.7c)-(7.7f) after linear feedback control. Then we have (i) $4\sigma^* \leq k_6$, (ii) $6\sigma^{*2} \leq k_5 + 4\Omega^2 - \frac{2\Omega}{\ell^*}k_4$, (iii) $4\sigma^{*3} \leq 3\Omega^2 k_6 - \frac{2\Omega}{\ell^*}k_3$, (iv) $\sigma^{*4} \leq 3\Omega^2(k_5 - 3\Omega^2)$.

In general, $\Omega \simeq 0.001$ and $\ell^* \geq 1000$ for expecting applications. It is observed from condition (iv) of Lemma 7.1 that large value of k_5 is needed to have σ^* big enough. For instance, let $\sigma^* = 0.1$, then we need to have $k_5 \geq 3300$ when $\Omega = 0.001$. Moreover, according to condtion (iii) of Lemma 7.1 we also need to have $k_6 \geq 1000$ (or $|k_3| \geq 2000$ when $\ell^* = 1000$). Thus, implied by Lemma 7.1, large linear feedback gains are necessary to make the eigenvalues of the Jacobian matrix (7.7c)-(7.7f) stable and far from the imaginary axis. The ill performances of the transient responses of the system given in Chapter 6, where small linear gains are used, might be attributed to this degenerate result.

It has been shown in Theorem 6.2 that the tension control force T given by $T = m(-U - k_3\theta - k_4\omega_\theta - k_5\tilde{\ell} - k_6 v - q_1\phi^2 - q_2\phi\omega_\phi - q_3\omega_\phi^2)$ will stabilize system (7.7), while linear gains k_i stabilize the eigenvalues of the Jacobian matrix (7.7c)-(7.7f) and the stability coefficient $\beta_2 < 0$, where (assuming $r_0 \gg \ell^*$)

$$\beta_2 \simeq \frac{9\Omega}{2(d_1^2 + d_2^2)}\{(\frac{-q_1 + a_5}{2} + (\frac{\ell^* + q_3}{2})a_1^2 - (k_5 - a_3)\frac{\ell^*}{4})d_1$$

$$+ \frac{a_1}{2}(-q_2 - \ell^* k_6)d_2\}, \tag{7.21}$$

with

$$d_1 = k_3 + \frac{\ell^* k_6}{2\Omega}(4a_1^2 - a_2^2) \tag{7.22}$$

$$d_2 = 2a_1(k_4 - 2\ell^*\Omega) + \frac{\ell^*(4a_1^2 - a_2^2)}{4a_1\Omega}(a_3 + 4a_1^2 - k_5). \tag{7.23}$$

According to Eqs. (7.21)-(7.23), the magnitude of β_2 decreases as linear feedback gains increase. Moreover, since $\Omega \simeq 0.001$, q_1 is observed to be the one of three quadratic feedbacks, which can drive $\beta_2 < 0$ with smallest magnitude. To improve the transient performance of system (7.7) by using Theorem 6.2, one might need to have large quadratic feedback gains, for instance, to have a better time response than the one of Example 6.3 q_1 must be greater than the previous design value, say it, 1500.

7.4. Stabilization via Center Manifold Reduction

Motivated by the observations given in Section 7.3.2, we design a new nonlinear stabilizing control law in this section for getting better performance for system (7.7) without using large linear and nonlinear feedback gains. It relies upon a linear feedback to preserve the two pairs of pure imaginary eigenvalues of the uncontrolled linearized model, instead of stabilizing one of pairs of pure imaginary eigenvalues. The linear feedback also provides the stability of the remaining two eigenvalues of (7.7). A locally invariant manifold for system (7.7) can then be derived by using Theorem 2.1, where Corollary 7.1 is applied to the design of a stabilizing control law.

7.4.1. Design of Stabilizing Control Laws

Let the tension control law T be as in (7.13) and suppose the two pairs of pure imaginary eigenvalues of the uncontrolled system linearization are preserved, then we can rewrite Eq. (7.15) as follows

$$(\lambda^2 + a_1^2)(\lambda^2 + a_2^2)(\lambda^2 + k_6\lambda + (k_5 - a_3)) = 0, \tag{7.24}$$

which implies that $k_3 = 0$ and $k_4 = 2\ell^*\Omega$. Moreover, according to Eq. (7.24) and Hurwitz stability criterion, the eigenvalues of the Jacobian matrix for state $(\tilde{\ell}, v)$ in (7.7) are stable if $k_5 > a_3$ and $k_6 > 0$.

As observed from the linearized model of system (7.14), there is a linear coupling term between ω_θ and v in the dynamics of $\dot{\omega}_\theta$. To apply the technique of center manifold reduction, i.e., Theorem 2.1, it is convenient to have linearized model in block diagonal form.

Let $\eta := (\phi, \omega_\phi, \theta, \omega_\theta)'$, $\xi := (\tilde{\ell}, v)'$, $\zeta := (x, y, z, w)' = \eta + P\xi$, $k_i = 0$, $i = 1, 2, 3$ and $k_4 = 2\ell^*\Omega$.

By choosing

$$P := \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \\ p_5 & p_6 \\ p_7 & p_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{2\Omega k_6(k_5 - a_3)}{k_h} & -\frac{2\Omega(k_5 - a_3 - a_2^2)}{k_h} \\ \frac{2\Omega(k_5 - a_3 - a_2^2)(k_5 - a_3)}{k_h} & -\frac{2\Omega k_6 a_2^2}{k_h} \end{pmatrix} \tag{7.25}$$

with

$$k_h = \ell^*\{a_2^2 k_6^2 + (k_5 - a_3 - a_2^2)^2\}, \tag{7.26}$$

we can rewrite system (7.7) in a block diagonal form as follows

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -k_5 + a_3 & -k_6 \end{pmatrix} \begin{pmatrix} \zeta \\ \xi \end{pmatrix} + \begin{pmatrix} 0 \\ g(\zeta, \xi) \\ r(\zeta, \xi) \\ s(\zeta, \xi) \\ 0 \\ \nu(\zeta, \xi) \end{pmatrix} \tag{7.27}$$

where

$$g(\zeta,\xi) = -2\Omega x(w - p_8 v - p_7\tilde{\ell} - \frac{2}{\ell^*}yv + a_{12}x\tilde{\ell}) - x(w - p_8 v - p_7\tilde{\ell})^2$$
$$+ a_6 x(z - p_6 v - p_5\tilde{\ell})^2 + \frac{2}{\ell^{*2}}y\tilde{\ell}v + a_8 x\tilde{\ell}^2 + a_7 x^3 + O(||(\zeta,\xi)||^4) \quad (7.28)$$

$$\nu(\zeta,\xi) = a_{13}\tilde{\ell}^2 + 2\Omega\tilde{\ell}(w - p_8 v - p_7\tilde{\ell}) + \ell^*(w - p_8 v - p_7\tilde{\ell})^2$$
$$+ a_4(z - p_6 v - p_5\tilde{\ell})^2 + \ell^* y^2 + a_5 x^2 + a_{15}\tilde{\ell}^3$$
$$+ \tilde{\ell}\{(w - p_8 v - p_7\tilde{\ell})^2 + a_{14}(z - p_6 v - p_5\tilde{\ell})^2 + y^2 + a_{11}x^2\}$$
$$- 2\ell^*\Omega x^2(w - p_8 v - p_7\tilde{\ell}) + u(\zeta,\xi) + O(||(\zeta,\xi)||^4) \quad (7.29)$$

$$r(\zeta,\xi) = p_6\nu(\zeta,\xi) \quad (7.30)$$

$$s(\zeta,\xi) = p_8\nu(\zeta,\xi) + \tilde{\ell}a_{12}(z - p_6 v - p_5\tilde{\ell}) + v\{\frac{2\Omega\tilde{\ell}}{\ell^{*2}} - \frac{2}{\ell^*}(w - p_8 v - p_7\tilde{\ell})\}$$
$$+ 2\Omega xy - v\{\frac{2\Omega}{\ell^{*3}}\tilde{\ell}^2 - \frac{2}{\ell^{*2}}\tilde{\ell}(w - p_8 v - p_7\tilde{\ell})\} + a_8\tilde{\ell}^2(z - p_6 v - p_5\tilde{\ell})$$
$$+ 2xy(w - p_8 v - p_7\tilde{\ell}) + a_9(z - p_6 v - p_5\tilde{\ell})^3$$
$$+ a_{10}x^2(z - p_6 v - p_5\tilde{\ell}) + O(||(\zeta,\xi)||^4). \quad (7.31)$$

According to Theorem 2.1, there exists a locally invariant manifold for system (7.27), which is given by the graph of a C^2 function $\xi = h(\zeta)$. As stated in Theorem 2.2, we can solve for the approximation of function h. Let this approximation of function h be given as follows

$$\tilde{\ell} = x(h_{\tilde{\ell},xx}x + h_{\tilde{\ell},xy}y + h_{\tilde{\ell},xz}z + h_{\tilde{\ell},xw}w) + y(h_{\tilde{\ell},yy}y + h_{\tilde{\ell},yz}z + h_{\tilde{\ell},yw}w)$$
$$+ h_{\tilde{\ell},zz}z^2 + h_{\tilde{\ell},zw}zw + h_{\tilde{\ell},ww}w^2 + O(||(x,y,z,w)||^3) \quad (7.32)$$

$$v = x(h_{v,xx}x + h_{v,xy}y + h_{v,xz}z + h_{v,xw}w) + y(h_{v,yy}y + h_{v,yz}z + h_{v,yw}w)$$
$$+ h_{v,zz}z^2 + h_{v,zw}zw + h_{v,ww}w^2 + O(||(x,y,z,w)||^3). \quad (7.33)$$

To employ the relationship (2.2), we can solve for $h_{\tilde{\ell},ij}$ and $h_{v,ij}$ for all $i,j \in \{x,y,z,w\}$. Also, according to Theorem 2.1, the stability of the full model of system (7.27) is known to be determined by the stability of the reduced fourth-order model for ζ (i.e., the submodel of (7.27)) only, with $\tilde{\ell}$ and v replaced by Eqs. (7.32) and (7.33), respectively. To apply Corollary 7.1 to the reduced model of system (7.27) by replacing $\tilde{\ell}$ and v with (7.32) and (7.33), respectively, the stability coefficients S_i given in Corollary 7.1 are obtained as follows

$$S_1 = \frac{1}{3+2a_1^2+3a_1^4}\{3a_1^2\tilde{g}_{yyy} + \tilde{g}_{xxy}$$

$$+ \frac{2\Omega}{4a_1^2 - a_2^2}(-a_2^2 r_{xy} + 2s_{xx} - 2a_1^2 s_{yy})\}, \tag{7.34}$$

$$S_2 = \frac{1}{3+2a_2^2+3a_2^4}\{3a_2^2\tilde{s}_{www} + 3\tilde{r}_{zzz} + \tilde{s}_{zzw} + a_2^2\tilde{r}_{zww}$$

$$+ s_{ww}(s_{zw} - 2a_2^2 r_{ww}) - r_{zw}(r_{zz} + a_2^2 r_{ww}) + \frac{1}{a_2^2}s_{zz}(s_{zw} + 2r_{zz})\} \tag{7.35}$$

$$S_3 = \frac{1}{(1+a_1^2)(1+a_2^2)}\{a_2^2\tilde{g}_{yww} + \tilde{g}_{yzz}\} = 0, \tag{7.36}$$

$$S_4 = \frac{1}{(1+a_1^2)(1+a_2^2)}\{a_1^2(\tilde{r}_{yyz} + \tilde{s}_{yyw}) + (\tilde{r}_{xxz} + \tilde{s}_{xxw})$$

$$+ \frac{1}{a_2^2}[(s_{xx} + a_1^2 s_{yy})(s_{zw} + 2r_{zz}) - a_2^2(r_{xx} + a_1^2 r_{yy})(r_{zw} + 2s_{ww})]$$

$$+ \frac{2\Omega}{4a_1^2 - a_2^2}(a_2^2 r_{xy} - 2s_{xx} + 2a_1^2 s_{yy})\}. \tag{7.37}$$

Here, tilde denote the new coefficients of the cubic terms after replacing $\tilde{\ell}$ and v with Eqs. (7.32) and (7.33), respectively.

As implied by Corollary 7.1 and Theorem 2.1, it is obvious to have following result.

Proposition 7.1. Let the applied tension control force T is as in (7.13) with $k_i = 0$, for $i = 1, 2, 3$ and $k_4 = 2\ell^*\Omega$. Then the origin is asymptotically stable for system (7.7) if $k_5 > a_3$, $k_6 > 0$, $S_1, S_2 < 0$ and $S_4 \leq 0$, where S_i are given in Eqs. (7.34)-(7.35) and (7.37).

■

As discussed in Section 7.3.2, at least one of four stable (or stabilized) eigenvalues will be very close to the imaginary axis, while one of the two pairs of nonzero pure imaginary eigenvalues are pushed to be stable by a linear feedback.

However, in the recent design, where a linear feedback is employed to preserve the two pairs of nonzero pure imaginary eigenvalues and to provide the stability of the remaining two system eigenvalues, there will be no such limitation on the stable eigenvalues. In general, the two stable eigenvalues can be placed to any positions in the open-left-half of complex plane by feedback gains k_5 and k_6. However, as shown in the next two sections, the values of the stability coefficients S_i strongly depend on the values of the linear gains k_5 and k_6. In order to prevent the high gains for the nonlinear controllers, the magnitudes of k_5, k_6 should not be too big, which implies that the two stable eigenvalues should not be placed too far from the imaginary axis.

7.4.2. Linear Feedback Stabilization

First, consider the case in which the tension control force T (given in (7.13)) is a linear state feedback, i.e, the nonlinear control input function $u(\zeta, \xi) = 0$. Denote by $S_{\mathcal{L},i}$ the parameters S_i given in (7.34)-(7.37) for the reduced model of linear control system (7.27), we have

$$\begin{aligned} S_{\mathcal{L},1} = & \frac{k_6}{H_1}\{32a_1^2\Omega^2(\ell^* a_1^2 - a_5) + 2(4a_1^2 - a_2^2)\cdot[\ell^{*2}a_1^2 a_{12} \\ & - \ell^*(a_{12}a_5 + 6a_1^4) + 6a_1^2 a_5]\}, \end{aligned} \tag{7.38}$$

$$\begin{aligned} S_{\mathcal{L},2} = & \frac{k_6}{H_2}\{2\ell^*(k_5 - a_3)[(k_5 - a_3 - a_2^2)^2 + a_2^2 k_6^2]\cdot[-\ell^{*2}a_{12}a_2^2 \\ & + \ell^*(a_{12}a_4 + 4a_2^4) - 4a_2^2 a_4] - 32\ell^*\Omega^2 a_2^4(k_6^2 + 4a_2^2)(-\ell^* a_2^2 - a_4) \\ & - (k_5 - a_3)^2\Omega^2[-12\ell^* a_2^2(-\ell^* a_2^2 + a_4) - 16a_4^2] \\ & - \Omega^2 a_2^2(k_5 - a_3)\cdot[-24\ell^* a_2^2(-\ell^* a_2^2 - 3a_4) + 32a_4^2]\}, \end{aligned} \tag{7.39}$$

$$\begin{aligned} S_{\mathcal{L},4} = & -\frac{\Omega^2 k_6 a_2^2}{H_3}\{4(k_5 - a_3)\cdot[(2a_1^2 - a_2^2)\ell^* + 2a_5] \\ & - 4(a_1^2\ell^* + a_5)(4a_1^2 - a_2^2)\}, \end{aligned} \tag{7.40}$$

where

$$H_1 = \ell^*(4a_1^2 - a_2^2)\{4a_1^2 k_6^2 + (k_5 - a_3 - 4a_1^2)^2\}, \tag{7.41}$$

$$H_2 = \ell^{*2}(k_5 - a_3)\{4a_2^4k_6^2(k_6^2 + k_5 - a_3) + 5a_2^2k_6^2(k_5 - a_3 - 2a_2^2)^2 + [(k_5 - a_3 - 2a_2^2)^2 - a_2^2(k_5 - a_3)]^2\}, \tag{7.42}$$

$$H_3 = -\ell^*(k_5 - a_3)(4a_1^2 - a_2^2)\{a_2^2k_6^2 + (k_5 - a_3 - a_2^2)^2\}. \tag{7.43}$$

It has been calculated that $S_{\mathcal{L},i} < 0$, $i = 1,2,4$ if $k_5 > a_3$ and $k_6 > 0$. Thus, we have following obvious result.

