

ABSTRACT

Title of dissertation: CAPACITY RESULTS FOR WIRELESS
NETWORKS: EFFECTS OF CORRELATION,
COOPERATION, AND INTERFERENCE

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Wireless communications has gained great popularity over the past decades. The wireless medium has many unique characteristics, which create new challenges as well as new opportunities in the communication problem. This thesis is devoted to the study of the ultimate performance limits of wireless communications. We study the effects of correlation, cooperation and interference in wireless communications from an information-theoretic perspective.

The main focus of the thesis is on capacity results for entirely wireless networks. Correlated data is an inherent part of wireless networks. We study the multiple access channel with a special form of correlated data, called common data, in fading. We obtain a characterization of the ergodic capacity region, and characterize the optimum power allocation schemes that achieve the rate tuples on the boundary of the capacity region.

In practical situations, correlated data manifests itself in more general forms than common data. We study a more general form of correlation by considering a sensor

network problem, where in addition to correlation, there is opportunity for cooperation. We first provide lower and upper bounds for the optimal performance of the sensor network under consideration. Then, we focus on the case where the underlying data satisfies some general conditions and evaluate the lower and upper bounds explicitly, and show that they are of the same order, for a wide range of power constraints. Thus, for these cases, we determine an order-optimal achievability scheme, which is separation-based, and identify the optimal performance.

Interference is unavoidable in wireless networks with multiple source-destination pairs. The capacity region of the interference channel is open except for some special cases, e.g., the discrete additive degraded interference channel. We generalize the capacity result for the discrete additive interference channel to a wider class of degraded interference channels, and provide a single-letter characterization for the capacity region.

The traditional interference channel is a simple model for four isolated nodes; and the need to modify the interference channel, so that it represents a stage of a multi-hop wireless network, is clear. We study a modified interference channel, the Gaussian Z-channel, and derive an achievable region and show that this region is almost equal to the capacity region by proving most of the converse. We also derive some additional lower and upper bounds for the capacity region of the Gaussian Z-channel.

Capacity Results for Wireless Networks: Effects of Correlation,
Cooperation and Interference

by

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DEDICATION

To my mother Jialan Su and my husband Wei...

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Chapter 1

Introduction

An information-theoretic view of the problem of communication involves the study of the ultimate performance limits of communication channels, i.e., utilizing the full capability of the communication channel to maximize the amount of information correctly conveyed. The results guide us in the design of communication systems, and inspire us to search for practical schemes that approach or reach the performance limits.

Wireless communications has gained great popularity over the past decades. The wireless medium, compared with the wired medium, has many unique characteristics. The signal strength decays rapidly with distance, the transmitted signal is affected by random fluctuations in the wireless channel, called fading, and all transmitted signals are heard by all receivers. These create new challenges as well as new opportunities in the communication problem: interference, cooperation, correlation, diversity and feedback.

Cellular networks and wireless LANs are special cases of wireless networks where only one hop of communication, to and from the base station, is wireless. The data generated by the users is independent, and the structure of the communication is pre-

determined and inflexible, in the sense that, the users can communicate only with the base stations and not among themselves. Information theoretically, uplinks and downlinks of single-cell cellular networks correspond to multiple access and broadcast channels, which have been very well studied. The capacity region of the multiple access channel was found by Liao [50] and Ahlswede [1]. In the case of fading, the sum capacity was found by Knopp and Humblet [44], and the entire capacity region was found by Tse and Hanly [79]. Yu *et al.* proposed an iterative waterfilling algorithm to compute the sum capacity of a multiple antenna multiple access channel [97]. The broadcast channel was first studied by Cover in [19] where an achievable region was found. The region was later proved to be the capacity region for degraded broadcast channels in [8,31]. The capacity region in fading was found in [49]. The sum capacity for the case of multiple antenna broadcast channel was found in [82,84,96], despite the fact that the multiple antenna broadcast channel is not degraded. Algorithms to compute the sum capacity of the broadcast channel explicitly were given in [39,95], and the entire capacity region was finally characterized in [86–88].

Recently, the research emphasis has shifted from cellular networks to networks which are entirely wireless, such as ad-hoc networks and sensor networks. In such wireless networks, the information is transmitted from the source nodes to the destination nodes through multiple hops of wireless communication. The fact that multiple source nodes communicate with multiple destination nodes through the same wireless medium makes the interference management problem much more difficult. At the same time, since such networks are expected to have more flexible structures, nodes will have the freedom of exploiting the over-heard information, and cooperate in var-

ious ways. In addition, correlated data arises naturally in such wireless networks. It arises mainly for three reasons: the observed data may be correlated (as in sensor networks) [4, 21, 76], the correlated data may be created by communication between the transmitters (as in user cooperation diversity) [73], and correlated data may result from decoding the data coming from the previous stages of a larger network (as in relaying and multi-hopping) [20, 23, 34, 72]. Hence, efficient means of exploiting correlated data, cooperating using over-heard information, and managing undesired interference in entirely wireless networks are of significant practical and theoretical importance.

In spite of recent progress, entirely wireless networks are not yet well understood. In this thesis, we tackle some of the new issues that arise in entirely wireless networks; namely, our focus will be on understanding certain aspects of correlated data, cooperation and interference.

Correlated data is an inherent part of wireless networks. Even in the simple multiple access channel, the optimal transmission of arbitrarily correlated data is an extremely difficult and open problem, with attempts made in [21, 25, 40, 41, 66]. Thus, in Chapter 2, we investigate correlated data by considering a simplified model for the correlation following Slepian and Wolf [75], which is called *common data*. In this multiple access channel, the two transmitters each have their individual messages, which will be denoted by W_1 and W_2 , respectively. Also, there is a common message W_0 , which is known to both transmitters. All three messages are independent. It can be seen that the data available at both transmitters are correlated through W_0 . The goal is to determine the rates, R_0 , R_1 and R_2 , at which all three messages

can be decoded with negligible error. The capacity will be a volume in the three dimensional space. Slepian and Wolf established the capacity region of the multiple access channel with common data for discrete memoryless channels in [75]. Prelov and van der Meulen gave the capacity expression for a Gaussian multiple access channel with common data in [67]. The characterization of the capacity region in [67] is implicit, in that the capacity region is expressed as a union of regions, and the boundary points on the capacity region are not determined explicitly. In the first part of Chapter 2, namely, in Section 2.2, we provide an explicit characterization for the capacity region and provide a simpler encoding/decoding scheme, compared to that mentioned in [75]; our encoding/decoding scheme is specially tailored for the Gaussian channel.

In the wireless medium, the presence of reflecting objects and scatterers in the environment creates fluctuations in the amplitude of the transmitted signal. This phenomenon is called fading. In information theory, the fading coefficients are modeled as channel side information. When the channel side information is known to both the transmitters and the receivers, it has been shown that, by adapting the transmission strategies according to the values of the side information, we may fully utilize the varying nature of the channel and convey more information. Hence, in the remainder of Chapter 2, we investigate optimal transmission strategies to combat fading when correlated data is present in the wireless network. More specifically, we concentrate on the case where there is fading in the multiple access channel with common data, and obtain a characterization of the ergodic capacity region. We also characterize the optimum power allocation schemes that achieve the rate tuples on

the boundary of the capacity region. In addition, we provide an iterative method for the numerical computation of the ergodic capacity region, and the optimum power control strategies.

In Chapter 2, we focus on correlated data in the special form of common data. However, in practical situations, correlated data manifests itself in more general forms. One practically interesting application is the sensor networks. Sensor networks typically compose of many sensor nodes and a collector node. The collector node is interested in some underlying random process, say temperature, over a limited region. Sensor nodes are deployed in the limited region in large numbers to distributedly sense the environment at their own locations and transmit the information to the collector node through the wireless medium. Due to the facts that the underlying random process is often correlated in space and the distances between near-by sensor nodes are very small, the data that the sensor nodes gather is often correlated. It is important to design the sensor network such that the collector node obtains accurate knowledge about the underlying environment by exploiting the correlatedness of the data gathered by the sensor nodes. In Chapter 3, we study the effects of correlation by considering a sensor network problem. More precisely, we investigate the optimal performance of a dense sensor network by studying the joint source-channel coding problem. The sensor network is composed of N sensors, where N is very large, and a single collector node. Each sensor node has the capability of taking noiseless samples from an underlying random process. Each node in the sensor network is equipped with one transmit and one receive antenna to transmit and receive signals through the wireless medium, i.e., all nodes hear a linear combination of the signals transmitted by

all other nodes at that time instant. The overall goal of the sensor network is to take measurements from a one-dimensional underlying random process $S(u)$, $0 \leq u \leq U_0$, code and transmit those measured samples to a collector node, which wishes to reconstruct the entire random process with as little distortion as possible; see Figure 1.1. Due to the existence of receive antennas at the sensor nodes and a transmit antenna at the collector node, the communication channel is a Gaussian cooperative multiple access channel with noisy feedback. We investigate the minimum achievable expected distortion and a corresponding achievability scheme when the underlying random process is Gaussian.

From an information theoretic point of view, our problem is a joint source-channel coding problem for lossy communication of correlated sources over a cooperative Gaussian multiple access channel with noisy feedback. This channel model contains both elements of correlation and cooperation. We have already mentioned that the optimal transmission of correlated data in forms other than common data is an extremely difficult problem and remains open. Furthermore, the optimal method of cooperation is not yet well understood. The simplest channel model that contains the element of cooperation is the relay channel. The relay channel contains a relay node that aids the communication of a transmitter-receiver pair. The relay channel was first proposed in [20], where achievability and converse results were provided, though, in general, they do not coincide. Since both correlation and cooperation are difficult open problems, a direct and closed-form expression for the optimal performance of sensor networks seems unlikely to be obtained. But since the number of sensor nodes is large, a weaker result that is of great practical interest, is the *order*

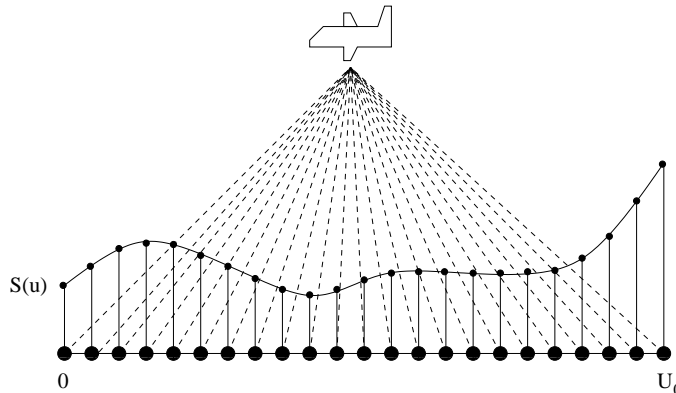


Figure 1.1: Sensor network.

optimal performance of the sensor network.

One branch of research on entirely wireless networks focuses on *large* wireless networks, which are made up of many nodes, where the number of nodes tends to infinity. In large networks, the capacity results need not be expressed in exact formulas; only the lower and upper bounds on the capacity need to be of the same order. The seminal paper of Gupta and Kumar [36] dealt with the network of many nodes in a fixed area. Messages traverse in the network in a multi-hop fashion, where relays decode the information using single user decoding techniques. Other works on the order performance of large wireless networks include [10, 34, 48, 94].

While the multi-hop wireless ad-hoc networks, where users transmit independent data and utilize single-user coding, decoding and forwarding techniques, do not scale successfully [36], Scaglione and Servetto [71] investigated the scalability of the sensor networks. Sensor networks, where the observed data is correlated, may scale successfully for two reasons: first, the correlation among the sampled data increases with the increasing number of nodes and hence, the amount of information the network needs to carry does not increase as fast as in ad-hoc wireless networks; and second,

correlated data facilitates cooperation, and may increase the information carrying capacity of the network. The goal of the sensor network in [71] was that each sensor reconstructs the data measured by all of the sensors using sensor broadcasting. In Chapter 3, we focus on the case where the reconstruction is required only at the collector node.

Marco et al. [59] is the first paper to formulate the sensor network problem considered in Chapter 3, where there is a single collector node which wishes to reconstruct the random process; see also [60]. The channel model used in [59] was similar to that used in [36], and is interference limited. The sensor encoders were limited to scalar quantization with entropy-rate coding. It was shown that the system performance becomes asymptotically poor as the number of sensors grows, i.e., the sensor network under consideration does not scale successfully. El Gamal [29] studied the same problem as in [59], but removed the constraint that the channel model is interference limited. By modelling the channel as a cooperative Gaussian multiple access channel, [29] showed that all spatially band-limited Gaussian processes can be estimated at the collector node, subject to any non-zero constraint on the mean squared distortion, i.e., the sensor network scales successfully. In Chapter 3, we study the minimum achievable expected distortion for space-limited, and thus, not band-limited, random processes, and we determine the rate at which the minimum achievable expected distortion decreases to zero as the number of nodes increases.

In Chapter 3, we first provide lower and upper bounds for the minimum achievable expected distortion for arbitrary Gaussian random processes whose Karhunen-Loeve expansion exists. Then, we focus on the case where the Gaussian random process

also satisfies some general conditions, such as the eigenvalues of its Karhunen-Loeve expansion decrease roughly inverse polynomially in order x , i.e., the k -th eigenvalue is roughly k^{-x} . For these random processes, we evaluate the lower and upper bounds explicitly, and show that they are of the same order, for a wide range of power constraints. Thus, for these random processes, under a wide range of power constraints, we determine an order-optimal achievability scheme, which is separation-based, and identify the minimum achievable expected distortion as a function of the number of nodes and the sum power constraint. We show that the minimum achievable expected distortion decreases to zero at the rate of $(\log NP(N))^{1-x}$, where $P(N)$ is the sum power constraint on the sensor nodes. In multi-user information theory, generally speaking, separation principle does not hold. However, in our case, we have found a scheme which is separation based, and is order-optimal.

In the first part of the thesis, i.e., in Chapters 2 and 3, we focused on the correlation and cooperation aspects of entirely wireless networks. In the remainder of this thesis, i.e., in Chapters 4 and 5, we will focus on the interference aspects of entirely wireless networks. Interference is unavoidable in wireless networks with multiple source-destination pairs. Since all transmissions share the same wireless medium, the desired information co-exists with undesired information in the received signal. Thus, a fundamental question that needs to be answered in order to optimize the achievable rates of wireless networks is: how should the interference be treated? To answer this question, we need to start with the investigation of the simplest model that carries the characteristics of the interference, namely, the interference channel [2]. On the other hand, such traditional interference channels are simple models for four isolated

nodes; and the need to modify the interference channel, so that it represents a stage of a multi-hop wireless network, is clear. Therefore, we studied the interference in two directions: in its traditional definition in Chapter 4, and in a modified version that reflects the fact that it is a part of a larger network in Chapter 5.

The traditional definition of the interference channel is a channel with two transmitter-receiver pairs, sharing the same communication medium [14]. The capacity region of the interference channel is open except for the special cases, for example, strong and very strong Gaussian interference channels [13, 70], additive degraded interference channels [5], a class of deterministic interference channels [28]. Some achievability and converse results were provided in [2, 14, 37, 68, 69, 81, 93], and [2, 15, 17, 45, 69], respectively. Currently, there are three known ways of treating interference: first, treating interference as noise, second, decoding interference while treating the useful information as noise and then subtracting it off, or third, time sharing the channel between the two transmitter-receiver pairs, e.g., as in TDMA (time division multiple access). Finding the capacity region and the optimal way to manage interference in a general interference channel is an extremely difficult problem, and has been open for more than thirty years. The simplest interference channel is the Z-interference channel or the degraded interference channel, where only one transmitter-receiver pair suffers from interference, i.e., the other transmitter-receiver pair sees, in effect, a clean channel.

Due to the power constraint imposed on the Gaussian interference channel, finding the capacity region of the Gaussian interference or Z-interference channel may be more difficult than the general discrete interference channels. Therefore, it is wise

to start the study of interference management with a discrete interference channel rather than the Gaussian case which involves the power constraint. For some discrete degraded interference channels, treating interference as noise is optimal [5]. Thus, to study interference management in discrete interference channels, the first question to answer is, under what conditions on the channel, is treating interference as noise optimal? In Chapter 4, we provide sufficient conditions on degraded interference channels such that treating interference as noise is optimal. We provide a single-letter characterization for the capacity region of a class of degraded interference channels. The class includes the additive degraded interference channel studied by Benzel [5] as a special case. We show that for the class of degraded interference channels studied, encoder cooperation does not increase the capacity region, and therefore, the capacity region of the class of degraded interference channels is the same as the capacity region of the corresponding degraded broadcast channel, which is known.

As mentioned before, there are clear needs to study modified versions of the interference channel such that they model building blocks of a larger network. To this end, achievability and converse results have been established for a number of modified interference channels, e.g., interference channel with common information [62], interference channel with cooperation [61], and interference channel with degraded message sets [92]. In Chapter 5, we follow the modified interference channel model proposed in [83], and study the Gaussian Z-channel. As mentioned before, an interference channel is a simple two-transmitter two-receiver network, where each transmitter has a message for only one of the receivers. A more general network structure is the X-channel [83], where the channel is the same as the interference channel except that

both transmitters have messages for both receivers. [83] has proposed a new multiuser model, called the Z-channel; see Figure 1.2. The Z-channel is a special case of the X-channel in that there is only one cross-over link and as a consequence, the transmitter that does not have a cross-over link has only one message to send. In [83], an achievable region for the Gaussian Z-channel is provided for the case of $\alpha > 1 + P_1$. In Chapter 5, we focus on the model of the Gaussian Z-channel where the cross-over link is weak, more specifically, $\alpha < 1$. We derive an achievable region and show that this region is almost equal to the capacity region by proving most of the converse. We also derive some lower and upper bounds on the capacity region. Finally, for the special case of $\alpha = 1$, we determine the capacity region exactly.

The rest of the thesis is organized as follows. In Chapter 2, we investigate the effects of correlation by studying a multiple access channel with common data. In the first part of Chapter 2, we focus on the case where there is no fading, and provide an explicit characterization of the capacity region and a simpler encoding/decoding scheme. In the remainder of Chapter 2, we concentrate on the case where there is fading in the system, and obtain a characterization of the ergodic capacity region. We also characterize the optimum power allocation schemes that achieve the rate tuples on the boundary of the capacity region. In addition, we provide an iterative method for the numerical computation of the ergodic capacity region, and the optimum power control strategies. In Chapter 3, we study correlated data in a more general form in the setting of sensor networks, where cooperation also comes into play. We investigate the optimal performance of dense sensor networks by providing separation-based lower and upper bounds for the minimum achievable expected distortion when the

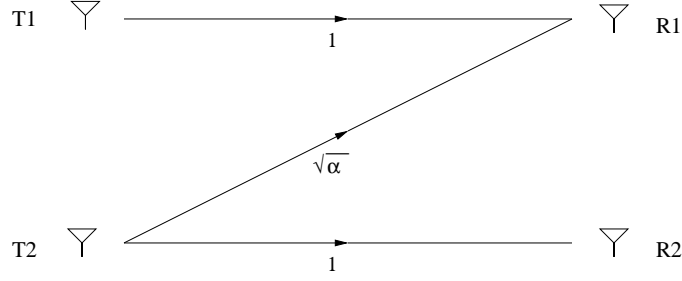


Figure 1.2: The Z-channel.

underlying random process is Gaussian. When the Gaussian random process satisfies some general conditions, we evaluate the lower and upper bounds explicitly, and show that they are of the same order for a wide range of power constraints. Thus, for these random processes, under these power constraints, we determine the rate at which the minimum achievable expected distortion decreases to zero as a function of the number of sensor nodes and the power constraint, and present a separation-based achievability scheme that is order optimal. In Chapters 4 and 5, we investigate the effects of interference in entirely wireless networks. In Chapter 4, we study the interference channel in its traditional definition and provide sufficient conditions on degraded interference channels such that treating interference as noise is optimal. We provide a single-letter characterization for the capacity region of a class of degraded interference channels, which was previously unknown. In Chapter 5, we study a modified version of the interference channel, and focus on the model of the Gaussian Z-channel where the cross-over link is weak. We derive an achievable region and show that this region is almost equal to the capacity region by proving most of the converse. We also derive some additional lower and upper bounds for the capacity region. Finally, in Chapter 6, we provide concluding remarks and suggestions for future work.

Chapter 2

Capacity Region and Optimum Power Control Strategies for Fading Gaussian Multiple Access Channels with Common Data

Correlated data arises naturally in many applications of wireless communications. In this chapter, we consider the transmission of correlated data in a multiple access channel (MAC). However, even in the simple MAC, finding capacity results for the transmission of arbitrarily correlated data is known to be extremely difficult [21, 25, 40, 41, 66]. Therefore, in this chapter, we constrain ourselves to a special kind of correlated data, correlated data in the sense of Slepian and Wolf [75], which we will call *common data*. In this MAC, the two transmitters each have their individual messages, which will be denoted by W_1 and W_2 , respectively. Also, there is a common message W_0 , which is known to both transmitters. All three messages are independent. The goal is to determine the rates, R_0 , R_1 and R_2 , at which all three messages can be decoded with negligible error. The capacity will be a volume in the three dimensional space. This model includes the traditional MAC as a special case, when $R_0 = 0$. It also includes the two-transmitter one-receiver point-to-point system as a special case,

when $R_1 = R_2 = 0$, except that we have individual power constraints for the two transmit antennas here, instead of a single sum power constraint as one would have in a point-to-point system [78].

The capacity region of the Gaussian MAC with common data with no fading is known [67, 75]. The characterization of the capacity region in [67] is implicit, in that the capacity region is expressed as a union of regions, and the boundary points on the capacity region are not determined explicitly. We first provide an explicit characterization for the capacity region and provide a simpler encoding/decoding scheme, compared to that mentioned in [75]; our encoding/decoding scheme is specially tailored for the Gaussian channel. We then concentrate on the case where there is fading in the channel and obtain a characterization of the ergodic capacity region. We also characterize the optimum power allocation schemes that achieve the rate tuples on the boundary of the capacity region. Finally, we provide an iterative method for the numerical computation of the ergodic capacity region, and the optimum power control strategies.

2.1 System Model

The Gaussian MAC we consider in this chapter has two transmitters and one receiver. Without fading, the inputs and the output are related as

$$Y = X_1 + X_2 + Z \tag{2.1}$$

where Z is a Gaussian random variable with zero-mean and unit-variance and independent of everything else. Transmitters 1 and 2 are subject to power constraints \bar{P}_1 and \bar{P}_2 , respectively. We have three independent messages W_0 , W_1 and W_2 , which are uniformly distributed in the sets $\{1, 2, \dots, 2^{nR_0}\}$, $\{1, 2, \dots, 2^{nR_1}\}$ and $\{1, 2, \dots, 2^{nR_2}\}$, respectively. Transmitter 1 knows W_0 and W_1 , and transmitter 2 knows W_0 and W_2 . Therefore, X_1 is a function of W_0, W_1 , and X_2 is a function of W_0, W_2 .

A rate triplet (R_1, R_2, R_0) is *achievable* if there exists a sequence of $((2^{nR_0} \times 2^{nR_1}, 2^{nR_0} \times 2^{nR_2}), n)$ codes with average probability of error approaching zero as n goes to infinity. Here, the probability of error is the probability that any of the three messages is decoded incorrectly. The *capacity region* is the closure of the set of achievable (R_1, R_2, R_0) .

With fading, the inputs and the output are related as

$$Y(k) = \sqrt{H_1(k)}X_1(k) + \sqrt{H_2(k)}X_2(k) + Z(k) \quad (2.2)$$

where $X_i(k)$ and $H_i(k)$ are the transmitted symbol and the fading process of user i , and $Z(k)$ is the zero-mean unit-variance Gaussian noise sample, at time k . $H_1(k)$ and $H_2(k)$ are jointly stationary and ergodic, and the stationary distribution has continuous density. $H_1(i)$ and $H_2(i)$, for all i , are independent of messages W_0, W_1, W_2 and $Z(k)$ for all k . The user signals are subject to average power constraints of \bar{P}_1 and \bar{P}_2 . We assume that both the transmitters and the receiver know $H_1(k)$ and $H_2(k)$ at time k , for all k . The *ergodic capacity region* is the closure of the set of

achievable rates in this scenario. For notational convenience, let $C(x) = \frac{1}{2} \log(1+x)$.

All logarithms are defined with respect to base e .

2.2 Capacity Region without Fading

The capacity region of the Gaussian MAC with common data is all triplets (R_1, R_2, R_0)

[67]

$$R_1 \leq C(\alpha \bar{P}_1) \quad (2.3)$$

$$R_2 \leq C(\beta \bar{P}_2) \quad (2.4)$$

$$R_1 + R_2 \leq C(\alpha \bar{P}_1 + \beta \bar{P}_2) \quad (2.5)$$

$$R_0 + R_1 + R_2 \leq C\left(\bar{P}_1 + \bar{P}_2 + 2\sqrt{(1-\alpha)(1-\beta)\bar{P}_1\bar{P}_2}\right) \quad (2.6)$$

for some α and β such that $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$.

An alternative representation of the capacity region is obtained by defining $P_1 = \alpha \bar{P}_1$, $P_2 = \beta \bar{P}_2$. With these definitions, the capacity region is all triplets (R_1, R_2, R_0) such that

$$R_1 \leq C(P_1) \quad (2.7)$$

$$R_2 \leq C(P_2) \quad (2.8)$$

$$R_1 + R_2 \leq C(P_1 + P_2) \quad (2.9)$$

$$R_0 + R_1 + R_2 \leq C(P_1 + P_2 + P_0) \quad (2.10)$$

for some $0 \leq P_1 \leq \bar{P}_1$, $0 \leq P_2 \leq \bar{P}_2$ and $P_0 = \left(\sqrt{\bar{P}_1 - P_1} + \sqrt{\bar{P}_2 - P_2} \right)^2$. For fixed P_1, P_2 , let $\mathcal{B}(P_1, P_2)$ denote the set of all rate triplets that satisfy (2.7)-(2.10). In the set $\mathcal{B}(P_1, P_2)$, certain points are of interest, which we define here: $Q = (0, 0, C(P_1 + P_2 + P_0))$, $S = (C(P_1), 0, C(P_1 + P_2 + P_0) - C(P_1))$, $T = (C(P_1), C(P_1 + P_2) - C(P_1), C(P_1 + P_2 + P_0) - C(P_1 + P_2))$ and the expressions for points V and U are the same as those for points S and T when the roles of users 1 and 2 are swapped. An example of $\mathcal{B}(P_1, P_2)$ and the corresponding points Q, S, T, U, V are shown in Figure 2.1. The capacity region is the union of $\mathcal{B}(P_1, P_2)$ over all P_1, P_2 satisfying $0 \leq P_1 \leq \bar{P}_1$ and $0 \leq P_2 \leq \bar{P}_2$.

