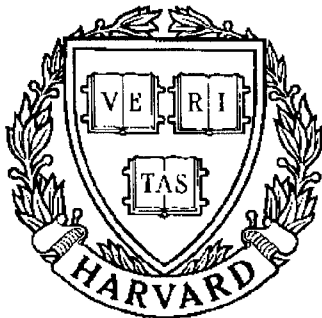


THESIS REPORT

Ph.D.



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Admission Control and Routing Issues in Data Networks

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**ADMISSION CONTROL AND ROUTING ISSUES
IN DATA NETWORKS**

by
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ABSTRACT

Title of Thesis: Admission Control and Routing Issues
in Data Networks
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In modern telecommunication and computer networks, there is an increasing demand to provide simultaneously a variety of services to heterogeneous traffic types with diverse characteristics and performance requirements. This has led to a need for understanding the basic structure that characterizes policies for efficiently allocating network resources, such as link capacity, to the various users. In this dissertation, we address some aspects of admission control and routing – key issues arising in the design and operation of integrated communication and computer networks. We begin by considering the problem of optimal admission control of messages arriving at a circuit-switched node. Next, the asymptotic behavior of such networks is investigated when the arrival intensities and the capacities of the network links are increasingly large. Finally a “combined” optimal admission-routing scheme at a simple network node is presented. A description of these problems is given below.

Two communication traffic streams with Poisson statistics arrive at a network node on separate routes. These streams are to be forwarded to their destinations *via* a common trunk. The two links leading to the common trunk have capacities C_1 and C_2 bandwidth units, respectively, while the capacity of the common trunk is C bandwidth units, where $C < C_1 + C_2$. Calls of either traffic type that are not admitted at the node are assumed to be discarded. An admitted call of either type will occupy, for an exponentially distributed random time, one bandwidth unit on

its forwarding link as well as on the common trunk. Our objective is to determine a scheme for the optimal dynamic allocation of available bandwidth among the two traffic streams so as to minimize a weighted blocking cost. The problem is formulated as a Markov decision process. By using dynamic programming principles, the optimal admission policy is shown to be of the “bang-bang” type, characterized by appropriate “switching curves.” The case of a general circuit-switched network, as well as numerical examples, are also presented.

Markov decision processes arise in a natural way in the optimal control of queuing systems in some of the problems considered in this thesis. In many cases the convexity of an optimal discounted cost associated with such processes plays a key role in the analysis. A method that ascertains the convexity of such optimal discounted costs is presented. The procedure relies on a straightforward examination of all possible state transitions of the underlying Markov decision process.

Next, the asymptotic behavior of circuit-switched networks is addressed. We first consider a class of simple circuit-switched nodes in the limiting situation where the link capacities and the offered traffic intensities are increased at the same rate. We assume that an incoming message is given access to the network only if none of the links constituting its route is saturated in capacity. The process of the normalized number of messages on each route is shown to converge in probability to a solution of a system of differential equations which possesses a unique stable point. Next, the difficulties encountered in extending this method to arbitrary circuit-switched networks are discussed. The usefulness of the method lies in its ability to provide a transient analysis of the limiting network and to determine its “most likely” steady state. A conjecture for a strong approximation concerning the limiting behavior of an arbitrary circuit-switched network is also given.

Finally, a problem of determining simultaneously optimal admission and routing policies at a data network node is considered. Specifically, a message arriving at the buffer of a node in a data network is to be transmitted over one of two

channels with different propagation times. Under suitably chosen criteria, two decisions have to be made: whether or not to admit an incoming message into the buffer, and under what conditions should the slower channel be utilized. A discounted infinite-horizon cost as well as an average cost are considered which consist of a linear combination of the blocking probability and the queuing delay at the buffer.

TO THE MEMORY OF MY BELOVED IOANNA
(1962 - 1990)

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but remain largely disregarded and forgotten. I would have quited several times in the past if she were not there to push me for one more time! In appreciation of her support and for everything she offered to me, I dedicate this small piece of work to her memory.

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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

The proliferation of digital communications traffic with increasingly diverse characteristics, performance requirements, and grades of service has created a recent surge of interest in the study of *integrated networks* capable of realizing an efficient sharing of facilities such as transmission and switching. This phenomenon is particularly evident in satellite and computer communication networks. Indeed it is expected that in communication networks of the future, large *integrated service digital networks* (ISDNs) will be designed to accommodate random demands for bandwidth usage from a population of heterogeneous users (i.e., users with different characteristics and performance requirements). This has led to an increasing need for an understanding of the basic structure that characterizes policies efficiently allocating various resources, such as link capacity, to the various users.

This thesis is concerned with some aspects of admission control and routing-key issues arising in the design and operation of integrated communication and computer networks – as is evident from the considerable literature in these areas. From the plethora of available references, we cite below only those of direct relevance to our work. In the realm of admission control, numerous proposals for providing integrated service have appeared heretofore in the literature, most of which concentrate on the integration of two traffic types, namely voice and data traffic. Furthermore, these proposals have assumed various forms, including the use of circuit switching for voice and packet switching for data [81,82]. As summarized in Konheim-Pickholtz [32], most of these schemes have focused on time-division multiplexing implementations. Typically, a shared communication resource, e.g., a trunk, is partitioned into *frames* of fixed duration ($\sim 125\mu s$). Each frame, in turn, is partitioned into *slots* (~ 24 slots in a frame). Each slot can accommodate a *packet*, which is the basic unit used for the transmission of both voice

and data in these schemes. In a *circuit-switched* scheme, a certain number of slots is allocated to a particular voice (or data) source during the entire duration of a “call”. Here, the integrated multiplexer has a *fixed boundary*; thus, voice packets are transmitted only in those slots that occupy a prespecified part of the frame, while the rest of the frame is used for the transmission of the data packets. In effect, a fixed boundary scheme is a *static* one which divides the channel into two independent subchannels.

Much work has also been done on *dynamic* bandwidth allocation to multiple traffic types. The schemes proposed usually focus on two traffic types, viz., voice and data, and employ a *movable* boundary. This scheme leads to an increased efficiency of channel usage by allowing idle channel capacity to be dynamically re-assigned from one type of traffic to the other; it is typified by the SENET network as described by Coviello-Vena [10]. The movable boundary scheme, in its basic form, employs the following format: a certain portion of each time-slotted frame is allocated to voice traffic, while data can use all of the remaining frame capacity including any unused voice capacity. Voice traffic on the other hand, cannot avail of any unused data capacity and operates as a *loss* system, i.e., voice messages which find no voice channels available upon arrival are assumed to be lost. Voice traffic is given priority over data traffic, and data can be queued. Several versions of moving boundary techniques have been proposed and analyzed by several authors, e.g., Fisher-Harris [11], and Leon-Garcia *et al* [83]. Gaver-Lehoczký [15] deal with a rather extreme but realistic situation in which data arrives in packets of short service duration, whereas voice traffic is less frequent but exhibits very long holding times. Arthurs-Stuck [2] have analyzed the theoretical performance limits of a variety of time-division multiplexing policies for models of synchronous and asynchronous traffic. Konheim-Pickholtz [32] and Lee *et al* [40] have studied various models for voice-data multiplexing when both voice and data are buffered. Another approach to voice-data integration is motivated by the fact that voice traffic is characterized by periods of silence, which constitute a significant percentage of the bandwidth allocated to it. Detection and utilization of such periods

for the transmission of either voice or data would certainly improve the performance of the network. The techniques of time-assignment speech interpolation, time-assignment data interpolation, and their variations [72,83], are based on this notion.

Kaufman [26] has studied the issue of blocking in a shared resource environment. Foshcini-Gopinath [12] have determined an optimal *coordinate convex* admission policy for a simple blocking node accommodating two different traffic types with similar bandwidth requirements. They showed that the admission strategy has a simple threshold structure. Ross-Tsang [61, 75] have showed similar results for traffic with different bandwidth requirements, and have devised algorithms for evaluating the blocking costs in these cases.

Dynamic programming techniques have been used by Maglaris-Schwartz [49] in attempting to derive a globally optimal policy for the control of a simple ISDN node. Using the same framework as [49] Stidham [73] has considered admission control policies for several simple queuing models. By using the theory of Markov decision processes and dynamic programming, he has demonstrated that the optimal admission control policies for all these models share the characteristic that they can be expressed in terms of a “switching curve”. Viniotis-Ephremides [79] have demonstrated a similar characterization of the optimal admission strategy at a simple ISDN node. Results in the same vein have been obtained by Christidou *et al* [9] for a cyclic interconnection of two queues, and by Lambadaris *et al* [39] for a circuit switched node. Hajek [16] has investigated the problem of optimally controlling two interacting queues.

Turning next to the class of routing problems, Lin-Kumar [41] have considered the task of routing messages arriving at a node among two channels (servers), one faster than the other. By minimizing the average queuing delay at the node buffer, they show that the optimal routing policy is characterized by a “threshold” on the size of the queue. Rosberg-Makowski [58] have treated a similar problem involving multiple servers under the assumption of light traffic. In a recent preprint [47], Luh

-Viniotis claim the optimality of a policy determined by multiple thresholds for the situation in [58] even with arbitrary arrival rates. Nain-Ross [53] consider the optimal assignment of a single server to multiple classes of traffic. In doing so, they minimize a linear combination of the average queue lengths of the various classes of traffic while simultaneously constraining the average queue length of a specific traffic class to lie below a specified value. Shwartz-Makowski [71] treat a similar problem with two types of traffic. Both [53,71] show the optimal assignment strategy to be a randomized one. Analogous constrained queuing optimization problems are treated also in [1,48].

Finally the emergence of high-speed optical communication networks offering access to a wide variety of very sophisticated and high-quality telecommunication services, has provided impetus for a research effort of a different hue: This effort concerns the development of approximate methods for the analysis and performance evaluation of *large* networks. In such networks optimal admission control and routing may not be the key issues because of the tremendous amounts of available bandwidth capacity. On the other hand, even the very light blocking exhibited by such networks is of great practical significance since the loss of a single packet during transmission may require a retransmission of an entire data stream—a process that is costly and incurs considerable delays. Furthermore, current telephone networks have become quite large and complex, and analytically intractable, leading to an increasing need for approximate analysis. For instance, the difficulty in computing the normalization constant in the product-form distribution associated with certain networks, has recently led to the use of integral transforms for approximating such normalizing constants [52]. A different approach due to Kelly [27] involves a procedure for evaluating the blocking probability in large circuit-switched networks. A similar study has been performed by Pittel [56]. Finally, strong approximations for a closed network of queues have been performed by Kogan *et al* in [30].

1.2 Organization of the Dissertation

This dissertation addresses certain issues relevant to a class of resource allocation problems, namely admission control and routing, that arise in wideband communication networks. We have selected specific problems which we hope will serve as insightful paradigms of network situations under certain assumptions and performance criteria. The initial concern is to understand the structural characteristics of optimal policies for bandwidth allocation and routing for a few simple analytical models. The optimal policies so obtained are described in qualitative terms which are usually unamenable to simple practical implementation. They are, however, useful in helping identify suboptimal yet efficient and potentially implementable policies. For large networks, when exact analysis is rarely feasible, this qualitative understanding can often be helpful in inferring approximate or asymptotic (with increasingly large arrival intensities and link capacities) behavior. Some work is presented on the asymptotic behavior of a network with a large number of nodes, based on which a few conjectures are drawn when such a network is circuit-switched.

In Chapter 2 we consider a circuit-switched node providing service to two different message traffic streams. We assume that the two traffic types are transmitted over two channels with capacities C_1 and C_2 frequency slots, and they eventually merge on a common trunk of capacity C *en route* their destination. The messages arrive according to a Poisson distributions with means λ_1 and λ_2 . Upon arrival of traffic-type i a slot is simultaneously allocated on the channel of capacity C_i and on the common trunk and the particular slot is occupied for a time interval that is exponentially distributed with parameter μ_i , $i = 1, 2$. Messages of either traffic type that are not admitted at the node are assumed to be lost. Given that a cost c_i is incurred for each type- i message that is discarded, we seek an optimal admission strategy that minimizes an average rejection cost over an infinite time horizon. Formulating the problem in terms of a two-dimensional Markov decision process, we show by an application of dynamic programming principles that

the optimal admission policy is characterized by monotone “switching curves” in the state space of the system. Finally an extension is given to circuit-switched networks with a specific structure.

In Chapter 3, we provide a criterion for the convexity of the optimal discounted cost associated with a class of Markov decision processes; such processes typically arise in the control of queuing systems, e.g., in Chapters 2 and 6. Specifically, for a Markov decision process with a given linear state evolution, the method ascertains the convexity of an associated infinite horizon optimal discounted cost by a straightforward examination of all possible state transitions of the process.

In Chapter 4, methods are presented for evaluating the blocking performance of circuit-switched networks. We focus on methods that involve easy calculations requiring minimal computing time. Next, simple bounds are derived for the blocking probability when the arrival intensities are small (*light traffic*). Such bounds, albeit for small arrival intensities, can be significant, particularly for high-speed optical networks where a loss by blocking is often costly. This chapter is not exhaustive and aims at introducing, rather than effectively solving, the problem of approximating the blocking behavior of circuit-switched networks.

Chapter 5 addresses the asymptotic behavior of circuit-switched networks. We first consider a class of simple circuit-switched nodes under the limiting situation where the link capacities and the offered traffic intensities are increased at the same rate. We assume an incoming message is given access to the network only if none of the links constituting its route is saturated in capacity. The process of the normalized number of messages on each route is shown to converge in probability to a solution of a system of differential equations which possesses a unique stable point. Next, the difficulties encountered in extending this method to arbitrary circuit-switched networks are discussed. The usefulness of the method lies in its ability to provide a transient analysis of the limiting network and to determine its “most likely” steady state. A conjecture for a strong approximation concerning the limiting behavior of an arbitrary circuit-switched network is given at the end

of the chapter.

Finally, Chapter 6 deals with the problem of the joint optimal admission and routing at a data network node. Specifically, a message arriving at the buffer of a node in a data network is to be transmitted over one of two channels with different propagation times. Under suitably chosen criteria, two decisions have to be made: Whether or not to admit an incoming message into the buffer, and under what conditions should the slower channel be utilized. A discounted infinite-horizon cost as well as an average cost are considered. These costs consist of a linear combination of the blocking probability and the queuing delay at the buffer.

We now provide a brief presentation of certain mathematical tools which are frequently employed in the subsequent chapters.

1.3 Mathematical Preliminaries

In this section we provide a brief introduction to the notions of (semi)-Markov decision processes and stochastic ordering. Our presentation is by no means complete; the interested reader may consult [6,7,8,22,42,43,44,64,65,69] for further details and proofs.

a) (Semi)-Markov decision processes

A semi-Markov decision process (SMDP) $(x_t, t \geq 0)$ is characterized by [42,43,64]:

- a state space \mathcal{S} in which $\mathbf{x}_t, t \geq 0$ takes its values;
- an action space $A \triangleq \prod_{s \in \mathcal{S}} A_s$, where A_s is the set of actions that are admissible at state s ;
- a law of motion Q ;
- a transition time T ;
- a cost C ;

Hereafter we assume that \mathcal{S} is countable and A_s is finite. Whenever the system represented by the process $(x_t, t \geq 0)$ is in state s and action $a \in A_s$ is chosen, the following occur:

- 1) The system moves to a new state selected according to the probability distribution $Q(\cdot|s, a)$,
- 2) Conditioned on the event that the new state is s' , the length of time it takes the system to move to state s' is a nonnegative random variable with probability distribution $T(\cdot|s, a, s')$,
- 3) Conditioned on the event that the new state is s' and the transition takes t time units, a cost $C(t|s, a, s')$ is incurred immediately after the transition is completed.

Remark: The cost $C(\cdot)$ need not be incurred all at once; it can be gradually incurred during the course of the transition.

A control strategy (CS) z is a sequence z_1, z_2, \dots of decision rules where the n th decision rule z_n dictates how to select an action in A at the completion of the $n-1$ th transition. Precisely, z_n is a conditional probability on A given the history of the process $h^n = (s_1, a_1, C_1, t_1, \dots, s_{n-1}, a_{n-1}, C_{n-1}, t_{n-1}, s_n)$ up to and including the time of the $n-1$ th transition. Hence, given the the history of the process h^n up to the time of the $n-1$ th transition, the n th action is chosen according to the distribution $z_n(\cdot|h^n)$. A CS z is said to be *stationary* if it is invariant with time and is always chosen with probability one, i.e., if the distribution $z_n(\cdot|h^n)$ is atomic and hence, identically 1 for a fixed action in A .

With every semi-Markov decision process and a given CS z , we associate two costs, namely a *discounted* and an *average* cost. Let $V(z, s, t)$ denote the total expected reward earned by time t when starting from state s . For a discount factor $\delta > 0$, the total δ -discounted cost over an infinite time horizon associated with a CS z is defined as

$$V^\delta(z, s) = \int_0^\infty e^{-\delta t} dV(z, s, t), \quad (1.1)$$

while the average cost over an infinite time horizon is defined as

$$\overline{V}(s, z) = \limsup_{t \rightarrow \infty} \frac{V(z, s, t)}{t}. \quad (1.2)$$

Finally, we define the optimal discounted and average costs $V^\delta(\cdot)$ and $\overline{V}(\cdot)$ by,

$$V^\delta(s) = \inf_z V^\delta(s, z),$$

and,

$$\overline{V}(s) = \inf_z \overline{V}(s, z).$$

If a policy exists which achieves either of the infima above, the policy is called *optimal* for the associated costs. Criteria for the existence of an optimal policy can be found in [42,69].

Costs similar to (1.1) and (1.2) can be defined for a finite horizon upto and including the n th transition. We denote by $V_n^\delta(\cdot)$, $\delta > 0$, the δ -discounted cost for a finite horizon of upto the time of the n th transition.

Let

$$\beta(s, a, s') \triangleq \int_0^\infty e^{-\delta t} dT(t|s, a, s'),$$

and

$$c(s, a, s') \triangleq \int_0^\infty C(t|s, a, s') dT(t|s, a, s')$$

be the expected discount factor and expected cost per transition, respectively, when action a is taken at state s leads to a transition to state s' .

Proposition 1.1: The cost functions $V^\delta(\cdot)$ and $V_n^\delta(\cdot)$ satisfy the following so called dynamic programming equations (DPE):

$$V^\delta(s) = \inf_{a \in A_s} \left\{ \sum_{s' \in \mathcal{S}} (c(s, a, s') + \beta(s, a, s') V^\delta(s')) Q(s'|s, a) \right\},$$

$$V_{n+1}^\delta(s) = \inf_{a \in A_s} \left\{ \sum_{s' \in \mathcal{S}} (c(s, a, s') + \beta(s, a, s') V_n^\delta(s')) q(s'|s, a) \right\}.$$

Remark: A considerable number of resource-allocation problems in queuing systems can be formulated using a cost per unit time (or *instantaneous cost*) while at state s and following a strategy z . Such a cost, denoted by $C(s, z)$, can be used in lieu of $C(\cdot)$ introduced in the definition of a SMDP. Relations (1.1) and (1.2), respectively, are now transformed into

$$V^\delta(s, z) = \mathbb{E}_s^z \int_0^\infty e^{-\delta t} C(s_t, z) dt$$

and

$$\bar{V}(s, z) = \limsup_{\sigma \rightarrow \infty} \mathbb{E}_s^z \int_0^\sigma \frac{C(s_t, z)}{\sigma} dt.$$

In the special case when the time distribution $T(\cdot)$ between transitions is exponential (memoryless), the associated process $(x_t, t \geq 0)$ is called a *Markov decision process* (MDP). In our presentation heretofore, we have dealt with a continuous time framework. In the study of MDP's describing certain queuing systems, it is often desirable to develop certain techniques that will convert a continuous-time MDP to a discrete time MDP, the discrete-time instants being the state transition times. Such a *uniformization* technique is illustrated in [38, page 54], by which an equivalent state process $(\hat{x}_t, t \geq 0)$ is constructed on \mathcal{S} such that,

$$\hat{x}_t = x_n, \text{ for } T_n \leq t < T_{n+1},$$

where T_n and T_{n+1} represent the transition times of a Poisson process with a uniform rate and $(x_n, n = 0, 1, \dots)$ is a suitable discrete time state process defined on \mathcal{S} . Moreover it can be shown that $x_t, t \geq 0$ and $\hat{x}_t, t \geq 0$ are equivalent in the sense that,

$$P(x_t = y) = P(\hat{x}_t = y)$$

for all $y = 1, 2, \dots$.

A study of $(x_t, t \geq 0)$, thus is equivalent to the studying $(\hat{x}_t, t \geq 0)$, and in particular of $(x_n, n = 0, 1, \dots)$. An important consequence of the uniformization is that the quantity $\beta(s, a, s')$ becomes independent of actions and states thus, thereby facilitating considerably the study of the dynamic programming equation. Further details of this technique will be introduced as and when needed during the course of the subsequent chapters; the interested reader may refer to [38] for a comprehensive treatment.

b) Stochastic order relations

We now introduce the notion of the stochastic ordering of random variables. Propositions are presented without proofs; for proofs and further details the reader may consult [22,65].

Definition 1.1: A \mathbb{R} -valued random variable X is said to be stochastically larger

than a \mathbb{R} -valued random variable Y , denoted $X \geq_{st} Y$, if

$$P(X > a) \geq P(Y > a) \quad (1.3)$$

for all a in \mathbb{R} . If X and Y have distribution functions F and G respectively, then (1.3) is equivalent to

$$1 - F(a) \geq 1 - G(a) \Leftrightarrow G(a) \geq F(a).$$

Proposition 1.2: The following equivalence is true:

$$X \geq_{st} Y \Leftrightarrow \mathbb{E}(f(X)) \geq \mathbb{E}(f(Y))$$

for all nondecreasing functions $f : \mathbb{R} \mapsto \mathbb{R}$ for which the expectations exist and are finite.

The following proposition plays an useful role in stochastic comparisons.

Proposition 1.3: If $X \geq_{st} Y$, then there exist random variables \overline{X} and \overline{Y} defined on a common probability space (Ω, \mathcal{F}, P) with the same distribution as X and Y , respectively, and such that

$$\overline{X} \geq \overline{Y} \quad (P \text{ a.s.})$$

Remark: Proposition (1.3) is also valid for stochastic comparisons of processes, i.e., the random variables in the proposition may be replaced by the processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$, such that $X_t \geq_{st} Y_t, t \geq 0$.

In the course of our study, whenever we use stochastic ordering or coupling arguments associated with certain random variables (processes), we shall always tacitly refer to the random variables (processes) defined on a common probability space in accordance with Proposition 1.3.

CHAPTER 2

OPTIMAL ADMISSION CONTROL OF TWO TRAFFIC TYPES AT A CIRCUIT-SWITCHED NETWORK NODE

2.1 Introduction

In modern telecommunication networks there is an increasing need to transmit simultaneously heterogeneous traffic types with diverse characteristics, performance requirements, and grades of service. This has resulted in a recent surge of interest in the study of *integrated systems* capable of realizing an efficient sharing of facilities such as transmission and switching. Indeed, it is expected that, in communication networks of the future, large *integrated service digital networks* (ISDN's) will be designed to accommodate random demands for bandwidth usage from a population of heterogeneous users.

Various schemes have been proposed to date for multiplexing several types of traffic on the same channel. Much of the work has concentrated on two types of traffic, namely voice and data [32]. Furthermore, the techniques addressed thus far can be classified into three broad categories:

1) *Complete Sharing Scheme*: In this scheme a call of a particular traffic type is always offered access to the network whenever sufficient bandwidth is available to accommodate it.

2) *Complete Partitioning Scheme*: In this technique, the available channel bandwidth at each node is partitioned and a portion of the bandwidth is dedicated to each traffic type.

3) *Moving Boundary Scheme* [33,49,50]: This scheme applies to two traffic types, namely voice calls and data packets. The total bandwidth is partitioned into two compartments. One compartment is typically allocated to voice traffic, while data can use the remaining compartment as well as any unused slots in the voice

compartment. On the other hand, voice traffic cannot use any unused data slots, and operates as a *loss* system, i.e., voice calls that are not accepted upon arrival are assumed to be lost.

In the schemes mentioned above, the customary objective is to control the multiplexer so as to maximize channel utilization or minimize blocking probability. Usually, the first call type, viz., voice, operates as a loss system while the second call type, viz., data, is queued. Also, much of the work to date involves *static* schemes for the control of the multiplexer, i.e., the allocation policy is chosen *a priori* for all time, and its performance analyzed. Much less work is available on *dynamic* control schemes, where the *best* allocation policy according to some criterion may vary in time; such a policy may be stationary (deterministic) or nonstationary.

In the study that follows, we consider the problem of allocating channel bandwidth to two communication traffic streams that arrive at a network node on two different routes and must be forwarded to their destinations *via* a common trunk. Calls of either traffic type that are not given admission at the node are assumed to be discarded. Our objective is to determine a scheme for the optimal dynamic allocation of available bandwidth among the two traffic streams so as to minimize a weighted blocking cost. The problem is formulated as a Markov decision process where the control actions consist of accepting or discarding a call at the instant of its arrival into the system. By using dynamic programming principles, we demonstrate that the optimal admission policy is of the “bang-bang” type characterized by two “switching-curves” in the state space of the system.

The current chapter is organized as follows. The control problem is formulated in section 2.2. Section 2.3 considers the discounted cost case and establishes key properties of the optimal discounted cost function; the associated optimal policy for this case is characterized in section 2.4. The average cost problem is addressed in section 2.5. Numerical results for a given link are given in section 2.6. Finally, in section 2.7 we consider the optimal admission control problem for a general

circuit-switched network.

2.2 A Description of the Problem

We consider an integrated scheme for providing service to two types of non-queuing traffic (e.g., voice and video) with different statistics and requiring different grades of service. We shall direct our attention to two traffic streams, one of each type, arriving at nodes 1 and 2, respectively (see Figure 2.1), for transmission to node 3 which subsequently directs the traffic onto a trunk. We assume that nodes 1 and 2 are connected to node 3 by means of two links of capacities C_1 and C_2 frequency slots, respectively, and that the trunk capacity is C slots. It is further supposed that each call (i.e., voice call or video message) occupies exactly one frequency slot on one of the two links and on the trunk.

When a call arrives at node 1 or 2, a decision is made at the node on either accepting it or blocking it. If accepted, the call is granted a slot *simultaneously* on the corresponding forwarding link as well as on the common trunk; if blocked it is assumed to be lost. These decisions are based on minimizing appropriate blocking costs associated with lost calls. We remark that deliberate blocking of a call of one type may be advantageous for the following reason: It may be worthwhile reserving an empty slot on the shared trunk for a call of the other type since blocking the latter (at a later time) would incur a greater cost.

The following statistical assumptions are made on the arrival and service times of the incoming traffic. Calls of type- i arrive in a Poisson stream of rate λ_i ; their corresponding service times are i.i.d. exponential random variables with mean μ_i , $i = 1, 2$. We shall assume without any loss of generality that $\mu_1 \leq \mu_2$ and that all arrival and service processes are mutually independent. The *state* of the system describing the distribution of the load at time $t \geq 0$ is defined by the two-dimensional vector $\mathbf{x}_t = (x_t^1, x_t^2)^T$ where x_t^1 and x_t^2 denote the number of calls in service of type-1 and type-2, respectively, such that $x_t^1 \leq C_1$, $x_t^2 \leq C_2$ and $x_t^1 + x_t^2 \leq C$. We further assume that $C < C_1 + C_2$. Then the state space of the system (see Figure 2.2) is the set:

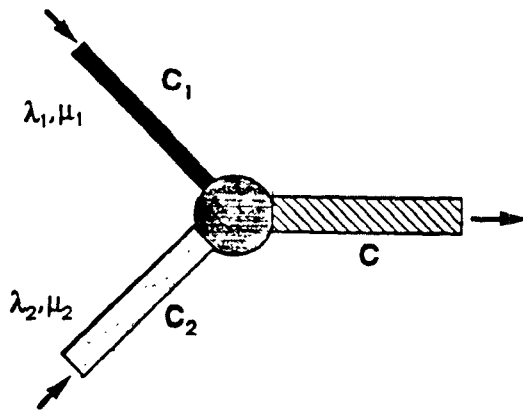


Figure 2.1

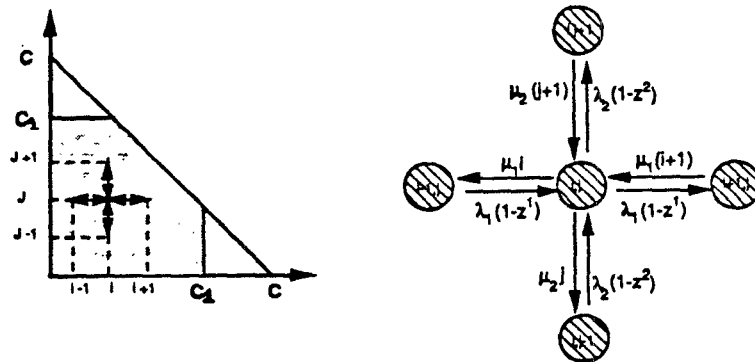


Figure 2.2

$$\mathcal{X} = \{\mathbf{x}_t = (x_t^1, x_t^2) : x_t^1, x_t^2 \in Z, 0 \leq x_t^1 \leq C_1, 0 \leq x_t^2 \leq C_2, x_t^1 + x_t^2 \leq C\}.$$

Transitions among the states in \mathcal{X} are described in terms of the operators A_i and D_i representing, respectively, an arrival or a departure of a message of type i , $i = 1, 2$. Thus, the operators $A_i : \mathcal{X} \rightarrow \mathcal{X}, D_i : \mathcal{X} \rightarrow \mathcal{X}, i = 1, 2$, are defined by:

$$A_1(x^1, x^2) = ((x^1 + 1)^*, x^2)$$

$$A_2(x^1, x^2) = (x^1, (x^2 + 1)^*)$$

$$D_1(x^1, x^2) = ((x^1 - 1)^+, x^2)$$

$$D_2(x^1, x^2) = (x^1, (x^2 - 1)^+),$$

where

$$((x^1 + 1)^*, x^2) = \begin{cases} (x^1 + 1, x^2) & \text{if } x^1 < C_1, x^1 + x^2 < C \\ x^1 & \text{otherwise,} \end{cases}$$

and $m^+ = \max\{0, m\}$.

