

Essay two addresses the problem of how to measure interest rate yield curve shift more realistically, and how to use these risk measures to hedge the interest rate risk of MBS. We use a Principal Components Analysis (PCA) approach to analyze historical interest rate data, and acquire the volatility factors we need in Heath-Jarrow-Morton interest rate model simulation. Then we propose a hedging algorithm to hedge MBS, based on PA gradient estimators derived upon these PCA factors. Our results show that the new hedging method can achieve much better hedging efficiency than traditional duration and convexity hedging.

Essay three addresses the application a new regression method on credit spread data. Previous research has shown that variables in traditional structural model have limited explanatory power in credit spread regression. We argue that this is partially due to the non-constancy of the credit spread gradients to state variables. We use a Random Coefficient Regression (RCR) model to accommodate this problem. The explanatory power increases dramatically with the new RCR model, without adding new independent variables. This is the first work to address the dependence between credit spread sensitivities and state variables of structural in a systematic way. Also our estimates are consistent with prediction from Merton's structural model.

THREE ESSAYS ON MORTGAGE BACKED SECURITIES:
HEDGING INTEREST RATE AND CREDIT RISKS

By

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Dedicated to

Huixian Jackie Xu, my lovely wife

Li Xueqing and Chen Zhixuan, my parents

Chuan Chen, my son

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All opinions expressed in this dissertation are not Fannie Mae's but mine own.

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LIST OF ABBREVIATIONS

A(t)	Amortization Factor at time t
AGE(t)	Aging Multiplier, a parameter to capture the aging effect in prepayment rate
ARM	Adjustable Rate Mortgage
B(t)	Balance at time t
BM(t)	Burnout Multiplier, a parameter to capture the burnout effect in prepayment rate
C(t)	Cash flow at time t
CMO	Collateralized Mortgage Obligation, a special type of MBS
CPR	Conditional Prepayment Rate, annualized prepayment rate
CS(t)	Credit Spread at maturity t
D(t)	Discounting Factor at time t
$f(0,t)$	Instantaneous forward rate starting from t observed at now
FNMA	Federal National Mortgage Association, also known as Fannie Mae
FD	Finite Difference
FHLMC	Federal Home Loan Mortgage Corporation, also known as Freddie Mac
FRM	Fixed Rate Mortgage
GNMA	Government National Mortgage Association, also known as Ginnie Mae
GLS	Generalized Least Square
GSE	Government-Sponsored Enterprise, mainly refers to Fannie Mae, Freddie Mac.

$H(t)$	Haircut at maturity t
HJM	Heath-Jarrow-Morton interest rate model, defined in Heath et al. [1992]
$IP(t)$	Interest Payment at time t
lev_t	Leverage at time t
LTV	Loan to Value ratio, an 80 LTV loan means the loan amounts for 80% of the property value
MBS	Mortgage-Backed Securities
$MM(t)$	Monthly Multiplier, a parameter to capture the seasonal effect in prepayment rate
$MP(t)$	Mortgage Monthly Payment at time t
OFHEO	Office of Federal Housing Enterprise Oversight, a government agency under Department of Housing and Urban Development, regulator of Fannie Mae and Freddie Mac.
OLS	Ordinary Least Square
PA	Perturbation Analysis
PCA	Principal Components Analysis
PDE	Partial Differential Equation
PMI	Primary Mortgage Insurance
$PP(t)$	Principal Prepayment at time t
$PV(t)$	Present Value of Cash flow at time t
$R(0,t)$	Spot rate for maturity t observed at now
$r(t)$	Short rate at time t
$r_{10}(t)$	10-year rate at time t

RCR	Random Coefficient Regression
REO	Real Estate Owned by the GSEs, in case borrower defaults
RI(t)	Refinance Incentive, a parameter to capture the refinance incentive effect in prepayment rate
SMM	Simple Monthly Mortality, monthly prepayment rate
SP(t)	Scheduled Principal Payment at time t
TPP(t)	Total Principal Payment at time t
Vega	The security price sensitivity to volatility
WAC	Weighted Average Coupon rate for MBS
WAM	Weighted Average Maturity for MBS
w.r.t.	with respect to

Chapter 1

Introduction

Mortgage-backed securities (MBS) have become increasingly important fixed income instruments, both because of their volume and the role they play in fund investment and portfolio management. However, there has not been a very comprehensive set of risk indicators to measure and manage the risks involved with MBS. Hedging the interest rate and credit risk of MBS remains a complicated problem in the fixed income industry. This dissertation develops a set of risk measures for interest rate risk and credit risk, and then attempts to hedge the risks effectively using such risk measures. Specifically, the dissertation consists of three essays addressing the following problems: efficiently estimating these new measures of interest rate risk of MBS, hedging MBS with these new measures, and hedging the credit risk of MBS with advanced models for credit spread regression.

The first essay is mainly positioned to answer the following research questions:

- How to measure the interest rate risk in a more comprehensive approach, rather than simply using the traditional duration¹ and convexity²?

¹ Duration is the first order derivative of the price of a fixed income security to interest rate, expressed as a percentage change, see Fabozzi [2001] for more details.

² Convexity is the second order derivative of the price of a fixed income security to interest rate, expressed as a percentage change, see Fabozzi [2001] for more details.

- How to efficiently estimate the risk measures if more factors are introduced into the measurement problem?

In answering these two questions, based on the results over a broad spectrum of mortgage products, we propose a new mortgage product, which could be attractive to MBS investors and mortgage borrowers.

The second essay tries to answer the following questions:

- What would be a realistic method to measure the term structure shift?
- How can we hedge MBS effectively with these measures, in a general interest rate model framework?

We use Principal Components Analysis (PCA) method to extract the empirical volatility factors of term structure, which provides some justification for the form of possible term structure shifts proposed in the first essay. Then we use the Heath-Jarrow-Morton model to incorporate the factors in developing new risk measures and show that the hedging effectiveness is far better than traditional duration/convexity hedging.

The third essay is related to credit risks MBS issuers incur when they purchase mortgage pool insurance from a third party, and attempts to answer the following questions:

- How to estimate the sensitivity of credit spread in a regression framework more effectively than a simple linear regression model?
- What implication does the model have on traditional structural models for credit spread?

We use a Random Coefficient Regression (RCR) model to build our regression model for credit spread changes. This model has explicit sensitivity measures dependent on state variables. We acquire much better explanatory power with this new model, without adding new independent state variables. Also our model supports the dependence of sensitivity of credit spread on state variables predicted by Merton's structural model (Merton [1974]).

1.1 Efficient Sensitivity Analysis of Mortgage Backed Securities

A mortgage-backed security (MBS) is a security collateralized by residential or commercial mortgage loans. An MBS is generally securitized, guaranteed and issued by three major MBS originating agencies: Ginnie Mae, Fannie Mae, and Freddie Mac. The cash flow of an MBS is generally the collected payment from the mortgage borrower, after the deduction of servicing and guaranty fees. However, the cash flows of an MBS are not as stable as that of a government or corporate coupon bond. Because the mortgage borrower has the prepayment option, mainly exercised when moving or refinancing, an MBS investor is actually writing a call option. Furthermore, the mortgage borrower also has the default option, which is likely to be exercised when the property value drops below the mortgage balance, and continuing mortgage payments would not be economically reasonable. In this case the guarantor is writing the borrower a put option, and the guarantor absorbs the cost. However, the borrower does not always exercise the options whenever it is financially optimal to do so, because there are always non-monetary factors associated with the home, like shelter, sense of stability, etc. And it is also very hard for the borrower to tell whether it is financially optimal to exercise these

options because of lack of complete and unbiased information, e.g., they may not be able to obtain an accurate home price, unless they are selling it. And there are also some other fixed/variable costs associated with these options, such as the commission paid to the real estate agent, the cost to initialize another loan, and the negative credit rating impact when the borrower defaults on a mortgage. All these factors contribute to the complexity of MBS cash flows. In practice, the cash flows are generally projected by complicated prepayment models, which are based on statistical estimation on large historical data sets. Because of the complicated behaviors of the MBS cash flow, due to the complex relationships with the underlying interest rate term structures, and path dependencies in prepayment behaviors, Monte Carlo simulation is generally the only applicable method to price MBS.

Associated with the uncertainty of cash flows are different kinds of risks. Treasury bonds only bear interest rate risk, whereas non-callable corporate bonds carry interest rate and credit risk. MBS are further complicated by prepayment risk (resulting from both voluntary prepayment and default). Thus risk management is especially critical for portfolios with large holdings in MBS. Duration and convexity are the main risk measurements for fixed income portfolio managers. Many practitioners use either the Macaulay duration, or modified duration (Kopprasch [1987]) to capture the MBS price sensitivity with respect to interest rate changes, but these duration measures assume a constant yield and known deterministic prepayment pattern, which is rarely the case in practice. So these two approaches to calculate duration can lead to serious errors when used in hedging. Golub [2001] proposed four different approaches to estimate the

duration: Percent of Price (POP), Option-Adjusted Duration (OAD), Implied Duration, and Coupon Curve Duration (CCD). The first two approaches apply parallel shifts in the yield curve, which is not a very realistic assumption. The latter two approaches require large numbers of previous or current accurate MBS prices that are comparable to the MBS whose duration is to be measured. This might not be practical for on-the-fly pricing and sensitivity analysis. Another drawback of these approaches is that they handle only duration and convexity, but not sensitivity to interest rate volatility. OAD method can estimate the vega (the price sensitivity to volatility) using a finite difference approach, which requires 3 simulations to estimate one gradient: the base, up and down cases. And non-parallel yield curve shifts require more parameters to characterize the shift. Thus, in the setting we consider in Chapter 2, to estimate the duration with respect to yield curve shift of 4 summed harmonic functions would require 9 ($2n+1$, $n=4$) simulations. To estimate vega requires 2 additional simulations. So estimating the duration and vega roughly increases the computational cost by a factor of 10. Calculating convexity would require 75 duration estimators to calculate 25 convexity estimators, increasing the simulation factor to 225. In other words, if one were to use 10,000 replications to estimate the MBS price, over 2.25 million simulations would be required to estimate the various sensitivities. Our work aims to decrease this computational burden dramatically.

Most literature on MBS has concentrated on prepayment model estimations, although some of the recent work has focused on computational efficiency, e.g., dimensionality reduction via Brownian bridge (Caflisch et al. [1997]), and quasi-Monte Carlo (Åkesson and Lehoczky [2000]). However, there is no work that we are aware of

that addresses efficient sensitivity analysis of MBS pricing and hedging. Related work in equities includes Fu and Hu [1995], Broadie and Glasserman [1996], Fu et al. [2000], [2001], and Wu and Fu [2001]. Perhaps the most relevant paper to our work is Glasserman [1999], which applied perturbation analysis (PA) method for caplet price sensitivity analysis. Yet most of these models involve only a single exercise decision with a one-time payoff, whereas an MBS is a pool of homogenous mortgages rather than an individual mortgage loan. So the cash flows exist until the maturity of the collateral, and they are highly path dependent, which makes sensitivity analysis of MBS more complicated.

The other relevant body of research literature analyzes the duration of different mortgage products. We know that adjustable rate mortgage (ARM) products will have a different response from fixed rate mortgage (FRM) products, due to ARMs' coupon-reset plan and different prepayment function. In a series of papers, Kau et al.[1990,1992,1993] priced the ARMs and performed some sensitivity analysis. Chiang [1997] applied a simple simulation scheme to estimate the modified duration of ARMs. Stanton [1999] calculated the duration of different indexed ARMs via a scheme like Kau's. However, most of these papers are based on solving models based on partial differential equation (PDE), using simplified assumptions that often miss essential features that can be captured by Monte Carlo simulation. The three major drawbacks of these models that make them impractical in the mortgage industry are the following:

- They assume borrowers exercise the prepayment option only when it is financially optimal to do so. This ignores the fact that people routinely prepay even in financially

adverse environments, e.g., house sales. Also seasoning and burnout effects are not considered.

- By solving the PDE, one can only obtain a set of present values of the MBS along the interest rate axis. By applying the finite difference method, duration of the MBS could be acquired. However, it provides no information about the discounting factor and cash flows along the time horizon. So you will have no information about how the interest shift affects different components of the present value.
- The PDE method generally uses one-factor interest rate model, which applies the same interest rate both for discounting and for the prepayment model, which ignores the difference between short-term and long-term interest rates.

In the first essay, we apply perturbation analysis (PA) to estimate the sensitivities of MBS. Our work makes the following contribution to MBS literature:

- We decompose any interest rate term structure change into four Fourier-like series, which is based on Fourier cosine series, and can better measure the yield curve shift;
- We derive PA estimators for these Fourier-like factors, as well as interest rate volatility, which can largely save computation effort. In our example, we calculate 5 duration estimators and 16 convexity estimators in our simulation, which would require 155 simulations using a conventional simulation scheme.
- Based on our comprehensive analysis of the sensitivity measures we calculated for a full spectrum of mortgage product, we propose a new mortgage product, which can potentially benefit both the MBS investor and mortgage borrower.

This essay is organized in the following manner. Section 1 describes the problem setting. We then derive the framework for PA in a general setting in section 2, without restrictions to any specific interest rate model or prepayment model. Then we consider the well-known Hull-White interest rate model (Hull and White [1993]) and a common prepayment model to derive the corresponding PA sensitivities for FRM and ARM products in section 3, assuming the shock of interest rate term structure takes the form of a series of trigonometric polynomial functions. Section 4 presents numerical examples, in which we compare the performance of FD and PA estimators, indicating that the PA estimator is at least as good as the FD estimator, while the computation cost is reduced dramatically. Section 5 gives the insights from our simulation results. Section 6 gives conclusions.

1.2 Hedging MBS in HJM Framework

There is a large body of literature on hedging with different interest risk measures, like first-order hedging with duration (Ilmanen [1992]), second order hedging with convexity (Kahn and Lochoff [1990], Lacey and Nawalkha [1993]), principal components hedging (Golub and Tilman [1997]), key rates hedging (Ho [1992]), level/slope/curvature hedging (Willner [1996]), etc. Yet there has not been a unifying effort in combining hedging the term structure together with hedging volatility factors.

This essay tries to extract the empirical volatility factors from historical term structure data, via principal components analysis (PCA), and apply these factors in a HJM framework for pricing MBS, while deriving the risk measures for hedging MBS. It makes the following contribution in the MBS literature:

- The first paper to hedge MBS with PCA factors empirically extracted from historical interest rate data;
- Hedging efficiency is proved to increase significantly, compared with traditional duration/convexity hedging.

This essay is organized in the following way. Section 1 gives the motivation for this research question. Section 2 describes the interest rate data set and PCA method we used to extract the volatility factors. Section 3 applies these factors in interest rate simulation within a HJM framework. Section 4 derives the PA estimators in the HJM framework. Section 5 gives the hedging algorithm for MBS, and Section 6 gives the performance analysis of this hedging method. Section 7 concludes the essay.

1.3 Hedging Credit Risk of MBS: A Random Coefficient

Approach

In order to hedge the credit risk of MBS, the MBS issuer sometimes needs to purchase pool insurance from a third party, beyond the protection of mortgage collateral, and primary mortgage insurance. In this case, it is important to model the credit risk of the third party. Recently there has been increased interest in some research papers to use regression method to determine what factors affect credit spread. Most of the papers, which use simple linear regression, found that variables in structural models lack explanatory power in such regression. We argue that the problem partially results from non-constant credit spread sensitivities to state variables.

We try to overcome the problem by proposing a Random Coefficient Regression (RCR) model. We collected data from multiple database, and constructed our data set. Our regression results show that our assumption of non-constancy of credit spread sensitivities is correct. As a result of improved regression, we improved adjusted R^2 to 28%, compared with 8% adjusted R^2 for a simple linear regression approach, using the same set of independent variables. Another important result of our RCR model is that it validated the relationship between credit spread sensitivities and state variables, which has been predicted by Merton's model.

This essay makes the following research contributions to the finance literature:

- The first paper to use the RCR method on credit spread data;

- The first paper to explicitly build a dependence relationship between credit spread sensitivity and state variables;
- The first paper to empirically validate the dependence relationship between credit spread sensitivity and state variables predicted by a structural model, such as Merton's model.

This essay is organized in the following way. Section 1 gives the motivation for this research question. Section 2 describes several previous papers on this topic. Section 3 gives a brief introduction to the Random Coefficient Regression model. Section 4 applies this model to changes of credit spread. Section 5 gives the data description used in the regression, and Section 6 gives the results analysis of this regression method. Section 7 concludes the essay.

Chapter 2

Efficient Sensitivity Analysis of MBS

2.1 Problem Setting

Generally the price of any security can be written as the net present value (NPV) of its discounted cash flows. Specifying the price of an MBS (here we consider only the pass-through MBS¹) is as follows:

$$P = E[V] = E\left[\sum_{t=0}^M PV(t)\right] = E\left[\sum_{t=0}^M d(t)c(t)\right], \quad (2.1)$$

where P is the price of the MBS,

V is the value of the MBS, which is a random variable, dependent on the realization of the economic scenario,

$PV(t)$ is the present value for cash flow at time t ,

$d(t)$ is the discounting factor at time t ,

$c(t)$ is the cash flow at time t ,

M is the maturity of the MBS.

¹A pass-through MBS is an MBS that passes through the principal and interest payments collected from a mortgage pool, minus the guaranty fee and servicing fee, to the MBS investor directly. This is in contrast to Collateralized Mortgage Obligations (CMOs), which have multiple tranches and pay the principal payments according to the seniorities of tranches. In this essay, we assume that mortgages in the MBS pool are homogenous.

Monte Carlo simulation is used to generate cash flows on many paths. By the strong law of large numbers, we have the following:

$$E[V] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N V_i, \quad (2.2)$$

where V_i is the value calculated out in path i .

The calculation of $d(t)$ is found from the short-term (risk-free) interest rate process,

$$d(t) = d(0,1)d(1,2)d(t-1,t) = \prod_{i=0}^{t-1} \exp(-r(i)\Delta t) = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\Delta t\right]\right\}, \quad (2.3)$$

where $d(i, i+1)$ is the discounting factor for the end of period $i+1$ at the end of period i ;

$r(i)$ is the short term rate used to generate $d(i, i+1)$, observed at the end of period i ;

Δt is the time step in simulation, generally monthly, i.e. $\Delta t = 1$ month.

An interest rate model is used to generate the short term-rate $r(i)$; then $d(t)$ is instantly available when the short-term rate path is generated.

The difficult part is to generate $c(t)$, the path dependent cash flow of MBS for month t , which is observed at the end of month t . From chapter 19 of Fabozzi [1993], we have the following formula for $c(t)$:

$$\begin{aligned} c(t) &= MP(t) + PP(t) = TPP(t) + IP(t); \\ MP(t) &= SP(t) + IP(t); \\ TPP(t) &= SP(t) + PP(t); \end{aligned} \quad (2.4)$$

where $MP(t)$: Scheduled Mortgage Payment for month t ;

$TPP(t)$: Total Principal Payment for month t ;

$IP(t)$: Interest Payment for month t ;

$SP(t)$: Scheduled Principal Payment for month t ;

$PP(t)$: Principal Prepayment for month t .

These quantities are calculated as follows:

$$\begin{aligned}MP(t) &= B(t-1) \left(\frac{WAC/12}{1 - (1 + WAC/12)^{-WAM+t}} \right); \\IP(t) &= B(t-1) \frac{WAC}{12}; \\PP(t) &= SMM(t)(B(t-1) - SP(t)); \\B(t) &= B(t-1) - TPP(t); \\SMM(t) &= 1 - \sqrt[12]{1 - CPR(t)};\end{aligned} \tag{2.5}$$

$B(t)$: The principal balance of MBS at end of month t ;

WAC^2 : Weighted Average Coupon rate for MBS;

WAM^3 : Weighted Average Maturity for MBS;

$SMM(t)$: Single Monthly Mortality for month t , observed at the end of month t ;

$CPR(t)$: Conditional Prepayment Rate for month t , observed at the end of month t .

In Monte Carlo simulation, along the sample path, $CPR(t)$ is the primary variable to be simulated. Everything else can be calculated out once $CPR(t)$ is known. Different prepayment models offer different $CPR(t)$, and it is not our goal to derive a new

² WAC is the weighted average mortgage rate for a mortgage pool, weighted by the balance of each mortgage.

³ WAM is the weighted average maturity in month for a mortgage pool, weighted by the balance of each mortgage.

prepayment model or compare existing prepayment models. Instead, our concern is, given a prepayment model, how can we efficiently estimate the price sensitivities of MBS against parameters of interest? Generally different prepayment models will lead to different sensitivity estimates, so it is at the user's discretion to choose an appropriate prepayment function, as our method for calculating the "Greeks" is universally applicable.

2.2 Derivation of General PA Estimators

If P , the price of the MBS, is a continuous function of the parameter of interest, say θ , and assuming the interchange of expectation and differentiation is permissible⁴, we have the following PA estimator by differentiating both sides of (2.1):

$$\frac{dP(\theta)}{d\theta} = \frac{dE[V(\theta)]}{d\theta} = E\left[\frac{d\sum_{t=1}^M PV(t, \theta)}{d\theta}\right] = E\left[\sum_{t=1}^M \frac{dPV(t, \theta)}{d\theta}\right], \quad (2.6)$$

$$\frac{d(PV(t, \theta))}{d\theta} = \frac{\partial d(t, \theta)}{\partial \theta} c(t, \theta) + \frac{\partial c(t, \theta)}{\partial \theta} d(t, \theta).$$

Now we have reduced the original problem from estimating the gradient of a sum function to estimating the sum of a bunch of gradients. Actually now we only need to estimate two gradient estimators, $\frac{\partial c(t, \theta)}{\partial \theta}$ and $\frac{\partial d(t, \theta)}{\partial \theta}$, at each time step.

⁴ Strictly speaking, sample pathwise continuity of price function with respect to θ will result in the interchange being valid.

2.2.1 Gradient Estimator for Cash Flow

We first derive $\frac{\partial c(t, \theta)}{\partial \theta}$. To simplify notation, we write $c(t)$ for $c(t, \theta)$.

A simplified expression for $c(t)$ is derived from (2.4) and (2.5) as follows:

$$\begin{aligned}
 c(t) &= MP(t) + PP(t) = MP(t) + [B(t-1) - SP(t)]SMM(t) \\
 &= MP(t) + \{B(t-1) - [MP(t) - IP(t)]\}SMM(t) \\
 &= MP(t)(1 - SMM(t)) + B(t-1)\left(1 + \frac{WAC}{12}\right)SMM(t) \\
 &= B(t-1)\frac{WAC/12}{1 - (1 + WAC/12)^{-WAM+t}}[1 - SMM(t)] + B(t-1)\left(1 + \frac{WAC}{12}\right)SMM(t) \\
 &= B(t-1)\{A(t)[1 - SMM(t)] + gSMM(t)\},
 \end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
 A(t) &= \frac{WAC/12}{1 - (1 + WAC/12)^{-WAM+t}}, \\
 g &= \left(1 + \frac{WAC}{12}\right).
 \end{aligned} \tag{2.8}$$

Then we can derive the gradient for $c(t)$, if WAC and t are independent⁵ of θ .

$$\frac{\partial c(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta} \{A(t)[1 - SMM(t)] + gSMM(t)\} + \frac{\partial SMM(t)}{\partial \theta} B(t-1)[-A(t) + g]. \tag{2.9}$$

This leads to recursive equations for calculation of the above gradient estimator from (2.5) and (3.2):

$$\begin{aligned}
 TPP(t) &= c(t) - IP(t) = c(t) - \frac{WAC}{12}B(t-1); \\
 B(t) &= B(t-1) - TPP(t) = B(t-1)\left(1 + \frac{WAC}{12}\right) - c(t) = B(t-1)g - c(t); \\
 \frac{\partial B(t)}{\partial \theta} &= \frac{\partial B(t-1)}{\partial \theta}g - \frac{\partial c(t)}{\partial \theta}.
 \end{aligned} \tag{2.10}$$

⁵ A fixed Rate Mortgage (FRM) would satisfy this assumption; however an Adjustable Rate Mortgage (ARM) will not, so we derive the gradient estimator for ARMs later in section 4.

Assuming we know that the initial balance is not dependent as θ , we have the initial conditions:

$$\begin{aligned}\frac{\partial B(0)}{\partial \theta} &= 0; \\ \frac{\partial c(1)}{\partial \theta} &= \frac{\partial SMM(1)}{\partial \theta} B(0)(-A(1) + g).\end{aligned}\tag{2.11}$$

This leads to the following

$$\frac{\partial B(1)}{\partial \theta} = \frac{\partial B(0)}{\partial \theta} g - \frac{\partial c(1)}{\partial \theta},\tag{2.12}$$

$$\frac{\partial c(2)}{\partial \theta} = \frac{\partial B(1)}{\partial \theta} \{A(2)(1 - SMM(2) + g SMM(2))\} + \frac{\partial SMM(2)}{\partial \theta} B(1)(-A(2) + g),\tag{2.13}$$

$$\frac{\partial c(t)}{\partial \theta}, t = 1, 2, \dots, M.$$

Thus the problem of calculating the gradient estimator of cash flow $c(t)$ is reduced to calculating:

$$\frac{\partial SMM(t)}{\partial \theta}, t = 1, \dots, M.$$

Since

$$SMM(t) = 1 - \sqrt[12]{1 - CPR(t)},$$

we have

$$\frac{\partial SMM(t)}{\partial \theta} = \frac{1}{12} (1 - CPR(t))^{-\frac{11}{12}} \frac{\partial CPR(t)}{\partial \theta}.\tag{2.14}$$

As discussed earlier, generally $CPR(t)$ is given in the form of a prepayment function, and there are four main types of prepayment functions (Fabozzi [2000]):

1. Arctangent Model: (An example from the Office of Thrift Supervision (OTS).)

$$CPR(t) = 0.2406 - 0.1389 \arctan\left(5.9518\left(1.089 - \frac{WAC}{r_{10}(t)}\right)\right). \quad (2.15)$$

2. $CPR(S,A,B,M)$ Model:

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t); \quad (2.16)$$

where $RI(t)$ is refinancing incentive;

$AGE(t)$ is the seasoning multiplier;

$MM(t)$ is the monthly multiplier, which is constant for a certain month;

$BM(t)$ is the burnout multiplier.

3. Prepayment models incorporating macroeconomic factors, i.e., the health of economics, housing market activity, etc.
4. Prepayment models for individual mortgages.

For the last two types of prepayment models, we do not have any explicitly stated functional forms, mainly because they are proprietary models in the mortgage industry. But since our approach is general for any type of prepayment function, we can derive the derivatives once we are given an explicit form for the prepayment function.

We would like to make one claim here: the $CPR(t)$ model we mentioned and we are going to use in Chapter 2 and Chapter 3 includes both voluntary prepayment (refinance, house turnover, and cash out) and involuntary prepayment (default). Because default only makes about 1% of total prepayment, and generally MBS issuer will guarantee the principal payment to the investor, in case borrower defaults, it is a reasonable not to model default separately. However, when analyzing MBS backed by high default loans, such as subprime mortgages, it is desirable to model voluntary prepayment and default separately.

2.2.2 Gradient Estimator for Discounting Factor

We have derived the gradient estimator of cash flow with respect to parameter θ .

Next, we derive the gradient estimator of the discounting factor $d(t)$.

We know that the discounting factor takes the following form from section 2, when the option adjusted spread (OAS) is not considered. For simplification, we write $d(t)$ as for $d(t, \theta)$:

$$d(t) = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\right]\Delta t\right\}. \quad (2.17)$$

Differentiating with respect to θ .

$$\frac{\partial d(t)}{\partial \theta} = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\right]\Delta t\right\} \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right)\Delta t = d(t) \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right)\Delta t. \quad (2.18)$$

From the gradient estimators for cash flow and discounting factor, we can easily get the gradient estimator of $PV(t)$:

$$\frac{d(PV(t, \theta))}{d\theta} = \frac{\partial d(t, \theta)}{\partial \theta} c(t, \theta) + \frac{\partial c(t, \theta)}{\partial \theta} d(t, \theta). \quad (2.19)$$

The last step would be to apply a specific prepayment model and interest rate model to arrive at the actual implemented gradient estimators. To illustrate the procedure, we carry out this exercise in its entirety for one setting in the following section.

2.3 Applying the Gradients

We choose our interest model to be the one-factor Hull-White (Hull and White [1990]) model, for its simplicity and easy calibration to market term structure. For the prepayment model, we consider a $CPR(S,A,B,M)$ model.

2.3.1 Interest Model Setup

In this section, we briefly discuss the model and the simulation scheme.

In the one-factor Hull-White interest rate model, the underlying process for the short-term rate $r(t)$ is given by

$$dr(t) = (\varphi(t) - ar(t))dt + \sigma dB(t), \quad (2.20)$$

where $B(t)$: a standard Brownian motion;

a : mean reverting speed, constant;

σ : standard deviation, constant;

$\varphi(t)$: chosen to fit the initial term structure, which is determined by

$$\varphi(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \quad (2.21)$$

$f(0,t)$: the instantaneous forward rate, which is determined by

$$R(0,t) = \frac{1}{t} \int_0^t f(0,u)du, \quad (2.22)$$

Differentiating both sides, with respect to t , we have

$$f(0,t) = t \frac{\partial R(0,t)}{\partial t} + R(0,t), \quad (2.23)$$

where $R(0,t)$: the continuous compounding interest rate from now to time t , i.e. the term structure.

In order to simplify the simulation process, the model can be re-parameterized from its original to the following:

$$dx(t) = -a(t)x(t)dt + \sigma dB(t), x(0) = 0; \quad (2.24)$$

$x(t)$ is determined by

$$a(t) = r(t) - x(t) = f(0, t) + \frac{\sigma^2}{2a} (1 - e^{-at})^2. \quad (2.25)$$

The process $x(t)$ is called an Ornstein-Uhlenbeck process, and its solution is given by

$$x(t) = \sigma e^{-at} \int_0^t e^{au} dB(u), \quad (2.26)$$

which is a Gaussian Markov process, and can also be represented as

$$x(t) = \sigma e^{-at} W\left(\frac{e^{2at} - 1}{2a}\right), \quad (2.27)$$

where $\{W(t), t \geq 0\}$ is also a Brownian motion.

In this case, the interest rate $r(t)$ can be represented in the following form:

$$r(t) = F(a(t) + g(t)W_{h(t)}), \quad (2.28)$$

where $a, g: R_+ \rightarrow R$ are continuous functions, and the functions $F: R \rightarrow R$ and $h: R_+ \rightarrow R$ are strictly increasing and continuous. From above we can see that

$$\begin{aligned} F(x) &= x; \\ a(t) &= f(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2; \\ g(t) &= \sigma e^{-at}; \\ h(t) &= \frac{e^{2at} - 1}{2a}. \end{aligned} \quad (2.29)$$

To simulate $r(t)$ given by above, we will first simulate

$$x(t) = g(t)W_{h(t)},$$

which is a Gaussian Markov process, and then compute the short-term interest rate by

$$r(t) = F(a(t) + x(t)).$$

For calculating the price of MBS, the short-term rate is not sufficient; the long-term rate process is also required, especially the 10-year Treasury rate, which is a deterministic function of $r(t)$ in the Hull-White model. Generally this is the case for short-term rate models, but not true for more complicated interest rate models, e.g., the HJM (Heath, Jarrow and Morton [1992]) model and the LIBOR forward rate model (Jamshidian[1997]). The long-term rate $R(t, T)$ is calculated from the following, :

$$\begin{aligned} P(t, T) &= e^{-R(t, T)(T-t)} = A(t, T)e^{-B(t, T)r(t)}; \\ B(t, T) &= \frac{1 - e^{-a(T-t)}}{a}; \\ \ln A(t, T) &= \ln \frac{P(0, T)}{P(0, t)} - B(t, T) \frac{\partial \ln P(0, t)}{\partial t} - \frac{\sigma^2}{4a^3} (e^{-aT} - e^{-at})^2 (e^{2at} - 1). \end{aligned} \quad (2.30)$$

$P(t, T)$ is the zero coupon bond price at time t , with face value \$1, matured at T .

Thus we can derive the $R(t, T)$ as following:

$$R(t, T) = -\frac{\ln A(t, T) - B(t, T)r(t)}{(T - t)}. \quad (2.31)$$

The standard (forward) path generation method for generating $x(t)$ is given by

$$\begin{aligned} x(t_{i+1}) &= \frac{g(t_{i+1})}{g(t_i)} g(t_i) W(h(t_i)) + g(t_{i+1}) [W(h(t_{i+1})) - W(h(t_i))] \\ &= \frac{g(t_{i+1})}{g(t_i)} x(t_i) + g(t_{i+1}) \sqrt{W(h(t_{i+1})) - W(h(t_i))} z_{i+1}, \end{aligned} \quad (2.32)$$

where $\{z_i\}$ is a series of independent standard normal random variables. In the special case where $x(t)$ is from the Hull-White model, we have

$$x(t_{i+1}) = e^{-a\Delta t_i} x(t_i) + \sigma \sqrt{\frac{1 - e^{-2a\Delta t_i}}{2a}} z_{i+1}, \quad (2.33)$$

where $\Delta t_i = t_{i+1} - t_i$.

2.3.2 Trigonometric Polynomial Shocks

There are multiple factors in the interest rate model that can change and affect the cash flows and discounting factor along the simulation path. The major changes could be the initial term structure $R(0,t)$ and the volatility σ .

The most common assumption for term structure change is a parallel shift on all maturities. However, this is often not an adequate model for the real world, where a shift in the term structure can take any shape. For example, short-term rates and long-term rates may change in opposite directions rather than in parallel. We consider a Fourier series decomposition of the term structure shift.

Our domain of concern is interest rates from time 0 to 30 years, since most mortgages are amortized in a 30-year term. So for example, we could assume the shift of term structure takes the following form:

$$\Delta R(0,t) = \sum_{n=0}^{\infty} \Delta_n \cos\left(\frac{n\pi t}{30}\right), \quad (2.34)$$

where Δ_n is the magnitude for the n^{th} Fourier function. Figure 4.1 depicts the first four trigonometric polynomial series. ($n=0,1,2,3$), which is all that we will consider in our model. When $n=0$, the shift is just like a parallel shift in term structure. When $n=1$, the short-term and long-term rates move in opposite directions. When $n=2$, the short-term and long-term rates move in the same direction, while the middle-term rate moves in the opposite direction. Thus we decompose any shift in the term structure into the Fourier functions by Fourier transform. If we have previously calculated the gradients with respect to the magnitude of each trigonometric polynomial function, we can apply these

gradients and get the corresponding changes in the cash flows and discounting factors, and hence the change in MBS prices.