We can interpret the capacity region in (2.7)-(2.10) in the following way. Transmitter 1 spends power P_1 for transmitting its individual message, W_1 , and the remaining power, $\bar{P}_1 - P_1$, for transmitting the common message, W_0 . Similarly, transmitter 2 spends power P_2 for transmitting its individual message, W_2 , and the remaining power, $\bar{P}_2 - P_2$, for transmitting the common message. Since the common message is known to both transmitters, the effective received power for the common message is P_0 , which may also be interpreted as the beamforming gain as in a two-transmitter one-receiver point-to-point system.

Both capacity region representations above are implicit in the sense that one has to vary some variables in their valid intervals and take the union of regions corresponding to each valid allocation of these variables in order to obtain the capacity region. Next, we seek an explicit characterization of the capacity region. Let the rate pair (R_1, R_2)

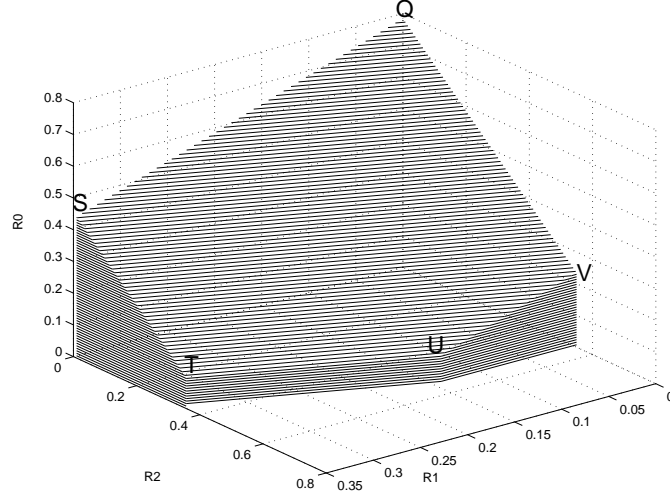


Figure 2.1: $\mathcal{B}(P_1, P_2)$.

be such that it satisfies the conditions

$$R_1 \leq C(\bar{P}_1), \quad R_2 \leq C(\bar{P}_2), \quad R_1 + R_2 \leq C(\bar{P}_1 + \bar{P}_2) \quad (2.11)$$

Let us define $c_1 = e^{2R_1} - 1$, $c_2 = e^{2R_2} - 1$ and $c = e^{2(R_1+R_2)} - 1$. Then, the powers P_1 and P_2 in representation (2.7)-(2.10) have to satisfy

$$P_1 \geq c_1, \quad P_2 \geq c_2, \quad P_1 + P_2 \geq c \quad (2.12)$$

For a fixed pair (R_1, R_2) , the largest possible R_0^* achievable is

$$R_0^* = \max_{P_1, P_2} C \left(\bar{P}_1 + \bar{P}_2 + 2\sqrt{(\bar{P}_1 - P_1)(\bar{P}_2 - P_2)} \right) - R_1 - R_2 \quad (2.13)$$

where the maximization in (2.13) is over all P_1, P_2 that satisfy (2.12). Note that (R_1, R_2, R_0^*) is on the boundary of the capacity region.

To solve the maximization problem in (2.13), it suffices to maximize $f(P_1, P_2) \triangleq (\bar{P}_1 - P_1)(\bar{P}_2 - P_2)$ subject to (2.12). Let P_1^* and P_2^* be the solution to this maximization problem. Then, (P_1^*, P_2^*) lies on the line $P_1 + P_2 = c$ since $f(P_1, P_2)$ is monotonically decreasing in both P_1 and P_2 . Hence, it suffices to maximize $f(P_1, P_2)$ subject to the constraints that $P_1 + P_2 = c$ and $c_1 \leq P_1 \leq c - c_2$. Given that $P_1 + P_2 = c$, $f(P_1, P_2)$ becomes a quadratic form and the validity of the following can be checked easily.

1. When $c_2 > \frac{\bar{P}_2 - \bar{P}_1 + c}{2}$,

$$P_1^* = c - c_2, \quad P_2^* = c_2 \quad (2.14)$$

Moreover, point U on $\mathcal{B}(P_1^*, P_2^*)$ is the (R_1, R_2, R_0^*) point.

2. When $c_1 > \frac{\bar{P}_1 - \bar{P}_2 + c}{2}$,

$$P_1^* = c_1, \quad P_2^* = c - c_1 \quad (2.15)$$

Moreover, point T on $\mathcal{B}(P_1^*, P_2^*)$ is the (R_1, R_2, R_0^*) point.

3. In all other cases,

$$P_1^* = \frac{\bar{P}_1 - \bar{P}_2 + c}{2}, \quad P_2^* = \frac{\bar{P}_2 - \bar{P}_1 + c}{2} \quad (2.16)$$

Moreover, some point on the line segment TU of $\mathcal{B}(P_1^*, P_2^*)$ is the (R_1, R_2, R_0^*) point.

This characterization is explicit because for a fixed rate pair (R_1, R_2) , we can calculate R_0^* such that (R_1, R_2, R_0^*) is on the boundary of the capacity region. With this characterization, we can easily plot the capacity region of the Gaussian MAC with common data. An example is shown in Figure 2.2 with $\bar{P}_1 = 2$ and $\bar{P}_2 = 1$.

It is interesting to note that all points on the capacity region are achieved by some point on the line segment TU of $\mathcal{B}(P_1, P_2)$ for some $0 \leq P_1 \leq \bar{P}_1, 0 \leq P_2 \leq \bar{P}_2$. All other points of $\mathcal{B}(P_1, P_2)$, for example, points Q , S and V are never on the boundary of the capacity region unless they coincide with point T or U .

Let us define $\mathcal{D}(P_1, P_2)$ to be the set of (R_1, R_2, R_0) such that

$$R_0 \leq C(P_0) \tag{2.17}$$

$$R_1 \leq C(P_1) \tag{2.18}$$

$$R_2 \leq C(P_2) \tag{2.19}$$

$$R_0 + R_1 \leq C(P_0 + P_1) \tag{2.20}$$

$$R_0 + R_2 \leq C(P_0 + P_2) \tag{2.21}$$

$$R_1 + R_2 \leq C(P_1 + P_2) \tag{2.22}$$

$$R_0 + R_1 + R_2 \leq C(P_0 + P_1 + P_2) \tag{2.23}$$

for a fixed P_1, P_2 and $P_0 = \left(\sqrt{\bar{P}_1 - P_1} + \sqrt{\bar{P}_2 - P_2} \right)^2$. In the set $\mathcal{D}(P_1, P_2)$, certain points are of interest, which we define here: Q , M , S , T , U and V are the points (R_1, R_2, R_0) where equations [(2.17), (2.20), (2.23)], [(2.17), (2.21), (2.23)], [(2.18), (2.20), (2.23)], [(2.18), (2.22), (2.23)], [(2.19), (2.22), (2.23)], [(2.19), (2.21), (2.23)]

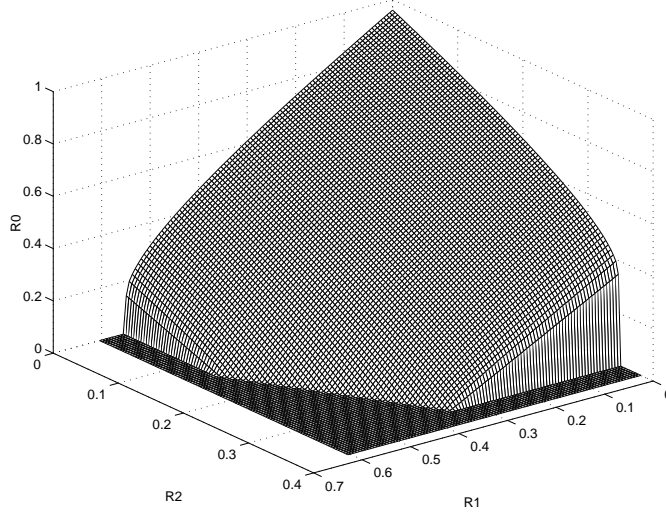


Figure 2.2: The capacity region of the Gaussian MAC with common data.

are all satisfied with equality, respectively. An example of $\mathcal{D}(P_1, P_2)$ and the corresponding points Q, M, S, T, U, V are shown in Figure 2.3.

Note that, for any given P_1 and P_2 , $\mathcal{D}(P_1, P_2)$ is a strict subset of $\mathcal{B}(P_1, P_2)$ since there are extra constraints involved in the definition of $\mathcal{D}(P_1, P_2)$. However, the capacity region of the Gaussian MAC with common data can also be written as the union of $\mathcal{D}(P_1, P_2)$ over all $0 \leq P_1 \leq \bar{P}_1$ and $0 \leq P_2 \leq \bar{P}_2$. This is because, the coordinates of the points on line segment TU of $\mathcal{B}(P_1, P_2)$ are exactly the same as those on line segment TU of $\mathcal{D}(P_1, P_2)$. Since only the line segment TU appears on the final capacity region, the union of $\mathcal{D}(P_1, P_2)$ over all $0 \leq P_1 \leq \bar{P}_1$ and $0 \leq P_2 \leq \bar{P}_2$ gives the same capacity region.

$\mathcal{D}(P_1, P_2)$ is very similar to the capacity region of the three-user Gaussian MAC with independent messages. This suggests that encoding and decoding schemes similar to those of the three-user Gaussian MAC with independent messages can be used to achieve the points on the boundary of the capacity region of the Gaussian MAC

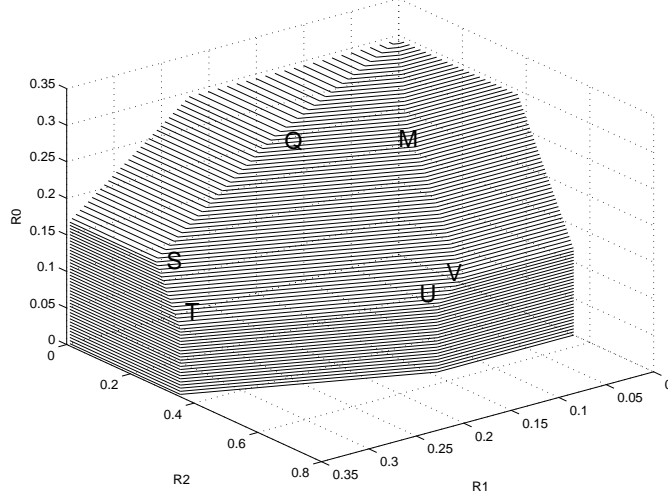


Figure 2.3: $\mathcal{D}(P_1, P_2)$.

with common data. To achieve a rate triplet (R_1, R_2, R_0) on the boundary of the capacity region, we first calculate P_1^* , P_2^* according to (2.14), (2.15) or (2.16). Depending on the values of (R_1, R_2, R_0) , we want to achieve either point T or U or some point on the line segment TU of region $\mathcal{D}(P_1^*, P_2^*)$. Points T and U can be achieved by successive decoding and, the remaining points on the line segment TU can be achieved by time sharing, just as in a three-user Gaussian MAC with independent messages.

More specifically, to achieve point T [similarly, point U], we generate three independent random codebooks C_0 , C_1 and C_2 of sizes $(2^{nR'_0}, n)$, $(2^{nR'_1}, n)$ and $(2^{nR'_2}, n)$, respectively, where (R'_1, R'_2, R'_0) is the coordinates of point T [similarly, point U]. Each entry of these codebooks is generated according to a zero-mean, unit-variance Gaussian random variable. When the messages to be transmitted are $W_0 = w_0, W_1 = w_1$ and $W_2 = w_2$, transmitter 1 transmits the sum of the w_0 th row of C_0 scaled by $\sqrt{P_1 - P_1^*}$ and the w_1 th row of C_1 scaled by $\sqrt{P_1^*}$, and transmitter 2 transmits the

sum of the w_0 th row of C_0 scaled by $\sqrt{\bar{P}_2 - P_2^*}$ and the w_2 th row of C_2 scaled by $\sqrt{P_2^*}$. The effective received power for W_0 , W_1 and W_2 are $P_0^* = \left(\sqrt{\bar{P}_1 - P_1^*} + \sqrt{\bar{P}_2 - P_2^*}\right)^2$, P_1^* and P_2^* , respectively. The receiver treats the received signal as if it comes from a three-user Gaussian MAC with independent messages, and successively decodes in the order of W_0 first, then W_2 , and finally W_1 [similarly, W_0 first, then W_1 , and finally W_2]. The encoding scheme proposed in [75] generates two large correlated codebooks, instead of three small independent codebooks as we do here. The decoding scheme proposed in [75] uses joint Maximum Likelihood (ML) detection of two codewords coming from the two large codebooks, while in our case, we can reduce the complexity by successive decoding, i.e., by applying ML detection to one codeword from a small codebook at a time, while treating other undecoded codewords as noise. If the aim is to achieve some interior point on the line segment TU , then time sharing is used between points T and U . This simpler encoding/decoding scheme is possible because we have a Gaussian channel.

Yet another way to write the capacity region, which will be useful in the development of the fading case in the next section, is the following. The capacity region is all triplets (R_1, R_2, R_0) such that inequalities (2.7)-(2.10) hold true for some P_1 , P_2 , $P_0 \geq 0$, $0 \leq \rho \leq 1$ such that $P_1 + \rho^2 P_0 = \bar{P}_1$ and $P_2 + (1 - \rho)^2 P_0 = \bar{P}_2$. This representation of the capacity region can be interpreted as follows: P_1 , P_2 and P_0 are the received powers for messages W_1 , W_2 and W_0 , respectively. In order for the received power for the common message to be P_0 , transmitter 1 spends $\rho^2 P_0$ power and transmitter 2 spends $(1 - \rho)^2 P_0$ power. Note that the two powers add up to less than P_0 which is to be expected because there is a beamforming gain for the common

message. Transmitter 1 spends a total of $P_1 + \rho^2 P_0$ power, and this must equal the power constraint \bar{P}_1 , and transmitter 2 spends a total of $P_2 + (1 - \rho)^2 P_0$ power and this must equal \bar{P}_2 . Here, ρ can be interpreted as the “portion” of the received power of the common message that comes from transmitter 1.

2.3 Capacity Region in Fading

Consider the system model in (2.2), in the simple case when $H_1(k) = h_1$ and $H_2(k) = h_2$ for all k . Using the representation of the capacity region with P_0, P_1, P_2 and ρ , the capacity region is the set of all triplets (R_1, R_2, R_0) such that inequalities (2.7)-(2.10) hold true for some $P_1, P_2, P_0 \geq 0, 0 \leq \rho \leq 1$ such that $\frac{1}{h_1} P_1 + \frac{\rho^2}{h_1} P_0 = \bar{P}_1$ and $\frac{1}{h_2} P_2 + \frac{(1-\rho)^2}{h_2} P_0 = \bar{P}_2$. Here, again, P_1, P_2 and P_0 are all received powers.

Now, we consider the case where the channel is time-varying and both the transmitters and the receiver track the channel perfectly. Let us denote the channel state as a vector $\mathbf{h} = [h_1, h_2]^T$. Let $\mathbf{p} = [p_1, p_2, p_0]^T$ be a mapping from the channel state space, \mathcal{H} , to the received power vector in \mathbb{R}_+^3 . Also, let us define ρ to be a mapping from \mathcal{H} to $[0, 1]$. Then, heuristically, when the channel state is \mathbf{h} , $\frac{p_1(\mathbf{h})}{h_1}$ is the power that transmitter 1 uses for W_1 , and $\frac{\rho(\mathbf{h})^2 p_0(\mathbf{h})}{h_1}$ is the power that transmitter 1 uses for W_0 . Similarly, $\frac{p_2(\mathbf{h})}{h_2}$ is the power that transmitter 2 uses for W_2 , and $\frac{(1-\rho(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2}$ is the power that transmitter 2 uses for W_0 . Let $\mathcal{C}_f(\mathbf{p}, \rho)$ be the set of (R_1, R_2, R_0) such

that

$$R_1 \leq E[C(p_1(\mathbf{h}))] \triangleq f_1(\mathbf{p}, \rho) \quad (2.24)$$

$$R_2 \leq E[C(p_2(\mathbf{h}))] \triangleq f_2(\mathbf{p}, \rho) \quad (2.25)$$

$$R_1 + R_2 \leq E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}))] \triangleq f_3(\mathbf{p}, \rho) \quad (2.26)$$

$$R_0 + R_1 + R_2 \leq E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h}))] \triangleq f_4(\mathbf{p}, \rho) \quad (2.27)$$

where the expectation is taken over the joint stationary distribution of the fading states h_1 and h_2 .

Theorem 2.1 *The ergodic capacity region of the fading Gaussian MAC with common data when perfect channel state information is available at the transmitters and the receiver is*

$$\mathcal{C}(\bar{P}_1, \bar{P}_2) = \bigcup_{(\mathbf{p}, \rho) \in \mathcal{F}} \mathcal{C}_f(\mathbf{p}, \rho) \quad (2.28)$$

where

$$\mathcal{F} = \left\{ (\mathbf{p}, \rho) : p_0(\mathbf{h}), p_1(\mathbf{h}), p_2(\mathbf{h}) \geq 0, 0 \leq \rho(\mathbf{h}) \leq 1, \forall \mathbf{h} \right. \\ \left. E \left[\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2 p_0(\mathbf{h})}{h_1} \right] \leq \bar{P}_1, E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] \leq \bar{P}_2 \right\} \quad (2.29)$$

A proof of Theorem 2.1 is given in Appendix 2.6.1.

To explicitly characterize the capacity region, we solve for the boundary surface of the capacity region. As in [79], the boundary surface of the capacity region $\mathcal{C}(\bar{P}_1, \bar{P}_2)$

is the closure of all points $\mathbf{R}^* = (R_1^*, R_2^*, R_0^*)$ such that \mathbf{R}^* is a solution to the problem

$$\max_{\mathbf{R}} \quad \mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0 \quad \text{subject to } \mathbf{R} \in \mathcal{C}(\bar{P}_1, \bar{P}_2) \quad (2.30)$$

for some $\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_0]^T \in \mathbb{R}_+^3$. This optimization problem is equivalent to

$$\max_{(\mathbf{R}, \tilde{P}_1, \tilde{P}_2)} \mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0 \quad \text{subject to } (\mathbf{R}, \tilde{P}_1, \tilde{P}_2) \in \mathcal{L}, \tilde{P}_1 \leq \bar{P}_1, \tilde{P}_2 \leq \bar{P}_2 \quad (2.31)$$

where

$$\mathcal{L} = \{(\mathbf{R}, \tilde{P}_1, \tilde{P}_2) : \tilde{P}_1, \tilde{P}_2 \in \mathbb{R}_+, \mathbf{R} \in \mathcal{C}(\tilde{P}_1, \tilde{P}_2)\} \quad (2.32)$$

Lemma 2.1 \mathcal{L} is a convex set.

A proof of Lemma 2.1 is given in Appendix 2.6.2.

Due to the convexity of \mathcal{L} , there exist lagrange multipliers $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T \in \mathbb{R}_+^2$ such that \mathbf{R}^* is a solution to the optimization problem

$$\max_{(\mathbf{R}, \tilde{P}_1, \tilde{P}_2) \in \mathcal{L}} \mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0 - \lambda_1 \tilde{P}_1 - \lambda_2 \tilde{P}_2 \quad (2.33)$$

Since $\mathcal{C}(\tilde{P}_1, \tilde{P}_2)$ is a union over $\mathcal{C}_f(\mathbf{p}, \rho)$, we first express $(\mathbf{R}, \tilde{P}_1, \tilde{P}_2)$ in terms of (\mathbf{p}, ρ) and then optimize over (\mathbf{p}, ρ) . It can be seen that the capacity region is unchanged

if we replace the two power constraint inequalities with equalities in (2.29). Hence,

$$\tilde{P}_1 = E \left[\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2 p_0(\mathbf{h})}{h_1} \right] \quad (2.34)$$

$$\tilde{P}_2 = E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] \quad (2.35)$$

Instead of considering all $\mathbf{R} \in \mathcal{C}(\tilde{P}_1, \tilde{P}_2)$, it suffices to consider $\mathbf{R} \in \mathcal{C}_f(\mathbf{p}, \rho)$ that maximizes $\mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0$ for each (\mathbf{p}, ρ) . Thus, we first focus on the following problem:

$$\max_{\mathbf{R}} \quad \mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0 \quad \text{subject to } \mathbf{R} \in \mathcal{C}_f(\mathbf{p}, \rho) \quad (2.36)$$

where $\mathcal{C}_f(\mathbf{p}, \rho)$ is a region with shape as in Figure 2.1. Due to the nature of $\mathcal{C}_f(\mathbf{p}, \rho)$, when $\mu_0 \geq \max(\mu_1, \mu_2)$, point $Q = [0, 0, f_4(\mathbf{p}, \rho)]$ achieves the maximum. When $\mu_1 \geq \mu_0 \geq \mu_2$, point $S = [f_1(\mathbf{p}, \rho), 0, f_4(\mathbf{p}, \rho) - f_1(\mathbf{p}, \rho)]$ achieves the maximum. When $\mu_1 \geq \mu_2 \geq \mu_0$, point $T = [f_1(\mathbf{p}, \rho), f_3(\mathbf{p}, \rho) - f_1(\mathbf{p}, \rho), f_4(\mathbf{p}, \rho) - f_3(\mathbf{p}, \rho)]$ achieves the maximum. When $\mu_2 \geq \mu_1 \geq \mu_0$, point $U = [f_3(\mathbf{p}, \rho) - f_2(\mathbf{p}, \rho), f_2(\mathbf{p}, \rho), f_4(\mathbf{p}, \rho) - f_3(\mathbf{p}, \rho)]$ achieves the maximum. When $\mu_2 \geq \mu_0 \geq \mu_1$, point $V = [0, f_2(\mathbf{p}, \rho), f_4(\mathbf{p}, \rho) - f_2(\mathbf{p}, \rho)]$ achieves the maximum. Hence, the optimization problem as defined in (2.36) is solved and the solution is expressed in terms of (\mathbf{p}, ρ) .

We are ready to solve the optimization problem in (2.33) now. According to the solution to the optimization problem in (2.36), we have five cases: 1) $\mu_0 \geq \max(\mu_1, \mu_2)$, 2) $\mu_1 \geq \mu_0 \geq \mu_2$, 3) $\mu_1 \geq \mu_2 \geq \mu_0$, 4) $\mu_2 \geq \mu_1 \geq \mu_0$ and 5) $\mu_2 \geq \mu_0 \geq \mu_1$. We will concentrate on the first three cases since case 4) is the same as case 3) and

case 5) is the same as case 2) by swapping indices 1 and 2.

1) When $\mu_0 \geq \max(\mu_1, \mu_2)$, the optimization problem in (2.33) is equivalent to

$$\min_{\mathbf{p} \geq \mathbf{0}, 0 \leq \rho \leq 1} E[-\mu_0 \log(1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})) + \lambda_1 \left(\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2}{h_1} p_0(\mathbf{h}) \right) + \lambda_2 \left(\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2}{h_2} p_0(\mathbf{h}) \right)] \quad (2.37)$$

Since the cost function is an expectation and the probability distributions are nonnegative, it suffices to consider the minimization for a fixed channel state $\mathbf{h} = (h_1, h_2)$, i.e.,

$$\min_{\mathbf{p}(\mathbf{h}) \geq \mathbf{0}, 0 \leq \rho(\mathbf{h}) \leq 1} -\mu_0 \log(1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})) + \lambda_1 \left(\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2}{h_1} p_0(\mathbf{h}) \right) + \lambda_2 \left(\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2}{h_2} p_0(\mathbf{h}) \right) \quad (2.38)$$

Though the cost function is not convex in $(\mathbf{p}(\mathbf{h}), \rho(\mathbf{h}))$, it is a quadratic function of $\rho(\mathbf{h})$ when $\mathbf{p}(\mathbf{h})$ is fixed. The optimal $\rho^*(\mathbf{h})$ is

$$\rho^*(\mathbf{h}) = \frac{\frac{h_1}{\lambda_1}}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}} \quad (2.39)$$

Since the dependencies of the cost functions on $\rho(\mathbf{h})$ in all three cases are the same, $\rho^*(\mathbf{h})$ is in fact the optimal solution for all three cases. Thus, we proceed with $\rho^*(\mathbf{h})$ in place of $\rho(\mathbf{h})$ and the problem becomes convex. We write the Karush-Kuhn-Tucker

(KKT) necessary conditions as follows:

$$-\frac{\mu_0}{1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})} + \frac{1}{\frac{h_1}{\lambda_1}} - \omega_1(\mathbf{h}) = 0 \quad (2.40)$$

$$-\frac{\mu_0}{1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})} + \frac{1}{\frac{h_2}{\lambda_2}} - \omega_2(\mathbf{h}) = 0 \quad (2.41)$$

$$-\frac{\mu_0}{1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})} + \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}} - \omega_0(\mathbf{h}) = 0 \quad (2.42)$$

$$p_1(\mathbf{h}), p_2(\mathbf{h}), p_0(\mathbf{h}), \omega_0(\mathbf{h}), \omega_1(\mathbf{h}), \omega_2(\mathbf{h}) \geq 0 \quad (2.43)$$

$$\omega_0(\mathbf{h})p_0(\mathbf{h}) = \omega_1(\mathbf{h})p_1(\mathbf{h}) = \omega_2(\mathbf{h})p_2(\mathbf{h}) = 0 \quad (2.44)$$

where $\omega_0(\mathbf{h})$, $\omega_1(\mathbf{h})$ and $\omega_2(\mathbf{h})$ are the complementary slackness variables. The KKTs have a unique solution and thus the solution is the global optimum. Let us define two regions in \mathbb{R}_+^2 ,

$$\mathcal{R}_1 = \{(x, y) : x + y \geq \frac{1}{\mu_0}\} \quad (2.45)$$

$$\mathcal{R}_2 = \{(x, y) : x + y < \frac{1}{\mu_0}\} \quad (2.46)$$

Then, the optimum solution is

$$p_0(\mathbf{h}) = \begin{cases} \mu_0(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}) - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{R}_1 \\ 0, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{R}_2 \end{cases} \quad (2.47)$$

$$p_1(\mathbf{h}) = 0 \quad (2.48)$$

$$p_2(\mathbf{h}) = 0 \quad (2.49)$$

The transmit powers can be found by dividing these received powers with corresponding channel gains. As seen from (2.48) and (2.49), in the case of $\mu_0 \geq \max(\mu_1, \mu_2)$, the transmitters use their entire power to transmit the common message; they do not allocate any power to transmit their individual messages. When $\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \geq \frac{1}{\mu_0}$, i.e., the combined channel is good enough, the transmitters transmit the common message using beamforming as if we have a two-transmitter one-receiver point-to-point system. When the channel is poor, i.e., $\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} < \frac{1}{\mu_0}$, the transmitters both keep silent and save their powers for better channel states. This is shown in Figure 2.4.