At this point, we provide a heuristic motivation of the nature of the control actions at nodes 1 and 2. Denoting by $z_t^i = z_t^i(\mathbf{x}_t)$ the probability of blocking an incoming call of type- i , $i = 1, 2$, arriving in the time interval $[t, t + dt)$, we must suitably select this probability based on a knowledge of \mathbf{x}_t . We refer to z_t^i as the *control action* taken at time t . If $a_i > 0$ is a blocking cost associated with the type- i traffic, the total cost incurred during $[t, t + dt)$ is $\sum_{i=1}^2 \lambda_i a_i z_t^i(\mathbf{x}_t) dt$. We can then write the *normalized cost per unit time* at time t with the system state being \mathbf{x}_t , as $c_t(\mathbf{x}_t) = z_t^1(\mathbf{x}_t) + a z_t^2(\mathbf{x}_t)$, where $a > 0$ (assuming, of course, that $\lambda_i > 0, i = 1, 2$).

Let $\delta > 0$ be the interest rate used for discounting future cost, i.e., the present value of a cost a incurred at time t is $ae^{-\delta t}$. Let $J_t^\delta(\mathbf{x})$ be the minimum “expected” total discounted-cost with respect to $z_t^i(\cdot)$, $i = 1, 2$, when the time horizon is $\{t : t \geq 0\}$ and the initial state is $\mathbf{x} = (x^1, x^2)$. Then, dynamic programming considerations

lead to the following optimality conditions for $J_t^\delta(\mathbf{x})$:

$$\begin{aligned}
J_{t+dt}^\delta(\mathbf{x}) = & \min_{0 \leq z_0^1, z_0^2 \leq 1} \{ z_0^1 dt + a z_0^2 dt + \\
& e^{-\delta dt} \left(\sum_{i=1}^2 (z_0^i \lambda_i J_t^\delta(\mathbf{x}) + \lambda_i (1 - z_0^i) J_t^\delta(A_i \mathbf{x}) + \right. \\
& \left. + x^i \mu_i J_t^\delta(D_i \mathbf{x})) dt \right. \\
& \left. + (1 - (x^1 \mu_1 + x^2 \mu_2) dt) J_t^\delta(\mathbf{x}) \right\} + o(dt).
\end{aligned}$$

It readily follows that:

$$\begin{aligned}
J_{t+dt}^\delta(\mathbf{x}) = & \min_{0 \leq z_0^1 \leq 1} \{ z_0^1 (1 - e^{-\delta dt} \lambda_1 (J_t^\delta(A_1 \mathbf{x}) - J_t^\delta(\mathbf{x}))) \} + \\
& + \min_{0 \leq z_0^2 \leq 1} \{ z_0^2 (a - e^{-\delta dt} \lambda_2 (J_t^\delta(A_2 \mathbf{x}) - J_t^\delta(\mathbf{x}))) \} \\
& + \text{terms not depending on } z_0^1, z_0^2.
\end{aligned}$$

Consequently $z_0^{1(2)} = 0$ (i.e., a type-1(2) call is accepted) if $J_t^\delta(A_{1(2)} \mathbf{x}) - J_t^\delta(\mathbf{x}) \leq e^{\delta dt}(a)/\lambda_{1(2)}$; otherwise $z_0^{1(2)} = 1$. Thus, we can associate with every state \mathbf{x} in \mathcal{X} a set of admissible actions $\mathcal{D} = \{0, 1\}^2$ with the understanding that an admissible action $\mathbf{z}_t(\mathbf{x})$ at state \mathbf{x} and at time t will have the form:

$$\mathbf{z}_t(\mathbf{x}) = (z_t^1(\mathbf{x}), z_t^2(\mathbf{x}))$$

where $z_t^i = 1$ or 0 according to whether an arriving call of type- i is rejected or accepted into the system. The *action space* is then defined as the product set D^S , and we represent an *admissible control strategy* (CS) as a D^S -valued stochastic process $(z_t, t \geq 0)$ where $z_t = (\mathbf{z}_t(\mathbf{x}), \mathbf{x} \in \mathcal{X})$. Hereafter, we shall use the abbreviated notation z for the CS $(z_t, t \geq 0)$. We denote by \mathcal{P} the set of all admissible control strategies. Further, for simplicity, we write z_t^i instead of $z_t^i(\mathbf{x}_t)$. Finally, observe that $((\mathbf{x}_t, \mathbf{z}_t), t \geq 0)$ is a Markov decision process with transition rates shown in Figure 2.2.

At this juncture, it is convenient to relate the continuous time Markov chain $(\mathbf{x}_t, t \geq 0)$ to a “suitable” discrete time chain by following the method of “uniformization” [44,77]. To this end, we first define the *total event rate* by:

$$\rho = \lambda_1 + \lambda_2 + C\mu_2.$$

Then, let $0 = t_0 < t_1 < t_2 \dots < t_n < \dots$ be the transition epochs (due to arrivals or departures) of the state process $(\mathbf{x}_t, t \geq 0)$. By suitably introducing “dummy” transitions as in [59,44], it follows that the interepoch intervals are i.i.d random variables with a common distribution determined by $\mathbb{P}(t_{k+1} - t_k > t) = e^{-t\rho}$, $t \geq 0$, $k = 1, 2, \dots$. Then it can be easily shown [59,77] that the δ -discounted expected cost accrued up to time t_n upon starting with initial state \mathbf{x} and following a policy z in \mathcal{P} , viz.,

$$\mathbb{E}_{\mathbf{x}}^z \left(\int_0^{t_n} e^{-\delta t} (z_t^1 + az_t^2) dt \right),$$

is equal to a cost of the form:

$$\mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=0}^{n-1} \beta^k (z_k^1 + az_k^2) \right),$$

with $z_k^i \triangleq z_{t_k}^i$ and $\beta = \rho/(\delta + \rho) < 1$. The last expectation is taken with respect to the probability distribution associated with a discrete time Markov decision process $(\mathbf{x}_k, k \geq 0)$ with transition probabilities:

$$\mathbb{P}(\mathbf{x}_{k+1} | \mathbf{x}_k, \mathbf{z}_k) \cdot \rho = \begin{cases} \lambda_1 & \text{if } \mathbf{x}_{k+1} = A_1 \mathbf{x}_k, z_k^1 = 0 \\ \lambda_1 & \text{if } \mathbf{x}_{k+1} = \mathbf{x}_k, z_k^1 = 1 \\ \lambda_2 & \text{if } \mathbf{x}_{k+1} = A_2 \mathbf{x}_k, z_k^2 = 0 \\ \lambda_2 & \text{if } \mathbf{x}_{k+1} = \mathbf{x}_k, z_k^2 = 1 \\ \mu_1 x^1 & \text{if } \mathbf{x}_{k+1} = D_1 \mathbf{x}_k \\ \mu_2 x^2 & \text{if } \mathbf{x}_{k+1} = D_2 \mathbf{x}_k \\ C\mu_2 - x^1 \mu_1 - x^2 \mu_2 & \text{if } \mathbf{x}_{k+1} = x_k, \mathbf{z}_k = 0. \end{cases} \quad (2.1)$$

Then for each initial state \mathbf{x} in \mathcal{X} , we can define,

$$J_n^\beta(\mathbf{x}) = \min_{z \in \mathcal{P}} \mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=0}^{n-1} \beta^k (z_k^1 + az_k^2) \right) \quad (2.2a)$$

as the n -step optimal β -discounted expected cost. Further, we introduce the infinite horizon optimal β -discounted expected cost:

$$J_n^\beta(\mathbf{x}) = \min_{z \in \mathcal{P}} \mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=0}^{\infty} \beta^k (z_k^1 + a z_k^2) \right) < +\infty. \quad (2.2b)$$

Since the underlying state space S is finite, it can be shown [64] that $\lim_{n \rightarrow \infty} J_n^\beta(\mathbf{x}) = J^\beta(\mathbf{x})$; moreover for the infinite horizon problem an optimal policy exists and it is stationary, i.e., the minimizing CS \hat{z} satisfies $\hat{\mathbf{z}}_t(\mathbf{x}_t) = \hat{\mathbf{z}}(\mathbf{x}_t)$, \mathbf{x}_t in \mathcal{X} , $t \geq 0$.

In terms of the discrete-time formulation, the Dynamic Programming equation can be written as follows (assume for simplicity that $\rho = 1$):

$$\begin{aligned} J_{k+1}^\beta(\mathbf{x}) = & \min_{\substack{\{z_0^i \in \{0,1\} \\ z_0^i = 1 \text{ if } x^i = C_i \\ \text{or } x^1 + x^2 = C, i=1,2\}}} \{ z_k^1 + a z_k^2 \\ & + \beta \lambda_1 (1 - z_k^1) J_k^\beta(A_1 \mathbf{x}) + \beta \lambda_1 z_k^1 J_k^\beta(\mathbf{x}) \\ & + \beta \lambda_2 (1 - z_k^2) J_k^\beta(A_2 \mathbf{x}) + \beta \lambda_2 z_k^2 J_k^\beta(\mathbf{x}) \\ & + \beta \mu_1 x^1 J_k^\beta(D_1 \mathbf{x}) + \beta \mu_2 x^2 J_k^\beta(D_2 \mathbf{x}) \\ & + \beta (C \mu_2 - x^1 \mu_1 - x^2 \mu_2) J_k^\beta(\mathbf{x}) \}, \end{aligned} \quad (2.3)$$

with $\mathbf{x} = (x^1, x^2)$ being the initial state and z_0^1, z_0^2 the corresponding actions at that state. Rearranging terms in (2.3), we get the following optimality criteria:

While at state \mathbf{x} such that $x^1 < C_1$ ($x^2 < C_2$), and $x^1 + x^2 < C$, an incoming type-1(2) call is blocked, i.e., $z_0^{1(2)} = 1$, if

$$J_k^\beta(A_{1(2)}(\mathbf{x})) - J_k^\beta(\mathbf{x}) \geq 1(a)/\beta \lambda_{1(2)}. \quad (2.4)$$

Furthermore, if $x^{1(2)} = C_{1(2)}$ or $x^1 + x^2 = C$, we set $z_0^{1(2)} = 1$.

2.3 Properties of the Optimal Discounted Cost Function

We now derive a few properties of the optimal n -step β -discounted cost function $J_n^\beta(\cdot)$ which will be employed in the next section to characterize the optimal policy for call acceptance or rejection.

Proposition 2.1: For each $x^1 \geq 0$, $x^2 \geq 0$, $J_n^\beta(\cdot, \cdot)$ is an increasing function of x^1, x^2 .

Proof: We will first show that

$$J_n^\beta(x^1 + 1, x^2) \geq J_n^\beta(x^1, x^2),$$

by using simple coupling arguments. We consider two identical systems starting with initial conditions $(x^1 + 1, x^2)$ and (x^1, x^2) , respectively. We apply the same control strategy to both systems, namely the one that is optimal for the n -step cost problem with initial conditions $(x^1 + 1, x^2)$. Whenever the system with initial condition $(x^1 + 1, x^2)$ admits a new call then it is feasible for the system, starting at (x^1, x^2) to do so. Let \mathbf{x}_k denote the state trajectory of the first system, and let $\sigma \triangleq \min(k : x_k^1 = 0)$. For $n \geq \sigma$, the states of the two systems coincide. Since we follow the same control strategy for both systems, elementary coupling arguments provide that

$$J_n^\beta(x^1 + 1, x^2) - J_n^\beta(x^1, x^2) \geq 0.$$

In a similar manner we show that $J_n^\beta(x^1, x^2 + 1) \geq J_n^\beta(x^1, x^2)$.

Proposition 2.2: For each $x^2 > 0$, $J_n^\beta(\cdot, x^2)$ is a convex function, i.e.,

$$J_n^\beta(x^1 + 1, x^2) - J_n^\beta(x^1, x^2) \geq J_n^\beta(x^1, x^2) - J_n^\beta(x^1 - 1, x^2). \quad (2.5)$$

A similar statement is true of $J_n^\beta(x^1, \cdot)$ for each $x^1 > 0$.

Proposition 2.2 can be established using linear programming techniques and duality theory in the manner of [59]. To this end, we shall need the following definitions:

A *sample path* ω^k (of arrivals and departures) is a sequence of k events, $k = 1, 2, \dots$, defined by

$$\omega^k = \{\omega_1, \omega_2, \dots, \omega_k\}, \omega_j \in \{A_1, A_2, D_1, D_2\}$$

$$j = 1, \dots, k,$$

with j representing the j^{th} arrival or departure epoch, and A_i, D_i denoting respectively an arrival or a departure of a type- i call, $i = 1, 2$. We define the *basic sample space*, Ω^k for the MDP to be the set of all sequences ω^k .

A *transition* ξ_k is specified by

$$\xi_k(\omega^k) = \begin{cases} (1, 0) & \text{if } \omega_k = A_1 \\ (0, 1) & \text{if } \omega_k = A_2 \\ (-1, 0) & \text{if } \omega_k = D_1 \\ (0, -1) & \text{if } \omega_k = D_2. \end{cases}$$

We can then express the evolution of the state trajectory corresponding to a policy z in \mathcal{P} , through the following recursive equation:

$$\mathbf{x}_0 = \mathbf{x}$$

$$\mathbf{x}_k(\omega^k) = \mathbf{x}_{k-1}(\omega^{k-1}) + \xi_k(\omega^k) - \text{diag } \xi_k(\omega^k) \mathbf{z}_k(\omega^k), \quad (2.6)$$

where \mathbf{x}_0 is the initial state and $\text{diag } \xi_k(\omega^k)$ is a 2×2 -diagonal matrix with diagonal elements $\xi_k(\omega^k)$. Solving the recursive state evolution equation (2.6), we obtain that

$$\mathbf{x}_k(\omega^k) = \mathbf{x} + \sum_{j=1}^k \xi_j(\omega^j) - \sum_{j=1}^k \text{diag } \xi_j(\omega^j) \mathbf{z}_j(\omega^j).$$

The n -step β -discounted cost corresponding to a control policy z in \mathcal{P} and with initial condition \mathbf{x} can be written as follows:

$$V_n^\beta(\mathbf{x}, z) = \mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=1}^n (\beta^k (z_k^1 \mathbf{1}(\omega_k = A_1) + a z_k^2 \mathbf{1}(\omega_k = A_2))) \right)$$

$$= \sum_{k=1}^n \sum_{\omega^k \in \Omega^k} \gamma_k(\omega^k) \mathbf{z}_k(\omega^k),$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, and $\gamma_k(\omega^k) = \beta^k(\mathbf{1}(\omega_k = A_1), a\mathbf{1}(\omega_k = A_2)) \mathbb{P}(\omega^k)$, with $\mathbb{P}(\omega^k)$ being the probability of the sample path ω^k .

The optimal n -step discounted cost (2.2a) then becomes

$$J_n^\beta(\mathbf{x}) = \min_{\mathbf{z}_k \in \{0,1\}^2} V_n^\beta(\mathbf{x}, z).$$

In an analogous manner, we define:

$$W_n^\beta(\mathbf{x}) = \min_{\mathbf{z}_k \in [0,1]^2} V_n^\beta(\mathbf{x}, z).$$

Then $W_n^\beta(\mathbf{x})$ is the (optimal) value function of a minimization problem of the form:

$$W_n^\beta(\mathbf{x}) = \min_{\{\mathbf{z}_k(\omega^k)\}_{k=1}^n} \sum_{k=1}^n \sum_{\omega^k \in \Omega^k} \gamma_k(\omega^k) \mathbf{z}_k(\omega^k),$$

such that

$$z_k^{1(2)}(\omega^k) = \begin{cases} 0 & \text{if } \omega_k = A_{2(1)} \text{ or } D_{2(1)}, \\ \in [0, 1] & \text{otherwise.} \end{cases} \quad (LP)$$

$$\omega^k \in \Omega^k,$$

$$k = 1, 2, \dots, n,$$

under the constraints

$$0 \leq \mathbf{x} + \sum_{j=1}^k \xi_j(\omega^j) - \sum_{j=1}^k \text{diag } \xi_j(\omega^j) \mathbf{z}_j(\omega^j) \leq \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

and

$$(1, 1)(\mathbf{x} + \sum_{j=1}^k \xi_j(\omega^j) - \sum_{j=1}^k \text{diag } \xi_j(\omega^j) \mathbf{z}_j(\omega^j)) \leq C.$$

This is a linear program in the finite array of variables $\{\mathbf{z}_k(\omega^k), \omega^k \in \Omega^k, 1 \leq k \leq n\}$. Since \mathbf{x} , the initial condition, enters linearly in the constraint equation, it can be shown [77] that $W_n^\beta(\cdot)$ is a convex function of \mathbf{x} . Note that, $W_n^\beta(\mathbf{x})$ cannot as

yet be associated with $J_n^\beta(\mathbf{x})$, since the variables $z_k^i(\omega^k)$ in (LP) can take values in the interval $[0,1]$. We now proceed to prove that there exists a solution $\mathbf{z}_k(\omega^k), \omega^k$ in $\Omega^k, 1 \leq k \leq n$, such that $\mathbf{z}_k(\omega^k)$ belongs to $\{0,1\}^2$. We first derive the necessary and sufficient conditions of optimality for the solutions of the linear program (LP). By duality theory [77, p. 50], $z^* = \{\mathbf{z}_k^*(\omega^k), \omega^k \in \Omega^k, 1 \leq k \leq n\}$ is an optimal solution of the (LP) above if and only if there exist nonnegative dual variables

$$\lambda_k^*(\omega^k) \in \mathbb{R}_+^2, \mu_k^*(\omega^k) \in \mathbb{R}_+^2, v_k^*(\omega^k) \in \mathbb{R}_+, 1 \leq k \leq n, \omega^k \in \Omega^k$$

such that (we drop in our notation the dependence of certain variables on ω^k to make the presentation simpler):

- 1) z^* is an optimal solution to the following unconstrained problem:

$$\begin{aligned} \min_{\left\{ \begin{array}{l} \mathbf{z}_k(\omega^k) \in [0,1] \\ k=1,2,\dots,n \\ \omega^k \in \Omega^k \end{array} \right\}} & \sum_{k=1}^n \sum_{\omega^k \in \Omega^k} (\gamma_k \mathbf{z}^k - \lambda_k^*(\mathbf{x} + \sum_{j=1}^k \xi_j - \sum_{j=1}^k \text{diag } \xi_j \mathbf{z}_j) \\ & + \mu_k^*(\mathbf{x} + \sum_{j=1}^k \xi_j - \sum_{j=1}^k \text{diag } \xi_j \mathbf{z}_j - (C_1, C_2)^T) \\ & + v_k^*((1,1)(\mathbf{x} + \sum_{j=1}^k \xi_j - \sum_{j=1}^k \text{diag } \xi_j \mathbf{z}_j) - C)). \end{aligned} \quad (2.7a)$$

- 2) If by $\{\mathbf{x}_k(\omega^k, z^*)\}_{k=1}^n$ we denote the state trajectory generated by z^* through (3.2), then

$$0 \leq \mathbf{x}_k(\omega^k, z^*) \leq (C_1, C_2)^T, \quad \mathbf{x}_k^1(\omega^k, z^*) + \mathbf{x}_k^2(\omega^k, z^*) \leq C. \quad (2.7b)$$

- 3) If $\lambda_k^{i*} > 0$, then $x_k^i(\omega^k, z^*) = 0, \quad i = 1, 2.$

$$\text{If } \mu_k^{i*} > 0, \text{ then } x_k^i(\omega^k, z^*) = C_i, \quad i = 1, 2. \quad (2.7c)$$

$$\text{If } v_k^* > 0, \text{ then } x_k^1(\omega^k, z^*) + x_k^2(\omega^k, z^*) = C.$$

The term being minimized in (2.7a) can be written as:

$$\sum_{k=1}^n \sum_{\omega^k \in \Omega^k} (\gamma_k(\omega^k) - \mathbf{c}_k(\omega^k)) z_k(\omega^k) + K,$$

where K does not depend on z , and

$$\mathbf{c}_k(\omega^k) = \left(\sum_{j=k}^n (\lambda_j^*(\omega^j) - \mu_j^*(\omega^j) - v_j^*(1, 1)) \right) \text{diag } \xi_k(\omega^k).$$

Hence, condition 1 above can be written more conveniently as

$$z_k^{*1(2)}(\omega^k) = 0 \text{ if } \omega_k = A_{2(1)} \text{ or } D_{2(1)};$$

otherwise,

$$z_k^i(\omega^k) = \begin{cases} 1 & \text{if } \gamma_k^i(\omega^k) - c_k^1(\omega^k) < 0 \\ 0 & \text{if } \gamma_k^i(\omega^k) - c_k^i(\omega^h) > 0 \\ \in [0, 1] & \text{if } \gamma_k^i(\omega^k) - c_k^i(\omega^h) = 0 \end{cases} \quad (2.8)$$

for $i = 1, 2$.

Lemma 2.1: Let $X = \{\mathbf{x} : \mathbf{x} = p_1 \xi(A_1) + p_2 \xi(A_2), p_1 \in (-\frac{1}{2}, \frac{1}{2}], p_2 \in [-\frac{1}{2}, \frac{1}{2})\}$.

Then

$$\begin{aligned} X - \{ \text{diag} \xi(\omega) \mathbf{z} \mid \mathbf{z} \in [0, 1]^2 \} &\subset X \cup \{X - \xi(\omega)\}, \\ \omega &\in \{A_1, A_2, D_1, D_2\}. \end{aligned}$$

Proof. The proof of the lemma is straightforward (see Figure 2.3).

Lemma 2.2: There is an integer-valued policy $z = \{\mathbf{z}_k(\omega^k) \mid \omega^k \in \Omega^k, 1 \leq k \leq n\}$ such that $\mathbf{z}_k(\omega^k) = \mathbf{z}_k^*(\omega^k)$ whenever the latter is integer-valued, and

$$\Delta_k \stackrel{\Delta}{=} (\mathbf{x}_k(\omega^k, z^*) - \mathbf{x}_k(\omega^k, z)) \in X,$$

for all ω^k in $\Omega^k, 1 \leq k \leq n$.

Proof. We use induction. Assume that Δ_k lies in X . Since:

$$\Delta_{k+1} = \Delta_k - \text{diag } \xi_{k+1}(\omega^{k+1}) \mathbf{z}_{k-1}^*(\omega^{k+1}(\omega^{k+1})) + \text{diag } \xi_{k+1}(\omega^{k+1}) \mathbf{z}_{k+1}(\omega^{k+1}),$$

if $\Delta_{k+1} = \Delta_k - \text{diag } \xi_{k+1}(\omega^{k+1}) \mathbf{z}_{k+1}^*(\omega^{k+1})$ lies in X , we choose $z_{k+1}^i(\omega^{k+1}) = 0$, $i = 1, 2$. If $\Delta_k - \text{diag } \xi_{k+1}(\omega^{k+1}) \mathbf{z}_{k+1}^*(\omega^{k+1})$ lies in $X - \xi(\omega^{k+1})$, and if

$\omega^{k+1} = A_{1(2)}$ (or $D_{1(2)}$), we choose $z_{k+1}^{1(2)}(\omega^{k+1}) = 1$ and $z_{k+1}^{2(1)}(\omega^{k+1}) = 0$, and in either case Δ_{k+1} belongs to X .

Remark: That $z_k^{1(2)}(\omega^k) = 1$ for $\omega_k = D_{1(2)}$ is not surprising. In this case we “disable” dummy departures so that $\mathbf{x}_k(\omega^k) \geq 0$ in order that the linear program (LP) may have a “feasible” solution.

Proposition 2.3: The integer-valued policy $\{z_k(\omega^k)\}_{k=1}^n$ in Lemma 2.2 is an optimal solution for the linear program (LP).

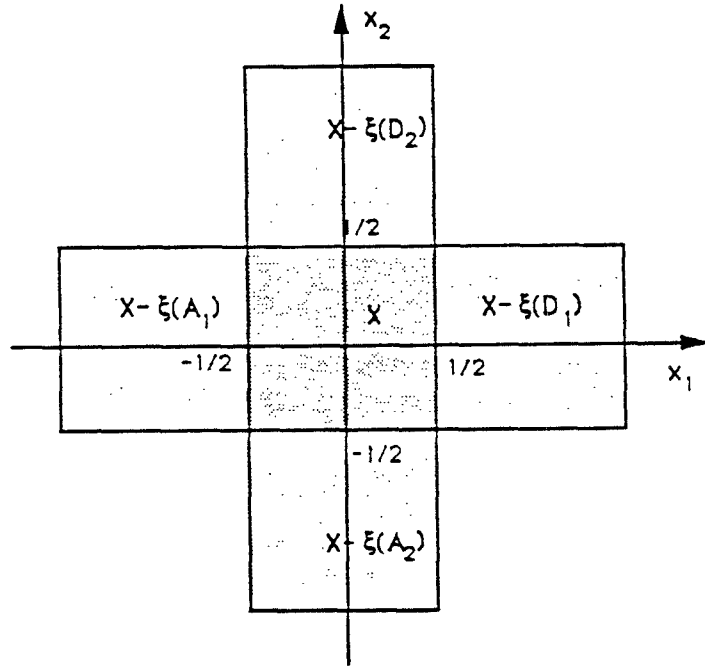


Figure 2.3

Proof: We show that the necessary conditions for optimality in (LP) are satisfied by the integer-valued policy $\{\mathbf{z}_k(\omega^k)\}_{k=1}^n$ in Lemma 2.2. Since $\mathbf{z}_k(\omega^k)$ is integer-valued when $\mathbf{z}_k^*(\omega^k)$ is, relation (2.8) and hence the minimization of (2.7a), automatically hold.

We now check the remaining two conditions of optimality, namely (2.7b) and (2.7c). We first show that $\mathbf{x}_k(\omega^k, z) \geq 0$. Suppose the opposite, and let $x_k^1(\omega^k, z) < 0$. Since $x_k^1(\omega^k, z)$ is integer-valued, $x_k^1(\omega^k, z) \leq -1$ and from Lemma 2.2 we have $x_k^1(\omega^k, z^*) = x_k^1(\omega^k, z) + p_1$ for a suitable p_1 in $(-\frac{1}{2}, \frac{1}{2}]$. Then, $x_k^1(\omega^k, z^*) \leq -1 + \frac{1}{2} = -\frac{1}{2} < 0$, a fact contradicting the feasibility of $x_k^1(\omega^k, z^*)$, and hence the optimality of z^* .

In a similar manner, we show that $\mathbf{x}_k(\omega^k, z) \leq (C_1, C_2)^T$. Assume that $x_k^1(\omega^k, z) > C_1$, whence $x_k^1(\omega^k, z) \geq C_1 + 1$; we then get $x_k^1(\omega^k, z^*) \geq C_1 + 1 - \frac{1}{2} = C_1 + \frac{1}{2} > C_1$, which is again a contradiction. Further, $x_k^1(\omega^k, z) + x_k^2(\omega^k, z) \leq C$ since in the opposite case $x_k^1(\omega^k, z^*) + x_k^2(\omega^k, z^*) > C + 1 - \frac{1}{2} - \frac{1}{2} = C$. Notice, however, that the last argument relies heavily on the fact that p_1 lies in $(-\frac{1}{2}, \frac{1}{2}]$ and p_2 lies in $[-\frac{1}{2}, \frac{1}{2})$, i.e., p_1 and p_2 cannot equal $\frac{1}{2}$ (or $-\frac{1}{2}$) simultaneously. Finally, we check the conditions (2.7c). We prove first that $v_k^* > 0$ implies $x_k^1(\omega^k, z) + x_k^2(\omega^k, z) = C$. Since z^* is an optimal solution of the linear program (LP), it is enough to show that $x_k^1(\omega^k, z^*) + x_k^2(\omega^k, z^*) = C$ implies $x_k^1(\omega^k, z) + x_k^2(\omega^k, z) = C$. To this end, we observe by Lemma 2.2 that for a suitable choice of p_1, p_2 ,

$$\begin{aligned} x_k^1(\omega^k, z) + x_k^2(\omega^k, z) &= x_k^1(\omega^k, z^*) + x_k^2(\omega^k, z^*) - p_1 - p_2 \\ &= C - p_1 - p_2 \in (C - \frac{1}{2}, C + \frac{1}{2}), \end{aligned}$$

and $x_k^1(\omega^k, z) + x_k^2(\omega^k, z) = C$, since the sum is integer-valued. Similarly it can be shown that $\lambda_k^{i*} > 0$ implies $x_k^i(\omega^k, z) = 0$, and $\mu_k^{i*} > 0$ implies $x_k^i(\omega^k, z^*) = C_i, i = 1, 2$. Since the necessary and sufficient conditions of optimality for (LP) are satisfied, the optimality of $z = \{\mathbf{z}_k(\omega^k), 1 \leq k \leq n, \omega^k \in \Omega^k\}$ is now evident.

Proof of Proposition 3.2: Since there exists an integer-valued solution $z = \{\mathbf{z}_k(\omega^k), 1 \leq k \leq n, \omega^k \in \Omega^k\}$ for (LP), if the initial condition \mathbf{x} is integer-valued,

it follows that $J_n^\beta(\mathbf{x}) = W_n^\beta(\mathbf{x})$. Therefore, the convexity of $W_n^\beta(\cdot)$ with respect to \mathbf{x} implies the same for $J_n^\beta(\cdot)$.

Further, since $W_n^\beta(\cdot)$ is the value function of a linear program, it is a piecewise linear function of \mathbf{x} [77, page 56], so that the following corollary holds:

Corollary 2.1: $J_n^\beta(\cdot)$ is a piecewise linear function of \mathbf{x} .

Proposition 2.4: For every $\mathbf{x} \geq 0$, $J_n^\beta(\cdot)$ is a “supermodular” function of \mathbf{x} , i.e.,

$$J_n^\beta(x^1 + 1, x^2 + 1) - J_n^\beta(x^1 + 1, x^2) \geq J_n^\beta(x^1, x^2 + 1) - J_n^\beta(x^1, x^2).$$

Proof: The proposition is a direct consequence of Corollary 2.1 and Proposition 2.1. The proof is provided in the appendix.

2.4 Determination of the Optimal Strategy

We now show that the optimal policy for call allocation is of the “bang-bang” type. Specifically we prove that for type-1 calls, there is a monotone *switching curve* which partitions the state space into two regions. One of them is a blocking region (i.e., blocking is optimal for all states belonging to the region) while the other is nonblocking. Analogous results hold for type-2 calls also.

We begin by making the following assertions:

Assertion 1: Assuming that state (x^1, x^2) is a blocking state for type-1 calls, all states (\bar{x}^1, x^2) with $\bar{x}^1 > x^1$ are also blocking states for type-1 calls.