Theorem 7.1. The origin is asymptotically stable for the controlled system (7.7) during station-keeping if the tension control law is governed by

$$T = -m(U + k_4\omega_\theta + k_5\tilde{\ell} + k_6 v), \tag{7.44}$$

with $k_4 = 2\ell^*\Omega$, $k_5 > a_3$ and $k_6 > 0$.

■

Suppose $r_0 \gg \ell$ during station-keeping, the values of $S_{\mathcal{L},i}$ can then be approximately obtained as follows

$$S_{\mathcal{L},1} \simeq \frac{-3968k_6\Omega^4}{13\{16\Omega^2k_6^2 + (k_5 - 19\Omega^2)^2\}} \tag{7.45}$$

$$S_{\mathcal{L},2} \simeq \frac{-144k_6\Omega^4}{H_4}\{3\Omega^2k_6^2 + (k_5 - 6.25\Omega^2)^2 + 1.4375\Omega^4\} \tag{7.46}$$

$$S_{\mathcal{L},4} \simeq \frac{-36k_6\Omega^4}{13\{3\Omega^2k_6^2 + (k_5 - 6\Omega^2)^2\}} \tag{7.47}$$

where

$$H_4 = 36k_6^2\Omega^4(k_6^2 + k_5 - 3\Omega^2) + 15k_6^2(k_5 - 9\Omega^2)^2 + \{(k_5 - 9\Omega^2)^2 - 6(k_5 - 3\Omega^2)\Omega^2\}^2. \tag{7.48}$$

In general, the magnitude of the angular velocity of the satellite Ω ($\simeq$ .001) is very small. As observed from Eqs. (7.45)-(7.48), the magnitudes of $S_{\mathcal{L},i}$, $i = 1,2,4$ are hence small for all $k_5 > a_3$ and $k_6 > 0$. As discussed in Section 4.2.3, the small magnitudes of $S_{\mathcal{L},i}$ might lead to the small decaying for the time response of system behavior. Simulation results given in Section 7.5 demonstrate this conclusion.

7.4.3. Nonlinear Feedback Stabilization

Next, consider the case in which the nonlinear control input function u is nonzero. Referring to Proposition 7.1, we have the nonlinear control function u being a quadratic-plus-cubic function as follows

$$\begin{aligned}u &= u_{xx}x^2 + u_{xy}xy + u_{xz}xz + u_{xw}xw + u_{yy}y^2 + u_{yz}yz + u_{yw}yw \\ &\quad + u_{zz}z^2 + u_{zw}zw + u_{ww}w^2 + u_{xxx}x^3 + (u_{xxy}y + u_{xxz}z + u_{xxw}w)x^2 \\ &\quad + (u_{xyy}x + u_{yyy}y + u_{yyz}z + u_{yyw}w)y^2 + u_{xyz}xyz + u_{xyw}xyw \\ &\quad + u_{xzw}xzw + u_{yzw}yzw + (u_{xzz}x + u_{yzz}y + u_{zzz}z + u_{zzw}w)z^2 \\ &\quad + (u_{xww}x + u_{yww}y + u_{zww}z + u_{www}\dot{w})w^2. \end{aligned} \tag{7.49}$$

Denote by $S_{\mathcal{N},i}$ the parameters S_i given in Eqs. (7.34)-(7.37) for the reduced model of nonlinear control system (7.27), we have

$$S_{\mathcal{N},1} = S_{\mathcal{L},1} + \rho_1 u_{yy} + \rho_2 u_{xx} + \rho_3 u_{xy}, \tag{7.50}$$

$$\begin{aligned}S_{\mathcal{N},2} &= S_{\mathcal{L},2} + \rho_4 u_{zzz} + \rho_5 u_{zww} + \rho_6 u_{zzw} + \rho_7 u_{www} + (\rho_8 u_{zz} + \rho_9 u_{zw} \\ &\qquad + \rho_{10})u_{zz} + (\rho_{11}u_{ww} + \rho_{12})u_{zw} + \rho_{13}u_{ww}^2 + \rho_{14}u_{ww},\end{aligned} \tag{7.51}$$

$$\begin{aligned}S_{\mathcal{N},4} &= S_{\mathcal{L},4} + \rho_{15}u_{yyz} + \rho_{16}u_{xxz} + \rho_{17}u_{yyw} + \rho_{18}u_{xxw} \\ &\qquad + (\rho_{19} + \rho_{20}u_{yy} + \rho_{21}u_{xx})u_{zz} + (\rho_{22}u_{yy} + \rho_{23}u_{xx} + \rho_{24})u_{zw} \\ &\qquad + (\rho_{25} + \rho_{26}u_{yy} + \rho_{27}u_{xx})u_{ww} + \rho_{28}u_{yy} + \rho_{29}u_{xy} + \rho_{30}u_{xx},\end{aligned} \tag{7.52}$$

where $S_{\mathcal{L},i}$ are defined in Eqs. (7.38)-(7.40) and ρ_i are functions of k_5 and k_6, for $i = 1, \cdots, 30$, as given in Appendix 7.A. It is observed from the expressions in Appendix 7.A that the values of $|\rho_i|$ decrease as the magnitude of k_5 (or k_6) increases.

The next result readily follows from Proposition 7.1.

Corollary 7.2. Suppose a nonlinear tension control law as in (7.13) is applied to system (7.7) with $k_i = 0$, $i = 1, 2, 3$ and $k_4 = 2\ell^*\Omega$, while the nonlinear control function u is as in (7.49). Then the origin is asymptotically stable for the system (7.7) during station-keeping if $k_5 > a_3$, $k_6 > 0$, $S_{\mathcal{N},1}, S_{\mathcal{N},2} < 0$ and $S_{\mathcal{N},4} \leq 0$.

As observed from Eqs. (7.50)-(7.52) and Corollary 7.2, only parts of the quadratic and cubic feedback gains contribute to the stability conditions for the system (7.7). Among them, u_{xx}, u_{xy} and u_{yy} play key roles in determining the magnitude and the sign of $S_{\mathcal{N},1}$, which corresponds to the stability of the out-of-plane angle ϕ; while the magnitudes and the signs for $S_{\mathcal{N},2}$ and $S_{\mathcal{N},4}$ are determined by some of quadratic and cubic feedback gains.

Referring to Eqs. (7.50)-(7.52), we can have the following obvious result from Corollary 7.2.

Theorem 7.2. Suppose the tension control law is governed by

$$\begin{aligned} T = m(&-U - 2\ell^*\Omega\omega_\theta - k_5\tilde{\ell} - k_6 v + u_{xx}x^2 + u_{xy}xy + u_{yy}y^2 \\ &+ u_{zz}z^2 + u_{zw}zw + u_{ww}w^2 + (u_{xxz}z + u_{xxw}w)x^2 + (u_{yyz}z \\ &+ u_{yyw}w)y^2 + (u_{zzz}z + u_{zzw}w)z^2 + (u_{zww}z + u_{www}w)w^2 \end{aligned} \tag{7.53}$$

during station-keeping. Then the origin is guaranteed to be asymptotically stable for the system (7.7) if $k_5 > a_3$, $k_6 > 0$ and the quadratic and cubic feedback gains u_{ij} and u_{ijk} in Eq. (7.53) are chosen such that $S_{\mathcal{N},1}, S_{\mathcal{N},2} < 0$ and $S_{\mathcal{N},4} \leq 0$, where $S_{\mathcal{N},i}$, $i = 1, 2, 4$ are given in Eqs. (7.50)-(7.52).

7.5. Simulation Results

A TSS with the same characteristics as the one in Section 6.4 is considered here: (i) nominal tether length $\ell^* = 100$ km, (ii) orbital radius $r_0 = 6598$ km, (iii) satellite mass $m = 170$ kg, (iv) orbital angular velocity $\Omega = 0.0011781$ radians/second.

Let the equilibrium point of (7.7) of interest be $x_0 = (0, 0, 0, 0, \ell^*, 0)'$. Simulation results for the case in which no external tension control force is applied to the system have been given in Figure 6.2. Here, we only present the results which illustrate the system dynamics for the various types of control studied in this chapter. Similar to Section 6.4, we choose the initial conditions $\phi = 0.01$ radians, $\theta = -0.01$ radians, and $\omega_\theta = \omega_\phi = 0$.

Example 7.5.1. (Linear stabilizing feedback control)

The tension controller is taken as $T = -m(U + 2\ell^*\Omega\omega_\theta + k_5\tilde{\ell} + k_6 v)$, with $k_5 = 18\Omega^2$, $k_4 = 0.00051$, and $U = 0.41019$. The control law is stabilizing, as can be checked using Theorem 7.1. Indeed, we have $S_{\mathcal{L},1} = -0.0452$, $S_{\mathcal{L},2} = -0.0047$, $S_{\mathcal{L},3} = 0$ and $S_{\mathcal{L},4} = -8.97 \times 10^{-6}$ for the closed-loop system. The time responses for the variables of ϕ, θ, and the deviation $\tilde{\ell}$ of the tether length are shown in Figure 7.1(a), (b) and (c), respectively. However, it is not easy to see in Figure 7.1(a) and 1(b) any decaying of the oscillations for the out-of-plane angle ϕ and in-plane angle θ. This may be attributed to the fact that $|S_{\mathcal{L},i}|$, $i = 1, 2, 4$ are too small. The applied tension force is also shown in Figure 7.1(d).

Example 7.5.2. (Linear-plus-cubic stabilizing feedback control)

Let the tension control law be of the form

$$T = m(-U - 2\ell^*\Omega\omega_\theta - k_5\tilde{\ell} - k_6 v + (u_{xxz}z + u_{xxw}w)x^2 + (u_{yyz}z + u_{yyw}w)y^2 + (u_{zzz}z + u_{zzw}w)z^2 + (u_{zww}z + u_{www}w)w^2, \qquad (7.54)$$

where $U = 0.41019$. By choosing $k_5 = 18\Omega^2$, $k_6 = 0.00051$, $u_{xxz} = 1000$, $u_{zzz} = 10000$ and letting the rest of cubic gains equal to zero, the in-plane angle θ is observed to decay in Figure 7.2(b) while the out-of-plane angle ϕ in Figure 7.2(a) still has no significant decaying. This result may be attributed to the fact of the large value of $|S_{\mathcal{N},2}|$ and the small value of $|S_{\mathcal{N},1}|$. As calculated, we have $S_{\mathcal{N},1} = -0.0452$, $S_{\mathcal{N},2} = -41671$, $S_{\mathcal{N},3} = 0$ and $S_{\mathcal{N},4} = -1389$ for the closed-loop system. The deviations of the tether length and the applied tension force are shown in Figure 7.2(c) and 7.2(d).

Example 7.5.3. (Linear-plus-quadratic-plus-cubic stabilizing feedback control)

Let the tension control law be of the form as given in Eq. (7.53) with $U = 0.41019$. Choose $k_5 = 20\Omega^2$, $k_6 = 0.0034$, $u_{zzz} = 10000$, $u_{xx} = -1500$ and let the rest of quadratic and cubic gains equal to zero, the out-of-plane angle

ϕ in Figure 7.3(a) is observed to decay significantly, while the in-plane angle in Figure 7.3(b) also has significant decaying. This might be attributed to the fact of the large values of $|S_{\mathcal{N},i}|$, for $i = 1, 2$. Indeed, we have $S_{\mathcal{N},1} = -10251$, $S_{\mathcal{N},2} = -32057$, $S_{\mathcal{N},3} = 0$ and $S_{\mathcal{N},4} = -187$ for the closed-loop system. The deviations of the tether length and the applied tension force are shown in Figure 7.3(c) and 7.3(d).

Example 7.5.4. (Linear-plus-quadratic-plus-cubic stabilizing feedback control)

A further example by using (7.53) as a tension control law is depicted in Figure 7.4. In this example, we set $k_5 = 18\Omega^2$, $k_6 = 0.00051$, $u_{xxz} = u_{zzz} = 1000$, $u_{xx} = -1000$ and let the rest of quadratic and cubic gains equal to zero. The stability coefficients $S_{\mathcal{N},i}$ are calculated as $S_{\mathcal{N},1} = -41859$, $S_{\mathcal{N},2} = -4167$, $S_{\mathcal{N},3} = 0$ and $S_{\mathcal{N},4} = -1415$ for the closed-loop system.

It is observed from the simulations in Figures 7.3 and 7.4 that the transient responses of the system by using the recent approach is superior to the ones given in Section 6.4. Moreover, the magnitudes of the nonlinear control gains and the variations of the in-plane angle and the tether length are much smaller than those in Section 6.4. The linear feedback gains used here are obviously small.