2) When $\mu_1 \geq \mu_0 \geq \mu_2$, the optimization problem in (2.33) is equivalent to

$$\min_{\mathbf{p} \geq \mathbf{0}, 0 \leq \rho \leq 1} E \left[-\mu_0 \log(1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})) - (\mu_1 - \mu_0) \log(1 + p_1(\mathbf{h})) \right. \\ \left. + \lambda_1 \left(\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2}{h_1} p_0(\mathbf{h}) \right) + \lambda_2 \left(\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2}{h_2} p_0(\mathbf{h}) \right) \right] \quad (2.50)$$

Following the same argument as in case 1), let us define four regions in \mathbb{R}_+^2 ,

$$\mathcal{S}_1 = \left\{ (x, y) : x \geq \frac{1}{\mu_1}, \frac{y}{x} < \frac{\mu_1}{\mu_0} - 1 \right\} \quad (2.51)$$

$$\mathcal{S}_2 = \left\{ (x, y) : x < \frac{1}{\mu_1}, x + y < \frac{1}{\mu_0} \right\} \quad (2.52)$$

$$\mathcal{S}_3 = \left\{ (x, y) : \frac{1}{x} - \frac{1}{x + y} \geq \mu_1 - \mu_0, x + y \geq \frac{1}{\mu_0} \right\} \quad (2.53)$$

$$\mathcal{S}_4 = \left\{ (x, y) : \frac{1}{x} - \frac{1}{x + y} < \mu_1 - \mu_0, \frac{y}{x} \geq \frac{\mu_1}{\mu_0} - 1, x + y \geq \frac{1}{\mu_0} \right\} \quad (2.54)$$

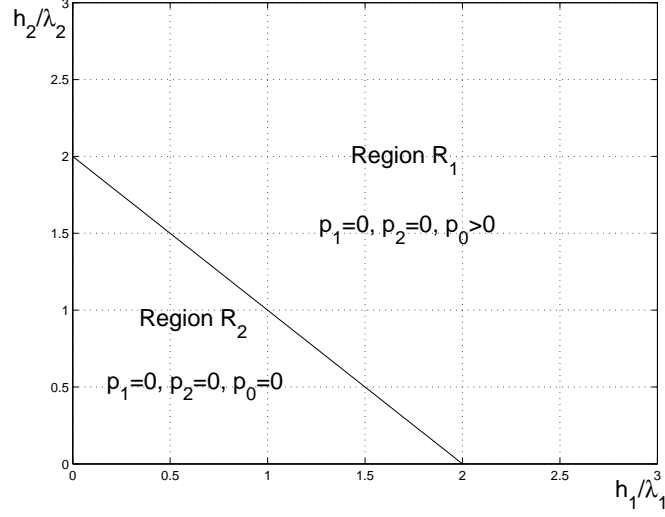


Figure 2.4: Power control policy in the case of $\mu_0 \geq \max(\mu_1, \mu_2)$.

Then, the optimal solution is

$$p_0(\mathbf{h}) = \begin{cases} \mu_0 \left(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \right) - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{S}_3 \\ \mu_0 \left(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \right) - (\mu_1 - \mu_0) \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \lambda_2} \right)^{-1}, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{S}_4 \\ 0, & \text{otherwise} \end{cases} \quad (2.55)$$

$$p_1(\mathbf{h}) = \begin{cases} \mu_1 \frac{h_1}{\lambda_1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{S}_1 \\ (\mu_1 - \mu_0) \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \lambda_2} \right)^{-1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{S}_4 \\ 0, & \text{otherwise} \end{cases} \quad (2.56)$$

$$p_2(\mathbf{h}) = 0 \quad (2.57)$$

Again, the transmit powers are found by dividing these with appropriate channel gains. As seen from (2.57), in the case of $\mu_1 \geq \mu_0 \geq \mu_2$, transmitter 2 never uses its power to transmit its individual message. When both channels are poor, no one transmits. When the channel of the first transmitter is much better than that

of the second transmitter, transmitter 1 transmits only its individual message and transmitter 2 keeps silent. When the channel of the second transmitter is much better than that of the first transmitter, both transmitters cooperate using beamforming to transmit the common message. When both channels are more or less equally good, both common message and individual message from transmitter 1 are transmitted. These regions are shown in Figure 2.5.

3) When $\mu_1 \geq \mu_2 \geq \mu_0$, the optimization problem in (2.33) is equivalent to

$$\begin{aligned} \min_{\mathbf{p} \geq \mathbf{0}, 0 \leq \rho \leq 1} E & \left[-\mu_0 \log(1 + p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h})) - (\mu_2 - \mu_0) \log(1 + p_1(\mathbf{h}) + p_2(\mathbf{h})) \right. \\ & - (\mu_1 - \mu_2) \log(1 + p_1(\mathbf{h})) + \lambda_1 \left(\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2}{h_1} p_0(\mathbf{h}) \right) \\ & \left. + \lambda_2 \left(\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2}{h_2} p_0(\mathbf{h}) \right) \right] \end{aligned} \quad (2.58)$$

Let us define eight regions in \mathbb{R}_+^2 ,

$$\mathcal{T}_1 = \left\{ (x, y) : x < \frac{1}{\mu_1}, y < \frac{1}{\mu_2}, x + y < \frac{1}{\mu_0} \right\} \quad (2.59)$$

$$\mathcal{T}_2 = \left\{ (x, y) : x + y \geq \frac{1}{\mu_0}, \frac{1}{y} - \frac{1}{x + y} \geq \mu_2 - \mu_0, \frac{1}{x} - \frac{1}{x + y} \geq \mu_1 - \mu_0 \right\} \quad (2.60)$$

$$\mathcal{T}_3 = \left\{ (x, y) : x \geq \frac{1}{\mu_1}, \frac{y}{x} < \min \left(\frac{\mu_1}{\mu_2}, \frac{\mu_1}{\mu_0} - 1 \right) \right\} \quad (2.61)$$

$$\mathcal{T}_4 = \left\{ (x, y) : y \geq \frac{1}{\mu_2}, \frac{x}{y} < \frac{\mu_2}{\mu_0} - 1, \frac{1}{x} - \frac{1}{y} \geq \mu_1 - \mu_2 \right\} \quad (2.62)$$

$$\mathcal{T}_5 = \left\{ (x, y) : x + y \geq \frac{1}{\mu_0}, \frac{1}{x} - \frac{1}{x + y} < \mu_1 - \mu_0, \frac{\mu_1}{\mu_0} - 1 \leq \frac{y}{x} < \sqrt{\frac{\mu_1 - \mu_0}{\mu_2 - \mu_0}} \right\} \quad (2.63)$$

$$\mathcal{T}_6 = \left\{ (x, y) : x + y \geq \frac{1}{\mu_0}, \frac{x}{y} \geq \frac{\mu_2}{\mu_0} - 1, \frac{1}{y} - \frac{1}{x + y} < \mu_2 - \mu_0, \frac{1}{x} - \frac{1}{y} \geq \mu_1 - \mu_2 \right\} \quad (2.64)$$

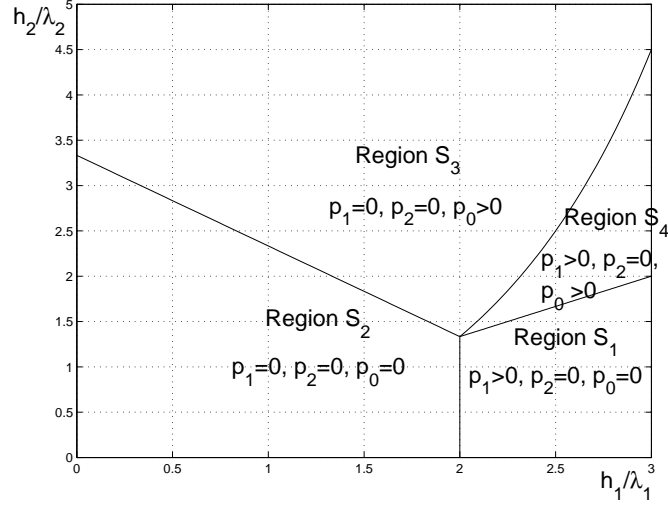


Figure 2.5: Power control policy in the case of $\mu_1 \geq \mu_0 \geq \mu_2$.

$$\mathcal{T}_7 = \left\{ (x, y) : y \geq \frac{1}{\mu_2}, \frac{1}{x} - \frac{1}{y} < \mu_1 - \mu_2, \frac{x}{y} < \min\left(\frac{\mu_2}{\mu_1}, \frac{\mu_2}{\mu_0} - 1\right) \right\} \quad (2.65)$$

$$\mathcal{T}_8 = \left\{ (x, y) : x + y \geq \frac{1}{\mu_0}, \frac{1}{y} - \frac{1}{x + y} < \mu_2 - \mu_0, \frac{1}{x} - \frac{1}{y} < \mu_1 - \mu_2, \right. \\ \left. \frac{\mu_2}{\mu_0} - 1 \leq \frac{x}{y} < \min \left(\sqrt{\frac{\mu_2 - \mu_0}{\mu_1 - \mu_0}}, \left(\frac{c + \sqrt{c^2 + 4}}{2} \right)^{-1} \right), \text{ where } c = \frac{\mu_1 - \mu_2}{\mu_0} \right\} \quad (2.66)$$

Then, the optimal solution is

$$p_0(\mathbf{h}) = \begin{cases} \mu_0 \left(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \right) - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{T}_2 \\ \mu_0 \left(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \right) - (\mu_1 - \mu_0) \left(\frac{1}{\frac{h_1}{\lambda_1}} - \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}} \right)^{-1}, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{T}_5 \\ \mu_0 \left(\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2} \right) - (\mu_2 - \mu_0) \left(\frac{1}{\frac{h_2}{\lambda_2}} - \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}} \right)^{-1}, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2} \right) \in \mathcal{T}_6 \cup \mathcal{T}_8 \\ 0, & \text{otherwise} \end{cases} \quad (2.67)$$

$$\begin{aligned}
p_1(\mathbf{h}) &= \begin{cases} \mu_1 \frac{h_1}{\lambda_1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_3 \\ (\mu_1 - \mu_0) \left(\frac{1}{\frac{h_1}{\lambda_1}} - \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}}\right)^{-1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_5 \\ (\mu_1 - \mu_2) \left(\frac{1}{\frac{h_1}{\lambda_1}} - \frac{1}{\frac{h_2}{\lambda_2}}\right)^{-1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_7 \cup \mathcal{T}_8 \\ 0, & \text{otherwise} \end{cases} \quad (2.68) \\
p_2(\mathbf{h}) &= \begin{cases} \mu_2 \frac{h_2}{\lambda_2} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_4 \\ (\mu_2 - \mu_0) \left(\frac{1}{\frac{h_2}{\lambda_2}} - \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}}\right)^{-1} - 1, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_6 \\ \mu_2 \frac{h_2}{\lambda_2} - (\mu_1 - \mu_2) \left(\frac{1}{\frac{h_1}{\lambda_1}} - \frac{1}{\frac{h_2}{\lambda_2}}\right)^{-1}, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_7 \\ (\mu_2 - \mu_0) \left(\frac{1}{\frac{h_2}{\lambda_2}} - \frac{1}{\frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}}\right)^{-1} - (\mu_1 - \mu_2) \left(\frac{1}{\frac{h_1}{\lambda_1}} - \frac{1}{\frac{h_2}{\lambda_2}}\right)^{-1}, & \text{if } \left(\frac{h_1}{\lambda_1}, \frac{h_2}{\lambda_2}\right) \in \mathcal{T}_8 \\ 0, & \text{otherwise} \end{cases} \quad (2.69)
\end{aligned}$$

As in the previous two cases, the transmit powers are found by dividing these with the corresponding channel gains. There are two subcases in the case of $\mu_1 \geq \mu_2 \geq \mu_0$. When $\frac{1}{\mu_1} + \frac{1}{\mu_2} \leq \frac{1}{\mu_0}$, i.e., μ_0 is very small, the common message never gets transmitted due to its small weight. When both channels are poor, no one transmits. When channel of the first transmitter is much better than that of the second transmitter, individual message W_1 is transmitted only. When channel of the second transmitter is much better than that of the first transmitter, individual message W_2 is transmitted only. When both channels are more or less equally good, both individual messages are transmitted. These regions are shown in Figure 2.6.

In the other subcase of $\frac{1}{\mu_1} + \frac{1}{\mu_2} > \frac{1}{\mu_0}$, all three messages get a chance to be transmitted. These regions are shown in Figure 2.7.

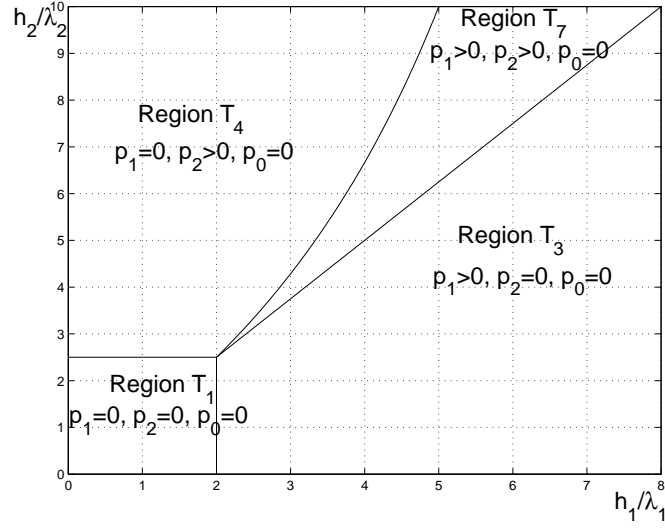


Figure 2.6: Power control policy in the case of $\mu_1 \geq \mu_2 \geq \mu_0$ and $\frac{1}{\mu_1} + \frac{1}{\mu_2} \leq \frac{1}{\mu_0}$.

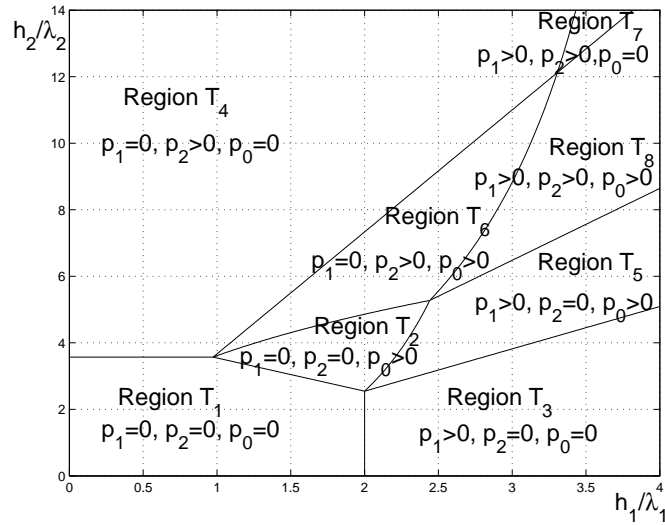


Figure 2.7: Power control policy in the case of $\mu_1 \geq \mu_2 \geq \mu_0$ and $\frac{1}{\mu_1} + \frac{1}{\mu_2} > \frac{1}{\mu_0}$.

Thus far, we have solved the optimization problem in (2.33) in terms of the lagrange multipliers $\boldsymbol{\lambda}$. Next, we need to solve for $\boldsymbol{\lambda}$. Since there is no duality gap, we will solve for $\boldsymbol{\lambda}$ by solving the dual problem, i.e., we will find $\boldsymbol{\lambda}$ that maximizes the dual function, $g(\boldsymbol{\lambda})$. The maximizer of the dual function enables the power policies to satisfy the power constraints with equalities due to the uniqueness of the optimal p_0, p_1, p_2, ρ for each given $\boldsymbol{\lambda}$. We will solve the dual problem by using the subgradient method [9]. For our problem,

$$u(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} E \left[\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho^*(\mathbf{h})^2 p_0(\mathbf{h})}{h_1} \right] - \bar{P}_1 \\ E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1-\rho^*(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] - \bar{P}_2 \end{bmatrix} \quad (2.70)$$

is a subgradient of the dual function and the set $\{\boldsymbol{\lambda} : \boldsymbol{\lambda} \geq 0, g(\boldsymbol{\lambda}) > -\infty\} = \{\boldsymbol{\lambda} : \boldsymbol{\lambda} > 0\}$. We start from an arbitrary point $\boldsymbol{\lambda}(0) \in \{\boldsymbol{\lambda} : \boldsymbol{\lambda} > 0\}$. At iteration k , we have available $\boldsymbol{\lambda}(k-1)$ from the previous iteration, and we compute (p_0, p_1, p_2, ρ) by setting $\boldsymbol{\lambda} = \boldsymbol{\lambda}(k-1)$. Then, using the (p_0, p_1, p_2, ρ) we obtained, we compute the subgradient vector $u(\boldsymbol{\lambda}(k-1))$ by equation (2.70) and update $\boldsymbol{\lambda}$ using

$$\boldsymbol{\lambda}(k) = \max[\boldsymbol{\lambda}(k-1) + s(k)u(\boldsymbol{\lambda}(k-1)), \boldsymbol{\epsilon}] \quad (2.71)$$

where $s(k)$ is a positive scalar stepsize at step k and $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2]^T$ is a positive vector very close to zero so that $\boldsymbol{\lambda}(k)$ stays in $\{\boldsymbol{\lambda} : \boldsymbol{\lambda} > 0\}$. We stop when both components of vector $u(\boldsymbol{\lambda}(k))$ are small enough. In [9], it is proved that for small enough step sizes, this algorithm converges.

Due to the strict concavity of the log function, the lagrange multipliers are unique.

The uniqueness of the lagrange multipliers ensures that the boundary rate triplet that solves (2.30) is unique for all $\boldsymbol{\mu}$ vectors except for the following three singular cases: $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_1 > \mu_0 = \mu_2 = 0$ and $\mu_2 > \mu_0 = \mu_1 = 0$. Thus, by varying the $\boldsymbol{\mu}$ vector over all possible values, and expressing the rates in limiting expressions for the singular cases, we obtain the entire boundary surface of the capacity region. In the process, we also obtain the power control policies that achieve the rate tuples on the boundary.

2.4 Simulations

In this section, we present simulation results for a two-user Gaussian MAC with common data in the presence of fading. The channel gains are assumed to be i.i.d. exponential with mean 1, independent across the two users. In our simulations, we use the subgradient method, and we picked the stepsize $s(k)$ by method (a) in [9, page 508].

In Figure 2.8, we show the ergodic capacity region of this two-user Gaussian MAC with common data in fading. The power constraints are $\bar{P}_1 = 2$ and $\bar{P}_2 = 1$. We calculated the rate triplets on the boundary of the capacity region by varying $\boldsymbol{\mu}$ over all possible values. It is straightforward to see that point R is the solution to case 1), which is independent of μ_0 . Points between R and S are the solutions to case 2). Points between T and U are solutions to subcase 1 of case 3) and case 4). Points between V and R are solutions to case 5). All points on the surface of $RSTUV$ are solutions to subcase 2 of case 3) and case 4). Surface YST is the singular case of

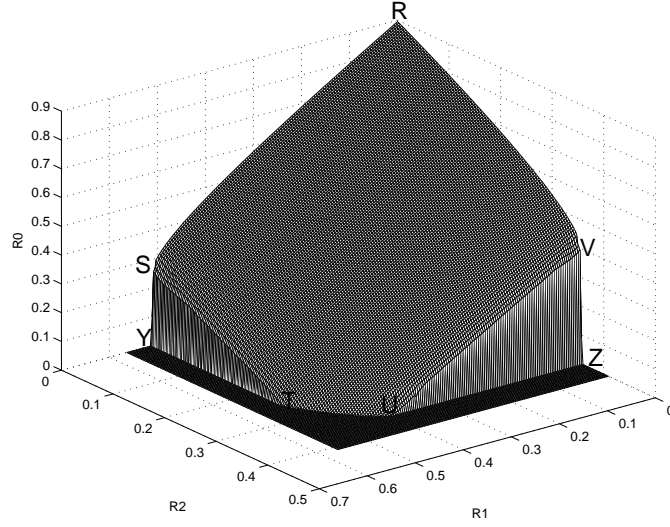


Figure 2.8: The ergodic capacity region of the Gaussian MAC with common data in fading.

$\mu_1 > \mu_0 = \mu_2 = 0$, and surface UVZ is the singular case of $\mu_2 > \mu_0 = \mu_1 = 0$.

We next compare the achievable rate $\mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0$ under different power allocation schemes. We choose $\mu_1 = 0.45$, $\mu_2 = 0.35$ and $\mu_0 = 0.2$ which corresponds to an interesting case where all three rates, R_0 , R_1 and R_2 , are non-zero, i.e., subcase 2 of case 3). In Figure 2.9, we plot the achievable rate as a function of the sum of the power constraints, i.e., $\bar{P}_1 + \bar{P}_2$. In this experiment, we assume that the power constraints are the same for both users, i.e., $\bar{P}_1 = \bar{P}_2$. The top-most curve in Figure 2.9 corresponds to the rate achieved by the optimum power allocation scheme we developed in this chapter. It is numerically solved by using the subgradient method. The “optimal channel-independent power control” curve corresponds to the solution

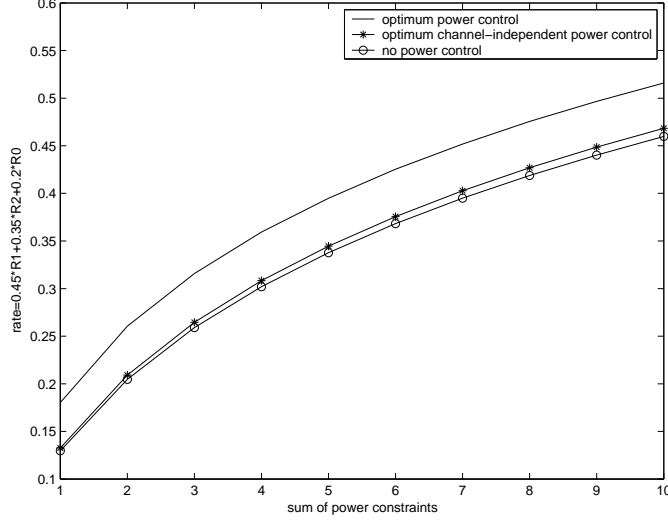


Figure 2.9: A weighted sum of rates with and without power control.

of the following problem

$$\begin{aligned}
\max_{0 \leq P_1 \leq \bar{P}_1, 0 \leq P_2 \leq \bar{P}_2} E & \left[\mu_0 \log \left(1 + h_1 \bar{P}_1 + h_2 \bar{P}_2 + 2\sqrt{h_1 h_2 (\bar{P}_1 - P_1)(\bar{P}_2 - P_2)} \right) \right. \\
& \left. + (\mu_2 - \mu_0) \log(1 + h_1 P_1 + h_2 P_2) + (\mu_1 - \mu_2) \log(1 + h_1 P_1) \right]
\end{aligned} \tag{2.72}$$

where we choose P_1 and P_2 to maximize the expectation in (2.72). Note that P_1 and P_2 are constants, and not functions of the channel realizations. This corresponds to the largest achievable rate $\mu_1 R_1 + \mu_2 R_2 + \mu_0 R_0$ when there is no channel state information at the transmitters, i.e., the transmitters only know the statistics of the channel gains. This maximization is solved numerically by searching over all admissible P_1 and P_2 . The lowest curve in Figure 2.9 corresponds to the case where we choose $P_1 = P_2 = P_0$, with $P_0 = \left(\sqrt{\bar{P}_1 - P_1} + \sqrt{\bar{P}_2 - P_2} \right)^2$. This corresponds to a case where the transmitters do not know the channel realizations or the channel

statistics. Consequently, the transmitters use “equal” powers for all three messages. For this instance, we see from Figure 2.9 that there is a relatively large performance gain due to adjusting the transmit powers according to the channel realizations. For this particular fading distribution, using optimum channel-independent power control provides only a small gain over choosing “equal” powers for all three messages.

2.5 Chapter Summary and Conclusions

In this chapter, we study the Gaussian MAC with common data. In the case of no fading, we provide an explicit characterization for the capacity region, and a simpler encoding/decoding scheme. In the case of fading, we characterize the ergodic capacity region, as well as the power control policies that achieve the rate tuples on the boundary of the capacity region. As expected, the common message enjoys a beamforming gain. The received power of the common message comes from both transmitters. In fact, the amount of power each transmitter spends for the common message is proportional to its channel gain at that time instant. Furthermore, the common message is only transmitted when both channels from the transmitters to the receiver are reasonably good.

The results of this chapter have been published in [51, 57].

2.6 Appendix

2.6.1 Proof of Theorem 2.1

The achievability part follows from an argument similar to [79] and thus is omitted.