Since the state (x^1, x^2) is a blocking state for type-1 calls, from the switching conditions (2.4) we have $J_n^\beta(x^1 + 1, x^2) - J_n^\beta(x^1, x^2) \geq J_n^\beta(x^1 + 1, x^2) - J_n^\beta(x^1, x^2) \geq 1/\beta\lambda_1$, and as a consequence the state $(x^1 + 1, x^2)$ is blocking thereby validating the assertion. An analogous result for type-2 calls can be similarly proved.

Assertion 2: Assuming that state (x^1, x^2) is a blocking state for type-1 calls, all states (x_1, \bar{x}_2) with $\bar{x}_2 > x_2$ are also blocking states for type-1 calls. The assertion is proved in a similar manner as assertion 1 by using the supermodularity property (Proposition 2.4) of $J_n^\beta(\cdot)$.

By combining assertions 1 and 2, we conclude that the optimal strategy minimizing the β -discounted n -step blocking cost (2.2a) is characterized by two monotone switching curves, one for each traffic type. Furthermore, since all the previous arguments are valid in the limit as $n \rightarrow \infty$, we can assert:

Proposition 2.5: The optimal policy for the blocking system under study with respect to an infinite horizon β -discounted blocking cost is characterized by two monotone (decreasing) switching curves, one for each traffic type (Figure 2.4).

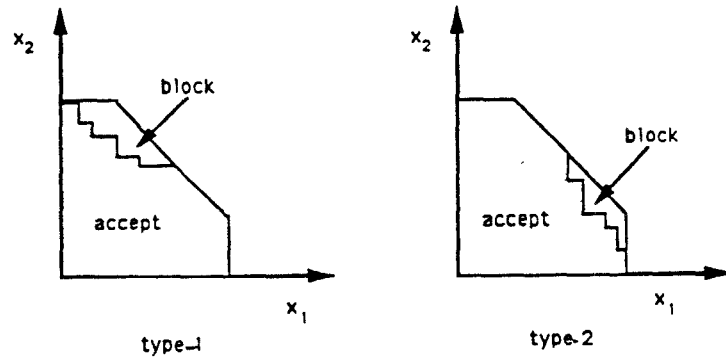


Figure 2.4

2.5 The Average Cost Case

In this section, we determine the structure of the optimal stationary policy with respect to an average cost criterion. To this end, we define the long-run average cost associated with a policy $z \in \mathcal{P}$ and starting with initial state \mathbf{x} , as:

$$V(\mathbf{x}, z) = \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbf{x}}^z \sum_{k=0}^n (z_k^1 + a z_k^2), \quad a > 0.$$

The minimum long-run average cost then is:

$$J_{av}(\mathbf{x}) = \min_{z \in \mathcal{P}} V(\mathbf{x}, z),$$

and the policy that achieves the minimum is an average cost optimal strategy.

From [64, Thm. 2.1 and 2.2] we conclude the following: Since the state space of our problem is finite for all discount factors $0 < \beta < 1$, the difference $|J^\beta(x^1, x^2) - J^\beta(0, 0)|$ is bounded. It follows that the average cost $J_{av}(\mathbf{x})$ is independent of the initial state \mathbf{x} and $J_{av} = \lim_{\beta \rightarrow 1} (1 - \beta)J^\beta(0, 0)$. Furthermore, there exists a bounded function $h(x^1, x^2)$ and a sequence of discount factors $\beta_n \rightarrow 1$ with $h(x^1, x^2) = \lim_{n \rightarrow \infty} (h^{\beta_n}(x^1, x^2) - h^{\beta_n}(0, 0))$, and satisfying the following DP-equation for the average cost:

$$\begin{aligned} J_{av} + h(\mathbf{x}) = & \min_{\substack{\{z_k^i \in \{0,1\} \\ z_k^i = 1 \text{ if } x^i = C_1 \\ \text{or } x^1 + x^2 = C\}}} \{z_k^1 + az_k^2 \\ & + \lambda_1(1 - z_k^1)h(A_1\mathbf{x}) + \lambda_1 z_k^1 h(\mathbf{x}) \\ & + \lambda_2(1 - z_k^2)h(A_2\mathbf{x}) + \lambda_2 z_k^2 h(\mathbf{x}) \\ & + \mu_1 x^1 h(D_1, \mathbf{x}) + \mu_2 x^2 h(D_2\mathbf{x}) \\ & + (C\mu_2 - x^1\mu_1 - x^2\mu_2)h(\mathbf{x})\}. \end{aligned}$$

Furthermore, there exists a stationary policy z that is average cost optimal and is the minimizer of the right side of the equation above. Obviously, $h(\cdot)$ has the same properties as $J^\beta(\cdot)$ for $0 < \beta < 1$, i.e., it is increasing, convex and supermodular. Switching conditions similar to (2.4) may be derived for $h(\cdot)$, and using the same arguments as for the discounted cost case, it can be shown that the average cost optimal strategy has the form of two monotone switching curves.

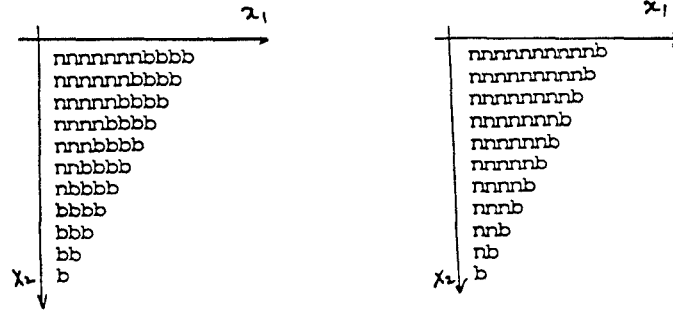
2.6 The Case of a Single Link

Similar results as before hold true for the problem associated with the optimal admission of two traffic types arriving at a common link having capacity C

frequency slots (i.e., when $C_1 > C$ and $C_2 > C$ in the model studied previously). Figure 2.5a illustrates the optimal admission policy for the β -discounted cost with $\lambda_1 = 10$, $\lambda_2 = 100$, $\mu_1 = 5$, $\mu_2 = 5$, $C = 10$, $\beta = 0.99$. In this example, the dynamic programming recursion (2.3) was iterated 200 times till the β -discounted cost converged. Then the optimal policy was evaluated through relations (2.4). In this particular case, and in many other similar ones, we observed from the computations that the traffic with the highest cost was never blocked (except on the boundaries of the state space). We were not able to provide a formal proof of this rather intuitively evident fact through dynamic programming equations.

We can gain some insight into the optimal admission policy for the simple link problem by examining the associated linear program (LP). Assuming that $\lambda_1 < \alpha\lambda_2$, and by the symmetry with respect to the variables $z_k^1(\omega_k)$, $z_k^2(\omega_k)$ of the constraint associated with the capacity C of the link, we conclude that if $z_1^1(A_1) = 0$ then $z_1^2(A_2) = 0$ since, in the opposite case, i.e. if $z_1^2(A_2) = 0$, we can interchange the values of z^1 , z^2 while still satisfying the constraint and simultaneously achieving a smaller increase in the cost function of the linear program. In a similar manner we can show that $z_1^2(A_2) = 1$ (call type-2 is blocked) implies $z_1^1(A_1) = 1$ (call type-1 is blocked) also. We have been yet unable to relate the optimal policy derived from the switching conditions (2.4) to the solution of the linear program (LP). Therefore, no valid conclusions about the the optimal policy can be derived by simply examining the behavior of the solution of (LP).

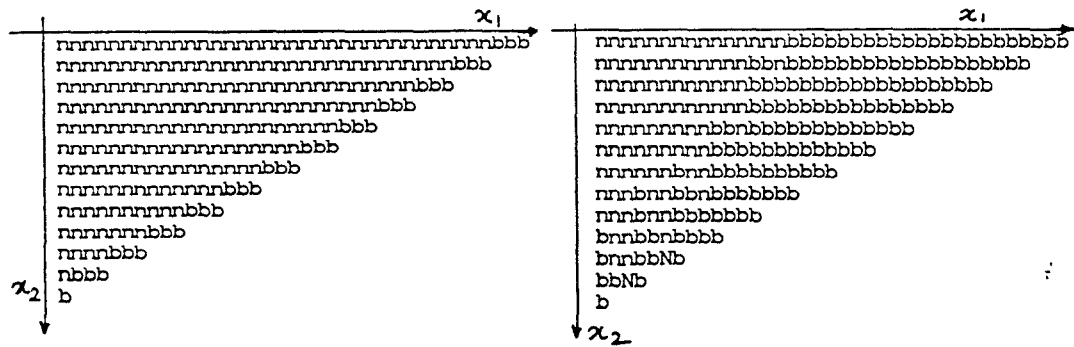
Another case of interest is the optimal admission policy when the traffic streams have different bandwidth requirements. Computations were performed for two streams arriving at a common link. In this case it is computationally demonstrated that the optimal policy for the β -discounted cost (Figure 2.5b) is not necessarily characterized by two monotone switching curves. As a consequence, the β -discounted cost may not be convex.



(a)

b: block a call, n: accept a call

$$\lambda_1 = 10, \lambda_2 = 100, \mu_1 = \mu_2 = 5, a = 5, C = 10$$



(b)

b: block a call, n: accept a call

Bandwidth ratio 1:3

$$\lambda_1 = 16, \lambda_2 = 55, \mu_1 = 0.5, \mu_2 = 8, a = 15, C = 36$$

Figure 2.5

2.7 The Optimal Admission Control Problem for a General Circuit Switched Network

We consider a circuit-switched network providing service to different traffic (call) types. The links between the nodes are labeled $j = 1, 2, \dots, J$, and each link j comprises C_j circuits (channels). Each call upon admission to the network is forwarded to its destination through a prespecified set of interconnected links which constitute a route. Let \mathcal{R} be the set of all routes in the network. We define the matrix $A = (a_{jr}, j = 1, 2, \dots, J, r \in \mathcal{R})$, where

$$a_{jr} = \begin{cases} 1 & \text{if a message on route } r \text{ uses a circuit of link } j \\ 0 & \text{otherwise.} \end{cases}$$

Assume that the calls requesting route r arrive according to a Poisson process with intensity λ_r . Moreover, the service time of each call (i.e., the time during which it is forwarded through route r) is exponentially distributed with parameter μ_r . A call requesting admission on route r is discarded if at least one link on the route r is saturated, i.e., has no free slots. We denote by \mathbf{C} the capacity vector, i.e., $\mathbf{C} = (C_1, \dots, C_J)^T$. Observe that the problem formulated in section 2.2 is a special case of the general problem with matrix A and capacity vector \mathbf{C} of the form:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} C_1 \\ C_2 \\ C \end{pmatrix}.$$

We study the equivalent discrete-time problem. As before, we define the state of the system at time instant k to be $\mathbf{x}_k = (x_k^r, r \in \mathcal{R})$ where x_k^r denotes the number of calls forwarded on route r at that time instant. Obviously, $\mathbf{A}\mathbf{x}_k \leq \mathbf{C}$, and we define the state space of the system to be $\mathcal{X} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{C}, \mathbf{x} \geq 0\}$. Recall that the time instants at which the system is observed correspond to state transition epochs (i.e., arrivals or departures). Given that a cost a_r is incurred for each call that is not given access to the network on route r , we seek an optimal admission strategy (in the same spirit as for the simple problem of section 2.2) minimizing

an infinite horizon (n step) β -discounted cost of the form:

$$\mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=1}^{\infty(n)} \beta^k \mathbf{a} \mathbf{z}_k \right),$$

where $\mathbf{a} = (a_r, r \in \mathcal{R})$, $a_k > 0$, $\mathbf{z}_k = (z_k^r, r \in \mathcal{R})$ and $z_k^r = 1(0)$ if an incoming call on route r is blocked (accepted).

We define the total event rate (i.e., the “uniformization” rate) out of a state to be:

$$\rho = \sum_{r \in \mathcal{R}} \lambda_r + \bar{\mathbf{x}} \cdot \boldsymbol{\mu},$$

where $\boldsymbol{\mu} = (\mu_i, i \in \mathcal{R})$ and $\bar{\mathbf{x}}$ is a solution to the following Linear Program:

$$\max_{\mathbf{x}} \boldsymbol{\mu} \mathbf{x},$$

such that:

$$\mathbf{A} \mathbf{x} \leq \mathbf{C}, \mathbf{x} \geq 0.$$

After normalizing all rates with respect to ρ (equivalently assuming $\rho = 1$), we can write the dynamic programming equation associated with the optimal n -step β -discounted cost $J_n^\beta(\mathbf{x})$. To this end we denote by \mathbf{e}_r the column-vector $(e_i, i \in \mathcal{R})$, with $e_i = 0$ for $i \neq r$ and $e_r = 1$. Moreover, we define the arrival and departure operators $A_r, D_r : \mathcal{X} \rightarrow \mathcal{X}$ as follows:

$$A_r(\mathbf{x}) = (\mathbf{x} + \mathbf{e}_r)^*, \quad D_r(\mathbf{x}) = (\mathbf{x} - \mathbf{e}_r)^*,$$

where

$$(\mathbf{x} + \mathbf{e}_r)^* = \begin{cases} \mathbf{x} + \mathbf{e}_r & \text{if } A(\mathbf{x} + \mathbf{e}_r) \leq \mathbf{C} \\ \mathbf{x} & \text{otherwise} \end{cases}, \quad (\mathbf{x} - \mathbf{e}_r)^+ = \begin{cases} \mathbf{x} - \mathbf{e}_r & \text{if } x_r \geq 1 \\ \mathbf{x} & \text{otherwise} \end{cases},$$

for r in \mathcal{R} . Then we can write:

$$\begin{aligned} J_{k+1}^\beta(\mathbf{x}) = & \min_{\substack{\{z_k^i \in [0,1], r \in \mathcal{R} \\ z_k^r = 1 \text{ if } A(\mathbf{x} + \mathbf{e}_r) > \mathbf{C}\}}} (\mathbf{a} \mathbf{z}_k + \beta \sum_{r \in \mathcal{R}} ((1 - z_k^r) \lambda_r J_k^\beta(A_r \mathbf{x}) + \lambda_r z_k^r J_k^\beta(\mathbf{x})) \\ & + x_r \mu_r J_k^\beta(D_r \mathbf{x}) + (\bar{\mathbf{x}}_r - x_r) \mu_r J_k^\beta(\mathbf{x})), \end{aligned}$$

and the following criterion for admission can be obtained:

If $A(\mathbf{x} + \mathbf{e}_r) \leq \mathbf{C}$ then an incoming call to route r is blocked ($z_k^r = 1$) if:

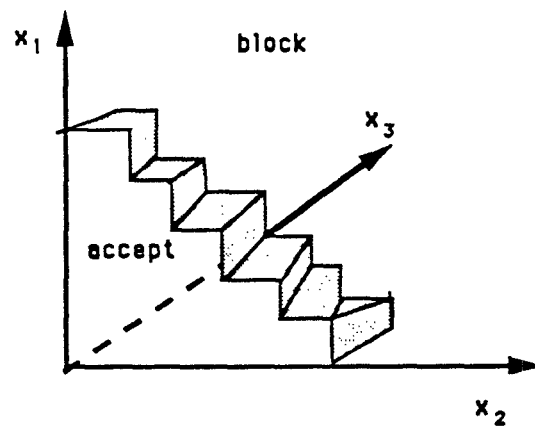
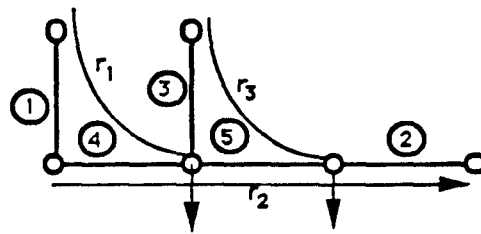
$$J_n^\beta(A_r \mathbf{x}) - J_n^\beta(\mathbf{x}) \geq \frac{a_r}{\beta \lambda_r}.$$

Unfortunately the convexity and supermodularity properties (cf. Propositions 2.2 and 2.4) for $J_n^\beta(\cdot)$ cannot be derived for an arbitrary network topology. This is due to the fact that an integer-valued solution to an associated linear program similar to (LP) cannot be found. For example, Proposition 2.3 fails to hold if we have a constraint of the form $x_k^1(\omega^k) + x_k^2(\omega^k) + x_k^3(\omega^k) \leq C$ for a system with 3 routes. As a consequence, the optimal strategy for the general problem cannot be shown to have a “switching” surface structure. Nevertheless, for the case where the matrix A has the simple form,

$$\mathbf{A} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \dots & \\ & & & & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 11 & & & & \\ & 11 & & & \\ & & 11 & & \\ & & & \dots & \\ & & & & 11 \end{pmatrix},$$

it can be easily verified that all the proofs of convexity and supermodularity of the optimal β -discounted cost hold true. In this case, the optimal admission strategy has a structure of a “monotone” switching surface, i.e., if $z^r(\mathbf{x}) = 1$ then $z^r(\mathbf{x} + \mathbf{e}_i) = 1$ for i in \mathcal{R} and $\mathbf{x} + \mathbf{e}_i$ in \mathcal{X} . Furthermore, similar results are also true for the long-run average cost problem.

A network with matrix A having the simplified form, as well as the associated switching surface for type-1 calls is shown in Figure 2.6.



Admission Policy for Traffic Type-1

	r_1	r_2	r_3	← routes
$A =$	1	0	0	①
	0	1	0	②
	0	0	1	③
	1	1	0	④
	0	1	1	⑤
				↑ links

Figure 2.6

Appendix 2.1

Proof of Proposition 2.4: In this section we prove that $J_n^\beta(\cdot)$ is a “supermodular” function. Specifically, we prove that a piecewise linear increasing function $W(\cdot)$ is supermodular, i.e.,

$$W(x^1 + 1, x^2 + 1) - W(x^1 + 1, x^2) \geq W(x^1, x^2 + 1) - W(x^1, x^2), \quad (A.1)$$

for $x^1 \geq 0, x^2 \geq 0$.

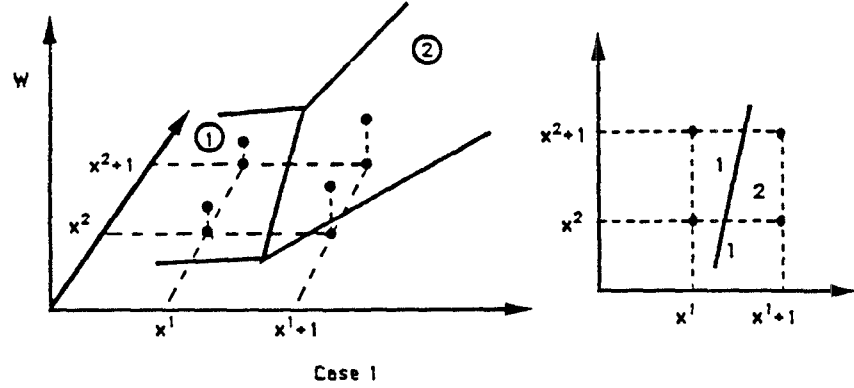


Figure 2.7

The proof consists of the following cases:

Case 1: Refer to Figure 2.7. Since $W(\cdot)$ is piecewise linear, its graph consists of intersecting planes. Assume that the points $(x^1, x^2, W(x^1, x^2))$ and $(x^1, x^2 + 1, W(x^1, x^2 + 1))$ belong to plane (1), while the other two points, i.e., $(x^1 + 1, x^2, W(x^1 + 1, x^2))$ and $(x^1 + 1, x^2 + 1, W(x^1 + 1, x^2 + 1))$ belong to plane (2). We now make a projection on the (x^1, x^2) -plane. The line l is the projection of the intersection of the planes (1) and (2) (Figure 2.7). Since $W(\cdot)$ is piecewise linear, we set $W(x^1, x^2) = A_1 x^1 + B_1 x^2 + C_1$, $W(x^1 + 1, x^2) = A_2(x^1 + 1) + B_2 x^2 + C_2$, $W(x^1, x^2 + 1) = A_1 x^1 + B_1(x^2 + 1) + C_1$, and $W(x^1 + 1, x^2 + 1) =$

$A_2(x^1 + 1) + B_2(x^2 + 1) + C_2$, where the subscript i in the group of coefficients (A_i, B_i, C_i) refers to plane $i = 1, 2$. Direct substitution in (A.1) gives $A_2 \geq A_1$, a valid fact due to the increasing nature of $W(\cdot)$.

For all subsequent cases, the reader is referred to Figure 2.8.

Case 2: By direct substitution in (A.1) we get:

$$A_1x^1 + B_1x^2 + C_1 \geq A_2x^1 + B_2x^2 + C_2.$$

This inequality is true due to the increasing property of $W(\cdot)$.

Case 3: By substitution in (A.1) we get :

$$A_2(x^1 + 1) + B_2(x^2 + 1) + C_2 \geq A_1(x^1 + 1) + B_1(x^2 + 1) + C_1,$$

in a manner similar to case 2.

Case 4: Inequality (A.1) is established since $B_2 \geq B_1$, similar to case 1.

In cases (5)-(9) we consider the intersection of 3 planes.

Case 5: By direct substitution in (A.1) we get,

$$A_2(x^1 + 1) + B_2(x^2 + 1) + C_2 - A_3(x^1 + 1) - B_3(x^2 + 1) - C_3 + A_3 \geq A_1,$$

since $A_3 \geq A_1$ and $A_2(x^1 + 1) + B_2(x^2 + 1) + C_2 \geq A_3(x^1 + 1) + B_3(x^2 + 1) + C_3$.

Cases (6) – (9) (see Figure 2.8) can be established in a similar manner. The case of four intersecting hyperplanes can easily be reduced to any of the previous cases.

Remark: For the proof of “supermodularity”, only the increasing nature and the piecewise linearity of $W(\cdot)$ are used; the convexity property is not needed.

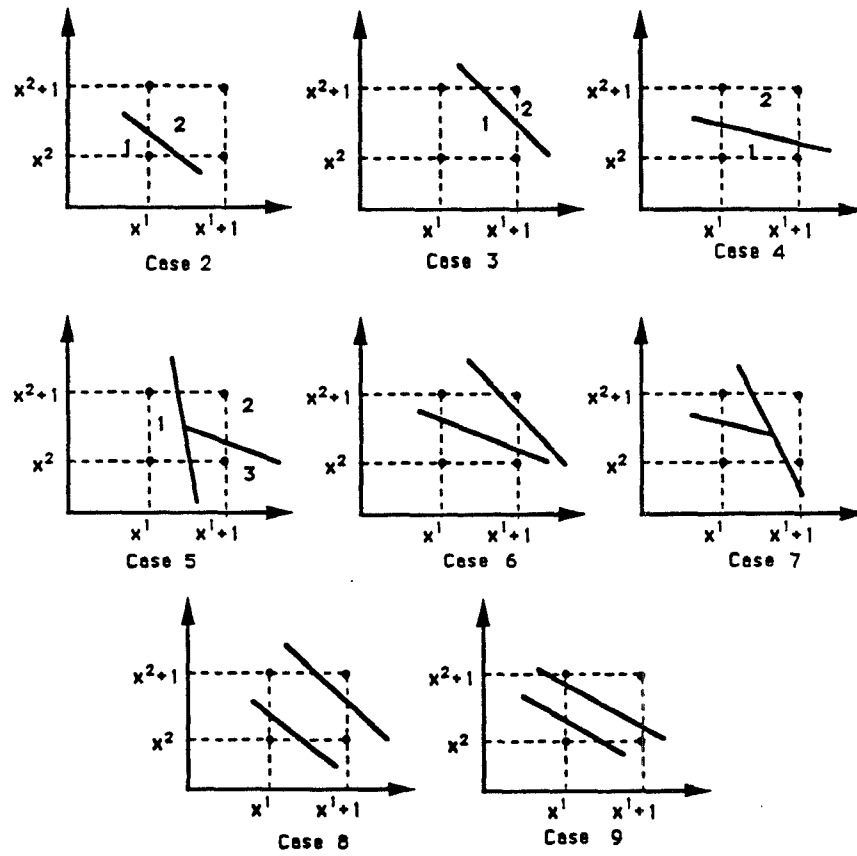


Figure 2.8

CHAPTER 3

THE CONVEXITY OF OPTIMAL DISCOUNTED COSTS FOR CERTAIN CLASSES OF MARKOV DECISION PROCESSES

3.1 Introduction

In many problems of optimal control of queuing networks, the convexity of the optimal discounted cost associated with an underlying Markov decision process plays a key role. A typical example is the simple blocking network studied in the preceding section. It is, therefore, desirable to investigate the possibility of devising a procedure that ascertains the convexity of an optimal discounted cost associated with an MDP mainly from the “macroscopic” properties of the process. To this end, we provide a brief study of the convexity property exhibited by the discounted optimal cost (over either a finite or an infinite time horizon) associated with a class of Markov decision processes. A key intermediate step relies on certain facts concerning linear programs and integral polytopes. We commence with a few relevant definitions and propositions.

3.2 When does a Linear Program have an Integer-Valued Optimal Solution? — A Sufficient Condition

Definition 3.1: A vector or matrix with elements in \mathbb{R} is called *integral* if its entries are all integers.

Definition 3.2: A set $\mathcal{P} \subset \mathbb{R}^n$ is called a polyhedron if

$$\mathcal{P} = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\},$$

where \mathbf{A} is a matrix and \mathbf{b} a vector both of suitable dimensions and consisting of \mathbb{R} -valued elements.

Definition 3.3: The integer hull \mathcal{P}_I of a polyhedron \mathcal{P} is defined as the convex hull of all integral vectors in \mathcal{P} . If $\mathcal{P}_I = \mathcal{P}$ then \mathcal{P} is called an *integral* polyhedron.

It follows that every face (and every corner) of an integral polyhedron contains integral vectors.

Definition 3.4: A real matrix \mathbf{A} is called *totally unimodular* if each subdeterminant of \mathbf{A} is 0, 1 or -1. In particular, each entry of a totally unimodular matrix is 0, 1, or -1.

The following propositions are stated *sans* proofs, which can be found in [68].

Proposition 3.1: If \mathbf{A} is a totally unimodular matrix and \mathbf{b} is an integral vector, then the polyhedron $\mathcal{P} = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\}$ is integral.

The following corollary is a direct consequence of the proposition.

Corollary 3.1: Let \mathbf{A} be a totally unimodular matrix and \mathbf{b} an integral vector. Then the linear program $\min \{\mathbf{cx} : \mathbf{Ax} \leq \mathbf{b}\}$, where \mathbf{c} is a real valued vector of appropriate dimension, has integral optimal solutions.

The next proposition provides a sufficient condition for a matrix \mathbf{A} to be totally unimodular.

Proposition 3.2: A matrix \mathbf{A} with entries 0, +1, -1 is totally unimodular if any arbitrary collection of columns of \mathbf{A} can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries 0, +1 or -1.

Remark: It is evident from Proposition 3.2 that the following procedure can be adopted to determine if a matrix \mathbf{A} is totally unimodular. This procedure will be employed in the analysis below.

- 1) Any given collection of columns of \mathbf{A} is split into subgroups.
- 2) The columns in each subgroup are signed and then added.
- 3) The resulting columns from the subgroups are again signed and added.
- 4) The resulting column from step (3) is a vector with entries 0, +1, or -1.

Steps (1)-(4) will be the key point of the analysis that follows.

3.3 Linear Programming Formulation of Markov Decision Processes

We consider a “discrete” set of events $\Omega = \{E_1, E_2, \dots, E_L\}$ along with a probability mass function $p_i, i = 1, \dots, L$, i.e., $\mathbb{P}(E_i) = p_i$; clearly $p_i \geq 0$, $\sum_{i=1}^L p_i = 1$. Let Ω^k be the k^{th} cartesian product of Ω , i.e., $\Omega^k = \{\omega^k : \omega^k = (\omega_1, \omega_2, \dots, \omega_k), \omega_i \in \Omega, i = 1, 2, \dots, k\}$. We then assume the induced probability mass function on Ω^k will be given by $\mathbb{P}(\omega^k) = \prod_{\ell=1}^k \mathbb{P}(\omega_\ell), \omega_i \in \Omega$. We can now define the following stochastic processes:

- A transition process $(\Xi_i, i = 1, 2, \dots)$, with $\Xi_i(\omega^i) = \Xi_i(\omega_i)$ and Ξ_i being an $N \times M$ matrix with entries 1, 0 or -1's.
- A process $(\mathbf{z}_i, i = 1, 2, \dots)$, with $\mathbf{z}_i(\omega^i) = \mathbf{z}_i(\omega_i)$ taking values in $\{0, 1\}^M$ (feasibility constraint).
- A process $(\mathbf{u}_i, i = 1, 2, \dots)$, with $\mathbf{u}_i(\omega^i) = \mathbf{u}_i(\omega_i)$, where \mathbf{u}_i is an N -dimensional integral vector.

We now define an N -dimensional integral vector-valued “state” process $(\mathbf{x}_k(\omega^k), k = 1, 2, \dots)$ according to the following recursion:

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{x}; \\ \mathbf{x}_{k+1}(\omega^{k+1}) &= \mathbf{x}_k(\omega^k) + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}) + \mathbf{u}_{k+1}(\omega^{k+1}) \end{aligned} \tag{3.1}$$

such that

$$\mathbf{0} \leq \mathbf{x}_k(\omega^k) \leq \mathbf{b} \quad k = 0, 1, \dots \tag{3.2}$$

where \mathbf{x} the initial “state” is assumed to be integral, and \mathbf{b} is a given N -dimensional integral vector.