The Fourier series have a serious drawback: they treat short-term rates the same as long-term rates. However, from experience, we know that the short-term rates generally change more frequency than long-term rates. So we would like to change the shape of the trigonometric polynomial function, which will concentrate more on the short-term rates, and keep the long-term rates relatively stable. The modified Fourier function that we adopt takes the following form:

$$\Delta R(0,t) = \sum_{n=0}^{\infty} \Delta_n \cos(n\pi(1 - e^{-t/T_0})), \quad (2.35)$$

where T_0 is a user-specified parameter of the modified Fourier shifts. The smaller T_0 is, the more likely short rates and long rates are going to act differently. See Figure 2.2 for the modified Fourier functions, where $T_0=10$. Comparing Figures 2.1 and 2.2, the modified Fourier series concentrate more on the changes with maturities less than T_0 , which is both desirable and easier for analytical purposes.

For a Fourier cosine series that has the following functional form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right), \quad (2.36)$$

the coefficients are given by a Fourier cosine transform:

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos\left(\frac{2n\pi t}{T}\right) dt, \quad n = 0,1,2,\dots \quad (2.37)$$

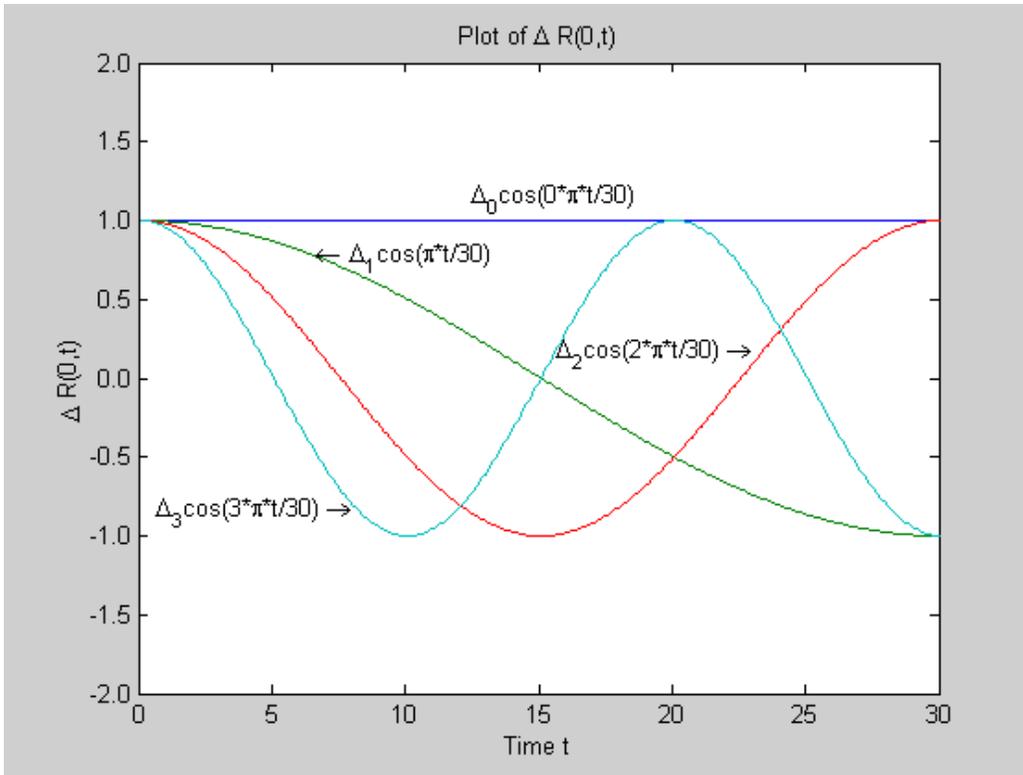


Figure 2.1 $\Delta R(0,t)$ with Original Fourier series

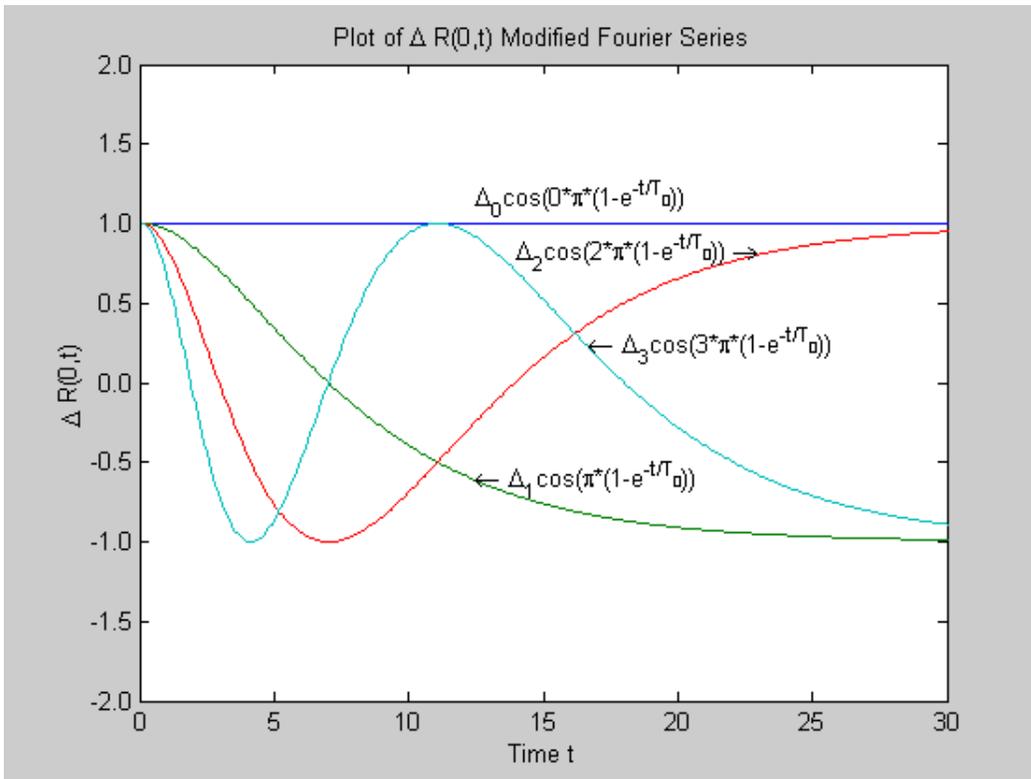


Figure 2.2 $\Delta R(0,t)$ with $T_0=10$ modified Fourier series

For our modified Fourier series, perform the following change of variables:

$$\frac{t'}{30} = (1 - e^{-t/T_0})$$

and substitute into the expression of $\Delta R(0,t)$ to get

$$\Delta R(0,t') = \sum_{n=0}^{\infty} \Delta_n \cos\left(\frac{2n\pi t'}{60}\right), \quad (2.38)$$

which is a standard Fourier cosine series, and we can use a Fourier transform to estimate the coefficients. In computer simulation, t is a vector of real time points, evenly distributed with sample function value $\Delta R(0,t)$, and t' is the mapped time point in a new time scale, which is not evenly distributed, with the same sample function value $\Delta R(0,t')$. However, in order to utilize the discrete cosine transform function provided in mathematical libraries, we need to resample $\Delta R(0,t')$ at even time intervals. This is carried out by interpolating the function of $\Delta R(0,t')$ on the t time scale. Figure 2.3 shows a sample of $\Delta R(0,t)$, $\Delta R(0,t')$ re-sampled on t , the coefficients estimated on the re-sampled $\Delta R(0,t')$, and the reconstructed Fourier series of $\Delta R(0,t)$.

From Figure 2.3, if we look at the upper left and lower right sub-figures, we can see that the reconstructed term structure matches the original sample very well, which validates our method for estimating the coefficients of the modified Fourier series.

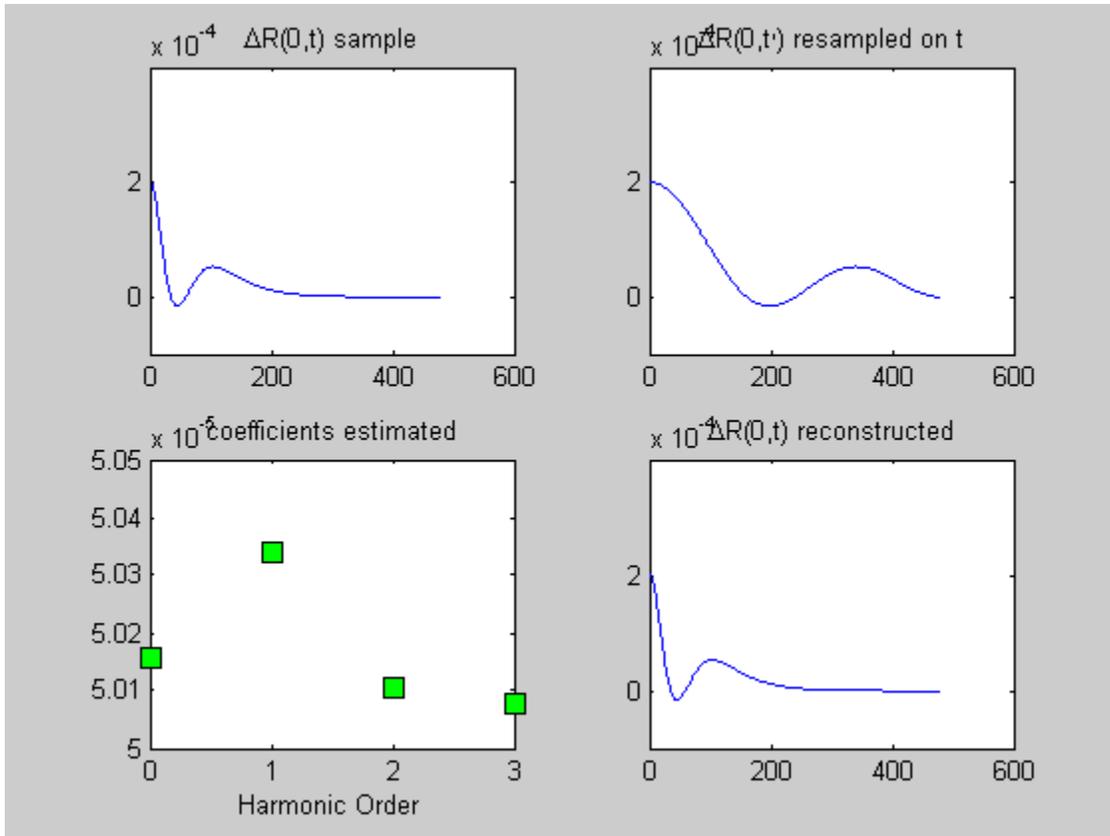


Figure 2.3 Coefficients Estimation for Modified Fourier series

2.3.3 Derivation of Gradients with respect to Modified Fourier Functions

Our major task in this section is to derive the gradient estimator with respect to to specific parameters in the interest rate model and prepayment model. Specifically, for the former, we are interested in the parameters of the modified Fourier functions ($\Delta_n, n=0, 1, 2, 3$).

First we derive the discounting factor gradient estimator. From (3.12), we know that in order to derive $\frac{\partial d(t)}{\partial \theta}$, we must first derive $\frac{\partial r(i)}{\partial \theta}$, $i=0, \dots, t-1$. Let us recall that in section 4.1, we have the following simulation scheme for short term rate $r(t)$:

$$r(t) = a(t) + x(t) .$$

So

$$\frac{\partial r(t)}{\partial \theta} = \frac{\partial a(t)}{\partial \theta} + \frac{\partial x(t)}{\partial \theta}, \quad (2.39)$$

where $\frac{\partial a(t)}{\partial \theta}$ and $\frac{\partial x(t)}{\partial \theta}$ are determined as the following in Hull-White model:

$$\begin{aligned} \frac{\partial a(t)}{\partial \theta} &= \frac{\partial f(0,t)}{\partial \theta}, \\ \frac{\partial x(t)}{\partial \theta} &= 0. \end{aligned} \quad (2.40)$$

We also know the relationship between $f(0,t)$ and $R(0,t)$ from (2.23), so $\frac{\partial f(0,t)}{\partial \theta}$

can be derived as:

$$\begin{aligned} \frac{\partial f(0,t)}{\partial \theta} &= \frac{\partial \left(t \frac{\partial R(0,t)}{\partial t} \right)}{\partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= \frac{\partial t}{\partial \theta} \frac{\partial R(0,t)}{\partial t} + t \frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= t \frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta}. \end{aligned} \quad (2.41)$$

Considering the changes in $R(0,t)$ which takes the form as in (2.38), we can get

the derivatives of $R(0,t)$ (θ taken to be Δ_n):

$$\begin{aligned} \frac{\partial R(0,t)}{\partial \Delta_n} &= \cos(n\pi(1 - e^{-t/T_0})), \\ \frac{\partial R^2(0,t)}{\partial t \partial \Delta_n} &= -\sin(n\pi(1 - e^{-t/T_0})) \frac{n\pi(-e^{-t/T_0})}{-T_0} = \sin(n\pi(1 - e^{-t/T_0})) \frac{n\pi(e^{-t/T_0})}{T_0}. \end{aligned} \quad (2.42)$$

We can get the derivatives of $r(i)$:

$$\frac{\partial r(i)}{\partial \Delta_n} = \frac{\partial f(0,t)}{\partial \Delta_n} = -t \sin(n\pi(1 - e^{-t/T_0})) \frac{n\pi(e^{-t/T_0})}{T_0} + \cos(n\pi(1 - e^{-t/T_0})). \quad (2.43)$$

And gradient estimator for discounting factor is also obtained, applying (2.18).

Next, we are going to derive the cash flow gradient estimator with respect to Δ_n .

From our derivation in section 3, we know that in order to get $\frac{\partial c(t)}{\partial \theta}$, we need to derive

$\frac{\partial CPR(t)}{\partial \theta}$ first. We use the second type of prepayment function, among the four

described in section 3. An example for this type of prepayment model is available from

the sample code at the Numerix homepage <http://www.numerix.com>.

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t); \quad (2.44)$$

where

$$RI(t) = 0.28 + 0.14 \arctan(-8.571 + 430(WAC - r_{10}(t-1)));$$

$$AGE(t) = \min(1, \frac{t}{30});$$

$$MM(t) = [0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.1, 1.18, 1.22, 1.23, 0.98],$$

starting from January, ending in December;

$$BM(t) = 0.3 + 0.7 \frac{B(t-1)}{B(0)};$$

$r_{10}(t)$ is the 10 year rate, observed at the end of period t , which is highly correlated with the prevailing mortgage rate.

From the formulas, only $RI(t)$ and $BM(t)$ depend on θ , when θ is not time t . Thus

we have the following formula for $\frac{\partial CPR(t)}{\partial \theta}$:

$$\frac{\partial CPR(t)}{\partial \theta} = \frac{\partial RI(t)}{\partial \theta} AGE(t)MM(t)BM(t) + RI(t)AGE(t)MM(t) \frac{\partial BM(t)}{\partial \theta}; \quad (2.45)$$

where

$$\frac{\partial RI(t)}{\partial \theta} = 0.14 \frac{(-430)}{1 + (-8.571 + 430(WAC - r_{10}(t-1)))^2} \frac{\partial r_{10}(t-1)}{\partial \theta};$$

$$\frac{\partial BM(t)}{\partial \theta} = 0.7 \frac{\partial B(t-1)}{\partial \theta} \frac{1}{B(0)}.$$

(2.46)

$\frac{\partial B(t)}{\partial \theta}$ is available, when $\frac{\partial c(t)}{\partial \theta}$ is calculated out, so the problem is reduced to

calculating $\frac{\partial r_{10}(t)}{\partial \theta}$. In the one-factor Hull-White framework, as we have discussed in

section 2.3.1, the long-term rate is a deterministic function of $r(t)$, so substituting $T=t+10$

for (2.30), we have

$$\begin{aligned}
P(t, t+10) &= e^{-R(t, t+10)(t+10-t)} = A(t, t+10)e^{-B(t, t+10)r(t)}; \\
B(t, t+10) &= \frac{1 - e^{-a(t+10-t)}}{a} = \frac{1 - e^{-10a}}{a}; \\
\ln A(t, t+10) &= \ln \frac{P(0, t+10)}{P(0, t)} - B(t, t+10) \frac{\partial \ln P(0, t)}{\partial t} - \frac{\sigma^2}{4a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1) \\
&= \ln P(0, t+10) - \ln P(0, t) - B(t, t+10) \frac{\partial \ln P(0, t)}{\partial t} - \frac{\sigma^2}{4a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1) \\
&= -R(0, t+10)(t+10) + R(0, t)t + \frac{1 - e^{-10a}}{a} R(0, t) - \frac{\sigma^2}{4a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1).
\end{aligned} \tag{2.47}$$

Since

$$\begin{aligned}
r_{10}(t) &= R(t, t+10) = -\frac{\ln A(t, t+10) - B(t, t+10)r(t)}{10} \\
&= -\frac{-R(0, t+10)(t+10) + R(0, t)t + \frac{1 - e^{-10a}}{a} R(0, t) - \frac{\sigma^2}{4a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1) - \frac{1 - e^{-10a}}{a} r(t)}{10},
\end{aligned} \tag{2.48}$$

$\frac{\partial r_{10}(t)}{\partial \theta}$ takes the following form, when θ is independent of σ and t :

$$\frac{\partial r_{10}(t)}{\partial \theta} = -\frac{-(t+10) \frac{\partial R(0, t+10)}{\partial \theta} + (t + \frac{1 - e^{-10a}}{a}) \frac{\partial R(0, t)}{\partial \theta} - \frac{1 - e^{-10a}}{a} \frac{\partial r(t)}{\partial \theta}}{10}. \tag{2.49}$$

Thus we have derived $\frac{\partial r_{10}(t)}{\partial \theta}$ as a function of $\frac{\partial R(t)}{\partial \theta}$ and $\frac{\partial r(t)}{\partial \theta}$ derived earlier.

2.3.4 Derivation of Gradients with respect to Volatility: Vega

The derivation is straightforward as in section 2.3.3; all we need to do is to substitute θ with σ , instead of Δ_n . In order to get $\frac{\partial d(t)}{\partial \sigma}$, we must first derive $\frac{\partial r(i)}{\partial \sigma}$.

Following the same logic in (2.40), we can get the vega of $r(t)$:

$$\begin{aligned}\frac{\partial a(t)}{\partial \sigma} &= \frac{\sigma}{a^2} (1 - e^{-at})^2, \\ \frac{\partial x(t)}{\partial \sigma} &= e^{-at} W\left(\frac{e^{2at} - 1}{2a}\right), \text{ so} \\ \frac{\partial r(t)}{\partial \sigma} &= \frac{\sigma}{a^2} (1 - e^{-at})^2 + e^{-at} W\left(\frac{e^{2at} - 1}{2a}\right).\end{aligned}\tag{2.50}$$

And vega of $d(t)$ would be:

$$\frac{\partial d(t)}{\partial \sigma} = d(t) \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \sigma}\right) \Delta t.\tag{2.51}$$

Now we derive $\frac{\partial c(t)}{\partial \sigma}$, which would require us to derive $\frac{\partial CPR(t)}{\partial \sigma}$ first, which has

the same form as in (2.45), while $\frac{\partial r_{10}(t)}{\partial \sigma}$ has the form of:

$$\begin{aligned}\frac{\partial r_{10}(t)}{\partial \sigma} &= \frac{-(t+10) \frac{\partial R(0, t+10)}{\partial \sigma} + (t + \frac{1-e^{-10a}}{a}) \frac{\partial R(0, t)}{\partial \sigma} - \frac{\sigma}{2a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1) - \frac{1-e^{-10a}}{a} \frac{\partial r(t)}{\partial \sigma}}{10} \\ &= \frac{\frac{\sigma}{2a^3} (e^{-a(t+10)} - e^{-at})^2 (e^{2at} - 1) + \frac{1-e^{-10a}}{a} \frac{\partial r(t)}{\partial \sigma}}{10}.\end{aligned}\tag{2.52}$$

2.3.5 Derivation of Second Order Gradients: Gamma

Another gradient that interests risk managers is convexity, or the gamma of MBS, which is the second order derivative of price against term structure shifts. Now we derive an estimator for the gamma.

In order to calculate the partial second order derivatives (Hessian matrix), we take

θ to be the vector, $\theta = [\Delta_1 \Delta_2 \Delta_3 \Delta_4 \sigma]'$. Differentiating (2.1)¹, we get

$$\begin{aligned}\frac{\partial P}{\partial \theta} &= \frac{\partial E[V]}{\partial \theta} = E \left[\sum_{t=0}^M \frac{\partial PV(t)}{\partial \theta} \right] = E \left[\sum_{t=0}^M \frac{\partial d(t)}{\partial \theta} c(t) + \frac{\partial c(t)}{\partial \theta} d(t) \right], \\ \frac{\partial^2 P}{\partial \theta^2} &= \frac{\partial^2 E[V]}{\partial \theta^2} = E \left[\sum_{t=0}^M \frac{\partial^2 PV^2(t)}{\partial \theta^2} \right] = E \left[\sum_{t=0}^M \frac{\partial d^2(t)}{\partial \theta^2} c(t) + \frac{\partial d(t)}{\partial \theta} \times \frac{\partial c(t)}{\partial \theta} + \frac{\partial c(t)}{\partial \theta} \times \frac{\partial d(t)}{\partial \theta} + \frac{\partial^2 c(t)}{\partial \theta^2} d(t) \right].\end{aligned}\tag{2.53}$$

where $\frac{\partial P}{\partial \theta} = \left[\frac{\partial P}{\partial \Delta_1} \frac{\partial P}{\partial \Delta_2} \frac{\partial P}{\partial \Delta_3} \frac{\partial P}{\partial \Delta_4} \frac{\partial P}{\partial \sigma} \right]'$, and $\frac{\partial^2 P}{\partial \theta^2}$ is a 5-by-5 matrix, whose (i, j)th

element is determined by $\frac{\partial^2 P}{\partial \theta_i \partial \theta_j}$, where θ_i and θ_j are the ith and jth elements of θ ,

respectively. The same notation will be used for gradients of other variables, i.e. $c(t)$, $d(t)$, $r(t)$, etc.

Since we have calculated $\frac{\partial c(t)}{\partial \theta}$ and $\frac{\partial d(t)}{\partial \theta}$ in previous sections, now the problem

is reduced to estimate $\frac{\partial^2 c(t)}{\partial \theta^2}$ and $\frac{\partial^2 d(t)}{\partial \theta^2}$. So we first derive the gamma for the

discounting factor $d(t)$. Differentiating (2.18), we get

$$\frac{\partial d^2(t)}{\partial \theta^2} = d(t) \times \sum_{i=0}^{t-1} \left(-\frac{\partial r^2(i)}{\partial \theta^2} \right) \times \Delta t + \frac{\partial d(t)}{\partial \theta} \times \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta} \right) \times \Delta t\tag{2.54}$$

Once we have $\frac{\partial^2 r(i)}{\partial \theta^2}$, the gamma of $d(t)$ is easily calculated. Now we derive the

gamma for cash flow $c(t)$. From (2.9), we can derive the following gamma equation:

¹ Again, we need the first order derivative to be pathwise continuous to make the interchange of expectation and differentiation permissible.

$$\begin{aligned}
\frac{\partial c^2(t)}{\partial \theta^2} &= \frac{\partial B^2(t-1)}{\partial \theta^2} \{A(t)[1 - SMM(t)] + gSMM(t)\} \\
&+ \left[\frac{\partial B(t-1)}{\partial \theta} \times \frac{\partial SMM(t)}{\partial \theta} + \frac{\partial SMM(t)}{\partial \theta} \times \frac{\partial B(t-1)}{\partial \theta} \right] [-A(t) + g] \\
&+ \frac{\partial^2 SMM(t)}{\partial \theta^2} B(t-1)[-A(t) + g].
\end{aligned} \tag{2.55}$$

And from (3.5), we can get the gamma of $B(t)$:

$$\frac{\partial B^2(t)}{\partial \theta^2} = \frac{\partial B^2(t-1)}{\partial \theta^2} g - \frac{\partial c^2(t)}{\partial \theta^2}. \tag{2.56}$$

Now we calculate gamma of $SMM(t)$:

$$\frac{\partial^2 SMM(t)}{\partial \theta^2}, t = 1, \dots, M.$$

As we know from (3.9), we have

$$\frac{\partial^2 SMM(t)}{\partial \theta^2} = \frac{11}{144} (1 - CPR(t)) \frac{23}{12} \left(\frac{\partial CPR(t)}{\partial \theta} \times \frac{\partial CPR(t)}{\partial \theta} \right) + \frac{1}{12} (1 - CPR(t)) \frac{11}{12} \frac{\partial^2 CPR(t)}{\partial \theta^2}. \tag{2.57}$$

$\frac{\partial CPR(t)}{\partial \theta}$ and $\frac{\partial^2 CPR(t)}{\partial \theta^2}$ will be prepayment model specific.

For discounting factors, if we choose the Hull-White one factor model, we have

the following:

$$\begin{aligned}
\frac{\partial r(i)}{\partial \theta} &= \left[\frac{\partial r(i)}{\partial \Delta_1} \quad \frac{\partial r(i)}{\partial \Delta_2} \quad \frac{\partial r(i)}{\partial \Delta_3} \quad \frac{\partial r(i)}{\partial \Delta_4} \quad \frac{\partial r(i)}{\partial \sigma} \right]; \\
\frac{\partial r^2(i)}{\partial \theta^2} &= \left[\frac{\partial r^2(i)}{\partial \theta_i \partial \theta_j} \right], 1 \leq i, j \leq 5.
\end{aligned} \tag{2.58}$$

From and (2.43) and (2.49), we can derive the following:

$$\begin{aligned}
\frac{\partial r^2(i)}{\partial \Delta_i \partial \Delta_j} &= 0; \\
\frac{\partial r^2(i)}{\partial \Delta_i \partial \sigma} &= 0; \\
\frac{\partial r^2(i)}{\partial \sigma^2} &= \frac{(1 - e^{-ai\Delta t})^2}{a^2}.
\end{aligned} \tag{2.59}$$

And the gamma of $d(t)$ would be

$$\begin{aligned}
\frac{\partial d^2(t)}{\partial \theta^2} &= d(t) \times \sum_{i=0}^{t-1} \left(-\frac{\partial r^2(i)}{\partial \theta^2}\right) \times \Delta t + \frac{\partial d(t)}{\partial \theta} \times \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right) \times \Delta t \\
&= \frac{\partial d(t)}{\partial \theta} \times \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right) \times \Delta t \\
&= \frac{\partial d(t)}{\partial \theta} \times \frac{\partial d(t)}{\partial \theta} / d(t)
\end{aligned} \tag{2.60}$$

For cash flows, based on the equations (2.45) and (2.48) in the $CPR(S, A, B, M)$ model, we have:

$$\begin{aligned}
\frac{\partial CPR^2(t)}{\partial \theta^2} &= AGE(t)MM(t) \left[\frac{\partial RI^2(t)}{\partial \theta^2} \times BM(t) + \frac{\partial RI(t)}{\partial \theta} \times \frac{\partial BM(t)}{\partial \theta} \right. \\
&\quad \left. + \frac{\partial BM(t)}{\partial \theta} \times \frac{\partial RI(t)}{\partial \theta} + RI(t) \frac{\partial BM^2(t)}{\partial \theta^2} \right]; \\
\frac{\partial RI^2(t)}{\partial \theta^2} &= \frac{\partial}{\partial r_{10}} \left(\frac{301/5}{1 + (-8.571 + 430(WAC - r_{10}(t-1)))^2} \right) \left(\frac{\partial r_{10}(t-1)}{\partial \theta} \times \frac{\partial r_{10}(t-1)}{\partial \theta} \right) + \\
&\quad \frac{301/5}{1 + (-8.571 + 430(WAC - r_{10}(t-1)))^2} \frac{\partial r_{10}^2(t-1)}{\partial \theta^2}; \\
\frac{\partial BM^2(t)}{\partial \theta^2} &= 0.7 * \frac{\partial B^2(t-1)}{\partial \theta^2} * \frac{1}{B(0)},
\end{aligned} \tag{2.61}$$

where we know from (2.58) that

$$\begin{aligned}
\frac{\partial r_{10}^2(t)}{\partial \theta_i \partial \theta_j} &= -\frac{-(t+10) \frac{\partial R^2(0, t+10)}{\partial \theta_i \partial \theta_j} + (t + \frac{1-e^{-10a}}{a}) \frac{\partial R^2(0, t)}{\partial \theta_i \partial \theta_j} - \frac{1-e^{-10a}}{a} \frac{\partial r^2(t)}{\partial \theta_i \partial \theta_j}}{10}; \\
\frac{\partial R^2(0, t)}{\partial \theta_i \partial \theta_j} &= 0; \\
\frac{\partial R^2(0, t+10)}{\partial \theta_i \partial \theta_j} &= 0; \\
\frac{\partial r_{10}^2(t)}{\partial \theta_i \partial \theta_j} &= \frac{\frac{1-e^{-10a}}{a} \frac{\partial r^2(t)}{\partial \theta_i \partial \theta_j}}{10}.
\end{aligned} \tag{2.62}$$

Finally, the gamma of price P given by equation (2.53) can be obtained from equations (2.60), (2.61), and (2.62).

2.3.6 Derivation of ARM PA estimators

In this section, we derive PA estimators for ARMs. We know FRMs only have two sources of uncertainty:

- Short-term rate $r(t)$, which affects the discounting factor $d(t)$, and
- Long-term rate $r_{10}(t)$, which determines the prepayment rate $CPR(t)$, and hence determines the cash flow $C(t)$.

ARMs introduce one more source of uncertainty, the coupon rate $WAC(t)$, which affects both the amortization schedule and the prepayment rate $CPR(t)$, and then affects the cash flow $C(t)$. Coupon rate is determined by many factors:

- The index rate. WAC resets to the index rate plus the margin periodically.
- Margin. The spread between the WAC and the index rate.
- Adjustment period. For fixed period (FP) ARMs, the first adjustment period is different from subsequent adjustment period.

- Period Cap/Floor. The maximum amount the WAC could increase/decrease from previous period.
- Lifetime Cap/Floor. The maximum/minimum coupon rate over the lifetime of the mortgage.

In order to derive the PA gradient estimator of $C(t)$ for ARM, we first need to derive the PA gradient estimator for $Index(t)$ and $WAC(t)$.

The most commonly used index rate is the 1-year Treasury rate. In the Hull-White model, it is an explicit function of short-term rate $r(t)$ and the term structure $R(0, t)$. As we have derived the function form of $r_{10}(t)$, we can derive the $r_{lag}(t)$ for any lag: (in this case, lag=1)

$$r_{lag}(t) = R(t, t + lag) = -\frac{\ln A(t, t + lag) - B(t, t + lag)r(t)}{lag}$$

$$= -\frac{-R(0, t + lag)(t + lag) - R(0, t)t - \frac{1 - e^{-lag*a}}{a}R(0, t) - \frac{1}{4a^3}(e^{-a(t+lag)} - e^{-at})^2(e^{2at} - 1) - \frac{1 - e^{-lag*a}}{a}r(t)}{lag}$$
(2.63)

Thus we have the PA gradient estimator of $\frac{\partial Index(t)}{\partial \theta}$ in following form:

$$\frac{\partial Index(t)}{\partial \theta} = \frac{\partial r_{lag}(t)}{\partial \theta} = -\frac{-(t + lag)\frac{\partial R(0, t + lag)}{\partial \theta} - (t + \frac{1 - e^{-lag*a}}{a})\frac{\partial R(0, t)}{\partial \theta} - \frac{1 - e^{-lag*a}}{a}\frac{\partial r(t)}{\partial \theta}}{lag}$$
(2.64)

The hard part is to get the $\frac{\partial WAC(t)}{\partial \theta}$ from $\frac{\partial Index(t)}{\partial \theta}$, because of the complicated rules to determine $WAC(t)$, based on all the factors mentioned above. Given $WAC(t-1)$, $Index(t)$,

$Margin, Period_Cap^2, Period_Floor, Life_Cap, Life_Floor, WAC(t)$ is determined as follows:

$WAC(t) = WAC(t - 1)$, if t is not an adjustment moment;

Otherwise,

$Effective_Floor = \min(Life_Floor, WAC(t-1) - Period_Floor)$;

$Effective_Cap = \max(Life_Cap, WAC(t-1) + Period_Cap)$;

$$WAC(t) = \begin{cases} Index(t) + Margin, & \text{if } Effective_Floor < Index(t) + Margin < Effective_Cap; \\ Effective_Floor, & \text{if } Effective_Floor \geq Index(t) + Margin; \\ Effective_Cap, & \text{if } Index(t) + Margin \geq Effective_Cap; \end{cases}$$

(2.65)

Figure 2.4 shows the relationship of WAC with Index.

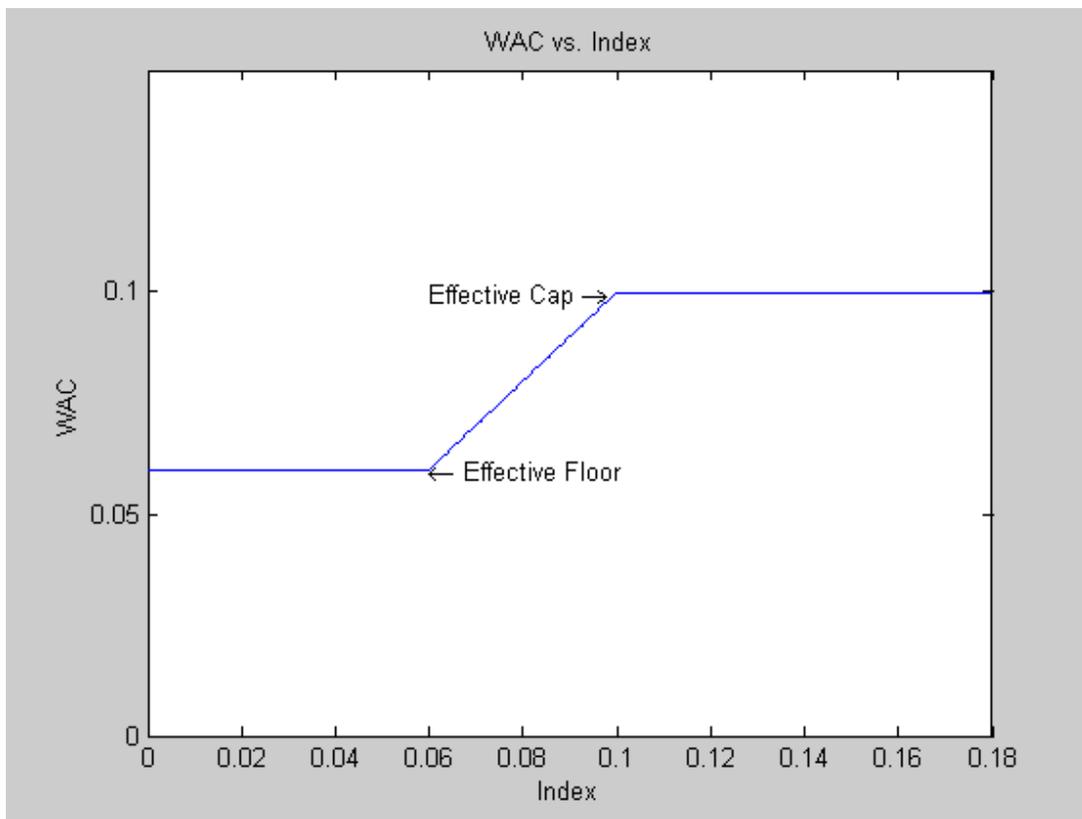


Figure 2.4 WAC as a function of Index

² $Life_Cap/Life_Floor$ are absolute numbers, while $Period_Cap/Period_Floor$ are relative.