For the converse, we develop a series of bounds on the achievable rates.

$$nR_1 = H(W_1 | \mathbf{H}^n) \quad (2.73)$$

$$\leq H(W_1 | Y^n, \mathbf{H}^n) + I(W_1; Y^n | \mathbf{H}^n) \quad (2.74)$$

$$\stackrel{(a)}{\leq} n\epsilon_n + I(W_1; Y^n | \mathbf{H}^n) \quad (2.75)$$

$$\stackrel{(b)}{\leq} n\epsilon_n + I(W_1; Y^n | W_0, X_2^n, \mathbf{H}^n) \quad (2.76)$$

where (a) follows from Fano's inequality [22] and (b) follows from the fact that W_1 and (W_0, X_2^n) are independent, conditioned on \mathbf{H}^n .

$$I(W_1; Y^n | W_0, X_2^n, \mathbf{H}^n) \stackrel{(c)}{\leq} I(X_1^n; Y^n | W_0, X_2^n, \mathbf{H}^n) \quad (2.77)$$

$$\stackrel{(d)}{\leq} \sum_{i=1}^n I(X_{1i}; Y_i | X_{2i}, W_0, \mathbf{H}_i) \quad (2.78)$$

$$= \sum_{i=1}^n \int_{\mathcal{H}} p_{\mathbf{H}_i}(\mathbf{h}) I(X_{1i}; Y_i | X_{2i}, W_0, \mathbf{h}) d\mathbf{h} \quad (2.79)$$

$$= \sum_{i=1}^n \int_{\mathcal{H}} p_{\mathbf{H}_i}(\mathbf{h}) \left(h \left(\sqrt{h_1} X_{1i} + Z_i | W_0, \mathbf{h} \right) - \frac{1}{2} \log(2\pi e) \right) d\mathbf{h} \quad (2.80)$$

where (c) follows from the data processing inequality [22] and (d) follows from the usual converse argument that upper bounds the mutual information of n -sequences by the sum of the mutual informations of the single letters, based on the fact that the channel is memoryless conditioned on the channel fading coefficients. In (2.80),

$h(\cdot)$ denotes the differential entropy.

$$h\left(\sqrt{h_1}X_{1i} + Z_i|W_0, \mathbf{h}\right) = E_s\left[h\left(\sqrt{h_1}X_{1i} + Z_i|W_0 = s, \mathbf{h}\right)\right] \quad (2.81)$$

$$\stackrel{(e)}{\leq} \frac{1}{2} \log(2\pi e) (h_1 E_s[V(X_{1i}|W_0 = s, \mathbf{h})] + 1) \quad (2.82)$$

where $V(\cdot)$ is the variance of a random variable and (e) follows from the fact that given the variance, Gaussian distribution maximizes the entropy, and applying Jensen's inequality [22] afterwards. Then,

$$R_1 \leq \frac{1}{n} \sum_{i=1}^n \int_{\mathcal{H}} p_{\mathbf{H}_i}(\mathbf{h}) \frac{1}{2} \log(1 + h_1 E_s[V(X_{1i}|W_0 = s, \mathbf{h})]) d\mathbf{h} + \epsilon_n \quad (2.83)$$

$$\stackrel{(f)}{=} \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{n} \sum_{i=1}^n \frac{1}{2} \log(1 + h_1 E_s[V(X_{1i}|W_0 = s, \mathbf{h})]) d\mathbf{h} + \epsilon_n \quad (2.84)$$

$$\stackrel{(g)}{\leq} \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log\left(1 + h_1 \frac{1}{n} \sum_{i=1}^n E_s[V(X_{1i}|W_0 = s, \mathbf{h})]\right) d\mathbf{h} + \epsilon_n \quad (2.85)$$

where in writing (f) we define \mathbf{H} to be a random variable whose distribution is the same as the stationary distribution of \mathbf{H}_i , and (g) follows from the concavity of the function $\log(1 + x)$.

$$\frac{1}{n} \sum_{i=1}^n E_s[V(X_{1i}|W_0 = s, \mathbf{h})] = \frac{1}{n} \sum_{i=1}^n (E_s[E[X_{1i}^2|W_0 = s, \mathbf{h}] - E^2[X_{1i}|W_0 = s, \mathbf{h}]]) \quad (2.86)$$

$$= \frac{1}{n} \sum_{i=1}^n E[X_{1i}^2|\mathbf{h}] - \frac{1}{n} \sum_{i=1}^n E_s[E^2[X_{1i}|W_0 = s, \mathbf{h}]] \quad (2.87)$$

Let us define $P_1(\mathbf{h}) = \frac{1}{n} \sum_{i=1}^n E[X_{1i}^2 | \mathbf{h}]$ and $(1 - \alpha(\mathbf{h})) P_1(\mathbf{h}) = \frac{1}{n} \sum_{i=1}^n E_s[E^2[X_{1i} | W_0 = s, \mathbf{h}]]$ and by definition, $0 \leq \alpha(\mathbf{h}) \leq 1$. Hence,

$$R_1 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_1 \alpha(\mathbf{h}) P_1(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.88)$$

Let us define $P_2(\mathbf{h}) = \frac{1}{n} \sum_{i=1}^n E[X_{2i}^2 | \mathbf{h}]$ and $(1 - \beta(\mathbf{h})) P_2(\mathbf{h}) = \frac{1}{n} \sum_{i=1}^n E_s[E^2[X_{2i} | W_0 = s, \mathbf{h}]]$. Then, a symmetric argument gives

$$R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_2 \beta(\mathbf{h}) P_2(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.89)$$

Following arguments similar to (2.73)-(2.84), we get an inequality akin to (2.85) as

$$\begin{aligned} & R_1 + R_2 \\ & \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log \left(1 + \frac{1}{n} \sum_{i=1}^n E_s \left[V \left(\sqrt{h_1} X_{1i} + \sqrt{h_2} X_{2i} | W_0 = s, \mathbf{h} \right) \right] \right) d\mathbf{h} + \epsilon_n \end{aligned} \quad (2.90)$$

$$\begin{aligned} & \stackrel{(h)}{=} \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log \left(1 + h_1 \frac{1}{n} \sum_{i=1}^n E_s \left[V \left(\sqrt{h_1} X_{1i} | W_0 = s, \mathbf{h} \right) \right] + \right. \\ & \quad \left. h_2 \frac{1}{n} \sum_{i=1}^n E_s \left[V \left(\sqrt{h_2} X_{2i} | W_0 = s, \mathbf{h} \right) \right] \right) d\mathbf{h} + \epsilon_n \end{aligned} \quad (2.91)$$

$$= \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_1 \alpha(\mathbf{h}) P_1(\mathbf{h}) + h_2 \beta(\mathbf{h}) P_2(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.92)$$

where (h) follows from the fact that, without loss of generality, we may consider encoders that depend only on the current channel state. Then, it follows that, conditioned on the common message W_0 and the current channel state $\mathbf{H}_i = \mathbf{h}$, X_{1i} and

X_{2i} are independent.

For the case of $R_0 + R_1 + R_2$, again, by following similar arguments, we get an inequality akin to (2.85) as

$$R_0 + R_1 + R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log \left(1 + \frac{1}{n} \sum_{i=1}^n V \left(\sqrt{h_1} X_{1i} + \sqrt{h_2} X_{2i} | \mathbf{h} \right) \right) d\mathbf{h} + \epsilon_n \quad (2.93)$$

Now,

$$\frac{1}{n} \sum_{i=1}^n V \left(\sqrt{h_1} X_{1i} + \sqrt{h_2} X_{2i} | \mathbf{h} \right) \quad (2.94)$$

$$\leq \frac{1}{n} \sum_{i=1}^n E \left[\left(\sqrt{h_1} X_{1i} + \sqrt{h_2} X_{2i} \right)^2 | \mathbf{h} \right] \quad (2.95)$$

$$= \frac{1}{n} \sum_{i=1}^n E_s \left[E \left[\left(\sqrt{h_1} X_{1i} + \sqrt{h_2} X_{2i} \right)^2 | W_0 = s, \mathbf{h} \right] \right] \quad (2.96)$$

$$= \frac{1}{n} \sum_{i=1}^n \left(h_1 E[X_{1i}^2 | \mathbf{h}] + h_2 E[X_{2i}^2 | \mathbf{h}] + 2\sqrt{h_1 h_2} E_s [E[X_{1i} X_{2i} | W_0 = s, \mathbf{h}]] \right) \quad (2.97)$$

$$\leq h_1 P_1(\mathbf{h}) + h_2 P_2(\mathbf{h}) + 2\sqrt{h_1 h_2} \frac{1}{n} \sum_{i=1}^n \left(E_s [E^2[X_{1i} | W_0 = s, \mathbf{h}]] \right)^{\frac{1}{2}} \left(E_s [E^2[X_{2i} | W_0 = s, \mathbf{h}]] \right)^{\frac{1}{2}} \quad (2.98)$$

$$\begin{aligned} &\leq h_1 P_1(\mathbf{h}) + h_2 P_2(\mathbf{h}) \\ &\quad + 2\sqrt{h_1 h_2 \left(\frac{1}{n} \sum_{i=1}^n E_s [E^2[X_{1i} | W_0 = s, \mathbf{h}]] \right) \left(\frac{1}{n} \sum_{i=1}^n E_s [E^2[X_{2i} | W_0 = s, \mathbf{h}]] \right)} \end{aligned} \quad (2.99)$$

$$= h_1 P_1(\mathbf{h}) + h_2 P_2(\mathbf{h}) + 2\sqrt{h_1 h_2 (1 - \alpha(\mathbf{h}))(1 - \beta(\mathbf{h})) P_1(\mathbf{h}) P_2(\mathbf{h})} \quad (2.100)$$

Hence,

$$R_0 + R_1 + R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log \left(1 + h_1 P_1(\mathbf{h}) + h_2 P_2(\mathbf{h}) + 2\sqrt{h_1 h_2 (1 - \alpha(\mathbf{h}))(1 - \beta(\mathbf{h})) P_1(\mathbf{h}) P_2(\mathbf{h})} \right) d\mathbf{h} + \epsilon_n \quad (2.101)$$

The power constraints of the system are:

$$\frac{1}{n} \sum_{i=1}^n X_{1i}^2 \leq \bar{P}_1, \quad \frac{1}{n} \sum_{i=1}^n X_{2i}^2 \leq \bar{P}_2 \quad \text{with probability 1} \quad (2.102)$$

Hence,

$$\int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) P_1(\mathbf{h}) d\mathbf{h} = \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{n} \sum_{i=1}^n E[X_{1i}^2 | \mathbf{h}] d\mathbf{h} = \frac{1}{n} \sum_{i=1}^n E[X_{1i}^2] \leq \bar{P}_1 \quad (2.103)$$

$$\int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) P_2(\mathbf{h}) d\mathbf{h} = \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{n} \sum_{i=1}^n E[X_{2i}^2 | \mathbf{h}] d\mathbf{h} = \frac{1}{n} \sum_{i=1}^n E[X_{2i}^2] \leq \bar{P}_2 \quad (2.104)$$

The rates triplets (R_1, R_2, R_0) have to satisfy

$$R_1 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_1 \alpha(\mathbf{h}) P_1(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.105)$$

$$R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_2 \beta(\mathbf{h}) P_2(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.106)$$

$$R_1 + R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log(1 + h_1 \alpha(\mathbf{h}) P_1(\mathbf{h}) + h_2 \beta(\mathbf{h}) P_2(\mathbf{h})) d\mathbf{h} + \epsilon_n \quad (2.107)$$

$$R_0 + R_1 + R_2 \leq \int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) \frac{1}{2} \log \left(1 + h_1 P_1(\mathbf{h}) + h_2 P_2(\mathbf{h}) + 2\sqrt{h_1 h_2 (1 - \alpha(\mathbf{h}))(1 - \beta(\mathbf{h})) P_1(\mathbf{h}) P_2(\mathbf{h})} \right) d\mathbf{h} + \epsilon_n \quad (2.108)$$

for some $\alpha(\mathbf{h})$ and $\beta(\mathbf{h})$ that map state space to $[0, 1]$ and $P_1(\mathbf{h})$ and $P_2(\mathbf{h})$ that satisfy

$$\int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) P_1(\mathbf{h}) d\mathbf{h} \leq \bar{P}_1 \quad (2.109)$$

$$\int_{\mathcal{H}} p_{\mathbf{H}}(\mathbf{h}) P_2(\mathbf{h}) d\mathbf{h} \leq \bar{P}_2 \quad (2.110)$$

We make the following variable changes:

$$p_1(\mathbf{h}) = h_1 \alpha(\mathbf{h}) P_1(\mathbf{h}) \quad (2.111)$$

$$p_2(\mathbf{h}) = h_2 \beta(\mathbf{h}) P_2(\mathbf{h}) \quad (2.112)$$

$$p_0(\mathbf{h}) = \left(\sqrt{h_1(1 - \alpha(\mathbf{h}))} P_1(\mathbf{h}) + \sqrt{h_2(1 - \beta(\mathbf{h}))} P_2(\mathbf{h}) \right)^2 \quad (2.113)$$

$$\rho(\mathbf{h}) = \frac{\sqrt{h_1(1 - \alpha(\mathbf{h}))} P_1(\mathbf{h})}{\sqrt{p_0(\mathbf{h})}} \quad (2.114)$$

Thus,

$$R_1 \leq E[C(p_1(\mathbf{h}))] \quad (2.115)$$

$$R_2 \leq E[C(p_2(\mathbf{h}))] \quad (2.116)$$

$$R_1 + R_2 \leq E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}))] \quad (2.117)$$

$$R_0 + R_1 + R_2 \leq E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h}))] \quad (2.118)$$

for some $\rho(\mathbf{h})$ that maps the state space to $[0, 1]$, and some $p_1(\mathbf{h})$, $p_2(\mathbf{h})$ and $p_0(\mathbf{h})$

that satisfy

$$E \left[\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho(\mathbf{h})^2 p_0(\mathbf{h})}{h_1} \right] \leq \bar{P}_1 \quad (2.119)$$

$$E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] \leq \bar{P}_2 \quad (2.120)$$

$$p_0(\mathbf{h}), p_1(\mathbf{h}), p_2(\mathbf{h}) \geq 0 \quad (2.121)$$

2.6.2 Proof of Lemma 2.1

Let $(\mathbf{R}^a, \tilde{P}_1^a, \tilde{P}_2^a)$ and $(\mathbf{R}^b, \tilde{P}_1^b, \tilde{P}_2^b)$ be two elements in set \mathcal{L} . To prove that set \mathcal{L} is convex, we need to show that for any $0 \leq \theta \leq 1$, $(\theta \mathbf{R}^a + (1 - \theta) \mathbf{R}^b, \theta \tilde{P}_1^a + (1 - \theta) \tilde{P}_1^b, \theta \tilde{P}_2^a + (1 - \theta) \tilde{P}_2^b)$ is in set \mathcal{L} .

For $i = a$ or b , $(\mathbf{R}^i, \tilde{P}_1^i, \tilde{P}_2^i) \in \mathcal{L}$ means that $\mathbf{R}^i \in \mathcal{C}_f(\mathbf{p}^i, \rho^i)$ for some (\mathbf{p}^i, ρ^i) such that

$$E \left[\frac{p_1^i(\mathbf{h})}{h_1} + \frac{\rho^i(\mathbf{h})^2 p_0^i(\mathbf{h})}{h_1} \right] \leq \tilde{P}_1^i, \quad E \left[\frac{p_2^i(\mathbf{h})}{h_2} + \frac{(1 - \rho^i(\mathbf{h}))^2 p_0^i(\mathbf{h})}{h_2} \right] \leq \tilde{P}_2^i \quad (2.122)$$

$$p_0^i(\mathbf{h}), p_1^i(\mathbf{h}), p_2^i(\mathbf{h}) \geq 0, \quad 0 \leq \rho^i(\mathbf{h}) \leq 1 \quad (2.123)$$

i.e., there exist $(\mathbf{p}^i(\mathbf{h}), \rho^i(\mathbf{h}))$ that satisfy (2.122) and (2.123) and

$$R_1^i \leq E [C (p_1^i(\mathbf{h}))] \quad (2.124)$$

$$R_2^i \leq E [C (p_2^i(\mathbf{h}))] \quad (2.125)$$

$$R_1^i + R_2^i \leq E [C (p_1^i(\mathbf{h}) + p_2^i(\mathbf{h}))] \quad (2.126)$$

$$R_0^i + R_1^i + R_2^i \leq E [C (p_1^i(\mathbf{h}) + p_2^i(\mathbf{h}) + p_0^i(\mathbf{h}))] \quad (2.127)$$

Let

$$p_1(\mathbf{h}) = \theta p_1^a(\mathbf{h}) + (1 - \theta)p_1^b(\mathbf{h}) \quad (2.128)$$

$$p_2(\mathbf{h}) = \theta p_2^a(\mathbf{h}) + (1 - \theta)p_2^b(\mathbf{h}) \quad (2.129)$$

$$p_0(\mathbf{h}) = \theta p_0^a(\mathbf{h}) + (1 - \theta)p_0^b(\mathbf{h}) \quad (2.130)$$

$$\rho_1(\mathbf{h}) = \sqrt{\frac{\theta p_0^a(\mathbf{h}) \rho^a(\mathbf{h})^2 + (1 - \theta)p_0^b(\mathbf{h}) \rho^b(\mathbf{h})^2}{\theta p_0^a(\mathbf{h}) + (1 - \theta)p_0^b(\mathbf{h})}} \quad (2.131)$$

$$1 - \rho_2(\mathbf{h}) = \sqrt{\frac{\theta p_0^a(\mathbf{h}) (1 - \rho^a(\mathbf{h}))^2 + (1 - \theta)p_0^b(\mathbf{h}) (1 - \rho^b(\mathbf{h}))^2}{\theta p_0^a(\mathbf{h}) + (1 - \theta)p_0^b(\mathbf{h})}} \quad (2.132)$$

It is straightforward to verify that $\rho_1(\mathbf{h}) \geq \rho_2(\mathbf{h})$ for all possible \mathbf{h} , θ , $p_0^a(\mathbf{h})$, $p_0^b(\mathbf{h})$, $\rho^a(\mathbf{h})$, $\rho^b(\mathbf{h})$. Due to the concavity of the log function,

$$\theta R_1^a + (1 - \theta)R_1^b \leq E[C(p_1(\mathbf{h}))] \quad (2.133)$$

$$\theta R_2^a + (1 - \theta)R_2^b \leq E[C(p_2(\mathbf{h}))] \quad (2.134)$$

$$(\theta R_1^a + (1 - \theta)R_1^b) + (\theta R_2^a + (1 - \theta)R_2^b) \leq E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}))] \quad (2.135)$$

$$\begin{aligned} &(\theta R_0^a + (1 - \theta)R_0^b) + (\theta R_1^a + (1 - \theta)R_1^b) + (\theta R_2^a + (1 - \theta)R_2^b) \leq \\ &E[C(p_1(\mathbf{h}) + p_2(\mathbf{h}) + p_0(\mathbf{h}))] \end{aligned} \quad (2.136)$$

Also, it is easy to check that

$$E \left[\frac{p_1(\mathbf{h})}{h_1} + \frac{\rho_1(\mathbf{h})^2 p_0(\mathbf{h})}{h_1} \right] \leq \theta \tilde{P}_1^a + (1 - \theta) \tilde{P}_1^b \quad (2.137)$$

$$E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho_1(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] \leq E \left[\frac{p_2(\mathbf{h})}{h_2} + \frac{(1 - \rho_2(\mathbf{h}))^2 p_0(\mathbf{h})}{h_2} \right] \leq \theta \tilde{P}_2^a + (1 - \theta) \tilde{P}_2^b \quad (2.138)$$

From (2.133)- (2.138), we see that $\theta \mathbf{R}^a + (1 - \theta) \mathbf{R}^b \in \mathcal{C}_f([p_1, p_2, p_0]^T, \rho_1)$. Also, $p_0, p_1, p_2 \geq 0, 0 \leq \rho_1 \leq 1$ and satisfy the power constraints of $\theta \tilde{P}_1^a + (1 - \theta) \tilde{P}_1^b$ and $\theta \tilde{P}_2^a + (1 - \theta) \tilde{P}_2^b$.

Hence,

$$\theta \mathbf{R}^a + (1 - \theta) \mathbf{R}^b \in \mathcal{C}(\theta \tilde{P}_1^a + (1 - \theta) \tilde{P}_1^b, \theta \tilde{P}_2^a + (1 - \theta) \tilde{P}_2^b) \quad (2.139)$$

and $(\theta \mathbf{R}^a + (1 - \theta) \mathbf{R}^b, \theta \tilde{P}_1^a + (1 - \theta) \tilde{P}_1^b, \theta \tilde{P}_2^a + (1 - \theta) \tilde{P}_2^b) \in \mathcal{L}$ as desired. Thus, \mathcal{L} is convex.

Chapter 3

Scaling Laws for Dense Gaussian Sensor Networks and the Order Optimality of Separation

In Chapter 2, we focused on correlated data which was in the special form of common data. However, in practical situations, correlated data manifests itself in more general forms. One practically interesting application is the sensor networks.

With recent advances in the hardware technology, small cheap nodes with sensing, computing and communication capabilities have become available. In practical applications, it is possible to deploy a large number of these nodes to sense the environment. In this chapter, we investigate the optimal performance of a dense sensor network by studying the joint source-channel coding problem. The sensor network is composed of N sensors, where N is very large, and a single collector node. Each sensor node has the capability of taking noiseless samples from an underlying random process. Each node in the sensor network is equipped with one transmit and one receive antenna to transmit and receive signals through the wireless medium, i.e., all nodes hear a linear combination of the signals transmitted by all other nodes at that time instant. The overall goal of the sensor network is to take measurements from an

underlying random process $S(u)$, $0 \leq u \leq U_0$, code and transmit those measured samples to a collector node, which wishes to reconstruct the entire random process with as little distortion as possible; see Figure 1.1. Due to the small distances between the sensor nodes and the correlation in the measured data, the underlying sensor samples are correlated, and due to the existence of receive antennas at the sensor nodes and a transmit antenna at the collector node, the communication channel is a Gaussian cooperative multiple access channel with noisy feedback. We investigate the minimum achievable expected distortion and a corresponding achievability scheme when the underlying random process is Gaussian.

El Gamal [29] showed that all spatially band-limited Gaussian processes can be estimated at the collector node, subject to any non-zero constraint on the mean squared distortion, i.e., the sensor network scales successfully. In this chapter, we study the minimum achievable expected distortion for space-limited, and thus, not band-limited, random processes, and we show the rate at which the minimum achievable expected distortion decreases to zero as the number of nodes increases. Also, in [29], it is assumed that the channel gains between the nodes are decreasing functions of the distance between them, without enforcing any upper bounds. This implies that, when the sensors are placed very densely, the channel gains between nearby sensors become unboundedly large. This physically impossible situation arises because although the channel model used in [29] is valid only in the far field of the transmitter, it is used for all distances. In this chapter, we have used a more realistic channel model, where we assume that the channel gains decrease with distance, however, they are lower and upper bounded.

Kashyap et al. [43] studied the source coding part of the problem investigated in this chapter. The paper showed that for any distortion constraint that is independent of N , the difference between the rate achievable by distributed source coding and the rate achievable by centralized source coding is bounded by a constant, independent of N . Though we study a joint source-channel coding problem, both our converse and achievability proofs are separation-based, and thus, we show a similar result: in the source coding part we show that the ratio between the rate achievable by distributed source coding and the rate achievable by centralized source coding is bounded by a constant, independent of N . In contrast to [43] where the distortion constraint is independent of N , we allow the distortion to go to zero as a function of N . Moreover, [43] deals with stationary Gaussian random processes, while we allow for nonstationarity of the underlying random process. It is not immediately evident whether the methods in [43] apply in the scenario considered in this chapter.

Neuhoff and Pradhan [64] studied the source coding part of the problem investigated in this chapter by allowing the random process to be unbounded in space. The sensors are densely as well as widely distributed. In this case, results from Grenander-Szego [35] were used. However, for the case of a finite interval, as considered in this chapter, such results cannot be used.

Gastpar and Vetterli [33] studied the case where the sensors observe a noisy version of a linear combination of L Gaussian random variables which all have the same variance, code and transmit those observations to a collector node, and the collector node reconstructs the L random variables. In [33], the expected distortion achieved by applying separation-based approaches was shown to be order worse than the lower

bound on the minimum achievable expected distortion. In this chapter, we study the case where the data of interest at the collector node is not a finite number of random variables, but a random process, which, using Karhunen-Loeve expansion, can be shown to be equivalent to a set of infinitely many random variables with varying variances. We assume that the sensors are able to take noiseless samples, but that each sensor observes only its own sample. Our upper bound on the minimum achievable distortion is also developed by using a separation-based approach, but it is shown to be of the same order as the lower bound, for a wide range of power constraints, for random processes that satisfy some general conditions.

We first provide lower and upper bounds for the minimum achievable expected distortion for arbitrary Gaussian random processes whose Karhunen-Loeve expansion exists. Then, we focus on the case where the Gaussian random process also satisfies some general conditions. For these random processes, we evaluate the lower and upper bounds explicitly, and show that they are of the same order, for a wide range of power constraints. Thus, for these random processes, under a wide range of power constraints, we determine an order-optimal achievability scheme, and identify the minimum achievable expected distortion as a function of the number of nodes and the sum power constraint. Our achievability scheme is separation-based: each sensor node first performs multi-terminal source coding [30], then, performs channel coding, and utilizes the cooperative nature of the wireless medium through the amplify-and-forward scheme [32]. In multi-user information theory, generally speaking, the separation principle does not hold. However, in our case, we have found a scheme which is separation based, and is order-optimal.

3.1 System Model

The collector node wishes to reconstruct a continuous random process $S(u)$, for $0 \leq u \leq U_0$, where u denotes the spatial position; $S(u)$ is assumed to be Gaussian with zero-mean and a continuous autocorrelation function $K(u, v)$. The N sensor nodes are placed on a straight line at positions $0 \leq u_1 \leq u_2 \leq \dots \leq u_N \leq U_0$, and observe samples

$$\mathbf{S}_N = \begin{bmatrix} S(u_1) & S(u_2) & \dots & S(u_N) \end{bmatrix}^T \quad (3.1)$$

For simplicity and to avoid irregular cases, we assume that the sensors are equally spaced, i.e.,

$$u_i = \frac{i-1}{N}U_0, \quad i = 1, 2, \dots, N \quad (3.2)$$

The distortion measure is the squared error,

$$d(S(u), \hat{S}(u)) = \frac{1}{U_0} \int_0^{U_0} (S(u) - \hat{S}(u))^2 du \quad (3.3)$$

Each sensor node and the collector node, denoted as node 0, is equipped with one transmit and one receive antenna. To simplify the presentation, from now until Section 3.6, we will assume that the collector node does not use its transmit antenna, and thus, there is no feedback in the system. We will allow the collector node to use its transmit antenna and provide feedback to the sensor nodes in Section 3.6, and show

that the results of the previous sections remain unchanged. The transmissions through the wireless medium are time slotted. The channel is assumed to be memoryless between the time slots. At any time instant, let X_i denote the signal transmitted by node i , and Y_j denote the signal received at node j . Let h_{ij} denote the channel gain from node i to node j . Then, the received signal at node j can be written as,

$$Y_j = \sum_{i=1, i \neq j}^N h_{ij} X_i + Z_j, \quad j = 0, 1, 2, \dots, N \quad (3.4)$$

where $\{Z_j\}_{j=0}^N$ is a vector of $N + 1$ independent and identically distributed, zero-mean, unit-variance Gaussian random variables. Therefore, the channel model of the network is such that all nodes hear a linear combination of the signals transmitted by all other nodes at that time instant. We assume that the channel gain h_{ij} is bounded, i.e.,

$$\bar{h}_l \leq h_{ij} \leq \bar{h}_u, \quad i = 1, \dots, N, \quad j = 0, 1, \dots, N \quad (3.5)$$

where \bar{h}_u and \bar{h}_l are positive constants independent of N . This model is very general and should be satisfied very easily. By the conservation of energy, $h_{ij}^2 \leq 1$, and since all nodes are within finite distances of each other, the channel gains should be lower bounded as well.