Remarks: We stress that the processes $(\Xi_k, k = 1, 2, \dots)$ and $(\mathbf{u}_k, k = 1, 2, \dots)$ are completely determined by $\omega^k, k = 1, 2, \dots$ and their role is to potentially alter the state process $(\mathbf{x}_k, k = 1, 2, \dots)$. We, therefore, refer to $(\Xi_k, k = 1, 2, \dots)$ and $(\mathbf{u}_k, k = 1, 2, \dots)$ as transition processes. While \mathbf{u}_k is uncontrolled, Ξ_k is “modulated” by \mathbf{z}_k . We refer to $(\mathbf{z}_k, k = 1, 2, \dots)$ as the *control* process (or

simply control) and taking values in $\{0, 1\}$, i.e., \mathbf{z}_k enables (or disables) Ξ_k . For all possible ω_{k+1} in Ω , we can then easily determine a transition probability law (induced by $\mathbb{P}(\cdot)$) of the state \mathbf{x}_{k+1} given \mathbf{x}_k , and the value of the control \mathbf{z}_{k+1} , so that the feasibility constraint (3.2) is met. In order to simplify the notation, we denote this transition probability by $\mathbb{P}(\mathbf{x}_{k+1}|\mathbf{x}_k, z)$. Obviously $((\mathbf{x}_k, \mathbf{z}_k), k = 1, 2, \dots)$ is a Markov decision process (MDP). We assume that an instantaneous cost of the form

$$C(\mathbf{x}_k, \mathbf{z}_k) = \mathbf{c}^T \mathbf{x}_k + \mathbf{d}^T \mathbf{z}_k, \quad \mathbf{c} > 0, \mathbf{d} > 0 \quad (3.3)$$

is incurred at each time instant k . We note here that $\mathbf{c} > 0, \mathbf{d} > 0$ mean component-wise positivity. As previously in Chapter 2, section 2.3, an optimal β -discounted cost over an infinite (finite) horizon, starting at \mathbf{x} can be defined as:

$$J_{(n)}^\beta(\mathbf{x}) = \min_z \mathbb{E}_{\mathbf{x}}^z \sum_{k=0}^{\infty(n)} \beta^k C(\mathbf{x}_k, \mathbf{z}_k) \quad (3.4)$$

where the minimization is performed with respect to the controls $\mathbf{z}_k \in \{0, 1\}^M$ under the constraint (3.2). The expectation in (3.4) is defined with respect to the stationary probability distribution of the MDP induced by the transition probability $\mathbb{P}(\mathbf{x}_{k+1}|\mathbf{x}_k, z)$.

Given that all the necessary and sufficient conditions for $J^\beta(\cdot)$ to exist [42,43] are satisfied by the MDP $(\mathbf{x}_k, k = 1, 2, \dots)$ in (3.4) we claim the following:

Proposition 3.3: For the MDP $(\mathbf{x}_k, k = 1, 2, \dots)$ with incurred per stage cost $C(\cdot, \cdot)$ (3.3), let Ξ be the matrix of the concatenation of columns of $(\Xi(\omega), \omega \in \Omega)$, such that no column is repeated (either with a positive or negative sign), and with no all-zero column. If a permanent sign assignment can be made to each column of Ξ such that the column formed by adding any arbitrary collection of signed columns contains -1, 0 or 1's, then $J_{(n)}^\beta(\cdot)$ is convex.

Proof: Following the approach of the previous section we formulate the minimization in (3.4), for a finite horizon of n -steps in terms of a linear program. For simplicity we omit the feasibility constraints on $\mathbf{z}_k(\omega^k)$ as also the constraint

$\mathbf{x} \leq \mathbf{b}$; the only constraint assumed to be $\mathbf{x}_k(\omega^k) \geq 0$, $\omega^k \in \Omega^k$, $k = 1, 2, \dots, n$. Hence, the linear program has the form:

$$\begin{aligned} & \min_{\{\mathbf{z}_k(\omega^k)\}_{k=1}^n} \sum_{k=1}^n \sum_{\omega^k} \gamma_k^T(\omega^k) \mathbf{z}_k(\omega^k) \\ & \text{such that :} \quad k=1, 2, \dots, n \\ & \quad \omega^k \in \Omega^k \quad (LP) \\ & \sum_{j=1}^k \Xi_j(\omega^i) \mathbf{z}_j(\omega^j) \geq \mathbf{x} - \sum_{j=1}^k \mathbf{u}_j(\omega^j) \quad (3.5) \end{aligned}$$

where $\gamma_k^T(\cdot)$ is a suitable vector not depending on \mathbf{z}_k . Note that the LHS of (3.5) is an integral vector (assuming that \mathbf{x} is integral). If the linear program (LP) were to accept an integral optimal solution, then since \mathbf{x} enters linearly in the constraint equations (3.5), it would easily follow [77] that $J_n^\beta(\cdot)$ is convex; Proposition 3.3 would then be valid. To check if (LP) accepts an integral solution, we rewrite the constraint equation (3.4) in matrix form as

$$\mathbf{A}\mathbf{z} \geq \tilde{\mathbf{b}}.$$

Since $\tilde{\mathbf{b}}$ is integral, we concentrate on the structure of matrix \mathbf{A} . The matrix \mathbf{A} has the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{22} & & & \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & & \\ \vdots & & \ddots & & \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \dots & \mathbf{A}_{n-1,n} & \mathbf{A}_{nn} \end{bmatrix},$$

where

$$\mathbf{A}_1 = \text{diag} [\Xi(E_1) \dots \Xi(E_L)],$$

$$\mathbf{A}_{k\ell} = \left[\begin{array}{cccc} \mathbf{A}_{k-1,\ell-1} & & & \\ & \mathbf{A}_{k-1,\ell-1} & & \\ & & \ddots & \\ & & & \mathbf{A}_{k-1,\ell-1} \end{array} \right] \Bigg\} k \text{ times } \begin{array}{l} \ell > 1 \\ k > 1 \end{array},$$

$$\mathbf{A}_{k1} = \left[\begin{array}{ccc} \left. \begin{array}{c} \Xi(E_1) \\ \vdots \\ \Xi(E_1) \end{array} \right\} L^k \text{ times} & & \\ & \left. \begin{array}{c} \Xi(E_2) \\ \vdots \\ \Xi(E_2) \end{array} \right\} L^k \text{ times} & \\ & & \ddots \\ & & \left. \begin{array}{c} \Xi(E_L) \\ \vdots \\ \Xi(E_L) \end{array} \right\} L^k \text{ times} \end{array} \right].$$

In Figure 3.1, the matrix \mathbf{A} is illustrated for $L = 3, n = 3$.

We now prove that the matrix \mathbf{A} is totally unimodular and hence (LP) accepts an integer-valued solution. To this end, we observe that any column of \mathbf{A} comprises once (or more than once) the same column of Ξ with the rest of the entries being 0's. Moreover if $\mathbf{A}_{.i}, \mathbf{A}_{.j}$ are two columns of \mathbf{A} with $i < j$ and both containing a common column of Ξ , then $\mathbf{A}_{.i}$ has no more nonzero elements than $\mathbf{A}_{.j}$. Assuming that Ξ is an $N \times M$ matrix and given a collection of columns of \mathbf{A} , we can form the subcollection $\mathcal{C}_1, \dots, \mathcal{C}_k, k \leq M$, so that \mathcal{C}_i contains columns comprising a column $\pm \xi_i$ of Ξ , and 0's. It is obvious that by adding or subtracting suitably the columns in each collection \mathcal{C}_i , we can eventually end up with a column vector \mathbf{c}_i containing the column ξ_i and 0's. By assigning to \mathbf{c}_i the signs for ξ_i given in Proposition 3.3, the sum of \mathbf{c}_i 's will contain 1, 0 or -1's, thus asserting that \mathbf{A} is a totally unimodular matrix according to Proposition 3.2 (via steps 1-4).

Remark: If the feasibility constraint for the control is included, i.e., $0 \leq \mathbf{z}_k \leq 1$, then similar arguments as above hold true for the system of constraints of the form:

$$\begin{pmatrix} \mathbf{A} \\ \dots \\ \mathbf{I} \\ \dots \\ -\mathbf{I} \end{pmatrix} \mathbf{z} \geq \begin{pmatrix} \tilde{\mathbf{b}} \\ \dots \\ \mathbf{0} \\ \dots \\ -\mathbf{1} \end{pmatrix},$$

where \mathbf{I} is a suitable unit matrix. Using analogous arguments, constraints of the form $\mathbf{a} \leq \mathbf{x}_k(\omega^k) \leq \mathbf{b}$, for suitable vectors \mathbf{a}, \mathbf{b} can also be handled. If in addition, constraints of the form $\mathbf{c} \leq \mathbf{B}\mathbf{x}_k(\omega^k) \leq \mathbf{d}$ for given (integral) vectors \mathbf{c}, \mathbf{d} and an integral matrix \mathbf{B} are imposed, then the criterion of Proposition 3.3 should also be checked for the matrix $\mathbf{B}\Xi$.

3.4 Applications and Conclusions

For the simple blocking network of Chapter 2 studied previously we have:

$$\Xi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Then $\mathbf{B}\Xi = \mathbf{B}$ and Proposition 3.3 (along with the subsequent remark) is satisfied by assigning the sign (+) to the first column of and the sign (-) to the second column of \mathbf{B} . Observe that for a matrix \mathbf{B} of the form:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & 1 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \end{bmatrix}$$

the same result is true, while for $\mathbf{B} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, Proposition 3.3 fails and the convexity of $J_{(n)}^\beta(\cdot)$ cannot be guaranteed. These results can be contrasted with the geometric approach used for proving the convexity of $J_{(n)}^\beta(\cdot)$ in the previous chapter.

Remark: It appears that Proposition 3.3 is rather restrictive since its hypothesis fails to hold whenever the matrix Ξ (or $\mathbf{B}\Xi$) has more than two 1's as entries in the same row. Nevertheless, in a considerable number of problems associated with simple queuing networks, Proposition 3.3 holds true. Unfortunately, it seems that whenever Proposition 3.3 fails so too do the geometric methods used in chapter 2 to show the existence of integer-valued optimal policies. This is most probably due to the close connection between the region X of Lemma 2.1 used in and the structure of matrix Ξ . In such cases, the proof of the convexity of $J_{(n)}^\beta(\cdot)$ (if true) is an open problem.

CHAPTER 4

UPPER BOUNDS FOR LINK BLOCKING PROBABILITIES IN CIRCUIT-SWITCHED NETWORKS

4.1 Introduction

Our studies thus far have focused mainly on determining the structural characteristics of optimal admission and resource allocation policies in a class of simple circuit-switched networks. A knowledge of the structure of the optimal policy provides qualitative information on system behavior. It can also serve as a useful guide in devising efficient (but suboptimal) resource allocation schemes which afford simple implementation. However, this structural information does not suggest any efficient computational procedure for a quantitative characterization of the optimal performance of the associated network. Furthermore, as seen in Chapter 2, for more complex network models even the structure of an optimal strategy can be difficult to characterize. Finally, it would be desirable to evaluate the performance of networks providing service to traffic types with different bandwidth requirements, e.g., ordinary voice calls and high quality video messages.

In this chapter, we consider a simple computational technique for evaluating upper bounds on the blocking probability associated with the links and routes of a simple circuit-switched network.

4.2 Description of the Problem

We consider a circuit-switched network with fixed routing. Let $\mathcal{R} = \{1, 2, \dots, R\}$ be the set of all possible routes and let $\mathcal{L} = \{1, 2, \dots, L\}$ be the set of the network links. We assume that link ℓ has a capacity of C_ℓ frequency slots, $\ell = 1, 2, \dots, L$, and let $\mathbf{C} = (C_1, C_2, \dots, C_L)$ be the vector of link capacities. The

topology of the network is characterized by the $L \times R$ -matrix \mathbf{A} , where:

$$(A)_{\ell r} \triangleq a_{\ell r} = \begin{cases} 0, & \text{if link } \ell \text{ does not lie on route } r; \\ \text{number} & \text{of slots used by a message on link } \ell \text{ while} \\ & \text{traversing route } r. \end{cases}$$

The following statistical assumptions are made. Arrivals at the network for transmission on route r are assumed to be Poisson with parameter λ_r . An arriving message is simultaneously granted $a_{\ell r}$ slots on each link ℓ belonging to route r ; if at least one such link is full when a message arrives then the message is blocked and assumed lost. Each message on route r has a random propagation time distributed according to a general distribution with mean μ_r , $r = 1, 2, \dots, R$.

An appropriate state description for the network at steady state is the vector $\mathbf{x} = (x_1, x_2, \dots, x_R)$ where x_r denotes the number of messages on route r at steady state. Let $\mathcal{J}(\mathbf{C}) = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{C}, \mathbf{x} \geq 0\}$. Then it is well-known [28] that at steady state, the stationary probability distribution of \mathbf{x} is given by,

$$P(\mathbf{x}) = \frac{1}{K(\mathbf{C})} \prod_{r \in \mathcal{R}} \frac{\rho_r^{x_r}}{x_r!}, \quad (4.1)$$

where,

$$K(\mathbf{C}) = \sum_{\mathbf{x} \in \mathcal{J}(\mathbf{C})} \prod_{r \in \mathcal{R}} \frac{\rho_r^{x_r}}{x_r!}, \quad (4.2)$$

and $\rho_r = \frac{\lambda_r}{\mu_r}$ is the *traffic load* on route r . Straightforward calculations [27] show that a message arriving for transmission on route r is blocked with probability

$$1 - K(\mathbf{C} - \mathbf{e}_r)K^{-1}(\mathbf{C}), \quad (4.3)$$

where \mathbf{e}_r is the unit vector having a “1” as its r th entry and of “0” elsewhere.

Although (4.3) defines a conceptually easy procedure for evaluating the blocking probability associated with each route in a circuit-switched network, it is computationally demanding especially when the network becomes large (i.e., when a large number of traffic types are forwarded in large capacity links). For large networks a measure of the blocking experienced on each route can be derived from

a knowledge of the blocking associated with the individual links constituting the route. If we denote by $q_\ell(j)$ the steady-state probability that the carried traffic on link ℓ occupies j slots then the blocking probability due to this link of a message traversing on route r is simply $\sum_{j=C_\ell-a_{\ell r}}^{C_\ell} q_\ell(j)$. By assuming that in relatively large networks, the links block almost independently [28], we can estimate the blocking probability of a message designated for route r by

$$1 - \prod_{\ell: a_{\ell r} \neq 0} \left(1 - \sum_{j=C_\ell-a_{\ell r}}^{C_\ell} q_\ell(j) \right). \quad (4.4)$$

Motivated by this fact we attempt to provide computational procedures for deriving bounds on the link blocking probability. We first present a few results on the blocking probability of a single link.

4.3 Upper Bounds on the Blocking Probability of a Single Link

We assume that R different traffic types pass through a single link with capacity C slots. (Note that for a single link, the R different routes of the previous section correspond naturally to R different traffic types). Let U be the random variable corresponding to the number of occupied slots in the channel at steady state. We refer to U as the *carried traffic* and set $P(U = y) = q(y)$, $y = 0, 1, \dots, C$. In [26,21] the following recursion for $q(\cdot)$ is shown to be true:

$$yq'(y) = \sum_{r=1}^R a_r \rho_r q'(y - a_r), \quad (4.5)$$

where $\rho_r = \frac{\lambda_r}{\mu_r}$ is the *load* offered by traffic type r , with $\sum_{y=1}^C q'(y) = 1$ and $q'(y) = 0$ for $y < 0$. As observed in [26], (4.5) defines a straightforward one-dimensional recursion. Moreover it is shown in [21] that (4.5) is satisfied in general by processes with stationary probability distributions of the form (4.1).

We define $W = \sum_{r=1}^R \mathbf{x}_r$ and temporarily assume that $C = \infty$, i.e., none of the arrivals ever blocked. If $\mu_W(s)$ denotes the log-moment generating function of

the random variable W , i.e.,

$$\mu_W(s) = \ln \left(\sum_{y=0}^{\infty} P(W = y) e^{sy} \right),$$

then it can be shown [20,21] that,

$$\mu_W(s) = \sum_{r=1}^R \rho_r a_r (e^{s a_r} - 1). \quad (4.6)$$

We refer to W as the *offered traffic*.

Since the probability mass functions of both W and U satisfy the same recursion (4.5) it follows that the quantities $P(W \leq y)$ and $P(U \leq y)$ must be proportional for $y = 0, 1, 2, \dots, C$. Using (4.5) and an initial condition $q'(0)$ we can then determine the proportionality constant as the ratio

$$\frac{\sum_{y=0}^{\infty} q'(y)}{\sum_{y=0}^C q'(y)} = \frac{1}{1 - \left(\frac{\sum_{y=C+1}^{\infty} q'(y)}{\sum_{y=0}^{\infty} q'(y)} \right)},$$

where it follows that

$$P(U \leq y) = (1 - P(W > C))^{-1} P(W \leq y).$$

for $y = 1, 2, \dots, C$. By assuming that we have a *light traffic* situation, i.e., $\sum_{r=1}^R a_r \rho_r \ll C$ we can disregard the term $P(W > C)$ as being close to 0 and claim $P(U \leq y) \approx P(W \leq y)$. The following light traffic bounds are derived in [20,21]:

$$P(U > y) \approx P(W > y) \leq e^{s^* y - \mu_W(s^*)} \quad (4.7)$$

$$\leq \left(s^* \sqrt{2\pi \frac{d}{ds} \mu_W(s^*)} \right)^{-1} e^{s^* y - \mu_W(s^*)}, \quad (4.8)$$

where s^* satisfies the equation $y = \frac{d}{ds} \mu_W(s) = \sum_{r=1}^R \rho_r a_r e^{a_r s}$.

Relation (4.7) is the well-known Chernoff bound, while (4.8) is an improved sharpened version of (4.7). The bounds (4.7) and (4.8) were first used for the analysis of blocking communication switches in [20,21]. Setting $y = C - a_r$, a bound is obtained for the blocking probability of a message of the r th traffic type on a single link, $r = 1, 2, \dots, R$.

4.4 Bounds on the Link Blocking Probabilities in a Circuit Switched Network

In this section we apply the computational procedures described above to the calculation of an upper bound on the link blocking probability in a circuit-switched network whose topology is characterized by a matrix \mathbf{A} introduced in section 4.2. We could begin by deriving a relation for a general circuit-switched network similar to (4.5). This derivation is performed in Appendix 4.1; however, this recursion is rather complicated and untractable. We can circumvent this difficulty by obtaining approximations *via* the following heuristic reasoning.

It is intuitively clear that the blocking associated with a given link in a circuit-switched network would be less severe than the blocking due to the same link if it were to constitute the whole network. Thus, the blocking probability due to each link in the network is bounded above by the blocking probability of the same link if it constituted a “single-link network.” In the latter case, traffic type- r is forwarded through the link ℓ if $a_{\ell r} \neq 0$, offers a load of $\rho_r = \frac{\lambda_r}{\mu_r}$, and requires $a_{\ell r}$ slots per message. The recursion (4.5) can be used to calculate $q_\ell(\cdot)$, which serves as an upper bound to the link probability distribution of the number of occupied slots on link ℓ when it belongs to the network. In the case of light traffic, the bounds (4.7) and (4.8) can then be calculated easily. In the case of *heavy traffic*, i.e., when $\sum_{r=1}^R \rho_r a_{\ell r} \gg C_\ell$ a *fluid* approximation may be utilized, i.e., a blocking probability of $1 - a_{\ell r} \rho_r C_\ell (\sum_{r=1}^R a_{\ell r} \rho_r)^{-1}$ is experienced by each traffic type r that is transmitted through link ℓ .

In closing this section, we reiterate that the method developed above has the

advantage of computational simplicity. The reader however, is cautioned that this approach is heuristic and should not be expected to work efficiently (i.e., provide sharp bounds) in all cases. Therefore, its usefulness may be restricted only to the purpose of evaluating first approximations of the link blocking probabilities.

4.5 A Computational Example and Conclusions.

The methods developed above for the evaluation of blocking probabilities are applied to the elementary network depicted in Fig. 4.1. For the sake of simplicity, we assume that each traffic-type requires one frequency slot on each link in its route. The *traffic load* is varied on the segment AD, and we assume throughout $\rho_1 = \rho_2 = \rho$. For $0 \leq \rho \leq 3$, i.e., between points A and B in Fig. 4.1, we provide the exact blocking associated with links 1,2 and 3, derived directly according to (4.3). Next, we compute the approximate blocking probabilities using the recursive approximation (4.5) by computing $q(C_\ell)$ as well the improved Chernoff bound (4.8). The results are plotted and compared in Fig. 4.2. We observe that the recursive approximation (4.5) agrees reasonably well with the exact blocking probabilities whereas the improved Chernoff bound gives acceptable tight values for very light traffic with $\rho < 0.7$ approximately.

For $3 \leq \rho \leq 7.5$, we do not plot the improved Chernoff bound since we expect that in this case the bound is not tight enough (Fig. 4.3). The recursive approximation gives reasonably tight bounds except in the case of link 2, but the blocking probability associated with this link is considerably less compared with the ones for links 1 and 3. Consequently the estimates on the blocking probabilities for each of the routes 1 and 2, will not be significantly affected when computed by using (4.4). Finally for $12 \leq \rho \leq 30$, i.e., when ρ varies on the segment CD in Fig. 1, we use the recursive approximation and the fluid approximation for the blocking probabilities, and compare again with the exact values; the bounds are observed not to be reasonably tight (Fig. 4.4).

We conclude with the following remarks.

1) The Chernoff bound approximation may be useful for very small traffic loads, e.g., for 1% – 5% of the capacity of the link under consideration. While blocking under such small traffic loads is highly unlikely, i.e., blocking becomes a *rare event*, it is important for modern very high speed optical networks that the associated probability be kept below a certain threshold. A reason for this is that blocking, and the subsequent loss of a single packet during a data transmission in a fast network may require retransmission of a complete data stream – a process that is costly and time consuming, especially when compared with the fast propagation times in an optical network.

2) Bounds for moderate traffic may be computed relatively easily by using the recursive approximation method.

3) Heavy traffic bounds for link blocking may not be tight enough.

4) We expect the tightness of the Chernoff bound to improve as the network becomes larger, i.e., the link capacities and the number of traffic types are increased suitably. Furthermore, techniques from the Theory of Large Deviations might be applied and sharper bounds may be obtained for link blocking for the case of light traffic.

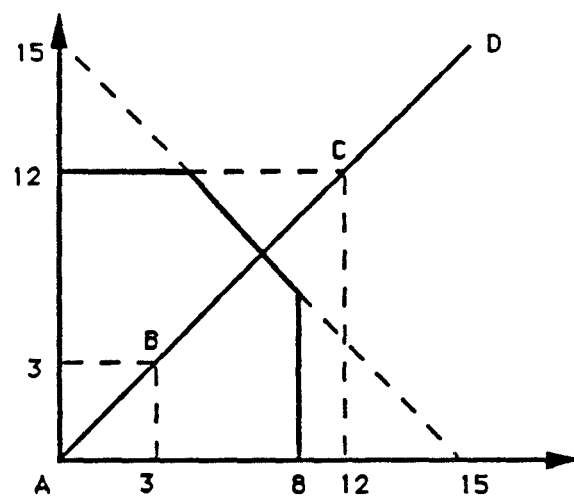


Figure 4.1

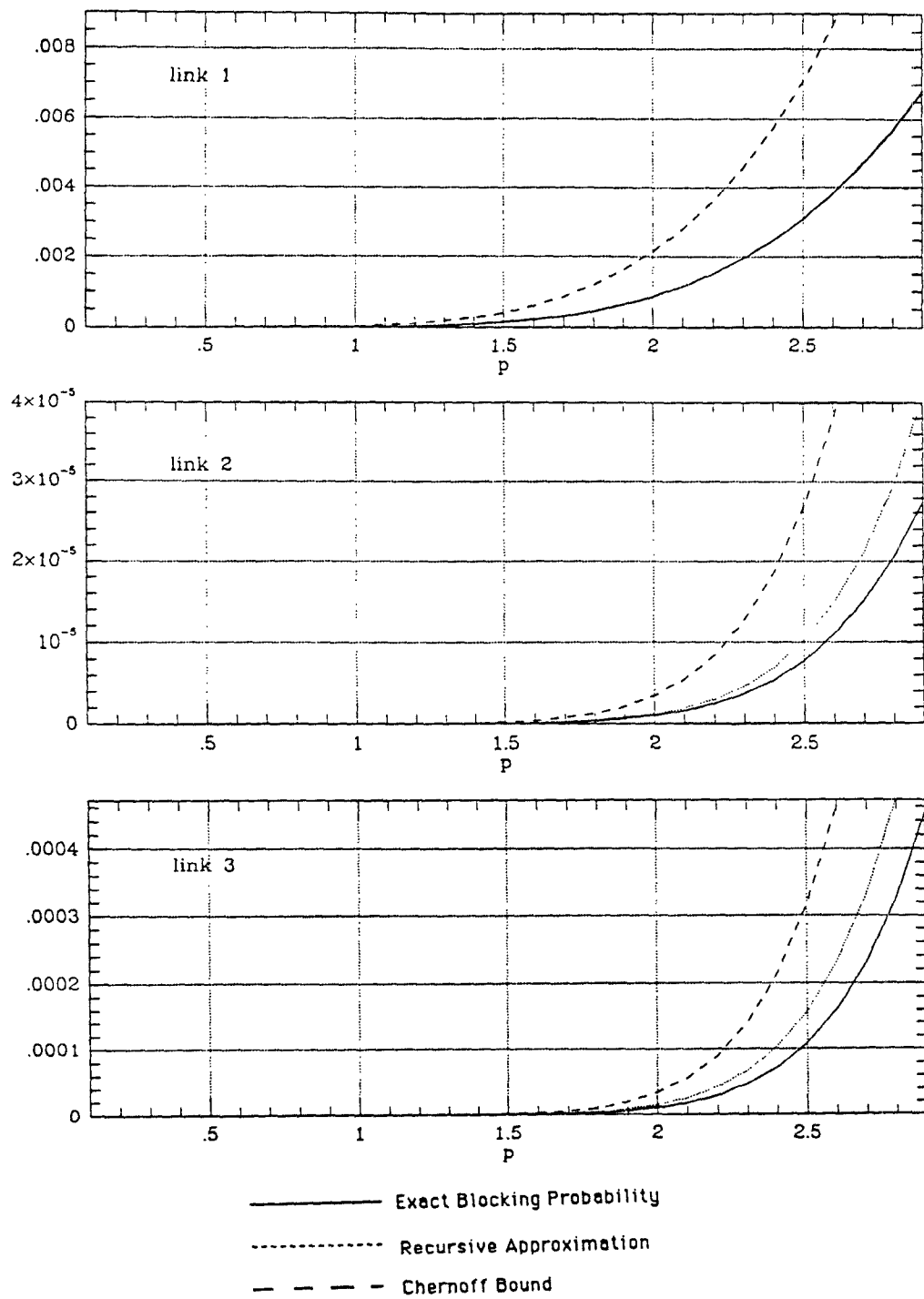


Figure 4.2
Link Blocking for $0 \leq \rho \leq 3$

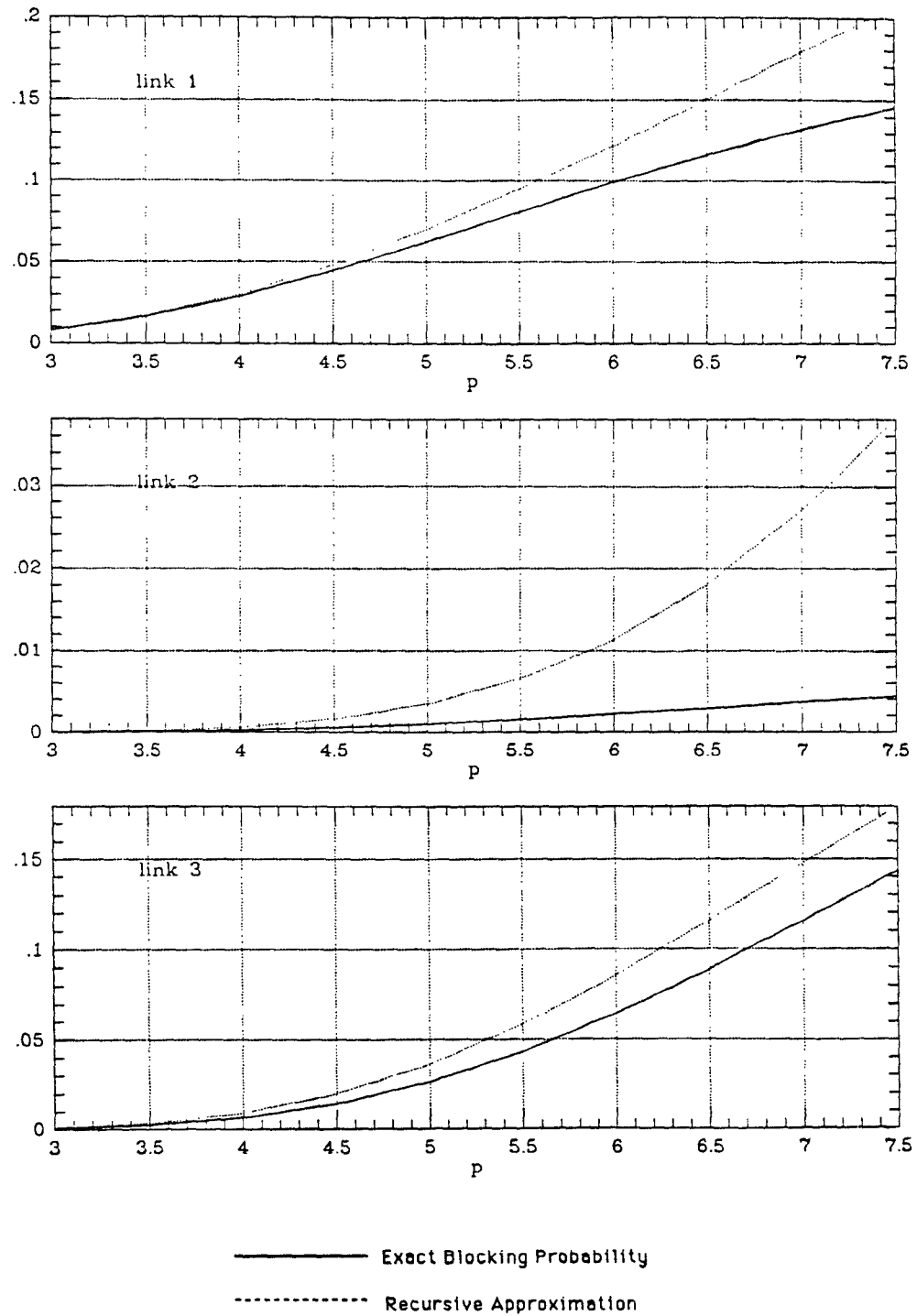


Figure 4.3
Link Blocking for $3 \leq \rho \leq 7.5$

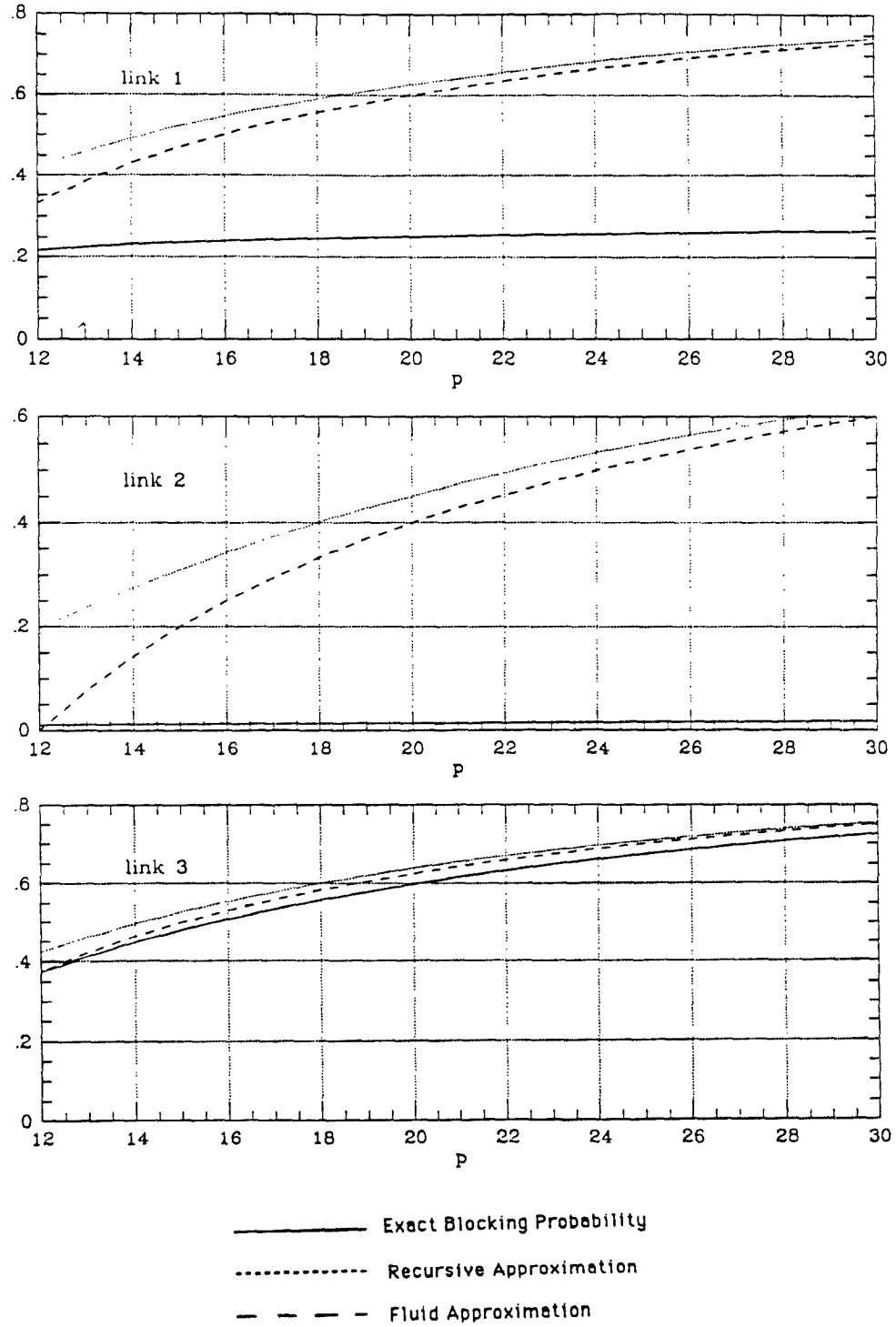


Figure 4.4
Link Blocking for $12 \leq \rho \leq 30$

Appendix 4.1

We now prove the analog of relation (4.5) for a general circuit-switched network.