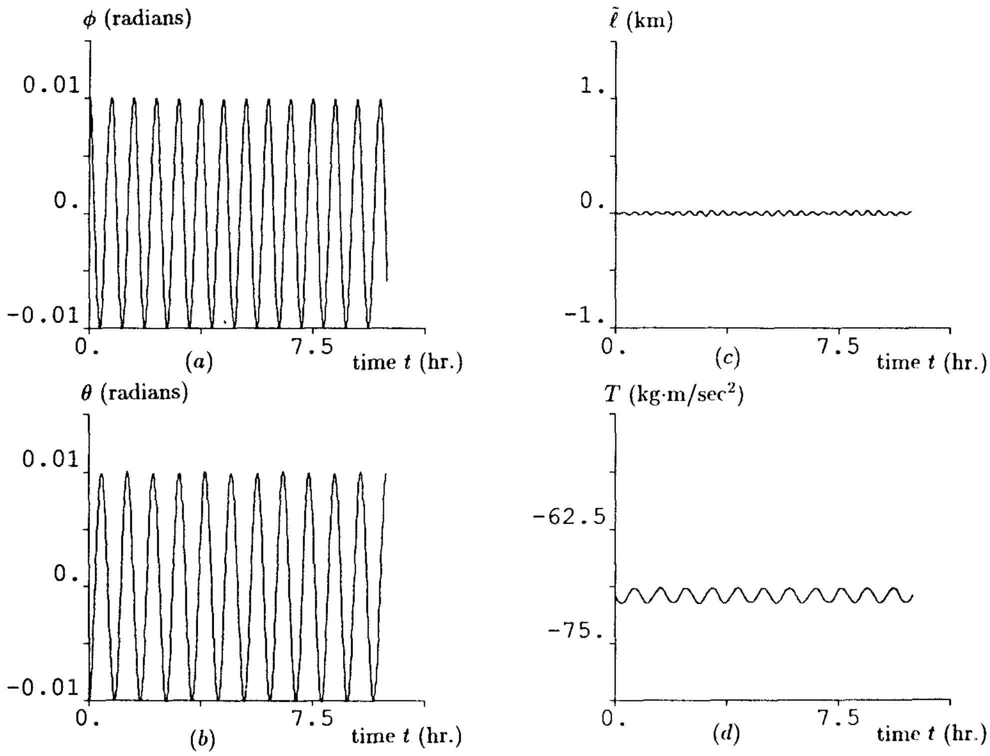

Figure 7.1. Simulation results for linear feedback system

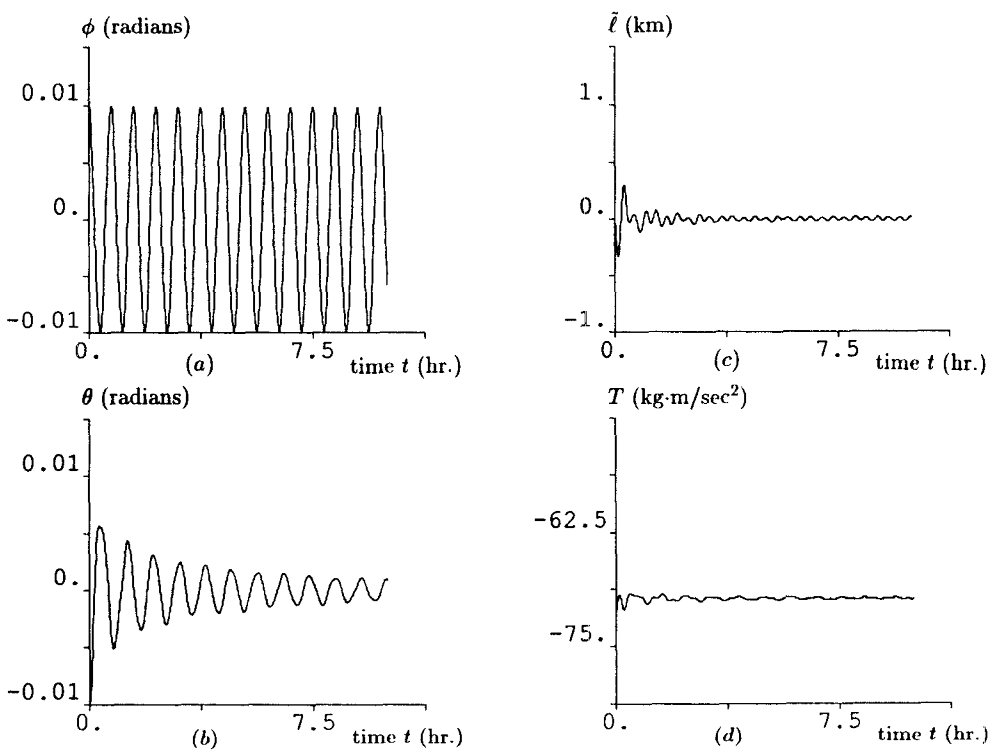

Figure 7.2. Simulation results for linear-plus-cubic feedback system

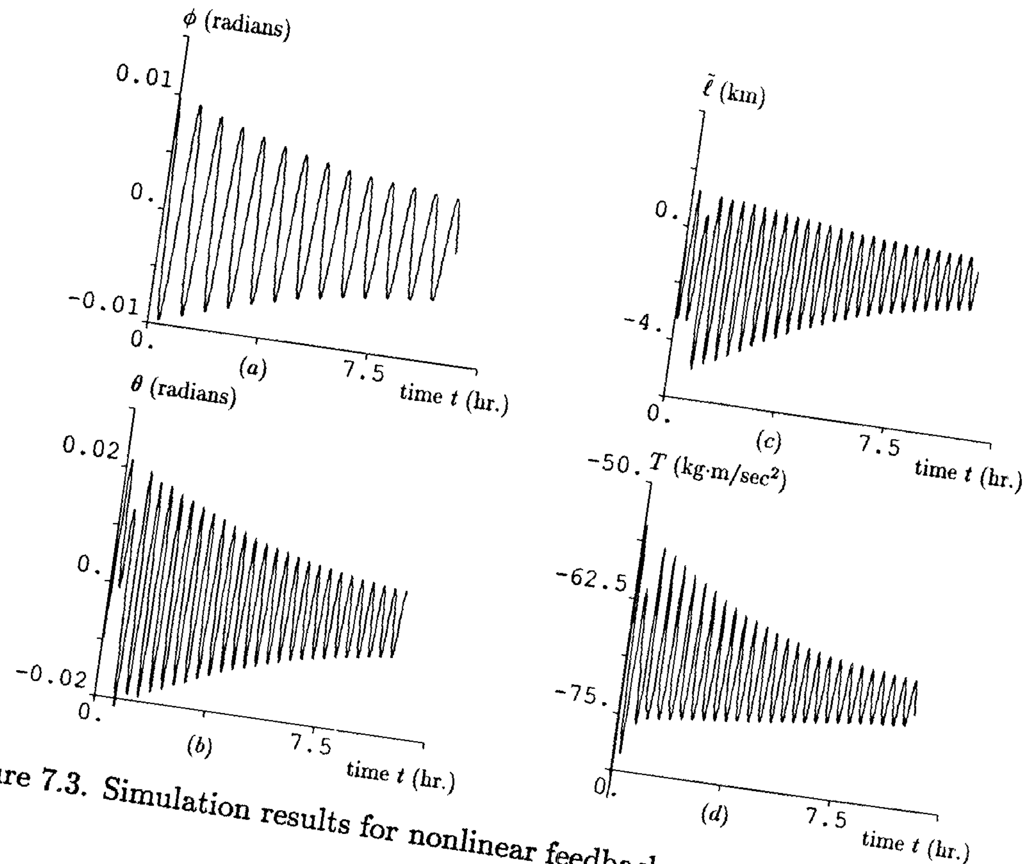

Figure 7.3. Simulation results for nonlinear feedback system in Example 7.5.3

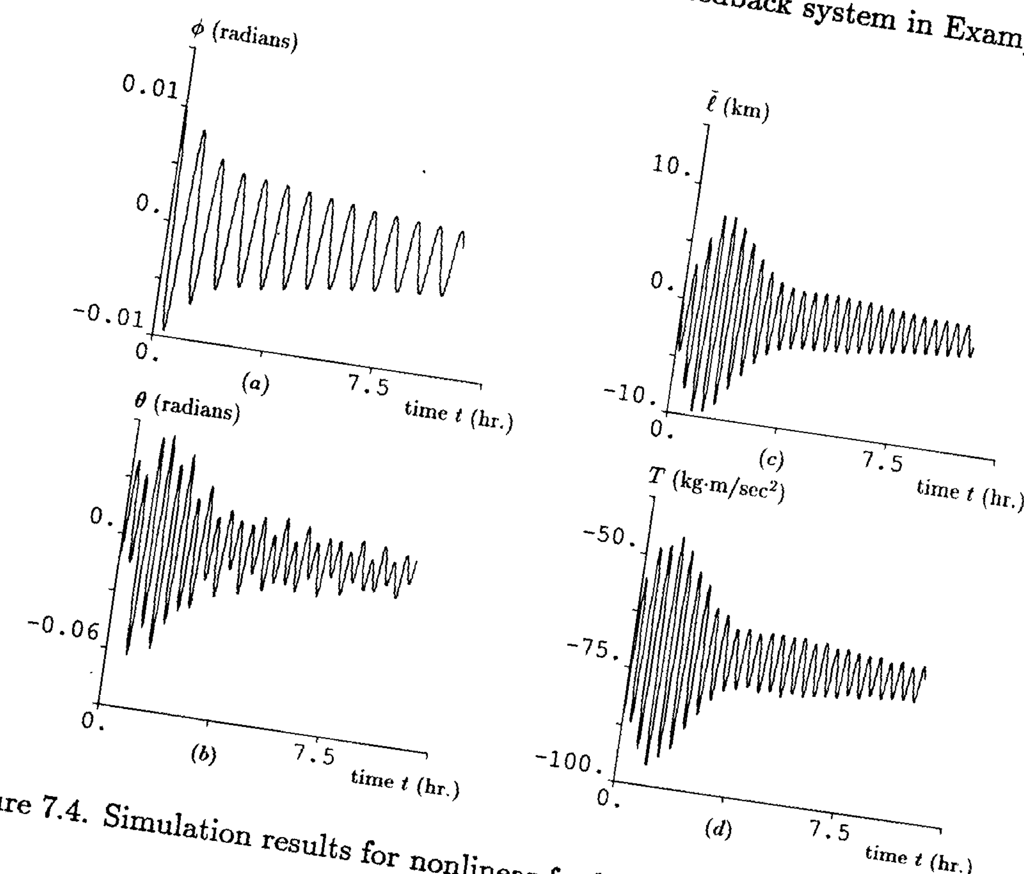

Figure 7.4. Simulation results for nonlinear feedback system in Example 7.5.4

7.6. Concluding Remarks

A new approach without using high linear and/or nonlinear feedback gains for the design of the stabilizing control laws for the TSS during station-keeping is presented in this chapter. A major difference between the recent approach and the previous method in Chapter 6 is that the stabilization of both the in-plane angle θ and the out-of-plane angle ϕ are obtained by using nonlinear stability criterion in this chapter, while linear stability criterion and Hopf bifurcation theorem were employed in the previous chapter to guarantee the stability of the in-plane angle θ and the out-of-plane angle ϕ, respectively.

It is found in this chapter that the quadratic feedback gains u_{xx}, u_{xy}, and u_{yy}, which correspond to the quadratic function of ϕ and ω_ϕ, play very important roles in determining the stability of the out-of-plane angle ϕ. This agrees with the one obtained in the previous chapter by using Hopf bifurcation theorem. To stabilize the in-plane angle θ, a linear feedback controller is designed in Section 6.3. However, in this chapter, such regulation of the in-plane angle θ is found to be achieved by a cubic feedback.

Although a purely linear feedback control can be designed to provide the asymptotic stability of the TSS during station-keeping, however, nonlinear feedbacks are needed in the stabilizing controller to get a better performance of the regulation (i.e., better time response). Specifically, a three steps control algorithm emerging from the approach proposed in this chapter can help to get better performance as: (i) use a linear feedback control of $\tilde{\ell}$ and v to stabilize the dynamics of the tether length, (ii) use a quadratic feedback control of ϕ and ω_ϕ to regulate the out-of-plane angle ϕ by letting $S_{\mathcal{N},1} < 0$, (iii) finally, use a cubic feedback control of θ, ω_θ, $\tilde{\ell}$ and v to regulate the in-plane angle θ by letting $S_{\mathcal{N},2} < 0$ and $S_{\mathcal{N},4} \leq 0$.

Appendix 7.A

The values of ρ_i, for $i = 1, \cdots, 30$ are as follows.

$$\rho_1 = \frac{1}{H_5} \cdot \{32a_1^2\Omega^2 + (-2a_{12}a_2^2 + 8a_1^2a_{12})\ell + (12a_1^2a_2^2 - 48a_1^4)\}a_1^2k_6$$

$$\rho_2 = -\frac{\rho_1}{a_1^2}$$

$$\rho_3 = \frac{1}{H_5} \cdot \{(-16a_1^2(a_3 - k_5) - 64a_1^4)\Omega^2 + ((a_{12}a_2^2 - 4a_1^2a_{12})(a_3 - k_5)$$

$$4a_1^2a_{12}a_2^2 - 16a_1^4a_{12})\ell + (-6a_1^2a_2^2 + 24a_1^4)(a_3 - k_5) - 24a_1^4a_2^2 + 96a_1^6\}$$

$$\rho_4 = \frac{1}{H_8} \cdot \{(24a_2^4 + 24a_2^2(a_3 - k_5))k_6^2 + 6(a_3 - k_5)^3$$

$$+ 54a_2^2(a_3 - k_5)^2 + 144a_2^4(a_3 - k_5) + 96a_2^6\}\ell\Omega$$

$$\rho_5 = \frac{1}{H_8} \cdot \{(8a_2^4(a_3 - k_5) + 8a_2^6)k_6^2 + 2a_2^2(a_3 - k_5)^3$$

$$+ 18a_2^4(a_3 - k_5)^2 + 48a_2^6(a_3 - k_5) + 32a_2^8\}\ell\Omega$$

$$\rho_6 = -\frac{1}{H_8} \cdot \{8a_2^4k_6^3 + (2a_2^2(a_3 - k_5)^2 + 16a_2^4(a_3 - k_5) + 32a_2^6)k_6\}\ell\Omega$$

$$\rho_7 = -\frac{1}{H_8} \cdot \{(6a_2^4(a_3 - k_5)^2 + 48a_2^6(a_3 - k_5) + 96a_2^8)k_6 + 24a_2^6k_6^3\}\ell\Omega$$

$$\rho_8 = -\frac{1}{H_6} \cdot \{(-32a_2^2(a_3 - k_5)^2 - 32a_2^4(a_3 - k_5))k_6^3 + (-8(a_3 - k_5)^4$$

$$- 72a_2^2(a_3 - k_5)^3 - 192a_2^4(a_3 - k_5)^2 - 128a_2^6(a_3 - k_5))k_6\}\Omega^2$$

$$\rho_9 = \frac{1}{H_6} \cdot \{-16a_2^4(a_3 - k_5)k_6^4 - (48a_2^6(a_3 - k_5) - 12a_2^2(a_3 - k_5)^3)k_6^2$$

$$+ 4(a_3 - k_5)^5 + 40a_2^2(a_3 - k_5)^4 + 132a_2^4(a_3 - k_5)^3$$

$$+ 160a_2^6(a_3 - k_5)^2 + 64a_2^8(a_3 - k_5)\}\Omega^2$$

$$\rho_{10} = -\frac{1}{H_6} \cdot \{(32a_2^6k_6^5 + (-36a_2^4(a_3 - k_5)^2 + 72a_2^6(a_3 - k_5) + 160a_2^8)k_6^3$$

$$+ (4a_2^2(a_3 - k_5)^4 - 56a_2^4(a_3 - k_5)^3 - 212a_2^6(a_3 - k_5)^2 - 24a_2^8(a_3 - k_5)$$

$$+ 128a_2^{10})k_6)\ell - 16a_2^2a_4(a_3 - k_5)^2k_6^3 + (8a_4(a_3 - k_5)^4 - 8a_2^2a_4(a_3 - k_5)^3$$

$$- 112a_2^4a_4(a_3 - k_5)^2 - +96a_2^6a_4(a_3 - k_5))k_6\}\Omega^2 + \{-2a_{12}a_2^4(a_3 - k_5)k_6^5$$

$$+ (-4a_{12}a_2^2(a_3-k_5)^3 - 8a_{12}a_2^4(a_3-k_5)^2 - 4a_{12}a_2^6(a_3-k_5))k_6^3$$
$$+ (-2a_{12}(a_3-k_5)^5 - 8a_{12}a_2^2(a_3-k_5)^4 - 12a_{12}a_2^4(a_3-k_5)^3$$
$$- 8a_{12}a_2^6(a_3-k_5)^2 - 2a_{12}a_2^8(a_3-k_5))k_6\}\ell^2 + \{8a_2^6(a_3-k_5)k_6^5$$
$$+ (16a_2^4(a_3-k_5)^3 + 32a_2^6(a_3-k_5)^2 + 16a_2^8(a_3-k_5))k_6^3 + (8a_2^2(a_3-k_5)^5$$
$$+ 32a_2^4(a_3-k_5)^4 + 48a_2^6(a_3-k_5)^3 + 32a_2^8(a_3-k_5)^2 + 8a_2^{10}(a_3-k_5))k_6\}\ell$$

$$\rho_{11} = \frac{1}{H_6}\cdot\{-16a_2^6(a_3-k_5)k_6^4 + (12a_2^4(a_3-k_5)^3 - 48a_2^8(a_3-k_5))k_6^2$$
$$+ 4a_2^2(a_3-k_5)^5 + 40a_2^4(a_3-k_5)^4 + 132a_2^6(a_3-k_5)^3$$
$$+ 160a_2^8(a_3-k_5)^2 + 64a_2^{10}(a_3-k_5)\}\Omega^2$$