Then we can derive the $\frac{\partial WAC(t)}{\partial \theta}$ as following:

$$\frac{\partial WAC(t)}{\partial \theta} = \frac{\partial WAC(t-1)}{\partial \theta}, \text{ if } t \text{ is not an adjustment moment;}$$

Otherwise,

$$Effective_Floor = \min(Life_Floor, WAC(t-1) - Period_Floor);$$

$$Effective_Cap = \max(Life_Cap, WAC(t-1) + Period_Cap);$$

$$\frac{\partial WAC(t)}{\partial \theta} = \begin{cases} \frac{\partial Index(t)}{\partial \theta}, & \text{if } Effective_Floor < Index(t) + Margin < Effective_Cap; \\ \frac{\partial WAC(t-1)}{\partial \theta} * I\{Effective_Floor > Life_Floor\}, & \text{if } Effective_Floor \geq Index(t) + Margin; \\ \frac{\partial WAC(t-1)}{\partial \theta} * I\{Effective_Cap < Life_Cap\}, & \text{if } Index(t) + Margin \geq Effective_Cap; \end{cases}$$

$$\text{where } I\{condition\} = \begin{cases} 1, & \text{when condition is true;} \\ 0, & \text{when condition is false.} \end{cases}$$

(2.66)

Note that the gradient is 0, when it is bounded by lifetime cap or floor, because a perturbation would not change the $WAC(t)$.

Next, we need to derive $\frac{\partial CPR(t)}{\partial \theta}$ for ARM, assuming ARM borrowers have the same prepayment behavior as FRM borrowers (which is not necessarily true, but it does not affect our analysis), so we are facing the same prepayment function as FRM30 as in (2.45).

$$\frac{\partial CPR(t)}{\partial \theta} \text{ will be affected because of the uncertainty of } WAC(t).$$

$$\begin{aligned}\frac{\partial CPR(t)}{\partial \theta} &= \frac{\partial RI(t)}{\partial \theta} AGE(t)MM(t)BM(t) + RI(t)AGE(t)MM(t)\frac{\partial BM(t)}{\partial \theta}; \\ \frac{\partial RI(t)}{\partial \theta} &= 0.14 \frac{430}{1 + (-8.571 + 430(WAC(t) - r_{10}(t-1)))^2} \left[\frac{\partial WAC(t)}{\partial \theta} - \frac{\partial r_{10}(t-1)}{\partial \theta} \right]; \\ \frac{\partial BM(t)}{\partial \theta} &= 0.7 \frac{\partial B(t-1)}{\partial \theta} \frac{1}{B(0)}.\end{aligned}\quad (2.67)$$

$$\begin{aligned}SMM(t) &= 1 - \sqrt[12]{1 - CPR(t)}; \\ \frac{\partial SMM(t)}{\partial \theta} &= \frac{1}{12} (1 - CPR(t))^{-\frac{11}{12}} \frac{\partial CPR(t)}{\partial \theta}.\end{aligned}\quad (2.68)$$

Also $C(t)$ will be affected by the introduced uncertainty in $WAC(t)$:

$$c(t) = B(t-1) \{A(t)[1 - SMM(t)] + g(t)SMM(t)\}, \quad (2.69)$$

where

$$\begin{aligned}A(t) &= \frac{WAC(t)/12}{1 - (1 + WAC(t)/12)^{-WAM+t}}; \\ g(t) &= \left(1 + \frac{WAC(t)}{12}\right);\end{aligned}\quad (2.70)$$

and

$$\begin{aligned}\frac{\partial c(t)}{\partial \theta} &= \frac{\partial B(t-1)}{\partial \theta} \{A(t)[1 - SMM(t)] + g(t)SMM(t)\} + \frac{\partial SMM(t)}{\partial \theta} B(t-1)[-A(t) + g(t)] \\ &\quad + B(t-1) \left\{ \frac{\partial A(t)}{\partial \theta} [1 - SMM(t)] + \frac{\partial g(t)}{\partial \theta} SMM(t) \right\},\end{aligned}\quad (2.71)$$

where

$$\begin{aligned}\frac{\partial A(t)}{\partial \theta} &= \left[\frac{1}{12} \frac{1}{1 - (1 + WAC(t)/12)^{-WAM+t}} + \right. \\ &\quad \left. \frac{1}{144} \frac{WAC(t)}{(1 - (1 + WAC(t)/12)^{-WAM+t})^2 (1 + WAC(t)/12)^{-WAM+t} \frac{-WAM+t}{1 + WAC(t)/12}} \right] \frac{\partial WAC(t)}{\partial \theta}; \\ \frac{\partial g(t)}{\partial \theta} &= \frac{1}{12} \frac{\partial WAC(t)}{\partial \theta};\end{aligned}\quad (2.72)$$

And the PA gradient estimator for balance $B(t)$ is as the following:

$$B(t) = B(t-1)\left(1 + \frac{WAC(t)}{12}\right) - c(t);$$
$$\frac{\partial B(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta} \left(1 + \frac{WAC(t)}{12}\right) - \frac{\partial c(t)}{\partial \theta} + \frac{B(t-1)}{12} \frac{\partial WAC(t)}{\partial \theta}. \quad (2.73)$$

The PA estimator for the discounting factor is unchanged, so we can get the modified

Fourier duration and volatility duration.

2.4 Numerical Example

2.4.1 Specification of Numerical Example

We need to specify two sets of data to price the mortgage: the mortgage data and the interest rate data, which includes the initial term structure and parameters for the interest rate model.

We price different mortgages to examine the different impacts that a term structure shift or change in volatility may have on different mortgage products.

The following data are fixed for all products:

Unpaid Balance/UPB = \$4,000,000;

WAM = 360 months.

Table 2.1 shows the difference between all the products. All the ARM products have the same subsequent adjustment period of 12 months, period cap/floor of 0.02, lifetime cap of initial WAC plus 0.06, and no lifetime floor.

Product	WAC	Index	Adjust First
FRM	0.07425	N/A	N/A
1 Year ARM	0.06425	Treasury 1 Year	12 month
3/1 FP ¹ ARM	0.07425	Treasury 1 Year	36 month
5/1 FPARM	0.07425	Treasury 1 Year	60 month
7/1 FPARM	0.07425	Treasury 1 Year	84 month
10/1 FPARM	0.07425	Treasury 1 Year	120 month
1 Year ARM ²	0.07425	Treasury 10 Year	12 month

Table 2.1 Product Specification for Mortgage Pricing

¹ FP ARM refers to Fixed Period ARM, which keep the coupon rate constant for a certain period, and then adjust periodically, generally once a year. So All the FP ARM products are the same, except different Adjust First date, which is the first coupon reset date.

² This ARM is not a mortgage product in the market at present, and is constructed for illustration purpose only. The following sections will discuss why we introduce this product, and what nice properties it has.

We use the same parameters for all the different products in order to have comparable results. Thus we set all the products to have the same coupon rate, except the first 1 year ARM with index of Treasury 1 year rate, which has a 100 basis points (bps) teaser rate. All the ARM products have the same characteristics, except for the Adjust First date, which is the feature that distinguishes these products.

Our initial term structure is the following:

$$f(0,t)=\ln(150+12t)/100, t=0,1,\dots,360.$$

This will produce an upward-sloping curve increasing gradually from 5% to 8.7% along 30 year maturity, and $R(0,t)$ is acquired by calculating the following:

$$R(0,t) = \frac{\int_0^t f(0,u)du}{t}, R(0,0) = r(0) = f(0,0); \quad (2.75)$$

which increases from 5%, to 7.78% gradually.

Our assumptions for interest rate model parameters are the following:

$a=0.1$; $\sigma=0.1$; $\Delta_n=0.00025$, $n=0,1,2,3$ (used in the FD gradient and gamma estimator calculation); $\Delta\sigma=0.00025$, (used in the FD vega estimator calculation).

2.4.2 Comparison of PA and FD gradient estimators

In order to test whether our PA gradient estimators are accurate, and are within the error tolerance range, we calculate the finite difference (FD) gradient estimators at the same time during our pricing process. This section will demonstrate the accuracy of our PA estimators of delta, vega, and gamma for FRM, as well as the delta and gamma for ARM.

Comparison of Modified Fourier Gradient Estimators for FRM

Figure 2.5 shows the FD estimator, PA estimator, their difference, and standard deviation of their difference for $\frac{\partial d(t)}{\partial \Delta_n}$. The four curves in each chart are specified as

following, which will be the convention for the rest of the paper:

Blue: Modified Fourier Order 1;

Green: Modified Fourier Order 2;

Red: Modified Fourier Order 3;

Cyan: Modified Fourier Order 4.

We can see that although these two estimators are pretty close, there exists a pattern in the difference of these two estimators. This will be explained later in the error analysis section.

Figure 2.6 shows the PA and FD gradient estimators for cash flow $c(t)$: they are pretty close, and the difference behaves as random noise. Based on $\frac{\partial c(t, \theta)}{\partial \theta}$ and $\frac{\partial d(t, \theta)}{\partial \theta}$, we can calculate $\frac{dPV(t, \theta)}{d\theta}$, and figure 5.3 shows us the $\frac{dPV(t)}{d\Delta_n}$. Figure 5.4 shows the 95% confidence interval for difference between PA and FD estimators of $\frac{dPV(t)}{d\Delta_n}$, and we can see that 0 is generally contained in the 95% confidence interval.

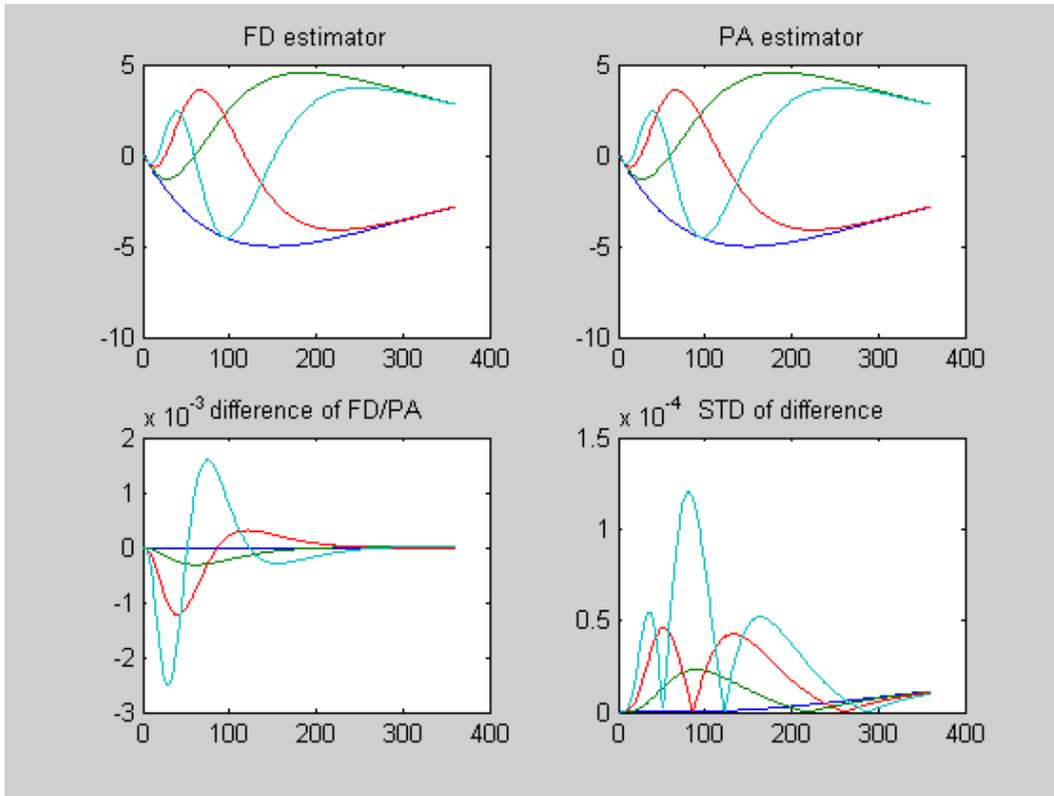


Figure 2.5 Gradient Estimator Comparison for $\partial d(t)/\partial \Delta_n$

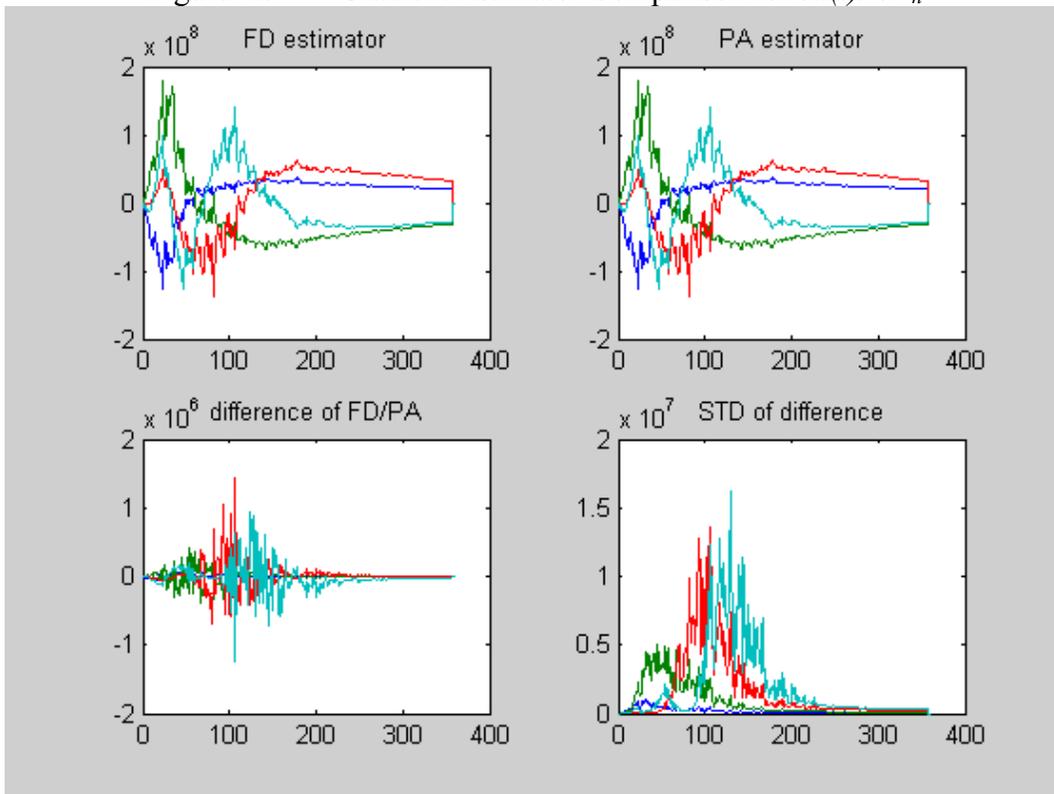


Figure 2.6 Gradient Estimator Comparison for $\partial c(t)/\partial \Delta_n$

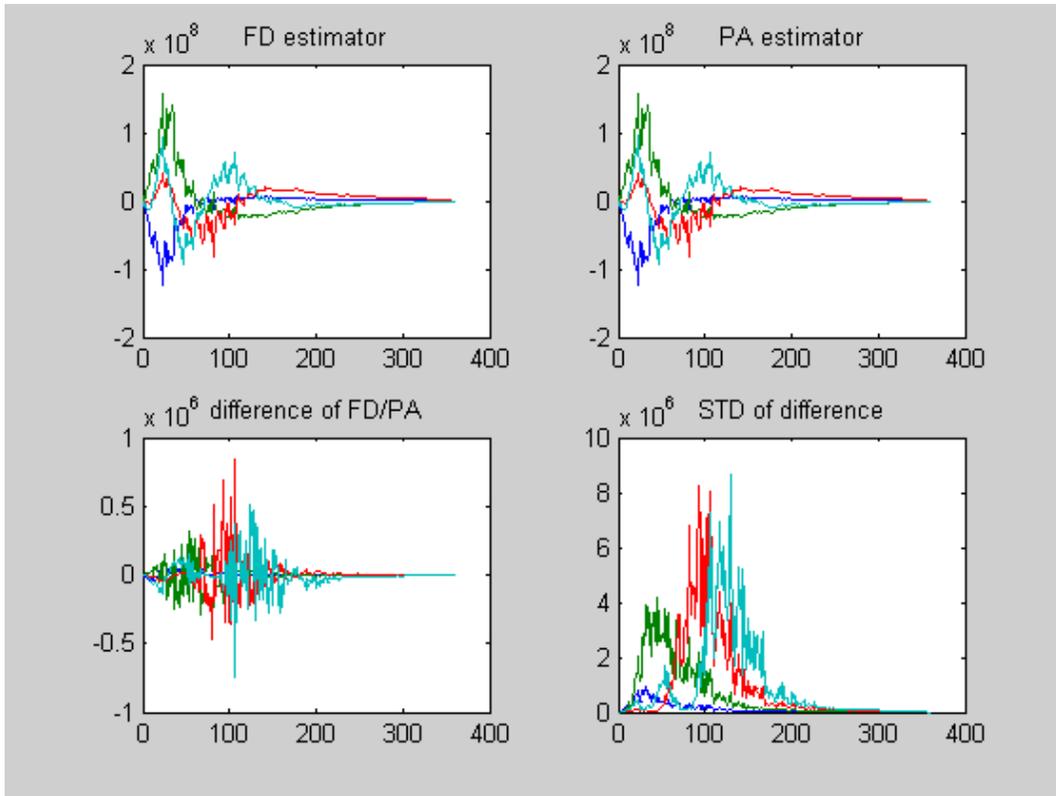


Figure 2.7 Gradient Estimator Comparison for $dPV(t)/d\Delta_n$

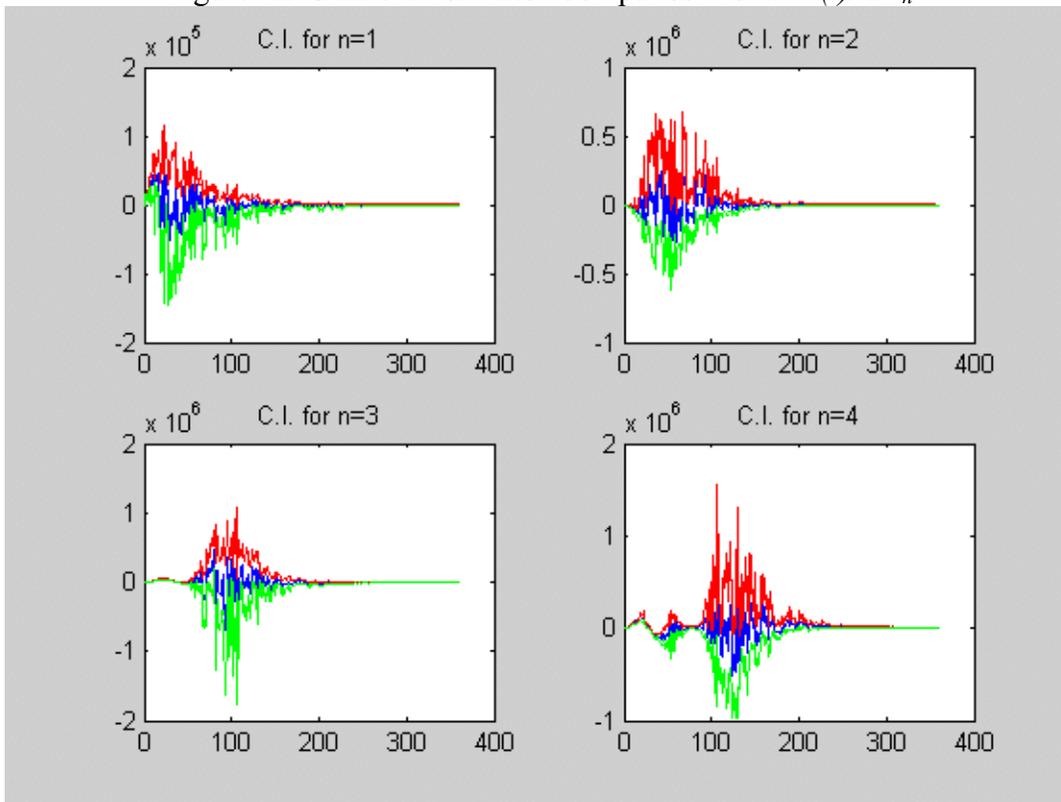


Figure 2.8 95% Confidence Interval for $dPV(t)/d\Delta_n$

Comparison of Vega Estimators for FRM

In this section, we also compare the FD and PA estimators for the gradient w.r.t. interest rate volatility: Vega. Figure 2.9 shows the FD estimator, PA estimator, their difference, and standard deviation of their difference for $\frac{\partial d(t)}{\partial \sigma}$. Also there exists a pattern in the difference of these two estimators. This will also be explained later in the error analysis section. Figure 2.10 shows the gradient estimators for cash flow $c(t)$: they are pretty close, and the difference behaves as random noise. Figure 2.11 shows us the $\frac{dPV(t)}{d\sigma}$, and figure 2.12 shows the 95% confidence interval for $\frac{dPV(t)}{d\sigma}$, and we can see that 0 is always contained in the 95% confidence interval.

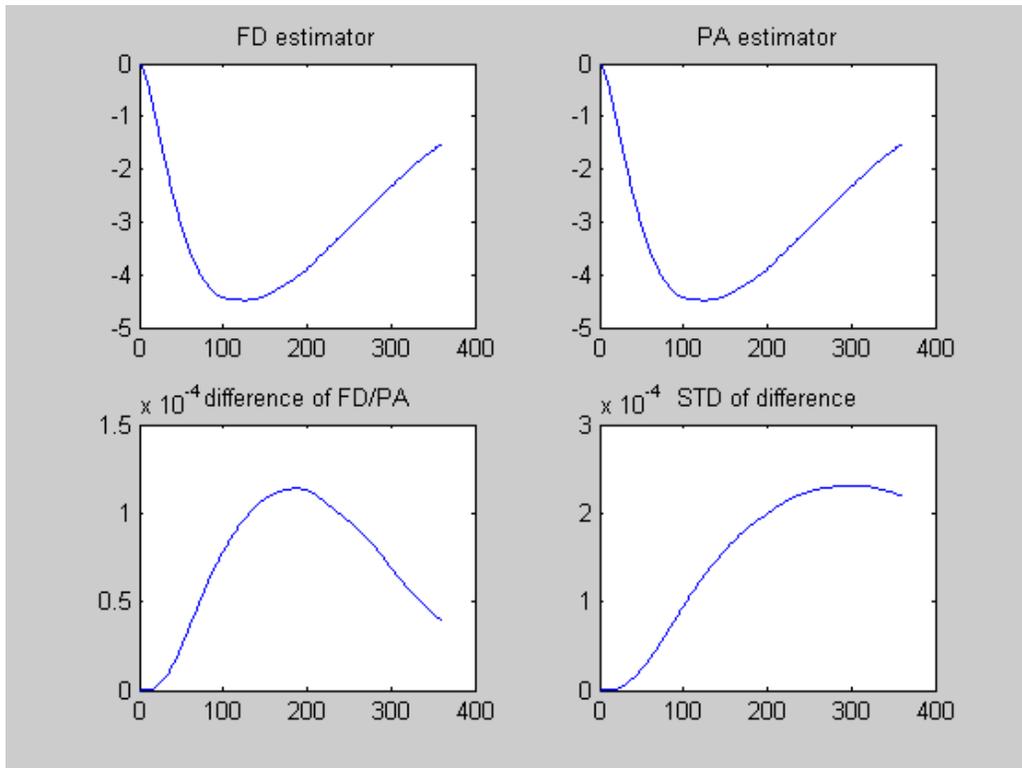


Figure 2.9 Gradient Estimator Comparison for $\frac{\partial d(t)}{\partial \sigma}$

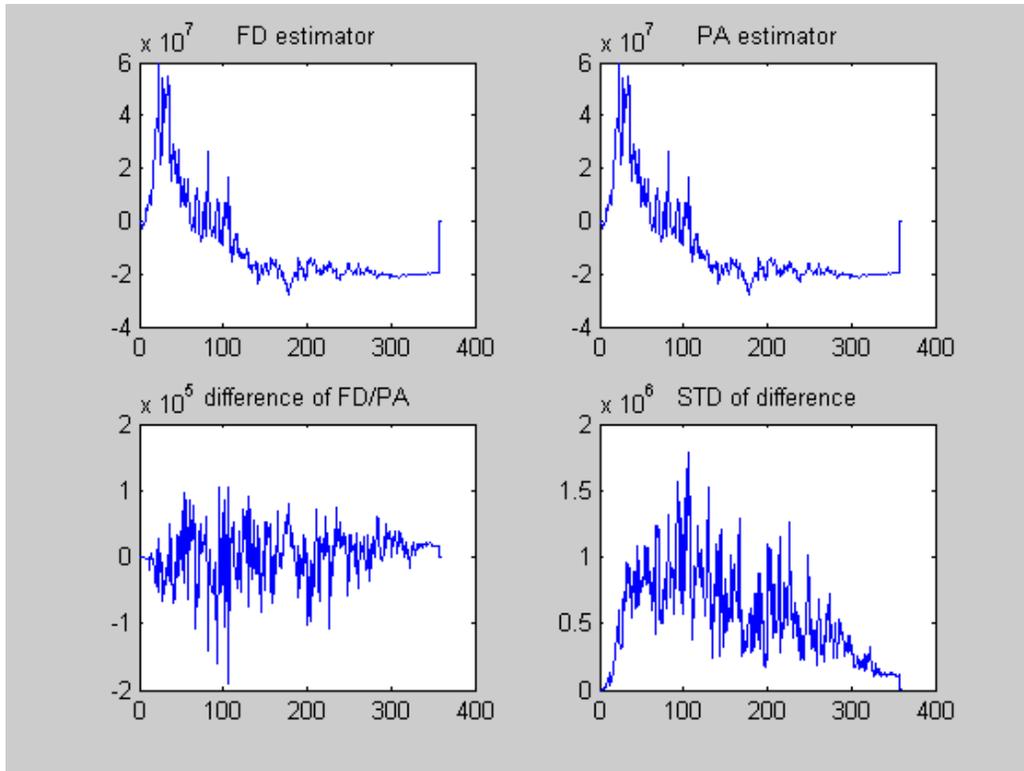


Figure 2.10 Gradient Estimator Comparison for $\hat{c}(t)/\partial\sigma$

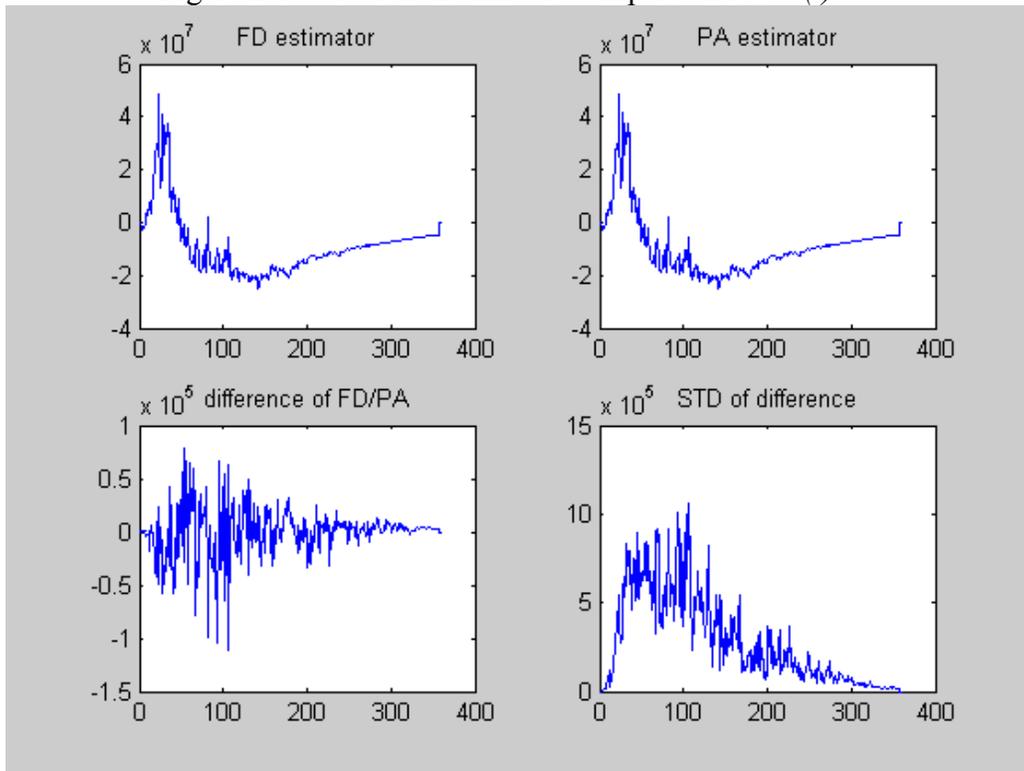


Figure 2.11 Gradient Estimator Comparison for $dPV(t)/d\sigma$

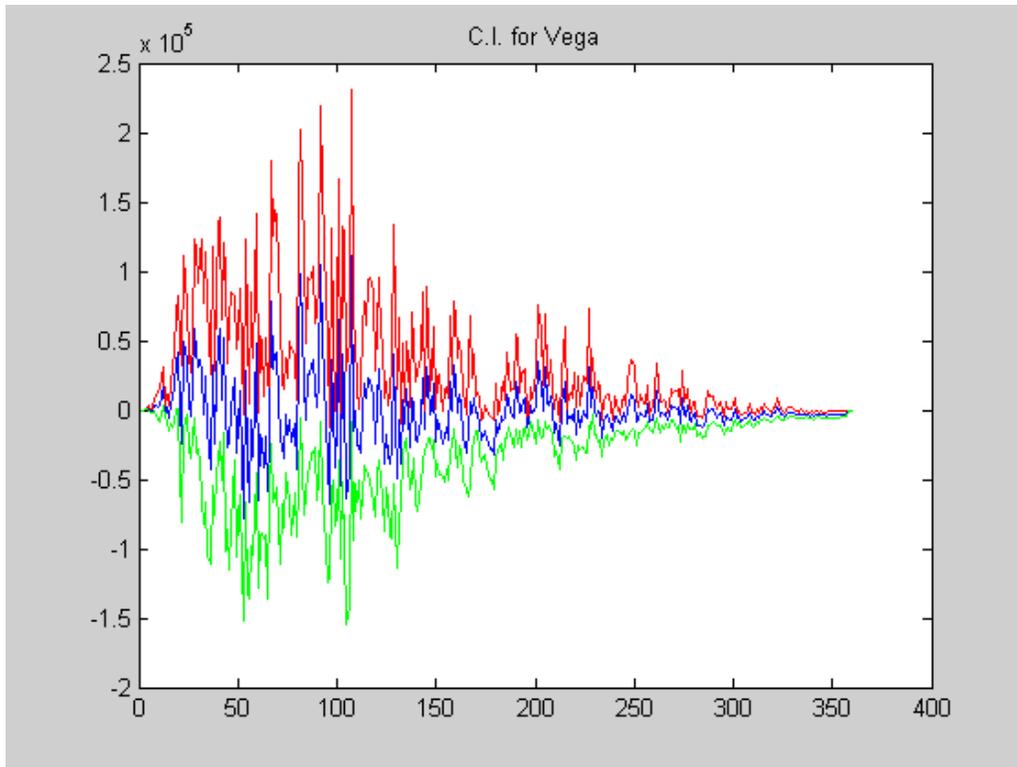


Figure 2.12 95% Confidence Interval for $dPV(t)/d\sigma$

Comparison of Gamma Estimators for FRM

For gamma estimation, $\theta = [\Delta_1 \Delta_2 \Delta_3 \Delta_4]'$. So $\frac{\partial^2 d(t)}{\partial \theta^2}$, $\frac{\partial^2 c(t)}{\partial \theta^2}$, or $\frac{\partial^2 PV(t)}{\partial \theta^2}$ is a 4x4 matrix. If we want to estimate this matrix by the FD method, we would need 144 points to estimate 48 first order derivatives and to estimate 16 second order derivatives.

The following figures show the FD, PA estimators, the difference and STD of difference for diagonal gamma elements.