Define

$$q_\ell(y, \mathbf{C}) = \sum_{\left\{ \mathbf{x} : \begin{array}{l} \mathbf{x} \in \mathcal{J}(\mathbf{C}) \\ \mathbf{A}_\ell \mathbf{x} = y \end{array} \right\}} P(\mathbf{x})$$

where \mathbf{A}_ℓ represents the ℓ th row of matrix \mathbf{A} . We first prove the following technical lemma.

Lemma A4.1: For each $r \in \mathcal{R}$, $\ell \in \mathcal{L}$ and $y \leq C_\ell$,

$$\rho_r q_\ell(y - a_{\ell r}, \mathbf{C} - \mathbf{A} \mathbf{e}_r) = \mathbb{E}[x_r | \mathbf{x} \in \mathcal{J}(\mathbf{C}), \mathbf{A}_\ell \mathbf{x} = y] q_\ell(y, \mathbf{C})$$

Proof: Since $\mathcal{J}(\mathbf{C})$ is a coordinate convex [26] set the rate balance equations for the associated R th dimensional Markov birth-death process can be written as

$$\rho_r \gamma_r(\mathbf{x}) P(\mathbf{x} - \mathbf{e}_r) = x_r P(\mathbf{x}), \quad r \in \mathcal{R} \quad (\text{A4.1})$$

where,

$$\gamma_r(\mathbf{x}) = \begin{cases} 1 & \text{if } x_r \geq 1; \\ 0 & \text{if } x_r = 0. \end{cases}$$

Defining the set

$$S_\ell(y, \mathbf{C}) = \{ \mathbf{x} : \mathbf{x} \in \mathcal{J}(\mathbf{C}), \mathbf{A}_\ell \mathbf{x} = y \},$$

we easily get from (A4.1) that

$$\rho_r \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C})} \gamma_r(\mathbf{x}) P(\mathbf{x} - \mathbf{e}_r) = \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C})} x_r P(\mathbf{x}). \quad (\text{A4.2})$$

We now simplify the left-side of (A4.2) as follows:

$$\rho_r \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C})} \gamma_r(\mathbf{x}) P(\mathbf{x} - \mathbf{e}_r) = \rho_r \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C}) \cap \{n_r \geq 1\}} P(\mathbf{x} - \mathbf{e}_r). \quad (\text{A4.3})$$

Since,

$$S_\ell(y, \mathbf{C}) = \{ \mathbf{x} : \mathbf{A}_\ell(\mathbf{x} - \mathbf{e}_r) = y - a_{\ell r}, \mathbf{A}(\mathbf{x} - \mathbf{e}_r) = \mathbf{C} - \mathbf{A} \mathbf{e}_r, x_r \geq 1, x_i \geq 0, i \neq r \}$$

the transformation

$$\hat{x}_i = (\mathbf{x} - \mathbf{e}_r)_i = \begin{cases} x_i & i \neq r; \\ x_r - 1 & i = r, \quad x_r \geq 1; \\ 0 & i = r, \quad x_r = 0; \end{cases} \quad i = 1, 2, \dots, R, \quad (\text{A4.4})$$

yields in (4A.3) that,

$$\begin{aligned} \rho_r \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C}) \cap \{n_r \geq 1\}} P(\mathbf{x} - \mathbf{e}_r) &= \\ = \rho_r \sum_{\mathbf{x} \in S_\ell(y - a_{\ell r}, \mathbf{C} - \mathbf{A}\mathbf{e}_r)} &= q_\ell(y - a_{\ell r}, \mathbf{C} - \mathbf{A}\mathbf{e}_r) \end{aligned}$$

for $r = 1, 2, \dots, R$, $\ell = 1, 2, \dots, L$ and $q_\ell(\mathbf{x}, \cdot) = 0$ for $\mathbf{x} < 0$.

We now simplify the right-side of (A4.2). Observe that,

$$\begin{aligned} P(\mathbf{x} | y \text{ slots are occupied in link } \ell) &= P(\mathbf{x} | \mathbf{x} \in S_\ell(y, \mathbf{C})) \\ &= \begin{cases} \frac{P(\mathbf{x})}{q_\ell(y, \mathbf{C})} & \mathbf{x} \in S_\ell(y, \mathbf{C}); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This leads to the following equality:

$$\begin{aligned} \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C})} x_r P(\mathbf{x}) &= \sum_{\mathbf{x} \in S_\ell(y, \mathbf{C})} (x_r P(\mathbf{x} | \mathbf{x} \in S_\ell(y, \mathbf{C}))) q_\ell(y, \mathbf{C}) \\ &= \mathbb{E}[x_r | \mathbf{x} \in S_\ell(y, \mathbf{C})] q_\ell(y, \mathbf{C}). \end{aligned} \quad (\text{A4.5})$$

The assertion of the lemma is established by (A4.4) and (A4.5).

The following lemma provides an analog of (4.5).

Lemma A4.2: For each $\ell \in \mathcal{L}$ and $y \leq C_\ell$, it holds that

$$\sum_{r \in \mathcal{R}} a_{\ell r} \rho_r q_\ell(y - a_{\ell r}, \mathbf{C} - \mathbf{A}\mathbf{e}_r) = y q_\ell(y, \mathbf{C})$$

with $q_\ell(\mathbf{x}, \cdot) = 0$ for $\mathbf{x} < 0$.

Proof: From Lemma A4.1 we have

$$\begin{aligned} \sum_{r \in \mathcal{R}} a_{\ell r} \rho_r q_\ell(y - a_{\ell r}, \mathbf{C} - \mathbf{A}\mathbf{e}_r) &= \mathbb{E}[\sum_{r \in \mathcal{R}} a_{\ell r} x_r | \mathbf{x} \in S_\ell(y, \mathbf{C})] q_\ell(y, \mathbf{C}) \\ &= y q_\ell(y, \mathbf{C}), \end{aligned}$$

which completes the proof.

CHAPTER 5

ASYMPTOTIC APPROXIMATIONS FOR CIRCUIT-SWITCHED NETWORKS

5.1 Introduction

In this chapter, we are concerned with strong approximations of the stochastic processes characterizing the states of certain blocking networks. The difficulty in computing the normalization constant in the product form distributions associated with certain networks has recently led to the investigation of asymptotic methods that describe, within a good approximation, the behavior of such “large” networks. For instance, in [29], approximations are provided for the blocking probabilities associated with a circuit-switched network. In [52], approximations involving integral transforms are given for certain closed Jackson networks; a similar study is performed in [56]. Finally, more general approximation methods are presented in [30,37].

Our analysis is performed for the case of a circuit-switched node similar to that introduced in Chapter 2, in the limiting regime where the link capacities and the offered traffic intensities increase at the same rate. Then the process of the normalized state (the state of the node was introduced earlier in Chapter 1) is shown to converge in probability to the solution of a system of ordinary differential equations. Furthermore, this system of ordinary differential equations has a unique equilibrium point whose stationary probability is maximal, i.e., it constitutes the “most likely state” of the system. In this respect, our work is strongly correlated with that of Kelly [28], who investigated the problem of blocking in a large general circuit-switched network. However, we have been unable to generalize our results to the arbitrary circuit-switched networks of [28]. Based on our results for simple circuit-switched networks (nodes), we have made an intuitively

appealing conjecture for the solution of the general case, but a formal proof is yet unavailable. Further, the analysis in this chapter provides an answer to the problem of describing the (approximate) transient behavior of a large system.

The work in the sections that follow was largely influenced by [29]. We commence with a few relevant mathematical preliminaries.

5.2 Mathematical Preliminaries

In this section, a brief collection of useful mathematical facts is presented. Proofs are omitted; the interested reader may consult [46,70] for further details.

All stochastic processes in this chapter are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supplied with a family $F = (\mathcal{F}_t, t \geq 0)$ of increasing, right-continuous sub- σ -algebras of \mathcal{F} (augmented by sets of measure 0) i.e.:

$$(\mathcal{F}_t \in F, t \geq 0), (t \geq s \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t) \text{ and } \left(\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t \right).$$

*

Let $T = (\tau_n, n \geq 1)$ be a sequence of stopping times with respect to the system $F = (\mathcal{F}_t, t \geq 0)$ with the following properties: $\tau_1 > 0$ (\mathbb{P} a.s.), $\tau_{n+1} > \tau_n$ (\mathbb{P} a.s.).

Definition 5.1: A random process $(N_t, t \geq 0)$ is a *counting* or *point* process if

$$N_t = \sum_{n \geq 1} 1(\tau_n \leq t), \quad t \geq 0,$$

where τ_n is in T , and $1(\cdot)$ denotes indicator function.

Remark: Often the sequence T of stopping times is said to be a random point process; both definitions are obviously equivalent.

Theorem 5.1 (Doob-Meyer decomposition): A point process $N = ((N_t, \mathcal{F}_t), t \geq 0)$, admits the unique (within stochastic equivalence) decomposition

$$N_t = m_t + A_t,$$

where $m = ((m_t, \mathcal{F}_t), t \geq 0)$ is a martingale and $A = ((A_t, \mathcal{F}_t), t \geq 0)$ is an increasing process.

We say that a random process $m = (m_t, \mathcal{F}_t), t < \sigma$, where σ is a stopping time with respect to F , is a σ -local martingale if there exists an increasing sequence of stopping times $\sigma_n, n \geq 1$, such that $\mathbb{P}(\sigma_n < \sigma_{n+1} < \sigma) = 1, \mathbb{P}(\lim_{n \rightarrow \infty} \sigma_n = \sigma) = 1$, and for each n the sequence $(m_{\min(t, \sigma_n)}, t < \sigma)$ forms a uniformly integrable martingale. For more details on this subject the interested reader is referred to [46, 70].

Remark: Set $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$. Then a more precise statement of Theorem 5.1 [46, vol. 2] requires that $m = ((m_t, \mathcal{F}_t), t \geq 0)$ be a τ_∞ -local martingale.

Example 5.1: Let $\pi = (\pi_t, \mathcal{F}_t^\pi), t \geq 0$, be a Poisson Process with parameter $\lambda > 0$. Here, $\mathcal{F}_t^\pi \triangleq \sigma(\pi_s, s \leq t)$ is referred to as the *natural filtration* of $(\pi_t, t \geq 0)$. It can be shown that $((\pi_t - \lambda t), \mathcal{F}_t^\pi)$ is a martingale, and as a consequence $m_t = \pi_t - \lambda t, A_t = \lambda t$.

Remark: Similar results apply to the case where the parameter λ is replaced by λ_t being a function of time or even a stochastic process. In this case: $m_t = \pi_t - \int_0^t \lambda_s ds, A_t = \int_0^t \lambda_s ds$.

Definition 5.2: The increasing process $A = ((A_t, \mathcal{F}_t), t \geq 0)$, appearing in the decomposition of Theorem 5.1 is called the *compensator* of the point process $N = ((N_t, \mathcal{F}_t), t \geq 0)$.

Definition 5.3: For each square integrable martingale $X = ((X_t, \mathcal{F}_t), t \geq 0)$ there exists a unique (to within stochastic equivalence) increasing process $\langle X \rangle = (\langle X \rangle_t, \mathcal{F}_t), t \geq 0$ such that

$$X_t^2 = \mu_t + \langle X \rangle_t$$

where $((\mu_t, \mathcal{F}_t), t \geq 0)$ is a martingale. $\langle X \rangle_t$ is called the *quadratic variation* of the process X .

In a similar manner, for two square integrable martingales we have the following definition.

Definition 5.4: For the square integrable martingale $X = ((X_t, \mathcal{F}_t), t \geq 0)$, $Y = ((Y_t, \mathcal{F}_t), t \geq 0)$, there exists a unique (up to stochastic equivalence) process $\langle XY \rangle_t$ which is the difference between two increasing processes, and a martingale $((\mu_t, \mathcal{F}_t), t \geq 0)$, such that:

$$X_t Y_t = \mu_t + \langle XY \rangle_t .$$

We call $\langle XY \rangle_t$ the quadratic cross-variation of the processes X , Y .

Theorem 5.2: (Lenglart's inequality)[70]: For the square-integrable martingale $M = ((M_t, \mathcal{F}_t), t \geq 0)$, with quadratic variation $\langle M \rangle = ((\langle M \rangle_t, \mathcal{F}_t), t \geq 0)$, for any stopping time τ adapted to $F = (\mathcal{F}_t, t \geq 0)$ and positive constants a, b , the following inequality is true:

$$\mathbb{P}(\sup_{0 \leq t \leq \tau} |M(t)| \geq a) \leq ba^{-2} + \mathbb{P}(\langle M \rangle_\tau \geq b). \quad (5.1)$$

5.3 Asymptotic Approximations for a Single Link

We begin our analysis by presenting a simple case of strong approximation techniques for a single link comprising C circuits (slots) and being fed by two traffic-streams. We assume that the traffic streams arrive according to a Poisson process with parameters λ_i , $i = 1, 2$, while their propagation time on the link are exponentially distributed with the same parameter (being set equal to 1 for convenience). We further assume that each traffic type occupies one circuit slot during transmission on the link. We shall consider the asymptotic regime where the arrival rates of the calls and the link capacities increase to infinity at the same rate, i.e., the parameters $\lambda = (\lambda_1, \lambda_2)$ and C , are replaced by $\lambda^N = (\lambda_1^N, \lambda_2^N)$ and C^N , respectively, where:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \lambda^N = \lambda, \quad i = 1, 2,$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} C^N = C.$$

We denote by \mathbf{x}_t^N the state vector of the system at time t (in the same manner as we did in Chapter 2, section 2.1), where $x_t^{i,N}$ represents the number of messages of traffic type i , $i = 1, 2$, that are transmitted on the link at time t .

Then the evolution of the state of the system can be written in terms of the following relation:

$$x_t^{i,N} = x_0^{i,N} + A_i(\lambda_i^N \int_0^t \mathbf{1} \{x_s^{1,N} + x_s^{2,N} < C\} ds) - D_i(\int_0^t x_s^{i,N} ds), \quad (5.2)$$

for $i = 1, 2$, where $x_0^{i,N}$ is an initial condition and $A_i(\cdot), D_i(\cdot)$, $i = 1, 2$ are independent Poisson point process representing the arrivals to and departures from the system. We remind the reader that these processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, as introduced in the previous section. The notation $A(v_t)$ defines a Poisson process with rate v_t ; since v_t may itself be a random process, we often refer to $A(v_t)$ as a doubly stochastic Poisson process [5]. Normalizing the state vector as in [37]:

$$\mathbf{z}_t^N = \frac{1}{N} \mathbf{x}_t^N.$$

Our goal is to approximate \mathbf{z}_t^N as $N \rightarrow \infty$. In particular, we would like to find a deterministic function \mathbf{z}_t which is a solution to a suitable integral equation, i.e., \mathbf{z}_t is of the form:

$$\mathbf{z}_t = \mathbf{z}_0 + \int_0^t f(\mathbf{z}_s) ds,$$

where \mathbf{z}_0 is an initial condition; $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ a piecewise continuous function; and the following relation is valid, namely

$$\mathbb{P}(\sup_{0 \leq t \leq T} \|\mathbf{z}_t^N - \mathbf{z}_t\| > \delta) \xrightarrow[N \rightarrow \infty]{} 0, \quad (5.3)$$

for any given time T , $\delta > 0$, and for a suitable metric $\|\cdot\|$. Relation (5.3) requires that as $N \rightarrow \infty$, with high probability almost all the sample paths $\mathbf{z}^N(\cdot)$ will be arbitrarily close to the deterministic trajectory $\mathbf{z}(\cdot)$.

Let $\mathcal{S} = \{\mathbf{x} : \mathbf{x} \geq 0, x^1 + x^2 \leq C\}$, assume that $\lambda_1 + \lambda_2 > C$, and let $\tau_c^N = \inf\{t : z_t^{1,N} + z_t^{2,N} = C\}$. The following proposition is true:

Proposition 5.1: If $\mathbf{z}_0^N, \mathbf{z}_0$ belong to $\text{int } \mathcal{S}$ and are such that

$$\mathbf{z}_0^N \xrightarrow{P} \mathbf{z}_0,$$

then for $\delta > 0$, and $t > 0$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t \wedge \tau_c^N} \|\mathbf{z}_s^N - \mathbf{z}_s\| > \delta\right) \xrightarrow{N \rightarrow \infty} 0.$$

where $\|\mathbf{x}\| = |x^1| + |x^2|$ for \mathbf{x} in \mathbb{R}^2 and \mathbf{z} is the solution to the ordinary differential equation

$$\dot{\mathbf{z}}_t = \underline{\lambda} - \mathbf{z}_t, \quad \text{with initial condition } \mathbf{z}_0.$$

Proof: The evolution equation (5.2) can be written in terms of $\mathbf{z}^N(\cdot)$ as:

$$z_t^{N,i} = z_0^{N,i} + \frac{1}{N} A_i(N\lambda_i t) - \frac{1}{N} D_i\left(N \int_0^t z_s^{N,i} ds\right),$$

for $0 \leq t \leq \tau_c^N$. We define the compensators of the Poisson point processes as

$$\tilde{A}_t^{i,N} = N\lambda_i t,$$

and

$$\tilde{D}_t^{i,N} = N \int_0^t z_s^{N,i} ds, \quad i = 1, 2,$$

(see also example 5.1 in previous section).

Then

$$M_t^{A_i,N} = A_i(N\lambda_i t) - \tilde{A}_t^{i,N}$$

$$M_t^{D_i,N} = D_i\left(N \int_0^t z_s^{N,i} ds\right) - \tilde{D}_t^{i,N}, \quad i = 1, 2,$$

are square-integrable martingales relative to $(\mathcal{F}_t, t \geq 0)$ with quadratic variations:

$$\langle M^{A_i, N} \rangle_t = \lambda_i N t$$

$$\langle M^{D_i, N} \rangle_t = N \int_0^t z_s^{N, i} ds, \quad i = 1, 2,$$

while for the cross-quadratic variations, we have

$$\langle M^{A_i, N} M^{D_j, N} \rangle_t = 0, \quad i, j = 1, 2$$

$$\langle M^{A_1, N} M^{A_2, N} \rangle_t = 0$$

$$\langle M^{D_1, N} M^{D_2, N} \rangle_t = 0,$$

since we have assumed that all processes $A_i(\cdot), D_i(\cdot)$, $i = 1, 2$ are mutually independent.

For the sake of convenience, define $U_t^{i, N} \triangleq z_t^{i, N} - z_t^i$, $i = 1, 2$.

Then

$$\begin{aligned} U_t^{i, N} &= \frac{1}{N} x_t^{i, N} - z_t^i \\ &= \frac{1}{N} \left(x_0^{i, N} + M_t^{A_i, N} - M_t^{D_i, N} + \lambda_i N t - \int_0^t z_s^{i, N} ds \right. \\ &\quad \left. - z_0 - \lambda_i t + \int_0^t z_s^i ds \right) \\ &= U_0^{i, N} + \frac{1}{N} (M_t^{A_i, N} - M_t^{D_i, N}) - \int_0^t U_s^{i, N} ds. \end{aligned}$$

Let

$$\mathbf{e}_t^N = \mathbf{U}_0^N + \frac{1}{N} (\mathbf{M}_t^{A, N} - \mathbf{M}_t^{D, N}),$$

where it follows that,

$$\|\mathbf{U}_t^N\| \leq \|\mathbf{e}_t^N\| + \int_0^t \|\mathbf{U}_s^N\| ds,$$

or

$$\|\mathbf{U}_t^N\| \leq \sup_{0 \leq s \leq t} \|\mathbf{e}_s^N\| + \int_0^t \|\mathbf{U}_s^N\| ds. \quad (5.3)$$

By applying the Bellman-Gronwall inequality to (5.3) we immediately get,

$$\|\mathbf{U}_t^N\| \leq \sup_{0 \leq s \leq t} \|\mathbf{e}_s^N\| e^t,$$

and as a consequence, for $\sigma > 0$, it holds that

$$\left\{ \sup_{0 \leq s \leq t \wedge \tau_c^N} \|\mathbf{U}_s^N\| \geq \sigma \right\} \subset \left\{ \sup_{0 \leq s \leq t} \|\mathbf{e}_s^N\| e^t \geq \sigma \right\}.$$

Therefore

$$\mathbb{P}\left(\sup_{0 \leq s \leq t \wedge \tau_c^N} \|\mathbf{U}_s^N\| \geq \delta\right) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} \|\mathbf{e}_s^N\| \geq \delta e^{-t}\right). \quad (5.4)$$

For the RHS of the inequality (5.4) above, it follows easily that

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq s \leq t} \|\mathbf{e}_s^N\| > \delta e^{-t}\right) &\leq \mathbb{P}\left(\sup_{0 \leq s \leq t} \left\{ \frac{1}{N} \|\mathbf{M}_s^{A,N} - \mathbf{M}_s^{D,N}\| \right\} + \|\mathbf{U}_0^N\| \geq \delta e^{-t}\right) \\ &= \mathbb{P}\left(\|\mathbf{U}_0^N\| + \sup_{0 \leq s \leq t} \frac{1}{N} \{ |M_s^{A_1,N} - M_s^{D_1,N}| + |M_s^{A_2,N} - M_s^{D_2,N}| \} \geq \delta e^{-t}\right) \\ &\leq \mathbb{P}\left(\|\mathbf{U}_0^N\| \geq \frac{\delta}{2} e^{-t}\right) + \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{A_1,N} - M_s^{D_1,N}| \right. \\ &\quad \left. + \sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{A_2,N} - M_s^{D_2,N}| \geq \frac{\delta}{2} e^{-t}\right) \\ &\leq \mathbb{P}\left(\|\mathbf{U}_0^N\| \geq \frac{\delta}{2} e^{-t}\right) + \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{A_1,N}| \geq \frac{\delta}{4} e^{-t}\right) \\ &\quad + \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{A_2,N}| \geq \frac{\delta}{4} e^{-t}\right) \\ &\quad + \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{D_1,N}| \geq \frac{\delta}{4} e^{-t}\right) + \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{D_2,N}| \geq \frac{\delta}{4} e^{-t}\right). \end{aligned}$$

Since $\mathbf{z}_0^N \xrightarrow{P} \mathbf{z}_0$ as $N \rightarrow \infty$ we trivially have that

$$\mathbb{P}\left(\|\mathbf{U}_0^N\| \geq \frac{\delta}{2} e^{-t}\right) \xrightarrow{N \rightarrow \infty} 0.$$

Moreover by Lengart's inequality (5.1), we conclude:

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} \left| \frac{1}{N} M_S^{A_i, N} \right| \geq \frac{\delta}{4} e^{-t}\right) \leq b \frac{16}{\delta^2} e^{2t} + \mathbb{P}\left(\frac{1}{N^2} < M^{A_i, N} >_t \geq b\right).$$

Since $\frac{1}{N^2} < M^{A_i, N} >_t = \frac{1}{N^2} \lambda_i t$, we can choose b suitably so that the quantity $\frac{16}{\delta^2} e^{2t}$ becomes arbitrarily small; can simultaneously choose N large enough so that $\frac{1}{N^2} \lambda_i t < b$. Similar arguments can be applied to the terms $\mathbb{P}(\sup_{0 \leq s \leq t} \frac{1}{N} |M_s^{D_i, N}| \geq \frac{\delta}{4} e^{-t})$, $i = 1, 2$. The proof of Proposition 5.1 is now evident.

Proposition 5.1 provides an approximation of the trajectory $\mathbf{z}_{(\cdot)}^N$ up to time τ_i^N , i.e., when $z_t^{1, N} + z_t^{2, N} = C$. In the next proposition, we analyze the case where $z_0^{1, N} + z_0^{2, N} \xrightarrow{P} C$. Precisely we have:

Proposition 5.2: If \mathbf{z}_0 is such that $z_0^1 + z_0^2 = C$, and

$$\mathbf{z}_0^N \xrightarrow{P} \mathbf{z}_0,$$

then for $\delta > 0$, and $t > 0$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t} \|\mathbf{z}_s^N - \mathbf{z}_s\| > \delta\right) \xrightarrow{N \rightarrow \infty} 0$$

where $\mathbf{z}_{(\cdot)}$ is the solution to the equation

$$\dot{\mathbf{z}}_t = \lambda \frac{C}{\|\lambda\|} - \mathbf{z}_t, \text{ with initial condition } \mathbf{z}_0.$$

Proof: As in the proof of Proposition 5.1 we have:

$$\tilde{A}_t^{i, N} = N \int_0^t \lambda_i \mathbf{1}\{z_s^{1, N} + z_s^{2, N} < C\} ds,$$

and

$$\tilde{D}_t^{i, N} = N \int_0^t z_s^{i, N} ds,$$

for $i = 1, 2$. Defining the martingales $M_t^{A_i, N}, M_t^{D_i, N}$, $i = 1, 2$, and U_t^N as previously, it follows

$$\begin{aligned} \|U_t^N\| \leq & \|U_0^N\| + \frac{1}{N}(\|M_t^A\| + \|M_t^D\|) + \|\lambda\| \int_0^t \mathbf{1}\{z_s^{1, N} + z_s^{2, N} < C\} ds \\ & - \lambda \frac{C}{\|\lambda\|} + \int_0^t \|U_s^N\| ds. \end{aligned} \quad (5.5)$$

We now show that

$$\int_0^t \mathbf{1}\{z_s^{1, N} + z_s^{2, N}\} ds \xrightarrow{N \rightarrow \infty} \frac{C}{\lambda_1 + \lambda_2}. \quad (5.6)$$

By a simple change of the variable of integration, we get

$$\begin{aligned} \int_0^t \mathbf{1}\{z_s^{1, N} + z_s^{2, N} < C\} ds &= \frac{1}{Nt} \int_0^{Nt} \mathbf{1}\{z_{s/n}^{1, N} + z_{s/n}^{2, N} < C^N\} ds \\ &= \frac{1}{Nt} \int_0^{Nt} \mathbf{1}\{y_s^N < C^N\} ds, \end{aligned}$$

where

$$y^N(s) = z_{t/N}^{1, N} + z_{t/N}^{2, N} = \frac{1}{N} x_{t/N}^{1, N} + \frac{1}{N} x_{t/N}^{2, N}.$$

We can claim the following:

Lemma 5.1: For $t > 0$,

$$\frac{1}{Nt} \int_0^{Nt} \mathbf{1}\{y_s^N = C\} ds \xrightarrow{N \rightarrow \infty} \left(1 - \frac{C}{\lambda_1 + \lambda_2}\right) t = \pi_C$$

Proof: The birth-death process $(y_t^N, t \geq 0)$ has its state space the set $\{0, \frac{1}{N}, \frac{2}{N}, \dots, C - \frac{1}{N}, C\}$ (i.e., the total number of states is $NC + 1$). In order to simplify the presentation, we rename the state space of $(y_t^N, t \geq 0)$ backwards, i.e., $\{C, C - \frac{1}{N}, \dots, \frac{2}{N}, \frac{1}{N}, 0\}$, and we prove that

$$\frac{1}{N} \int_0^{Nt} \mathbf{1}\{y_s^N = 0\} ds \xrightarrow{N \rightarrow \infty} \left(1 - \frac{C}{\lambda_1 + \lambda_2}\right) t.$$

The rates of $(y_s^N, t \geq 0)$ and the rearranged state space are shown in Fig. 5.1a.

We now consider the birth death process $(v_t^N, t \geq 0)$ with an infinite state space, death rates equal to $\lambda = \lambda_1 + \lambda_2$, and birth rate equal to C (fig. 5.1b).