$$\rho_{12} = \frac{1}{H_6}\cdot((-24a_2^6(a_3-k_5)k_6^4 + (24a_2^4(a_3-k_5)^3 - 28a_2^6(a_3-k_5)^2$$
$$- 136a_2^8(a_3-k_5))k_6^2 + 20a_2^4(a_3-k_5)^4 + 120a_2^6(a_3-k_5)^3$$
$$+ 180a_2^8(a_3-k_5)^2 + 80a_2^{10}(a_3-k_5))\ell - 16a_2^4a_4(a_3-k_5)k_6^4$$
$$+ (36a_2^2a_4(a_3-k_5)^3 + 16a_2^4a_4(a_3-k_5)^2 - 80a_2^6a_4(a_3-k_5))k_6^2$$
$$+ 4a_4(a_3-k_5)^5 + 56a_2^2a_4(a_3-k_5)^4 + 228a_2^4a_4(a_3-k_5)^3$$
$$+ 304a_2^6a_4(a_3-k_5)^2 + 128a_2^8a_4(a_3-k_5))\Omega^2 + ((a_{12}a_2^4(a_3-k_5)^2$$
$$+ 4a_{12}a_2^6(a_3-k_5))k_6^4 + (2a_{12}a_2^2(a_3-k_5)^4 + 12a_{12}a_2^4(a_3-k_5)^3$$
$$+ 18a_{12}a_2^6(a_3-k_5)^2 + 8a_{12}a_2^8(a_3-k_5))k_6^2 + a_{12}(a_3-k_5)^6$$
$$+ 8a_{12}a_2^2(a_3-k_5)^5 + 22a_{12}a_2^4(a_3-k_5)^4 + 28a_{12}a_2^6(a_3-k_5)^3$$
$$+ 17a_{12}a_2^8(a_3-k_5)^2 + 4a_{12}a_2^{10}(a_3-k_5))\ell^2 + ((-4a_2^6(a_3-k_5)^2$$
$$- 16a_2^8(a_3-k_5))k_6^4 + (-8a_2^4(a_3-k_5)^4 - 48a_2^6(a_3-k_5)^3$$
$$- 72a_2^8(a_3-k_5)^2 - 32a_2^{10}(a_3-k_5))k_6^2 - 4a_2^2(a_3-k_5)^6$$
$$- 32a_2^4(a_3-k_5)^5 - 88a_2^6(a_3-k_5)^4 - 112a_2^8(a_3-k_5)^3$$
$$- 68a_2^{10}(a_3-k_5)^2 - 16a_2^{12}(a_3-k_5))\ell$$

$$\rho_{13} = -\frac{1}{H_6}\cdot\{(32a_2^8(a_3-k_5) + 32a_2^6(a_3-k_5)^2)k_6^3 + (8a_2^4(a_3-k_5)^4$$
$$+ 72a_2^6(a_3-k_5)^3 + 192a_2^8(a_3-k_5)^2 + 128a_2^5 10(a_3-k_5))k_6\}\Omega^2$$

$$\rho_{14} = -\frac{1}{H_6} \cdot \{(32a_2^8 k_6^5 + (52a_2^6(a_3 - k_5)^2 + 120a_2^8(a_3 - k_5) + 160a_2^{10})k_6^3$$

$$+ (-4a_2^4(a_3 - k_5)^4 + 72a_2^6(a_3 - k_5)^3 + 356a_2^8(a_3 - k_5)^2$$

$$+ 408a_2^{10}(a_3 - k_5) + 128a_2^{12})k_6\}\ell + \{80a_2^4 a_4(a_3 - k_5)^2$$

$$+ 64a_2^6 a_4(a_3 - k_5))k_6^3 + (8a_2^2 a_4(a_3 - k_5)^4 + 152a_2^4 a_4(a_3 - k_5)^3$$

$$+ 496a_2^6 a_4(a_3 - k_5)^2 + 352a_2^8 a_4(a_3 - k_5))k_6\}\Omega^2$$

$$+ \{2a_{12}a_2^6(a_3 - k_5)k_6^5 + (4a_{12}a_2^4(a_3 - k_5)^3 + 8a_{12}a_2^6(a_3 - k_5)^2$$

$$+ 4a_{12}a_2^8(a_3 - k_5))k_6^3 + (2a_{12}a_2^2(a_3 - k_5)^5 + 8a_{12}a_2^4(a_3 - k_5)^4$$

$$+ 12a_{12}a_2^6(a_3 - k_5)^3 + 8a_{12}a_2^8(a_3 - k_5)^2 + 2a_{12}a_2^{10}(a_3 - k_5))k_6\}\ell^2$$

$$+ \{-8a_2^8(a_3 - k_5)k_6^5 + (-16a_2^6(a_3 - k_5)^3 - 32a_2^8(a_3 - k_5)^2$$

$$- 16a_2^{10}(a_3 - k_5))k_6^3 + (-8a_2^4(a_3 - k_5)^5 - 32a_2^6(a_3 - k_5)^4$$

$$- 48a_2^8(a_3 - k_5)^3 - 32a_2^{10}(a_3 - k_5)^2 - 8a_2^{12}(a_3 - k_5))k_6\}\ell$$

$$\rho_{15} = \frac{2}{H_9} \cdot a_1^2(k_5 - a_2^2 - a_3)\Omega$$

$$\rho_{16} = \frac{2}{H_9} \cdot (k_5 - a_2^2 - a_3)\Omega$$

$$\rho_{17} = \frac{2}{H_9} \cdot (a_1^2 a_2^2 k_6 \Omega)$$

$$\rho_{18} = \frac{2}{H_9} \cdot (a_2^2 k_6 \Omega)$$

$$\rho_{19} = \frac{1}{H_7} \cdot \{8(4a_1^2 - a_2^2)(k_5 - a_2^2 - a_3)(a_3 - k_5)(\ell a_1^2 + a_5)k_6\Omega^2\}$$

$$\rho_{20} = \frac{1}{H_7} \cdot \{8(4a_1^2 - a_2^2)(k_5 - a_2^2 - a_3)(a_3 - k_5)a_1^2 k_6 \Omega^2\}$$

$$\rho_{21} = \frac{\rho_{20}}{a_1^2}$$

$$\rho_{22} = \frac{1}{H_7} \cdot \{-4a_1^2(4a_1^2 - a_2^2)(a_3 - k_5)[-a_2^2 k_6^2 + (k_5 - a_2^2 - a_3)^2]\Omega^2\}$$

$$\rho_{23} = \frac{\rho_{22}}{a_1^2}$$

$$\rho_{24} = \rho_{23}(\ell a_1^2 + a_5)$$

$$\rho_{25} = -a_2^2\rho_{19}$$

$$\rho_{26} = -a_2^2\rho_{20}$$

$$\rho_{27} = \frac{\rho_{26}}{a_1^2}$$

$$\rho_{28} = -\frac{1}{H_7}\cdot\{(8a_1^2a_2^4(a_3-k_5)+4a_1^2a_2^6-16a_1^4a_2^4)k_6^3$$

$$+(8a_1^2a_2^2(a_3-k_5)^3+(20a_1^2a_2^4+16a_1^4a_2^2)(a_3-k_5)^2$$

$$+(16a_1^2a_2^6-32a_1^4a_2^4)(a_3-k_5)+4a_1^2a_2^8-16a_1^4a_2^6)k_6\}\ell\Omega^2$$

$$\rho_{29} = \frac{1}{H_7}\cdot\{(4a_2^4(a_3-k_5)^2+4a_2^6(a_3-k_5))k_6^2+4a_2^2(a_3-k_5)^4$$

$$+12a_2^4(a_3-k_5)^3+12a_2^6(a_3-k_5)^2+4a_2^8(a_3-k_5)\}\ell\Omega^2$$

$$\rho_{30} = -\frac{1}{H_7}\cdot\{(-8a_2^4(a_3-k_5)+4a_2^6-16a_1^2a_2^4)k_6^3$$

$$+(-8a_2^2(a_3-k_5)^3-(16a_12a_2^2+12a_2^4)(a_3-k_5)^2$$

$$-32a_1^2a_2^4(a_3-k_5)+4a_2^8-16a_1^2a_2^6)k_6\}\ell\Omega^2$$

where

$$H_5 = (4a_1^2-a_2^2)\{4a_1^2k_6^2+((a_3-k_5)+4a_1^2)^2\}\ell$$

$$H_6 = (a_3-k_5)\{-4a_2^6k_6^6+(-9a_2^4(a_3-k_5)^2-24a_2^6(a_3-k_5)-24a_2^8)k_6^4$$

$$+(-6a_2^2(a_3-k_5)^4-36a_2^4(a_3-k_5)^3-90a_2^6(a_3-k_5)^2$$

$$-96a_2^8(a_3-k_5)-36a_2^{10})k_6^2-(a_3-k_5)^6-12a_2^2(a_3-k_5)^5-54a_2^4(a_3-k_5)^4$$

$$-116a_2^6(a_3-k_5)^3-129a_2^8(a_3-k_5)^2-72a_2^{10}(a_3-k_5)-16a_2^{12}\}\ell^2$$

$$H_7 = (4a_1^2-a_2^2)(a_3-k_5)(-a_2^2k_6^2-((a_3-k_5)+a_2^2)^2)^2\ell^2$$

$$H_8 = \{4a_2^4k_6^4+(5a_2^2(a_3-k_5)^2+16a_2^4(a_3-k_5)+20a_2^6)k_6^2$$

$$+(a_3-k_5)^4+10a_2^2(a_3-k_5)^3+33a_2^4(a_3-k_5)^2+40a_2^6(a_3-k_5)+16a_2^8\}\ell^2$$

$$H_9 = -\{a_2^2k_6^2+(k_5-a_2^2-a_3)^2\}\ell.$$

CHAPTER

EIGHT

CONSTANT ANGLE CONTROL FOR DEPLOYMENT AND RETRIEVAL OF TSS

In this chapter, we continue our study of the control of tethered satellite systems, specifically, the deployment and retrieval control for the Tethered Satellite System (TSS). A constant angle control method is hypothesized for subsatellite deployment and retrieval. It is proved that this control law results in stable deployment but unstable retrieval. An enhanced control law for deployment is also proposed, which entails the use of the constant angle method followed by a station-keeping control law once the tether length is sufficiently near the desired value. Finally, simulations are given to illustrate the conclusions.

8.1. Introduction

Arnold [5] proposed a constant angle method for deployment and retrieval of the subsatellite of the tethered satellite system. In [5], the satellite and subsatellite are modeled as point masses and the tether is assumed massless and of length small compared with the radius of the satellite's orbit. Based on these assumptions, Arnold obtained an approximate model of the TSS by applying the gravity-gradient method and argued that the constant angle scheme would result in stable deployment and unstable retrieval.

One goal of this chapter is to give a proof of the validity of these conclusions. Viewing the tether length as an input variable for deployment and retrieval of the TSS, a constant in-plane angle control scheme is considered, which is based on the mathematical model derived in Chapter 6. Within this setting, we prove stability of constant-angle deployment and instability of constant-angle retrieval. This is achieved through the construction of appropriate Liapunov-like functions and by appealing to the finite-time stability theory. A new control strategy for deployment of the subsatellite is also proposed. This control law consists of the constant angle scheme followed by the stabilizing station-keeping control proposed in Section 6.3.

Finally, simulation results are given to demonstrate the analytical conclusions of this chapter.

8.2. Results on Finite-Time Stability

From the basic definitions and conditions for finite-time stability given in Section 2.5, some extended results are proposed in this section. Then these finite-time stability criteria are applied to prove the instability of constant angle retrieval in Section 8.4 and the stability of constant angle deployment in Section 8.5. Note that, the norm used throughout this chapter is the Euclidean norm.

Consider a system given by

$$\dot{x} = f(t, x), \tag{8.1}$$

where $f : \Gamma \times I\!R^n \to I\!R^n$ and $\Gamma := [t_0, t_0 + \mathcal{T})$ for some $t_0 \in I\!R$, $\mathcal{T} > 0$. Let x_0 denote the initial condition of (8.1) at t_0, and let $\phi(t; t_0, x_0)$ be the solution of (8.1) at time t satisfying the initial condition. The basic definitions and conditions of the finite time stability are given in Section 2.5. Those conditions depend on the known bounds, say, α, β and γ. But, stability properties of a system may be investigated without reference to the specific bounds on the states (i.e., α, β and γ). In the following lemma and theorem, two sufficient conditions are introduced for this type of stability. These provide a means for finding the associated bounds α, β, γ. Lemma 8.1 gives a sufficient condition for uniform finite-time stability. Theorem 8.1 then gives a relationship among $\mathcal{T}, \alpha, \beta$ and γ providing a sufficient condition for contractive stability.

Lemma 8.1. System (8.1) is uniformly finite-time stable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ for any given α and β with

$$0 < \alpha < \beta\sqrt{\frac{k_1}{k_2}} \leq \beta \leq r, \tag{8.2}$$

if there exist an $r > 0$ and a continuously differentiable function $V(t,x)$ with

$$\begin{aligned} &\dot{V}(t,x) \leq 0, \\ &k_1||x||^2 \leq V(t,x) \leq k_2||x||^2, \end{aligned} \tag{8.3}$$

for all $x \in \overline{B(r)}$, $t \in \Gamma$. Here, $0 < k_1 \leq k_2$.

Proof. The result follows directly from condition (2.34) of Lemma 2.8.

■

In the next theorem, we introduce a condition on (8.1) and a relationship among $\mathcal{T}, \alpha, \beta$ and γ guaranteeing finite-time *contractive* stability.

Theorem 8.1. System (8.1) is contractively stable with respect to (α, β, γ, Γ, $||\cdot||$) for any triple α, β, γ with

$$\alpha\sqrt{\frac{k_2}{k_1}\cdot\exp(-\frac{k_3}{k_2}\mathcal{T})} \leq \gamma < \alpha < \sqrt{\frac{k_1}{k_2}}\beta < \beta \leq r \tag{8.4}$$

if there exist an $r > 0$ and a continuously differentiable function $V(t,x)$ satisfying the conditions

$$k_1||x||^2 \leq V(t,x) \leq k_2||x||^2, \tag{8.5}$$

$$k_3||x||^2 \leq -\dot{V}(t,x), \tag{8.6}$$

for all $x \in \overline{B(r)}$, $t \in \Gamma$. Here, $k_i > 0$, $i = 1, 2, 3$, and the time interval length $\mathcal{T}$ is such that

$$\mathcal{T} > \frac{k_2}{k_3}\cdot\ln\frac{k_2}{k_1}. \tag{8.7}$$

Proof: Condition (8.6) implies that

$$\dot{V}(t,x) \leq 0, \quad \text{for all} \quad x \in \overline{B(\beta)}, \quad t \in \Gamma.$$

Hence, it is implied by Lemma 8.1 that (8.1) is uniformly finite-time stable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ for any α, β satisfying condition (8.4). Next, we prove quasi-contractive stability of the system. From conditions (8.5) and (8.6), we have

$$\dot{V}(t,x) \leq -\frac{k_3}{k_2}V(t,x), \quad \text{for all } x \in \overline{B(r)}, \quad t \in \Gamma.$$

Hence,

$$V(t,\phi(t;t_0,x_0)) \leq V(t_0,x_0)\exp(-\frac{k_3}{k_2}(t-t_0)), \text{ for all } x_0 \in \overline{B(r)}, \ t \in \Gamma.$$

Then it follows from (8.5) that

$$||\phi(t;t_0,x_0)||^2 \leq \frac{k_2}{k_1}||x_0||^2 \exp(-\frac{k_3}{k_2}(t-t_0)), \text{ for all } x_0 \in \overline{B(r)}, \ t \in \Gamma.$$

Thus, there exists a $t_1 \in \Gamma$ such that $||\phi(t;t_0,x_0)|| < \gamma$, for all $t \in [t_1, t_0 + \mathcal{T})$ when conditions (8.4) and (8.7) hold. According to Definition 2.6, system (8.1) is hence quasi-contractively stable with respect to $(\alpha, \gamma, \Gamma, ||\cdot||)$ for any α, γ satisfying (8.4).

■

According to Definition 2.4 for the finite-time instability, a sufficient condition is given in the next lemma for finding the possible value for α such that system (8.1) is finite-time unstable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$, when the time interval $\mathcal{T}$ and the bound for the trajectory β are given.

Lemma 8.2. System (8.1) is finite-time unstable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ for any $\alpha > ||x_0||$ and $\beta \leq r$, if there exist an $r > 0$, $||x_0|| < r$ and a continuously differentiable function $V(t,x)$ with $V(t_0,x_0) > 0$ and satisfying the following conditions:

$$V(t,x) \leq k_2||x||^2 \text{ and } k_1||x||^2 \leq \dot{V}(t,x) \quad \text{for all } x \in \overline{B(r)}, \quad t \in \Gamma, \tag{8.8}$$

$$\text{and} \quad \mathcal{T} > \frac{k_2^2 r^2 - k_2 V(t_0, x_0)}{k_1 V(t_0, x_0)}, \tag{8.9}$$

where k_1, k_2 are two positive real numbers.