We assume that all sensors share the sum power constraint $P(N)$. The two most interesting cases for $P(N)$ are $P(N) = NP_{\text{ind}}$, where each sensor has its individual power constraint P_{ind} , and $P(N) = P_{\text{tot}}$, where the sum power constraint is a constant

P_{tot} and does not depend on the number of sensor nodes. In the latter case, when more and more sensor nodes are deployed, the individual power of each sensor node decreases as P_{tot}/N . Our goal is to determine the scheme that achieves the minimum achievable expected distortion

$$\frac{1}{U_0} \int_0^{U_0} E \left[\left(S(u) - \hat{S}(u) \right)^2 \right] du \quad (3.6)$$

at the collector node for a given sum power constraint $P(N)$, and also to determine the rate at which this distortion goes to zero as a function of the number of sensor nodes and the sum power constraint.

Next, we give a more precise definition of our problem. Each sensor node observes a sample of a sequence of spatial random processes $\{S^{(l)}(u)\}_{l=1}^n$, where index l denotes time, u denotes the spatial position, and n is the block length of the sequence of random processes, and also the delay parameter, which may be a function of N , the number of sensor nodes. The sequence of spatial random processes $\{S^{(l)}(u)\}_{l=1}^n$ is assumed to be i.i.d. in time. For now, we assume that n channel uses are allowed for n realizations of the random process; the case where we allow the number of channel uses and the number of realizations to differ will be treated in Section 3.6. At time instant m , sensor node j transmits

$$X_j(m) = F_j^{(m)}(\{S^{(l)}(u_j)\}_{l=1}^n, \{Y_j^{(l)}\}_{l=n+1}^{m-1}), \quad m = n+1, n+2, \dots, 2n, \\ j = 1, 2, \dots, N \quad (3.7)$$

i.e., after waiting n time slots to obtain a block of observations, the sensor node transmits, at time m , a signal that is a function of its observations of the entire block of random process samples and also the signal it received before time m . We are interested in the performance in the information-theoretic sense and hence, we allow the delay n to be arbitrarily large. By the assumption of sum power constraint, we have

$$\frac{1}{n} \sum_{m=1}^n \sum_{j=1}^N E[X_j^2(m)] \leq P(N) \quad (3.8)$$

The collector node reconstructs the random process as

$$\{\hat{S}^{(l)}(u), u \in [0, U_0]\}_{l=1}^n = G(Y_0^{(n+1)}, Y_0^{(n+2)}, \dots, Y_0^{(2n)}) \quad (3.9)$$

For fixed encoding functions of the nodes $\{F_j^{(m)}\}_{m=2n, j=N}^{m=n+1, j=1}$ and the decoding function of the collector node G , the achieved expected distortion averaged over time is

$$D^N = \frac{1}{n} \sum_{l=1}^n E \left[d \left(S^{(l)}(u), \hat{S}^{(l)}(u) \right) \right] \quad (3.10)$$

and we are interested in the smallest achievable expected distortion over all encoding and decoding functions where n is allowed to be arbitrarily large.

In this chapter, our purpose is to understand the behavior of the minimum achievable expected distortion when the number of sensor nodes is very large. We introduce the big- O , big- Ω and big- Θ notations. We say that f is $O(g)$, and g is $\Omega(f)$, if there exist constants c and k , such that $|f(N)| \leq c|g(N)|$ for all $N > k$; we say that f is

$\Theta(g)$ if $f(N)$ is both $O(g)$ and $\Omega(g)$. All logarithms are defined with respect to base e , and $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x . $(\mathbf{x})_i$ and $\|\mathbf{x}\|$ denote the i -th element and the Euclidean norm of vector \mathbf{x} , respectively. $\|\mathbf{A}\|_2$ denotes the spectral norm of matrix \mathbf{A} , which is defined as the square root of the largest eigenvalue of matrix $\mathbf{A}^T \mathbf{A}$ [38].

3.2 A Class of Gaussian Random Processes

For a Gaussian random process $S(u)$ with a continuous autocorrelation function, we perform the Karhunen-Loeve expansion [65],

$$S(u) = \sum_{k=0}^{\infty} \bar{S}_k \phi_k(u) \quad (3.11)$$

to obtain the ordered eigenvalues $\{\lambda_k\}_{k=0}^{\infty}$, and the corresponding set of orthonormal eigenfunctions $\{\phi_k(u), u \in [0, U_0]\}_{k=0}^{\infty}$.

Let \mathcal{A} be the set of Gaussian random processes on $[0, U_0]$ with continuous autocorrelation functions, that satisfy the following conditions:

1. There exist nonnegative constants d_l , d_u , and nonnegative integers c_l , c_u , $K_0 \geq c_u + 1$ and two sequences of numbers $\{\lambda'_k\}_{k=0}^{\infty}$ and $\{\lambda''_k\}_{k=0}^{\infty}$ defined as

$$\lambda'_k = \begin{cases} \lambda_k, & k \leq K_0 \\ \frac{d_l}{(k+c_l)^x}, & k > K_0 \end{cases} \quad (3.12)$$

and

$$\lambda_k'' = \begin{cases} \lambda_k, & k \leq K_0 \\ \frac{d_u}{(k-c_u)^x}, & k > K_0 \end{cases} \quad (3.13)$$

for some constant $x > 1$, such that

$$\lambda_k' \leq \lambda_k \leq \lambda_k'' \quad (3.14)$$

The condition that $x > 1$ is without loss of generality, because for all continuous autocorrelations, the eigenvalues decrease faster than k^{-1} .

2. In addition to continuity, $K(u, v)$ satisfies the Lipschitz condition of order $1/2 < \alpha \leq 1$, i.e., there exists a constant $B > 0$ such that

$$|K(u_1, v_1) - K(u_2, v_2)| \leq B \left(\sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2} \right)^\alpha \quad (3.15)$$

for all $u_1, v_1, u_2, v_2 \in [0, U_0]$.

3. For $k = 0, 1, \dots$, the function $\phi_k(v)$ and the function $K(u, v)\phi_k(v)$ as a function of v satisfy the following condition: there exist positive constants $B_1, B_2, B_3, B_4, \beta \leq 1, \gamma \leq 1$, and nonnegative constant τ , independent of k , such that

$$|\phi_k(v_1) - \phi_k(v_2)| \leq B_3(k + B_4)^\tau |v_1 - v_2|^\gamma \quad (3.16)$$

and

$$|K(u, v_1)\phi_k(v_1) - K(u, v_2)\phi_k(v_2)| \leq B_2(k + B_1)^\tau |v_1 - v_2|^\beta \quad (3.17)$$

for all $u, v_1, v_2 \in [0, U_0]$.

The reasons why these conditions are needed for the explicit evaluation of the lower and upper bounds on the minimum achievable expected distortion will be clear from the proofs. Here, we provide some intuition as to why they are needed. Condition 1 states that we consider random processes that have eigenvalues λ_k which decrease at a rate of approximately k^{-x} . The rate of decrease in the eigenvalues is an indication of how the randomness of the random process is distributed upon the eigenfunctions. For example, a small x means that the randomness is distributed rather evenly upon all eigenfunctions, while a large x means that the randomness is mostly concentrated upon a subset of eigenfunctions. Thus, the minimum achievable expected distortion depends crucially on the rate of decrease parameter x . The lower (upper) bound on the eigenvalues in (3.14) will be used to calculate the lower (upper) bound on the minimum achievable expected distortion. Conditions 2 and 3 are needed because instead of the random process itself that is of interest to the collector node, the collector node, at best, can know only the sampled values of the random process. How well the entire process can be approximated from its samples is of great importance in obtaining quantitative results. Lipschitz conditions describe the quality of this approximation well. By condition 3, we require the variation in the eigenfunction ϕ_k to be no faster than k^τ . We note that the well-known trigonometric basis satisfies

this condition.

We also note that our conditions are quite general. Many random processes satisfy these conditions, including the Gauss-Markov process, Brownian motion process, centered Brownian bridge, etc. For example, a Gauss-Markov process, also known as the Ornstein-Uhlenbeck process [80, 85], is defined as a random process that is stationary, Gaussian, Markovian, and continuous in probability. It is known that the autocorrelation function of this process is [11, 24, 42]

$$K(u, v) = \frac{\sigma^2}{2\eta} e^{-\eta|u-v|} \quad (3.18)$$

The Karhunen-Loeve expansion of the Gauss-Markov process yields the eigenfunctions

$$\{\phi_k(u)\}_{k=0}^{\infty}$$

$$\phi_k(u) = b_k \left(\cos \sqrt{\frac{\sigma^2}{\lambda_k} - \eta^2} u + \frac{\eta}{\sqrt{\frac{\sigma^2}{\lambda_k} - \eta^2}} \sin \sqrt{\frac{\sigma^2}{\lambda_k} - \eta^2} u \right) \quad (3.19)$$

where $\{\lambda_k\}_{k=0}^{\infty}$ are the corresponding eigenvalues and b_k are positive constants chosen such that the eigenfunctions $\phi_k(u)$ have unit energy. It can be shown that $\{\lambda_k\}_{k=0}^{\infty}$ may be bounded as

$$\lambda'_k \leq \lambda_k \leq \lambda''_k \quad (3.20)$$

where $\{\lambda'_k\}_{k=0}^\infty$ is defined as

$$\lambda'_k = \begin{cases} \lambda_k, & k \leq K_0 \\ \frac{\sigma^2 U_0^2}{(k+1)^2 \pi^2}, & k > K_0 \end{cases} \quad (3.21)$$

with $K_0 = \max\left(2, \left\lfloor \frac{\eta^2 U_0^2}{\pi^2} - \frac{3}{4} \right\rfloor\right)$, and $\{\lambda''_k\}_{k=0}^\infty$ is defined as

$$\lambda''_k = \begin{cases} \lambda_k, & k \leq K_0 \\ \frac{\sigma^2 U_0^2}{(k-1)^2 \pi^2}, & k > K_0 \end{cases} \quad (3.22)$$

Thus, we observe that the Gauss-Markov process satisfies the conditions defined in this section with $x = 2$, $d_l = d_u$ and $\alpha = \beta = \tau = \gamma = 1$.

The lower and upper bounds on the minimum achievable expected distortion will be calculated using $\{\lambda'_k\}_{k=0}^\infty$ and $\{\lambda''_k\}_{k=0}^\infty$, respectively. Some properties of $\{\lambda'_k\}_{k=0}^\infty$ and $\{\lambda''_k\}_{k=0}^\infty$ which will be used in later proofs are stated in Lemmas 3.5 and 3.6 and proved in Appendix 3.8.1.

3.3 A Lower Bound on the Achievable Distortion

3.3.1 Arbitrary Gaussian Random Processes

A lower bound is obtained by assuming that all of the sensor nodes know the random process exactly, i.e., $S(u)$, $u \in [0, U_0]$, and the sensor network forms an N -transmit 1-receive antenna point-to-point system to transmit the random process to the collector node. Let C_u^N be the capacity of this point-to-point system in nats per channel use

and $D_p(R)$ be the distortion-rate function of the random process $S(u)$ [6]. In this point-to-point system, the separation principle holds, and therefore

$$D^N \geq D_p(C_u^N) \quad (3.23)$$

To evaluate $D_p(C_u^N)$, we first find the distortion-rate function, $D_p(R)$, of $S(u)$ [6, Section 4.5] as,

$$R(\theta) = \sum_{k=0}^{\infty} \max \left(0, \frac{1}{2} \log \left(\frac{\lambda_k}{\theta} \right) \right) \quad (3.24)$$

and

$$D(\theta) = U_0^{-1} \sum_{k=0}^{\infty} \min(\theta, \lambda_k) \quad (3.25)$$

where θ is an intermediate variable used to describe the distortion-rate function. The distortion-rate function $D_p(R)$ characterizes the minimum achievable expected distortion when we use R nats per source realization to describe the random process.

Next, we find C_u^N , the capacity of the N -transmit 1-receive antenna point-to-point system [78] as,

$$C_u^N = \frac{1}{2} \log \left(1 + \sum_{i=1}^N h_{i0}^2 P(N) \right) \quad \text{nats/channel use} \quad (3.26)$$

To see how C_u^N changes with N , using (3.26) and (3.5), we can lower and upper bound

C_u^N as

$$\frac{1}{2} \log (1 + \bar{h}_l^2 NP(N)) \leq C_u^N \leq \frac{1}{2} \log (1 + \bar{h}_u^2 NP(N)) \quad (3.27)$$

For arbitrary Gaussian random processes, a lower bound on the minimum achievable expected distortion is

$$D_l^N = D_p(C_u^N) \quad (3.28)$$

3.3.2 The Class of Gaussian Random Processes in \mathcal{A}

Next, we evaluate $D_p(C_u^N)$ for the class of Gaussian random processes in \mathcal{A} . Based on the structure of the eigenvalues in (3.12) and (3.14), and the properties of $\{\lambda'_k\}_{k=0}^\infty$ in Lemma 3.5 in Appendix 3.8.1, the rate-distortion function of the random process satisfies the following lemma.

Lemma 3.1 *For Gaussian random processes in \mathcal{A} , for any constant $0 < \kappa < 1$, we have*

$$R(\theta) \geq \frac{\kappa x d_l^{\frac{1}{x}}}{2} \theta^{-\frac{1}{x}} \quad (3.29)$$

$$D(\theta) \geq \kappa \left(1 + \frac{\kappa}{x-1} \right) \frac{d_l^{\frac{1}{x}}}{U_0} \theta^{1-\frac{1}{x}} \quad (3.30)$$

when θ is small enough .

A proof of Lemma 3.1 is provided in Appendix 3.8.2. Using Lemma 3.1, and recognizing the facts that $D(\theta)$ is a nondecreasing function of θ , and $R(\theta)$ is a strictly

decreasing function of θ when $\theta < \lambda_1$, i.e., its inverse function $\theta(R)$ exists when R is large enough, we have the next theorem which presents a lower bound for the distortion-rate function of the random process.

Theorem 3.1 *For Gaussian random processes in \mathcal{A} , for any constant $0 < \kappa < 1$, we have*

$$D_p(R) \geq \kappa \left(1 + \frac{\kappa}{x-1}\right) \left(\frac{\kappa x}{2}\right)^{x-1} \frac{d_l}{U_0} R^{1-x} \quad (3.31)$$

when R is large enough.

We will divide our discussion into two separate cases based on the sum power constraint, $P(N)$. For the first case, $P(N)$ is such that

$$\lim_{N \rightarrow \infty} \frac{1}{NP(N)} = 0 \quad (3.32)$$

The cases $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$ are included in $P(N)$ satisfying (3.32). From (3.27), we see that in this case C_u^N increases monotonically in N . Hence, when N is large enough, C_u^N will be large enough such that Theorem 3.1 holds. Hence, for any constant $0 < \kappa < 1$, a lower bound on the minimum achievable expected distortion is

$$D_l^N = D_p(C_u^N) \geq \kappa^2 \left(1 + \frac{\kappa}{x-1}\right) (\kappa x)^{x-1} \frac{d_l}{U_0} \left(\frac{1}{\log(NP(N))}\right)^{x-1} \quad (3.33)$$

when N is large enough. Hence, when sum power constraint $P(N)$ satisfies (3.32), the minimum achievable distortion is

$$\Omega \left(\left(\frac{1}{\log(NP(N))} \right)^{x-1} \right) \quad (3.34)$$

For the second case, $P(N)$ is such that (3.32) is not satisfied. In this case, C_u^N is either a constant independent of N or goes to zero as N goes to infinity. The minimum achievable distortion does not go to zero with increasing N .

Therefore, for all possible sum power constraints $P(N)$, the minimum achievable distortion is

$$\Omega \left(\min \left(\left(\frac{1}{\log(NP(N))} \right)^{x-1}, 1 \right) \right) \quad (3.35)$$

When the sum power constraint $P(N)$ grows almost exponentially with the number of nodes, the lower bound on the minimum achievable expected distortion in (3.35) decreases inverse polynomially with N . Even though this provides excellent distortion performance, it is impractical since sensor nodes are low energy devices and it is often difficult, if not impossible, to replenish their batteries. When the sum power constraint $P(N)$ is such that (3.32) is not satisfied, the transmission power is so low that the communication channels between the sensors and the collector node are as if they do not exist. From (3.35), the lower bound on the estimation error in this case is on the order of 1, which is equivalent to the collector node blindly estimating $S(u) = 0$ for all $u \in [0, U_0]$. Even though the consumed sum power $P(N)$

is very low in this case, the performance of the sensor network is unacceptable; even the lower bound on the minimum achievable expected distortion does not decrease to zero with the increasing number of nodes. For practically meaningful sum power values, including the cases of $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$, the lower bound on the minimum achievable expected distortion in (3.35) decays to zero at the rate of

$$\frac{1}{(\log N)^{x-1}} \quad (3.36)$$

3.4 An Upper Bound on the Achievable Distortion

3.4.1 Arbitrary Gaussian Random Processes

Any distortion found by using any achievability scheme will serve as an upper bound for the minimum achievable expected distortion. We consider the following separation-based achievable scheme. First, we perform multi-terminal rate-distortion coding at all sensor nodes using [30, Theorem 1]. After obtaining the indices of the rate-distortion codes, we transmit the indices as independent messages using the amplify-and-forward method introduced in [32]. The distortion obtained using this scheme will be denoted as D_u^N .

First, we determine an achievable rate region for the communication channel from the sensor nodes to the collector node. The channel in its nature is a multiple access channel with potential cooperation between the transmitters. The capacity region for this channel is not known. We get an achievable rate region for this channel by using the idea presented in [32].

Theorem 3.2 *When the sum power constraint $P(N)$ is such that there exists an $\epsilon > 0$ where*

$$\lim_{N \rightarrow \infty} P(N)N^{\frac{1}{2}-\epsilon} > 1 \quad (3.37)$$

for any constant $0 < \kappa < 1$, the following rate region is achievable,

$$\sum_{i=1}^N R_i^N \leq \kappa \nu \log(NP(N)) \triangleq C_a^N \quad \text{nats/channel use} \quad (3.38)$$

where R_i^N is the rate achievable from sensor i to the collector node, ν is a positive constant independent of N ,

$$\nu = \min \left(\frac{\epsilon}{1+2\epsilon}, \frac{1}{4} \right) \quad (3.39)$$

when N is large enough. Otherwise, the sum rate is bounded by a nonnegative constant as $N \rightarrow \infty$.

A proof of Theorem 3.2 is provided in Appendix 3.8.3. Theorem 3.2 shows that when the sum power constraint is such that (3.37) is satisfied, the achievable rate increases with N . Furthermore, the achievable rate is the same as the upper bound on the achievable sum rate in (3.26) order-wise. Otherwise, the achievable sum rate is either a positive constant or decreases to zero, which will result in poor estimation performance at the collector node. The achievability scheme proposed in the proof of Theorem 3.2 incurs a delay that is proportional to the number of sensor nodes.

From a practical point of view, it is desirable to have achievability schemes that perform better in terms of the latency. In this chapter, we propose *an* achievability scheme that meets the lower bound order-wise, and leave the issue of developing better achievability schemes to future work.

Since the achievable rate region developed above is only characterized by the sum rate constraint, in the source coding part, for a fixed distortion constraint, we only need to characterize the achievable sum rate, rather than the achievable rate region. We apply [30, Theorem 1], generalized to N sensor nodes in [16, Theorem 1], to obtain an achievable sum rate-distortion point. The achievability scheme is an indirect version of the achievability scheme developed by Berger and Tung [7].

Theorem 3.3 *For all Gaussian random processes, the following sum rate and distortion are achievable,*

$$D_a^N(\theta) = \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \frac{U_0}{N} \boldsymbol{\rho}_N^T(u) (\boldsymbol{\Sigma}'_N + \theta \mathbf{I})^{-1} \boldsymbol{\rho}_N(u) \right) du \quad (3.40)$$

$$R_a^N(\theta) = \sum_{k=0}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \quad (3.41)$$

where

$$\boldsymbol{\rho}_N(u) = \left[K(u, 0) \quad K\left(u, \frac{U_0}{N}\right) \quad K\left(u, \frac{2U_0}{N}\right) \cdots \quad K\left(u, \frac{(N-1)U_0}{N}\right) \right]^T \quad (3.42)$$

and

$$\Sigma_N = E[\mathbf{S}_N \mathbf{S}_N^T] \quad (3.43)$$

$$= \begin{bmatrix} K(0,0) & K\left(0, \frac{U_0}{N}\right) & \cdots & K\left(0, \frac{(N-1)U_0}{N}\right) \\ K\left(\frac{U_0}{N}, 0\right) & K\left(\frac{U_0}{N}, \frac{U_0}{N}\right) & \cdots & K\left(\frac{U_0}{N}, \frac{(N-1)U_0}{N}\right) \\ \vdots & \vdots & \vdots & \vdots \\ K\left(\frac{(N-1)U_0}{N}, 0\right) & K\left(\frac{(N-1)U_0}{N}, \frac{U_0}{N}\right) & \cdots & K\left(\frac{(N-1)U_0}{N}, \frac{(N-1)U_0}{N}\right) \end{bmatrix} \quad (3.44)$$

and $\Sigma'_N = \frac{U_0}{N} \Sigma_N$ and $\mu_0^{(N)'}, \mu_1^{(N)'}, \dots, \mu_{N-1}^{(N)'}$ are the eigenvalues of Σ'_N .

A proof of Theorem 3.3 is provided in Appendix 3.8.4.

We further evaluate $D_a^N(\theta)$ in the next lemma.

Lemma 3.2 *For all Gaussian random processes, we have*

$$D_a^N(\theta) \leq 2A^{(N)} + B^{(N)} + D_b^N(\theta) \quad (3.45)$$

where $A^{(N)}$, $B^{(N)}$ and $D_b^N(\theta)$ are defined as

$$\begin{aligned} A^{(N)} &= \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(K(u, u) - K\left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0\right) \right) du \\ &\quad + \frac{2}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(\rho_N\left(\frac{i-1}{N}U_0\right) - \rho_N(u) \right)_i du \end{aligned} \quad (3.46)$$

and

$$B^{(N)} = \frac{2}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left\| \rho_N\left(\frac{i-1}{N}U_0\right) - \rho_N(u) \right\| du \quad (3.47)$$

and

$$D_b^N(\theta) = \frac{1}{U_0} \sum_{k=0}^{N-1} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} \quad (3.48)$$

respectively.

A proof of Lemma 3.2 is provided in Appendix 3.8.5. Lemma 3.2 tells us that the expected distortion achieved by using the separation-based scheme is upper bounded by the sum of three types of distortion. The first two types of distortion, $A^{(N)}$ and $B^{(N)}$, have nothing to do with the rate and only depend on how well the samples approximate the entire random process. The third distortion, $D_b^N(\theta)$, depends on the rate through variable θ .

The function $R_a^N(\theta)$ is a strictly decreasing function of θ , thus, the inverse function exists, which we will denote as $\theta_a^N(R)$. Let us define $D_a(R)$ as the composition of the two functions $D_a^N(\theta)$ and $\theta_a^N(R)$, i.e.,

$$D_a(R) = D_a^N(\theta_a^N(R)) \quad (3.49)$$

An upper bound on the minimum achievable distortion, i.e., the achievable distortion by the separation-based scheme described above, is

$$D_u^N = D_a(C_a^N) \quad (3.50)$$

where C_a^N is defined in (3.38).

We will perform this calculation when the underlying random process is in \mathcal{A} .

3.4.2 The Class of Gaussian Random Processes in \mathcal{A}

We analyze the three types of distortion in (3.45) for Gaussian random processes in \mathcal{A} . We will focus on $A^{(N)}$ and $B^{(N)}$ in Lemma 3.3, and on $D_b^N(\theta)$ in Lemma 3.4.

Lemma 3.3 *For Gaussian random processes in \mathcal{A} , we have*

$$A^{(N)} = O(N^{-\alpha}) \quad (3.51)$$

$$B^{(N)} = O(N^{\frac{1}{2}-\alpha}) \quad (3.52)$$

A proof of Lemma 3.3 is provided in Appendix 3.8.6. The result depends crucially on condition 2 in the definition of \mathcal{A} in Section 3.2, i.e., the smoothness of the autocorrelation function $K(u, v)$. Note that since $1/2 < \alpha \leq 1$, both $A^{(N)}$ and $B^{(N)}$ decrease to zero inverse polynomially as N goes to infinity.

It remains to calculate the functions $R_a^N(\theta)$ and $D_b^N(\theta)$ for random processes in \mathcal{A} . To do so, we need some properties of $\{\mu_k^{(N')}\}_{k=0}^{N-1}$ which are stated in Lemmas 3.7 and 3.8 and proved in Appendix 3.8.7. Lemma 3.7 is of great importance, as it serves as a tool to link $\{\mu_k^{(N')}\}_{k=0}^{N-1}$ to $\{\lambda_k\}_{k=0}^\infty$, which is used in the derivation of the lower bound in Section 3.3, through the lower and upper bounds $\{\lambda'_k\}_{k=0}^\infty$ and $\{\lambda''_k\}_{k=0}^\infty$. Armed with the properties of $\mu_k^{(N')}$, λ'_k and λ''_k in Lemmas 3.5, 3.6, 3.7 and 3.8 in Appendices 3.8.1 and 3.8.7, we can show the following lemma. First, we define two sequences ϑ_L^N

and ϑ_U^N , which are functions of N , that satisfy

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\min(\frac{x\gamma}{2\tau}, \frac{\alpha x}{x-1}, \frac{\beta x}{x+\tau+1})}} = 0, \quad \lim_{N \rightarrow \infty} \vartheta_U^N = 0 \quad (3.53)$$

Lemma 3.4 *For Gaussian random processes in \mathcal{A} , for any constant $0 < \kappa < 1$, lower and upper bounds for the function $R_a^N(\theta)$ are*

$$\frac{\kappa x d_l^{\frac{1}{x}}}{4} \theta^{-\frac{1}{x}} \leq R_a^N(\theta) \leq \frac{d_u^{\frac{1}{x}} (x^2 - (1 - \log 2)x + (1 - \log 2))}{2(x-1)\kappa^2} \theta^{-\frac{1}{x}} \quad (3.54)$$

and an upper bound for the function $D_b^N(\theta)$ is

$$D_b^N(\theta) \leq \frac{d_u^{\frac{1}{x}} (1 + \kappa^2(x-1))}{\kappa^3(x-1)U_0} \theta^{1-\frac{1}{x}} \quad (3.55)$$

for $\theta \in [\vartheta_L^N, \vartheta_U^N]$ and N large enough.