Trivially we get

$$y_t^N \leq_{st} v_t^N,$$

and as a consequence,

$$\lim_{N \rightarrow \infty} \frac{1}{Nt} \int_0^{Nt} \mathbf{1}\{y_s^N = 0\} ds \geq \lim_{N \rightarrow \infty} \frac{1}{Nt} \int_0^{Nt} \mathbf{1}\{v_s^N = 0\} ds = \pi_c.$$

For any $\epsilon > 0$, we choose $C' > 0$ such that

$$\pi_c < \pi_{c'} < \pi_c + \epsilon.$$

Fix a number $k > 0$, and take $C - \frac{k}{M} > C'$ for suitably large M . Consider the following birth-death processes with death rate $\lambda = \lambda_1 + \lambda_2$:

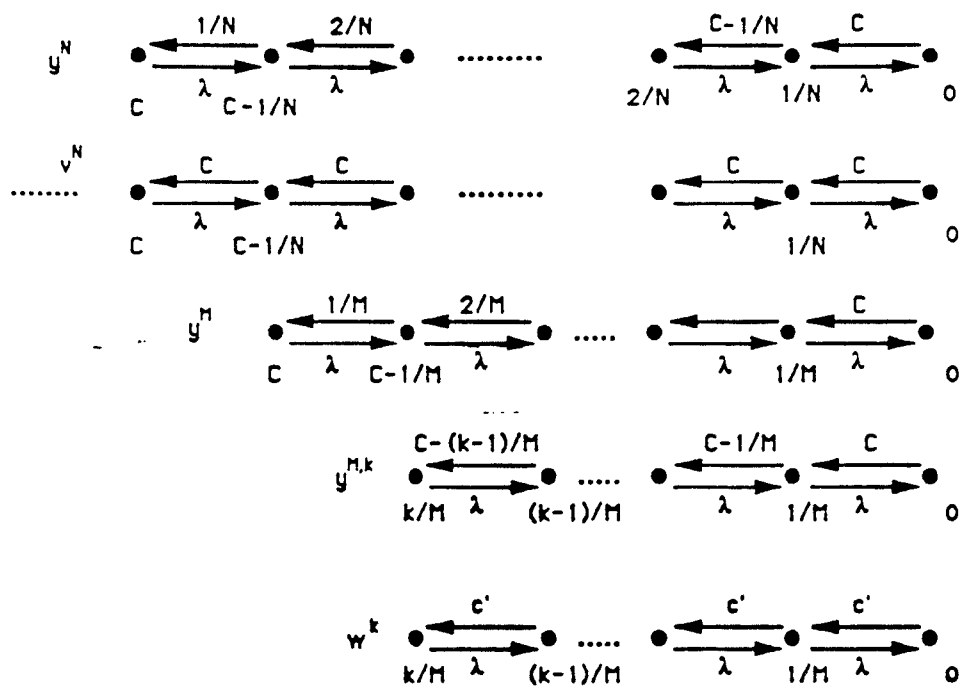


Figure 5.1

- $(y_t^M, t \geq 0)$, with state space $\{C, C - \frac{1}{M}, \dots, 1, 0\}$, and birth rate at state i being $C - \frac{i}{M}$ (fig. 5.1c);
- $(y_t^{M,k}, t \geq 0)$, with state space $\{\frac{k}{M}, \frac{k-1}{M}, \dots, \frac{1}{M}, 0\}$, and birth rate at state i being $C - \frac{i}{N}$ (fig. 5.1d);
- $(w_t^k, t \geq 0)$, with state space $\{\frac{k}{M}, \frac{k-1}{M}, \dots, \frac{1}{M}, 0\}$, and birth rate C' (fig. 5.1e).

By using straightforward coupling arguments, we get

$$y_t^M \geq_{st} y_t^{M,k} \geq_{st} w_t^k,$$

so that

$$\lim_{M \rightarrow \infty} \frac{1}{Mt} \int_0^{Mt} \mathbf{1}\{y_s^M = 0\} ds \leq \lim_{M \rightarrow \infty} \frac{1}{Mt} \int_0^{Mt} \mathbf{1}\{w_s^k = 0\} ds \quad \forall k > 0.$$

Taking $k \rightarrow \infty$ we get for every $\epsilon > 0$ that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_0^{Nt} \mathbf{1}\{y_s^N = 0\} ds < \pi_{c'} t < (\pi_c + \epsilon)t,$$

so that the lemma (as well as (5.6)) is proved.

From (5.6) we have

$$\|\lambda(\int_0^t \mathbf{1}\{z_s^{1,N} + z_s^{2,N} < C\} ds - \frac{c}{\lambda_1 + \lambda_2})\| \rightarrow 0.$$

All the remaining terms in (5.5) are treated similarly as in the proof of Proposition 1. The proof of Proposition 5.2 is thus complete. In fig. 5.2, the phase plane of the deterministic equation for \mathbf{z}_t is shown; there is one stable equilibrium point, namely $\lambda \frac{C}{\lambda_1 + \lambda_2}$.

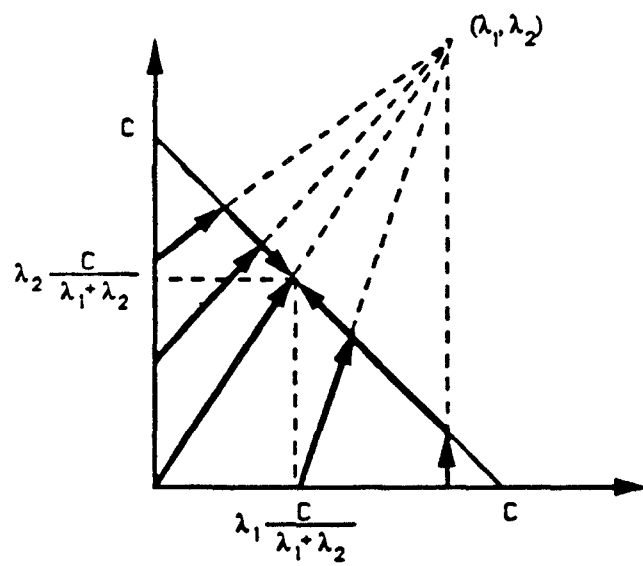


Figure 5.2

5.4 Asymptotic Approximations for a Simple Network

We now extend the results of the previous section to the simple circuit-switched network studied earlier in Chapter 2 (fig. 2.1). In this case, we also assume that the capacities of the links, and the arrival rates, increase at the same rate as done previously, while $\mu_1^N = \mu_2^N = 1$. The notation is identical to that of section 5.3. We assume that the initial condition \mathbf{z}_0 is such that $z_0^1 < C_1$, $z_0^2 < C_2$, $z_0^1 + z_0^2 < C$ and also $\mathbf{z}_0^N \xrightarrow{P} \mathbf{z}_0$. Then, for the period during which the process is in the interior of the convex set characterizing the state space, the differential equation for \mathbf{z}_t is:

$$\dot{\mathbf{z}}_t = \lambda - \mathbf{z}_t, \text{ with initial condition } \mathbf{z}_0.$$

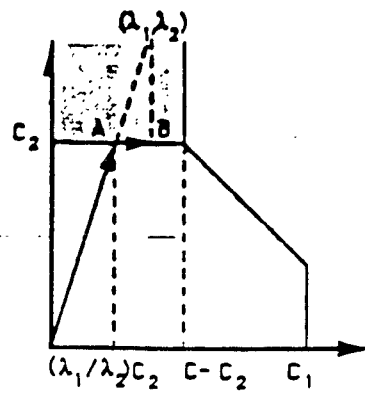
We now consider the following cases (assuming that time is initialized, i.e., $t = 0$):

Case 1: (λ_1, λ_2) is such that $\lambda_2 \geq C_2, \lambda_2 \leq C - C_2$ (fig. 5.3a):

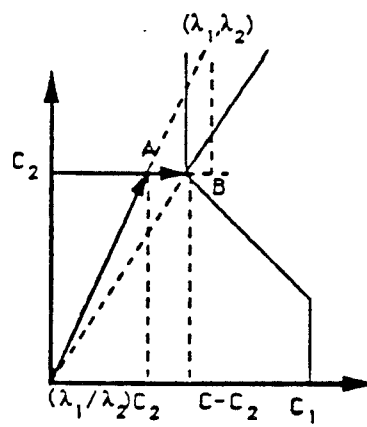
It can be proved using identical techniques as in the proof of Proposition 5.2 that \mathbf{z}_t moves from point A to point B according to equation:

$$\dot{z}_t^1 = \lambda_1 - z_t^1, \quad \dot{z}_t^2 = 0,$$

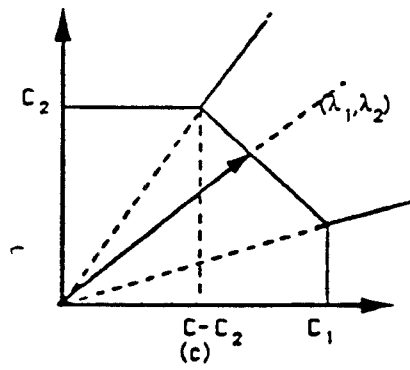
with initial condition $(\frac{\lambda_1}{\lambda_2} C_1, C)$.



(a)



(b)



(c)

Figure 5.3

For $t \rightarrow \infty$, $\mathbf{z}_t \rightarrow \mathbf{z}_\infty = (\lambda_1, C_2)$ which is a unique stable equilibrium point.

Case 2: (λ_1, λ_2) satisfies $\lambda_1 \geq C - C_2$, $\frac{\lambda_2}{\lambda_1} \geq \frac{C_2}{C - C_2}$ (fig. 5.3b):

The equations describing the motion of \mathbf{z}_t from point A to point B are:

$$\begin{aligned}\dot{z}_t^1 &= \lambda_1 - z_t^1 \\ \dot{z}_t^2 &= 0,\end{aligned}$$

with initial condition $(\frac{\lambda_1}{\lambda_2} C_1, C)$. The final stable equilibrium point of \mathbf{z}_t is point B with coordinates $(C - C_2, C_2)$.

Case 3: (λ_1, λ_2) satisfy $\frac{\lambda_2}{\lambda_1} \geq \frac{C_1}{C - C_1}$, $\frac{\lambda_2}{\lambda_1} \leq \frac{C_2}{C - C_2}$ (fig. 5.3c):

The results are identical to those derived for the case of a single link in section 5.3.

Remarks: 1) The hitting time τ introduced in section 5.3 can be estimated from the differential equation describing the motion of $\mathbf{z}(t)$. For the case of the simple network of section 5.4, similar estimates can be drawn; moreover the order with which each link saturates can be determined.

2) Unfortunately for $\mu_1^N \neq \mu_2^N$, the technique applied for the proof of Proposition 5.2 is not applicable. Nevertheless, Proposition 5.1 is true for $\mu_1^N \neq \mu_2^N$ with slight modifications on the differential equation for $\mathbf{z}(t)$. The complete analysis of this case is an open problem.

3) We focus attention at the original processes \mathbf{x}_t^N . At steady state, this process possess a stationary probability mass function of the familiar product form:

$$\mathbb{P}(x^{1,N}, x^{2,N}) = C \frac{(\lambda_1^N)^{x^{1,N}}}{x^{1,N}!} \cdot \frac{(\lambda_2^N)^{x^{2,N}}}{x^{2,N}!}.$$

As in [29], we would like to compute the state which carries the highest probability at steady state, i.e., the “most likely state.” To this end, we maximize the logarithm of $\mathbb{P}(x^{1,N}, x^{2,N})$, and use Stirling’s formula: $\log n! \simeq n \log n - n$. The problem of

finding the most likely state for the case of a single link reduces to the following nonlinear maximization:

$$\begin{aligned} \max \sum_{i=1}^2 (x^{i,N} \ln \lambda_i^N - x^{i,N} \ln x^{i,N} + x^{i,N}) \\ \text{such that : } x^{i,N} \geq 0 \quad i = 1, 2, \quad (ML) \\ \sum_{i=1}^2 x^{i,N} \leq C^N. \end{aligned}$$

Letting $y \geq 0, z \geq 0$ the associated Lagrangian is

$$L(\mathbf{x}^N, z, y) = \sum_{i=1}^2 (x^{i,N} \ln \lambda_i^N - x^{i,N} \ln x^{i,N} + x^{i,N}) + y(C^N - \sum_{i=1}^2 x^{i,N} - z),$$

for $i = 1, 2$. We now consider the Lagrangian (dual) problem:

$$\begin{aligned} \max L(\mathbf{x}^N, z, y) \\ \text{such that : } \mathbf{x}^N \geq 0. \quad (L) \end{aligned}$$

The solution of problem (L) depends on y, z . Here y is a Lagrange multiplier while z is a slack variable.

The dual problem (L) will provide a solution $\bar{\mathbf{x}}^N$ which is optimal for the original problem (ML) when the following complementary slackness conditions are satisfied

$$\begin{aligned} y &\geq 0, \\ y \cdot z &= 0 \quad (CS) \\ z &= C - (x^{1,N} + x^{2,N}) \geq 0. \end{aligned}$$

By differentiating the Lagrangian, we can trivially show that the solution of the program (L) (and hence (ML)) has the form:

$$\bar{\mathbf{x}}^N = \lambda^N e^{-y}.$$

The unique stable equilibrium point of the equation characterizing the motion of \mathbf{z}_t in Proposition 5.2 is

$$\mathbf{z}_\infty^N = \frac{\mathbf{x}_\infty^N}{N} = \lambda \frac{C}{\lambda_1 + \lambda_2}.$$

We now prove that \mathbf{x}_∞^N is a “most likely state”. The complementary slackness variable y is found to be

$$y = -\ln \frac{C}{\lambda_1 + \lambda_2} > 0.$$

Moreover, $x_\infty^{1,N} + x_\infty^{2,N} = C^N \Rightarrow z = 0 \Rightarrow y \cdot z = 0$, showing that \mathbf{x}_∞^N is indeed a most likely state. In a similar manner, the stable equilibria found in cases (1-3) in Section 5.4 (figure 5.3) may be shown to be “most likely states” for the respective two-dimensional Markov processes.

4) As a last remark, we would like to mention that the same equilibria (or “most likely states”) for the systems studied heretofore are given by a differential inclusion of the form [3].

$$\dot{\mathbf{z}}_t \in \bigcap_{\epsilon > 0} \overline{\text{co}} f(\mathbf{z}(t) + \epsilon B), \quad (5.7)$$

where B is the unit ball in \mathbb{R}^2 , $f(\mathbf{z}(t) + \epsilon B)$ is the function giving all the directions for $\dot{\mathbf{z}}_t$ in a neighborhood of \mathbf{z}_t (at time t), and $\overline{\text{co}}$ denotes closure of the convex hull. Hence, if \mathbf{z}_t is the interior of the convex set characterizing the state space, the only direction given by $f(\cdot)$ is $\lambda - \mathbf{z}_t$, and the inclusion is identical to the differential equation in Proposition 5.1. Further, if \mathbf{z}_t is on the boundary of the state space, certain components of the vector λ are disabled. As an example, if the component disabled is λ_1 , the directions provided by $f(\cdot)$ will be $\lambda - \mathbf{z}_t, (0, \lambda_2) - \mathbf{z}_t$. Unfortunately, we have been unable to determine a closer relation between the solutions of the differential inclusion (5.7), and the process $\mathbf{z}^N(\cdot)$.

5.5 A Conjecture for a General Network

The problem of asymptotic approximations for the case of a general circuit-switched network, whose topology is characterized by a routing matrix \mathbf{A} (section

2.7), is not fully solved. Nevertheless, some observations from the cases already studied suggest an intuitively appealing conjecture for the solution of the general problem.

We focus attention on the simple network studied in section 5.4, and consider the case depicted in figure 5.3b. Assuming that $\mathbf{z}_0 = 0$, the differential equation describing the motion of \mathbf{z}_t up to point A when link 2 saturates, is $\dot{\mathbf{z}}_t = \lambda - \mathbf{z}$, i.e., \mathbf{z}_t is attracted by the point λ , which can trivially be shown to be the “most likely state” for the unconstrained process \mathbf{z}_t^N (see fig. 5.4). When \mathbf{z}_t reaches the boundary for which $z_t^2 = C_2$, it is then attracted by the point (C_2, λ_1) which is the “most likely state” for the process \mathbf{z}_t^N if only the constraint associated with link 2 were present. The differential equation for \mathbf{z}_t is given by: $\dot{z}_t^1 = \lambda_1 - z_t^1$, $\dot{z}_t^2 = 0$, with initial condition being the point A with coordinates $(\frac{\lambda_1}{\lambda_2}C_2, C_2)$. When \mathbf{z}_t reaches point B it remains there, and it can be shown that point B is the “most likely state” for \mathbf{z}_t^N if the constraints associated with the links of capacities C_2 and C alone are present. The previous observation suggests the following method for deriving the differential equation characterizing the motion of \mathbf{z}_t for the case of an arbitrary circuit switched network. Assuming that \mathbf{z}_0 is in the interior of the set $\mathcal{S} = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{C}, \mathbf{x} \geq 0\}$, the differential equation for \mathbf{z}_t up to the first time it hits the boundary of \mathcal{S} is $\dot{\mathbf{z}}_t = \lambda - \mathbf{z}$. Assuming that \mathbf{z}_t meets the faces of \mathcal{S} associated with the constraints for links $\ell_{i1}, \ell_{i2}, \dots, \ell_{ik}$, we determine the “most likely state” of \mathbf{z}_t^N , denoted by $\bar{\mathbf{z}}$, as if only links $\ell_{i1}, \dots, \ell_{ik}$ were constituting the network. The differential equation for \mathbf{z}_t is then $\dot{\mathbf{z}}_t = \bar{\mathbf{z}} - \mathbf{z}_t$ up to the point where a new constraint is saturated and a new “most likely state” is computed. This argument proceeds inductively, and the final “most likely state” of \mathbf{z}_t^N is achieved in finite time. One important issue that remains to be proved is that if \mathbf{z}_t saturates a certain number of constraints, then all these constraints will be saturated in the future. This simply means that if \mathbf{z}_t ends up at the intersection of certain hyperplanes that constitute faces of \mathcal{S} , then it will never escape from this intersection. This claim remains an open problem.

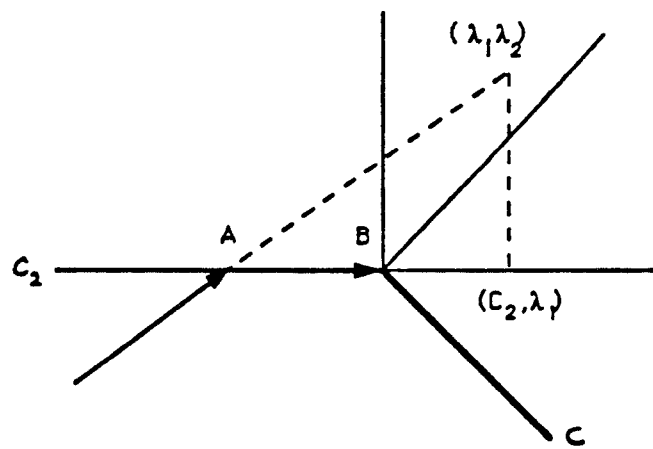


Figure 5.4

CHAPTER 6

JOINTLY OPTIMAL ADMISSION AND ROUTING CONTROL AT A NETWORK NODE

6.1 Introduction

Admission control and routing are key issues arising in the design and operation of communication and computer networks, and have received considerable attention in recent years. The admission control problem entails a determination of efficient policies for allowing incoming messages to gain access to network facilities. The routing problem involves selecting paths from several alternatives in the network along which accepted messages can be efficiently forwarded to their destinations.

Numerous studies of admission control and routing problems at a single node or at several nodes of a network can be found in the literature. Decisions for allowing messages into a network have customarily been based on an appropriate minimization of a blocking cost in conjunction with a cost for queuing delays in the buffers at the nodes. Routing strategies, on the other hand, have typically been determined using the queuing delays at the buffers as the measure of performance. We cite below some of the studies relevant to our work; this list is by no means exhaustive.

Stidham [74] has considered admission control policies for several simple queuing models. The optimal admission control policies for all these models share the characteristic that they can be expressed in terms of a “switching curve.” Viniotis-Ephremides [79] have demonstrated a similar characterization of the optimal admission strategy at a simple node in an Integrated Services Digital Network (ISDN). Results in the same vein have been obtained by Christidou *et al* [9] for a cyclic interconnection of two queues, and by Lambadaris *et al* [39] for a circuit-switched node. Hajek [16] has investigated the problem of optimally controlling

two interacting queues.

In the realm of relevant routing problems, Lin-Kumar [41] have considered the task of routing messages arriving at a node among two channels (servers), one faster than the other. By minimizing the average queuing delay at the node buffer, they show that the optimal routing policy is characterized by a “threshold” on the size of the queue. Rosberg-Makowski [58] have treated a similar problem involving multiple servers under the assumption of light traffic. In a recent preprint, Luh-Viniotis [47] claim the optimality of a policy determined by multiple thresholds for the situation in [58] even with arbitrary arrival rates. Nain-Ross [53] consider the optimal assignment of a single server to multiple classes of customers. In doing so, they minimize a linear combination of the average queue lengths of the various classes of customers while simultaneously constraining the average queue length of a specific customer class to lie below a specified value. Shwartz-Makowski [71] treat a similar problem with two types of customers. Both [53,71] show the optimal assignment strategies to be randomized.

In what follows, we combine the elements of the admission control and routing problems at a simple node of a communication network similar to that studied in Lin-Kumar[41]. To our knowledge, this is the first determination of *simultaneously optimal policies* for flow control and routing. In our model, a message arriving at a buffer is to be transmitted over one of two channels with different propagation times. Under suitably chosen criteria, two decisions have to be made: whether or not to admit an incoming message into the buffer, and under what conditions should the slower channel be utilized. A discounted infinite-horizon cost as well as an average cost are considered which consist of a linear combination of the blocking probability and the queuing delay at the buffer.

Beginning with the discounted cost case, we formulate the optimal control task as a Markov Decision problem. It is first shown from Lippman [42, 43] that an optimal policy exists for admission and routing which is stationary in nature. Next, properties of the optimal cost function are derived using arguments which rely heavily on sample path methods [80], as well as on the techniques introduced

in chapter 3. The said properties are then used to demonstrate that the optimal admission and routing strategies are characterized almost completely by “switching curves.” Finally, we show that the average-cost problem also yields similar results, by following the approach of Sennott [69].

The remainder of this chapter is organized as follows. The problem is formulated in section 6.2. Section 6.3 considers the discounted cost case and establishes key properties of the optimal discounted cost function. The associated optimal policy for this case is characterized in section 6.4; to do so, we need some convexity properties of the optimal discounted cost which are established in section 6.5. Finally, the average cost problem is addressed in section 6.6.

6.2 Problem Statement and Preliminaries

The model under consideration is shown in Figure 6.1. We focus our attention on a single node of a communication network providing service to a stream of message packets that arrive according to a Poisson distribution with parameter λ . The packets (customers) are stored in a buffer (queue), and subsequently are to be routed through one of channels (servers) 1 or 2 which have propagation times that are exponentially distributed with parameters μ_i , $i = 1, 2$. We assume that propagation over channel 1 is faster than that over channel 2, i.e., $\mu_1 > \mu_2$, and that channel 1 is non-idling. Furthermore, in order to ensure that the number of packets in the buffer remains bounded we shall assume the standard stability condition, $\lambda < \mu_1 + \mu_2$.

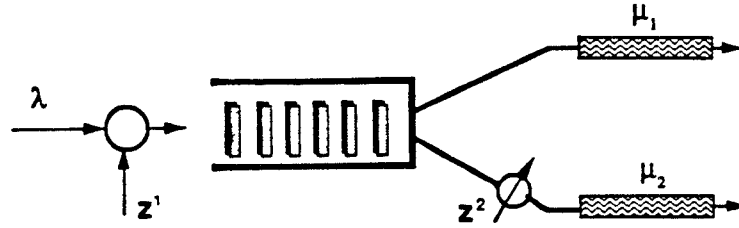


Figure 6.1

The objective is the following: We wish to simultaneously control the admission of packets to the buffer, as well as their subsequent allocation to the two channels; this will be done in such a way as to minimize a weighted sum of the probability of rejecting admission of an arriving packet to the buffer and the delay experienced by the packets in the queue. This problem can be precisely formulated in terms of a Markov Decision Process (MDP) [17,38,64] as follows.

The *state* of the system at time $t, t \geq 0$, is defined by a stochastic process $(\mathbf{x}_t, t \geq 0)$, describing the evolution of the total load of the system as well as of the status of the slower channel, where $\mathbf{x}_t = (x_t^1, x_t^2)$ takes values in the *state space* $\mathcal{S} = (0, 0) \cup (Z^+ \times \{0, 1\})$, with

$$x_t^1 = \text{total number of packets in the system (including the two channels)},$$

$$x_t^2 = \begin{cases} 0 & \text{if channel 2 is empty of packets} \\ 1 & \text{if channel 2 is forwarding a packet,} \end{cases}$$

at time t . Observe that the process $(\mathbf{x}_t, t \geq 0)$ is piecewise constant; next, we associate with each state \mathbf{x} in \mathcal{S} a set of admissible actions $\mathcal{D} = \{0, 1\}^2$. Thus, an admissible action $\mathbf{z}_t(\mathbf{x})$ in state \mathbf{x} at time t , with values in \mathcal{D} will have the form

$$\mathbf{z}_t(\mathbf{x}) = (z_t^1(\mathbf{x}), z_t^2(\mathbf{x}))$$

where $z^1 = 1$ or 0 according to whether an arriving packet is accepted into the buffer or is rejected, and $z^2 = 1$ or 0 according to whether or not the slower channel 2 is activated.

Defining the *action space* to be the product set $\mathcal{A} = \mathcal{D}^{\mathcal{S}}$, we can now represent an *admissible control strategy* (CS) as an \mathcal{A} -valued stochastic process $(z_t, t \geq 0)$, where $z_t = (\mathbf{z}_t(\mathbf{x}), \mathbf{x} \in \mathcal{S})$. Hereafter, we shall use the abbreviated notation z for the CS $(z_t, t \geq 0)$. Let \mathcal{P} denote the set of all admissible control strategies.

A *law of motion* corresponding to a CS z is specified by a transition probability $\mathbb{P}(\mathbf{x}'|\mathbf{x}, z_t)$, $\mathbf{x}, \mathbf{x}' \in \mathcal{S}, t \geq 0$, denoting the conditional probability that the system moves to state \mathbf{x}' at time t^+ when the action $\mathbf{z}_t(\mathbf{x})$ is applied to it at time t while in state \mathbf{x} .

Our objective is to find a CS z in \mathcal{P} minimizing the following cost:

$$\limsup_{T \rightarrow \infty} \mathbb{E}_{\mathbf{x}}^z \left(\frac{1}{T} \int_0^T (1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1) dt \right), \quad \gamma > 0, \quad (P1)$$

where $\mathbb{E}_{\mathbf{x}}^z$ denotes expectation with respect to the probability measure induced by the CS z on the process $(\mathbf{x}_t, t \geq 0)$ with initial state \mathbf{x} at $t = 0$. If such a minimizing CS exists, we shall refer to it as the optimal strategy for the *unconstrained average cost problem* (P1).

A key step involves the discounted cost problem associated with (P1). Namely we wish to find a CS z in \mathcal{P} for which the following discounted cost [17,64] is minimized:

$$\limsup_{T \rightarrow \infty} \mathbb{E}_{\mathbf{x}}^z \left(\int_0^T e^{-\delta t} (1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1) dt \right), \quad \delta > 0, \quad \gamma > 0. \quad (P2)$$

If such a minimizing CS exists, it is called the optimal strategy for the *discounted cost* problem (P2).

We conclude this section by introducing two special classes of relevant CS's. An admissible CS which is an i.i.d. stochastic process will be called a *stationary randomized strategy (SRS)*. Furthermore, if the common distribution of the SRS z has all its mass concentrated at some point in \mathcal{A} , we shall refer to it as a *stationary strategy (SS)*. Let $\mathcal{P}_S \subset \mathcal{P}$ denote the set of all SS's.

We shall see below that problems (P1) and (P2) are closely related. We consider first the discounted cost problem (P2).

6.3 The Discounted Cost Problem: Existence of a Stationary Optimal Policy

We begin our treatment of the discounted cost problem (P2) by asserting that an optimal CS exists which, furthermore, is stationary. The assertion is made upon verifying the conditions in the hypothesis of Lippman [43, p.1238]. To this end, first observe that the cost incurred in state $\mathbf{x}_t = (x_t^1, x_t^2)$ at time t has at most a linear growth with respect to x_t^1 , i.e.,

$$1 - z_t^1(\mathbf{x}_t) + \gamma x_t^1 \leq 1 + \gamma x_t^1. \quad (6.1)$$

Next, the inter-arrival and inter-departure times of the packets are exponentially distributed. Furthermore, the action set \mathcal{D} is finite. The assumptions of [43, Thm. 1 p. 1239] are thereby satisfied, leading to the following.

Theorem 6.1: (Lippman) [43] An optimal strategy for the discounted problem (P2) exists, and furthermore, is stationary.