Proof: Since $V(t_0, x_0) > 0$, there exists a $\delta > 0$ such that $V(t_0, x_0) = k_2\delta^2$. Thus, according to the assumption of $\dot{V}(t,x) \geq 0$ we then have

$$V(t, \phi(t; t_0, x_0)) \geq V(t_0, x_0) \geq k_2\delta^2, \quad \text{for all } t \in \Gamma,$$

which (from (8.8)) implies that $||\phi(t; t_0, x_0)|| \geq \delta$ for all $t \in \Gamma$.

Thus, for any given $\beta \leq r$ and $\alpha > ||x_0||$, we have

$$\begin{aligned} V(t_0 + \mathcal{T}, \phi(t_0; t_0, x_0)) &\geq V(t_0, x_0) + k_1\delta^2\mathcal{T} \\ &> k_2 r^2 \geq k_2\beta^2 \end{aligned} \tag{8.10}$$

when the time interval $\mathcal{T}$ satisfies condition (8.9). Thus, there exists a $t_1 \in \Gamma$ such that $||\phi(t_1; t_0, x_0)|| = \beta$. The conclusion is hence implied by Definition 2.4.

■

8.3. Constant In-Plane Angle Control

The stability and stabilization of the TSS during station-keeping have been studied in the previous two chapters. In this chapter, we focus on the control of the deployment and retrieval of the subsatellite of the TSS. By viewing ℓ as an external control input, we can write the state equations for the system (6.6)-(6.8) as follows:

$$\dot{\theta} = \omega_\theta \tag{8.11}$$

$$\dot{\omega}_\theta = -\frac{2\dot{\ell}}{\ell}(\omega_\theta + \Omega) + 2\tan\phi(\omega_\theta + \Omega)\omega_\phi - \frac{\Omega^2 r_0 \sin\theta}{\ell\cos\phi}(1 - \frac{r_0^3}{r_m^3}) \tag{8.12}$$

$$\dot{\phi} = \omega_\phi \tag{8.13}$$

$$\dot{\omega}_\phi = -\frac{2\dot{\ell}}{\ell}\omega_\phi - \frac{1}{2}\sin(2\phi)(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0}{\ell}\cos\theta\sin\phi(1 - \frac{r_0^3}{r_m^3}). \tag{8.14}$$

At an equilibrium point $(\theta^*, \omega_\theta^*, \phi^*, \omega_\phi^*)$ of (8.11)-(8.14), if one exists, we have $\omega_\theta^* = \omega_\phi^* = 0$, $\dot{\ell}$ must satisfy (from Eq. (8.12))

$$\dot{\ell} = -\frac{\Omega r_0}{2\cos\phi^*}\sin\theta^*\left(1 - \frac{r_0^3}{(r_m^*(\ell))^3}\right), \tag{8.15}$$

and ϕ^* must satisfy either

$$\sin\phi^* = 0, \quad \text{or} \tag{8.16a}$$

$$\cos\phi^* = -\frac{r_0}{\ell}\left(1 - \frac{r_0^3}{(r_m^*(\ell))^3}\right)\cos\theta^*, \tag{8.16b}$$

where

$$r_m^*(\ell) := (r_0^2 + \ell^2 + 2r_0\ell\cos\theta^*\cos\phi^*)^{1/2}. \tag{8.17}$$

Remark 8.1. In fact, only the case $\sin\phi^* = 0$ is realistic. To see this, briefly consider the possibility (8.16*b*), which, using (8.15), would imply that at equilibrium $\dot{\ell}$ obeys

$$\dot{\ell} = \frac{\Omega\ell}{2}\tan\theta^*. \tag{8.18}$$

Since $-\frac{\pi}{2} \le \phi^* \le \frac{\pi}{2}$, we have $\cos\phi^* \ge 0$ (see Figure 6.1). Considering the possibilities $0 < \phi^* \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \phi^* < 0$ separately, and referring to Figure 6.1 for the relative magnitudes of $r_m^*(\ell)$ and r_0, we find that the left and right sides of (8.16*b*) are then of opposite sign unless they both vanish. Hence, we obtain $\phi^* = \theta^* = \pm\frac{\pi}{2}$, implying $\dot{\ell}$ of (8.18) would be infinite.

In view of the Remark, we let $\phi^* = 0$. Eq. (8.15) now implies that, at equilibrium, ℓ satisfies

$$\dot{\ell} = -\frac{\Omega r_0}{2}\left(1 - \frac{r_0^3}{(\hat{r}_m^*(\ell))^3}\right)\sin\theta^*, \tag{8.19}$$

where

$$\hat{r}_m^*(\ell) := (r_0^2 + \ell^2 + 2r_0\ell\cos\theta^*)^{1/2}. \tag{8.20}$$

This control law, which is a *constant in-plane angle control method*, has the feature that it results in the existence of an equilibrium point of (8.11)-(8.14). Moreover, the associated equilibrium point of system (8.11)-(8.14) will then be $(\theta^*, 0,0,0)$, where θ^* is the desired in-plane angle.

8.4. Stability Analysis of the TSS During Retrieval

Suppose for simplicity that $\dot{\ell} < 0$ throughout retrieval. From Eq. (8.19) we have

$$\dot{\ell} < 0 \iff -\frac{\Omega r_0}{2}\left(1 - \frac{r_0^3}{(\hat{r}_m^*(\ell))^3}\right)\sin\theta^* < 0.$$

Denote by ℓ_i the initial (pre-retrieval) tether length. Then the condition for $\dot{\ell} < 0$ is that θ^* satisfies either $0 < \theta^* < \frac{\pi}{2}$ or $-\pi < \theta^* < \theta_1$ (see Figure 8.1), where $\theta_1 = \theta_1(\ell_i)$ is such that

$$\cos\theta_1 = -\frac{\ell_i}{2r_0}, \quad -\pi < \theta_1 < -\frac{\pi}{2}. \tag{8.21}$$

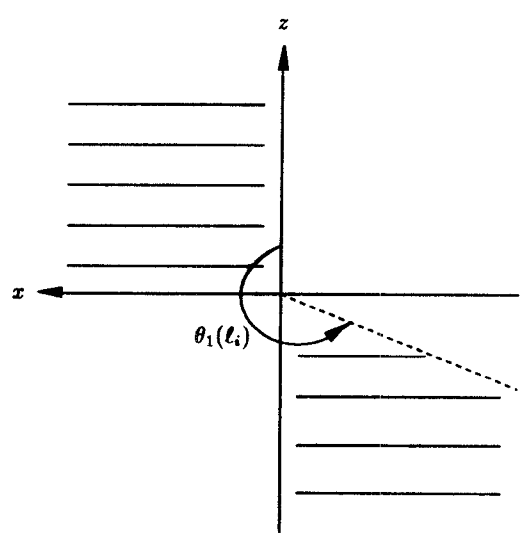

Figure 8.1. Retrieval regions for θ^* with $\phi^* = 0$

Thus, the set of the candidates of the desired in-plane angle θ^* for constant angle retrieval is given as follows

$$S_r := \{\theta | 0 < \theta < \frac{\pi}{2} \quad \text{or} \quad -\pi < \theta < \theta_1(\ell_i)\}, \tag{8.22}$$

where $\theta_1(\ell_i)$ is defined in (8.21). From the discussion above, we have $\dot{\ell} < 0$ and $\ell > 0$ during retrieval. In addition, $\dot{\ell} = 0$ occurs only at $\ell = 0$. Hence, ℓ must approach 0 asymptotically.

Denoting $\tilde{\theta} := \theta - \theta^*$ and $x := (\tilde{\theta}, \omega_\theta, \phi, \omega_\phi)'$, we can then write the series expansion of system (8.11)-(8.14) at the equilibrium point $(\theta^*, 0, 0, 0)$ as follows

$$\dot{\tilde{\theta}} = \omega_\theta \tag{8.23a}$$

$$\dot{\omega}_\theta = n_1(t)\tilde{\theta} + n_3(t)\omega_\theta + f_1(t, x) \tag{8.23b}$$

$$\dot{\phi} = \omega_\phi \tag{8.23c}$$

$$\dot{\omega}_\phi = n_2(t)\phi + n_3(t)\omega_\phi + f_2(t, x) \tag{8.23d}$$

where

$$n_1(t) := 2\Omega\frac{\dot{\ell}}{\ell}\cot\theta^* + \frac{3\Omega^2 r_0^5}{(\hat{r}_m^*(\ell))^5}\sin^2\theta^*, \tag{8.24}$$

$$n_2(t) := -\Omega^2 + 2\Omega\frac{\dot{\ell}}{\ell}\cot\theta^*, \tag{8.25}$$

$$n_3(t) := -2\frac{\dot{\ell}}{\ell}, \tag{8.26}$$

$$\begin{aligned} f_1(t, x) = &-\frac{2\dot{\ell}}{\ell}\Omega + 2\tan\phi(\omega_\theta + \Omega)\omega_\phi \\ &- \frac{\Omega^2 r_0 \sin(\theta^* + \tilde{\theta})}{\ell\cos\phi}(1 - \frac{r_0^3}{\hat{r}_m^3}) - n_1(t)\tilde{\theta}, \end{aligned} \tag{8.27}$$

$$\begin{aligned} f_2(t, x) = &-\frac{1}{2}\sin(2\phi)(\omega_\theta + \Omega)^2 - \frac{\Omega^2 r_0}{\ell}\cdot \\ &\cos(\theta^* + \tilde{\theta})\sin\phi(1 - \frac{r_0^3}{\hat{r}_m^3}) - n_2(t)\phi, \end{aligned} \tag{8.28}$$

with

$$\hat{r}_m := (r_0^2 + \ell^2 + 2r_0\ell\cos(\theta^* + \tilde{\theta})\cos(\phi))^{1/2}. \tag{8.29}$$

Here, we have

$$\lim_{||x||\to 0}\sup_{t\geq 0}\frac{|f_i(t, x)|}{||x||} = 0, \quad \text{for } i = 1, 2.$$

It is shown below that the system (8.11)-(8.14) is not only unstable as the tether length ℓ approachs 0, but also unstable in the sense of *finite-time stability* during the process of constant in-plane angle retrieval, where the tether length ℓ might not be small. Details are given as follows.

First, by invoking an instability criterion given in Lemma 8.3 below, we can prove the equilibrium point $(\theta^*, 0, 0, 0)$ is unstable for retrieval when the tether length ℓ approachs 0.

Lemma 8.3 (e.g., [86]) Consider a system

$$\dot{\xi} = A_0 \xi + F(t, \xi), \tag{8.30}$$

where A_0 is a constant matrix and F is continuous differentiable with $F(t,0) = 0$ and

$$\lim_{||\xi|| \to 0} \sup_{t \geq 0} \frac{||F(t,\xi)||}{||\xi||} = 0.$$

Then the equilibrium point $\xi = 0$ of (8.30) is unstable if at least one of the eigenvalues of A_0 has a positive real part.

■ .

Now, we can apply Lemma 8.3 to show the instability of system (8.11)-(8.14) as the tether length ℓ approachs 0 in the next theorem.

Theorem 8.2. Let the tether length ℓ be governed by the constant in-plane angle retrieval law as in (8.19). Then $(\theta^*$,0,0,0) is an unstable equilibrium point of the system (8.11)-(8.14) as the tether length approachs 0.

Proof: Denote by $\epsilon := \frac{\ell}{r_0}$ the ratio of the tether length with respect to the radius of the satellite's orbit. We can combine the dynamics of $\dot{\ell}$ as in (8.19) with (8.23) to describe the behavior of the TSS during constant in-plane angle retrieval by a mathematical model as follows

$$\begin{pmatrix} \dot{x} \\ \dot{\epsilon} \end{pmatrix} = A_0 \begin{pmatrix} x \\ \epsilon \end{pmatrix} + F(x, \epsilon), \tag{8.31}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3\Omega^2(\sin^2\theta^* - \cos^2\theta^*) & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\Omega^2(1+3\cos^2\theta^*) & a_0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{a_0}{2} \end{pmatrix} \quad (8.32)$$

and $a_0 := 3\Omega\cos\theta^*\sin\theta^*$. Here, $F(x,\epsilon)$ denotes a purely nonlinear function with $F(0,0) = 0$. Since $a_0 > 0$ for all the desired in-plane angle $\theta^* \in S_r$ (given in (8.22)), there are at least two of the eigenvalues of the constant matrix A_0 (given in (8.32)) having positive real parts. According to Lemma 8.3, the origin is hence unstable for (8.31). Referring to the foregoing discussion, we know that the tether length ℓ approachs 0 asymptotically. Thus, the state variable ϵ is stable in system (8.31). Consequently, the only possibility for the unstable state variables in (8.31) are then some of the elements of x, which implies that (θ^*,0,0,0) is an unstable equilibrium for the system (8.11)-(8.14) as the tether length approachs 0.

■

In the rest of this section, we intend to apply Lemma 8.2 to show that the system (8.23) is finite-time unstable during the process of the constant angle retrieval.

We choose

$$V(\tilde{\theta},\omega_\theta,\phi,\omega_\phi) = p_1\tilde{\theta}^2 + 2p_2\tilde{\theta}\omega_\theta + p_3\omega_\theta^2 + p_4\phi^2 + 2p_5\phi\omega_\phi + p_6\omega_\phi^2, \quad (8.33)$$

as a testing function for proving the finite-time instability of system (8.23), where p_i are constant scalars for $i = 1,\cdots,6$.

By taking the derivative of V along the trajectory of system (8.23), we have

$$\begin{aligned}\dot{V}(\tilde{\theta},\omega_\theta,\phi,\omega_\phi) = 2\{&p_2 n_1(t)\tilde{\theta}^2 + (p_1 + p_2 n_3(t) + p_3 n_1(t))\tilde{\theta}\omega_\theta \\ &+ (p_2 + p_3 n_3(t))\omega_\theta^2 + p_5 n_2(t)\phi^2 + (p_5 + p_6 n_3(t))\omega_\phi^2 \\ &+ (p_4 + p_5 n_3(t) + p_6 n_2(t))\phi\omega_\phi + (p_2\tilde{\theta} + p_3\omega_\theta)f_1(t,x) \\ &+ (p_5\phi + p_6\omega_\phi)f_2(t,x)\}. \end{aligned} \quad (8.34)$$

In order to apply Lemma 8.2 to study the instability of system (8.23), we make the following arrangement. First, we choose p_1, p_4 such that

$$p_1 = -\,3p_2\Omega\cos\theta^*\sin\theta^* - 3p_3\Omega^2(\sin^2\theta^* - \cos^2\theta^*), \tag{8.35}$$

$$p_4 = -\,3p_5\Omega\cos\theta^*\sin\theta^* + p_6\Omega^2(1 + 3\cos^2\theta^*), \tag{8.36}$$

for any given desired in-plane angle $\theta^* \in S_r$ and given p_i for $i = 2, 3, 5, 6$, where S_r is as in (8.22).

Then it is obvious to have the following result.

Proposition 8.1. There exist an $r > 0$ and scalars $k_1, k_2 > 0$ such that condition (8.8) of Lemma 8.2 holds if p_1 and p_2 are as in (8.35)-(8.36) and the remaining scalars p_i satisfy the following requirements:

(i) $p_2 n_1(t) > 0$; $p_2 + p_3 n_3(t) > 0$; $4p_2 n_1(t)(p_2 + p_3 n_3(t)) > (p_3 v_1(t) + p_2 v_3(t))^2$,

(ii) $p_5 n_2(t) > 0$; $p_5 + p_6 n_3(t) > 0$; $4p_5 n_2(t)(p_5 + p_6 n_3(t)) > (p_6 v_2(t) + p_5 v_3(t))^2$,

for all $t \in \Gamma = [0, \mathcal{T})$, where $n_i(t)$ for $i = 1, 2, 3$ are as in (8.24)-(8.26) and

$$v_1(t) = n_1(t) - 3\Omega^2(sin^2\theta^* - \cos^2\theta^*), \tag{8.37}$$

$$v_2(t) = n_2(t) + \Omega^2(1 + 3\cos^2\theta^*), \tag{8.38}$$

$$v_3(t) = n_3(t) - 3\Omega\cos\theta^*\sin\theta^*. \tag{8.39}$$

■

Remark 8.2. It is observed from Eq. (8.24) that we have the following conditions to provide the definiteness of $n_1(t)$ along the constant angle retrieval:

(i) $n_1(t) > 2\Omega^2(\sin^2\theta^* - \cos^2\theta^*)$, for $\frac{\pi}{4} < \theta^* < \pi$ or $\frac{5\pi}{4} < \theta^* < \theta_1$ if $\ell < \frac{r_0}{10}(\sin^2\theta^* - \cos^2\theta^*)$,

(ii) $n_1(t) < 2\Omega^2(\sin^2\theta^* - \cos^2\theta^*)$, for $0 < \theta^* < \frac{\pi}{4}$ or $\pi < \theta^* < \frac{5\pi}{4}$ if $\ell < -\frac{r_0}{10}(\sin^2\theta^* - \cos^2\theta^*)$,

(iii) $n_1(t) < -\frac{7\Omega^2}{2\sqrt{2}} \cdot \frac{\ell}{r_0}$ when $\theta^* = \frac{\pi}{4}$ as well as $n_1(t) > \frac{7\Omega^2}{2\sqrt{2}} \cdot \frac{\ell}{r_0}$ when $\theta^* = \frac{5\pi}{4}$.