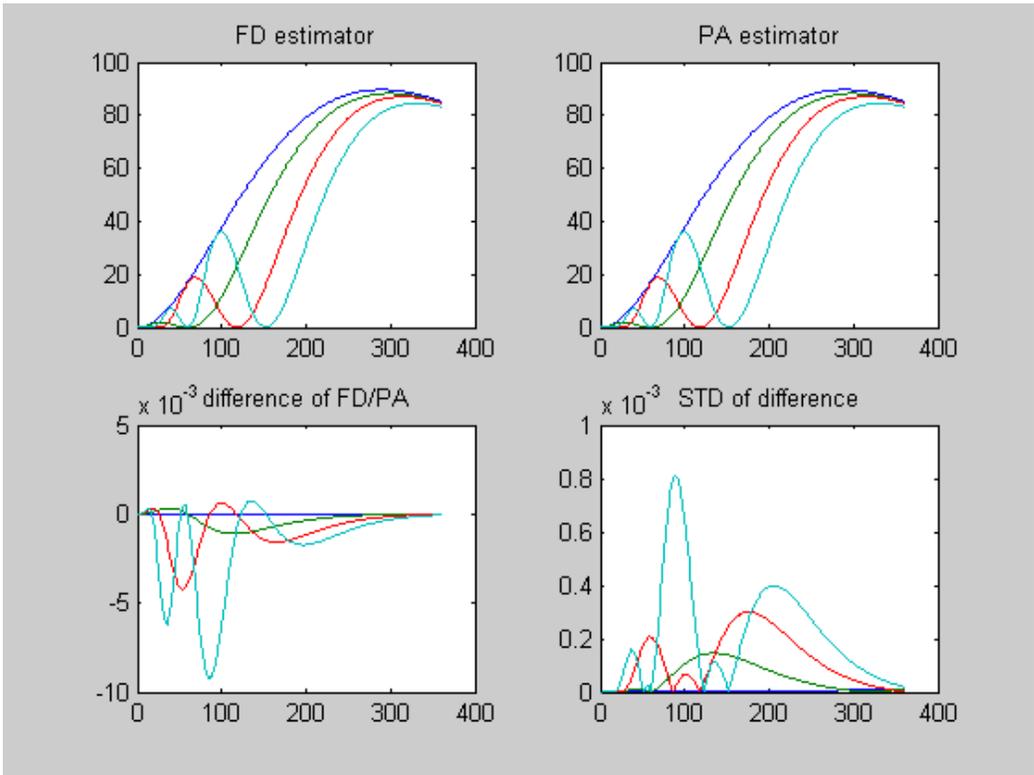


Figure 2.13 gamma estimators for $\frac{\partial^2 d(t)}{\partial \Delta_i^2}, i=1, 2, 3, 4$

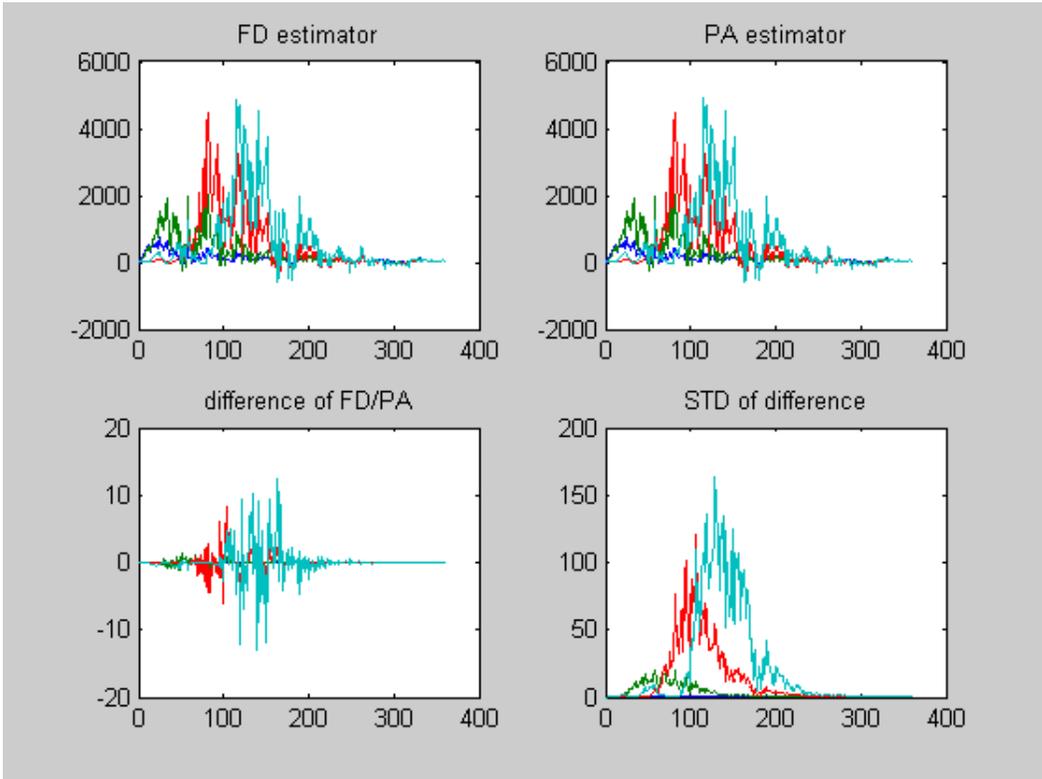


Figure 2.14 gamma estimators for $\frac{\partial^2 CPR(t)}{\partial \Delta_i^2}, i=1, 2, 3, 4$

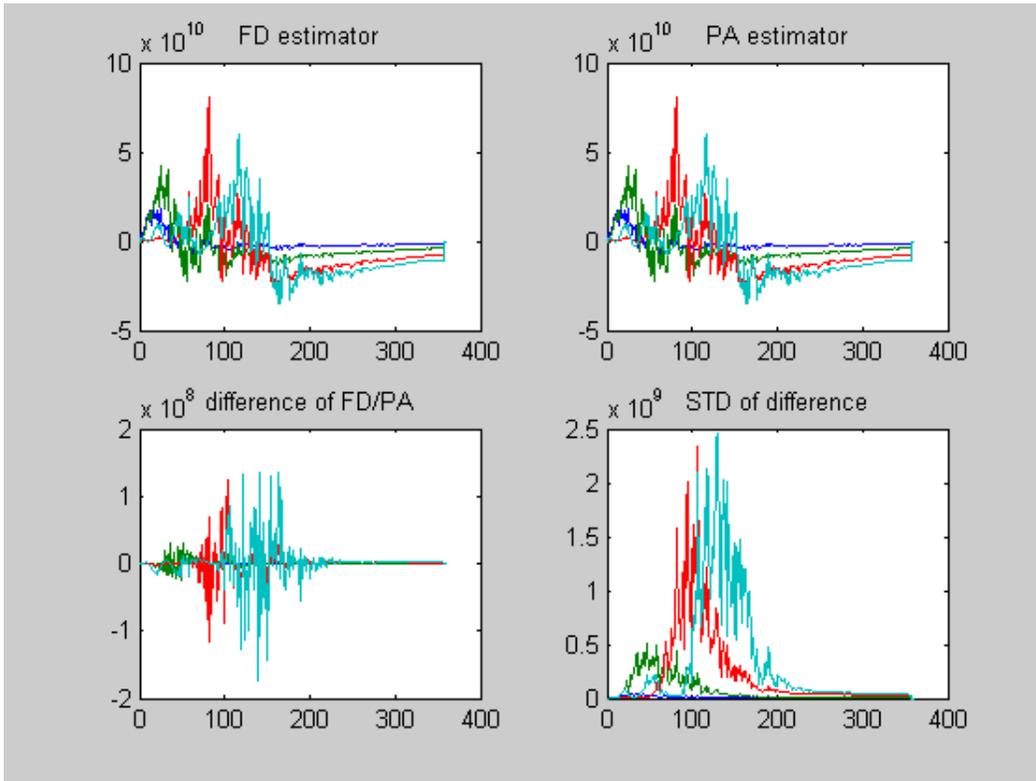


Figure 2.15 gamma estimators for $\frac{\partial^2 CF(t)}{\partial \Delta_i^2}$, $i=1, 2, 3, 4$

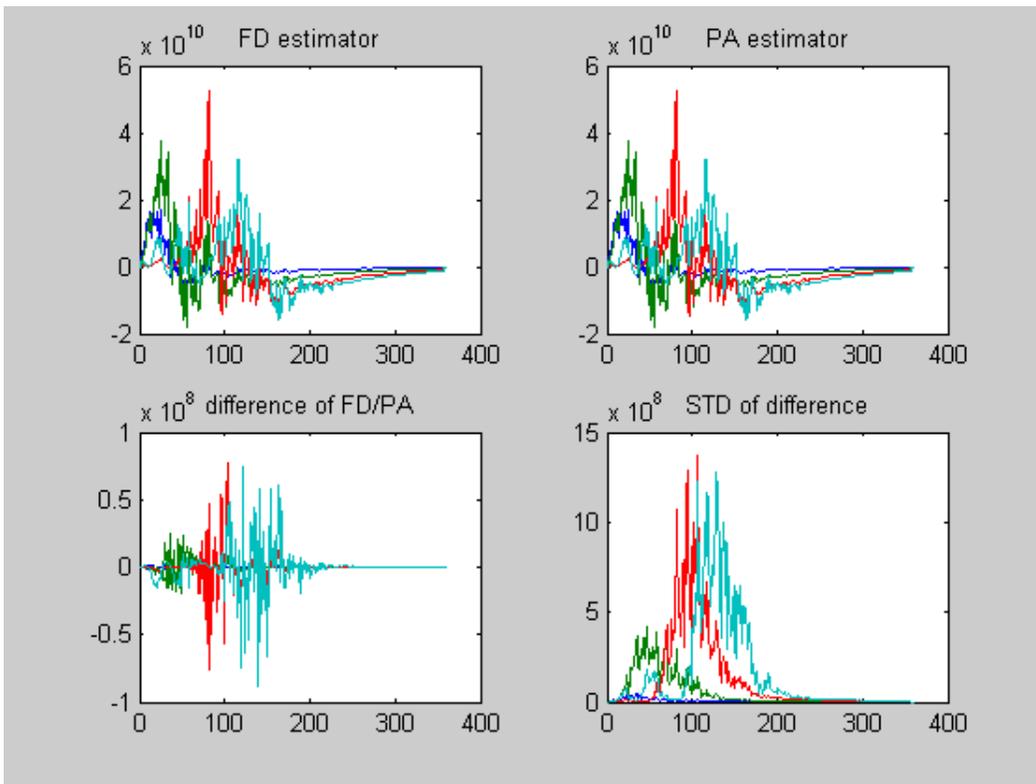


Figure 2.16 gamma estimators for $\frac{\partial^2 PV(t)}{\partial \Delta_i^2}$, $i=1, 2, 3, 4$

Comparison of ARM gradient estimators

For ARM products, we basically have the same set of PA gradient estimators to compare with FD gradient estimators, with one additional set of estimators for

$\frac{\partial WAC(t)}{\partial \Delta_i}$ (figure 2.17). To illustrate the accuracy of our simulation in a brief way, we

only show the FD and PA gradient estimator comparison for one ARM product, 1-Year ARM with index of 1-Year Treasury rate, adjusted annually.

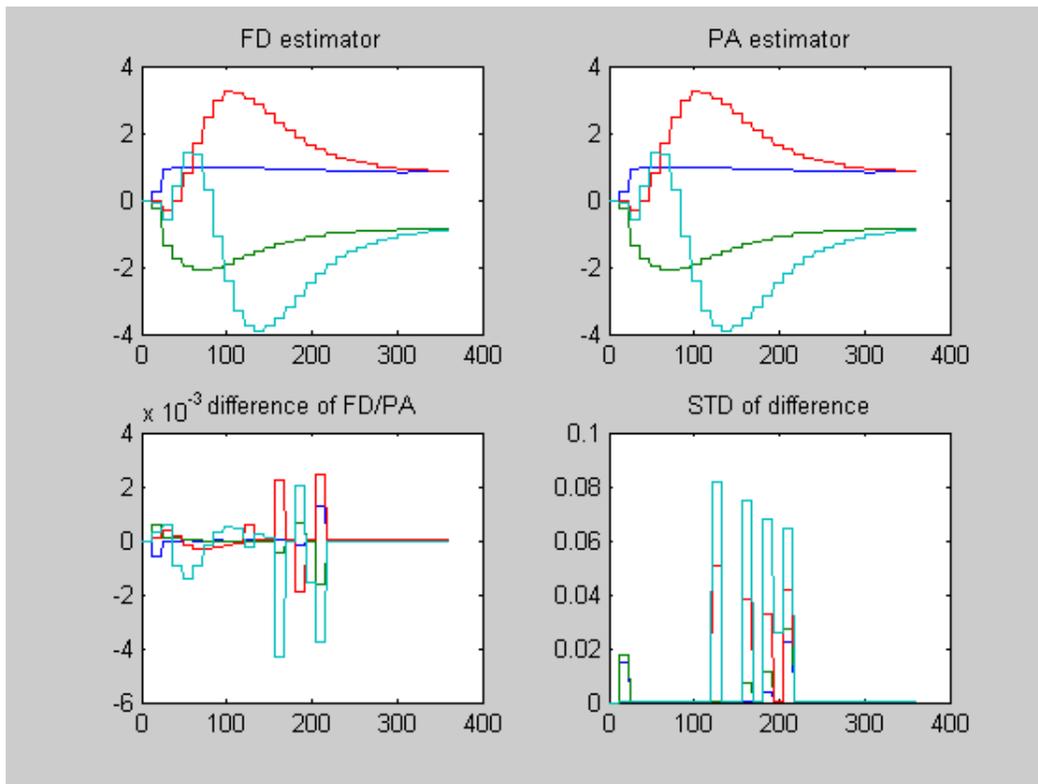


Figure 2.17 Gradient Estimator Comparison for $\frac{\partial WAC(t)}{\partial \Delta_i}$, $i=1, 2, 3, 4$

Figures 2.18 and 2.19 show the FD/PA gradient estimator comparison for $\frac{\partial PV(t)}{\partial \Delta_i}$ and $\frac{\partial PV(t)}{\partial \sigma}$ for this ARM product, respectively.

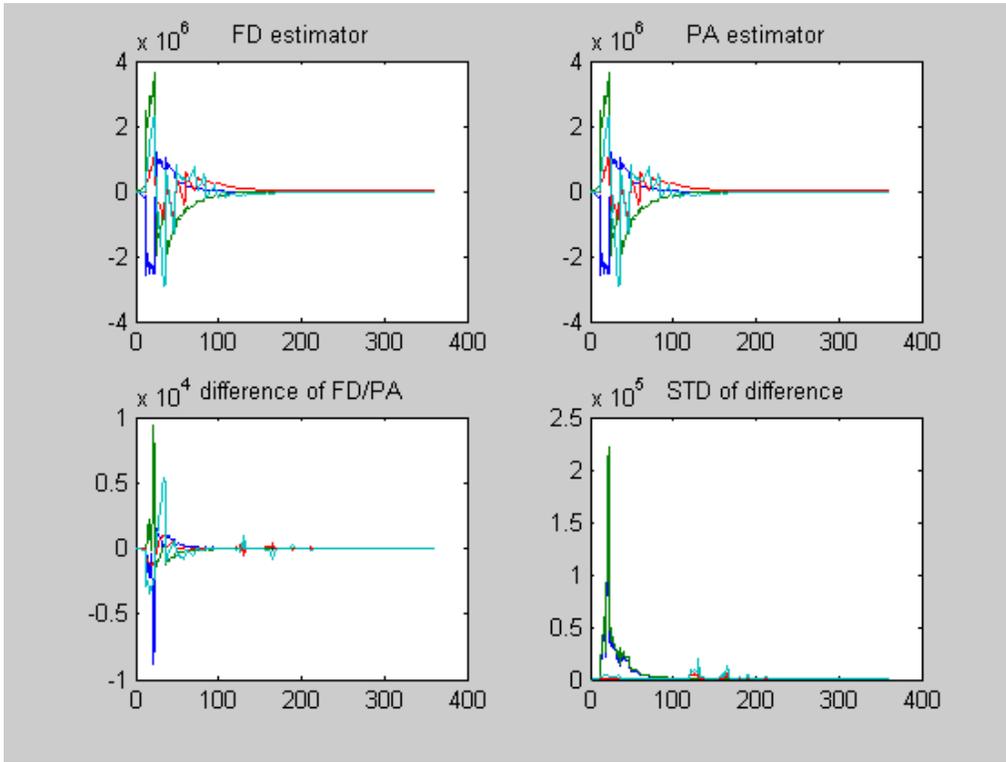


Figure 2.18 Gradient Estimator Comparison for $\frac{\partial PV(t)}{\partial \Delta_i}$, $i=1, 2, 3, 4$

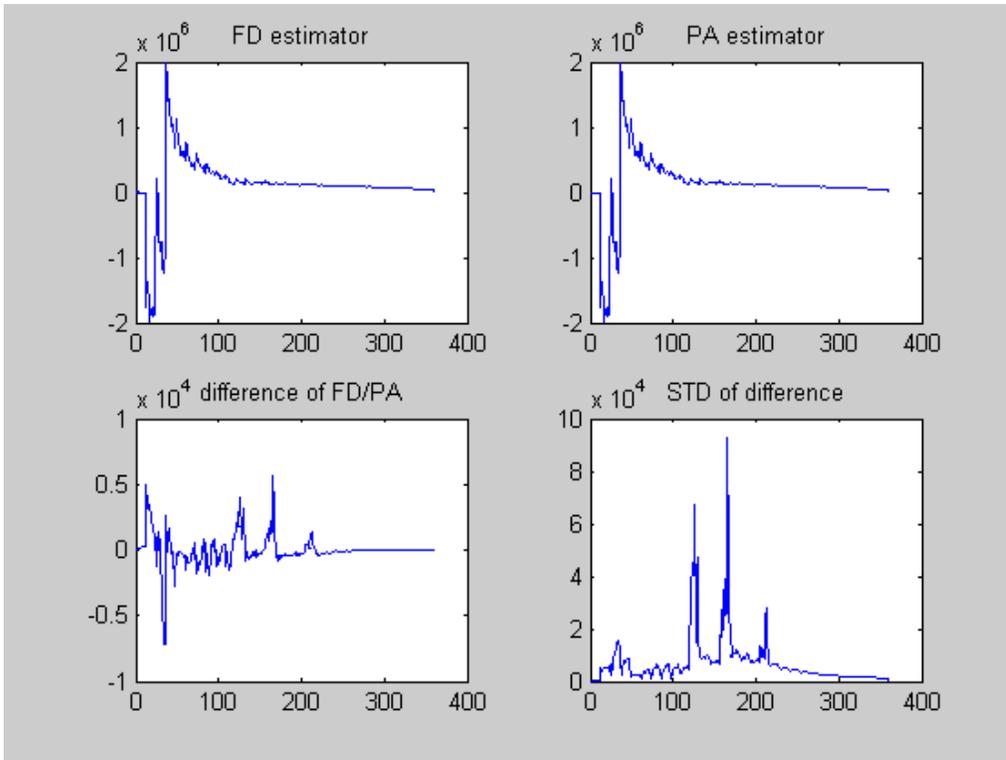


Figure 2.19 Gradient Estimator Comparison for $\frac{\partial PV(t)}{\partial \sigma}$

2.4.3 Result Analysis

Efficiency Analysis

In financial practice, people are more interested in duration, which is the percentage change for a security, once there is a minor shift in one parameter, which mathematically is expressed as

$$duration = \frac{dNPV(\theta)}{d\theta} \frac{1}{NPV}. \quad (2.76)$$

Actually, there should be a minus sign before the expression, since the original duration of fixed income securities measures the percentage price drop resulting from an increase in the interest rate. Yet for our analytical purpose, we do not need the duration always to be positive, since from the following numbers, we see that durations can also be negative. Table 2.2 shows the FD and PA durations for FRM30, their 95% confidence interval, and the error range of the mean.

Fourier Order	0	1	2	3	Vega
PA estimator	-6.4816±0.1017	3.1012±0.1860	-0.5705±0.1817	0.6269±0.1189	-6.7567±0.6712
FD estimator	-6.4814±0.1017	3.1001±0.1860	-0.5695±0.1816	0.6259±0.1188	-6.7565±0.6712
Absolute Error	-0.0002	0.0011	-0.001	0.001	-0.0002
Relative Error	0.0031%	0.0355%	0.1753%	0.1595%	0.0030%

Table 2.2 Comparison of PA/FD Duration

We can see that the error size is very small, and the 95% confidence intervals are almost the same. Thus from the accuracy point of view, we can use PA estimator to replace FD estimator without causing too much problem. And the improvement in computation efficiency is enormous. The FD duration estimator works in the following way:

$$\frac{dNPV(t, \theta)}{d\theta} * \frac{1}{NPV} = \frac{NPV(\theta + \Delta\theta) - NPV(\theta - \Delta\theta)}{2\theta} \frac{1}{NPV(\theta)}. \quad (2.77)$$

Thus for each parameter, we need two additional simulations. In our case, we need $2 \times 5 + 1 = 11$ simulations to estimate the FD duration. However, by PA estimator, we only need one simulation. Ignoring the costs of middle steps, and middle variables, we can reduce the computational time by $10/11$, or 90.9%. When we consider the second order derivative, gamma, the computational efficiency improves even more.

The following table shows the comparison of convexity estimators for FRM.

Convexity is calculated as following:

$$convexity = \frac{d^2 NPV(\theta)}{d\theta^2} \frac{1}{NPV}. \quad (2.78)$$

As we have mentioned earlier, we only estimated part of the FD gamma estimators, via using the PA delta estimators. Because to fully estimate one set of 25 gamma estimators, we would need to simulate 225 times to get all of them. And each element is a 360 by 300 (time length by simulation path) matrix.

So from the above analysis, we can see that by the conventional FD method, to estimate one full set of duration and convexity estimators with 5 free variables, would require 11 plus 225 simulations. Since we achieve almost the same accuracy by a single simulation in PA analysis, the simulation cost is reduced roughly by more than 99.5%. However, we also need to contemplate the introduced costs of intermediate variables as a tradeoff of the PA method.

Convexity =	Gamma/Mortgage Value				
Mortgage Value =	4.22E+08				
FD estimator	0	1	2	3	Vol
0	-246.5944896	N/A	N/A	N/A	N/A
1	N/A	-1871.407927	N/A	N/A	N/A
2	N/A	N/A	-1854.492905	N/A	N/A
3	N/A	N/A	N/A	-2000.544882	N/A
Vol	N/A	N/A	N/A	N/A	-4751.605032
PA estimator					
0	-246.6418706	951.9319609	-646.2296558	161.8535453	919.0969179
1	951.9319609	-1871.360546	1251.356282	-233.9200682	-1435.431523
2	-646.2296558	1251.356282	-1854.208619	1106.51252	1223.425174
3	161.8535453	-233.9200682	1106.51252	-2000.92393	-715.5006989
Vol	919.1916799	-1435.407832	1223.377793	-715.4533179	-4755.158608
Fourier Order	0	1	2	3	Vol
PA estimator	-246.5944896	-1871.407927	-1854.492905	-2000.544882	-4751.605032
FD estimator	-246.6418706	-1871.360546	-1854.208619	-2000.92393	-4755.158608
Absolute Error	0.047381014	-0.047381014	-0.284286087	0.379048115	3.553576082
Relative Error	0.0192%	0.0025%	0.0153%	0.0189%	0.0748%

Table 2.3 Comparison of Convexity Estimators

We did all the simulations on a Pentium III 800 MHz processor, with 512 MB memory, in Matlab Release 12.0 under Windows 2000. Here is the simulation comparison.

Method	FD	PA
Memory Required	17 MB	54 MB
Simulation Time for 300 paths	115.5	765.8
Number of Duration Measures	5	5
Simulation required for estimating Duration	11	1
Number of Convexity Measures	25	25
Simulation required for estimating Convexity	225	1
Total Simulation	236	1
Total Expected Simulation Time	27257.7	765.8
Efficiency Improvement		97.2%

Table 2.4 Comparison of Computing Costs

Accuracy Analysis

In order to validate the predictive power of our PA estimator, we setup a test case to compare the predicted percentage change in the MBS price with the real percentage change.

The test case is set up as following:

$$\begin{aligned}
 \text{Perturbed_}R(0,t) &= R(0,t) + \sum_{n=0}^3 \Delta_n \cos(n\pi(1 - e^{-t/T_0})); \\
 \Delta_n &= 5e - 5, n = 0,1,2,3; \\
 \Delta\sigma &= 5e - 5.
 \end{aligned} \tag{2.79}$$

$$\begin{aligned}
 \frac{\Delta NPV}{NPV} &= \frac{\text{Perturbed_}NPV - NPV}{NPV} \\
 &= -5.0474e - 004 .
 \end{aligned}$$

While the predicted change in NPV is calculated as following:

$$\begin{aligned}
 \frac{\Delta NPV}{NPV} &\approx \frac{(\frac{\partial NPV}{\partial \theta})' \times \Delta \theta}{NPV} + \frac{1}{2} \frac{\Delta \theta' \times (\frac{\partial^2 NPV}{\partial \theta^2}) \times \Delta \theta}{NPV} \\
 &= \text{duration}' \times \Delta \theta + \frac{1}{2} \Delta \theta' \times \text{convexity} \times \Delta \theta \\
 &= -5.0414e - 004
 \end{aligned} \tag{2.80}$$

where $\Delta \theta = [\Delta_1 \ \Delta_2 \ \Delta_3 \ \Delta_4 \ \Delta \sigma]$.

We can see that the relative error by using both duration and convexity measures is only 0.0056%, while using duration measures only would produce a relative error of 0.1403%. So this test validates the predictive power of our PA gradient estimators. In the next section, we are going to show that PA estimator not only is more efficient than FD estimator, but also is a more accurate estimator.

Error Analysis

Figure 2.5 and 2.9 show that there exists a pattern in the difference of gradient estimator of discounting factor $d(t)$. Actually this has two reasons: the calculation of forward rates $f(0,t)$ and the finite difference estimator of $d(t)$. This could be verified by figure 2.9, which shows the difference of FD and PA $\frac{\partial f(0,t)}{\partial \Delta_n}$ estimators.

We know that in the Hull-White model, $f(0,t)$ is determined by (2.23). However, generally we do not have an explicit function form for $R(0,t)$. Instead, we only have discrete points for term structure, so $R(0,t)$ is estimated by interpolation. And $f(0,t)$ is further estimated by calculating the difference between adjacent points on $R(0,t)$ as $\frac{\partial R(0,t)}{\partial t}$, which is not so accurate. The detailed calculation is given below.

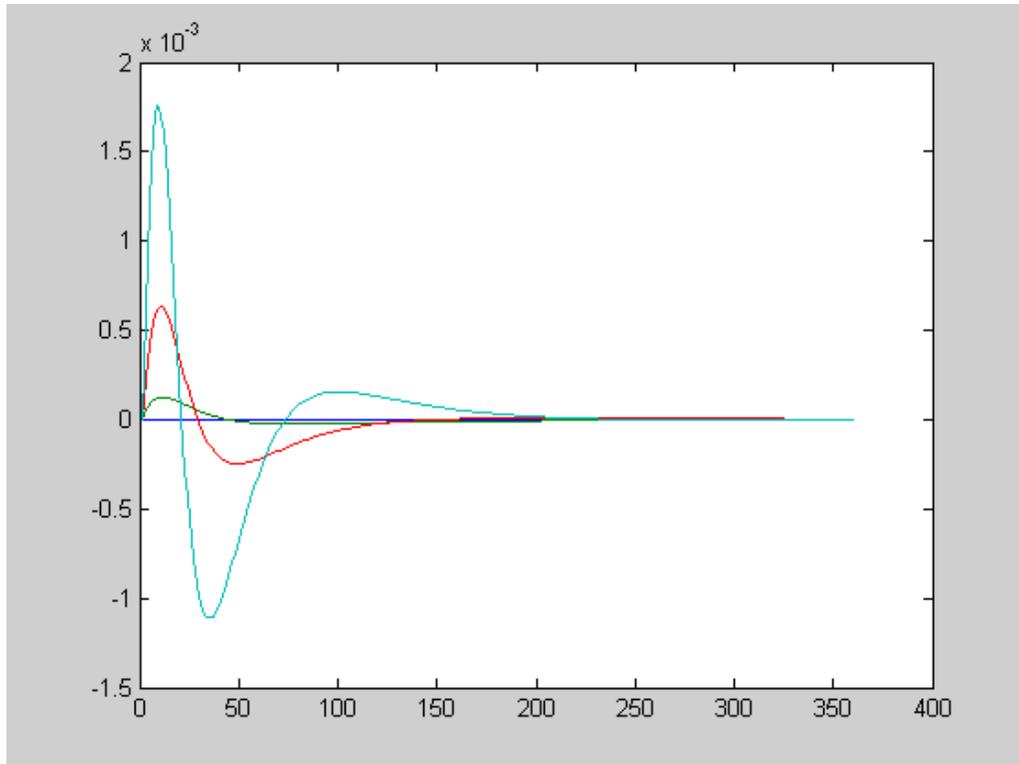


Figure 2.20 Difference of FD/PA $\frac{\partial f(0,t)}{\partial \Delta_n}$ estimators

$$\frac{\partial R(0,t)}{\partial t} = \begin{cases} \frac{R(0,\Delta t) - R(0,0)}{\Delta t}, t = 0 \\ \frac{R(0,t + \Delta t) - R(0,t - \Delta t)}{2\Delta t}, 0 < t < T \\ \frac{R(0,T) - R(0,T - \Delta t)}{\Delta t}, t = T, T \text{ is the maximum term} \end{cases} \quad (2.81)$$

So using FD method to calculate the $f(0,t)$ will result inaccuracy in FD estimator of $\frac{\partial r(t)}{\partial \Delta_n}$, and this will result inaccuracy in $d(t)$. Also we know that $d(t)$ takes the

following form:

$$d(t) = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\right]\Delta t\right\}, \text{ and}$$

$$\frac{\partial d(t)}{\partial \theta} = \exp\left\{-\left[\sum_{i=0}^{t-1} r(i)\right]\Delta t\right\} \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right)\Delta t = d(t) \sum_{i=0}^{t-1} \left(-\frac{\partial r(i)}{\partial \theta}\right)\Delta t. \quad (2.82)$$

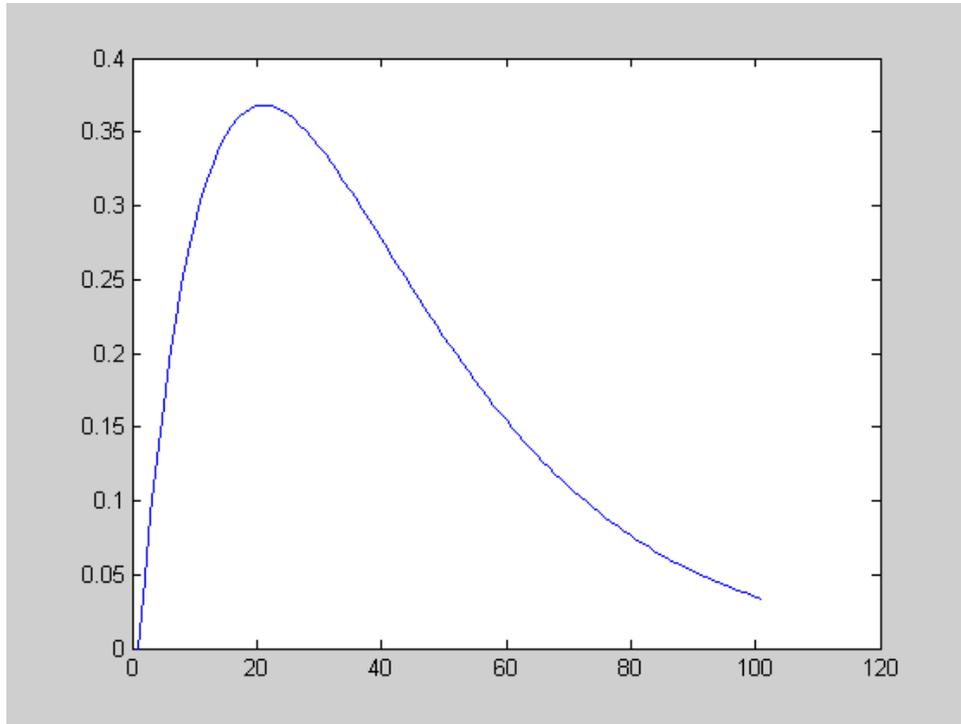


Figure 2.21 Fuction of xe^{-x}

However, when we use FD method to estimate the first order derivative of e^{-x} , the FD estimator is always greater in the absolute value, because e^{-x} is a convex function. So FD estimator of $\frac{\partial d(t)}{\partial \theta}$ is always biased, the bias decreases as the FD step width reduces. The bias increases linearly, while $d(t)$ decreases exponentially. As a result, the bias takes the form of xe^{-x} . Compare the difference of FD/PA $\frac{\partial d(t)}{\partial \sigma}$ gradient estimators and the figure of xe^{-x} as in Figure 2.14, which resembles the error pattern very closely.

For the PA method, $\frac{\partial f(0,t)}{\partial \Delta_n}$ is estimated by the following formula,

$$\frac{\partial f(0,t)}{\partial \Delta_n} = \cos(n\pi(1 - e^{-t/T_0})) + \sin(n\pi(1 - e^{-t/T_0})) \frac{n\pi(e^{-t/T_0})}{T_0} \quad (2.83)$$

which does not involve the FD estimation of $\frac{\partial R(0,t)}{\partial t}$. And $\frac{\partial d(t)}{\partial \theta}$ is directly estimated using its analytical form of first order derivative. So the PA estimator is more accurate than the FD estimator.

2.5 Interpretation of the Results

In this section, we briefly present the durations for various mortgage products, which show different trends for modified Fourier duration of different order. And we try to interpret how the modified Fourier shocks of different order would affect the discounting factors and the cash flows, and then the present value (PV) of the mortgage. Then we analyze the relationship of mortgage prepayment option and mortgage duration. Based on these analysis, we propose a potential new ARM product, which could reduce the duration over any of the existing mortgages, while having a less volatile index than most existing mortgages. This product would benefit both the investors who want to reduce the interest risk, and the mortgage borrowers who want to have a fairly stable coupon rate.

2.5.1 Overview of the Results

The following table shows the durations for various ARM and FRM products we specified and priced in section 2.4:

Fourier Order	0	1	2	3	Vega
ARM TSY 1	-1.7761	-4.313	6.5674	-2.3822	-3.4618
FP 3/1 ARM	-2.8441	-2.9642	7.1814	-3.4784	-4.1601
FP 5/1 ARM	-3.8514	-1.1609	5.9506	-5.3355	-5.2456
FP 7/1 ARM	-4.3054	-0.3064	4.9272	-5.4472	-5.6651
FP 10/1 ARM	-5.4256	1.6401	1.7592	-2.6933	-6.6163
FRM30	-6.4816	3.1012	-0.5705	0.6269	-6.7567

Table 2.5 Durations of Different Products

The relation can be better illustrated with the figure 6.1. The zeroth order modified Fourier duration (with respect to Δ_0) is the same as Option Adjusted Duration (OAD), which measures the price percentage change to a parallel interest term structure shift. Other modified Fourier durations are the same measure, with respect to other interest

term structure changes. Vega measures the price percentage change to an interest volatility change. As we can see, for OAD and Vega, the most important hedge measures, FRM30 has the highest numbers, and 1-Year ARM has the lowest. For everything between pure FRM and pure ARM, there exists a monotonic relationship with the product's approximation to an FRM30. For example, the Fixed Period 10/1 ARM is more like an FRM30 than a Fixed Period 7/1 ARM, so it has higher OAD, and higher Vega.

This means that ARM products have a lower interest risk than FRM products, since an ARM borrower takes more interest risk than an FRM borrower. This result is consistent with Kau et al. [1990, 1992, 1993] and Chiang [1997].

However, an interesting phenomenon is that the first order modified Fourier duration (with respect to Δ_1) actually decreases, and changes sign as volatility of the coupon rate decreases. This indicates that an opposite move of the long-term and short-term rates would not only affect ARMs with a different magnitude, but also has a reverse effect from FRMs. Here is the explanation for this. The first order modified Fourier duration models the following changes in term structure:

- Short-term rate increases;
- Intermediate term rate (e.g. 10 year rate) doesn't change, or moves only a little bit;
- Long-term rate decreases.