A proof of Lemma 3.4 is provided in Appendix 3.8.8. The proof of Lemma 3.4 uses conditions 1, 2 and 3 in Section 3.2. Let us define a sequence ϑ_{LL}^N , which is a function of N , that satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_{LL}^N N^{\min(\frac{x\gamma}{2\tau}, \frac{(\alpha-1/2)x}{x-1}, \frac{\beta x}{x+\tau+1})}} = 0 \quad (3.56)$$

Combining (3.45), (3.51), (3.52), (3.54) and (3.55), we have the following theorem.

Theorem 3.4 *For Gaussian random processes in \mathcal{A} , for any constant $0 < \kappa < 1$,*

the achievable distortion-rate function, $D_a(R)$, is upper bounded as

$$D_a(R) \leq \frac{d_u(1 + \kappa^2(x-1))(x^2 - (1 - \log 2)x + (1 - \log 2))^{x-1}}{U_0 \kappa^{2x+2} 2^{x-1} (x-1)^x} R^{1-x} \quad (3.57)$$

for R in the interval of

$$\left[\frac{d_u^{\frac{1}{x}} (x^2 - (1 - \log 2)x + (1 - \log 2))}{2(x-1)\kappa^2} (\vartheta_U^N)^{-\frac{1}{x}}, \frac{\kappa x d_l^{\frac{1}{x}}}{4} (\vartheta_{LL}^N)^{-\frac{1}{x}} \right] \quad (3.58)$$

when N is large enough.

A proof of Theorem 3.4 is provided in Appendix 3.8.9. This theorem shows that when R is in the interval (3.58), the achievable distortion-rate function is the same as the lower bound on the distortion-rate function in (3.31) order-wise.

Theorem 3.5 *For Gaussian random processes in \mathcal{A} , when the sum power constraint satisfies (3.37) and*

$$\lim_{N \rightarrow \infty} \frac{NP(N)}{e^{N^{\min(\frac{\gamma}{2\tau}, \frac{2\alpha-1}{2(x-1)}, \frac{\beta}{x+\tau+1})}}} = 0 \quad (3.59)$$

an upper bound on the minimum achievable expected distortion, or equivalently, the achievable rate in the separation-based scheme, is

$$D_u^N = D_a(C_a^N) \quad (3.60)$$

$$\leq \frac{d_u(1 + \kappa^2(x-1))(x^2 - (1 - \log 2)x + (1 - \log 2))^{x-1}}{U_0 \kappa^{3x+1} 2^{x-1} (x-1)^x \nu^{x-1}} \left(\frac{1}{\log(NP(N))} \right)^{x-1} \quad (3.61)$$

when N is large enough.

A proof of Theorem 3.5 is provided in Appendix 3.8.10. Theorem 3.5 implies that, when the sum power constraint satisfies (3.37) and (3.59), the minimum achievable expected distortion is

$$O\left(\left(\frac{1}{\log(NP(N))}\right)^{x-1}\right) \quad (3.62)$$

For the interesting cases of $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$, the upper bound on the minimum achievable expected distortion decays to zero at the rate of

$$\frac{1}{(\log N)^{x-1}} \quad (3.63)$$

When the sum power constraint is such that (3.37) is not satisfied, an upper bound on the minimum achievable expected distortion is $\Theta(1)$.

3.5 Comparison of the Lower and Upper Bounds for Gaussian Random Processes in \mathcal{A}

3.5.1 Order-wise Comparison of Lower and Upper Bounds

In this section, we compare the lower bound in (3.35) and the upper bound in (3.62). When the sum power constraint is large, i.e., $P(N)$ is so large that (3.59) is not satisfied, our methods in finding the upper bound do not apply. Even though our lower bound in (3.35) is valid, we have not shown whether the lower and upper bounds

meet. However, in this case, $P(N)$ is larger than $\frac{e^{N^{\min(\frac{\gamma}{2\tau}, \frac{2\alpha-1}{2(x-1)}, \frac{\beta}{x+\tau+1})}}}{N}$, and this region of sum power constraint is not of practical interest.

When the sum power constraint is medium, i.e., $P(N)$ is in the wide range of $N^{-1/2+\epsilon}$ to $\frac{e^{N^{\min(\frac{\gamma}{2\tau}, \frac{2\alpha-1}{2(x-1)}, \frac{\beta}{x+\tau+1})}}}{N}$, our lower and upper bounds do meet and the minimum achievable expected distortion is

$$D^N = \Theta \left(\left(\frac{1}{\log(NP(N))} \right)^{x-1} \right) \quad (3.64)$$

One possible order-optimal achievability scheme is a separation-based scheme, which uses distributed rate-distortion coding as described in [30] and optimal single-user channel coding with amplify-and-forward method as described in [32]. In fact, when the sum power constraint is medium, as shown in (3.31) and (3.57), lower and upper bounds on the distortion-rate function, $D_p(R)$ and $D_a(R)$ coincide order-wise. In addition, as shown in (3.27) and (3.38), the lower and upper bounds on the achievable sum rate, C_a^N and C_u^N , coincide order-wise as well. The practically interesting cases of $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$ fall into this region of medium sum power constraint. In both of these cases, the minimum achievable expected distortion decreases to zero at the rate of

$$\frac{1}{(\log N)^{x-1}} \quad (3.65)$$

Hence, the sum power constraint $P(N) = P_{\text{tot}}$ performs as well as $P(N) = NP_{\text{ind}}$ “order-wise”, and therefore, in practice we may prefer to choose $P(N) = P_{\text{tot}}$. In fact,

we can decrease the sum power constraint to $P(N) = \Theta(N^{-1/3})$ and the minimum achievable distortion will still decrease to zero at the rate in (3.65).

When the sum power constraint is small, i.e., $P(N)$ ranges from N^{-1} to $N^{-1/2}$, our lower and upper bounds do not meet. Our lower bound in (3.35) decreases to zero as $\frac{1}{(\log N)^{x-1}}$ but our upper bound is a non-zero constant. The main discrepancy between our lower and upper bounds comes from the gap between the lower and upper bounds on the sum capacities, C_a^N and C_u^N , for a cooperative multiple access channel. In fact, when the sum power constraint is small, as shown in (3.31) and (3.57), lower and upper bounds on the distortion-rate function, $D_p(R)$ and $D_a(R)$ still coincide order-wise. This sum power constraint region should be of practical interest, because in this region, the sum power constraint is quite low, and yet the lower bound on the distortion is of the same order as one would obtain with any $P(N)$ which increases polynomially with N . Hence, from the lower bound, it seems that this region potentially has good performance. However, our separation-based upper bound does not meet the lower bound, and whether the lower bound can be achieved remains an open problem.

When the sum power constraint is very small, i.e., $P(N)$ is less than N^{-1} , our lower and upper bounds meet and the minimum achievable expected distortion is a constant that does not decrease to zero with increasing N . This case is not of practical interest because of the unacceptable distortion.

In the case of Gauss-Markov random process, we have $x = 2$ and $\alpha = \beta = \tau = \gamma = 1$. Inserting these values into the above results, we see that in the medium sum power constraint region, i.e., $P(N)$ is in the wide range of $N^{-1/2+\epsilon}$ to $\frac{e^{N^{1/4}}}{N}$, the minimum

achievable expected distortion is

$$D^N = \Theta \left(\frac{1}{\log(NP(N))} \right) \quad (3.66)$$

For the Gauss-Markov random process, in the cases of $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$, the minimum achievable expected distortion decreases to zero at the rate of

$$\frac{1}{\log N} \quad (3.67)$$

3.5.2 Comparison of the Constants in the Lower and Upper Bounds

Though the lower and upper bounds meet order-wise in a wide range of sum power constraints, the constants in front of them are different and we aim to compare these constants for various sum power constraints in this section.

Combining (3.33) and (3.61), when $P(N)$ satisfies (3.37) and (3.59), the minimum distortion D^N satisfies

$$\begin{aligned} \kappa^2 \left(1 + \frac{\kappa}{x-1} \right) (\kappa x)^{x-1} \frac{d_l}{U_0} \left(\frac{1}{\log(NP(N))} \right)^{x-1} &\leq D^N \\ &\leq \frac{d_u(1 + \kappa^2(x-1))(x^2 - (1 - \log 2)x + (1 - \log 2))^{x-1}}{U_0 \kappa^{3x+1} 2^{x-1} (x-1)^x \nu^{x-1}} \left(\frac{1}{\log(NP(N))} \right)^{x-1} \end{aligned} \quad (3.68)$$

Note that κ can be made as close to 1 as possible for large enough N . Let $\pi(x, \nu)$ be the ratio of the constant in the lower bound and the constant in the upper bound

when N is large enough. Then,

$$\pi(x, \nu) = \frac{d_l}{d_u} (2\nu)^{x-1} \left(\frac{x^2 - x}{x^2 - (1 - \log 2)x + (1 - \log 2)} \right)^{x-1} \quad (3.69)$$

Here, x is a parameter of the underlying Gaussian random process and ν depends on the sum power constraint of the sensor nodes, $P(N)$. It is straightforward to see that since from (3.39), $\nu \leq 1/4$, $\pi(x, \nu)$ is a monotonically decreasing function of x for a fixed ν . Hence, we conclude that the constants in front of the lower and upper bounds differ more as x gets large. Since x is an indication of how concentrated the randomness of the random process is, this means that the more evenly distributed the randomness, the more the constants in the lower and upper bounds meet. For a fixed underlying random process, i.e., for a fixed x , $\pi(x, \nu)$ is a decreasing function of ν . This means that the less the sum power constraint we have, the more different the constants will be.

In the Gauss-Markov random process, $x = 2$, and $d_l = d_u$. When $P(N) = NP_{\text{ind}}$ and $P(N) = P_{\text{tot}}$, the ratio of the two constants is

$$\pi(2, 1/4) = \frac{1}{3 + \log 2} \simeq 0.2708 \quad (3.70)$$

When $P(N) = \Theta(N^{-\omega})$, $0 < \omega < \frac{1}{2}$, the ratio of the two constants is

$$\pi\left(2, \frac{1}{2} - \frac{1}{4} \frac{1}{1 - \omega}\right) = \left(\frac{1}{2} - \frac{1}{4} \frac{1}{1 - \omega}\right) \frac{4}{3 + \log 2} \quad (3.71)$$

For example, when $P(N) = \Theta(N^{-1/3})$, the ratio of the constants is

$$\pi(2, 1/8) = \frac{1}{2}\pi(2, 1/4) \simeq 0.1354 \quad (3.72)$$

3.6 Further Remarks

We have shown that the minimum achievable expected distortion behaves order-wise as

$$\Theta\left(\left(\frac{1}{\log(NP(N))}\right)^{x-1}\right) \quad (3.73)$$

Due to the order-optimality of separation, this result can be generalized straightforwardly to several other scenarios.

The result in (3.73) still holds when we allow the collector node to use its transmit antenna with an arbitrary power constraint. The collector node, using its transmit antenna, can send some form of feedback to the sensor nodes. However, the lower bound on the minimum distortion remains unchanged in this case, because in deriving our lower bound, we assumed that all sensor nodes know the entire random process, thus, forming a point-to-point system. In a point-to-point system, feedback, perfect or not, does not change the capacity. Meanwhile, our upper bound is still valid, as in this achievable scheme, we choose not to utilize the feedback link. Hence, our result in (3.73) remains valid.

The result in (3.73) still holds when we allow K channel uses per realization of the random process, where K is a constant independent of N . This is because both

lower and upper bounds are derived using separation-based schemes. The minimum achievable distortion still behaves as (3.73), and the number K will only effect the constant in front. Due to the same reasoning, the minimum achievable distortion behaves as (3.73) when we allow multiple transmit and receive antennas at each node, as long as the number of antennas on each node is a constant, independent of N .

The assumption of the polynomial decay of the eigenvalues play a key role for the separation principle to hold order-wise. For example, when the eigenvalues decrease exponentially, i.e., the k -th eigenvalue is roughly e^{-k} , the distortion rate function of the lower bound is

$$D_p(R) = \Theta \left(\sqrt{R} e^{-2\sqrt{R}} \right) \quad (3.74)$$

Thus, in the lower bound the distortion goes to zero almost exponentially with the rate R , as opposed to the polynomial decrease in R as in (3.31). It can be shown, using the exact same proof techniques as those used in Section 3.4.2, that the achievable distortion-rate function is the same order as (3.74), for a wide range of sum power constraints. However, in the channel coding part, the converse and the achievability of the sum rate meet only order-wise, i.e., the lower and upper bounds on the sum rate are of the form $c_1 \log(NP(N))$ and $c_2 \log(NP(N))$ where $c_1 < c_2$. The difference in the constants in the lower and upper bounds on the sum rate will cause an order difference in the distortion, i.e., $\sqrt{c_1 \log(NP(N))} e^{-2\sqrt{c_1 \log(NP(N))}}$ is strictly of a larger order than $\sqrt{c_2 \log(NP(N))} e^{-2\sqrt{c_2 \log(NP(N))}}$ for $c_1 < c_2$. Hence, when the

underlying random process is such that the eigenvalues decrease exponentially, separation principle does not hold, even order-wise. This agrees with the observation made in Section 3.5.2 that the constants in front of the lower and upper bounds differ more as x gets large.

For simplicity, we have considered only one dimensional spatial random processes. We expect the generalization to two dimensional random fields to be straightforward, but nonetheless tedious. Our results do not generalize straightforwardly when the samples that the sensor nodes obtain are subject to noise. Since the lower bound of assuming all sensors know the entire random process would remain the same with or without noise, the lower bound becomes too loose. Hence, the optimal performance under the noisy sensor scenario remains open.

3.7 Chapter Summary and Conclusions

In this chapter, we investigated the performance of dense sensor networks by studying the joint source-channel coding problem. We provided separation-based lower and upper bounds for the minimum achievable expected distortion when the underlying random process is Gaussian. When the random process satisfies some general conditions, such as polynomial decrease rate of the ordered eigenvalues of the random process, i.e., the k -th eigenvalue is roughly k^{-x} , we evaluated the lower and upper bounds explicitly, and showed that they are both of order $\frac{1}{(\log(NP(N)))^{x-1}}$ for a wide range of sum power constraints ranging from $N^{-\frac{1}{2}+\epsilon}$ to $\frac{e^{N^{\min(\frac{\gamma}{2\tau}, \frac{2\alpha-1}{2(x-1)}, \frac{\beta}{x+\tau+1})}}}{N}$. In the most interesting cases when the sum power constraint is a constant or grows linearly

with N , the minimum achievable expected distortion decreases to zero at the rate of $\frac{1}{(\log N)^{x-1}}$. For random processes that satisfy these general conditions, under these power constraints, we have found an order-optimal scheme that is separation-based, and is composed of distributed rate-distortion coding [30] and amplify-and-forward channel coding [32].

The results of this chapter were published in [54, 55], and have been accepted for publication in [53].

3.8 Appendix

3.8.1 Some properties of λ'_k and λ''_k

In this subsection, we provide two lemmas which characterize some properties of $\{\lambda'_k\}_{k=0}^\infty$ and $\{\lambda''_k\}_{k=0}^\infty$, defined in (3.12) and (3.13), which will be useful in deriving our main results.

Lemma 3.5 *For any constant $0 < \kappa < 1$, we have*

$$\sum_{k=\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l + 1 \right\rfloor}^{\infty} \lambda'_k \geq \frac{\kappa d_l^{\frac{1}{x}}}{(x-1)} \theta^{1-\frac{1}{x}} \quad (3.75)$$

and

$$\sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor} \frac{1}{2} \log \left(\frac{\lambda'_k}{\theta} \right) \geq \frac{\kappa x d_l^{\frac{1}{x}}}{2} \theta^{-\frac{1}{x}} \quad (3.76)$$

when θ is small enough.

Lemma 3.6 *For any constant $0 < \kappa < 1$, we have*

$$\sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}+c_u} \right\rfloor+1}^{\infty} \lambda_k'' \leq \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa} \theta^{1-\frac{1}{x}} \quad (3.77)$$

and

$$\sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}+c_u} \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\lambda_k''}{\theta} \right) \leq \left(\frac{\log 2 + x}{2\kappa} \right) d_u^{\frac{1}{x}} \theta^{-\frac{1}{x}} \quad (3.78)$$

when θ is small enough.

Proof of Lemma 3.5

We will first prove (3.75).

$$\sum_{k=\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}-c_l+1} \right\rfloor}^{\infty} \lambda_k' = \sum_{k=\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}-c_l+1} \right\rfloor}^{\infty} \frac{d_l}{(k+c_l)^x} \quad (3.79)$$

$$\geq \frac{d_l}{x-1} \frac{1}{\left[\left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} + 1 \right]^{x-1}} \quad (3.80)$$

$$\geq \frac{\kappa d_l^{\frac{1}{x}}}{(x-1)} \theta^{1-\frac{1}{x}} \quad (3.81)$$

where (3.79) is true when θ is small enough, more specifically, when $\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l + 1 \right\rfloor > K_0$. We have (3.80) because of the inequality

$$\sum_{k=n}^{\infty} \frac{1}{k^x} \geq \int_n^{\infty} \frac{1}{y^x} dy = \frac{1}{(x-1)n^{x-1}} \quad (3.82)$$

and (3.81) is true when θ is small enough, i.e., for any $0 < \kappa < 1$, there exists a $\theta_0(\kappa) > 0$ such that when $0 < \theta \leq \theta_0(\kappa)$, (3.81) is true.

Next, we will prove (3.76).

$$\begin{aligned} & \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor \sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor} \frac{1}{2} \log \left(\frac{\lambda'_k}{\theta} \right) \\ &= \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{\lambda_k}{\theta} \right) + \sum_{k=K_0+1}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor} \frac{1}{2} \log \left(\frac{d_l}{(k+c_l)^x \theta} \right) \end{aligned} \quad (3.83)$$

$$\begin{aligned} &= \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{\lambda_k}{d_l} \right) + \frac{1}{2} \left(\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor - c_l + 1 \right) \log \left(\frac{d_l}{\theta} \right) - \frac{x}{2} \log \left(\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor! \right) \\ &\quad + \frac{x}{2} \log((K_0 + c_l)!) \end{aligned} \quad (3.84)$$

$$\begin{aligned} &\geq \frac{1}{2} \left(\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor - c_l + 1 \right) \log \left(\frac{d_l}{\theta} \right) - \frac{x}{2} \left(\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor + \frac{1}{2} \right) \log \left[\left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right] \\ &\quad + \frac{x}{2} \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor - \frac{x}{24 \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor} + \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{\lambda_k}{d_l} \right) + \frac{x}{2} \log((K_0 + c_l)!) - \frac{x}{4} \log(2\pi) \end{aligned} \quad (3.85)$$

$$\geq \frac{x}{2} \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor + \frac{x}{2} \left(-c_l + \frac{1}{2} \right) \log \left[\left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right] - \frac{x}{24 \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor} + c_3 \quad (3.86)$$

$$\geq \frac{\kappa x d_l^{\frac{1}{x}}}{2} \theta^{-\frac{1}{x}} \quad (3.87)$$

where (3.83) is true when θ is small enough, more specifically, when $\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor > K_0$, and (3.85) follows by using Stirling's approximation,

$$n! < \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}} \quad (3.88)$$

(3.86) follows because c_3 is a constant, independent of θ , defined as

$$c_3 \triangleq \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{\lambda_k}{d_l} \right) + \frac{x}{2} \log((K_0 + c_l)!) - \frac{x}{4} \log(2\pi) \quad (3.89)$$

and (3.87) is true when θ is small enough, i.e., for any $0 < \kappa < 1$, there exists a $\theta_1(\kappa) > 0$ such that when $0 < \theta \leq \theta_1(\kappa)$, (3.87) is true.

Therefore, for any $0 < \kappa < 1$, (3.75) and (3.76) hold when θ is small enough.

Proof of Lemma 3.6

We will first prove (3.77).

$$\sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{\infty} \lambda_k'' = \sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{\infty} \frac{d_u}{(k - c_u)^x} \quad (3.90)$$

$$= \frac{d_u}{(x-1) \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} \right\rfloor \right)^{x-1}} \quad (3.91)$$

$$\leq \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa} \theta^{1-\frac{1}{x}} \quad (3.92)$$

where (3.90) follows when θ is small enough, more specifically, when $\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1 > K_0$. In obtaining (3.91) we used

$$\sum_{k=n}^{\infty} \frac{1}{k^x} \leq \int_{n-1}^{\infty} \frac{1}{y^x} dy = \frac{1}{(x-1)(n-1)^{x-1}} \quad (3.93)$$

and (3.92) follows when θ is small enough, i.e., for any $0 < \kappa < 1$, there exists a $\theta_2(\kappa) > 0$ such that when $0 < \theta \leq \theta_2(\kappa)$, (3.92) is true.

Next, we will prove (3.78).

$$\begin{aligned} & \left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\lambda_k''}{\theta} \right) \\ &= \sum_{k=0}^{K_0} \frac{1}{2} \log \left(1 + \frac{\lambda_k}{\theta} \right) + \sum_{k=K_0+1}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{d_u}{(k - c_u)^x \theta} \right) \end{aligned} \quad (3.94)$$

$$\leq \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{2\lambda_k}{\theta} \right) + \sum_{k=K_0+1}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(\frac{2d_u}{(k - c_u)^x \theta} \right) \quad (3.95)$$

$$\begin{aligned} &= \sum_{k=0}^{K_0} \frac{1}{2} \log \left(\frac{2\lambda_k}{\theta} \right) + \sum_{k=c_u+1}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(\frac{2d_u}{(k - c_u)^x \theta} \right) - \sum_{k=c_u+1}^{K_0} \frac{1}{2} \log \left(\frac{2d_u}{(k - c_u)^x \theta} \right) \\ & \quad (3.96) \end{aligned}$$

$$\begin{aligned} &= \left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} \right\rfloor \frac{1}{2} \log 2 - \frac{x}{2} \log \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} \right\rfloor! \right) + \frac{1}{2} \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} \right\rfloor \right) \log \frac{d_u}{\theta} \\ & \quad + \frac{c_u + 1}{2} \log \frac{d_u}{\theta} + c_1 \end{aligned} \quad (3.97)$$

$$\begin{aligned}
&\leq \left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor \frac{1}{2} \log 2 - \frac{x}{4} \log(2\pi) - \frac{x}{2} \left(\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor + \frac{1}{2} \right) \log \left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor \\
&\quad + \frac{x}{2} \left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor - \frac{x}{24 \left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor + 2} + \frac{1}{2} \left(\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} \right\rfloor \right) \log \frac{d_u}{\theta} + \frac{c_u + 1}{2} \log \frac{d_u}{\theta} + c_1
\end{aligned} \tag{3.98}$$

$$\leq \left(\frac{\log 2 + x}{2\kappa} \right) d_u^{\frac{1}{x}} \theta^{-\frac{1}{x}} \tag{3.99}$$

where (3.94) is true when θ is small enough, more specifically, when $\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor > K_0$. We have (3.95) because

$$\frac{d_u}{(k - c_u)^x \theta} > 1 \tag{3.100}$$

for all k between $K_0 + 1$ and $\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor$, and when θ is small enough such that

$$\theta \leq \lambda_k, \quad k = 1, 2, \dots, K_0 \tag{3.101}$$

We have (3.97) because we defined

$$c_1 \triangleq \sum_{k=1}^{K_0} \frac{1}{2} \log \frac{2\lambda_k}{d_u} - \sum_{k=c_u+1}^{K_0} \frac{1}{2} \log \frac{2}{(k - c_u)^x} \tag{3.102}$$

We used Stirling's approximation,

$$n! > \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} \tag{3.103}$$

to obtain (3.98), and (3.99) follows when θ is small enough, i.e., for any $0 < \kappa < 1$,

there exists a $\theta_3(\kappa) > 0$ such that when $0 < \theta \leq \theta_3(\kappa)$, (3.99) is true.

Therefore, for any $0 < \kappa < 1$, (3.77) and (3.78) hold when θ is small enough.

3.8.2 Proof of Lemma 3.1

For any $0 < \kappa < 1$, when θ is small enough, the results of Lemma 3.5 hold.

From (3.24), we have

$$R(\theta) = \sum_{k=0}^{\infty} \max \left(0, \frac{1}{2} \log \left(\frac{\lambda_k}{\theta} \right) \right) \quad (3.104)$$

$$\geq \sum_{k=0}^{\infty} \max \left(0, \frac{1}{2} \log \left(\frac{\lambda'_k}{\theta} \right) \right) \quad (3.105)$$

$$= \sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta} \right)^{\frac{1}{x}} - c_l \right\rfloor} \frac{1}{2} \log \left(\frac{\lambda'_k}{\theta} \right) \quad (3.106)$$

$$\geq \frac{\kappa x d_l^{\frac{1}{x}}}{2} \theta^{-\frac{1}{x}} \quad (3.107)$$

where in (3.105) we have used the definition of sequence λ'_k in (3.12) and the observation in (3.14). (3.106) follows when θ is small enough, more specifically, when $\theta < \lambda_{K_0}$ and $\left\lfloor \left(\frac{d_l}{\theta} \right)^{\frac{1}{x}} - c_l \right\rfloor > K_0$. (3.107) follows from (3.76) in Lemma 3.5.