Similarly, from Eqs. (8.25)-(8.26), we have $n_2(t) < -\Omega^2$ and $n_3(t) > 2\Omega\cos^*\sin^*$ along the constant angle retrieval if $\ell < \frac{2r_0}{15}\tan\theta^*$.

■

By employing Proposition 8.1, Remark 8.2 and Lemma 8.2, we can prove the finite-time instability of system (8.23) along the constant angle retrieval as in the next theorem.

Theorem 8.3. Let the tension control law be as in (8.19). Then there exists an $\beta > 0$ such that system (8.23) is finite-time unstable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ for any $\alpha \leq \beta$ along the retrieval, if the time interval $\mathcal{T}$ is large enough as well as the desired in-plane angle θ^* and the tether length satisfy either of the following relationships:

(i) if $\frac{\ell}{r_0} < \min\{\frac{1}{10}(-\sin^2\theta^* + \cos^2\theta^*), \frac{2}{15}\tan\theta^*, \frac{1}{18}\sin 2\theta^*\sqrt{\cos 2\theta^*}\}$ when $0 < \theta^* < \frac{\pi}{4}$ or $\pi < \theta^* < \frac{5\pi}{4}$,

(ii) if $\frac{\ell}{r_0} < \min\{\frac{1}{100}, \frac{2}{15}\tan\theta^*\}$ when $\theta^* = \frac{\pi}{4}$ or $\theta^* = \frac{5\pi}{4}$, or

(iii) if $\frac{\ell}{r_0} < \min\{\frac{1}{10}(\sin^2\theta^* - \cos^2\theta^*), \frac{2}{15}\tan\theta^*, \frac{1}{18}\sin 2\theta^*\sqrt{-\cos 2\theta^*}\}$ when $\frac{\pi}{4} < \theta^* < \pi$ or $\frac{5\pi}{4} < \theta^* < \theta_1(\ell_i)$, where $\theta_1(\ell_i)$ is as in (8.21).

Proof: In the following, we will employ the observations given in Remark 8.2 to construct the scalars p_i for $i = 1, \cdots, 6$ such that the sufficient conditions of Proposition 8.1 hold. Then we apply Lemma 8.2 to prove the conclusion of theorem. Details of this are given as follows.

First, by choosing $p_5 = -\Omega\cos\theta^*\sin\theta^*$, $p_6 = 1$ and p_4 being calculated by (8.36), we can check that the condition (ii) of Proposition 8.1 will hold for each $\theta^* \in S_r$ with the assumption of $\ell < \frac{2r_0}{15}\tan\theta^*$.

It is more complicated to construct the scalars p_i, for $i = 1, 2, 3$ such that condition (i) of Proposition 8.1 holds. Indeed, in the following, we choose different values for the scalars p_i under the different situations of the desired in-plane angle. For instance, we choose $p_2 = -\Omega\cos\theta^8\sin\theta^*$, $p_3 = 1$ and p_1 is calculated by using the formula (8.35) for the cases in which the desired in-plane angle θ^* satisfies the condition: $0 < \theta^* < \frac{\pi}{4}$ or $\pi < \theta^* < \frac{5\pi}{4}$. It is not difficult to check that these p_i's satisfy the condition (i) of Proposition 8.1 by invoking the observations in Remark 8.2, while $\frac{\ell}{r_0} < \min\{\frac{1}{10}(-\sin^2\theta^* + \cos^2\theta^*), \frac{1}{18}\sin 2\theta^*\sqrt{\cos 2\theta^*}\}$.

For the cases in which the desired in-plane angle $\theta^* = \frac{\pi}{4}$ and $\theta^* = \frac{5\pi}{4}$, we can choose $p_2 = -\Omega\cos\theta^*$, $p_3 = 1$ and p_1 is calculated by using Eq. (8.35). Similarly, the condition (i) of Proposition 8.1 holds when $\ell < \frac{r_0}{100}$.

Let $p_2 = \Omega\cos\theta^*\sin\theta^*$, $p_3 = 1$ and p_1 be calculated from (8.35). By invoking the observations in Remark 8.2 and assuming that $\frac{\ell}{r_0} < \min\{\frac{1}{10}(\sin^2\theta^* - \cos^2\theta^*), \frac{1}{18}\sin 2\theta^*\sqrt{-\cos 2\theta^*}\}$, we can then show that these p_i's satisfy the condition (i) of Proposition 8.1 for the cases in which the desired in-plane angle θ^* satisfies the condition: $\frac{\pi}{4} < \theta^* < \pi$ or $\frac{5\pi}{4} < \theta^* < \theta_1(\ell_i)$, where $\theta_1(\ell_i)$ is as in (8.21).

As implied by Proposition 8.1, there exists an $r > 0$ and real scalars $k_1, k_2 > 0$ such that condition (8.8) holds. Moreover, according to the foregoing choice of $p_3 = p_6 = 1$, we always have an initial state $x_0 = (0, \omega_\theta^0, 0, \omega_\phi^0)'$ of system (8.23) arbitrarily close to the origin such that $V(t, x_0) > 0$, where function V is as in (8.33). Thus, as implied by Lemma 8.2, system (8.23) is finite-time unstable with respect to $(\alpha, \beta, \Gamma, ||\cdot||)$ for any given $\alpha \le \beta \le r$ during the constant angle retrieval, if the time interval $\mathcal{T}$ is large enough and one of the conditions (i)-(iii) of Theorem 8.3 holds.

■

Note that, it is not difficult to observe from the proof of Lemma 8.2 that not only system (8.23) is finite-time unstable as claimed in Theorem 8.3, the state disturbances of system (8.23) will also diverge as long as the time interval $\mathcal{T}$ is large enough.

8.5. Stability Analysis of the TSS During Deployment

In this section, we consider the application of the constant in-plane angle strategy of Section 8.3 to subsatellite deployment. For simplicity, suppose that

$\dot{\ell} > 0$ for all $t \geq t_0$. Since $\dot{\ell}$ is always positive in this consideration, one might expect that the tether length ℓ increases without bound. In reality, only a finite final tether length is meaningful for deployment. Stability of the TSS is hence only considered in a finite time interval, where standard Liapunov stability criteria cannot be employed. Results from finite-time stability shall be applied to study the behavior of the TSS during deployment. Especially, the contractive stability criteria are used to study the stability of the TSS during constant in-plane angle deployment. In addition to the proof of finite-time stability of deployment, a switching type control law combining constant angle deployment and station-keeping control is also proposed to achieve asymptotic stability. Details of this are given as follows.

In the following discussion, we consider the deployment of the subsatellite of the tethered satellite system. For simplicity, let $\dot{\ell} > 0$ throughout deployment. By Eq. (8.19), we have

$$\dot{\ell} > 0 \iff -\frac{\Omega r_0}{2}(1 - \frac{r_0^3}{(\hat{r}_m^*(\ell))^3}) \sin\theta^* > 0.$$

From the discussion above and Eq. (8.20), the condition on θ^* for $\dot{\ell} > 0$ is that either $\theta_2(\ell_f) < \theta^* < \pi$, or $-\frac{\pi}{2} < \theta^* < 0$ (see Figure 8.2), where $\theta_2(\ell_f)$ solves

$$\cos\theta_2 = -\frac{\ell_f}{2r_0}, \quad 0 < \theta_2 < \pi, \tag{8.40}$$

and ℓ_f is the desired post-deployment tether length.

Two strategies for deployment are considered here. The first consists of the constant in-plane angle control law for deployment, and the second involves the constant in-plane angle control law followed by a stabilizing station-keeping control once the desired in-plane angle is close enough to 0 radians or π radians.

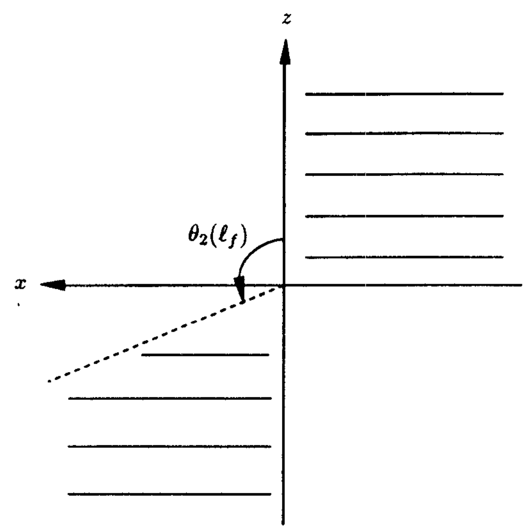

Figure 8.2. Deployment regions for θ^* with $\phi^* = 0$

Strategy 8.1: Constant Angle Control Only

We now consider application of the constant in-plane angle control law discussed above to subsatellite deployment. In the following, ℓ_f denotes the desired final tether length and ℓ_i denotes the initial tether length, which accounts for support by a boom.

From (8.19), we have

$$\begin{aligned}\frac{\dot{\ell}}{\ell} &= -\frac{\Omega r_0 \sin\theta^*}{2\ell}(1 - \frac{r_0}{(\hat{r}_m^*(\ell))^3}) \\ &= -\frac{\Omega \sin\theta^*}{2} \cdot \frac{r_0}{\ell} \cdot \frac{(\hat{r}_m^*(\ell))^6 - r_0^6}{(\hat{r}_m^*(\ell))^3[r_0^3 + (\hat{r}_m^*(\ell))^3]} \\ &\geq -\Omega \sin\theta^* \cos\theta^* > 0 \end{aligned} \tag{8.41}$$

for any $\theta^* \in S_d$ and $\ell_i \leq \ell \leq \frac{1}{22} r_0$, where (in radian measure)

$$S_d := \{\theta^* | -0.68 \leq \theta^* < 0, \text{ or } 2.5 \leq \theta^* < \pi\}.$$

Hence, $\frac{\dot{\ell}}{\ell}$ is bounded below for all $\ell_i \leq \ell \leq \ell_f \leq \frac{1}{22} r_0$ and $\theta^* \in S_d$, and similarly for $\dot{\ell}$. Thus, for any $\ell_f > \ell_i$, ℓ will increase past ℓ_f at some $\mathcal{T} > 0$. Theorem 8.4 below asserts that the system will be finite-time contractively stable during deployment over the interval $[t_0, t_0 + \mathcal{T})$, near the equilibrium point $(\theta^*, 0, 0, 0)$ with $\theta^* \in S_d$.

Theorem 8.4. Suppose $\Omega < 1$, $\ell_f \leq \frac{1}{22} r_0$, and $\Gamma := [t_0, t_0 + \mathcal{T})$. There is an $r > 0$ such that system (8.11)-(8.14) is finite-time contractively stable with respect to $(\alpha, \beta, \gamma, \Gamma, ||\cdot||)$ at the equilibrium point $(\theta^*, 0, 0, 0)$ for any α, β, γ and $\mathcal{T}$ satisfying (8.4) and (8.7), if either of the following two conditions on the desired in-plane angle θ^* holds:

(i) $-0.68 \leq \theta^* < 0$,

(ii) $2.5 \leq \theta^* < \pi$.

Proof: Denote $m := \Omega \sin 2\theta^*$. It is clear from (8.41) that

$$-\frac{2\dot{\ell}}{\ell} \leq m < 0, \qquad \text{for all} \quad t \in \Gamma,$$

if either (i) or (ii) holds and $\ell \leq \ell_f \leq \frac{1}{22} r_0$. Invoking the finite-time stability criterion given in Theorem 8.1, the stability of the TSS during constant angle deployment can be proved as follows.

Using a general construction [54] for a class of second order linear time-variant system, we prove the finite-time contractive stability of (8.11)-(8.14) during deployment by employing the Liapunov-like function

$$\begin{aligned} V(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) = (\frac{2\dot{\ell}}{\ell} + \frac{n_1(t)}{m})\tilde{\theta}^2 + 2\tilde{\theta}\omega_\theta - \frac{1}{m}\omega_\theta^2 \\ + (\frac{2\dot{\ell}}{\ell} + \frac{n_2(t)}{m})\phi^2 + 2\phi\omega_\phi - \frac{1}{m}\omega_\phi^2, \end{aligned} \tag{8.42}$$

where $n_i(t)$ for $i = 1, 2$ are given in (8.24)-(8.25).

Then corresponding to the original system (8.11)-(8.14), we have

$$\begin{aligned} \dot{V}(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) = n_4(t)\tilde{\theta}^2 + n_5(t)\phi^2 + 2(1 + \frac{2\dot{\ell}}{m\ell}) \cdot (\omega_\theta^2 + \omega_\phi^2) \\ + 2(\tilde{\theta} - \frac{\omega_\theta}{m}) f_1(t, x) + 2(\phi - \frac{\omega_\phi}{m}) f_2(t, x), \end{aligned} \tag{8.43}$$

where $x := (\tilde{\theta}, \omega_\theta, \phi, \omega_\phi)'$,

$$n_4(t) = 2n_1(t) + \frac{d}{dt}\{\frac{2\dot{\ell}}{\ell}\} + \frac{1}{m} \cdot \frac{dn_1(t)}{dt}, \tag{8.44}$$

$$n_5(t) = 2n_2(t) + \frac{d}{dt}\{\frac{2\dot{\ell}}{\ell}\} + \frac{1}{m} \cdot \frac{dn_2(t)}{dt}, \tag{8.45}$$

and $f_i(t,x)$ for $i = 1,2$ are given in (8.27)-(8.28).

First, consider the case in which θ^* satisfies condition (i). After some calculations using Eqs. (8.24)-(8.25) and (8.42), we find that there exist $k_{1,1}$, $k_{1,2} > 0$ (given in the Appendix 8.A) such that

$$k_{1,1}||x||^2 \leq V(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) \leq k_{1,2}||x||^2, \quad \text{for all} \quad t \in \Gamma. \tag{8.46}$$

Moreover, by choosing $k_{1,3} := 0.132\Omega^2$ and

$$r = \sup_{t\in\Gamma}\{\ ||x||\ :\ |f_1| \leq \frac{mk_{1,3}||x||}{2(m-1)} \quad \text{and} \quad |f_2| \leq \frac{mk_{1,3}||x||}{2(m-1)}\}, \tag{8.47}$$

we have

$$-\dot{V}(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) \geq k_{1,3}||x||^2, \quad \text{for all} \quad t \in \Gamma, \quad x \in \overline{B(r)}. \tag{8.48}$$

Thus, conditions (8.5)-(8.6) are satisfied and the conclusion follows from Theorem 8.1.