In this scenario, people with a short-term ARM, e.g. 1-year ARM are burnt the hardest, so they are going to refinance anyway, even if the prevailing mortgage rate does not change a lot. This will create huge prepayment, and reduce the NPV of the ARM mortgage. People with FRM, on the other hand, have no incentive to refinance, since the refinance mortgage rate (highly correlated with 10-Year Treasury rate) does not change a

lot. This will make the future cash flow more stable and valuable, since they are discounted at a lower long-term interest rate, and increase the NPV of the FRM mortgage.

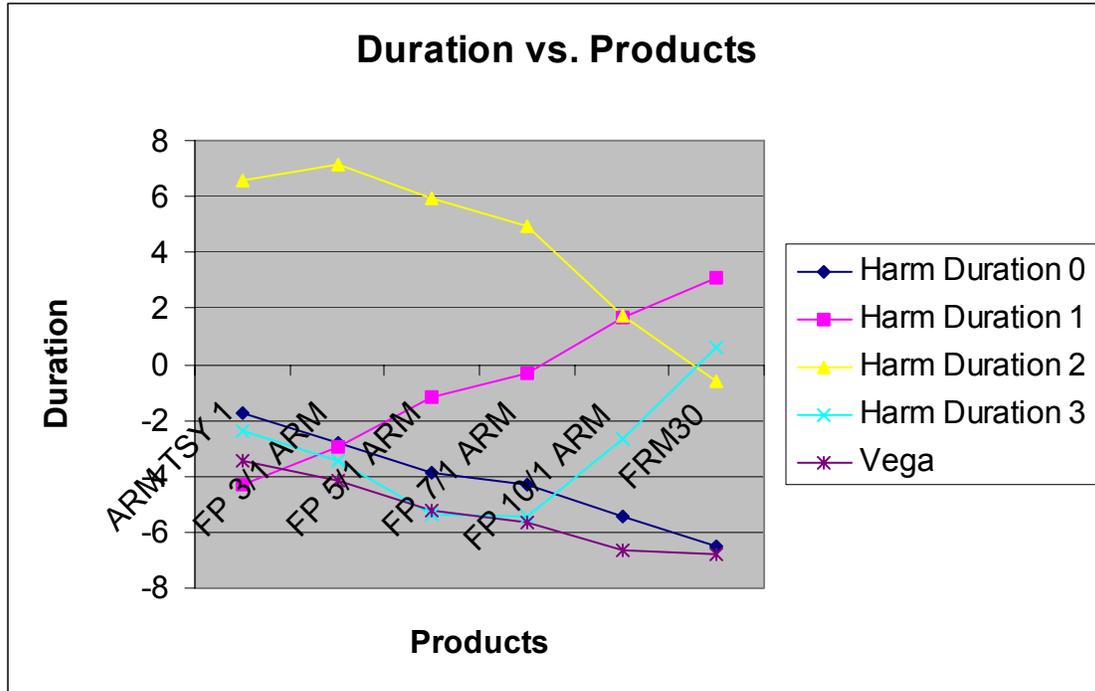


Figure 2.22 Duration vs. Products

The above analysis is based mainly on intuition, and does not show how will this term structure shock affect the discounting factors, cash flows, and NPV of MBS. In the following section, we will see what effect each one of the modified Fourier functions has on these components of MBS for various mortgage products.

2.5.2 Modified Fourier Shock Impact

The following 8 charts will show different modified Fourier shocks on term structure $R(0,t)$, and their impact on $d(t)$, $CF(t)$, and $PV(t)$.

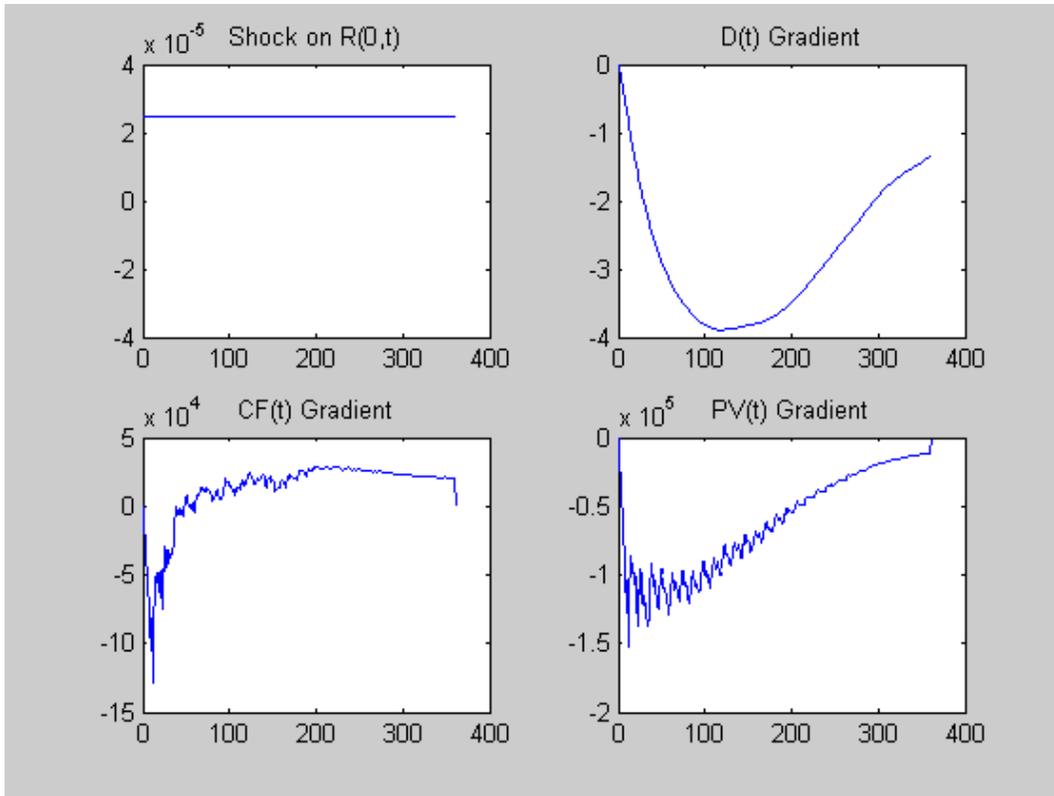


Figure 2.23 The Impact of Modified Fourier Function Order 0 on FRM30

Explanation: A parallel shift in the upward slope term structure will have a negative impact on the discounting factor. Also people are less likely to prepay in the near future, which reduces the cash flow in the short term, and increase the cash flow in the long term a little bit. However, the overall effect of such a shift on present value is negative, and thus reduces the NPV of this MBS.

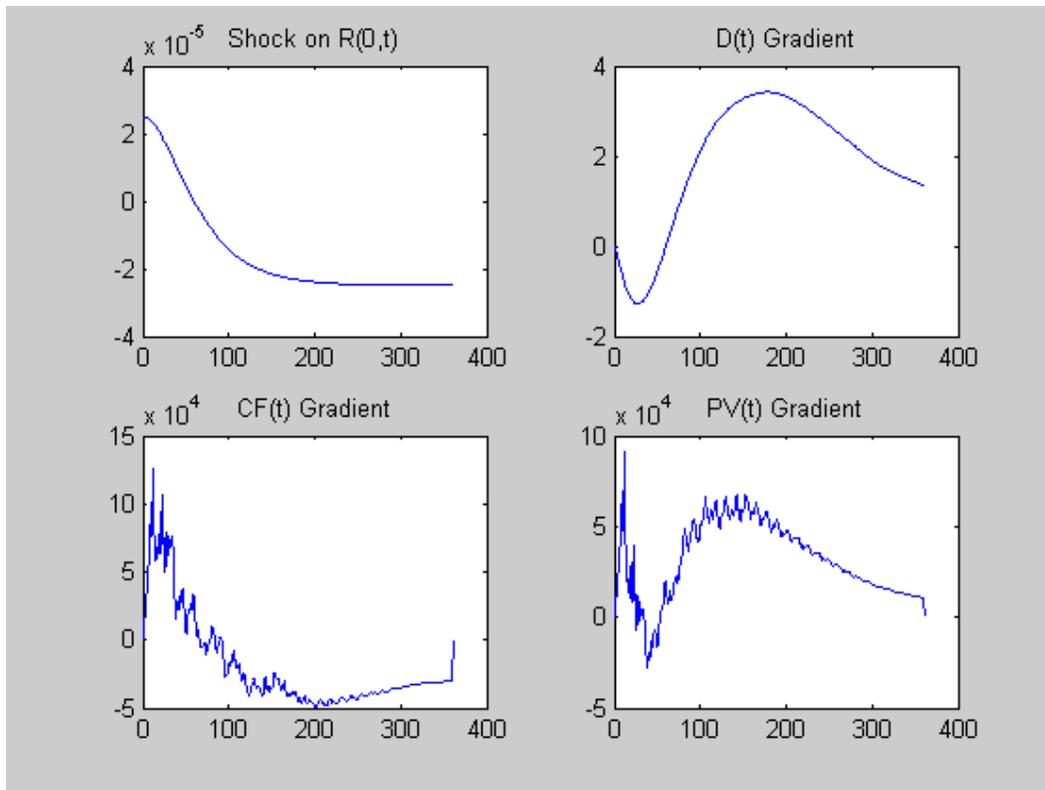


Figure 2.24 The Impact of Modified Fourier Order 1 on FRM30

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: decrease it in the short term, but increase it in the long term. Also people are more likely to prepay in the near future, which increases the cash flow in the short term, and reduces the cash flow in the long term a little bit. However, the overall effect of such a shift on present value is positive, and thus increases the NPV of this MBS.

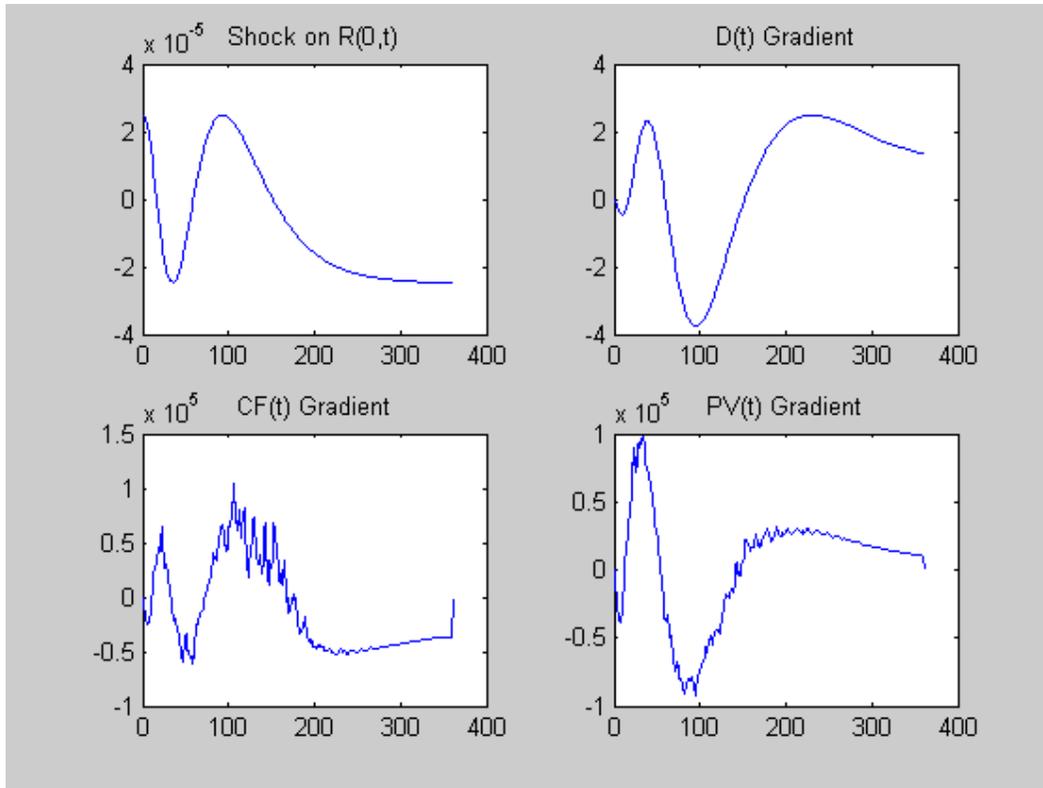


Figure 2.25 The Impact of Modified Fourier Order 2 on FRM30

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: increase it in the middle term, but decrease it in the long term. There is little incentive for people to prepay in the near future, and they will also cling to their current coupon rate in the middle term, because at that time the refinance rate will increase. However, the overall effect of such a shift on present value is cancelled out, and has little impact on the NPV of this MBS.

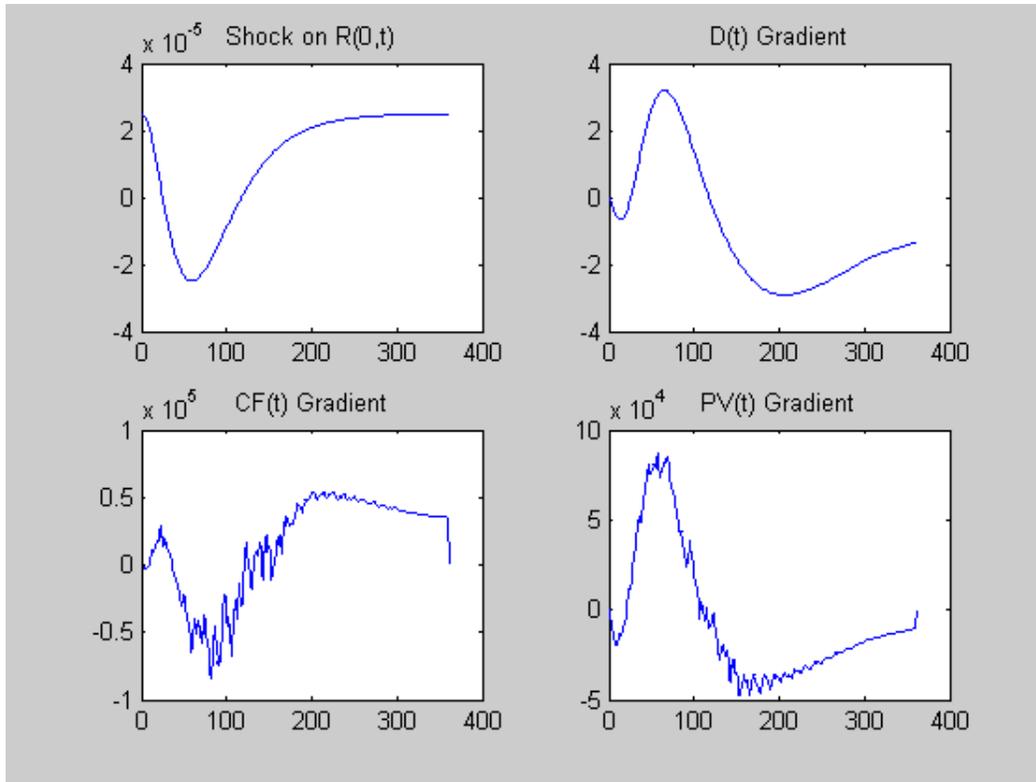


Figure 2.26 The Impact of Modified Fourier Order 3 on FRM30

Explanation: same as Modified Fourier Order 2

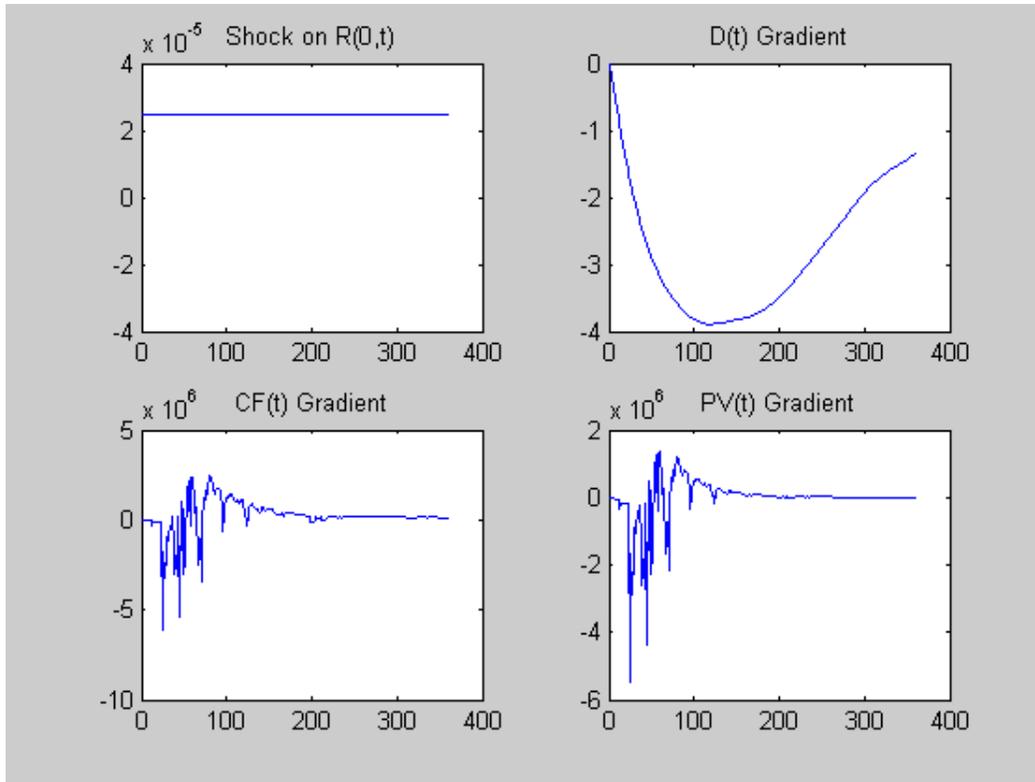


Figure 2.27 The Impact of Modified Fourier Order 0 on ARM TSY 1

Explanation: A parallel shift in the upward slope term structure will have a negative impact on the discounting factor. Also people with ARM are less likely to prepay in the near future, because they have a lower ARM rate than the refinance. Yet they will start prepay in the middle term, because short term rate at that time will increase, due to the upward slop term structure. This behavior will reduce the cash flow in the short term, and increase the cash flow in the middle term. However, the overall effect of such a shift on present value is negative, and thus reduces the NPV of this MBS. Yet the impact will be much smaller than that for FRM.

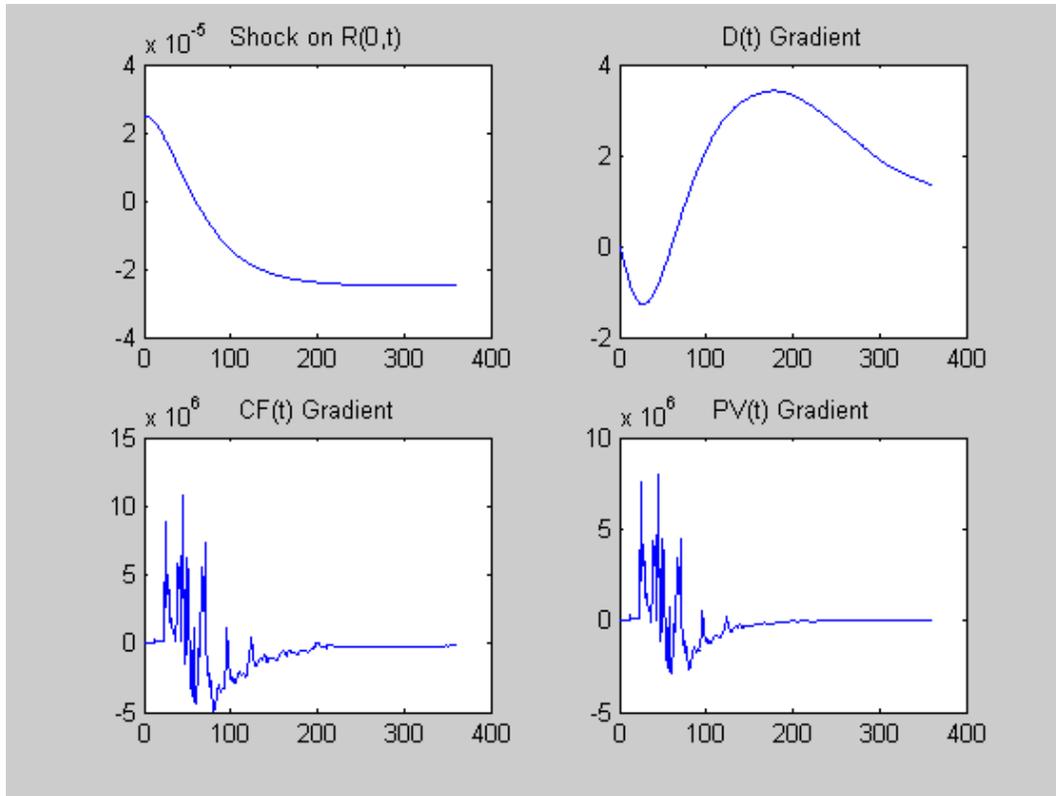


Figure 2.28 The Impact of Modified Fourier Order 1 on ARM TSY 1

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: decrease it in the short term, but increase it in the long term. Also people are more likely to prepay in the near future, which increase the cash flow in the short term, and reduce the cash flow in the long term a little bit. The overall effect of such a shift on present value is negative, and thus decreases the NPV of this MBS.

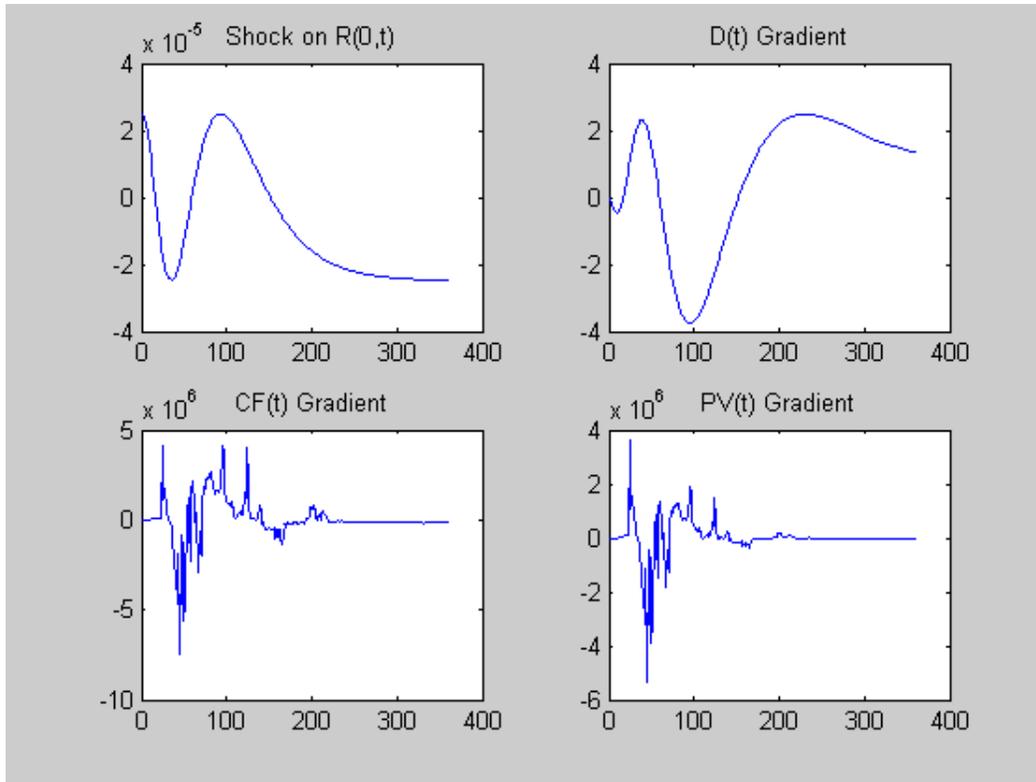


Figure 2.29 The Impact of Modified Fourier Order 2 on ARM TSY 1

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: increase it in the middle term, but decrease it in the long term. People will cling to their low ARM rate for the first few years, but then start to prepay in the middle term, since short term rate will increase at that time. The overall effect of such a shift on present value is positive, due to the increase cash flow and discounting factor in the middle term.

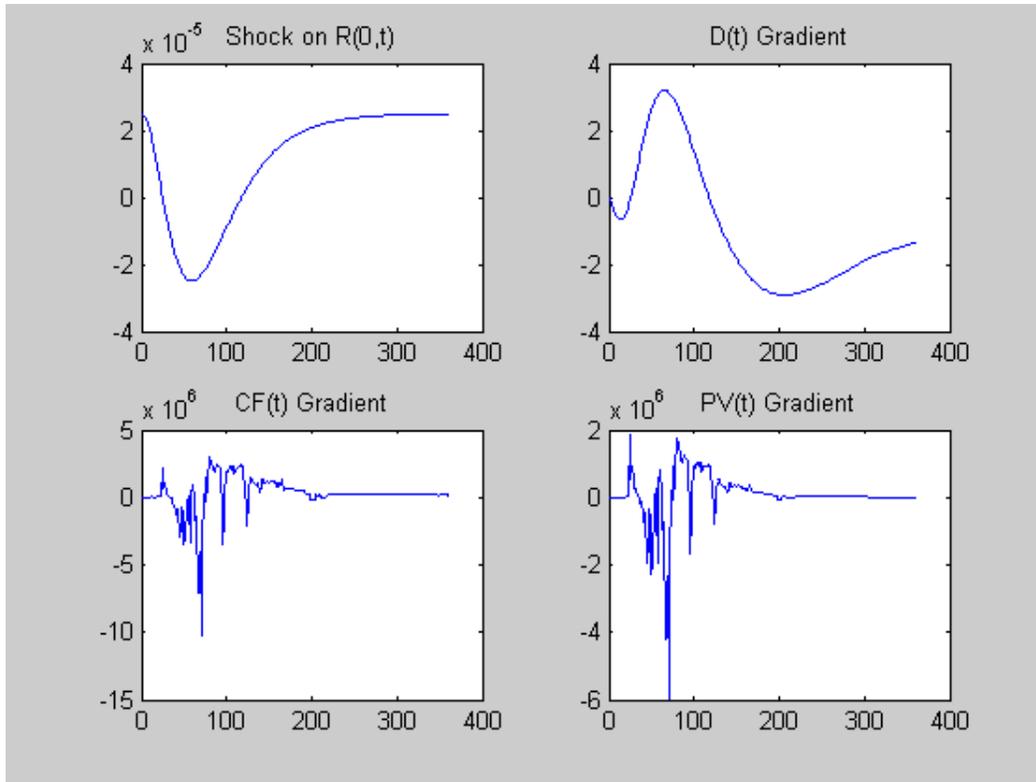


Figure 2.30 The Impact of Modified Fourier Order 3 on ARM TSY 1

Explanation: much like Modified Fourier Order 3, yet because the reverse effect of discounting factor, the overall effect will be negative.

6.3 Potential New ARM Product

Duration is used to measure the interest risk of a fixed income security. The higher the duration is, the more interest risk that security bears. From the investor's perspective, she will benefit if interest rates fall, and suffer if interest rates climb, if the security is non-callable (no prepayment option). From the mortgage borrower's point of view, he will exercise his prepayment option if interest rates drop, and thus reduce the benefit for the investor. He will be able to lock in the low mortgage rate (for FRM), in case interest rates climb, and thus hurt the investor more. However, for the ARM borrower, he benefits from the rate drop, so he does not prepay like the FRM; thus the

MBS investor will also benefit. And he also pays the high coupon rate when interest rates increase, and the ARM MBS investor will not suffer like the FRM MBS investors. From this perspective, the ARM should have a lower duration compared to FRM.

ARM borrower's coupon rate fluctuates with the current interest rate, which is correlated with the prevailing mortgage rate. Because of this, she will have less incentive to prepay when interest rate drops. So the prepayment option value for a FRM borrower will be larger than that of an ARM borrower. This is compatible with the market, where FRM mortgages are sold with the highest rate (borrower pays for the valuable prepayment option), and ARM, that adjust most frequently are offered with the lowest rate.

In option theory, we know that option value generally increases as the volatility of underlying asset increases. However, from the above analysis, we also know that the option value for a FRM is generally greater than for an ARM, while an ARM bears a more volatile coupon rate than a FRM. This looks like a contradiction to the option-volatility relationship. In fact, it's not, because the underlying asset of a prepayment option is not its coupon rate, but the difference between the coupon rate and the prevailing mortgage rate. In most cases, the more volatile the coupon rate is, the less the difference will be, and the less valuable the option will be. However, the borrower does not like the volatility, which put her at risk when interest rate jumps. The investor, on the other hand, does not like the prepayment, which reduce her investment value. It seems that no product can both reduce the coupon rate volatility and the prepayment option at the same time. Is this true? We will see that we can achieve both goals in a potential new ARM product.

We have mentioned that the underlying asset for the prepayment function is the spread between the coupon rate and the prevailing mortgage rate. So an ARM bearing a volatile index does not necessarily indicate a less volatile spread. From historical data, we know that 10-Year Treasury rate is highly correlated with conventional (FRM30) mortgage rate. Figure 6.10 shows the two rates for the period between 1971 and 2001. The correlation calculated is 97.9%. Figure 6.10 also shows the 10-Year Treasury Rate and the 1-Year Treasury rate, which is the most commonly used index in ARM. As we can see, the 1-Year Treasury rate is relatively more volatile than the 10-Year Treasury Rate. The calculated standard deviation is 2.7890 for the 1-Year Treasury rate, and 2.6309 for the 10-Year Treasury rate. However, the standard deviation of spread of FRM30 vs. TSY10 is 0.58, compared with the standard deviation of spread of FRM30 vs. TSY1 at 1.16. Obviously 10-Year Treasury rate has a lower volatility and also a lower volatile spread. The spread between conventional mortgage rate (FRM30) and 10-Year Treasury rate and the spread between conventional mortgage rate (FRM30) and 1-Year Treasury rate are also shown, which indicates that ARM with index of 1-Year Treasury rate has a more volatile spread.

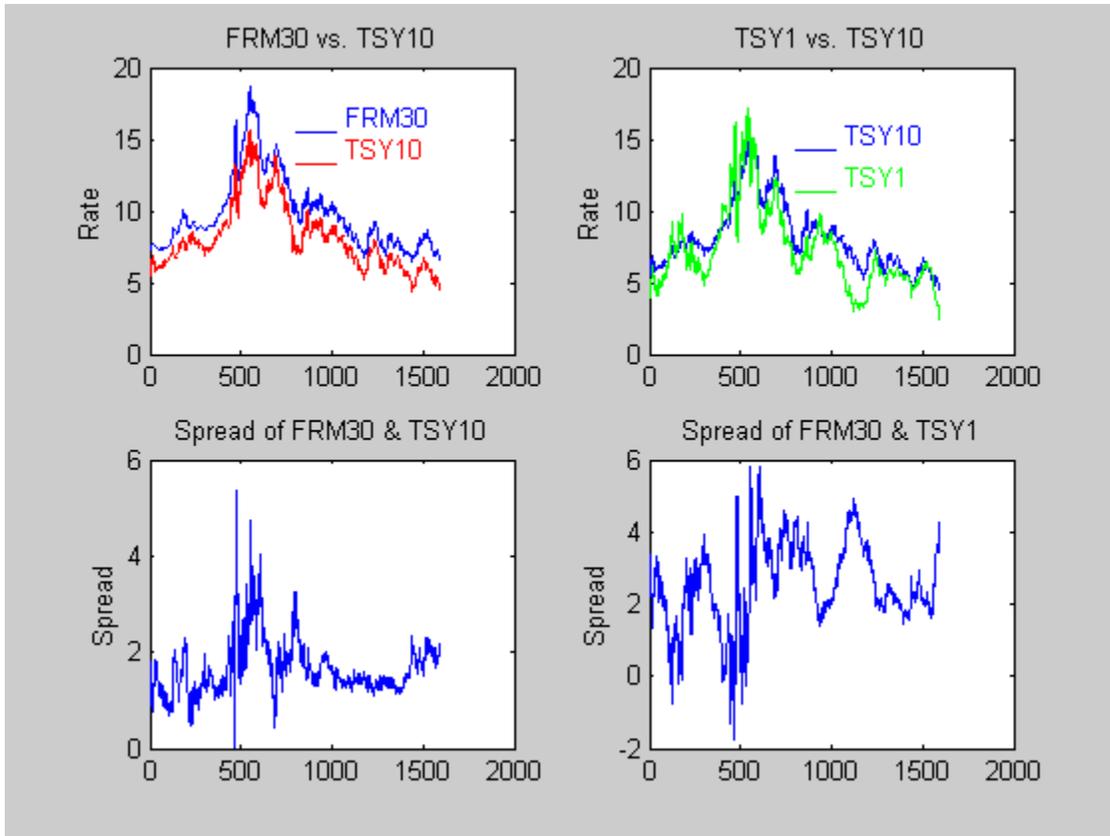


Figure 2.31 10-Year T Rate, 1-Year T Rate, and mortgage rate

Thus if we construct an ARM with index of 10-Year Treasury rate, and reset it more frequently, we could expect a lower duration. So we construct such an ARM with the adjustment period of 12 months. This ARM does not exist at present; it is for illustration purposes only. We then got the modified Fourier duration measures as following:

Fourier Order	0	1	2	3	Vega
ARM TSY 10	-1.2741	-4.0635	3.1819	-0.4894	-1.8855

Figure 2.32 shows the new ARM product's duration against duration of other mortgage products we calculated earlier. We compare this set of durations with table 6.1,

and we can see that this product has the smallest durations for modified Fourier function order 0 and 3, as well as for vega. The durations for modified Fourier function order 1 and 2 are not very high. And we know that generally when there is a shock on the term structure, the biggest magnitude would be that of the first-order modified Fourier function, and volatility is also a big impact. So this product would actually have the least percentage change during a common term structure shift, which satisfies the needs of investors.