From (3.25), we have

$$D(\theta) = U_0^{-1} \sum_{k=0}^{\infty} \min(\theta, \lambda_k) \quad (3.108)$$

$$\geq U_0^{-1} \sum_{k=0}^{\infty} \min(\theta, \lambda'_k) \quad (3.109)$$

$$= U_0^{-1} \sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor} \theta + U_0^{-1} \sum_{k=\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l + 1 \right\rfloor}^{\infty} \lambda'_k \quad (3.110)$$

$$\geq U_0^{-1} \left(\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} \right\rfloor - c_l + 1 \right) \theta + U_0^{-1} \frac{\kappa d_l^{\frac{1}{x}}}{(x-1)} \theta^{1-\frac{1}{x}} \quad (3.111)$$

$$\geq \kappa \left(1 + \frac{\kappa}{x-1} \right) \frac{d_l^{\frac{1}{x}}}{U_0} \theta^{1-\frac{1}{x}} \quad (3.112)$$

where in (3.109) we have used the definition of sequence λ'_k in (3.12) and the observation in (3.14). (3.110) follows when θ is small enough, more specifically, when $\theta < \lambda_{K_0}$ and $\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l + 1 \right\rfloor > K_0$. (3.111) follows from (3.75) in Lemma 3.5. (3.112) is true for small enough θ , i.e., for any $0 < \kappa < 1$, there exists a $\theta_4(\kappa) > 0$ such that when $0 < \theta \leq \theta_4(\kappa)$, (3.112) is true.

Therefore, for any $0 < \kappa < 1$, (3.29) and (3.30) hold when θ is small enough.

3.8.3 Proof of Theorem 3.2

We will show that each sensor node i can achieve a rate of C_a^N , while the other sensor nodes have rate zero, then by the time sharing argument [22], we can achieve the rate

region of

$$\sum_{i=1}^N R_i^N \leq C_a^N \quad (3.113)$$

We will consider the transmission of the data of node i . All other sensor nodes have no data to transmit and are helping with the communication between sensor node i and the collector node. Node i codes its message using capacity achieving single-user coding techniques with codeword length \bar{n} . Each codeword symbol requires two time slots. In the first time slot, node i transmits its codeword symbol using power $P(N)$. All other nodes remain silent, and receive a noisy version of node i 's transmitted signal. The collector node ignores its received signal, which is suboptimal but eases calculation and does not affect the scaling law of the achievable rate. In the second time slot, all sensor nodes, except node i , amplify and forward what they have received in the previous time slot to the collector node using a sum power constraint $P(N)$. The collector node, after $2\bar{n}$ time slots, decodes using capacity achieving single-user decoding techniques. The scheme described satisfies the sum power constraint of $P(N)$. Now, we calculate the rate achievable with this scheme. In the first time slot, sensor node j receives

$$Y_j = h_{ij}X_i + Z_j, \quad i, j = 1, 2, \dots, N, \quad j \neq i \quad (3.114)$$

and in the second time slot, sensor node j transmits

$$X_j = \beta_{ij} Y_j \quad (3.115)$$

$$= \beta_{ij} h_{ij} X_i + \beta_{ij} Z_j, \quad i, j = 1, 2, \dots, N, \quad j \neq i \quad (3.116)$$

where β_{ij} is the scaling coefficient of node j when it amplifies the signal it received from node i . In order to satisfy the sum power constraints, $\{\beta_{ij}\}_{j=1, j \neq i}^N$ have to satisfy

$$\sum_{j=1, j \neq i}^N \beta_{ij}^2 (h_{ij}^2 P(N) + 1) \leq P(N), \quad i = 1, 2, \dots, N \quad (3.117)$$

The collector node receives

$$Y_0 = \sum_{j=1, j \neq i}^N h_{j0} X_j + Z_0 \quad (3.118)$$

$$= \left(\sum_{j=1, j \neq i}^N \beta_{ij} h_{ij} h_{j0} \right) X_i + \left(\sum_{j=1, j \neq i}^N h_{j0} \beta_{ij} Z_j \right) + Z_0 \quad (3.119)$$

Therefore, the achievable rate is,

$$\frac{1}{4} \log \left(1 + \frac{\left(\sum_{j=1, j \neq i}^N \beta_{ij} h_{ij} h_{j0} \right)^2 P(N)}{\sum_{j=1, j \neq i}^N (\beta_{ij} h_{j0})^2 + 1} \right) \quad (3.120)$$

where we have $\frac{1}{4}$ because we used two time slots to transmit one codeword symbol.

We choose

$$\beta_{ij} = \zeta h_{ij} h_{j0} \quad (3.121)$$

where, in order to satisfy the power constraint, the constant ζ must satisfy

$$\zeta^2 \leq \frac{P(N)}{\left(\sum_{j=1, j \neq i}^N h_{ij}^4 h_{j0}^2\right) P(N) + \left(\sum_{j=1, j \neq i}^N h_{ij}^2 h_{j0}^2\right)} \quad (3.122)$$

We can choose ζ as

$$\zeta^2 = \frac{P(N)}{\bar{h}_u^6 N P(N) + \bar{h}_u^4 N} \quad (3.123)$$

Thus, from (3.120), a lower bound on the achievable rate is

$$\frac{1}{4} \log \left(1 + \frac{\zeta^2 \left(\sum_{j=1, j \neq i}^N h_{ij}^2 h_{j0}^2 \right)^2 P(N)}{\zeta^2 \left(\sum_{j=1, j \neq i}^N h_{ij}^2 h_{j0}^4 \right) + 1} \right) \geq \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8 \zeta^2 (N-1)^2 P(N)}{\bar{h}_u^6 \zeta^2 N + 1} \right) \triangleq C_b^N \quad (3.124)$$

Clearly, rate C_b^N can be achievable by any node i . We have

$$C_b^N = \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8 (P(N))^2 \frac{(N-1)^2}{N}}{2\bar{h}_u^6 P(N) + \bar{h}_u^4} \right) \quad (3.125)$$

$$\geq \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8 (P(N))^2 N}{4\bar{h}_u^6 P(N) + 2\bar{h}_u^4} \right) \quad (3.126)$$

where the last step follows when N is large enough such that $\frac{(N-1)^2}{N} > \frac{N}{2}$.

When $P(N)$ is such that

$$\lim_{N \rightarrow \infty} \frac{1}{P(N)} = 0 \quad (3.127)$$

for any $0 < \kappa < 1$, we have,

$$C_b^N \geq \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8 (P(N))^2 N}{8\bar{h}_u^6 P(N)} \right) \quad (3.128)$$

$$= \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8}{8\bar{h}_u^6} NP(N) \right) \quad (3.129)$$

$$\geq \frac{\kappa}{4} \log (NP(N)) \quad (3.130)$$

for N large enough, i.e., there exists $N_1(\kappa) > 0$, such that when $N > N_1(\kappa)$, (3.128)

and (3.130) are true.

When $P(N)$ is such that

$$\lim_{N \rightarrow \infty} P(N) = l \quad (3.131)$$

and l is a number that satisfies $0 < l < \infty$, fix some small $l_0 > 0$, there exists an $N_2(l_0) > 0$ such that when $N > N_2(l_0)$, we have,

$$l - l_0 < P(N) < l + l_0 \quad (3.132)$$

Hence, when $N > N_2(l_0)$, for any $0 < \kappa < 1$,

$$C_b^N \geq \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8 (l - l_0)}{4\bar{h}_u^6 (l + l_0) + 2\bar{h}_u^4} P(N)N \right) \quad (3.133)$$

$$\geq \frac{\kappa}{4} \log (NP(N)) \quad (3.134)$$

where the last step follows when N is large enough, i.e., when there exists an $N_3(\kappa) >$

0, such that when $N > \max(N_2(l_0), N_3(\kappa))$, (3.134) is true.

When $P(N)$ is such that

$$\lim_{N \rightarrow \infty} P(N) = 0 \quad (3.135)$$

and there exists a constant $0 < \epsilon < \frac{1}{2}$, such that

$$\lim_{N \rightarrow \infty} P(N)N^{\frac{1}{2}-\epsilon} > 1 \quad (3.136)$$

we have, for $0 < \kappa < 1$,

$$C_b^N \geq \frac{1}{4} \log \left(1 + \frac{\bar{h}_l^8}{4\bar{h}_u^4} (P(N))^2 N \right) \quad (3.137)$$

$$\geq \frac{\kappa}{4} \log ((P(N))^2 N) \quad (3.138)$$

$$= \frac{\kappa}{4} \log(NP(N)) + \frac{\kappa}{4} \log(P(N)) \quad (3.139)$$

$$\geq \frac{\kappa}{4} \frac{4\epsilon}{1+2\epsilon} \log(NP(N)) \quad (3.140)$$

where the last step follows from

$$\frac{\kappa}{4} \left(1 - \frac{4\epsilon}{1+2\epsilon} \right) \log(NP(N)) + \frac{\kappa}{4} \log(P(N)) = \frac{\kappa}{4} \frac{2}{1+2\epsilon} \log(P(N)N^{\frac{1}{2}-\epsilon}) \geq 0 \quad (3.141)$$

when N is large enough, i.e., there exists an $N_4(\kappa) > 0$, such that when $N > N_4(\kappa)$,

(3.137), (3.138) and (3.141) are true, and therefore, (3.140) is true.

Thus, combining all possible cases of $P(N)$, we see that when $P(N)$ is such that

there exists a constant $\epsilon > 0$, such that

$$\lim_{N \rightarrow \infty} P(N)N^{\frac{1}{2}-\epsilon} > 1 \quad (3.142)$$

for any $0 < \kappa < 1$, the following rate C_a^N from sensor node i to the collector node is achievable,

$$C_a^N = \kappa \nu \log(NP(N)) \quad (3.143)$$

where constant ν is

$$\nu = \min \left(\frac{\epsilon}{1+2\epsilon}, \frac{1}{4} \right) \quad (3.144)$$

when N is large enough.

Since the achievable rate C_a^N is achievable for any sensor i , by a time sharing argument, the region

$$\sum_{i=1}^N R_i^N \leq C_a^N \quad (3.145)$$

is achievable.

For all other $P(N)$, from (3.126), we see that the achievable sum rate approaches a positive constant or zero as N goes to infinity.

3.8.4 Proof of Theorem 3.3

We restate the generalization of [30, Theorem 1], which appeared in [16, Theorem 1] for N sensor nodes below. This provides us with an achievable sum rate-distortion point, since the sum rate constraint is always tight [16].

Theorem 3.6 *[16, 30] A rate-distortion sum rate R_c and distortion D_c are achievable if there exist random variables T_1, T_2, \dots, T_N with*

$$(S(u), u \in [0, U_0], S_{\{i\}^c}, T_{\{i\}^c}) \rightarrow S_i \rightarrow T_i, \quad i = 1, 2, \dots, N \quad (3.146)$$

and an estimator function

$$\hat{S}(u) = g(T_1, T_2, \dots, T_N) \quad (3.147)$$

such that

$$R_c \geq I(S_1, S_2, \dots, S_N; T_1, T_2, \dots, T_N) \quad (3.148)$$

$$D_c \geq E[d(S(u), g(T_1, T_2, \dots, T_N))] \quad (3.149)$$

where random variables $\{S_i\}_{i=1}^N$ are defined as $S_i = S(u_i)$, $i = 1, 2, \dots, N$.

We obtain an achievable rate-distortion point when we specify the relationship between $(S(u), \{S_i\}_{i=1}^\infty, \{T_i\}_{i=1}^\infty)$ as

$$T_i = S_i + W_i, \quad i = 1, 2, \dots, N \quad (3.150)$$

where $W_i, i = 1, 2, \dots, N$, are i.i.d. Gaussian random variables with zero-mean and variance σ_D^2 and independent of everything else. Here, we can adjust σ_D^2 to achieve various feasible rate-distortion points [30].

We choose the MMSE estimator to estimate $S(u)$ from observations $\{T_k\}_{k=1}^N$. Hence, the achieved distortion is

$$D_c^N(\sigma_D^2) = \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \boldsymbol{\rho}_N^T(u) (\boldsymbol{\Sigma}_N + \sigma_D^2 \mathbf{I})^{-1} \boldsymbol{\rho}_N(u) \right) du \quad (3.151)$$

The sum rate required to achieve this distortion is

$$\begin{aligned} R_c^N(\sigma_D^2) &= I(S_1, S_2, \dots, S_N; T_1, T_2, \dots, T_N) \\ &= \frac{1}{2} \log \det \left(\mathbf{I} + \frac{1}{\sigma_D^2} \boldsymbol{\Sigma}_N \right) \end{aligned} \quad (3.152)$$

$$= \sum_{k=0}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)}}{\sigma_D^2} \right) \quad (3.153)$$

where $\mu_0^{(N)}, \mu_1^{(N)}, \dots, \mu_{N-1}^{(N)}$ are the eigenvalues of $\boldsymbol{\Sigma}_N$.

Next, let $\theta = \frac{U_0}{N} \sigma_D^2$, $\boldsymbol{\Sigma}'_N = \frac{U_0}{N} \boldsymbol{\Sigma}_N$ and therefore, $\mu_k^{(N)'} = \frac{U_0}{N} \mu_k^{(N)}$. We define two functions of θ as

$$R_a^N(\theta) \triangleq R_c(\sigma_D^2) = \sum_{k=0}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \quad (3.154)$$

and

$$D_a^N(\theta) \triangleq D_c^N(\sigma_D^2) = \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \frac{U_0}{N} \boldsymbol{\rho}_N^T(u) (\boldsymbol{\Sigma}'_N + \theta \mathbf{I})^{-1} \boldsymbol{\rho}_N(u) \right) du \quad (3.155)$$

and by definition, sum rate $R_a^N(\theta)$ and distortion $D_a^N(\theta)$ are achievable for an arbitrary Gaussian random process.

3.8.5 Proof of Lemma 3.2

Using the matrix inversion lemma [38],

$$(\Sigma'_N + \theta \mathbf{I})^{-1} = \Sigma'^{-1}_N - \Sigma'^{-1}_N \left(\frac{1}{\theta} \mathbf{I} + \Sigma'^{-1}_N \right)^{-1} \Sigma'^{-1}_N \quad (3.156)$$

we have

$$\begin{aligned} D_a^N(\theta) &= \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \frac{U_0}{N} \boldsymbol{\rho}_N^T(u) \Sigma'^{-1}_N \boldsymbol{\rho}_N(u) \right) du \\ &\quad + \frac{1}{N} \int_0^{U_0} \boldsymbol{\rho}_N^T(u) \Sigma'^{-1}_N \left(\frac{1}{\theta} \mathbf{I} + \Sigma'^{-1}_N \right)^{-1} \Sigma'^{-1}_N \boldsymbol{\rho}_N(u) du \end{aligned} \quad (3.157)$$

$$= D_s^{(N)} + D^{(N)}(\theta) \quad (3.158)$$

where we have defined

$$D_s^{(N)} \triangleq \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \frac{U_0}{N} \boldsymbol{\rho}_N^T(u) \Sigma'^{-1}_N \boldsymbol{\rho}_N(u) \right) du \quad (3.159)$$

$$D^{(N)}(\theta) \triangleq \frac{1}{N} \int_0^{U_0} \boldsymbol{\rho}_N^T(u) \Sigma'^{-1}_N \left(\frac{1}{\theta} \mathbf{I} + \Sigma'^{-1}_N \right)^{-1} \Sigma'^{-1}_N \boldsymbol{\rho}_N(u) du \quad (3.160)$$

We continue evaluating $D^{(N)}(\theta)$,

$$\begin{aligned}
D^{(N)}(\theta) &= \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(\boldsymbol{\rho}_N^T \left(\frac{i-1}{N}U_0 \right) - \boldsymbol{\Delta}_i^T(u) \right) \boldsymbol{\Sigma}'_N^{-1} \left(\frac{1}{\theta} \mathbf{I} + \boldsymbol{\Sigma}'_N^{-1} \right)^{-1} \\
&\quad \boldsymbol{\Sigma}'_N^{-1} \left(\boldsymbol{\rho}_N \left(\frac{i-1}{N}U_0 \right) - \boldsymbol{\Delta}_i(u) \right) du \quad (3.161)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{U_0} \sum_{i=1}^N \left(\left(\frac{1}{\theta} \mathbf{I} + \boldsymbol{\Sigma}'_N^{-1} \right)^{-1} \right)_{(i,i)} \\
&\quad - 2 \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(\left(\frac{1}{\theta} \mathbf{I} + \boldsymbol{\Sigma}'_N^{-1} \right)^{-1} \boldsymbol{\Sigma}'_N^{-1} \boldsymbol{\Delta}_i(u) \right)_i du \\
&\quad + \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \boldsymbol{\Delta}_i^T(u) \boldsymbol{\Sigma}'_N^{-1} \left(\frac{1}{\theta} \mathbf{I} + \boldsymbol{\Sigma}'_N^{-1} \right)^{-1} \boldsymbol{\Sigma}'_N^{-1} \boldsymbol{\Delta}_i(u) du \quad (3.162)
\end{aligned}$$

where $\boldsymbol{\Delta}_i(u)$ is defined as

$$\boldsymbol{\Delta}_i(u) = \boldsymbol{\rho}_N \left(\frac{i-1}{N}U_0 \right) - \boldsymbol{\rho}_N(u) \quad (3.163)$$

for $\frac{i-1}{N}U_0 \leq u \leq \frac{i}{N}U_0$, and (3.162) follows based on the fact that

$$\boldsymbol{\rho}_N^T \left(\frac{i-1}{N}U_0 \right) \boldsymbol{\Sigma}'_N^{-1} = \frac{N}{U_0} \mathbf{e}_i \quad (3.164)$$

where \mathbf{e}_i is the row vector whose i -th entry is 1 and all other entries are 0.

The eigenvalues of $\boldsymbol{\Sigma}'_N^{-1} \left(\frac{1}{\theta} \mathbf{I} + \boldsymbol{\Sigma}'_N^{-1} \right)^{-1} \boldsymbol{\Sigma}'_N^{-1}$ are

$$\frac{\theta}{\mu_k^{(N)'} + \theta \mu_k^{(N)'}} \frac{1}{\mu_k^{(N)'}} \quad k = 0, 1, \dots, N-1 \quad (3.165)$$

which are smaller than the corresponding eigenvalues of $\Sigma_N'^{-1}$, i.e., $\frac{1}{\mu_k^{(N)'}}$. Thus, the third term in (3.162) is bounded by

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \Delta_i^T(u) \Sigma_N'^{-1} \left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \Delta_i(u) du &\leq \\ \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \Delta_i^T(u) \Sigma_N'^{-1} \Delta_i(u) du &\quad (3.166) \end{aligned}$$

To further upper bound the third term in (3.162), we write

$$D_s^{(N)} = \frac{1}{U_0} \int_0^{U_0} \left(K(u, u) - \frac{U_0}{N} \rho_N^T(u) \Sigma_N'^{-1} \rho_N(u) \right) du \quad (3.167)$$

$$\begin{aligned} &= \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(K(u, u) - \frac{U_0}{N} \left(\rho_N^T \left(\frac{i-1}{N}U_0 \right) - \Delta_i(u)^T \right) \right. \\ &\quad \left. \Sigma_N'^{-1} \left(\rho_N \left(\frac{i-1}{N}U_0 \right) - \Delta_i(u) \right) \right) du \quad (3.168) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(K(u, u) - K \left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0 \right) \right) du \\ &\quad + \frac{2}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} (\Delta_i(u))_i du - \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \Delta_i(u)^T \Sigma_N'^{-1} \Delta_i(u) du \quad (3.169) \end{aligned}$$

$$= A^{(N)} - \frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \Delta_i(u)^T \Sigma_N'^{-1} \Delta_i(u) du \quad (3.170)$$

where $A^{(N)}$ is defined in (3.46). Then, we have the third term in (3.162) upper bounded by $A^{(N)}$ because of (3.166), (3.170) and the fact that $D_s^{(N)}$ is non-negative, i.e.,

$$\frac{1}{N} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \Delta_i^T(u) \Sigma_N'^{-1} \left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \Delta_i(u) du \leq A^{(N)} \quad (3.171)$$

Furthermore, we can see from (3.170) that

$$D_s^{(N)} \leq A^{(N)} \quad (3.172)$$

Now, we evaluate the second term in (3.162). Since,

$$\left| \left(\left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \Delta_i(u) \right)_i \right| \leq \left\| \left(\left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \Delta_i(u) \right)_i \right\| \quad (3.173)$$

$$\leq \left\| \left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \right\|_2 \cdot \|\Delta_i(u)\| \quad (3.174)$$

$$= \max_{0 \leq k \leq N-1} \left(\mu_k^{(N)'} \right)^{-1} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} \|\Delta_i(u)\| \quad (3.175)$$

$$\leq \|\Delta_i(u)\| \quad (3.176)$$

Therefore, the second term in (3.162) is bounded by

$$\left| -2 \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N} U_0}^{\frac{i}{N} U_0} \left(\left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \Sigma_N'^{-1} \Delta_i(u) \right)_i du \right| \leq B^{(N)} \quad (3.177)$$

where $B^{(N)}$ is defined in (3.47). Finally, the first term in (3.162) can be written as

$$\frac{1}{U_0} \sum_{i=1}^N \left(\left(\frac{1}{\theta} \mathbf{I} + \Sigma_N'^{-1} \right)^{-1} \right)_{(i,i)} = \frac{1}{U_0} \sum_{k=0}^{N-1} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} \triangleq D_b^N(\theta) \quad (3.178)$$

where the last step is by the definition of $D_b^N(\theta)$ in (3.48). Hence, for an arbitrary Gaussian random process, by (3.158), (3.162), (3.171), (3.172), (3.177) and (3.178),

we have shown that

$$D_a^N(\theta) \leq 2A^{(N)} + B^{(N)} + D_b^N(\theta) \quad (3.179)$$

3.8.6 Proof of Lemma 3.3

We first consider $A^{(N)}$.

$$\begin{aligned} A^{(N)} &= \frac{1}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(K(u, u) - K\left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0\right) \right) du \\ &\quad + \frac{2}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left(\rho_N\left(\frac{i-1}{N}U_0\right) - \rho_N(u) \right) du \end{aligned} \quad (3.180)$$

$$\leq B \left(2^{\frac{\alpha}{2}} + 2 \right) U_0^\alpha \frac{1}{N^\alpha} \quad (3.181)$$

$$= \Theta \left(N^{-\alpha} \right) \quad (3.182)$$

where (3.181) follows from condition 2 in Section 3.2. Using similar ideas, we have

$$B^{(N)} = \frac{2}{U_0} \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left\| \rho_N\left(\frac{i-1}{N}U_0\right) - \rho_N(u) \right\| du \quad (3.183)$$

$$\leq 2BU_0^\alpha \frac{N^{\frac{1}{2}}}{N^\alpha} \quad (3.184)$$

$$= \Theta \left(N^{\frac{1}{2}-\alpha} \right) \quad (3.185)$$

3.8.7 Some properties of $\mu_k^{(N)'}$

Lemma 3.7 *For all Gaussian random processes in \mathcal{A} , let $K_1(N)$ be a sequence of numbers that satisfies*

$$\lim_{N \rightarrow \infty} \frac{1}{K_1(N)} = 0 \quad (3.186)$$

$$\lim_{N \rightarrow \infty} \frac{K_1(N)^{2\tau}}{N^\gamma} = 0 \quad (3.187)$$

$$\lim_{N \rightarrow \infty} \frac{K_1(N)^{x+1+\tau}}{N^\beta} = 0 \quad (3.188)$$

Then, for each k such that $k \leq K_1(N)$, there exists an eigenvalue $\mu^{(N)'}$, different for each k , of Σ'_N such that

$$\left| \mu^{(N)' } - \lambda_k \right| \leq d_1 \frac{(k + B_7)^\tau}{N^\beta} \quad (3.189)$$

for some $d_1 > 0$ and some positive integer B_7 , both independent of k and N , when N is large enough.

Lemma 3.7 shows that the convergence of $\mu_k^{(N)'}$ to λ_k is not uniform, and the approximation of $\mu_k^{(N)'}$ using λ_k is accurate only when $k \ll N^{\frac{\gamma}{2\tau}}$ and $\lambda_k \gg d_1 \frac{(k+B_7)^\tau}{N^\beta}$.

When the conditions of Lemma 3.7 are satisfied, we label the $\mu^{(N)'}$ that satisfies (3.189) to be $\mu_k^{(N)'}$ for $k \leq K_1(N)$. The remaining $N - K_1(N)$ eigenvalues of $\mu^{(N)'}$ will be labelled according to the order from large to small.

Lemma 3.8 *For all Gaussian random processes in \mathcal{A} , let two sequences ϑ_L^N and ϑ_U^N*

satisfy (3.53). Then, for any constant $0 < \kappa < 1$, we have

$$\sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{N-1} \mu_k^{(N)'} \leq \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa^2} \theta^{1-\frac{1}{x}} \quad (3.190)$$

when $\theta \in [\vartheta_L^N, \vartheta_U^N]$ and N is large enough, and ϑ_L^N and ϑ_U^N satisfies (3.53).

Lemma 3.8 shows that the sum of the eigenvalues that do not converge to λ_k for $k = 0, 1, \dots, \left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor$ is the same order as $\sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{\infty} \lambda_k''$ as calculated in (3.77).