In the sequel we replace $\mathbf{z}_t(\mathbf{x}_t)$ by \mathbf{z}_t for notational convenience. Furthermore, in view of Theorem 6.1, we restrict attention to stationary CS's and define the δ -discounted cost starting with initial state \mathbf{x} associated with the problem (P2) by

$$J^{\gamma, \delta}(\mathbf{x}) \triangleq \min_{z \in \mathcal{P}_S} \mathbb{E}_{\mathbf{x}}^z \left(\int_0^\infty e^{-\delta t} (1 - z_t^1 + \gamma x_t^1) dt \right), \quad \delta > 0, \quad \gamma > 0. \quad (6.2)$$

The minimum cost in (6.2) can be expressed in an alternative form which facilitates further analysis. To this end let $0 = t_0 < t_1 < t_2 < \dots < t_n \dots$ be the (random) instants in time denoting transition epochs of the system state $(\mathbf{x}_t, t \geq 0)$, where each transition epoch represents either an arrival of a packet into, or a departure of a packet from, the system. It is convenient to introduce at this point the δ -discounted expected cost over the time-horizon $[0, t_n)$, with initial state \mathbf{x} , and following a control strategy z in \mathcal{P}_S , namely,

$$V_n^{\gamma, \delta}(\mathbf{x}, z) \triangleq \mathbb{E}_{\mathbf{x}}^z \left(\int_0^{t_n} e^{-\delta t} (1 - z_t^1 + \gamma x_t^1) dt \right). \quad (6.3)$$

Let

$$J_n^{\gamma, \delta}(\mathbf{x}) = \min_{z \in \mathcal{P}_S} V_n^{\gamma, \delta}(\mathbf{x}, z), \quad n = 0, 1, \dots,$$

$$J_\infty^{\gamma, \delta}(\mathbf{x}) = \lim_{n \rightarrow \infty} J_n^{\gamma, \delta}(\mathbf{x}).$$

We now show that the minimum cost in (6.2) has the alternative expression

$$J^{\gamma, \delta}(\mathbf{x}) = J_\infty^{\gamma, \delta}(\mathbf{x}) \quad (6.4)$$

for every initial state \mathbf{x} . First observe from [43, Theorem 1], that $J^{\gamma, \delta}(\cdot)$ is the unique solution to the following functional Dynamic Programming equation:

$$J^{\gamma, \delta}(\mathbf{x}) = \min_{z \in \mathcal{P}_S} \left(1 - z^1(\mathbf{x}) + \gamma x^1 + \sum_{\mathbf{x}' \in \mathcal{S}} \beta_\delta(\mathbf{x}, z, \mathbf{x}') J^{\gamma, \delta}(\mathbf{x}') Pr(\mathbf{x}' | \mathbf{x}, z) \right), \quad (6.5)$$

where $\mathbf{x} = (x^1, x^2)$, and $\beta_\delta(\mathbf{x}, z, \mathbf{x}')$ is an expected discount factor of the form

$$\beta_\delta(\mathbf{x}, z, \mathbf{x}') = \int_0^\infty e^{-\delta \xi} dT(\xi | \mathbf{x}, z, \mathbf{x}'),$$

with $T(\cdot|\mathbf{x}, z, \mathbf{x}')$ denoting the probability distribution function of the random time it takes the system to go from state \mathbf{x} to \mathbf{x}' under the CS z .

Next, using Dynamic Programming arguments, it can be easily shown that $J_n^{\gamma, \delta}(\cdot)$ satisfies the following recursion:

$$J_{n+1}^{\gamma, \delta}(\mathbf{x}) = \min_{z \in \mathcal{P}} \left(1 - z^1(\mathbf{x}) + \gamma x^1 + \sum_{\mathbf{x}' \in \mathcal{S}} \beta_\delta(\mathbf{x}, z, \mathbf{x}') J_n^{\gamma, \delta}(\mathbf{x}') Pr(\mathbf{x}'|\mathbf{x}, z) \right). \quad (6.6)$$

Since $J_{n+1}^{\gamma, \delta}(\cdot) \geq J_n^{\gamma, \delta}(\cdot)$, we have that $J_\infty^{\gamma, \delta}(\mathbf{x}) = \lim_{n \rightarrow \infty} J_n^{\gamma, \delta}(\mathbf{x})$ exists. Moreover $J_\infty^{\gamma, \delta}(\mathbf{x})$ is the unique solution to the contraction mapping (6.6). This observation, together with the fact that $J^{\gamma, \delta}(\mathbf{x})$ is the unique solution to the same contraction mapping (6.5), yields that $J_\infty^{\gamma, \delta}(\mathbf{x}) = J^{\gamma, \delta}(\mathbf{x})$.

We now derive a few properties of the cost functions $J_n^{\gamma, \delta}(\cdot)$ and $J_\infty^{\gamma, \delta}(\cdot)$ which will be employed in the next section to characterize the optimal policy for the discounted cost problem (P2). In order to avoid repetition, the notation $J_\star^{\gamma, \delta}(\cdot)$ will be used to represent $J_n^{\gamma, \delta}(\cdot)$, $n = 0, 1, \dots$, as well as $J_\infty^{\gamma, \delta}(\cdot)$, as appropriate. Note that these properties are valid for every $\gamma > 0$, $\delta > 0$.

Proposition 6.1: For each x^2 , $J_\star^{\gamma, \delta}(\cdot, x^2)$ is nondecreasing.

Proof: We shall first prove that $J_n^{\gamma, \delta}(x^1 + 1, x^2) \geq J_n^{\gamma, \delta}(x^1, x^2)$ for all x^1 using the following coupling argument. Consider two similar systems starting with initial conditions (x^1, x^2) and $(x^1 + 1, x^2)$, respectively. Couple the arrival and service processes of both systems; further, in both cases, follow the optimal CS for the latter system starting with initial state $(x^1 + 1, x^2)$. Denoting this strategy by $z = (z^1, z^2)$, let $(\tilde{\mathbf{x}}_t)$ and (\mathbf{x}_t) , $0 \leq t < t_n$, be the corresponding trajectories of the systems starting at (x^1, x^2) and $(x^1 + 1, x^2)$, respectively. Define a stopping time by $\tau = [\min(t : \mathbf{x}_t = \tilde{\mathbf{x}}_t)] \wedge t_n$. We see that

$$\int_0^{t_n} e^{-\delta t} (1 - z_t^1 + \gamma x_t^1) dt - \int_0^{t_n} e^{-\delta t} (1 - z_t^1 + \gamma \tilde{x}_t^1) dt \geq \int_0^\tau \gamma e^{-\delta t} dt \geq 0$$

so that

$$\begin{aligned}
J_n^{\gamma,\delta}(x^1 + 1, x^2) &= \mathbb{E}_{(x^1+1, x^2)}^z \int_0^{t_n} e^{-\delta t} (1 - z_t^1 + \gamma x_t^1) dt \\
&\geq \mathbb{E}_{(x^1, x^2)}^z \int_0^{t_n} e^{-\delta t} (1 - z_t^1 + \gamma \tilde{x}_t^1) dt \\
&\geq J_n^{\gamma,\delta}(x^1, x^2).
\end{aligned}$$

The proof of the claim is completed by letting $n \rightarrow \infty$, when it readily follows that $J_\infty^{\gamma,\delta}(x^1 + 1, x^2) \geq J_\infty^{\gamma,\delta}(x^1, x^2)$.

Proposition 6.2: For every $\delta > 0$, there exists an integer $\bar{x} = \bar{x}(\delta)$ such that:

$$J^{\gamma,\delta}(\bar{x}, 0) \geq J^{\gamma,\delta}(\bar{x}, 1). \quad (6.7)$$

Proof: Suppose that for some $\delta > 0$ and for all x the reverse inequality holds, i.e.,

$$J^{\gamma,\delta}(x, 0) < J^{\gamma,\delta}(x, 1).$$

We shall show that this supposition leads to a contradiction. The proof employs coupling arguments *a la* Walrand [80]. Consider a system with initial state $(x, 0)$, where x is a positive integer. Consider a second system which is similar but has initial state $(x, 1)$. Couple the arrival and service processes of the two systems, and apply to each the optimal strategy, denoted $z = (z^1, z^2)$, associated with the first system (i.e., with initial state $(x, 0)$). Let (\mathbf{x}_t) and $(\tilde{\mathbf{x}}_t)$ respectively represent the corresponding state trajectories. By the supposition above, observe that in view of the stationarity of z , the first system never forwards a message through the slower channel.

Letting $V^{\gamma,\delta}(x, 1) = \mathbb{E}_{(x,1)}^z (\int_0^\infty e^{-\delta t} (1 - z_t^1 + \gamma \tilde{x}_t^1) dt)$, it is clear that $J^{\gamma,\delta}(x, 1) \leq V^{\gamma,\delta}(x, 1)$. Let $\tau = \min(t : x_t^1 = 0)$ and let σ be an exponential random variable with mean μ_2^{-1} which represents the packet propagation time on the slower channel. Then

$$\begin{aligned}
J^{\gamma,\delta}(x, 0) - J^{\gamma,\delta}(x, 1) &\geq J^{\gamma,\delta}(x, 0) - V^{\gamma,\delta}(x, 1) \\
&= \gamma \mathbb{E}_{(x,0)}^z (1[\sigma \leq \tau] \int_\sigma^\tau e^{-\delta t} dt - 1[\sigma > \tau] \int_0^\sigma e^{-\delta t} dt) \\
&\triangleq \varphi(x, \delta).
\end{aligned}$$

Next consider an $M|M|1$ system starting with x initial packets, with *no arrivals*, and with the service time distribution being exponential with parameter μ_1 . Let $\tilde{\tau}$ denote the time at which this system empties. Define

$$\tilde{\varphi}(x, \delta) = \gamma \mathbb{E}(1[\sigma \leq \tilde{\tau}] \int_{\sigma}^{\tilde{\tau}} e^{-\delta t} dt - 1[\sigma > \tilde{\tau}] \int_0^{\sigma} e^{-\delta t} dt).$$

Since $\tau \geq_{st} \tilde{\tau}$, it is clear that $\varphi(x, \delta) > \tilde{\varphi}(x, \delta)$ for all x . Then our supposition is contradicted if we show that $\tilde{\varphi}(x, \delta)$ is non-negative for some x suitably large. To this end, we observe that as x increases $\tilde{\tau}$ increases stochastically, so that $1[\sigma \leq \tilde{\tau}]$ increases while $1[\sigma > \tilde{\tau}]$ decreases, both in the stochastic sense. Noting that $\mathbb{P}(\sigma > \tilde{\tau}) = 1 - \mathbb{P}(\sigma \leq \tilde{\tau})$ goes to 0 as x increases, it is clear that there exists an integer $\bar{x} = \bar{x}(\delta)$ such that $\tilde{\varphi}(x, \delta) > 0$.

Proposition 6.3: For each x , $J_{\star}^{\gamma, \delta}(x, 0) \leq J_{\star}^{\gamma, \delta}(x + 1, 1)$.

Proof: The proof is similar to that of the previous proposition. Let z be the optimal policy associated with the system starting with initial condition $(x + 1, 1)$. If σ is an exponential random variable with mean μ_2^{-1} , we easily get:

$$J_n^{\gamma, \delta}(x + 1, 1) - J_n^{\gamma, \delta}(x, 0) \geq \gamma \mathbb{E}_{(x+1,1)}^z \int_0^{\tau} e^{-\delta t} dt \geq 0$$

where $\tau = \min(t_n, \sigma)$. Similar arguments hold true for the case $n \rightarrow \infty$.

Proposition 6.4: $J_{\star}^{\gamma, \delta}(1, 1) \geq J_{\star}^{\gamma, \delta}(1, 0)$.

Proof: We use the same argument as in [80, p. 133]. Let σ_1 and σ_2 be random variables representing the propagation times of a packet on the fast and slow channels respectively. Clearly, we can choose $\sigma_2 = \frac{\mu_2}{\mu_1} \sigma_1$. Consider two similar systems, the first starting with initial condition $(1, 1)$, and the second with initial condition $(1, 0)$. Denote by z the optimal policy associated with the first system. For the second system, we follow the policy \tilde{z} constructed as follows: Whenever z activates the fast channel (i.e., accepts messages in the system), \tilde{z} enables the slower channel.

We need only consider two cases. In the first case, the fast channel of the second system is transmitting, and hence, so is the slow channel of the first system. Then the strategies z and \tilde{z} , as defined above, will result in the same state trajectories for the two systems, and hence, the two systems will incur identical costs. In the second case, the fast channel of the second system is idle while the slow channels of both systems are busy. By introducing a dummy packet on the fast channel of the second system, the system states are coupled. As the dummy packet incurs no cost, the assertion is established in this case too.

Finally, we introduce two more propositions. Their proofs are more involved than those of the previous propositions, and will be provided in section 6.5 below.

Proposition 6.5(Concavity): For each $n \geq 0$, $\delta > 0$, and $x^2 = 0, 1$, $J_*^{\gamma, \delta}(\cdot, x^2)$ is a concave function, i.e.,

$$J_*^{\gamma, \delta}(x^1 + 1, x^2) - J_*^{\gamma, \delta}(x^1, x^2) \geq J_*^{\gamma, \delta}(x^1, x^2) - J_*^{\gamma, \delta}(x^1 - 1, x^2) \quad (6.8)$$

for all $x^1 > 1$.

Proposition 6.6: For each $n \geq 0$, $\delta > 0$,

$$J_*^{\gamma, \delta}(x^1 + 1, 1) - J_*^{\gamma, \delta}(x^1, 0) \geq J_*^{\gamma, \delta}(x^1, 1) - J_*^{\gamma, \delta}(x^1 - 1, 0) \quad (6.9)$$

for all $x^1 > 1$.

6.4 An Optimal Policy for the Discounted Cost Problem

In this section we derive the form of the optimal strategy associated with the β -discounted cost problem (P2). This is done below in two steps.

The first step entails converting the original continuous-time problem (P2) into its discrete-time equivalent by the standard procedure of “uniformization” [38,59]. We recall from section 3 that $0 = t_0 < t_1 < t_2 < \dots < t_n \dots$ are the (random) instants in time denoting transition epochs of the system state. By suitably introducing dummy departures as in [12,17], the inter-epoch intervals are seen to be i.i.d. random variables with distribution

$$\mathbb{P}[t_{k+1} - t_k > t] = e^{-t(\lambda + \mu_1 + \mu_2)} \quad (6.10)$$

for $k = 0, 1, \dots$. Consider the discrete time system obtained as in [38,59] by sampling the original continuous-time system at its transition epochs. To this end, we introduce the notation $\mathbf{x}_k \triangleq \mathbf{x}_{t_k}$ and $\mathbf{z}_k \triangleq \mathbf{z}(\mathbf{x}_{t_k})$ and define

$$\beta = \frac{\lambda + \mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2 + \delta}, \quad (6.11)$$

whence $0 < \beta < 1$. The β -discounted cost incurred by the n -step discrete time system for the CS z is defined [38,59] as

$$\tilde{V}_n^{\gamma, \beta}(\mathbf{x}, z) \triangleq \mathbb{E}_{\mathbf{x}}^z \sum_{k=0}^{n-1} \beta^k (1 - z_k^1 + \gamma x_k^1).$$

It then follows that (cf. (6.4))

$$V_n^{\gamma, \delta}(\mathbf{x}, z) = \frac{1 - \beta}{\delta} \tilde{V}_n^{\gamma, \beta}(\mathbf{x}, z). \quad (6.12)$$

Let

$$\tilde{V}^{\gamma, \beta}(\mathbf{x}, z) \triangleq \lim_{n \rightarrow \infty} \tilde{V}_n^{\gamma, \beta}(\mathbf{x}, z).$$

We can now state the minimization problem (P2) in terms of a discrete-time problem of equivalent cost as follows. Define the minimum β -discounted-cost for the n -step and infinite horizon discrete-time systems, respectively, by

$$\tilde{J}_n^{\gamma, \beta}(\mathbf{x}) \triangleq \min_{z \in \mathcal{P}} \tilde{V}_n^{\gamma, \beta}(\mathbf{x}, z) \quad (6.13)$$

and

$$\tilde{J}^{\gamma, \beta}(\mathbf{x}) \triangleq \min_{z \in \mathcal{P}} \tilde{V}^{\gamma, \beta}(\mathbf{x}, z). \quad (6.14)$$

Letting

$$\tilde{J}_{\infty}^{\gamma, \beta}(\mathbf{x}) \triangleq \lim_{n \rightarrow \infty} \tilde{J}_n^{\gamma, \beta}(\mathbf{x}) \quad (6.15)$$

it can be easily deduced, as in section 6.3, that

$$\tilde{J}_{\infty}^{\beta, \gamma}(\mathbf{x}) = \tilde{J}^{\beta, \gamma}(\mathbf{x})$$

for every initial condition \mathbf{x} . Finally, the equivalence, in the sense of optimal discounted cost, between (P2) and the discrete-time formulation above follows readily from (6.12) and (6.15) by noting that

$$J^{\gamma,\delta}(\mathbf{x}) = \frac{1-\beta}{\delta} \tilde{J}^{\gamma,\beta}(\mathbf{x}). \quad (6.16)$$

Thus, it suffices to restrict attention hereafter to the discrete-time β -discounted cost problem defined by (6.14).

We can now proceed to the second step associated with problem (P2) by developing the Dynamic Programming equations for the problem in (6.14). The notation is considerably simplified by introducing the following quantities:

$$\begin{aligned} A_i &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(i+1,0) - \tilde{J}^{\gamma,\beta}(i,0) \right) - 1, \quad i \geq 0 \\ B_i &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(i,1) - \tilde{J}^{\gamma,\beta}(i,0) \right) + \beta\mu_1 \left(\tilde{J}^{\gamma,\beta}(i-1,1) - \tilde{J}^{\gamma,\beta}(i-1,0) \right), \quad i \geq 2 \\ B_1 &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(1,1) - \tilde{J}^{\gamma,\beta}(1,0) \right), \\ B_0 &= 0, \\ C_i &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i+1,0) + \tilde{J}^{\gamma,\beta}(i,0) - \tilde{J}^{\gamma,\beta}(i,1) \right), \quad i \geq 1 \\ C_0 &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(1,1) - \tilde{J}^{\gamma,\beta}(1,0) \right), \\ D_i &= \lambda\beta \left(\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i,1) \right) - 1, \quad i \geq 1 \\ E_i &= \beta\mu_2 \left(\tilde{J}^{\gamma,\beta}(i-1,1) - \tilde{J}^{\gamma,\beta}(i-1,0) \right), \quad i \geq 2 \\ E_1 &= 0. \end{aligned}$$

Furthermore, the following observations are be useful:

- (i) A_i and D_i are increasing functions of i , $i \geq 0$, by the convexity of $\tilde{J}^{\gamma,\beta}(\cdot, x^2)$ (cf. Proposition 6.5 and (6.16)).
- (ii) For every $i \geq 1$, $A_i + C_i = D_i$. This follows directly from the definition of A_i , B_i , C_i .

(iii) For $i \geq 1, A_i \leq D_{i+1}$. This follows from Proposition 6.6 and (6.16), since

$$\tilde{J}^{\gamma,\beta}(i+1,1) - \tilde{J}^{\gamma,\beta}(i,0) \geq \tilde{J}^{\gamma,\beta}(i,1) - \tilde{J}^{\gamma,\beta}(i-1,0),$$

or

$$\tilde{J}^{\gamma,\beta}(i+2,1) - \tilde{J}^{\gamma,\beta}(i+1,1) \geq \tilde{J}^{\gamma,\beta}(i+1,0) - \tilde{J}^{\gamma,\beta}(i,0),$$

whence the assertion results.

Referring to the state transition diagram in Figure 6.2, the Dynamic Programming equations can now be written as follows:

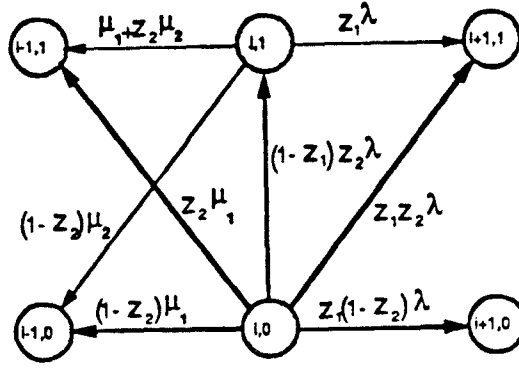


Figure 6.2

For $i > 1$:

$$\begin{aligned} \tilde{J}^{\gamma,\beta}(i,0) = & 1 + \gamma i + \min_{z^1, z^2 \in \{0,1\}} \{z^1 A_i + z^2 B_i + z^1 z^2 C_i\} + \\ & + \beta \left(\mu_1 \tilde{J}^{\gamma,\beta}(i-1,0) + \lambda \tilde{J}^{\gamma,\beta}(i,0) + \mu_2 \tilde{J}^{\gamma,\beta}(i,0) \right) \\ \tilde{J}^{\gamma,\beta}(i,1) = & 1 + \gamma i + \min_{z^1, z^2 \in \{0,1\}} \{z^1 D_i + z^2 E_i\} + \\ & + \beta \left(\mu_2 \tilde{J}^{\gamma,\beta}(i-1,0) + \mu_1 \tilde{J}^{\gamma,\beta}(i-1,1) + \lambda \tilde{J}^{\gamma,\beta}(i,1) \right). \end{aligned} \tag{6.17}$$

For $i = 1$, we get:

$$\begin{aligned}
\tilde{J}^{\gamma,\beta}(1,0) = & 1 + \gamma + \min_{z^1, z^2 \in \{0,1\}} \{z^1 A_1 + z^2 B_1 + z^1 z^2 C_1\} + \\
& + \beta \left(\mu_1 \tilde{J}^{\gamma,\beta}(0,0) + \lambda \tilde{J}^{\gamma,\beta}(1,0) + \mu_2 \tilde{J}^{\gamma,\beta}(1,0) \right); \\
\tilde{J}^{\gamma,\beta}(1,1) = & 1 + \gamma + \min_{z^1 \in \{0,1\}} \{z^1 D_1\} + \beta \left(\mu_2 \tilde{J}^{\gamma,\beta}(0,0) + \lambda \tilde{J}^{\gamma,\beta}(1,1) \right. \\
& \left. + \mu_1 \tilde{J}^{\gamma,\beta}(1,1) \right).
\end{aligned} \tag{6.18}$$

Finally, for $i = 0$, we have:

$$\tilde{J}^{\gamma,\beta}(0,0) = 1 + \min_{z^1, z^2 \in \{0,1\}} \{z^1 A_0 + z^1 z^2 C_0\} + \beta \tilde{J}^{\gamma,\beta}(0,0). \tag{6.19}$$

The optimal control actions taken at states $(i,0)$ and $(i,1)$, respectively, can be seen from (6.17) – (6.19) to be determined by the minimization with respect to (z^1, z^2) of the functions:

$$f_i^0(z^1, z^2) \triangleq z^1 A_i + z^2 B_i + z^1 z^2 C_i, \quad i \geq 0, \tag{6.20}$$

and

$$f_i^1(z^1, z^2) \triangleq z^1 D_i + z^2 E_i, \quad i \geq 1. \tag{6.21}$$

Before proceeding with the minimization, it is instructive to consider the nature of the optimal cost functions $\tilde{J}^{\gamma,\beta}(\cdot, 0)$ and $\tilde{J}^{\gamma,\beta}(\cdot, 1)$. From propositions 6.1-6.6 it is evident that the forms and relative values of these two cost functions will be as depicted in Figure 6.3a; furthermore, they will intersect at most at two points. Consider first the case where they intersect at one point (Figure 6.3a). (Subsequently, in this section it will be shown that $\tilde{J}^{\gamma,\beta}(\cdot, 0)$ and $\tilde{J}^{\gamma,\beta}(\cdot, 1)$ cannot intersect at more than one point).

We commence with the actions taken at state $(i,0)$. There are four cases to be considered.

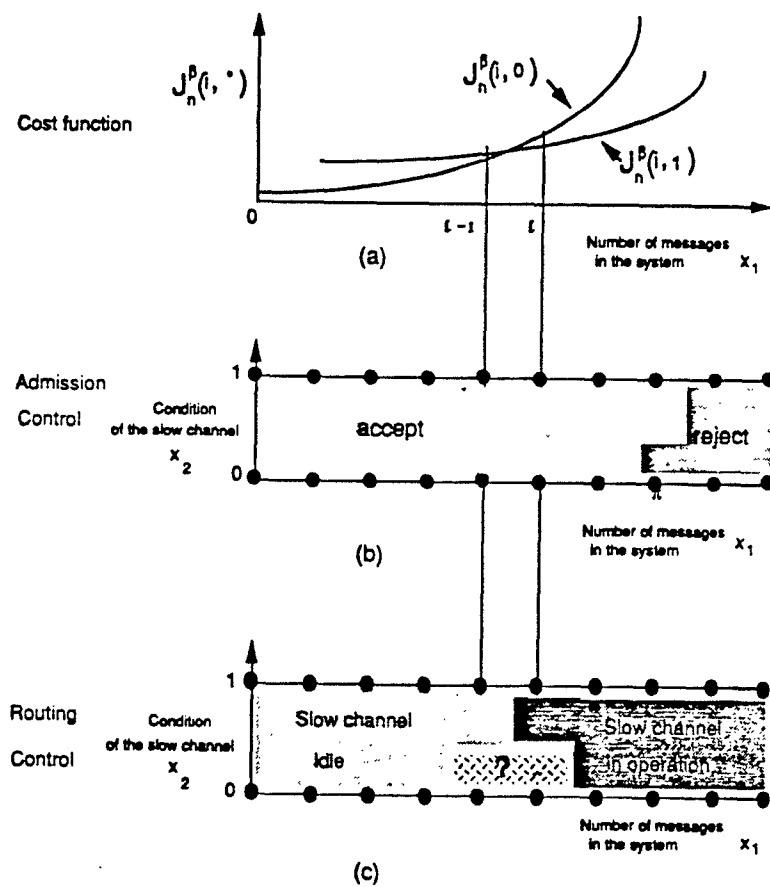


Figure 6.3

Case (i): $i < \bar{i} - 1$ (see Figure 6.3a). The values of $f_i^0(z^1, z^2)$ for the four possible choices of (z^1, z^2) are:

$$\begin{aligned} & 0 \text{ for } z^1 = 0, z^2 = 0; \\ & A_i \text{ for } z^1 = 1, z^2 = 0; \\ & B_i \text{ for } z^1 = 0, z^2 = 1; \\ & A_i + B_i + C_i \text{ for } z^1 = 1, z^2 = 1. \end{aligned}$$

Since in this case $B_i \geq 0$ and $B_i + C_i \geq 0$, the choice is between $(z^1 = 0, z^2 = 0)$ and $(z^1 = 1, z^2 = 0)$ according to whether $A_i \geq 0$ or $A_i < 0$. Thus, the optimal strategy disables the slower channel and accepts (resp. blocks) an incoming message at the buffer if $A_i < 0$ (resp. $A_i \geq 0$).

Case (ii): $i > \bar{i}$: In this case it is easily verified that $B_i \leq 0$ and $B_i + C_i \leq 0$, so that the choice is between $(z^1 = 0, z^2 = 1)$ and $(z^1 = 1, z^2 = 1)$, the corresponding values of $f_i^0(z^1, z^2)$ being B_i and $A_i + B_i + C_i$, respectively. Thus, the optimal policy keeps the slower channel active and accepts (resp. blocks) an incoming message if $A_i + C_i < 0$ (resp. $A_i + C_i \geq 0$).

Case (iii): $i = \bar{i} - 1$. Since $B_i \geq 0, C_i \geq 0$, the possible choices are $(z^1 = 0, z^2 = 0)$, $(z^1 = 1, z^2 = 0)$ and $(z^1 = 1, z^2 = 1)$, with the corresponding values of $f_i^0(z^1, z^2)$ being 0, A_i , and $A_i + B_i + C_i$, respectively. Hence, if $A_i < 0$, then $z^1 = 1$ (i.e., the optimal policy accepts an incoming message), while $z^2 = 1$ (resp. 0) if $B_i + C_i < 0$ (resp. $B_i + C_i \geq 0$). Finally, if $A_i \geq 0$, then the action pair $(z^1 = 0, z^2 = 0)$ is optimal if $A_i + B_i + C_i \geq 0$, while the pair $(z^1 = 1, z^2 = 1)$ is optimal if $A_i + B_i + C_i < 0$ (so that $A_i + C_i < 0$).

Case (iv): $i = \bar{i}$. Here, all four combinations of the action pairs are possible.

The optimal control actions taken at states $(i, 1)$ are easily determined in a similar manner by the signs of D_i and E_i . If $D_i < 0$ (resp. ≥ 0), $E_i < 0$ (resp. ≥ 0), then $(z^1 = 1, z^2 = 1)$ (resp. $(z^1 = 0, z^2 = 0)$) minimize $f_i^0(z^1, z^2)$.

We are now in a position to characterize the optimal admission policy at the buffer; this is done in Propositions 6.7-6.10.

Proposition 6.7: The optimal admission policy is characterized by a switching curve (see Figure 6.3b).

Proof: We first show $z^1(i, 0) = 1 \Rightarrow z^1(i', 0) = 1$ for all $i' < i$, i.e., if the optimal policy admits an incoming message into the buffer at state $(i, 0)$, it must also do so at states $(i', 0)$, $i' < i$. Suppose $i > \bar{i}$ (cf. case (ii) above). If $z^1(i, 0) = 1$, then $A_i + C_i < 0$. From the convexity of $\tilde{J}^{\gamma, \beta}(\cdot, 0)$, it follows that for all $\bar{i} < i' \leq i$, $A_{i'} + C_{i'} < 0$, so that $z^1(i', 0) = 1$. Moreover, for $i' < i$, $A_{i'} < 0$ (by observation (iii) earlier in this section). It is then evident that $A_{\bar{i}}$ or $A_{\bar{i}} + B_{\bar{i}} + C_{\bar{i}}$ will be the possible minima of $f_{\bar{i}}^1(z^1, z^2)$, so that $z^1(\bar{i}, 0) = 1$. Next, $A_{\bar{i}-1} < 0$ so that $z^1(\bar{i}-1, 0) = 1$ (cf. case (iii)). Finally for $i' < \bar{i}-1$, again $A_{i'} < 0$, whence $z^1(i', 0) = 1$ (cf. case (i)). Similar arguments show that $z^1(i, 0) = 1$ for $i \leq \bar{i}$ would imply $z^1(i', 0) = 1$ for all $i' < i$.

Lastly, it follows in a straightforward manner that $z^1(i, 1) = 1$ implies $z^1(i', 1) = 1$ for $i' < i$. Indeed, if $z^1(i, 1) = 1$, then $D_i < 0$ whence $D_{i'} < 0$ for $i' < i$ by the convexity of $\tilde{J}^{\gamma, \beta}(\cdot, 1)$; consequently, $z^1(i', 1) = 1$.

Proposition 6.8: If the optimal policy accepts an incoming message at state $(i, 1)$, then it also does so at states $(i', 0)$, $i' < i$, i.e., $z^1(i, 1) = 1$ implies $z^1(i', 0) = 1$.

Proof: Since $z^1(i, 1) = 1$, it follows that $D_{i'} < 0$ and $A_{i'} < 0$ for all $i' < i$, so that $z^1(i', 0) = 1$.

Proposition 6.9: For $i > \bar{i}$, $z^1(i, 1) = 0$ iff $z^1(i, 0) = 0$.

Proof: The proof is obvious by the fact that $D_i = A_i + C_i < 0$.

Next, we characterize the optimal routing strategy, governing the activation of the slower channel.