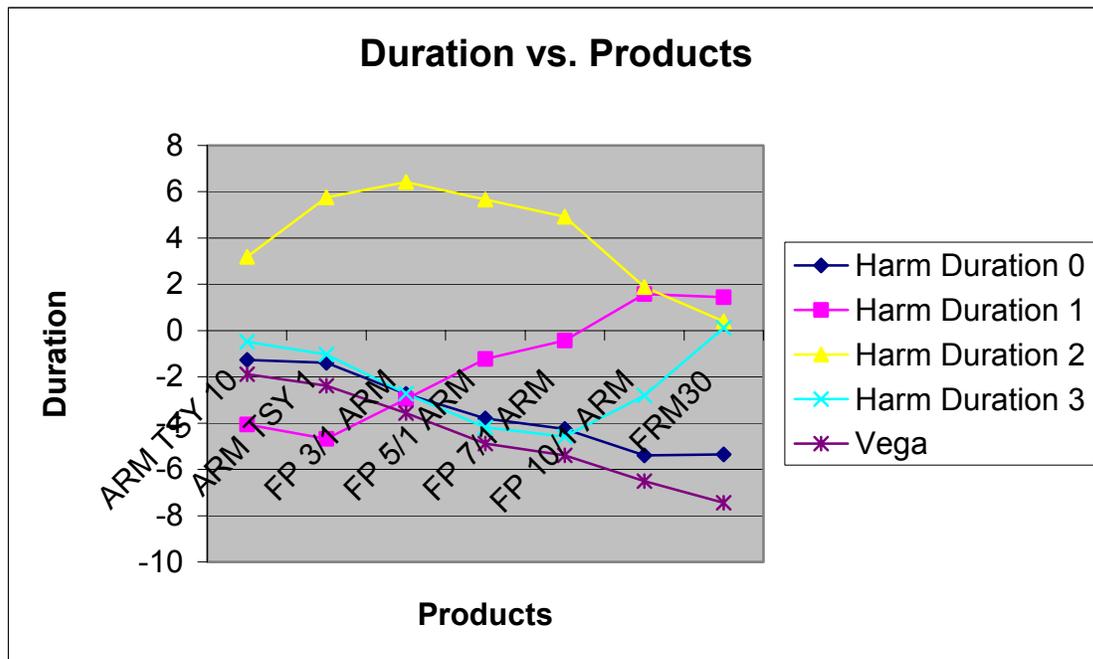


Figure 2.32 New ARM TSY 10 Durations

So we could predict that if there exist such a mortgage, it would have the least refinancing incentive, which would be a better product to suit investors' needs, and it will also have a less volatile index, which suits borrowers' needs.

2.6 Conclusion

This paper applies perturbation analysis (PA) method to estimate MBS sensitivities. The sensitivity estimators include most interest risk measures like duration (equivalent to delta), convexity (equivalent to gamma), and vega. MBS products covered includes fixed rate mortgages (FRMs) and adjustable rate mortgages (ARMs).

We first derive a general framework to derive the PA estimators of MBS, without restriction to MBS type, interest rate model, or prepayment model. Then we apply the PA estimator to both FRM and ARM products, in the setup of a one-factor Hull-White model and a commonly used prepayment model. We compare the PA estimators with finite difference (FD) estimators, and find that PA method can achieve at least the same accuracy as FD method, with a much lower computational cost. In the case we presented, the computational time is reduced by 95.7%, while the memory requirement increases only by a factor of 3, which can be handled by current computer technology with ease. Then we analyze the results of PA estimated sensitivity measures for various MBS products. We justify why and how different term structure shock would affect FRM and ARM differently. Based these analysis, we propose a potential new ARM product which could benefit both the MBS investor and the mortgage borrower.

Future research includes applying this method to other MBS-like securities, since the PA method proposed in section 3 is a very general framework. These include other asset-backed securities, e.g. securities backed by student loans, car loans, credit card receivables. It is pretty straightforward to expand this framework to those securities, since all that is required is to apply a specific interest rate model and prepayment model.

Another area for further research is to incorporate more complicated prepayment and/or default models into the MBS pricing scheme. For MBS investors, the major concerns are price sensitivities to interest changes, which we have covered in detail. However, the MBS guarantor/insurer and issuer might have other concerns, e.g., how will the interest rate change affect the default behavior of the mortgage borrowers? Our framework would be able to serve this purpose as well. By applying the default model that same way as we apply a prepayment model, the default cash flow will take the place of payment cash flow, so the default cost sensitivities could be easily estimated.

Chapter 3

Hedging MBS in HJM Framework

3.1 Motivation

As we have pointed out in our first essay, short term rate and long term rate do not always move in the same direction, it is sometimes misleading to use the conventional interest rate risk measures like duration and convexity to hedge fixed income instruments, especially MBS.

One recent event can illustrate this point very well. In July 2003, Federal Reserve lowered the short term interest rate by another 25 bps, yet just in one month, the long term 10 year rate jumped upward for more than 100 bps. Part of the reason is that the rate deduction is lower than market expectation, and market responded with a selling wave in the bond section. So using a duration measure, which assumes the yield curve moves in parallel, will produce significant hedging error.

It is natural to hedge against the factors of which any yield curve shift can be decomposed. We use a series of exponentially decaying modified Fourier series to approximate any interest rate change in our first essay. However, this is purely for the generality of modeling convenience, and there is no empirical evidence that such a series provides a good match of the actual yield curve shift.

A lot of literature studying the dynamics of interest rates found that there are three major factors affecting the yield curve: level, slope, and curvature. A common method to estimate these factors is Principal Components Analysis (PCA). See details in Litterman and Scheikman [1991], Litterman, Scheikman, and Weiss [1991], Knez, Litterman, and Scheikman [1994], Nunes and Webber [1997]. Despite the abundance of research on identifying the various factors affecting bond prices, there has been little research on hedging these factors effectively. Golub and Tilman [1999] compared different risk measures, like PCA, VaR, and key rate duration for yield curve risk, but did not give hedging performance for these different measures.

In mortgage industry, practitioners generally use effective duration, and empirical duration in hedging. Goodman and Ho [1999] examined the performance of three different hedge ratios: effective duration, empirical duration, and option-implied duration, which is acquired from forward option for a given pass-through MBS. They found that the average hedging error for a monthly hedge could reach 120 bps in an 18-month period. And for a daily hedging, it is 25 bps in the same time period. They concluded that option-implied duration performs the best. However, it does not always outperform the other hedging measure, and the difference is small. Hayre and Chang [1999] compared effective duration and empirical duration, and found that effective duration calculated from OAS model are generally longer than empirical duration, and they challenged a few assumptions for effective duration calculation in OAS mode. To cite a few, the parallel yield curve shifts, absence of convexity, etc. These are also issues we addressed in our

paper, but in a more systematic way. They proposed a combined duration, which is a effective duration adjusted for correlations between changes in the yield and prices of MBS in recent market data, i.e., a combination of effective duration and empirical duration.

There has not been a unifying framework in hedging MBS with factors affecting the yield curve shifts, and we would like to pursue in this direction, since we are pretty confident about its effectiveness in reducing the hedging error, and/or reducing hedging frequency. In order to incorporate these factors into MBS hedging strategy, we need to choose an interest rate model, which can handle these factors readily. HJM model is a good choice, because it is basically driven by volatility structure, and the volatility factors can take any shape, which easily accommodate the PCA factors we identified from historical data.

In the rest of this essay, we discuss how to get volatility factors from historical interest rate data via the PCA method. In section 3, we are give the detailed implementation of HJM model with these estimated volatility factors. Then we derive the PA estimators for hedging MBS, which is very similar to Chapter 2, and we will not go into details to derive PA estimator for each state variable. In section 5, we give the detailed hedging algorithm with these hedging measures, and we discuss the performance of our hedging method in section 6. Section 7 concludes the essay, and gives potential future research directions.

3.2 Estimation of Volatility Factors via PCA

The Principal Components Analysis method is generally used to find the explanatory factors that maximize successive contributions to the variance, effectively explaining variations as a diagonal matrix. This method has been used in yield curve analysis for more than 10 years, see Litterman and Scheinkman [1991], Steeley [1990], Carverhill and Strickland [1992]. Here we give a brief description of PCA method applied in yield curve analysis:

1. Suppose we have observation of interest rates $r_i(\tau_j)$ at time $t_i, i=1, 2, \dots, n+1$, for different maturity dates τ_j .
2. Calculate the difference $d_{i,j} = r_{t_{i+1}}(\tau_j) - r_{t_i}(\tau_j)$, where the $d_{i,j}$ are regarded as observations of a random variable, d_j , that measures the successive variations in the term structure.
3. Find the covariance matrix $\Sigma = \text{cov}(d_1, \dots, d_k)$. Write $\Sigma = \{\Sigma_{i,j}\}$, where $\Sigma_{i,j} = \text{cov}(d_i, d_j)$.
4. Find an orthogonal matrix P such that $P' = P^{-1}$ and $P\Sigma P' = \text{diag}(\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \dots \geq \lambda_k$.
5. The column vectors of P are the principal components.
6. Using P , each observation of d_j can be decomposed into a linear combination of the principal components. By setting $e_i = p_i' d_j$, where p_i is the i^{th} column of P , we can find e_i , which is the corresponding coefficient for principal component $i, i=1, \dots, k$. A

small change in e_i will cause the term structure to alter by a multiple of p_i along the time horizon.

We use the weekly data of nominal zero coupon yield from January 1997 to October 2001 as the term structure data. All data were retrieved from Professor McLulloch's web site at the Department of Economics, Ohio State University, at <http://econ.ohio-state.edu/jhm/ts/ts.html>. For each observation date, interest rates are provided for maturities in monthly increments from the instantaneous rate to the 40-year rate, providing a total of 481 interest rates as principal components. Table 3.1 lists the eigen-values and % variance explained by the first ten factors, and Figure 3.1 graphs the shapes of the first four factors.

Factor	Eigenvalue	Explained(%)	Cumulative(%)
1	16.38	75.824	75.824
2	4.41	20.432	96.257
3	0.72	3.335	99.592
4	0.087	0.40	99.995
5	0.00088	0.0041	99.999
6	8.67E-05	0.00040	99.9996
7	1.59E-05	7.4E-05	99.99966
8	4.20E-06	1.9E-05	99.99968
9	4.03E-06	1.9E-05	99.99970
10	3.67E-06	1.7E-05	99.99972

Table 3.1. Statistics for Principal Components

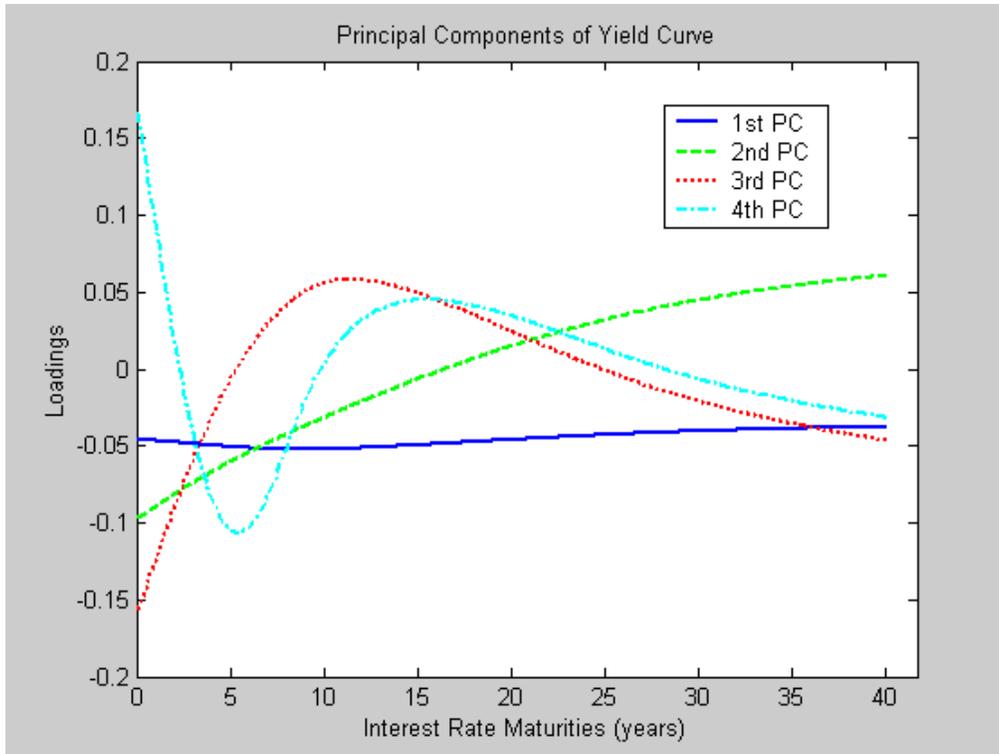


Figure 3.1 The first four principal components

The statistics indicate that the first three factors explain about 99.6% of the yield curve changes, and the first four factors explain about 99.995% of the total variance of yield curve. These results are similar to findings by Litterman and Scheikman [1991], and Nunes and Webber [1997]. Figures 3.2 and 3.3 plot the matching results with three and four factors, respectively, for a monthly yield curve shift, as well as for an annual shift. The figures indicate that four factors provide a substantially improved match, both for the short term and the long term, over three factors, so in our model we will use four factors. Thus, hedging against these factors will lead to a considerably more stable portfolio, thereby reducing hedging transactions and its associated costs.

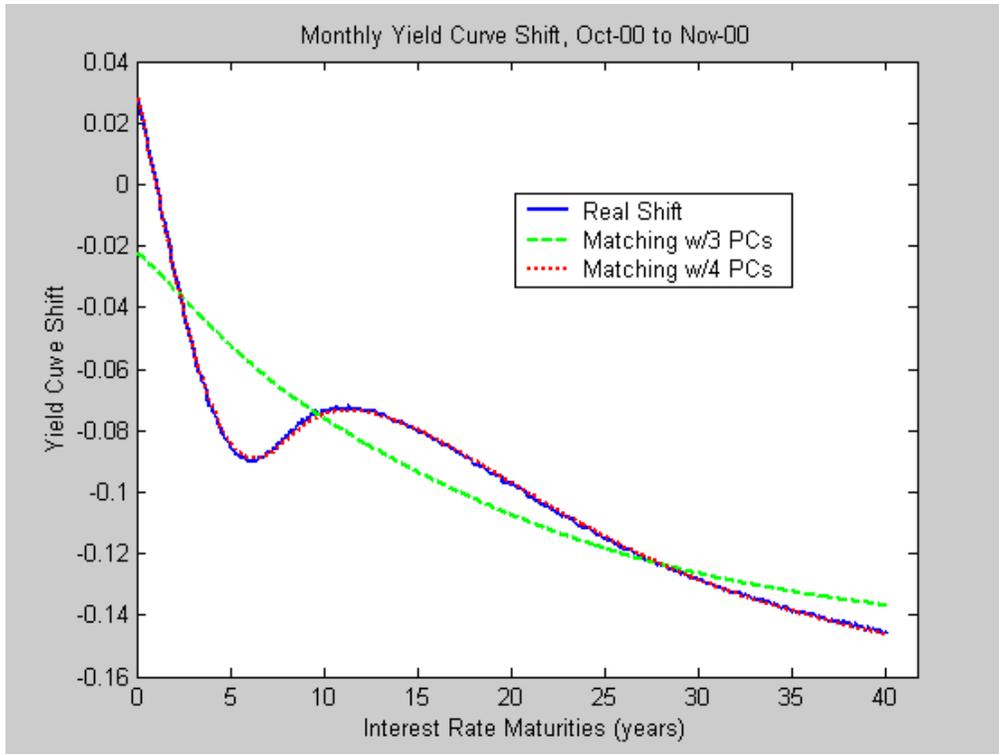


Figure 3.2 Match monthly yield curve shift

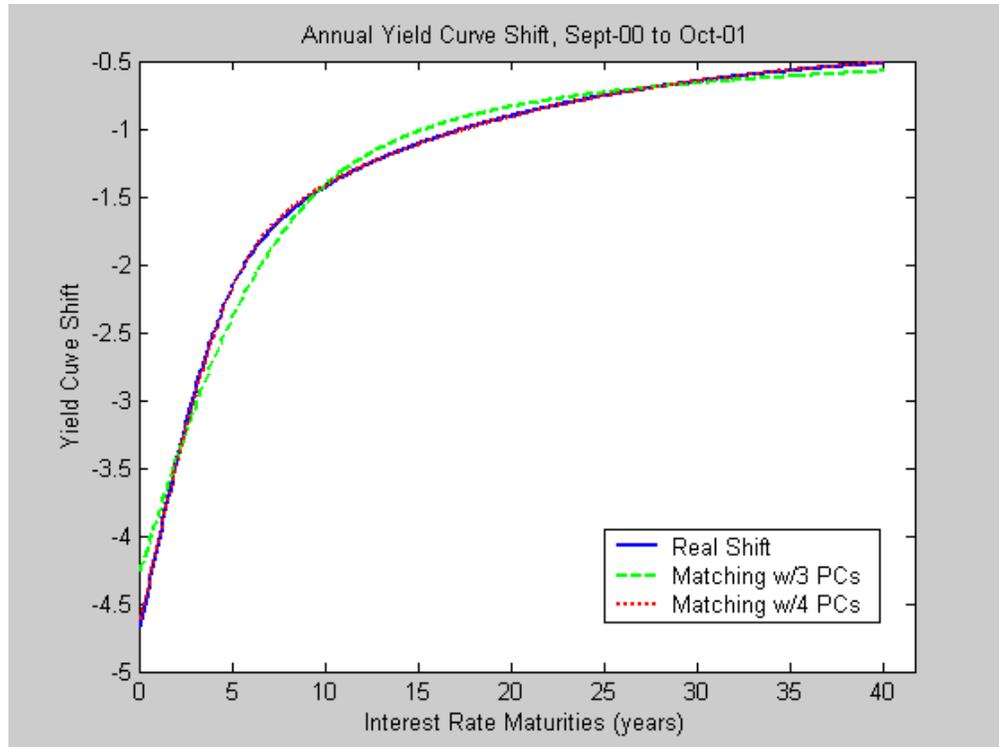


Figure 3.3 Match annual yield curve shift

3.3 Simulation in HJM Framework

This section gives the detailed implementation of HJM model, using the volatility factors identified in PCA analysis.

We know that, in a multifactor HJM framework, the dynamics of instantaneous forward rate looks like:

$$df(t, T) = m(t, T, \Omega_t) dt + \sum_{k=1}^N \sigma_k(t, T, \Omega_t) dZ_k(t), \quad (3.1)$$

where under no arbitrage assumption, the drift term is determined by volatility structure.

$$m(t, T, \Omega_t) dt = \sum_{k=1}^N \sigma_k(t, T, \Omega_t) \int_t^T \sigma_k(t, \tau, \Omega_t) d\tau. \quad (3.2)$$

Assume our volatility functions take the following form:

$$\sigma_k(t, T, \Omega_t) = \beta_k PC_k(t, T), \quad (3.3)$$

where $PC_i(t, T)$ is the principal components we get in last section;

β_i is a parameter to be calibrated to market price of interest rate derivatives.

Detailed Implementation:

1. Input data include $f(0, T)$, the instantaneous forward curve, and $\sigma_k(t, T)$, which has a specified functional form fitting into our PCA factors.
2. Start loop for maturity, if we need 10 year rate for 30 years, we need maturity at 40 year;
3. Start of time step loop;
4. Start of τ loop, to calculate $\sigma_k(t, \tau)$ from t to T ;

5. Calculate $\sigma_k^*(t, T) = \int_t^T \sigma_k(t, \tau) d\tau$, using numerical integration technique;

6. Calculate $m(t, T) = \sigma_k(t, T) * \sigma_k^*(t, T)$;

7. Advance $f(t, T)$ one more step, in our simulation, one month increment:

$$f(t + \Delta t, T) = m(t, T)\Delta t + \sum_k \sigma_k(t, T)z_{t,k};$$

where $z_{t,k}$ is a series of independent standard normal random variables.

8. End of time step loop;

9. Short rate $r(t)=f(t,t)$; Long rate $r_{10}(t) = \frac{\int_t^{t+10} f(t, \tau) d\tau}{10}$;

10. End of maturity step loop.

3.4 Deriving PA estimators in HJM Framework

Following the logic in Chapter 2, we only need to derive the PA estimator for short rate $r(t)$ and 10-year rate $r_{10}(t)$, since our prepayment model and valuation model are totally dependent on these two factors.

If we assume that in a short period of time, the principal components for yield curve volatility are going to be constant, then any interest rate yield curve shift can be decomposed of these principal components, which is to say:

$$\Delta R(0,t) = \sum_k \Delta_k PC_k(t), \quad (3.4)$$

which is analogous to (2.35). Following the same logics as in (2.51), we can have the following:

$$\begin{aligned} \frac{\partial f(0,t)}{\partial \theta} &= \frac{\partial \left(t \frac{\partial R(0,t)}{\partial t} \right)}{\partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= \frac{\partial t}{\partial \theta} \frac{\partial R(0,t)}{\partial t} + t \frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= t \frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta}, \\ \frac{\partial f(0,t)}{\partial \Delta_k} &= t \frac{\partial PC_k(t)}{\partial t} + PC_k(t). \end{aligned} \quad (3.5)$$

We know that in HJM framework:

$$r(t) = f(t,t) \quad , \quad (3.6)$$

So

$$\frac{\partial r(t)}{\partial \Delta_k} = \frac{\partial f(t,t)}{\partial \Delta_k}. \quad (3.7)$$

We also know:

$$df(t, T) = m(t, T, \Omega_t) dt + \sum_{i=1}^N \sigma_i(t, T, \Omega_t) dZ_i(t). \quad (3.8)$$

When $T=t+dt$

$$f(t+dt, t+dt) = f(t, t+dt) + df(t, t+dt). \quad (3.9)$$

So

$$\frac{\partial f(t+dt, t+dt)}{\partial \Delta_k} = \frac{\partial f(t, t+dt)}{\partial \Delta_k} + \frac{\partial [df(t, t+dt)]}{\partial \Delta_k}, \quad (3.10)$$

$$\frac{\partial [df(t, t+dt)]}{\partial \Delta_k} = \frac{\partial m(t, t+dt)}{\partial \Delta_k} + \sum_{i=1}^N \frac{\partial \sigma_i(t, t+dt)}{\partial \Delta_k} dZ_i(t). \quad (3.11)$$

If we rewrite the drift term as:

$$m(t, T) = \sum_{k=1}^N \sigma_k(t, T) * \sigma_k^*(t, T), \quad (3.12)$$

where

$$\sigma_k^*(t, T) = \int_t^T \sigma_k(t, \tau) d\tau. \quad (3.13)$$

Then gradient of $m(t, T)$ can be written as

$$\frac{\partial m(t, T)}{\partial \Delta_k} = \sum_{k=1}^N \left\{ \frac{\partial \sigma_i(t, T)}{\partial \Delta_k} * \sigma_i^*(t, T) + \sigma_i(t, T) * \frac{\partial \sigma_i^*(t, T)}{\partial \Delta_k} \right\}. \quad (3.14)$$

From the form of $\sigma_k(t, T)$

$$\sigma_k(t, T, \Omega_t) = \beta_k PC_k(t, T). \quad (3.15)$$

There is no direct relationship between $\sigma_i(t, T)$ and Δ_k ¹, so the above gradients are zero. This gives us

$$\frac{\partial f(t+dt, t+dt)}{\partial \Delta_k} = \frac{\partial f(t, t+dt)}{\partial \Delta_k} = \dots = \frac{\partial [f(0, d+dt)]}{\partial \Delta_k}. \quad (3.16)$$

For the same reason, we can derive the 10 year rate gradient as:

$$\frac{\partial r_{10}(t)}{\partial \Delta_k} = \frac{\int_t^{t+10} \frac{\partial f(t, \tau)}{\partial \Delta_k} d\tau}{10}. \quad (3.17)$$

And follow the same logic, we can get the gradients of discounting factors, prepayment rate, cash flows, present values, etc.

¹ Although observed Δ_k and β_k might have a positive correlation, i.e., when the volatility is high, the observed shift also might have bigger magnitude. But they have total different meaning, β_k is the parameter to calibrate to market price, and Δ_k is the observed shift in yield curve.

3.5 Hedging MBS in HJM Framework

This section gives a detailed implementation of our hedging algorithm.

Security to be hedged: MBS

Hedging Instruments: Portfolio of {MBS, Treasury bonds with different maturities}

Hedging Method: Dynamic hedging using PCA duration. vs. Conventional duration and convexity hedging

Hedging Parameters: PCA duration

Hedging Error: The net present value of the portfolio, which has initial value of zero

Hedging Efficiency: Reduce hedging Error

Hedging Strategy: Construct a portfolio, consisting of MBS and various T-notes, bonds, with 0 face value. Duration matched to 0. Rebalance at each time period to match the hedging parameters; compare the results with duration and convexity hedging.

There are two issues we need to pay special attention to, in order to effectively execute the hedging strategy.

Issue 1: With the coupon payment and prepayment of MBS, what needs to be done with this extra cash flow?

Answer: Use this cash flow to rebalance the portfolio, basically to change the weights of Treasury bonds holdings. If the position is short in MBS, and long in Treasury bonds, we need to sell the Treasury bonds to honor the MBS payment.

Issue 2: Some Treasury bonds used to hedging the MBS will expire before the MBS maturity date. This will hurt the capacity of available hedging instruments.

Answer: We only hedge the MBS for a short period of time, e.g. 3 years, and then we can use Treasury bonds with greater or equal to 3 years maturities. Another solution is to introduce an extra hedging instrument when there is one expiring at that period.

Hedging Framework

1. At time 0, get the MBS price, gradients (PCA duration) by simulation (360x300 simulation needed). Zero coupon Treasury bonds price and gradients should be directly available from the yield curve, and the PCA factors;
2. Construct the portfolio, by shorting MBS to finance Treasury bonds; match the duration, and get the corresponding weights;
3. At time 1, use HJM model to update the yield curve, then get the new price and gradients of MBS as well as those of Treasury bonds;
4. Use MBS payment to rebalance portfolio (MBS payment is deterministic upon the last period yield curve);
5. Repeat 3, 4 for next month, till the end of hedging period;
6. Check the effectiveness of hedging strategy.

Implementation of Hedging MBS with Treasury Bonds

1. Get mortgage information;
2. Get historical yield curve data;
3. Get Principal Components Factors;
4. Start clock for hedging period: $m=0$
5. Calculate $MBS_Price(m)$, $MBS_Duration(m)_{4 \times 1}$, $Payment(m)$, $PrincipalPayment(m)$;

6. Choose hedging instrument: $Treas_Portfolio=[12\ 36\ 60\ 84\ 120]$, each element represent months to maturity;
7. Calculate Treasury bond price $Treas_Price(m)_{5 \times 1}$, 5 hedging components are needed because of 5 factors to hedge: Price, and Duration $_{4 \times 1}$.

$Treas_Duration(m)_{4 \times 5}$.

8. Solving for hedging ratio $W(m)$:

$$\begin{bmatrix} Treas_price(m)' \\ Treas_Duration(m) \end{bmatrix}_{5 \times 5} W(m)_{5 \times 1} = \begin{bmatrix} MBS_price(m) \\ MBS_Duration(m) \end{bmatrix}_{5 \times 1}, \text{ if } m=0; \quad (3.18)$$

$$\begin{bmatrix} Treas_price(m)' \\ Treas_Duration(m) \end{bmatrix}_{5 \times 5} W(m)_{5 \times 1} = \begin{bmatrix} Treas_price(m)'W(m-1) - MBS_payment(m-1) \\ MBS_Duration(m) \end{bmatrix}_{5 \times 1}, \text{ if } m>0. \quad (3.19)$$

9. Calculate hedging error:

$$error(m) = MBS_price(m) - Treas_price(m)'W(m-1) + MBS_payment(m-1)$$

10. Update loan.UPB=loan.UPB-PrincipalPayment(m);
11. Update loan.WAM=loan.WAM-1/12;
12. Update $Treas_Portfolio=Treas_Portfolio-1$;
13. $m=m+1$, go back to 5 until hedging period ends.

3.6 Hedging Performance Analysis

In this section, we compare the hedging performance of our PCA-based hedging and traditional duration and convexity based hedging for a FRM30 MBS instrument.

The principal balance of the MBS is \$4 million. We are selling short this MBS at the market price, and use the proceeds to buy treasury bonds. Initial net present value of the hedging portfolio is zero. Every month, we try to rebalance the portfolio, and we sell part of our bonds to meet the payment obligation of the MBS. Hedging error is defined as the net present value of current portfolio at each time point.

We carry on this practice for 22 months, during which our PCA estimation does not change dramatically. We repeat the hedging practice for 25 simulations, which is relatively few, because the simulation scheme takes an extremely long time. The PCA-based hedging takes around 40 CPU hours to finish, while the duration and convexity based hedging takes 120 CPU hours to complete the task.

Figure 3.4 shows the hedging performance of three PCA factors, while Figure 3.5 shows the hedging performance of duration and convexity hedging. We can see that the standard deviation of PCA-based hedging ranges from \$4000 to \$20000, which is 10 bps to 50 bps for a \$4 million portfolio. Consider the standard deviation of duration and convexity based hedging, which ranges from \$60000 to \$200000, i.e. 150 bps to 500 bps of the hedging balance. The hedging improvement is obvious.

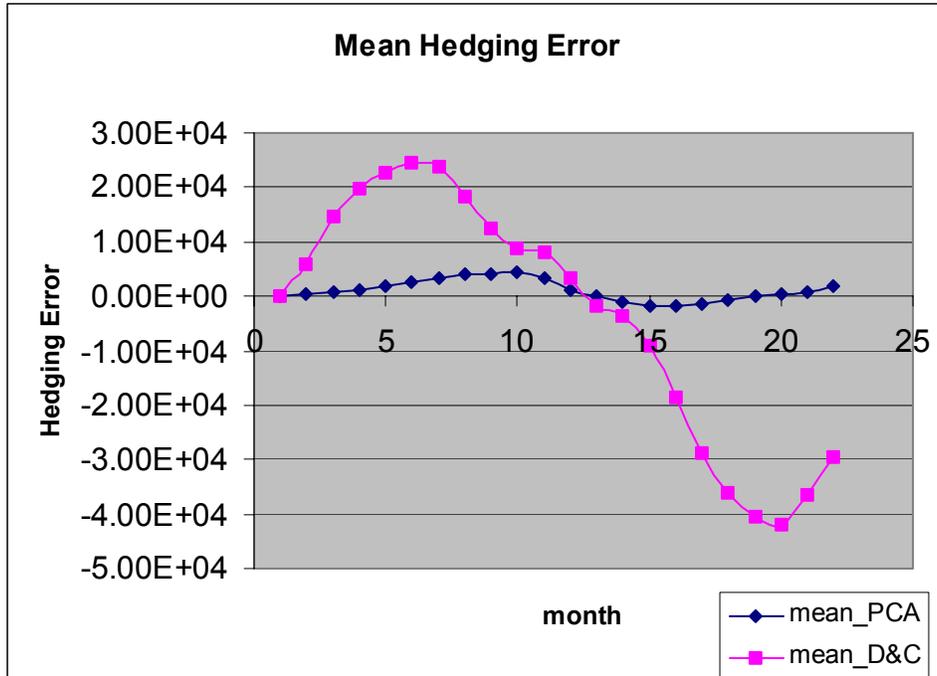


Figure 3.4 Mean Hedging Error of PCA vs. D&C

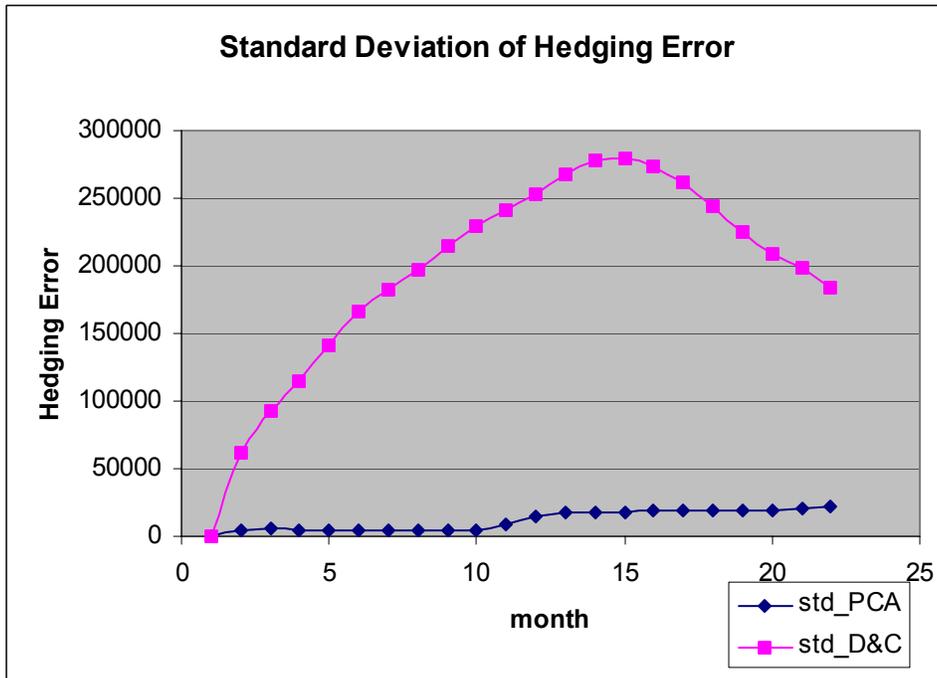


Figure 3.5 STD of Hedging Error: PCA vs. D&C

3.7 Conclusion

In this essay, we proposed a new method to hedge the interest risk of MBS, based on PCA factors estimated from historical interest rate data. We estimated the PA estimators for hedging MBS, and implemented the hedging with a dynamically rebalancing portfolio of MBS and Treasury bonds. We achieved much better hedging efficiency, compared with traditional hedging, not only in the measure of mean hedging error, but also in the standard deviation of hedging error. We made the following contribution:

- A unified hedging framework for hedging yield curve shift and volatility factors;
- Improved hedging efficiency compared with traditional duration and convexity based hedging. Our monthly hedging get very close results to daily hedging with traditional hedging method.

We would like to pursue in the following directions for our future research:

- Apply this hedging method to more sophisticated prepayment models, and analyze the robustness of this hedging algorithm;
- Improve computational efficiency of the algorithm, which is now very time consuming.

Chapter 4

Hedging the Credit Risk of MBS: A Random Coefficient Approach

4.1 Motivation

In our previous two chapters, we have assumed that the credit risk of the MBS is totally absorbed by the MBS issuer, and the MBS investor only needs to hedge the interest rate risk due to voluntary prepayment, including housing turnover and refinancing. This assumption is reasonable since in the secondary market for conforming mortgages, the three major MBS issuers, Ginnie Mae, Fannie Mae, Freddie Mac¹, all promise that they will guarantee the principal payment when there is a default event incurred on the mortgage borrower's side. The MBS issuers have the following methods to mitigate the credit risk:

- Mortgage Collateral: Basically when a default occurs, the collateral property will become REO(Real Estate Owned), and the issuer can foreclose the mortgage and sell the property, and recover whatever is left;
- Primary Mortgage Insurance (PMI): If a borrower initiate a loan with LTV greater than 80%, she will be required to purchase mortgage insurance. If default occurs, the mortgage insurance company pays the owner of the mortgage whatever is promised in the insurance contract, generally 35% for a 95 LTV loan, and 20% for a 85 LTV loan;

¹ However, the credit risks of these agencies are different. Ginnie Mae is guaranteed by the full faith and credit of the United States government. Both Fannie Mae and Freddie Mac have \$2.2 billion line of credit with the Treasury department. Also they receive an implicit guarantee from the government, since most

- Credit Enhancement: The MBS issuer can purchase additional insurance from a mortgage insurance company for a mortgage pool. This deal is also called pool insurance, or backend credit enhancement. It is not necessarily purchased from the same company that provides PMI in the mortgage pool. There is generally an auction among several insurance companies, and the bidder with the most competitive price will be awarded the contract.