Similarly, for the case in which θ^* satisfies condition (ii), we have

$$k_{2,1}||x||^2 \leq V(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) \leq k_{2,2}||x||^2, \quad \text{for all} \quad t \in \Gamma, \tag{8.49}$$

where $k_{2,1}$, $k_{2,2} > 0$ are also specified in the Appendix 8.A. By choosing $k_{2,3} := 0.0442\Omega^2$ and

$$r = \sup_{t\in\Gamma}\{\ ||x||\ :\ |f_1| \leq \frac{mk_{2,3}||x||}{2(m-1)} \quad \text{and} \quad |f_2| \leq \frac{mk_{2,3}||x||}{2(m-1)}\}, \tag{8.50}$$

we guarantee that

$$-\dot{V}(t, \tilde{\theta}, \omega_\theta, \phi, \omega_\phi) \geq k_{2,3}||x||^2, \quad \text{for all} \quad t \in \Gamma, \quad x \in \overline{B(r)}. \tag{8.51}$$

The conclusion again follows from Theorem 8.1.

■

The finite-time contractive stable regions of the desired in-plane angle θ^* for constant in-plane angle deployment are given in Theorem 8.4. In addition, a relationship between the time-interval $\mathcal{T}$, the bound of initial disturbances and the final contracted region is set up in Theorem 8.1. Furthermore, the simulation results given in Section 8.6.2 show that the criteria given in Theorem 8.4 are not vacuous.

Strategy 8.2: Station-Keeping Control Included

A tension control law has been designed in Chapter 6 to regulate the tether length, while ensuring the out-of-plane angle $\phi = 0$ and the in-plane angle $\theta = 0$ or $(\theta = \pi)$. Combining the result of Theorem 8.4 with the station-keeping control strategies of Theorems 6.1 and 6.2, a switching control law for deployment is constructed as follows:

Step 1. Apply the constant angle control law (8.19) for the first step subsatellite deployment, in which the desired in-plane angle θ^* satisfies the conditions of Theorem 8.4 and is close to 0 (or π).

Step 2. Apply the tension control law given in Theorem 6.1 (or Theorem 6.2) once the tether length is sufficiently near the desired length ℓ_f.

Theorem 8.4 implies that the initial disturbance in the state of the TSS can be attenuated. Specifically, with the desired in-plane angle sufficiently near 0 or π, the system state can be steered to the domain of attraction of the station-keeping control mode in Step 1. Hence, the tether length will be regulated to the desired length upon switching to the station-keeping stabilization control when the tether length is sufficiently near the desired value. Simulation results of a typical system given in Section 8.6.2 demonstrate the asymptotic stability of the TSS using this algorithm.

8.6. Simulation Results

Many simulation examples for tethered satellite systems in the station-keeping mode have been presented in Section 6.4. In this section, we present simulation results only for deployment and retrieval.

A TSS with following characteristics is considered :

- Orbital radius $r_0 = 6598$ km,
- Subsatellite mass $m = 170$ kg,
- Orbital angular velocity $\Omega = 0.0011781$ rad/sec.

In the following discussion, $\tilde{\theta} = \theta - \theta^*$ denotes the differential of the in-plane angle, ℓ_f denotes the desired final tether length, $\tilde{\ell} = \ell - \ell_f$ denotes the differential of the tether length and T denotes the applied tension control force.

8.6.1. Retrieval

As discussed in Section 8.4, the set of candidate in-plane angles for constant angle retrieval S_r is as in (8.22). Let the initial state of the system be $\phi = 0.01$, $\tilde{\theta} = -0.01$, and $\omega_\theta = \omega_\phi = 0$. The initial tether length ℓ_i is assumed to be 10 km. It is observed from Figures 8.3 and 8.4 that the equilibrium point $(\theta^*, 0, 0, 0)$ is unstable during retrieval with a desired in-plane angle of $\theta^* = -3.0$ and $\theta^* = -1.6$, respectively. As mentioned in Chapter 6, since the tether is not in reality rigid, the applied tension control force cannot be positive (to rule out compression). However, Figure 8.4(d) shows that a positive tension control force T occurs during some time interval. Thus, if a constant angle control law is applied during retrieval, then not only will the system be unstable, but tether compression may also occur.

It is also found that when the desired in-plane angle satisfies $\theta^* \in S_1 := \{-2.1 < \theta^* < \theta_1(\ell_i)\}$ for constant angle retrieval, the applied tension control force T can assume positive values during some time-intervals, i.e., compression may occur. The system response for $\theta^* = -2.1$ and constant angle retrieval is depicted in Figure 8.5, where T is found (see Figure 8.5(d)) to at times be very close to 0 but is never positive.

Similar simulation results are found for the region $0 < \theta^* < \frac{\pi}{2}$ for constant angle retrieval. The equilibrium point $(\theta^*, 0, 0, 0)$ is found to be unstable during retrieval and compression of the tether may occur in case $1.0 < \theta^* < \frac{\pi}{2}$. The system responses are not shown.

8.6.2. Deployment

According to Theorem 8.4, the set of candidate θ^* for stable deployment is

$$S_d = \{\theta | -0.68 \leq \theta < 0, \quad \text{or} \quad 2.5 \leq \theta < \pi\}.$$

Let the initial disturbance of the system be $\phi = 0.01$, $\tilde{\theta} = -0.01$, and $\omega_\theta = \omega_\phi = 0$. The initial tether length is assumed to be $\ell_i = 10$ m, which is provided by a boom. First, the system response during deployment (applying constant in-plane angle control only) are depicted in Figures 8.6 and 8.7, with $\theta^* = -0.68$, and $\theta^* = 2.5$, respectively. It is observed from the system responses that, for instance, the differential of the in-plane angle $\tilde{\theta}$ and the out-of-plane angle ϕ decay during deployment.

The switching control strategy, which involves both constant angle control and station-keeping control, is applied to deploy a subsatellite from the satellite with the desired final tether length $\ell_f = 10$ km. The first example concerns deploying the subsatellite upward (i.e., away from the Earth) by applying constant angle control with $\theta^* = -0.015$ for the first 260,500 seconds, and applying the station-keeping control thereafter. The applied tension control force for station-keeping is governed by

$$T = -m(U + h_1\tilde{\ell} + h_2\dot{\ell}), \tag{8.52}$$

where $U = 0.041019$, $h_1 = 3.1\Omega^2$ and $h_2 = 0.0034$. The responses of the system during constant angle deployment are shown in Figure 8.8. At time $t = 260,500$ seconds, we have

- out-of-plane angle $\phi = -7.01636 \times 10^{-6}$, and $\dot{\phi} = 1.70633 \times 10^{-8}$ rad/sec

- in-plane angle $\theta = -0.0150051$,and $\dot{\theta} = 5.61812 \times 10^{-10}$ rad/sec
- actual tether length $\ell = 9.97617$ km and $\dot{\ell} = 2.63603 \times 10^{-4}$ km/sec.

With these values, the applied tension control law is switched to the station-keeping control and governed by Eq. (8.52). The system responses governed by (8.52) are depicted in Figure 8.9.

Another example for deploying the subsatellite downward (i.e., toward the Earth) is implemented by applying constant angle control for the first 235,300 seconds with $\theta^* = 3.125$, then switched to the station-keeping control governed by Eq. (8.52). At time $t = 235,300$ seconds, we have

- out-of-plane angle $\phi = -2.01378 \times 10^{-6}$, and $\dot{\phi} = -2.35517 \times 10^{-8}$ rad/sec
- in-plane angle $\theta = 3.1250$, and $\dot{\theta} = 8.96566 \times 10^{-9}$ rad/sec
- actual tether length $\ell = 9.88531$ km and $\dot{\ell} = 2.90670 \times 10^{-4}$ km/sec.

The system responses are shown in Figures 8.10 and 8.11.

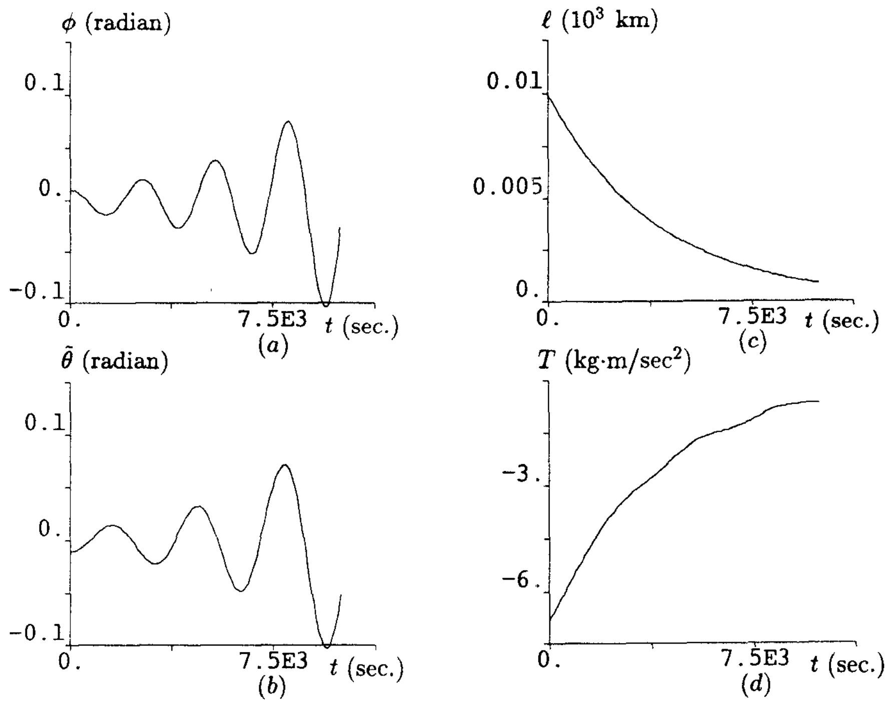

Figure 8.3. Simulation results for constant angle retrieval with $\theta^* = -3.0$ radians

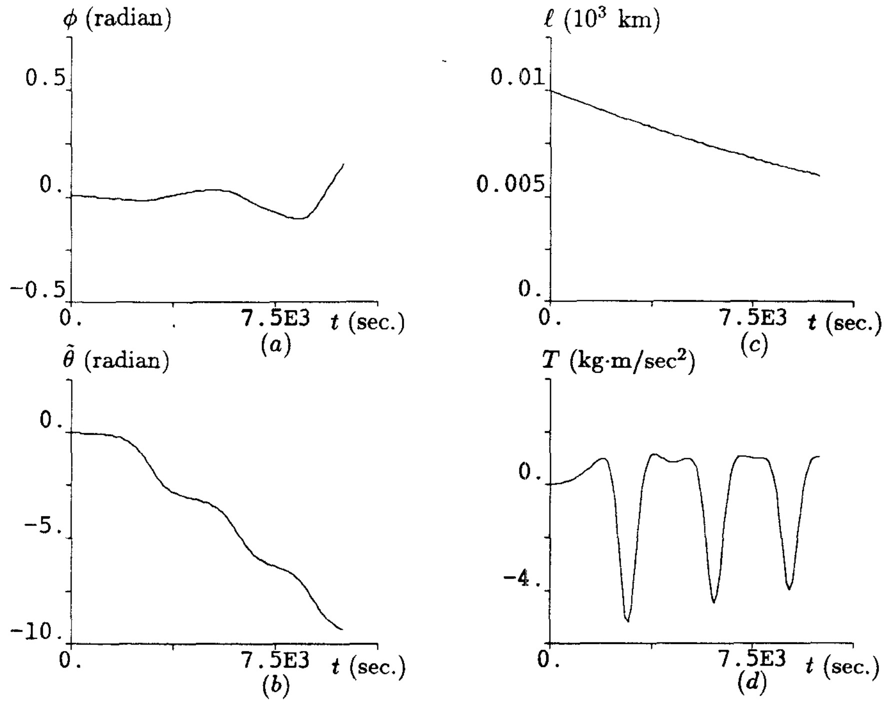

Figure 8.4. Simulation results for constant angle retrieval with $\theta^* = -1.6$ radians

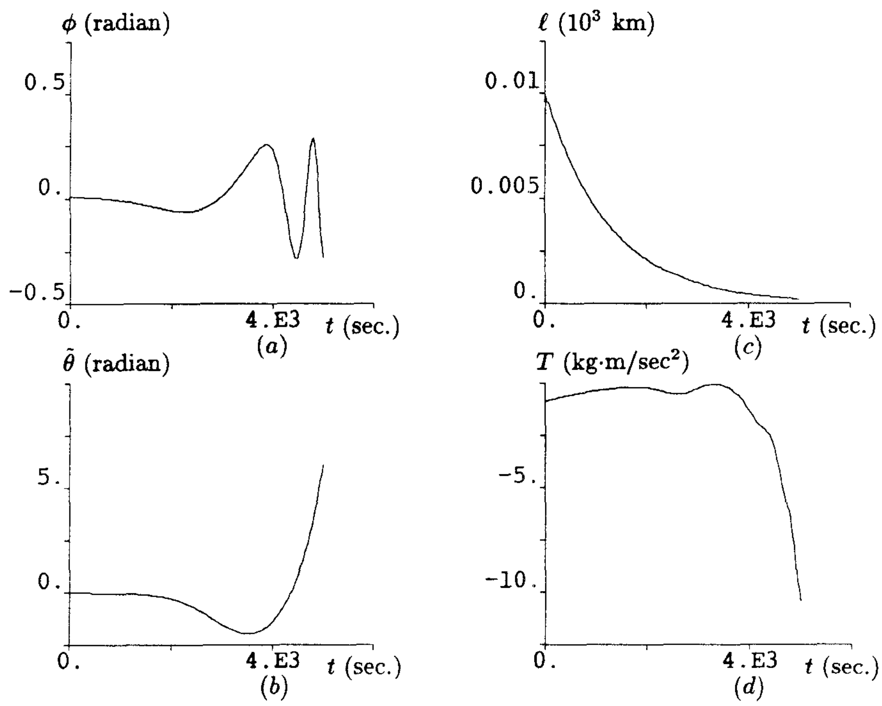

Figure 8.5. Simulation results for constant angle retrieval with $\theta^* = -2.1$ radians

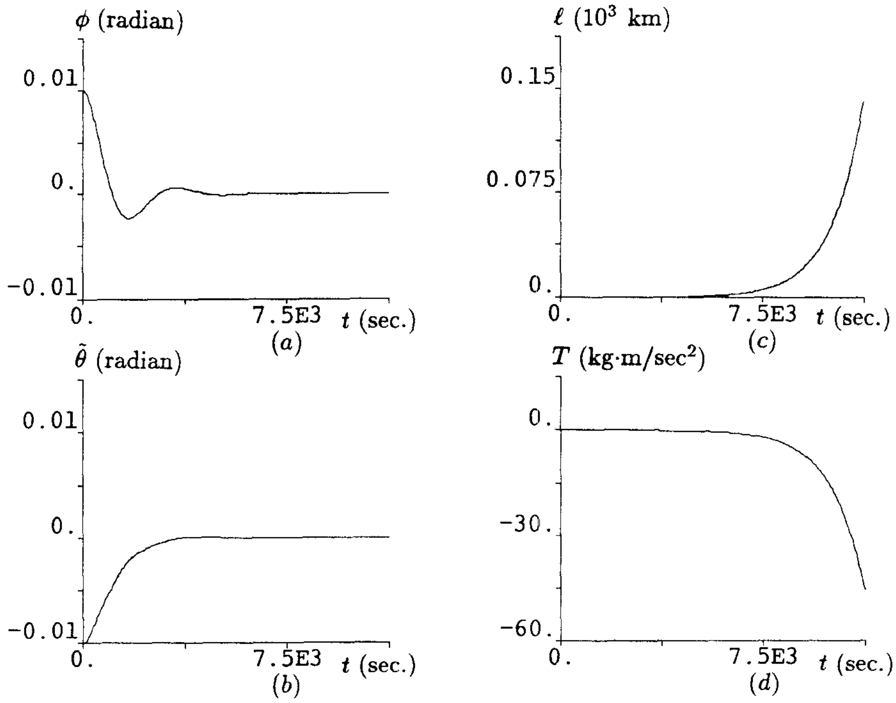

Figure 8.6. Simulation results for constant angle deployment with $\theta^* = -0.68$ radians