Proposition 6.10: For all states $(i, 1)$ with $i \geq \bar{i}$ (resp. $i < \bar{i}$), the optimal routing

strategy yields $z^2(i, 1) = 1$ (resp. 0). Furthermore, for all states $(i, 0)$ with $i > \bar{i}$ (resp. $i < \bar{i} - 1$), it provides that $z^2(i, 0) = 1$ (resp. 0) (Figure 6.3c).

Proof: The proof is immediate from cases (i) and (ii).

Finally, we show that the discrete-time optimal costs $\tilde{J}^{\gamma, \delta}(\cdot, 0)$ and $\tilde{J}^{\gamma, \delta}(\cdot, 1)$ do not intersect at more than one point. We provide the proof in continuous time for the optimal costs $J^{\gamma, \delta}(\cdot, 0)$ and $J^{\gamma, \delta}(\cdot, 1)$ but the result is obviously true for the discrete time costs as well by virtue of (4.7).

Proposition 6.11: For every $\delta > 0$, there exists $\bar{x} = \bar{x}(\delta)$ such that

$$J^{\gamma, \delta}(x, 0) \geq J^{\gamma, \delta}(x, 1) \quad (6.22)$$

for $x \geq \bar{x}$.

Proof: We assume that the proposition is not true and show that this leads to a contradiction. Because $J^{\gamma, \delta}(\cdot, x^2)$ is increasing and convex, there exists $\bar{y} > \bar{x}$ such that $J^{\gamma, \delta}(x, 0) < J^{\gamma, \delta}(x, 1)$ for all $x > \bar{y}$.

By using the same argument as in case (i) above, we can show that for all $x > \bar{y}$ the optimal policy keeps the slower channel idle. As in Proposition 6.2 of section 6.3, we consider two identical systems starting with initial conditions $(x, 0)$ and $(x, 1)$, respectively. Let σ be an exponential random variable with mean μ_2^{-1} . For both systems we follow the optimal policy associated with that starting at $(x, 0)$. The following cases are of interest:

a) If the optimal policy employs the second channel at time $\tau = \min(t : z^2(\mathbf{x}_t) = 1)$, then

$$J^{\gamma, \delta}(x, 0) - J^{\gamma, \delta}(x, 1) \geq \gamma \mathbb{E}(1[\sigma \leq \tau] \int_{\sigma}^{\infty} e^{-\delta, t} dt) \geq 0$$

which contradicts our assumption.

b) If the optimal policy never uses the slower channel, the same approach as in proposition 6.2 results in a contradiction if x is chosen large enough.

6.5 Convexity of the Optimal Cost

We now provide a proof of Proposition 6.5 dealing with the convexity of $\tilde{J}^{\gamma,\beta}(\mathbf{x})$. In view of (6.4), it suffices to show that for each $n \geq 0$, $\beta < 1$, and $x^2 = 0, 1$

$$\tilde{J}_n^{\gamma,\beta}(x^1 + 1, x^2) - \tilde{J}_n^{\gamma,\beta}(x^1, x^2) \geq \tilde{J}_n^{\gamma,\beta}(x^1, x^2) - \tilde{J}_n^{\gamma,\beta}(x^1 - 1, x^2)$$

for all $x^1 \geq 1$. In order to do so we shall use the techniques introduced in chapter 3 (namely Proposition 3.3).

Let A , D_1 , and D_2 represent respectively the events of an arrival of a packet at the buffer, and the departures of a packet on channels 1 and 2. Next, let $\Omega^k = \{\omega^k(\omega_1, \dots, \omega_k) : \omega_i \in \{A, D_1, D_2\}\}$, $k = 1, 2, \dots, n$, represent the collection of all events corresponding to arrivals and departures of packets at the transition epochs during the interval $[0, t_k]$. On Ω_k we define the following transition matrix:

$$\Xi_k(\omega^k) = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \text{if } \omega_k = A \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \text{if } \omega_k = D_1 \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} & \text{if } \omega_k = D_2. \end{cases}$$

We can then write the evolution of the state of the system as follows:

$$\begin{bmatrix} x_{k+1}^1(\omega^{k+1}) \\ x_{k+1}^2(\omega^{k+1}) \end{bmatrix} = \begin{bmatrix} x_k^1(\omega^k) \\ x_k^2(\omega^k) \end{bmatrix} + \Xi_{k+1}(\omega^{k+1}) \begin{bmatrix} z_{k+1}^1(\omega^{k+1}) \\ z_{k+1}^2(\omega^{k+1}) \\ z_{k+1}^3(\omega^{k+1}) \end{bmatrix},$$

for $k = 0, 1, \dots$, where $z_k^1(\omega^k)$ and $z_k^2(\omega^k)$ correspond respectively, as in section 6.2, to the actions of admission of a message into the system and the activation of channel 2 at transition epoch t_k ; $z_k^3(\omega^k)$ takes the value 1 or 0 at t_k according to whether or not a “dummy” departure [6,12,17] occurs on channel 2. We can easily see that the matrix Ξ of Proposition 3.3 in Chapter 3 is a 2×2 -unit matrix. As a result, the requirements of Proposition 3.3 are trivially satisfied and the convexity of $J_n^{\gamma,\beta}(\cdot, x^2)$ (as well as $J_*^{\gamma,\delta}(\cdot, x^2)$) follow. Another proof of the convexity of the aforementioned costs is provided in Appendix 6.1.

6.6 The Average Cost Problem

The average cost problem (P1) entails a minimization of the expected average cost per unit time [19,57,69,74]. In this section, we determine the optimal stationary policy for the average cost problem (P1) by associating it with the discounted cost problem (P2). Moreover, the optimal average cost for problem (P1) can be expressed as follows by using the standard procedure of uniformization earlier employed in section 6.4; thus

$$\tilde{J}_{av}(\mathbf{x}) = \min_{z \in \mathcal{P}} \tilde{V}(\mathbf{x}, z)$$

and

$$\tilde{V}(\mathbf{x}, z) = \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mathbf{x}}^z \left(\sum_{k=0}^n 1 - z_k^1(\mathbf{x}_k) + \gamma x_k^1 \right)$$

where, with a slight abuse of notation, z denotes a policy that is not necessarily stationary.

However, the following lemma proposes a stationary policy which is a candidate for average-cost optimality. Furthermore, this stationary policy will be seen to arise as a limit of optimal policies associated with the discounted cost problem (P2). The lemma can be found in [19] and [69].

Lemma 6.4: Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of discount factors converging to 1. Let $\{z_{\beta_n}\}_{n=1}^{\infty}$ be the associated sequence of (stationary) optimal policies for the discounted-cost problem. Then there exists a subsequence $\{\beta_{n'}\}$ and a stationary policy z which is the limit point of $z_{\beta_{n'}}$.

Although the lemma has been proved in [19,69], for the sake of completeness we present a brief proof below along with some observations.

Proof of Lemma 6.4: The finiteness of the action set $\mathcal{D} = \{0, 1\}^2$ enables it to be viewed as a compact topological space with a discrete topology, where every subset of \mathcal{D} is simultaneously open and closed. Further, the associated topological basis formed by these open sets is finite. By the Tychonoff theorem [67], the countable product space $\mathcal{A} = \mathcal{D}^{\mathcal{S}}$ is also compact under the product discrete topology. Since

the basis for \mathcal{A} under the same topology is countable and since \mathcal{A} is normal [67], it is also metrizable by Uryshon's lemma [7]. Consequently, \mathcal{A} is sequentially compact, i.e., every sequence $\{z_{\beta_n}\}$ (of stationary policies) has a convergent subsequence $\{z_{\beta_{n'}}\}$ converging to a stationary policy z in the following sense: for every \mathbf{x} in \mathcal{S} there exists an integer $N(\mathbf{x})$ such that $z_{\beta_{n'}}(\mathbf{x}) = z(\mathbf{x})$ for $n' \geq N(\mathbf{x})$.

Next, we establish that the stationary policy z of the previous lemma will have the form derived in section 6.4. In particular, we show that $z^1(x^1, x^2) = 0$ implies $z^1(\bar{x}^1, x^2) = 0$ for $\bar{x}^1 \geq x^1$. If this were not true, suppose that $z^1(\bar{x}^1, x^2) = 1$ for $\bar{x}^1 > x^1$. Then since $z_{\beta_{n'}} \rightarrow z$, we conclude that there exist $N(x^1, x^2)$, $\bar{N}(\bar{x}^1, x^2)$ such that $z_{\beta_{n'}}^1(x^1, x^2) = 0$ for all $n' \geq N$, and $z_{\beta_{n'}}^1(\bar{x}^1, x^2) = 1$ for all $n' \geq \bar{N}$. By choosing $k = \max\{N, \bar{N}\}$ we see that $z_{\beta_k}^1(x^1, x^2) = 0$ and $z_{\beta_k}^1(\bar{x}^1, x^2) = 1$ for $\bar{x}^1 \geq x^1$ which contradicts Proposition 6.7 of section 6.4. A similar argument can be applied to the routing control z^2 .

It only remains to establish the optimality of the stationary policy z of Lemma 6.4 for the average cost problem. To this end, we consider the following two cases determined by the nature of the admission control z^1 .

1) Assuming that $z^1(x^1, x^2) = 0$ for some finite x^1 , we conclude that the underlying Markov Decision Process is “essentially” a finite, irreducible chain and, hence, ergodic. If $p_z(\mathbf{x})$ is the associated stationary probability distribution under the policy z we obviously have $\sum_{\mathbf{x} \in \mathcal{S}} p_z(\mathbf{x})(1 - z^1(\mathbf{x}) + \gamma x^1) < \infty$. Moreover, since $\tilde{J}^{\gamma, \beta}(x^1, x^2)$ is increasing in x^1, x^2 (Propositions 6.1, 6.3), we have that $\tilde{J}^{\gamma, \beta}(x^1, x^2) - \tilde{J}^{\gamma, \beta}(0, 0) \geq 0$ for all β, γ, x^1, x^2 . Hence the following theorem from [69] follows:

Theorem 2: The policy z from lemma 4 is optimal for the average cost problem (P1). Furthermore the average cost is given by:

$$\tilde{J}_{av} = \lim_{\beta \rightarrow 1} (1 - \beta) \tilde{J}^{\beta, \gamma}(\mathbf{x}),$$

not depending on the initial state \mathbf{x} .

2) Assuming that $z^1(x^1, x^2) = 1$ for all x^1 , the problem reduces to that studied by Lin and Kumar [10], where the routing control is shown to be of the threshold type.

6.7 Concluding Remarks

In studying the problem of simultaneous admission and routing at a network node, it has been assumed that if the system is nonempty, then the faster channel is nonidle. This assumption is intuitively appealing and might even be “optimal” for the costs under consideration.

A harder and as yet unsolved problem appears to be that which seeks a minimal blocking probability under an explicit constraint on the average delay in the system. On the basis of Ross [19], it seems reasonable to conjecture that the optimal policy will no longer be stationary but rather a randomized one.

Appendix 6.1

In this section we establish Propositions 6.5 and 6.6 (cf. section 6.3). In order to do so, we shall employ a technique introduced in [59], wherein the discounted-cost, discrete time problem is first suitably transformed into a Linear Program.

We begin by artificially “enlarging” the state space of the system, redefining the state at instant k (corresponding to the transition epoch t_k) in terms of a triple $\mathbf{x}_k = (x_k^1, x_k^2, x_k^3)$ where

x_k^1 = number of packets in the buffer and on channel 1 at the transition epoch t_k (and not the total number in the system, as defined earlier);

x_k^2 = number of packets transfered from the buffer to channel 2 upto transition epoch t_k ;

x_k^3 = number of packets that have departed on channel 2 upto transition epoch t_k .

This new state description is intended solely for the proofs at this section, and should cause no confusion. It subsumes the state description of section 4, since the total number of messages in the system at time instant k is given by $x_k^1 + x_k^2 - x_k^3$ while the condition of the second channel is simply $x_k^2 - x_k^3$. Clearly, $x_k^1 \geq 0$ and $x_k^2 - x_k^3$ belongs to $\{0, 1\}$. In terms of the new state description, the n -step β -discounted cost for the discrete-time problem with initial state x corresponding to a CS z in \mathcal{P} is given by

$$\tilde{V}_n^\beta(\mathbf{x}, z) = E_{\mathbf{x}}^z \sum_{k=0}^{n-1} \beta^k \left((1 - z_k^1(\omega^k)) + \gamma(x_k^1(\omega^k) + x_k^2(\omega^k) - x_k^3(\omega^k)) \right), \quad (\text{A6.1})$$

and the corresponding optimal cost by

$$\tilde{J}_n^{\gamma, \beta}(\mathbf{x}) = \min_{z \in \mathcal{P}} \tilde{V}_n^{\gamma, \beta}(\mathbf{x}, z). \quad (\text{A6.2})$$

Let A , D_1 , D_2 represent, respectively, the events of an arrival of a packet at the buffer, and the departures of a packet on channels 1 and 2. Next, let $\Omega^k =$

$\{\omega^k(\omega_1, \dots, \omega_k) : w_i \in \{A, D_1 D_2\}\}, \quad k = 1, 2, \dots, n$ represent the collection of all events corresponding to arrivals and departures of packets at the transition epochs during the interval $[0, t_k]$. On Ω_k we define the following transition matrix:

$$\Xi_k(\omega^k) = \begin{cases} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } \omega_k = A \\ \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } \omega_k = D_1 \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } \omega_k = D_2. \end{cases}$$

We can then write the evolution of the system state as

$$\begin{bmatrix} x_{k+1}^1(\omega^{k+1}) \\ x_{k+1}^2(\omega^{k+1}) \\ x_{k+1}^3(\omega^{k+1}) \end{bmatrix} = \begin{bmatrix} x_k^1(\omega^k) \\ x_k^2(\omega^k) \\ x_k^3(\omega^k) \end{bmatrix} + \Xi_{k+1}(\omega^{k+1}) \begin{bmatrix} z_{k+1}^1(\omega^{k+1}) \\ z_{k+1}^2(\omega^{k+1}) \\ z_{k+1}^3(\omega^{k+1}) \end{bmatrix}, \quad (\text{A6.3})$$

for $k = 0, 1, \dots$, where $z_k^1(\omega^k)$ and $z_k^2(\omega^k)$ correspond, respectively (as in section 6.2), to the actions of admission to the buffer, and the activation of channel 2, at transition epoch t_k , and $z_k^3(\omega^k)$ takes the value 1 or 0 at t_k according to whether or not a “dummy” departure [38,43,59] occurs on channel 2. Equation (A6.3) can be solved recursively to yield

$$\mathbf{x}_k(\omega^k) = \mathbf{x} + \sum_{j=1}^k \Xi_j(\omega^j) \mathbf{z}_j(\omega^j), \quad (\text{5.4})$$

where the notation $\mathbf{z}_k(\omega^k)$, $k = 1, 2, \dots$, is obvious. We can now rewrite (A6.1) as

$$\tilde{V}_n^{\beta, \gamma}(\mathbf{x}, z) = \mathbb{E}_{\mathbf{x}}^z \sum_{k=0}^{n-1} \beta^k (I(\omega^k = A)(1 - z_k^1) + \gamma(x_k^1 + x_k^2 - x_k^3)).$$

Upon substituting (A6.4) in the equation above, we get

$$\tilde{V}_n^{\beta, \gamma}(\mathbf{x}, z) = \sum_{k=1}^n \sum_{\omega^k \in \Omega^k} c_k^1(\omega^k) z_k^1(\omega^k) + c_k^2(\omega^k) z_k^2(\omega^k) + c$$

where

$$c_k^1(\omega^k) = \mathbb{P}(\omega^k) \beta^k [\Xi_k^{11}(\omega^k) \gamma \sum_{j=k}^n \beta^j - I(\omega^k = A)],$$

$$c_k^2(\omega^k) = -\mathbb{P}(\omega^k) \beta^k \Xi_k^{33}(\omega^k) \gamma \sum_{j=k}^n \beta^j,$$

and

$$c = \lambda \sum_{k=1}^n \beta^k + \sum_{k=1}^n \gamma \beta^k [x^1 + x^2 - x^3].$$

The minimization problem (A6.2) can now be transformed into the following Linear Program:

$$W_n(\mathbf{x}) = \min_{\{\mathbf{z}_k(\omega^k)\}_{k=1}^n} \sum_{k=1}^n \sum_{\omega^k \in \Omega^k} c_k^1(\omega^k) z_k^1(\omega^k) + c_k^2(\omega^k) z_k^3(\omega^k) + c,$$

such that for each ω^k in Ω , and $k = 1, 2, \dots, n$

$$z_k^i(\omega^k) \in [0, 1], \quad i = 1, 2, 3 \quad (\text{feasibility constraint}) \quad (\text{LP})$$

$$x^1 + \sum_{j=1}^k \Xi_j^{(1)}(\omega^j) \mathbf{z}_j(\omega^j) \geq 0,$$

and

$$0 \leq x^2 - x^3 + \sum_{j=1}^k (\Xi_j^{(2)}(\omega^j) - \Xi_j^{(3)}(\omega^j)) \mathbf{z}_j(\omega^j) \leq 1$$

where $\Xi_j^{(i)}(\omega^j)$ denotes the i th row of the matrix $\Xi_j(\omega^j)$.

Remark: The first constraint is associated with the feasibility of the control actions. The second and third constraints, respectively, imply that the number of packets in the buffer and on the slow channel should be nonnegative, and, moreover, that the second channel cannot forward more than one packet at any time.

Lemma A6.1: For $0 < \beta < 1$, $n = 0, 1, \dots$, $W_n^\beta(\cdot)$ is a convex piecewise linear function.

Proof: Since the quantity \mathbf{x} enters linearly in the constraints of (LP), the lemma follows directly from the theory of Linear Programming [77, page 56].

Lemma A6.2: The Linear Program (LP) accepts an integer solution, i.e., $z_k^i(\omega^k)$ belongs to $\{0, 1\}$, $i = 1, 2, 3$ for every $k = 1, 2, \dots, n$.

Proof: We denote by z^* the optimal strategy that solves the linear program (LP). By using duality theory [77, page 50] as it applies to linear programming, we conclude that z^* is an optimal solution for (LP) iff there exist suitable vector-valued variables $\underline{\lambda}_k^*(\omega^k)$ in \mathbb{R}^3 such that $\underline{\lambda}_k^*(\omega^k) \geq 0$ (componentwise), and the following conditions are satisfied for $k = 1, 2, \dots, n$, ω^k in Ω^k : (We drop below the dependence of certain variables on ω^k to make the presentation simpler.)

c1) z^* is a solution to the following program:

$$\min_z \sum_{k=1}^n \sum_{\omega^k} (c_k^1 z_k^1 + c_k^2 z_k^3 - \lambda_k^{*1} x_k^1 - \lambda_k^{*2} (x_k^2 - x_k^3) + \lambda_k^{*3} (x_k^2 + x_k^3 - 1)).$$

c2) The state trajectory generated by z^* , denoted $\mathbf{x}_k(z^*)$, should satisfy,

$$x_k^1(z^*) \geq 0 \quad \text{and} \quad 0 \leq x_k^2(z^*) - x_k^3(z^*) \leq 1.$$

c3) If $\lambda_k^{*1} > 0$, then $x_k^1 = 0$. Further, if $\lambda_k^{*2(3)} > 0$ then $x_k^2 + x_k^3 = 0(1)$.

The cost function in c1) can be transformed (after a simple change of the variables of summation) to:

$$\begin{aligned} \min_z \sum_k \sum_{\omega^k} & \left(c_k^1 z_k^1 + c_k^2 z_k^3 - \sum_{j=k}^n \lambda_j^{*1} \right) \Xi_k^{(1)} \mathbf{z}_k \\ & + \left(\sum_{j=k}^n (\lambda_j^{*3} - \lambda_j^{*2}) (\Xi_k^{(2)} - \Xi_k^{(3)}) \right) \mathbf{z}_k + \text{terms independent of } z \\ & = \min_z \sum_{k=1}^n \sum_{\omega^k} \mathbf{d}_k(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) \mathbf{z}_k + \text{terms independent of } z \end{aligned}$$

where $\mathbf{d}_k(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k)$ is defined in an obvious manner.

We then conclude immediately that

$$\mathbf{z}_k^{*i} = \begin{cases} 1 & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) < 0 \\ 0 & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) > 0 \\ \in [0, 1] & \text{if } d_k^i(\mathbf{c}_k, \underline{\lambda}_k^*, \omega_k) = 0 \end{cases} \quad i = 1, 2, 3. \quad (\text{A6.5})$$

Henceforth, suppose that the initial condition \mathbf{x} is integer-valued. For $k = 1, 2, \dots, n$ let $z^*, \underline{\lambda}^*$ satisfy the optimality conditions c1), c2) and c3). We shall use z^* to construct an integer-valued policy z that is optimal, i.e., satisfies the abovementioned conditions. To this end, we provide the following lemma.

Lemma A6.3: Consider the following region in \mathbb{R}^3 :

$$X = \left\{ p_1 e_1 + p_2 e_2 + p_3 e_3, \quad p_i \in \left(-\frac{1}{2}, \frac{1}{2} \right] \right\},$$

where $e_1 = (1, 0, 0)^T, e_2 = (-1, +1, 0)^T, e_3 = (0, 0, -1)^T$. Let $\xi^i(\omega), i = 1, 2, 3$ be the i th column of the matrix $\Xi(\omega)$. Then

$$\{X + \Xi(\omega)\mathbf{z}, \quad \mathbf{z} \in [0, 1]^3\} \subset X \cup \left\{ \bigcup_{\substack{i, j \in \{1, 2, 3\}, i \neq j \\ \omega \in \{A, D_1, D_2\}}} X + \xi^i(\omega) + \xi^j(\omega) \right\}.$$

Proof: The proof is straightforward and, hence, omitted.

Proposition A6.1: There is an integer-valued (i.e., $\{0, 1\}$ -valued) policy $z = (z_k(\omega^k), k = 1, 2, \dots, n)$ such that $\mathbf{z}_k^*(\omega^k) = \mathbf{z}_k(\omega^k)$, where the latter is integer-valued, and for all ω^k in Ω^k and $k > 1$, it holds that

$$\Delta_k \triangleq (\mathbf{x}_k(\omega^k, z^*) - \mathbf{x}_k(\omega^k, z)) \in X.$$

Proof: The proof is by induction. Suppose for some $k \geq 0$ that Δ_k is in X .

Then, it follows that

$$\Delta_{k+1} = \Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}^*(\omega^{k+1}) - \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}).$$

If $z_{k+1}^*(\omega^{k+1})$ is integer-valued, then obviously $z_{k+1}^* = z_{k+1}$. Else, by lemma A6.1, either $(\Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}))$ is in X , so that we set $z_{k+1}^i = 0$ ($i = 1, 2, 3$), or $(\Delta_k + \Xi_{k+1}(\omega^{k+1})\mathbf{z}_{k+1}(\omega^{k+1}))$ is in $(X + \xi_{k+1}^i(\omega^{k+1}) + \xi_{k+1}^j(w^{k+1}))$, when we choose $z_{k+1}^i = z_{k+1}^j = 1$ and $z_{k+1}^\ell = 0$, $\ell \neq i, j$. In either case Δ_{k+1} belongs to X .

Proposition A6.2: The integer-valued policy z constructed above is optimal, i.e, it satisfies conditions c1), c2) and c3).

Proof: Condition c1) is trivially satisfied, since $z_k^i(w^k) = z_k^{*i}(w^k)$ whenever z_k^{*i} is integer valued. We now check the feasibility conditions c2). We wish to show that $x_k^1 \geq 0$ and $x_k^2 - x_k^3 \geq 0$. Suppose this were not true. Then, since z_k^1, z_k^2, z_k^3 equal 0 or 1, we clearly have $x_k^1 \leq -1$ and $x_k^2 + x_k^3 \leq -1$. Since Δ_k lies in X , we conclude that $x_k^{*1} = x_k^1 + p_1 - p_2$ for p_1, p_2 in $(-\frac{1}{2}, \frac{1}{2}]$. Hence, $x_k^{*1} < -1 + \frac{1}{2} + \frac{1}{2} = 0$, which is clearly a contradiction since x_k^{*1} is known to be optimal (and hence feasible).

Similarly, since Δ_k belongs to X , we readily see that:

$$(x_k^{*2}, x_k^{*3}) = (x_k^2, x_k^3) + (p_2, p_3) \quad p_1, p_2 \in (-\frac{1}{2}, \frac{1}{2}]$$

and

$$x_k^{*2} - x_k^{*3} \leq -1 + p_2 - p_3 < -1 + \frac{1}{2} + \frac{1}{2} = 0,$$

which again lead to a contradiction.

Next, we must show that $x_k^2 - x_k^3 \leq 1$. As before, if this were not true we must have $x_k^2 - x_k^3 \geq 2$. Using the same arguments as before, $x_k^{*2} - x_k^{*3} \geq 2 + p_2 - p_3 > 2 - \frac{1}{2} - \frac{1}{2} > 1$, clearly a contradiction.

Finally, we establish the complementary slackness conditions c3). It is enough to show that if $x_k^{*1} = 0$ then $x_k^1 = 0$. In a similar way we must show that $x_k^{*2} - x_k^{*3} = 0(1)$ implies $x_k^2 + x_k^3 = 0(1)$. As before, we have that $x_k^1 = x_k^{*1} - (p_1 - p_2) = p_2 - p_1$ belongs to $(-1, 1)$ so that $x_k^1 = 0$, since x_k^1 is integer-valued. All other cases in c3) can be treated in a similar manner.

At this point, the proof of Proposition A6.5 is evident. Returning to the old state description of the system, if $\mathbf{x} = (x^1, x^2)$ is a point with integer coordinates, then since z , the solution of the linear program (LP), is integer-valued, i.e., belongs to $\{0, 1\}$, and $J_n^{\beta, \gamma}(x^1, x^2)$ is unique for each n , we conclude that:

$$\tilde{J}_n^{\beta, \gamma}(x^1, x^2) = W_n(x^1 - x^2, x^2, 0),$$

for $x^1 \geq 1$, and x^2 in $\{0, 1\}$; furthermore, $\tilde{J}_n^{\beta, \gamma}(x^1, x^2)$ inherits the convexity (with respect to the argument x^1) of $W_n(x^1 - x^2, x^2, 0)$ for every n . Hence, $\tilde{J}^{\beta, \gamma}(x^1, x^2)$ is also convex with respect to x^1 .

Next, from [77, page 56] we have that $W_n^{\beta, \gamma}(x^1, x^2, 0)$ is a piecewise linear function. Furthermore, by using arguments similar to those in Proposition 1, it can be shown that it is an increasing function in x^1 and x^2 , and hence it holds that

$$W_n(x^1 + 1, 1, 0) - W_n(x^1 + 1, 0, 0) \geq W_n(x^1, 1, 0) - W_n(x^1, 0, 0). \quad (\text{A6.6})$$

Proposition A6.2 now follows immediately. Property (A6.6) is known as “super-modularity”; a detailed proof of (A6.6) can be found in Appendix 2.1.

CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

This thesis has been motivated by the advent of large-scale Integrated Service Digital Networks (ISDN's), and has focussed on the study of a class of admission control and routing problems. Specifically, we addressed two problems in the general area of optimal control of queuing systems. The structure of an optimal admission control scheme was derived for a simple blocking node carrying two different traffic types in an ISDN environment. A generalization for certain circuit-switched networks was also proposed. Further, a jointly optimal admission and routing scheme was studied for the case of a network node carrying queueable traffic. These problems were modeled by suitably employing the theory of Markov decision processes, incorporating discounted and average costs that consisted of a blocking penalty in conjunction with cost for queuing delay.

The convexity property of the aforementioned discounted costs – a key issue in the study of such problems – was established by using certain facts from the theory of Linear Programming. Furthermore, a procedure was developed that ascertains the convexity property of discounted costs associated with a class of Markov decision processes that often arise in optimization problems in the context of queuing systems.

Next, we investigated simple numerical procedures for quantifying the blocking behavior of circuit-switched networks. Namely, we developed bounds for the blocking probability associated with each link of the network. Particular attention was given to the case of light traffic – a situation that assumes a special significance in high speed optical networks.

Finally, we focused on the asymptotic behavior of certain circuit-switched nodes and derived strong approximations of the state trajectory of the node when the arrival intensities and the capacities of the links are increasingly large. A conjecture for an arbitrary circuit-switched network was also given.

We conclude this thesis with various possible applications, as well as some directions for future research.

Our analyses in the area of optimal routing and resource allocation problems have thus far have made the simplifying assumption that the imposed costs are weighted sums comprising a blocking penalty or a queuing delay. In practice, however, constrained problems are of particular interest; for instance, the minimization of an average blocking cost is desired under the additional constraint that the average queuing delay does not exceed a prespecified value. We conjecture that optimal control schemes for such cases comprise suitably randomized policies. In a similar vein, modern ISDN's must be capable of handling traffic with dissimilar bandwidth requirements. Computations (fig. 2.5) show that in this case the optimal resource allocation strategy will no longer be characterized by switching curves. It may be possible though, to derive optimal randomized policies of a particularly simple form. For example, a candidate strategy would be a threshold policy along with an appropriate randomization imposed on its boundary. Further extensions include problems incorporating deadlines on the waiting time of messages in the buffer; messages that are queued for duration exceeding a prespecified amount of time are assumed to be lost, incurring at the same time a suitable cost. Simple situations of this sort are studied in [4].

From a practical viewpoint, it is of interest to enlarge the scope of the aforementioned problems by also allowing sub-optimal control policies. Sub-optimal policies are often simple to implement and relatively efficient. Important intuition for devising such policies may be may be obtained from the structural studies of optimal control of the simple queuing systems in Chapters 2 and 6 respectively. Furthermore, it would be of practical interest to consider situations where some parameters of the system under control are unknown and possibly time-varying (e.g., arrival or service rates). In these situations, the theory of estimation and adaptive control may prove useful in determining certain control strategies.

An important and different class of problems that naturally arise at this point

concern the study and appropriate use of efficient numerical methods for evaluating and comparing the performance of specific queuing systems operating under various control regimes. In some instances a queuing network may operate under limiting regimes, for example, in the regime of very small arrival rates (e.g., optical high speed networks), or large arrival rates (e.g., congested networks). Some computational techniques for such cases were presented in Chapters 4 and 5 and many interesting extensions are possible. For example, it would be desirable to attempt to characterize the manner of blocking in high speed optical networks (i.e., under the assumption of very light traffic), and also to determine relatively sharp upper and lower bounds on the blocking probabilities of the various routes comprising such a network. Techniques from the theory of Large Deviations seem to offer hope in these cases. Finally the conjecture for a strong approximation of the state process of an arbitrary circuit-switched network (cf. section 5.5) is still an open and challenging problem.

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