When hedging the credit risk of the MBS with credit enhancement from a third party, the issuer is now exposed to the credit risk of the counter party. In order to hedge the credit risks effectively and efficiently, we not only need to model the default behavior of the mortgage borrower, but also need to understand the credit worthiness of the counter party. The credit worthiness of a given counter party for a given time horizon is generally called a haircut². We need to model the haircuts of the counter party to perform the following tasks:

- Calculate the insurance premium, i.e., the purchase price for the insurance policy, to be paid. Apparently, a company with lower credit risk should be charging higher fees, and vice versa, since lower credit risk means better insurance policy.
- Estimate the credit loss, and report it to external investors and regulators.

Currently the Office of Federal Housing Enterprise Oversight (OFHEO), regulator of Fannie Mae and Freddie Mac, requires both GSEs to report their risk-based

market participants believe that federal government will interfere whenever any of these two giant GSEs steps in financial distress.

² This term is used to determine the reduction applied to promised payment, due to credit risk, e.g., a 25% haircut means that the promised payment needs to be reduced to 75%.

capital calculated by pre-specified haircuts for different rated counter parties.

With the implementation of Basel Accord II³, internal credit risk models could be used to calculate the haircuts, and in-house model for calculating the counter party credit risk is of extreme importance in reporting the credit risks.

We below show that a haircut is actually a credit risk measure similar to credit spread. And estimation of a haircut is equivalent to estimation of credit spread. Suppose we need to take the haircut $H(t)$ for a promised future payment of \$1, what would be the price for this promised payment? In risk neutral probability, the price should be:

$$P = \exp(-r(0,t) * t) [1 - H(t)]$$

Where $r(0,t)$ is the spot rate for maturity t .

If the promised payment can be viewed as a zero-coupon defaultable bond with face value of \$1, its price is given by

$$P = \exp\{-[r(0,t) + CS(t)] * t\}, \text{ where } CS(t) \text{ is the credit spread for maturity } t.$$

Apparently haircut and credit spread have the following one-to-one relationship:

$$H(t) = 1 - \exp(-CS(t) * t)$$

Once we estimated the credit spread of the third party's defaultable bonds, we will get the haircut we need to impose on the insurance contract automatically. So it is of critical importance that we have a good estimation for the credit spread changes of the counter party. There are generally two ways to model the dynamics of the credit spread:

³ Basel Accord II is the new international banking regulation rule proposed by Basel Committee, which will be implemented before 2006. It gives more flexibility in treating credit risk, and internal credit risk models can be used in calculating risk-based capital requirement, which is the capital a financial institution needs to reserve in order to alleviate the credit risk exposure to counter parties.

theoretical approach and empirical approach. There has been a lot of published work on the literature on the theoretical part of credit risk modeling: either using structural models (Merton [1974], Longstaff and Schwartz [1995], Collin-Dufresne and Goldstein [2001]), or reduced form (hazard rate) models (Duffie and Singleton [1999], Madan and Unal [2000]). In empirical work, different models are estimated and fitted with market data, and the performances of these models are compared in recent papers, e.g. Eom, Helwege, and Huang [2003]. Recently there has been interest in using regression to determine the factors affecting credit spread changes, because neither structural nor reduced form models can handle the large number of factors affecting credit spread changes. With a flourishing credit derivative market, there is a great need for identifying the factors that affect credit spread, in order to find possible financial instruments to hedge credit derivatives written on credit spreads.

The main model used in these researches is the simple linear regression model, e.g., Duffee [1998], Collin-Dufresne, Goldstein and Martin [2001], Huang and Kong [2003]. However, these models generally do not offer very compelling results. In this essay, we identify the theoretical drawbacks of this type of models, and address these problems with a new approach: the Random Coefficient Regression (RCR) model, which we can handle the non-constancy phenomena of credit spread sensitivities.

The rest of this essay is organized as follows. We first give a literature review in the following section; specifically we are going to discuss several important papers. In section 3, we introduce the random coefficient regression (RCR) model is given and then

apply the model to estimate the dynamics of credit spread changes, using variables from the simplest structural model. Description of the data is given in section 5, and the regression results are discussed in the next section. We show that our assumption about non-linearity and non-constancy of credit spread changes are well supported by the regression results, also the regression results are consistent with theoretical structural models, such as Merton [1974]. In the last part of this essay, we give conclusions and possible future research directions.

4.2 Literature Review

There has been a lot of recent interest on identifying the key factors affecting credit spread. One approach is to add macroeconomic variables into the traditional structural model. However, by adding new state variables, the model not only becomes more complicated in the form, but also harder to identify empirical evidence to improve pricing and hedging practice. Another approach is to concentrate on regression models. Because of the simplicity and convenience in incorporating any new state variables, regression is gaining popularity in empirical research for credit spread modeling. Regression models can be divided into two categories: regression on credit spread changes and regression on credit spread levels.

Of the first category, there are three major papers: Duffee [1998], Collin-Dufresne, Goldstein and Martin [2001], and Huang and Kong [2003].

Duffee [1998] did the pioneer work on credit spread changes regression. He analyzed the credit spread data indexed by different industry, rating group, and maturity. He used only the interest rate level and slope in the regression, and found that there is a significant negative correlation between short rate change and credit spread change. He achieved an average adjusted R^2 around 17%.

Collin-Dufresne et al. [2001] performed similar analysis, but on a lot more variables. They divided corporate bond data by leverage ratio, rating, and maturity, and performed multiple regressions. Among many regression models in the literature, this

model appears to be the most complicated. Their basic model included six basic explanatory variables: leverage, interest rate level, interest rate slope, VIX, S&P, jump probability. They achieved around 25% adjusted R^2 . They then performed principal components analysis on the residual and found that over 75% variations are due to the first component. Then they introduced new variables. The total number of variables in final regression is 19, and the adjusted R^2 improved only to 34%. Eventually they acknowledged that they could not identify the factor that contributes to the 75% residual variation, within all the proxy they constructed for liquidity, etc. They claim that the single factor driving the credit spread variation could be attributed to local demand/supply fluctuation. Interestingly, while they introduce new variables, none of these new variables are bond specific; most of them are macroeconomic variables.

Huang and Kong [2003] criticized Collin-Dufresne et al. [2001] for not having chosen the best proxies for state variables. So they performed regression on credit spread changes, with similar explanatory variables, while testing multiple proxies for each variable among eight independent variables, and choosing the best one. Also they choose to work with credit spread index OAS data (which they claim as cleaner credit spread) of rating and maturity group. They achieved adjusted R^2 of more than 40% for 5 out of 9 groups. However, the number of observations for each index is merely 67. There is no theoretical support as to why certain proxies for a state variable should perform better than other proxies. And using index data in a short time period might have alleviated the problem.

Table 4.1 gives an itemized comparison of the three papers.

	Duffee [1998]	Collin-Dufresne et al. [2001]	Huang and Kong [2003]
Category	Industry, rating, maturity	Leverage, rating, maturity	Rating and Maturity, total=9
	{All sectors, Industrial, Utility, Financial}	{<15%, 15-25%, 25-35%, 45-55%, 55%}	Investment Grade: {AA-AAA, BBB-A}
	{Aaa, Aa, A, Baa}	{AAA, AA, A, BBB, BB, B}	{1-10 yr, 10-15 yr, 15+ yr}
	{2-7, 7-15, 15-30}	{long (>12 yr), short (<9 yr)}	High Yield: {BB, B, C}
Data Type	Mean corporate yield vs. corresponding Treasury yield(self constructed index)	Corporate yield vs. corresponding Treasury yield	Index
OAS?	N	N	Y
Data Description		No option embedded, > 4 yr maturity.	
Data Range	Monthly, Jan-85 to March-95	Monthly, July-88 to Dec-97	Monthly, Jan-97 to July-02
observations		At least 25 observations for each bond	67 observations for each index
Adj. R-square	Around 17%	19% to 25% by leverage ratio	>40% for 5 out of 9
		17% to 34% by rating group	67% for B
		34% after additional variables	60% for BB

Table 4.1 Comparison of three papers on credit spread regression

Clearly we can see that all these three papers try to improve the explanatory power by either adding more state variables or cherry-picking different proxies for the same state variables. The regression model is fundamentally the same, and the improvement is marginal.

Of the second category of credit spread level regression, one major paper is Campbell and Taksler [2003]. They claim that equity volatility in the regression is almost as good as the credit rating variable. What they used in the regression is the excess return (equity return minus market return) volatility for the last 180 trading days, not the historical volatility, or implied volatility from options market.

This paper falls into the first category by modeling credit spread changes on individual bonds. We believe there are several benefits focusing on changes:

- Credit spread changes are more relevant to the modeling of credit spread dynamics, since regression on credit spread levels will have a large intercept portion, which is not very informational, because we know that there is always some credit premium associated with the corporate bond yields;
- Regression in credit spread changes is more useful in developing a hedging framework, since we can estimate the sensitivities of the credit spread changes to interest rate, leverage of the company. These sensitivities can be used to derive hedge ratios.
- Individual bond data contain far more information than the indexed data. All the firm-relevant data could enter the modeling, especially the leverage, which is a very important factor in any structural model.

However, realizing the drawbacks from simple linear regression, we adopt a more flexible approach: Random Coefficient Regression model. Although the RCR model is not new in statistics (Hildreth and Houck [1968]), it has rarely been used in financial research.

4.3 Introduction to Random Coefficient Model

The most frequently used linear model in statistics might be the following:

$$y_i = X_i\beta + \varepsilon_i, \quad (4.1)$$

where y_i is the observed response of dependent variables;

X_i is the vector of explanatory variables;

β is the vector of coefficients of the linear model;

ε_i is the error term, and $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$.

For time series data, like those we frequently encounter in financial econometrics, it can be written as:

$$y_t = \sum_k x_{tk} \beta_k + \varepsilon_t, \quad t = 1, 2, \dots, T.$$

where y_t is the observed random variable, x_{kt} are known explanatory variables, β_k are unknown constants to be estimated, and ε_t are the error terms, independently and identically distributed with mean zero, and finite variance. If exact tests of significance are desired, the error terms, ε_t , are typically assumed to be normal.

In some applications, the constancy of the coefficients, β_k , in consecutive observations may not hold. For example, a particular β_k represents the response of credit spread change for a bond to interest rate, which depends on the demand/supply ratio and market liquidity premium. If both demand/supply ratio and market liquidity premium are relatively stable, the assumption of constancy for β_k might be a tolerable approximation. However, if demand/supply ratio and market liquidity premium vary, but are not

observed, assuming β_k as the mean of a random response rate may be better than assuming the response rate to be constant.

Consider the following simple extension of model (4.1):

$$y_i = X_i \beta_i + \varepsilon_i$$

where

$$\beta_i = \beta + v_i;$$

$$E[v_i] = 0, \quad E[v_i v_i'] = \Gamma, \quad \text{and } v_i \text{ is uncorrelated with } \varepsilon_i.$$

As before, y_i is still the observed random dependent variable and X_i are known values of independent variables. In this extension, β is the mean response of the dependent variable to the independent variables and $(\beta + v_i)$ is the actual response rate in the i th observation. Combining terms, we have the model:

$$\begin{aligned} y_i &= X_i \beta + (\varepsilon_i + X_i v_i) \\ &= X_i \beta + w_i \end{aligned} \tag{4.2}$$

where

$$E[w_i] = 0,$$

$$E[w_i w_i'] = \sigma_i^2 I + X_i \Gamma X_i' = \Pi_i$$

An important difference between random coefficient and simple regression model is that the simple linear model assumes the sample is relatively homogeneous. Therefore, if the estimate for β is zero, then X will be concluded to have no effect on the dependent variable. However, random coefficients may indicate that the effect the results

from cancellation of positive effects on some observations with negative effects on other observations. As a result, the randomness of coefficient provides better explanation power even if the mean of the coefficient (β) is neutral. Dielman, Nantell, and Wright [1980] emphasized that random coefficient models are very useful in analyzing pooled cross-sectional and time series data.

In more complex cases, β_i can be parametrically expressed. For example, β_i can be a linear function of several independent variables. While this specification involves additional assumptions, it is essentially the same as the previous simple extension. This functional form is appealing in some cases, especially when there is theoretical basis for the relationship between β_i and those independent variables, and the relationship are of interests of researchers.

There are several ways to estimate the model. Details will be given in the next section.

4.4 Random Coefficient Model for Credit Spread Changes

Huang and Kong [2003] mention that the low explanatory power of theoretical determinants, documented in Collin-Dufresne et al. [2001], could be due to two reasons. The first reason is that the explanatory variables may not be the best proxies to measure the changes in default risk. The second reason is that the current existing corporate bond pricing model might miss some important systematic risk factors. We have a different opinion as to why the simple linear regression model lacks explanatory power.

We think the fundamental cause lies in the underlying assumptions of the regression model. When the regression (4.1) is estimated, there is an assumption that the coefficients are fixed. That is, the marginal effect for one unit change in X_i has the same effect (β) on y_i regardless of the characteristics of instance i .

Suppose credit spread (CS_t) is a complex function of interest rate (r_t), firm leverage (lev_t), firm asset volatility (σ_t) and other state variables (X_t), which is compatible with most structural models. Given

$$CS_t = CS(lev_t, r_t, \sigma_t, \{X_t\}),$$

we can derive the first order approximation for the change of credit spread:

$$\Delta CS \cong \frac{\partial CS}{\partial lev} \Delta lev_t + \frac{\partial CS}{\partial r} \Delta r_t + \frac{\partial CS}{\partial \sigma} \Delta \sigma_t + \frac{\partial CS}{\partial X} \Delta X_t \quad (4.3)$$

In a short time period, if all these state variables, lev , r , σ , X , do not change dramatically, a simple linear regression model could be used to estimate the coefficients. However, over a relatively long time period, as in Duffee [1998] and Collin-Dufresne et al. [2001],

which uses data spanning a 10-year period, this assumption is no longer valid. Because all these gradients themselves are functions of each state variable, they are going to change as well. Specifically, we have:

$$\begin{aligned}\frac{\partial CS}{\partial lev} &= f(lev_t, r_t, \sigma_t, X_t) \\ \frac{\partial CS}{\partial r} &= g(lev_t, r_t, \sigma_t, X_t) \\ \frac{\partial CS}{\partial \sigma} &= h(lev_t, r_t, \sigma_t, X_t)\end{aligned}$$

Merton [1974] in his seminal paper has calculated the credit spread for a zero-coupon corporate bond in the following form:

$$\begin{aligned}H &= R(\tau) - r = -\frac{1}{\tau} \ln \left\{ \Phi[h_2(d, \sigma^2 \tau)] + \frac{1}{d} \Phi[h_1(d, \sigma^2 \tau)] \right\} \\ h_1(d, \sigma^2 \tau) &= -\left[\frac{1}{2} \sigma^2 \tau - \ln(d) \right] / \sigma \sqrt{\tau} \\ h_2(d, \sigma^2 \tau) &= -\left[\frac{1}{2} \sigma^2 \tau + \ln(d) \right] / \sigma \sqrt{\tau}\end{aligned}$$

where H is the credit spread;

d is a debt ratio measure, defined as $d = Be^{-r\tau}/V$;

B is the face value of the debt;

V is the asset value of the firm;

σ is the volatility for the corporate asset process;

τ is the maturity of the zero-coupon bond.

Then he calculates the credit spread gradient to most state variables as follows:

$$H_r \equiv H_d \frac{\partial d}{\partial r} = -g[d, T] < 0;$$

$$H_d \equiv \frac{1}{\tau d} g[d, T] > 0; \Rightarrow H_{lev} = H_d \frac{\partial d}{\partial lev} = \frac{1}{\tau d} g[d, T] * e^{-r\tau} = \frac{1}{\tau * lev} g[d, T]$$

$$H_{\sigma^2} \equiv \frac{1}{2\sqrt{T}} g[d, T] \frac{\Phi'(h_1)}{\Phi(h_1)} > 0;$$

where $g[d, T]$ is the ratio of instantaneous bond return volatility to instantaneous asset return volatility, and is defined as:

$$g(d, T) = \frac{\sigma_y}{\sigma} = \frac{VF_V}{F} = \Phi[h_1(d, T)] / (P[d, T]d)$$

$P[d, T]$ is the price ratio of the defaultable bond to risk-free bond, which is defined as:

$$P[d, T] = \Phi[h_2(d, T)] + \frac{1}{d} \Phi[h_1(d, T)]$$

$$T = \sigma^2 \tau.$$

Clearly we can see that all these gradients are time varying. If we assume $g[d, T]$ is constant, or estimate it as ratio of excess return on bond to excess return on asset, then model can be estimated in a simple form.

To summarize, using a simple linear regression to estimate these coefficients over a long time horizon can lead to poor results. By adopting a random coefficient method, the model (4.3) can be restated as:

$$\begin{aligned} \Delta CS &= \alpha^{\Delta CS} + \beta_{\Delta r}^{\Delta CS} \Delta r + \beta_{\Delta lev}^{\Delta CS} \Delta lev + \beta_{\Delta \sigma}^{\Delta CS} \Delta \sigma + \varepsilon^{\Delta CS} \\ \beta_{\Delta r}^{\Delta CS} &= \alpha^r + \beta_r^r r + \beta_{lev}^r lev + \beta_{\sigma}^r \sigma + \beta_T^r T + \varepsilon^r \\ \beta_{\Delta lev}^{\Delta CS} &= \alpha^{lev} + \beta_r^{lev} r + \beta_{lev}^{lev} lev + \beta_{\sigma}^{lev} \sigma + \beta_T^{lev} T + \varepsilon^{lev} \\ \beta_{\Delta \sigma}^{\Delta CS} &= \alpha^{\sigma} + \beta_r^{\sigma} r + \beta_{lev}^{\sigma} lev + \beta_{\sigma}^{\sigma} \sigma + \beta_T^{\sigma} T + \varepsilon^{\sigma} \end{aligned} \quad (4.4)$$

Rewriting the original model, we have:

$$\begin{aligned}
\Delta CS &= \alpha^{\Delta CS} + (\alpha^r + \beta_r^r r + \beta_{lev}^r lev + \beta_\sigma^r \sigma + \beta_T^r T + \varepsilon^r) \Delta r + \\
&(\alpha^{lev} + \beta_r^{lev} r + \beta_{lev}^{lev} lev + \beta_\sigma^{lev} \sigma + \beta_T^{lev} T + \varepsilon^{lev}) \Delta lev + \\
&(\alpha^\sigma + \beta_r^\sigma r + \beta_{lev}^\sigma lev + \beta_\sigma^\sigma \sigma + \beta_T^\sigma T + \varepsilon^\sigma) \Delta \sigma + \varepsilon^{\Delta CS} \\
&= \alpha^{\Delta CS} + \alpha^r \Delta r + \alpha^{lev} \Delta lev + \alpha^\sigma \Delta \sigma + \\
&\beta_r^r r \Delta r + \beta_{lev}^r lev \Delta r + \beta_\sigma^r \sigma \Delta r + \beta_T^r T \Delta r + \\
&\beta_r^{lev} r \Delta lev + \beta_{lev}^{lev} lev \Delta lev + \beta_\sigma^{lev} \sigma \Delta lev + \beta_T^{lev} T \Delta lev + \\
&\beta_r^\sigma r \Delta \sigma + \beta_{lev}^\sigma lev \Delta \sigma + \beta_\sigma^\sigma \sigma \Delta \sigma + \beta_T^\sigma T \Delta \sigma + \\
&\varepsilon^r \Delta r + \varepsilon^{lev} \Delta lev + \varepsilon^\sigma \Delta \sigma + \varepsilon^{\Delta CS}
\end{aligned} \tag{4.5}$$

with standard assumptions:

$$E[\varepsilon_i^r] = E[\varepsilon_i^{lev}] = E[\varepsilon_i^\sigma] = E[\varepsilon_i^{\Delta CS}] = 0,$$

$E[\varepsilon_i^r \varepsilon_i^{\Delta CS}] = E[\varepsilon_i^{lev} \varepsilon_i^{\Delta CS}] = E[\varepsilon_i^\sigma \varepsilon_i^{\Delta CS}] = 0$, which basically states that the error terms are uncorrelated.

The coefficients in the original model now are random, and have their own specifications.

The difference between our model and the simple linear regression model exists not only in the specification of coefficients, but also in the difference in the assumption of the error terms. The homoscedasticity assumption in a simple linear regression is relaxed in our model.

Apparently, the OLS estimates for β in model (4.5) are still consistent under the assumption stated above because $X_i \beta$ and ε^{sum} (sum of all error terms) are uncorrelated.

However, the estimates are no longer efficient⁴. Both Feasible Generalized Least Square (FGLS) and the White robust estimator can provide consistent and efficient estimates (Greene [1997]). We tried both methods in our application. The difference between the two estimates is small. We only report the White estimates, because FGLS estimates involve additional weights from the variance-covariance matrix. If the form of the heteroscedasticity and parameters involved are known, then FGLS will be a better choice; otherwise, the White estimator, which is robust to unknown heteroscedasticity, is certainly appealing, because the weights introduced by FGLS may add additional variation into the slope estimates.

⁴ Efficiency of estimators: an unbiased estimator $\hat{\theta}_1$ is more efficient than another unbiased estimator $\hat{\theta}_2$ if the sampling variance of $\hat{\theta}_1$ is less than that of $\hat{\theta}_2$. That is, $Var[\hat{\theta}_1] < Var[\hat{\theta}_2]$.

4.5 Data Description

We extract data from three databases: Warga bond database⁵, CRSP⁶, and COMPUSTAT⁷ for different financial data.

Warga database, which is also known as the Lehman Brothers Fixed Income Database, contains the most comprehensive bond data for academia. We only choose those bond that satisfy the following standards:

1. Dealer quoted price, instead of matrix price, since it has been pointed out that matrix price could produce some problems (Sarig and Warga [1989]);
2. At least 30 consecutive observations;
3. Non-callable and non-putable. This would eliminate the optionality-induced premium in the yield spread;
4. Bond with maturity greater than four years, since it is well known structural model is less accurate for short maturity bonds.

Based on these standards, we end up with credit spread time series for 728 bonds, with 45627 observations. We have bond price, yield to maturity, maturity date, and duration data from this database. These data are used later to construct the credit spread.

⁵ The Warga Fixed Income Securities Database (FISD) for academia is a collection of publicly offered U.S. Corporate and Agency bond data. Produced by LJS Global Information Services, Inc., this fixed income database engine is used by Reuters/Telerate and Bridge/EJV. These vendors collectively account for 83% of trader screens.

⁶ The CRSP Database provides access to NYSE, AMEX and Nasdaq daily and monthly securities prices, as well as to other historical data related to over 20,000 companies. The data is produced, and updated quarterly, by the Center for Research in Security Prices (CRSP), a financial research center at the Graduate School of Business at The University of Chicago.

⁷ The Standard & Poor's COMPUSTAT® databases contain financial, statistical, and market data for different regions of the world. The databases are searched using Standard & Poor's Research Insight® software, which enables data queries, retrieval, manipulation and analysis. The software includes predefined sets for searching different types of data and allows the user to generate this data using predefined reports.

We acquired the equity data from CRSP database. The equity data is linked with bond data via the CRSP permno (permanent number) index. We retrieved the daily equity data for 322 companies from January 1987 to March 1998. These data are used later to construct the mark-to-market equity, as well as stock return volatility.

COMPUSTAT database provided us with the balance sheet information. It is also linked to the CRSP database via the permno index. We retrieved the quarterly balance sheet data for the same 322 companies from January 1987 to March 1998. Then we interpolated the total asset value and total liability value for the months between. These data are used later to construct leverage ratio.

Here we provide a brief description for the data we constructed in the regression.

Treasury curve is constructed by using linear interpolation. The treasury rate source is the constant maturity Treasury (CMT) rate of H.15 release from the Federal Reserve web site. We use the 3-year, 5-year, 7-year, 10-year, 30-year treasury rates. 20-year treasury rate is disregarded because its discontinuity for the observation period. Interest rate level is defined as 10-year Treasury rate.

Credit Spread is calculated as the difference between bond yield and treasury rate with the same maturity⁸. As a convention, only quoted price are used, excluding callable and puttable bonds. The bond yield we use is the yield to maturity. Data ranges from July 1988 to March 1998.

Firm leverage is calculated by the following formula:

$$\text{leverage} = \frac{\text{Total Liability}}{\text{Market Value of Equity} + \text{Total Liability}}$$

Total liability in each quarter is acquired from Compustat database; data in between months are interpolated linearly. Market value of equity is acquired by multiplying stock price with shares outstanding. Firm leverage is an important factor in structural models to calculate distance to default. However, different researchers have been using different numbers to calculate leverage ratio, e.g. Collin-Dufresne et al. [2001] uses the book value of debt to calculate leverage, and Moody's KMV is using short-term debt to calculate default probability.

Volatility: We considered three different measures for volatility:

1. VIX, which is the volatility index as a weighted average of eight implied volatilities of near-the-money options on the OEX (S&P 100) index. This volatility measure is identical to the Collin-Dufresne et al. [2001] paper.
2. Simple estimated standard deviation of last 20 daily returns, for the corresponding company's common stock.
3. Excess return volatility for last 180 trading days return. It is the standard deviation of the last 180 trading day's excess return, which is defined as the return minus market return (S&P 500 return). This volatility measure is identical to the Campbell and Taksler [2003] paper.

The effect of these three different volatility measures will be discussed in later sections.

⁸ Treasury rate with the same maturity is linearly interpolated from adjacent CMT rates.

4.6 Results Analysis

We are going to discuss the regression results of our new model in this section. First, we compare the coefficients of simple linear regression model with RCR model, and examine the assumption of dependence between credit spread sensitivities and state variables. Second, we examine the regression results for different rating and maturity groups. In the last subsection, we are going to examine the assumption of non-constancy of credit spread sensitivities.

4.6.1 Dependence of Credit Spread Sensitivities to State Variables

In this section, we are going to discuss the regression results of our RCR model, compared with simple linear model. Table 4.2 shows the coefficient estimation for both models, and their t-values. Applying White robust estimator, regression is performed on individual bond and the average statistics⁹ are reported. In the simple linear regression model, we can find that the sensitivity measures to interest rate change, leverage change, and volatility changes are significant, and the signs and magnitudes of coefficients are consistent with structural models and regression results in previously mentioned papers.

In the new model, we find that the following newly constructed interactive variables are significant (with $|t| > 2$):

$$r\Delta r, r\Delta\sigma, \sigma\Delta\sigma, T\Delta\sigma.$$

⁹ We followed the convention in Collin-Dufresne et al. [2001] to report these statistics. The reported coefficient values are average of the regression estimates for the coefficient on each variable. The t-statistics are calculated by dividing each reported coefficient by the standard deviation of the N estimates and scaled by \sqrt{N} .

Linear Model	Variable	beta	Standard error	t
	intercept	-0.006	0.02	-9.45
	Δr	-0.058	0.21	-7.37
	Δlev	0.977	4.77	5.53
	$\Delta \sigma$	0.005	0.01	8.48
	Adj. r^2	0.077		
RCR Model	Variable	beta		t
	intercept	0.05	0.002	-9.23
	Δr	18.47	0.68	-1.95
	Δlev	295	10.9	-1.38
	$\Delta \sigma$	1.32	0.049	-1.42
	$r\Delta r$	0.60	0.022	6.02
	$lev\Delta r$	18.59	0.69	-0.42
	$\sigma\Delta r$	0.09	0.003	1.53
	$T\Delta r$	0.84	0.031	1.42
	$r\Delta lev$	12.44	0.46	-0.45
	$lev\Delta lev$	308	11.4	0.91
	$\sigma\Delta lev$	1.68	0.062	-0.32
	$T\Delta lev$	19.41	0.72	1.44
	$r\Delta \sigma$	0.07	0.002	-3.02
	$lev\Delta \sigma$	1.54	0.057	-0.35
	$\sigma\Delta \sigma$	0.01	0.000	3.58
	$T\Delta \sigma$	0.11	0.004	2.30
	N	728		
	Adj. r^2	0.297		

Table 4.2 Comparison of RCR vs. linear model

These significant interactive terms mean that the level of state variables has a significant impact on the sensitivity of credit spread to these state variables. For example, a positive coefficient for $r\Delta r$ means that when interest rate increases, the sensitivity of credit spread to interest rate change should decrease (because the sensitivity of credit spread to interest rate is negative). In other words, in a higher interest rate environment, credit spread will be less sensitive to interest rate, given everything else unchanged. For the same reason, a negative $r\Delta \sigma$ coefficient means that in a higher interest environment, credit spread will be less sensitive to volatility, when everything else is kept constant.

Also a positive $\sigma\Delta\sigma$ coefficient would stand for high volatility sensitivity in high volatility environment. The next table summarizes the relationship we found between levels of state variables and credit spread sensitivities.

Sign of beta	$\frac{\partial CS}{\partial r}$ (<0)	$\frac{\partial CS}{\partial lev}$ (>0)	$\frac{\partial CS}{\partial \sigma}$ (>0)
Interest	+	Not Significant (N/S)	-
Leverage	N/S	N/S	N/S
Volatility	N/S	N/S	+
Maturity	N/S	N/S	+

Table 4.3 Relationship between state variables and credit spread sensitivities

These findings validate our assumption that sensitivity should be dependent on state variables. Also we would like to compare these coefficients to structural models, and to validate whether these findings are consistent with theoretical models. We take the most straightforward structural model for credit spread, the Merton [1974] model, for which we have given the derivatives of the credit spread with respect to state variables in section 4.4. Although it is possible to derive the second order derivative of credit spread to validate the relationship we found are consistent with structural model or not, we prefer to demonstrate this in a static analysis, as Merton did in the paper, which will be more intuitive. The following charts show the results of our static analysis.

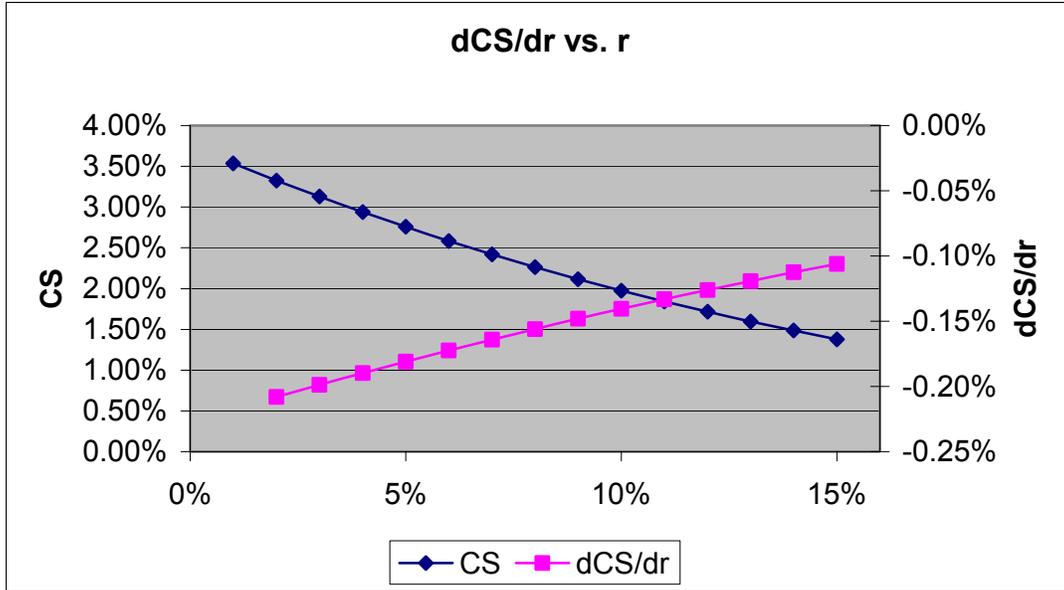


Figure 4.1 Credit Spread vs. Risk-free Rate

Figure 4.1 shows the credit spread and credit spread sensitivity to interest rate at different interest rate level. This zero-coupon bond is evaluated with 30% leverage, 30% asset volatility and 5-year maturity, which is pretty representative. We can see that while the Merton model predicts the credit spread will be decreasing while the interest rate increases, the credit spread sensitivity to interest rate is an increasing function. However, since the sensitivity measure itself is negative, being an increasing function actually means reduced sensitivity at higher interest rate level, which is consistent with our findings.

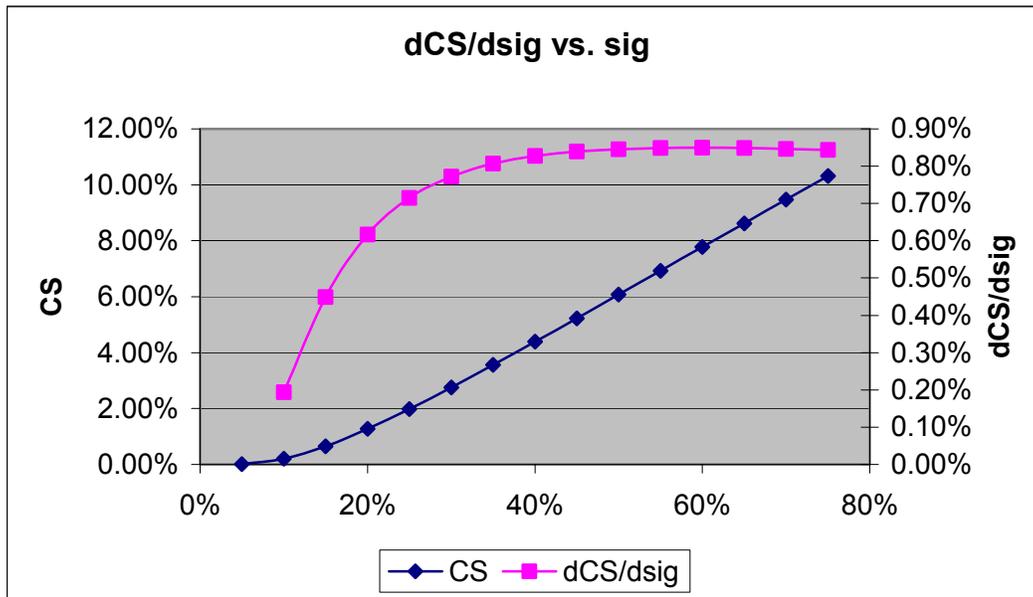


Figure 4.2 Credit Spread vs. Volatility

Figure 4.2 depicts the credit spread and its sensitivity to volatility at different volatility level. The bond is evaluated at 5% risk-free rate, with 30% leverage and 5-year maturity. From Figure 4.2, we find that credit spread is an increasing function of volatility, and sensitivity to volatility is an increasing function for the most volatility spectrum, from 5% to 60%, and after that, is pretty flat with a slight trend of decreasing.