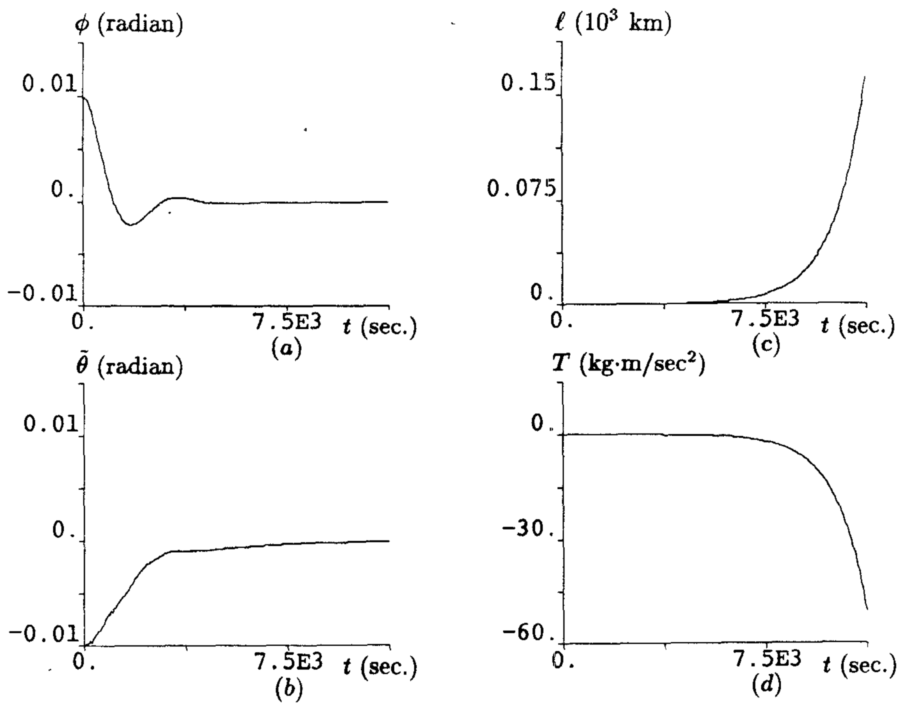

Figure 8.7. Simulation results for constant angle deployment with $\theta^* = 2.5$ radians

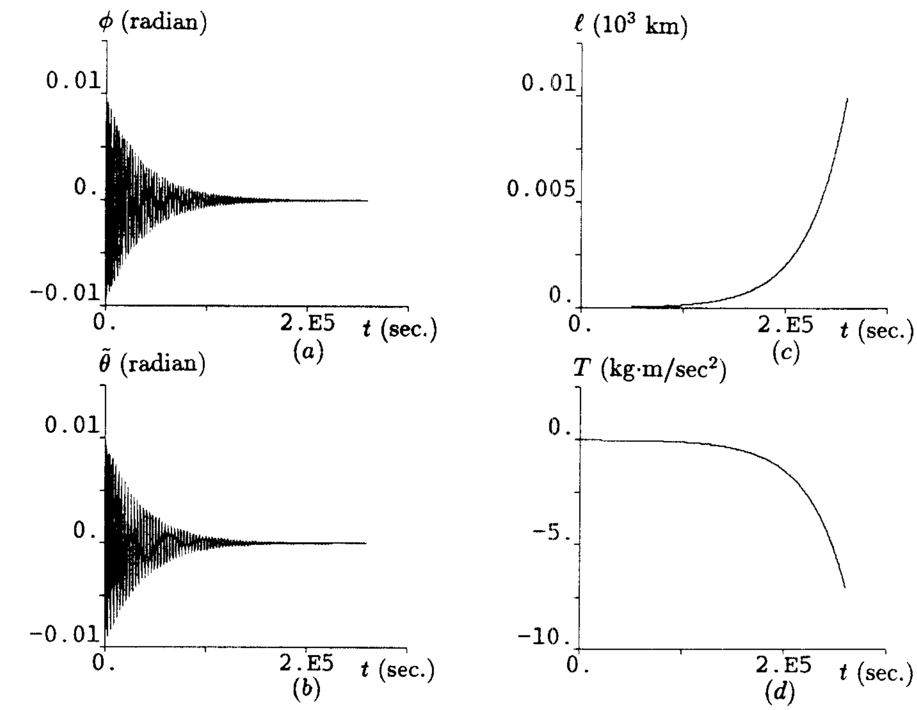

Figure 8.8. Simulation results for constant angle deployment with $\theta^* = -0.015$ radians

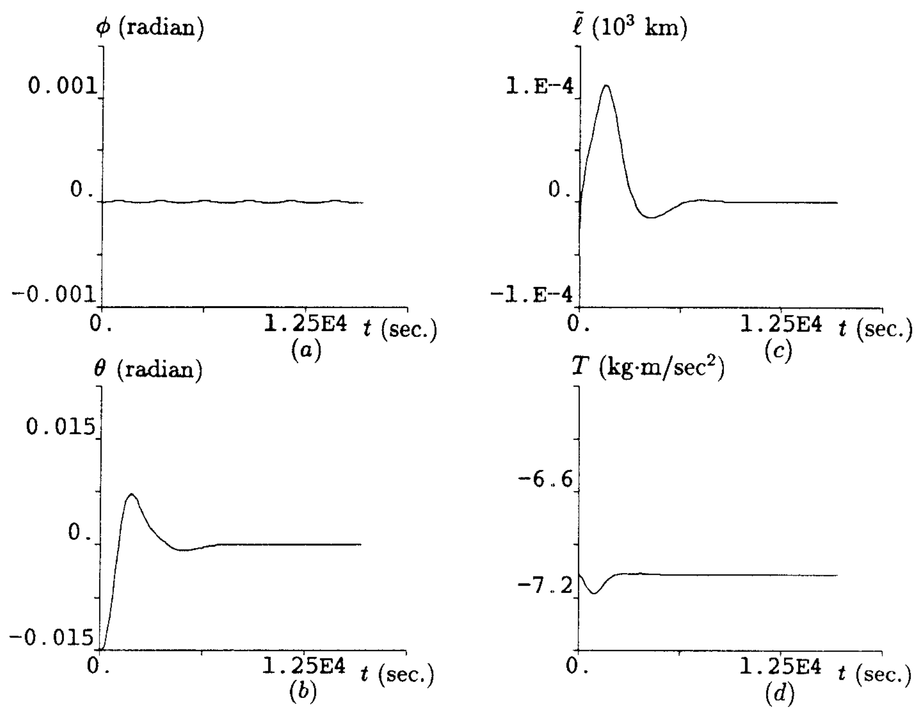

Figure 8.9. Simulation results for station-keeping with $\theta^* = 0$ radians

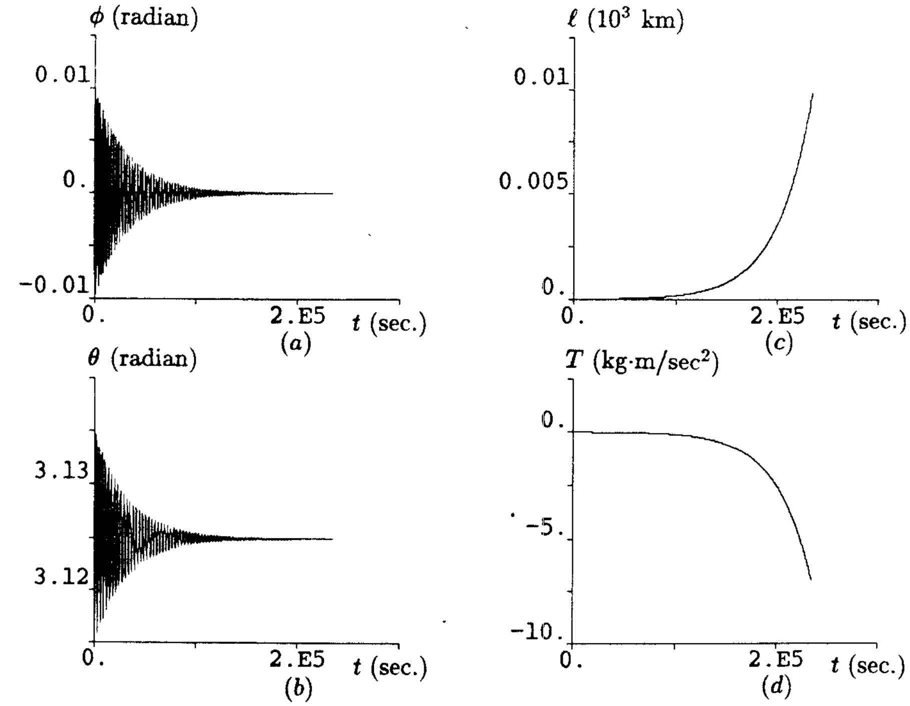

Figure 8.10. Simulation results for constant angle deployment with $\theta^* = 3.125$ radians

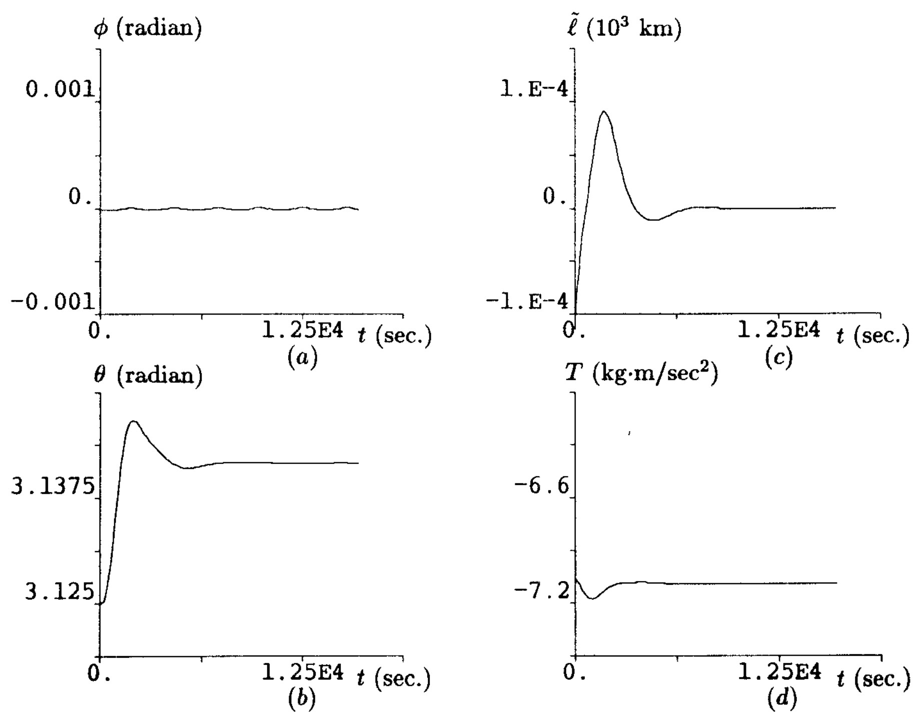

Figure 8.11. Simulation results for station-keeping with $\theta^* = \pi$ radians

Appendix 8.A

The values of $k_{i,j}$, $i = 1, 2$ and $j = 1, 2$ are given as below.

$$k_{1,1} = \frac{0.295575407 \csc^2 2\theta^*}{k_{1,2}}$$

$$k_{1,2} = \frac{l_1 + l_2 + \sqrt{(l_1 + l_3)^2 + 4}}{2}$$

$$k_{2,1} = \frac{0.498328311 \csc^2 2\theta^*}{k_{2,2}}$$

$$k_{2,2} = \frac{l_4 + l_5 + \sqrt{(l_4 + l_6)^2 + 4}}{2}$$

where

$$l_1 = -3\Omega \cos\theta^* \sin\theta^* - 0.5\Omega \sin\theta^*$$

$$l_2 = -\frac{\Omega^2(3\cos^2\theta^* + 0.5\cos\theta^*) + 1}{\Omega \sin 2\theta^*}$$

$$l_3 = -\frac{\Omega^2(3\cos^2\theta^* + 0.5\cos\theta^*) - 1}{\Omega \sin 2\theta^*}$$

$$l_4 = -3.68961\Omega \cos\theta^* \sin\theta^*$$

$$l_5 = -\frac{3.6896061\Omega^2 \cos^2\theta^* + 1}{\Omega \sin 2\theta^*}$$

$$l_6 = -\frac{3.6896061\Omega^2 \cos^2\theta^* - 1}{\Omega \sin 2\theta^*}.$$

CHAPTER NINE

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

Among the topics studied in this dissertation is a detailed consideration of the applications of the Center Manifold Theorem to the stability analysis and stabilization of nonlinear critical systems. For these systems, the Jacobian matrix has eigenvalues lying on the imaginary axis; while the remaining eigenvalues are either stable or stabilizable by feedbacks. The feedback stabilizing control laws have been designed for both linearly controllable and linearly uncontrollable cases for the critical modes (i.e., for the eigenvalues which lie on the imaginary axis). A linear transformation has been introduced to play a key role in linear and linear-plus-quadratic feedback designs for the linearly uncontrollable case, facilitating application of the Center Manifold Theorem to system stabilization. In the linearly controllable case, we have focused on the design of purely nonlinear feedback stabilizing control laws.

The stabilization of two simple critical cases in which the linearized system model has one zero eigenvalue or a pair of nonzero pure imaginary eigenvalues have been obtained to demonstrate the applications. Moreover, a nonlinear transformation, the so-called "normal form formulation", was employed to supplement the study of the stability and stabilization of nonlinear systems with compound criticalities. The compound criticalities considered in this dissertation are the cases in which the linearized system model has two zero eigenvalues with geometric multiplicity one; one zero eigenvalue and one pair of nonzero pure imaginary eigenvalues; or two distinct pairs of nonzero pure imaginary eigenvalues, along with the assumption of the remaining eigenvalues being stable or stabilizable. We have obtained the stability conditions and have designed feedback stabilizing control laws in terms of the original nonlinear system dynamics for these critical systems, as an alternative to those given in terms of system dynamics in normal form [10]. Moreover, our results do not restrict the dimensionality of the noncritical modes.

In this thesis, we have also proposed a technique to construct Liapunov function candidates for general nonlinear critical systems in which the center manifold reduction technique is employed to simplify the complexity of the design. To demonstrate the applications of the proposed technique, we have constructed stability conditions for the simple critical cases and those for the compound critical cases by using the proposed composite Liapunov function approach. The stability results were found to agree with those obtained by using normal function reduction. Furthermore, families of Liapunov functions for these critical cases have also been obtained. The center manifold reduction results presented in this thesis may be easily coded using a symbolic algebra package.

In the practical applications, we have studied the mathematical model and the control of the Tethered Satellite Systems (TSS). A point-mass model of the TSS was derived based on several simplifying assumptions. Linear and/or nonlinear state feedback stabilizing control laws for the TSS during the station-keeping mode have been obtained by using the Hopf bifurcation theorem. It was found that such stabilizing control laws can also be implemented by using center manifold reduction. Another approach, using application of center manifold reduction for the stabilization of double critical systems whose linearized model has two distinct pairs of nonzero pure imaginary eigenvalues, was also proposed to guarantee the stability of the TSS during station-keeping and to improve the system performances. Simulations indicate that the transient response of the system by using this new method is superior to that obtained from the design via the Hopf bifurcation stability criterion. The stability of the constant in-plane angle control for the deployment and the retrieval of the subsatellite of the TSS have also been addressed in this thesis. By invoking the "finite-time stability" criteria, we have proved that the TSS will be unstable during retrieval, but stable during deployment. A new switching type controller, which combines the constant in-plane angle control and the station-keeping control, was also designed to guarantee the asymptotic stability for subsatellite deployment.

To further extend the researches covered in this thesis, we note several possible directions. First, the stabilization techniques proposed in this thesis using center manifold reduction can be applied to study the local stabilization of parametrized families nonlinear systems, specifically, bifurcating systems and multiple time-scale systems. Secondly, the proposed method for constructing families of Liapunov functions can be used to study the optimization-based nonlinear controller design (for instance, the requirements of optimal transient performance and the largest attraction domains), specifically, for the critical nonlinear systems. A third possible direction for future research is to further study the stabilization and control of the TSS. Application of existing control techniques and development of new control ideas in this area are important. In this thesis, we only focused on the stability analysis and the control for the simple point-mass model of the TSS. More complicated models, which include flexibility, mass of the tether and other possible factors, should be addressed in future research.

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