This result is consistent with our findings.

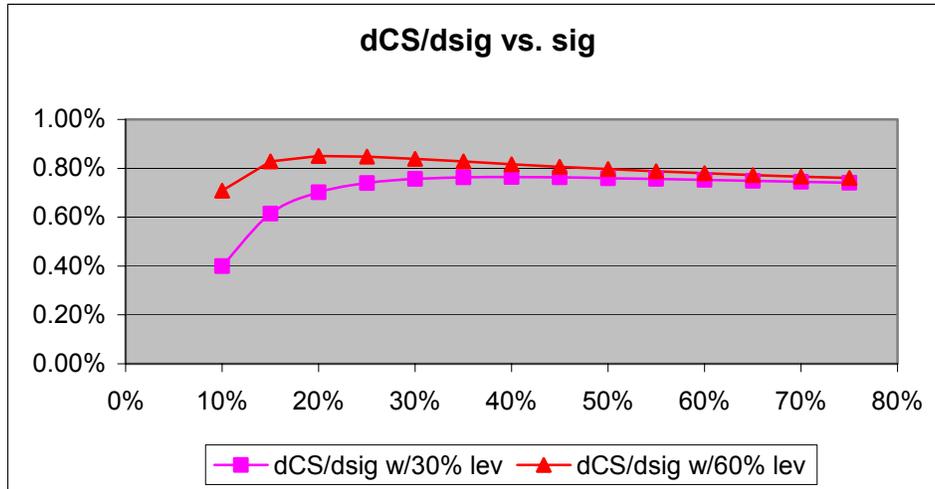


Figure 4.3 Sensitivity to Volatility at different Leverage

In order to test the robustness of the relationship between credit spread sensitivity to volatility and volatility itself, we choose two different setting for maturity and leverage. Figure 4.3 shows the credit sensitivity to volatility for a zero coupon bond with maturity of 15 year, and leverage of 30%, and 60%, at 5% interest rate level. We can see increased maturity make the curve more flat, compared with Figure 4.2. Also increasing leverage makes the yield more flat as well. Since the vast majority of our bonds have maturity less than 15 years, and leverage below 60%, we think the estimated positive coefficient of volatility on sensitivity to volatility is a valid prediction for the majority of these bonds.

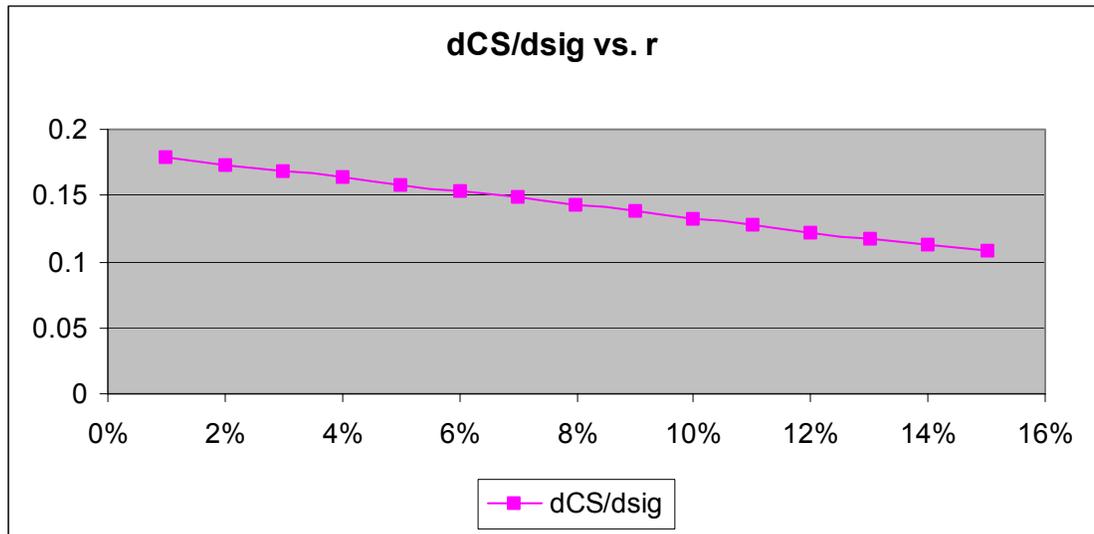


Figure 4.4 Sensitivity to Volatility vs. Interest Rate

Figure 4.4 shows the credit sensitivity to volatility for a zero coupon bond with maturity of 5 year, and leverage of 30%, volatility of 30%, at different interest rate levels. The Merton [1974] model predicts it to be a decreasing function in interest rate, which means that the higher the interest rate is, the lower the sensitivity to volatility will be. The chart is consistent with our coefficient estimator of $r\Delta\sigma$, which is negative.

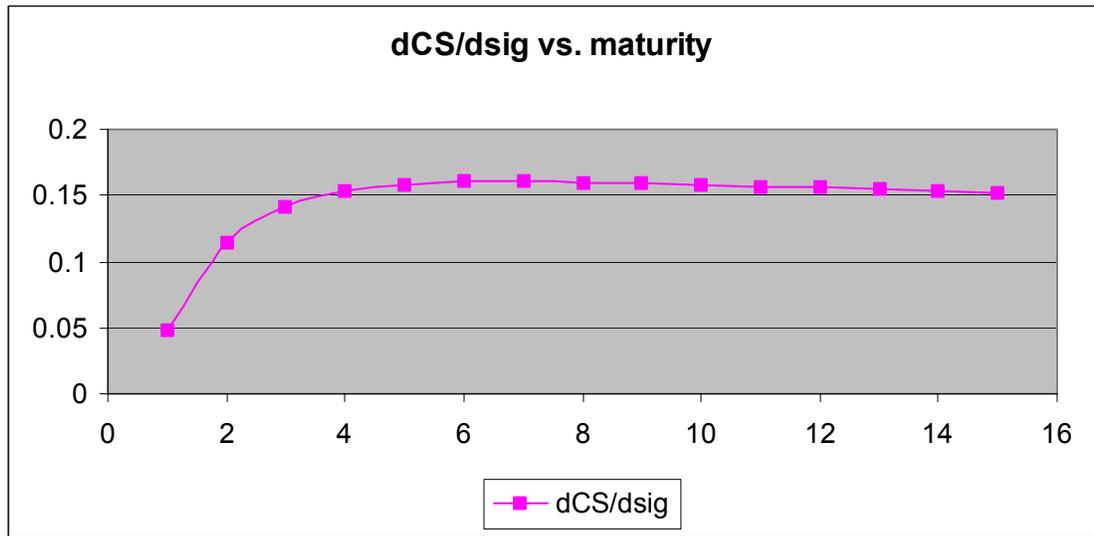


Figure 4.5 Sensitivity to Volatility vs. Maturity

Figure 4.5 shows the credit sensitivity to volatility for a zero coupon bond with leverage of 30%, volatility of 30%, at 5% rate level, with different maturities. It shows how maturity change would affect the credit spread sensitivity to volatility. The sensitivity will increase rapidly with respect to maturity till 8 years, and then decrease slightly after maturity passed 8 years. In our estimation, the coefficient is positive, but the significance is not very strong. We predict it is a mixed result of the rapid increasing and slow decreasing. Also Merton's model is based on zero coupon bond, so if applied to coupon bond, the maturity might be better replaced with duration measure, which is significantly shorter than the maturity. That would explain why we have the coefficient estimator to be positive, which is more inclined to the shorter end of the maturity.

4.6.2 Results by Rating and Maturity

In this section, we show results for different rating and maturity groups. We have five rating groups: AAA-AA, A, BBB, BB, B and others (below rating B or not rated), and three maturity groups: LONG (maturity >12 years), MEDIUM (12 years >maturity>8 years), and SHORT (maturity < 4years). The total number of combinations is 15.

	AA LONG			AA MEDIUM			AA SHORT		
	beta	std_error	t	beta	std_error	t	beta	std_error	t
intercept	-0.01	0.001	-5.13	-0.003	0.003	-1.33	-0.01	0.003	-3.34
Δr	-0.53	1.01	-0.52	0.72	0.49	1.48	0.43	0.46	0.94
Δlev	8.41	13.12	0.64	8.20	5.13	1.60	2.88	11.59	0.25
$\Delta \sigma$	-0.39	0.19	-2.00	-0.12	0.06	-2.01	-0.24	0.16	-1.51
$r\Delta r$	0.10	0.02	5.85	0.05	0.04	1.20	-0.03	0.05	-0.66
$lev\Delta r$	0.31	0.59	0.53	-0.01	0.86	-0.01	0.07	0.71	0.09
$\sigma\Delta r$	0.01	0.00	1.61	-0.02	0.01	-1.85	0.01	0.02	0.75
$T\Delta r$	0.00	0.02	-0.01	-0.07	0.03	-2.25	-0.07	0.06	-1.28
$r\Delta lev$	-0.32	0.38	-0.84	-0.04	0.48	-0.08	0.93	0.67	1.38
$lev\Delta lev$	4.42	16.88	0.26	-3.94	14.31	-0.28	-39.72	30.33	-1.31
$\sigma\Delta lev$	-0.06	0.09	-0.68	-0.22	0.16	-1.37	0.17	0.26	0.64
$T\Delta lev$	-0.15	0.51	-0.30	-0.39	0.65	-0.60	-0.07	1.17	-0.06
$r\Delta \sigma$	-0.01	0.002	-2.49	-0.004	0.004	-0.95	0.002	0.01	0.25
$lev\Delta \sigma$	0.07	0.06	1.08	-0.03	0.17	-0.20	0.34	0.24	1.42
$\sigma\Delta \sigma$	0.00	0.00	0.98	0.002	0.001	1.97	-0.001	0.002	-0.67
$T\Delta \sigma$	0.01	0.01	2.03	0.01	0.003	4.61	0.02	0.01	1.34
N	42			10			53		
Adj. r^2	0.266			0.270			0.233		

Table 4.4 RCR coefficients for AA-AAA group

For the rating group of AA-AAA, we found that the model performs better (with higher R^2) in long maturity group than short maturity group. This is consistent with previous regression model (Duffee [1998]). Also the average explanatory power for this group is below average, which is also consistent with previous regression results. AAA bonds are counted in this group because of the limited numbers in each AAA maturity group.

	A_LONG			A_MEDIUM			A_SHORT		
	beta	std_error	t	beta	std_error	t	beta	std_error	t
intercept	-0.01	0.002	-4.24	-0.01	0.003	-3.13	-0.01	0.003	-3.73
Δr	-1.66	0.68	-2.43	-1.67	0.77	-2.16	-0.33	0.30	-1.10
Δlev	16.91	10.05	1.68	-2.19	7.53	-0.29	4.91	7.65	0.64
$\Delta \sigma$	0.00	0.10	0.00	0.08	0.09	0.85	-0.01	0.05	-0.32
$r\Delta r$	0.11	0.03	3.91	0.12	0.04	3.38	0.11	0.04	2.96
$lev\Delta r$	0.01	0.51	0.02	1.14	0.92	1.24	-0.02	0.44	-0.05
$\sigma\Delta r$	0.01	0.00	1.94	0.02	0.01	2.78	-0.01	0.01	-1.44
$T\Delta r$	0.02	0.02	1.05	0.01	0.05	0.15	-0.04	0.03	-1.47
$r\Delta lev$	-0.21	0.37	-0.57	-0.33	0.76	-0.44	-0.86	0.57	-1.51
$lev\Delta lev$	-1.52	9.49	-0.16	-12.68	14.33	-0.88	-21.83	11.84	-1.84
$\sigma\Delta lev$	-0.01	0.07	-0.19	-0.10	0.19	-0.54	0.13	0.09	1.45
$T\Delta lev$	-0.60	0.45	-1.35	1.26	0.75	1.68	1.56	0.62	2.51
$r\Delta \sigma$	-0.01	0.00	-3.50	-0.02	0.01	-3.19	-0.005	0.003	-1.69
$lev\Delta \sigma$	-0.05	0.06	-0.79	-0.02	0.12	-0.20	-0.03	0.06	-0.45
$\sigma\Delta \sigma$	0.001	0.0005	2.39	-0.0001	0.001	-0.09	0.002	0.001	2.13
$T\Delta \sigma$	0.002	0.003	0.59	0.01	0.01	1.12	0.01	0.01	1.28
N	124			35			145		
Adj. r^2	0.332			0.298			0.187		

Table 4.5 RCR coefficients for A group

For the rating group of A, we also found that the performance of our RCR model deteriorates as the maturity decreases. The average explanatory power for long and medium maturity is above and near average (R^2 of 28%), which is also consistent with previous literature (Duffee [1998]).

	BBB_LONG			BBB_MEDIUM			BBB_SHORT		
	beta	std_error	t	beta	std_error	t	beta	std_error	t
intercept	-0.0175	0.00341	-5.1259	0.0040	0.00686	0.5792	-0.0150	0.00317	-4.7205
Δr	-1.4606	1.30092	-1.1228	0.4031	3.15080	0.1279	0.4132	0.67383	0.6108
Δlev	6.3313	14.42570	0.4389	-25.1286	19.48469	-1.2897	-3.3511	15.10461	-0.2210
$\Delta \sigma$	0.1922	0.16763	1.1464	0.1464	0.37627	0.3890	0.0037	0.07100	0.0517
$r\Delta r$	0.1302	0.03744	3.4778	0.0416	0.09990	0.4168	0.1056	0.09241	1.1383
$lev\Delta r$	-0.4979	0.71171	-0.6996	-1.3209	3.10094	-0.4260	-1.2636	1.04115	-1.2089
$\sigma\Delta r$	0.0309	0.00940	3.2876	-0.0354	0.02867	-1.2358	-0.0130	0.00842	-1.5402
$T\Delta r$	0.0260	0.05094	0.5094	0.1095	0.15163	0.7223	0.0049	0.13122	0.0373
$r\Delta lev$	0.3805	0.71955	0.5288	1.4138	1.85106	0.7638	-1.8690	1.80103	-1.0337
$lev\Delta lev$	7.5851	13.19033	0.5751	34.6082	17.93499	1.9296	-0.8355	20.67328	-0.0403
$\sigma\Delta lev$	0.1919	0.14222	1.3493	-0.2079	0.29622	-0.7018	-0.2272	0.33345	-0.6786
$T\Delta lev$	-0.4527	0.70592	-0.6413	0.0419	2.22403	0.0189	3.9359	2.19255	1.7880
$r\Delta \sigma$	-0.0187	0.00617	-3.0315	0.0065	0.01899	0.3438	0.0014	0.00847	0.1649
$lev\Delta \sigma$	0.0901	0.08254	1.0912	-0.1715	0.30587	-0.5607	-0.0434	0.14752	-0.2927
$\sigma\Delta \sigma$	-0.0004	0.00069	-0.5319	0.0033	0.00455	0.7291	0.0031	0.00164	1.8640
$T\Delta \sigma$	-0.0044	0.00747	-0.5856	-0.0151	0.03758	-0.4024	-0.0032	0.00888	-0.3582
N	72			24			126		
Adj. r^2	0.3159			0.1873			0.2032		

Table 4.6 RCR coefficients for BBB group

For rating group BBB, we found that the performance of our RCR model deteriorates as the maturity decreases. The average explanatory power for long and maturity is above average, which is also consistent with previous literature (Duffee [1998]). The regression results for the BBB medium group is far below average, and almost none of these variables are statistically significant, which we suspect is due to limited data problem (only 24 bonds available.)

	BB_LONG			BB_MEDIUM			BB_SHORT		
	beta	std_error	t	beta	std_error	t	beta	std_error	t
intercept	-0.06	0.01	-11.45	N/A	N/A	N/A	-0.07	0.01	-5.68
Δr	-5.18	1.57	-3.29	N/A	N/A	N/A	-4.61	2.35	-1.96
Δlev	21.89	13.93	1.57	N/A	N/A	N/A	-87.91	36.79	-2.39
$\Delta \sigma$	-0.28	0.08	-3.75	N/A	N/A	N/A	0.48	0.35	1.40
$r\Delta r$	0.37	0.04	8.32	N/A	N/A	N/A	0.34	0.16	2.17
$lev\Delta r$	-1.29	2.21	-0.59	N/A	N/A	N/A	1.03	2.03	0.51
$\sigma\Delta r$	0.03	0.01	2.96	N/A	N/A	N/A	0.01	0.04	0.16
$T\Delta r$	0.18	0.15	1.19	N/A	N/A	N/A	0.15	0.16	0.90
$r\Delta lev$	0.90	0.70	1.29	N/A	N/A	N/A	-1.99	2.28	-0.87
$lev\Delta lev$	27.62	13.75	2.01	N/A	N/A	N/A	66.40	38.54	1.72
$\sigma\Delta lev$	-0.78	0.28	-2.80	N/A	N/A	N/A	-0.10	0.83	-0.12
$T\Delta lev$	-1.72	0.66	-2.62	N/A	N/A	N/A	7.07	3.14	2.25
$r\Delta \sigma$	-0.01	0.00	-3.69	N/A	N/A	N/A	-0.01	0.02	-0.48
$lev\Delta \sigma$	-0.48	0.14	-3.43	N/A	N/A	N/A	-0.66	0.44	-1.51
$\sigma\Delta \sigma$	0.002	0.001	1.35	N/A	N/A	N/A	0.01	0.002	3.79
$T\Delta \sigma$	0.03	0.01	3.83	N/A	N/A	N/A	-0.01	0.03	-0.44
N	24			1			27		
Adj. r^2	0.123			N/A			0.104		

Table 4.7 RCR coefficients for BB group

For the rating group BB, we also see that model became worse when the maturity decreases. And for BB medium group, there is only one bond, so we cannot draw any reasonable conclusion about variable significance. Again the explanatory power is both low for long and short maturity, which could be contributed to limited bond numbers in both categories.

	B_LONG			B_MEDIUM			B_SHORT		
	beta	std_error	t	beta	std_error	t	beta	std_error	t
intercept	N/A	N/A	N/A	N/A	N/A	N/A	-0.056	0.060	-0.935
Δr	N/A	N/A	N/A	N/A	N/A	N/A	-29.005	15.404	-1.883
Δlev	N/A	N/A	N/A	N/A	N/A	N/A	30.750	309.773	0.099
$\Delta \sigma$	N/A	N/A	N/A	N/A	N/A	N/A	0.350	2.174	0.161
$r\Delta r$	N/A	N/A	N/A	N/A	N/A	N/A	1.422	0.609	2.336
$lev\Delta r$	N/A	N/A	N/A	N/A	N/A	N/A	12.422	10.146	1.224
$\sigma\Delta r$	N/A	N/A	N/A	N/A	N/A	N/A	0.075	0.183	0.409
$T\Delta r$	N/A	N/A	N/A	N/A	N/A	N/A	1.372	1.538	0.892
$r\Delta lev$	N/A	N/A	N/A	N/A	N/A	N/A	46.626	20.449	2.280
$lev\Delta lev$	N/A	N/A	N/A	N/A	N/A	N/A	-419.950	376.543	-1.115
$\sigma\Delta lev$	N/A	N/A	N/A	N/A	N/A	N/A	-3.616	7.040	-0.514
$T\Delta lev$	N/A	N/A	N/A	N/A	N/A	N/A	22.297	18.849	1.183
$r\Delta \sigma$	N/A	N/A	N/A	N/A	N/A	N/A	-0.291	0.277	-1.047
$lev\Delta \sigma$	N/A	N/A	N/A	N/A	N/A	N/A	4.618	5.316	0.869
$\sigma\Delta \sigma$	N/A	N/A	N/A	N/A	N/A	N/A	-0.069	0.069	-0.990
$T\Delta \sigma$	N/A	N/A	N/A	N/A	N/A	N/A	-0.162	0.137	-1.186
N	1			1			8		
Adj. r^2	N/A			N/A			0.500		

Table 4.8 RCR coefficients for B and other group

The total number of bonds in B and other group are very limited, so we cannot make judgments about model performance in each maturity group.

Overall, our model performs best for the A and BBB groups, as well as for longer maturities. These findings are consistent with Duffee [1998], as well as with theoretical structural models for credit spreads.

The following table shows the significance level of previously identified interactive terms in each rating and maturity group.

Interactive Terms	$r\Delta r$	$r\Delta\sigma$	$\sigma\Delta\sigma$	$T\Delta\sigma$
AA_LONG	+	-	Not Significant (N/S)	+
AA_MEDIUM	+	-	N/S	+
AA_SHORT	N/S	N/S	N/S	+
A_LONG	+	-	+	N/S
A_MEDIUM	+	-	N/S	N/S
A_SHORT	+	-	+	+
BBB_LONG	+	-	N/S	N/S
BBB_MEDIUM	N/S	N/S	N/S	N/S
BBB_SHORT	+	N/S	+	N/S
BB_LONG	+	-	+	+
BB_SHORT	+	N/S	+	N/S
B_SHORT	+	-	N/S	N/S

Table 4.9 Summary of RCR coefficients

Not surprisingly, we found that for the interactive terms, which were constructed in the RCR model, the significance levels and signs of the coefficient estimators are very consistent for each rating and maturity group. If we remove the three groups with one sample each, the $r\Delta r$ term is significant for 10 of 12 groups, which means that the interest rate level has a positive impact on the credit spread sensitivity on interest rate, no matter what rating group or maturity category. Also the impact of interest rate level on credit spread sensitivity on volatility is very consistent.

4.6.3 Non-Constancy of Credit Spread Sensitivities

The non-constancy of credit spread sensitivities would naturally be embedded in their dependence on state variables in the RCR model. However, we would like to see how they change over time, and compare it to the simple linear regression sensitivity estimators, and find out why the RCR estimators provide better accuracy.

Let's take one bond as example, the bond with CUSIP of "001765AE", one of American Airlines' long term bonds, and depict its random coefficients and constant coefficients. The following three figures show the comparison of regression coefficients with respect to interest rate changes, leverage changes, and volatility changes.

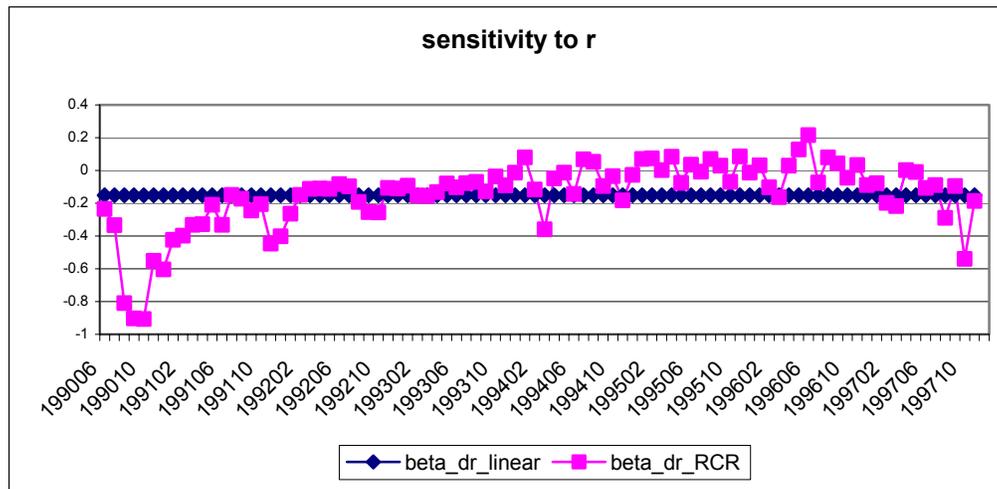


Figure 4.6 Coefficient for Δr in RCR vs. Linear Model

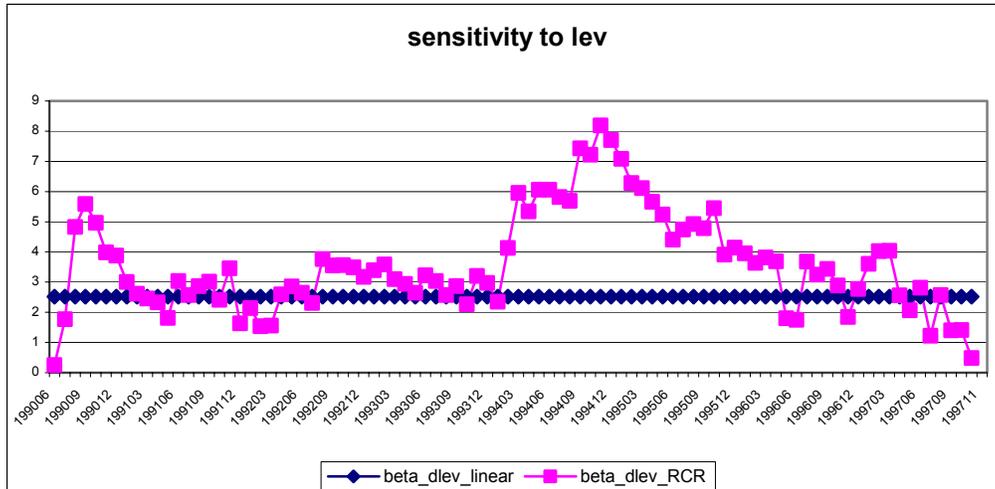


Figure 4.7 Coefficient for Δvol in RCR vs. Linear Model

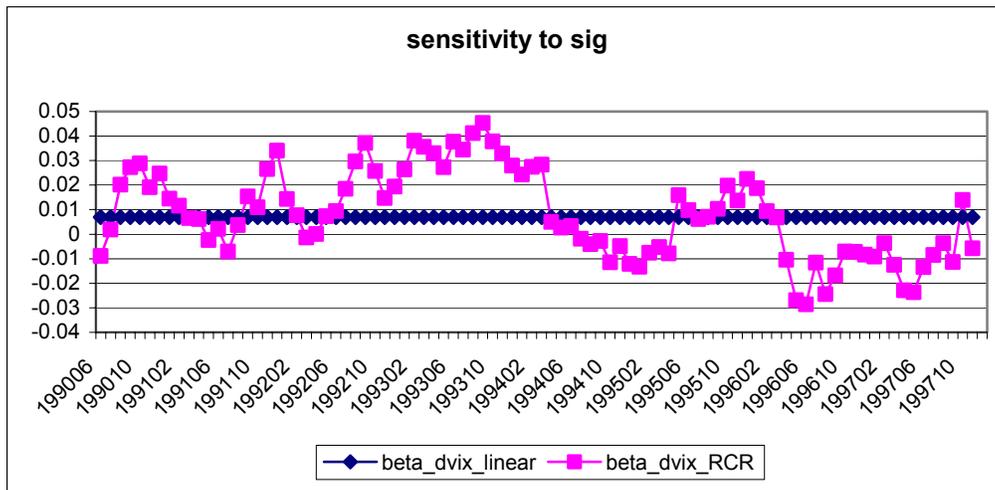


Figure 4.8 Coefficient for Δlev in RCR vs. Linear Model

There are three major findings from the graphs:

1. Sensitivity to Δr does change over time. In Merton's model, increased interest rate would increase the risk neutral drift term, thus decrease the default probability, and shrink the credit spread. In reality, Fed generally lowers interest rate to stimulate economy when there is a recession, which is the case during 1990-1992. Generally higher credit spreads are observed during a recession. That is the main reason for negative correlation between interest rate and credit spread. However,

what about during times of economic recovery or boom? It would be interesting to compare Figure 4.6 which depicts the sensitivity of credit spread to interest rate, to Figure 4.9, the history of 3-month Treasury rate, a close reflection of Fed's policy on funding rate. When the economy is recovering, lowering interest rate would have less effect on credit spread. That is exactly the case we found during 1992-1993, when the Fed continued lowering the short interest rate, and the sensitivity of credit spread to the interest rate is close to zero. Also when economy is booming, the Fed is likely to raise the interest rate, and that seems to have little effect on the credit spread. That is likely the case for 1994-1995.

2. Sensitivity to $\Delta\sigma$ also changes over time. In structural model, increase in volatility would increase the default probability, and thus widen the credit spread. However, comparing Figure 4.8 with 4.10: the history of VIX volatility index, we found that while volatility is high both in the early 90's and the late 90's, their impact on credit spread sensitivity are quite different. One explanation for this could be that during a recession, volatility is a bad thing, because it is likely that the volatility is a result of dropping equity, and investors will be really concerned with a volatility spike. However, when the economy is booming, it is likely that high volatility is introduced by rising stock prices, and investors are less likely to require high credit spread for this "good volatility".

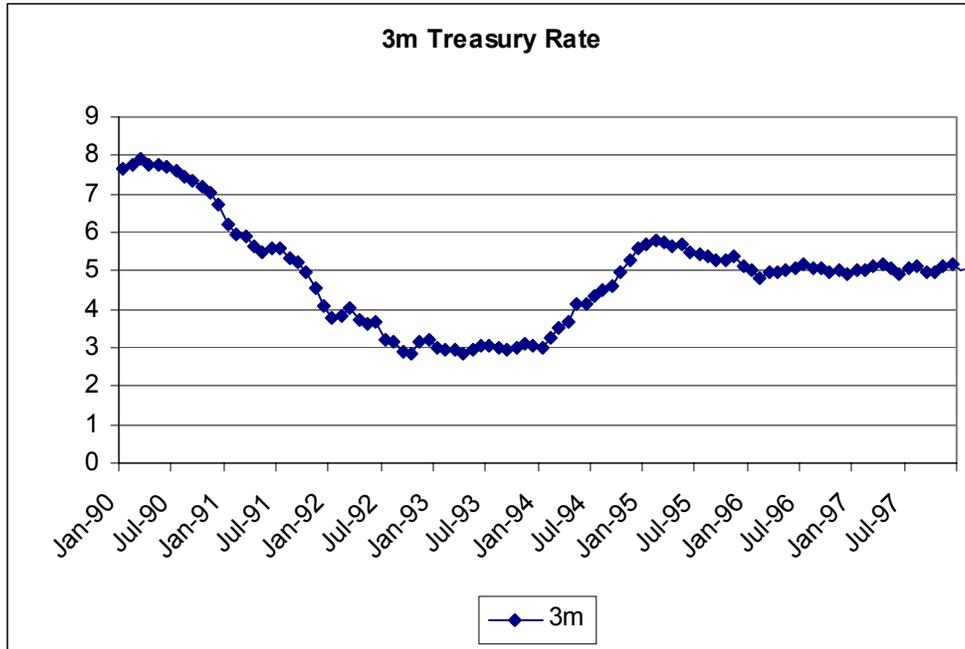


Figure 4.9 Three-Month Treasury Rate from 1990 to 1997

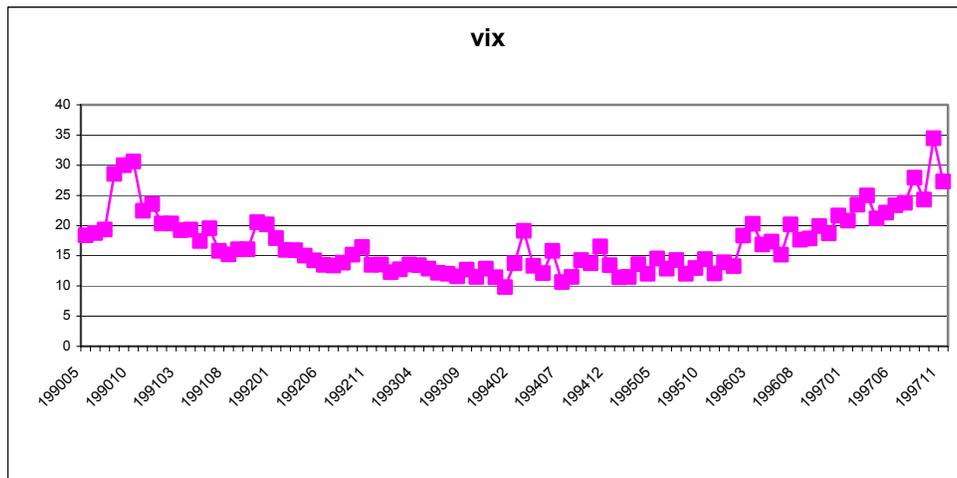


Figure 4.10 VIX index from 1990 to 1997

Of the three volatility measures we used, our results show that VIX is better than both history volatility and excess volatility, which is unanticipated. Originally we thought that since VIX is a broad market volatility index, replacing it with company specific volatility should improve our results. The reason for this phenomenon might be that credit spread response is more sensitive to market perception of risk than to historical

volatility. Also we tried the excess return volatility, which Campbell and Taksler (2003) claims to have significant explanatory power in regression of credit spread **levels**. The results are disappointing, and the adjusted R^2 is comparable to historical volatility, but not as good as VIX index. The reason might be that the credit spread itself already has a build-in premium associated with the standard deviation of excess return, but the change of credit spread is not sensitive to its change, so the regression on credit spread **levels** and **changes** will have different explanation.

4.7 Conclusions and future work

By using the RCR approach, we not only model the dynamics of credit spread sensitivities in a more consistent way with current structural model, but also achieve more explanatory power than simple linear model. Our contributions are the following:

1. The first paper to use RCR model on credit spread data;
2. The first paper to explicitly model the credit spread sensitivities with dependence against state variables, and empirically validate the dependence relationship predicted by Merton's model;
3. Higher explanatory power is achieved without adding new independent state variables. In this case, we increased the adjusted R^2 from 8% to 30%.

Obviously, there are still some unanswered questions remaining in our work. We would like to pursue future research in the following directions:

1. We can see from our results analysis from section 4.5, the theoretical sensitivity changes are not always linear with respect to state variables (Figure 4.5), and when there is a strong non-linear relationship, our predictions of coefficient are generally weaker. So can we change the functional form in the regression model for sensitivity parameters and achieve better explanatory power? And which structural model should we adopt in selecting the functional form? It will be interesting to compare the regression results for different functional forms of credit spread sensitivity from different structural models.
2. What would be a better asset volatility proxy than the VIX index? We think that the option implied volatility for each company's stock option might be a better

indicator of the market perception of risk. However, how to convert the equity volatility to asset volatility? One way to look into this might be to look at the combined bond return volatility of the specific company, which means that we need to group the bonds of the same company, instead of doing individual regression on each bond issued.

3. It would be interesting to analyze the residuals of the regression error, and find whether there exists any pattern to discover hidden significant drivers for credit spread.

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