Proof of Lemma 3.7

By definition, λ_k for any k satisfies

$$\lambda_k \phi_k \left(\frac{l-1}{N} U_0 \right) = \int_0^{U_0} K \left(\frac{l-1}{N} U_0, v \right) \phi_k(v) dv, \quad \forall l = 1, 2, \dots, N \quad (3.191)$$

We rewrite the right hand side of (3.191) by

$$\frac{U_0}{N} \sum_{i=1}^N K \left(\frac{l-1}{N} U_0, \frac{i-1}{N} U_0 \right) \phi_k \left(\frac{i-1}{N} U_0 \right) + \epsilon_N^k \left(\frac{l-1}{N} U_0 \right) \quad \forall l = 1, 2, \dots, N \quad (3.192)$$

where $\epsilon_N^k \left(\frac{l-1}{N} U_0 \right)$ is defined as

$$\sum_{i=1}^N \int_{\frac{i-1}{N} U_0}^{\frac{i}{N} U_0} \left(K \left(\frac{l-1}{N} U_0, v \right) \phi_k(v) - K \left(\frac{l-1}{N} U_0, \frac{i-1}{N} U_0 \right) \phi_k \left(\frac{i-1}{N} U_0 \right) \right) dv \quad (3.193)$$

Using (3.191) and (3.192), we have for any $l = 1, 2, \dots, N$,

$$\lambda_k \phi_k \left(\frac{l-1}{N} U_0 \right) = \frac{U_0}{N} \sum_{i=1}^N K \left(\frac{l-1}{N} U_0, \frac{i-1}{N} U_0 \right) \phi_k \left(\frac{i-1}{N} U_0 \right) + \epsilon_N^k \left(\frac{l-1}{N} U_0 \right) \quad (3.194)$$

Let us define vector $\mathbf{a}_k^{(N)}$ of length of N by defining its l -th element to be $\sqrt{\frac{U_0}{N}} \epsilon_N^k \left(\frac{l-1}{N} U_0 \right)$ and vector $\mathbf{b}_k^{(N)}$ of length of N by defining its l -th element to be $\sqrt{\frac{U_0}{N}} \phi_k \left(\frac{l-1}{N} U_0 \right)$, we have in matrix form

$$\lambda_k \mathbf{b}_k^{(N)} = \mathbf{\Sigma}'_N \mathbf{b}_k^{(N)} + \mathbf{a}_k^{(N)} \quad (3.195)$$

The links between the eigenvalues of $\mathbf{\Sigma}'_N$ and the eigenvalues of $K(u, v)$, i.e., the λ_k s, will be determined using (3.195). To do this, we first bound three quantities, $\left\| \mathbf{a}_k^{(N)} \right\|$, $\left\| \mathbf{b}_k^{(N)} \right\|$, $\left| \mathbf{b}_m^{(N)T} \mathbf{b}_l^{(N)} \right|$ for $k, m, l \leq K_1(N)$ and $m \neq l$.

Now, we analyze the norm of $\mathbf{a}_k^{(N)}$. From the definition of $\epsilon_N^k \left(\frac{l-1}{N} U_0 \right)$ in (3.193), we have

$$\begin{aligned} & \left| \epsilon_N^k \left(\frac{l-1}{N} U_0 \right) \right| \\ & \leq \sum_{i=1}^N \int_{\frac{i-1}{N} U_0}^{\frac{i}{N} U_0} \left| K \left(\frac{l-1}{N} U_0, v \right) \phi_k(v) - K \left(\frac{l-1}{N} U_0, \frac{i-1}{N} U_0 \right) \phi_k \left(\frac{i-1}{N} U_0 \right) \right| dv \end{aligned} \quad (3.196)$$

$$\leq B_2 U_0^{1+\beta} \frac{(k + B_1)^\tau}{N^\beta} \quad (3.197)$$

where (3.197) follows because the random process satisfies condition 3 in Section 3.2.

Thus, the norm of vector $\mathbf{a}_k^{(N)}$ is bounded by

$$\left\| \mathbf{a}_k^{(N)} \right\| \leq B_2 U_0^{3/2+\beta} \frac{(k + B_1)^\tau}{N^\beta} \quad (3.198)$$

Now, we will calculate the norm of vector $\mathbf{b}_k^{(N)}$. We write

$$1 = \int_0^{U_0} \phi_k^2(u) du = \sum_{i=1}^N \frac{U_0}{N} \phi_k^2\left(\frac{i-1}{N} U_0\right) + \delta_N^k \quad (3.199)$$

where δ_N^k is defined as

$$\delta_N^k = \sum_{i=1}^N \int_{\frac{i-1}{N} U_0}^{\frac{i}{N} U_0} \left(\phi_k^2(u) - \phi_k^2\left(\frac{i-1}{N} U_0\right) \right) du \quad (3.200)$$

We first upper bound $|\phi_k(u)|$, for $u \in [0, U_0]$. Let $F_k(s)$ be defined as

$$\int_0^s \phi_k^2(u) du \quad (3.201)$$

Then, by the mean value theorem on interval $[0, U_0]$, we have that there exists a $U' \in [0, U_0]$, such that

$$1 = F_k(U_0) - F_k(0) = \phi_k^2(U') \quad (3.202)$$

Hence, using condition 3 in Section 3.2, we have

$$|\phi_k(u) - \phi_k(U')| \leq B_3(k + B_4)^\tau U_0^\gamma, \quad u \in [0, U_0] \quad (3.203)$$

Thus,

$$|\phi_k(u)| \leq B_3(k + B_4)^\tau U_0^\gamma + 1, \quad u \in [0, U_0] \quad (3.204)$$

which means

$$\max_{u \in [0, U_0]} |\phi_k(u)| \leq B_3(k + B_4)^\tau U_0^\gamma + 1 \quad (3.205)$$

Using (3.16), we have for any $v_1, v_2 \in [0, U_0]$,

$$|\phi_k^2(v_1) - \phi_k^2(v_2)| = |\phi_k(v_1) + \phi_k(v_2)| |\phi_k(v_1) - \phi_k(v_2)| \quad (3.206)$$

$$\leq 2 \max_{v \in [0, U_0]} |\phi_k(v)| B_3(k + B_4)^\tau |v_1 - v_2|^\gamma \quad (3.207)$$

$$\leq 2 (B_3(k + B_4)^\tau U_0^\gamma + 1) B_3(k + B_4)^\tau |v_1 - v_2|^\gamma \quad (3.208)$$

where (3.208) follows from (3.205). The approximation error, δ_N^k satisfies

$$|\delta_N^k| \leq \sum_{i=1}^N \int_{\frac{i-1}{N}U_0}^{\frac{i}{N}U_0} \left| \phi_k^2(u) - \phi_k^2\left(\frac{i-1}{N}U_0\right) \right| du \quad (3.209)$$

$$\leq U_0^{1+\gamma} \frac{2 (B_3(k + B_4)^\tau U_0^\gamma + 1) B_3(k + B_4)^\tau}{N^\gamma} \quad (3.210)$$

$$\leq U_0^{1+\gamma} \frac{2 (B_3(K_1(N) + B_4)^\tau U_0^\gamma + 1) B_3(K_1(N) + B_4)^\tau}{N^\gamma} \quad (3.211)$$

$$\leq B_5 \quad (3.212)$$

where (3.210) follows from (3.208), and (3.212) is due to the fact that $K_1(N)$ satisfies

(3.186) and (3.187), for a fixed constant B_5 that satisfies $0 < B_5 < 1$, Then, there

exists an integer $N_0 > 0$, such that for $N \geq N_0$,

$$U_0^{1+\gamma} \frac{2(B_3(K_1(N) + B_4)^\tau U_0^\gamma + 1) B_3(K_1(N) + B_4)^\tau}{N^\gamma} \leq B_5 \quad (3.213)$$

Finally, by the definition of $\mathbf{b}_k^{(N)}$, we have

$$\|\mathbf{b}_k^{(N)}\| = \sqrt{\sum_{i=1}^N \frac{U_0}{N} \phi_k^2\left(\frac{i-1}{N} U_0\right)} = \sqrt{1 - \delta_N^k} \quad (3.214)$$

where (3.214) follows from (3.199). From (3.212), we have

$$\sqrt{1 - B_5} \leq \|\mathbf{b}_k^{(N)}\| \leq \sqrt{1 + B_5} \quad (3.215)$$

Next, we show that based on the orthogonality of the eigenfunctions of $\phi_k(t)$, the sampled version $\mathbf{b}_k^{(N)}$ s are almost orthogonal. Using (3.16), we have

$$|\phi_m(v_1)\phi_l(v_1) - \phi_m(v_2)\phi_l(v_2)| \quad (3.216)$$

$$= |\phi_m(v_1)\phi_l(v_1) - \phi_m(v_1)\phi_l(v_2) + \phi_m(v_1)\phi_l(v_2) - \phi_m(v_2)\phi_l(v_2)| \quad (3.217)$$

$$\leq |\phi_m(v_1)\phi_l(v_1) - \phi_m(v_1)\phi_l(v_2)| + |\phi_m(v_1)\phi_l(v_2) - \phi_m(v_2)\phi_l(v_2)| \quad (3.218)$$

$$\leq \max_{v_1 \in [0, U_0]} |\phi_m(v_1)| |\phi_l(v_1) - \phi_l(v_2)| + \max_{v_2 \in [0, U_0]} |\phi_l(v_2)| |\phi_m(v_1) - \phi_m(v_2)| \quad (3.219)$$

$$\begin{aligned} &\leq (B_3(m + B_4)^\tau U_0^\gamma + 1) B_3(l + B_4)^\tau |v_1 - v_2|^\gamma \\ &\quad + (B_3(l + B_4)^\tau U_0^\gamma + 1) B_3(m + B_4)^\tau |v_1 - v_2|^\gamma \end{aligned} \quad (3.220)$$

$$= (2B_3^2(m + B_4)^\tau (l + B_4)^\tau U_0^\gamma + B_3(l + B_4)^\tau + B_3(m + B_4)^\tau) |v_1 - v_2|^\gamma \quad (3.221)$$

where (3.220) follows from (3.205). Let m and l be two different integers, that belong to $\{1, 2, \dots, N\}$. Then, we have

$$0 = \int_0^{U_0} \phi_m(u) \phi_l(u) du = \sum_{i=1}^N \frac{U_0}{N} \phi_m \left(\frac{i-1}{N} U_0 \right) \phi_l \left(\frac{i-1}{N} U_0 \right) + \varepsilon_N^{m,l} \quad (3.222)$$

Then, we have

$$\left| \varepsilon_N^{m,l} \right| \leq \sum_{i=1}^N \int_{\frac{i-1}{N} U_0}^{\frac{i}{N} U_0} \left| \phi_m(v) \phi_l(v) - \phi_m \left(\frac{i-1}{N} U_0 \right) \phi_l \left(\frac{i-1}{N} U_0 \right) \right| dv \quad (3.223)$$

$$\leq U_0^{1+\gamma} \frac{2B_3^2(m+B_4)^\tau(l+B_4)^\tau U_0^\gamma + B_3(l+B_4)^\tau + B_3(m+B_4)^\tau}{N^\gamma} \quad (3.224)$$

$$\leq 2U_0^{1+\gamma} \frac{B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+\gamma} + B_3(K_1(N) + B_4)^\tau}{N^\gamma} \quad (3.225)$$

$$\leq \frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \quad (3.226)$$

where (3.224) follows from (3.221), (3.226) follows when N is large enough due to the fact that $K_1(N)$ satisfies (3.186), i.e., there exists an integer N_2 such that when $N > N_2$, (3.226) is true. The right hand side of (3.226) converges to zero as N goes to infinity due to the fact that $K_1(N)$ satisfies (3.187). We have

$$\left| \mathbf{b}_m^{(N)^T} \mathbf{b}_l^{(N)} \right| = \left| \sum_{i=1}^N \frac{U_0}{N} \phi_m \left(\frac{i-1}{N} U_0 \right) \phi_l \left(\frac{i-1}{N} U_0 \right) \right| = \left| \varepsilon_N^{m,l} \right| \quad (3.227)$$

$$\leq \frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \quad (3.228)$$

which means that vectors $\mathbf{b}_m^{(N)}$ and $\mathbf{b}_l^{(N)}$ become more orthogonal as N gets larger.

Now, we are ready to establish the link between the eigenvalues of Σ'_N and λ_k .

From (3.195), we have

$$\left\| \mathbf{b}_k^{(N)} \right\| \leq \left\| (\boldsymbol{\Sigma}'_N - \lambda_k \mathbf{I})^{-1} \right\|_2 \left\| \mathbf{a}_k^{(N)} \right\| \quad (3.229)$$

$$= \left(\min_{0 \leq m \leq N-1} \left| \mu_m^{(N)'} - \lambda_k \right| \right)^{-1} \left\| \mathbf{a}_k^{(N)} \right\| \quad (3.230)$$

Thus, we have

$$\min_{0 \leq m \leq N-1} \left| \mu_m^{(N)'} - \lambda_k \right| \leq \frac{\left\| \mathbf{a}_k^{(N)} \right\|}{\left\| \mathbf{b}_k^{(N)} \right\|} \quad (3.231)$$

$$\leq \frac{B_2 U_0^{3/2+\beta} (k + B_1)^\tau}{\sqrt{1 - B_5} N^\beta} \quad (3.232)$$

$$= d_0 \frac{(k + B_1)^\tau}{N^\beta} \quad (3.233)$$

where (3.233) follows by defining d_0 as

$$d_0 = \frac{B_2 U_0^{3/2+\beta}}{\sqrt{1 - B_5}} \quad (3.234)$$

Hence, for $k = 0, 1, 2, \dots, K_1(N)$, there exists an eigenvalue $\mu^{(N)'}$ of $\boldsymbol{\Sigma}'_N$ such that

$$\left| \mu^{(N)'} - \lambda_k \right| \leq d_0 \frac{(k + B_1)^\tau}{N^\beta} \quad (3.235)$$

when N is large enough, more specifically, when $N \geq \max(N_0, N_2)$.

For $k = 0, 1, 2, \dots, K_1(N)$, if we label the $\mu^{(N)'}$ that satisfies (3.235) to be $\mu_k^{(N)'}$,

then when λ_k for different k s are sufficiently close, more specifically,

$$|\lambda_m - \lambda_l| \leq 2d_0 \frac{(K_1(N) + B_1)^\tau}{N^\beta}, \quad m, l \leq K_1(N), m \neq l \quad (3.236)$$

$\mu_m^{(N)'}$ and $\mu_l^{(N)'}$, though labelled differently, might be the same eigenvalue of Σ'_N , which is undesirable. If we relax the minimum distance of $d_0 \frac{(k+B_1)^\tau}{N^\beta}$, we will be able to eliminate this problem. Thus, we will next show that for $k = 0, 1, 2, \dots, K_1(N)$, there exists an eigenvalue $\mu^{(N)'}$ of Σ'_N , different for each k , such that

$$\left| \mu^{(N)' } - \lambda_k \right| \leq (2\bar{\chi} + 1) \sqrt{d_2} d_0 \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta} \quad (3.237)$$

when N is large enough, where we define $\bar{\chi} \triangleq \max(K_0 + 1 + c_u + c_l, 2c_u + 2c_l + 1)$ and constant d_2 as the largest root of the following second-order equation

$$(1 - B_5)d_2^2 - 2((1 - B_5) + 3\bar{\chi}(1 + B_5))d_2 + (1 - B_5) + 2\bar{\chi}(1 + B_5) = 0 \quad (3.238)$$

It can be checked that both roots of the above equation are real, and the largest root is a positive constant, strictly larger than $\frac{2\bar{\chi}(1+B_5)}{1-B_5} + 1$, that is a function of $\bar{\chi}$ and B_5 .

First, let us define a cluster of λ s. We say that χ λ s are a cluster, where with no loss of generality, we may label these λ s $\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+\chi-1}$, if

$$\lambda_{k+l} - \lambda_{k+l+1} \leq 2\sqrt{d_2} d_0 \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta}, \quad 0 \leq k \leq K_1(N), \quad l = 0, 1, \dots, \chi - 1 \quad (3.239)$$

Note here that whether the λ s are in a cluster depends on N . Next, we prove that the number of λ s within a cluster is upper bounded by $\bar{\chi}$ when N is large enough.

For $k > K_0$, we have

$$\frac{d}{(k + c_l)^x} \leq \lambda_k \leq \frac{d}{(k - c_u)^x} \quad (3.240)$$

$$\frac{d}{(k + 2c_l + c_u + 1)^x} \leq \lambda_{k+c_u+c_l+1} \leq \frac{d}{(k + c_l + 1)^x} \quad (3.241)$$

Hence, for every $k \geq K_0$, the distance between λ_k and $\lambda_{k+c_u+c_l+1}$ satisfies

$$\lambda_k - \lambda_{k+c_u+c_l+1} \geq \frac{d}{(k + c_l)^x} - \frac{d}{(k + c_l + 1)^x} \quad (3.242)$$

which is a non-increasing function of k . Thus, for all $K_0 < k \leq K_1(N)$, the distance between λ_k and $\lambda_{k+c_u+c_l+1}$ satisfies

$$\lambda_k - \lambda_{k+c_u+c_l+1} \geq \frac{d}{(K_1(N) + c_l)^x} - \frac{d}{(K_1(N) + c_l + 1)^x} \quad (3.243)$$

$$= \frac{d}{(K_1(N) + c_l)^x} \left(1 - \left(1 - \frac{1}{K_1(N) + c_l + 1} \right)^x \right) \quad (3.244)$$

$$\geq \frac{d}{(K_1(N) + c_l)^x} \left(x \frac{1}{K_1(N) + c_l + 1} - \frac{x(x-1)}{2} \frac{1}{(K_1(N) + c_l + 1)^2} \right) \quad (3.245)$$

$$= \frac{xd}{(K_1(N) + c_l)^{x+1}} - \frac{x(x-1)d}{2(K_1(N) + c_l)^{x+2}} \quad (3.246)$$

$$\geq \frac{xd}{2(K_1(N) + c_l)^{x+1}} \quad (3.247)$$

$$> 2\sqrt{d_2}d_0 \frac{(K_1(N) + \bar{\chi} + B_1)^\tau}{N^\beta} \quad (3.248)$$

where (3.247) is true when N is large enough due to the fact that $K_1(N)$ satisfies (3.186), i.e., there exists an integer N_3 , such that when $N > N_3$, (3.247) is true, and (3.248) is true when N is large enough, due to the fact that $K_1(N)$ satisfies (3.188), i.e., there exists an integer N_4 , such that when $N > N_4$, (3.248) is true.

Hence, for all $K_0 < k \leq K_1(N)$, when N is large enough, more specifically, when $N > \max(N_3, N_4)$, due to the sufficient distance between λ_k and $\lambda_{k+c_u+c_l+1}$, shown in (3.248), they cannot be in the same cluster. Hence, we may conclude that for large enough N , the size of a cluster is at most $\bar{\chi}$, which is a finite number. Let the eigenvalues and the corresponding eigenvectors of Σ'_N be $\mu_i^{(N)'}$ and $\mathbf{u}_i^{(N)}$, $i = 0, 1, \dots, N-1$, with arbitrary labelling of the eigenvalues and eigenvectors. Following from (3.195), we have

$$\sum_{i=0}^{N-1} \left(\lambda_k - \mu_i^{(N)'} \right) \mathbf{u}_i^{(N)} \mathbf{u}_i^{(N)T} \mathbf{b}_k^{(N)} = \mathbf{a}_k^{(N)} \quad (3.249)$$

We take the norm squared on both sides, and due to the orthogonality of eigenvectors $\mathbf{u}_i^{(N)}$, we have

$$\sum_{i=0}^{N-1} \left(\lambda_k - \mu_i^{(N)'} \right)^2 \left(\mathbf{u}_i^{(N)T} \mathbf{b}_k^{(N)} \right)^2 = \left\| \mathbf{a}_k^{(N)} \right\|^2, \quad k = 0, 1, 2, \dots \quad (3.250)$$

and we also have

$$\sum_{i=0}^{N-1} \left(\mathbf{u}_i^{(N)T} \mathbf{b}_k^{(N)} \right)^2 = \left\| \mathbf{b}_k^{(N)} \right\|^2, \quad k = 0, 1, 2, \dots \quad (3.251)$$

Let $\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+\bar{\chi}-1}$ be a cluster, and from previous arguments, we know

$\chi \leq \bar{\chi}$. Furthermore, we are only interested in the first $K_1(N) + 1$ eigenvalues, and therefore $k + \chi - 1 \leq K_1(N)$. We will prove by contradiction. Suppose that only ς number of $\mu_i^{(N)'}s$ are within distance

$$\sqrt{d_2}d_0 \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta} \quad (3.252)$$

from any of the $\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+\chi-1}$, with $1 \leq \varsigma < \chi$, we will show that there is a contradiction, and therefore, we can conclude that our assumption that $\varsigma < \chi$ number of $\mu_i^{(N)'}s$ are within distance (3.252) from any of the $\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+\chi}$ is not correct.

Let us label the $\mu^{(N)'}$ that are within distance (3.252) from any of the $\lambda_k, \lambda_{k+1}, \dots, \lambda_{k+\chi}$ $\mu_0^{(N)'}, \mu_1^{(N)'}, \dots, \mu_{\varsigma-1}^{(N)'}$. Before we dive into the details, let us first explain the basic idea of the proof. $\mathbf{u}_0^{(N)}, \mathbf{u}_1^{(N)}, \dots, \mathbf{u}_{\varsigma-1}^{(N)}$ form the basis of a ς dimensional subspace. On the other hand, $\mathbf{b}_k^{(N)}, \mathbf{b}_{k+1}^{(N)}, \dots, \mathbf{b}_{k+\chi-1}^{(N)}$ are almost orthogonal, according to (3.228), and roughly form the basis of a χ dimensional subspace. Since all other $\mu_i^{(N)'}$ s, for $i = \varsigma, \varsigma + 1, \dots, N - 1$, are farther than distance (3.252) away, by Wedin's theorem in perturbation theory [77], the angle between $\mathbf{b}_{k+l}^{(N)}$ and the subspace is small, for all $l = 0, 1, \dots, \chi - 1$. But this is not possible, since ς is strictly smaller than χ . Now, we proceed with the rigorous proof.

Note that $\varsigma \geq 1$ because we have already proved (3.235). Based on (3.233), the distance in (3.252) satisfies

$$\sqrt{d_2}d_0 \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta} \geq \frac{\sqrt{d_2} \|\mathbf{a}_{k+l}^{(N)}\|}{\|\mathbf{b}_{k+l}^{(N)}\|}, \quad l = 0, 1, \dots, \chi - 1 \quad (3.253)$$

Then, based on (3.250), we have

$$\left(\frac{\sqrt{d_2} \|\mathbf{a}_{k+l}^{(N)}\|}{\|\mathbf{b}_{k+l}^{(N)}\|} \right)^2 \sum_{i=\varsigma}^{N-1} \left(\mathbf{u}_i^{(N)T} \mathbf{b}_{k+l}^{(N)} \right)^2 \leq \left\| \mathbf{a}_{k+l}^{(N)} \right\|^2, \quad l = 0, 1, \dots, \chi - 1 \quad (3.254)$$

Hence, we have

$$\sum_{i=\varsigma}^{N-1} \left(\mathbf{u}_i^{(N)T} \mathbf{b}_{k+l}^{(N)} \right)^2 \leq \frac{\|\mathbf{b}_{k+l}^{(N)}\|^2}{d_2}, \quad l = 0, 1, \dots, \chi - 1 \quad (3.255)$$

Together with (3.251), we have

$$\sum_{i=0}^{\varsigma-1} \left(\mathbf{u}_i^{(N)T} \mathbf{b}_{k+l}^{(N)} \right)^2 \geq \frac{(d_2 - 1) \|\mathbf{b}_{k+l}^{(N)}\|^2}{d_2}, \quad l = 0, 1, \dots, \chi - 1 \quad (3.256)$$

Since the $\mathbf{u}_i^{(N)}$ form a complete set of orthonormal basis in \mathbb{R}^N , we can write $\mathbf{b}_{k+l}^{(N)}$ as

$$\mathbf{b}_{k+l}^{(N)} = \sum_{i=0}^{\varsigma-1} \alpha_{k+l,i} \mathbf{u}_i^{(N)} + \mathbf{v}_{k+l}^{(N)}, \quad l = 0, 1, \dots, \chi - 1 \quad (3.257)$$

where $\mathbf{v}_{k+l}^{(N)}$ is orthogonal to $\mathbf{u}_i^{(N)}$, for $i = 1, 2, \dots, \varsigma$. If we take the expression of $\mathbf{b}_{k+l}^{(N)}$ in (3.257) and plug it in (3.256), we get

$$\sum_{i=0}^{\varsigma-1} (\alpha_{k+l,i})^2 \geq \frac{(d_2 - 1) \|\mathbf{b}_{k+l}^{(N)}\|^2}{d_2}, \quad l = 0, 1, 2, \dots, \chi - 1 \quad (3.258)$$

From (3.257), we get

$$\|\mathbf{b}_{k+l}^{(N)}\|^2 = \sum_{i=0}^{\varsigma-1} (\alpha_{k+l,i})^2 + \|\mathbf{v}_{k+l}^{(N)}\|^2, \quad l = 0, 1, \dots, \varsigma - 1 \quad (3.259)$$

Hence, we conclude that

$$\|\mathbf{v}_{k+l}^{(N)}\|^2 \leq \frac{\|\mathbf{b}_{k+l}^{(N)}\|^2}{d_2}, \quad l = 0, 1, \dots, \varsigma - 1 \quad (3.260)$$

Furthermore, from (3.257), we have

$$\mathbf{b}_{k+m}^{(N)T} \mathbf{b}_{k+l}^{(N)} = \sum_{i=0}^{\varsigma-1} \alpha_{k+m,i} \alpha_{k+l,i} + \mathbf{v}_{k+m}^{(N)T} \mathbf{v}_{k+l}^{(N)}, \quad m, l = 0, 1, \dots, \varsigma - 1, m \neq l \quad (3.261)$$

Hence, we have

$$\sum_{i=0}^{\varsigma-1} \alpha_{k+m,i} \alpha_{k+l,i} = \mathbf{b}_{k+m}^{(N)T} \mathbf{b}_{k+l}^{(N)} - \mathbf{v}_{k+m}^{(N)T} \mathbf{v}_{k+l}^{(N)}, \quad m, l = 0, 1, \dots, \varsigma - 1, m \neq l \quad (3.262)$$

and for $m, l = 0, 1, \dots, \varsigma - 1, m \neq l$, we have

$$\left| \sum_{i=0}^{\varsigma-1} \alpha_{k+m,i} \alpha_{k+l,i} \right| \leq \left| \mathbf{b}_{k+m}^{(N)T} \mathbf{b}_{k+l}^{(N)} \right| + \left| \mathbf{v}_{k+m}^{(N)T} \mathbf{v}_{k+l}^{(N)} \right| \quad (3.263)$$

$$\leq \frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} + \left\| \mathbf{v}_{k+m}^{(N)} \right\| \left\| \mathbf{v}_{k+l}^{(N)} \right\| \quad (3.264)$$

$$\leq \frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} + \frac{\left\| \mathbf{b}_{k+m}^{(N)} \right\| \left\| \mathbf{b}_{k+l}^{(N)} \right\|}{d_2} \quad (3.265)$$

$$\leq \frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} + \frac{1 + B_5}{d_2} \quad (3.266)$$

$$\leq \frac{2(1 + B_5)}{d_2} \quad (3.267)$$

where (3.264) follows from (3.228) when $N > N_2$, (3.265) follows from (3.260), (3.266)

follows from (3.215) when $N > N_0$, and (3.267) follows when N is large enough, due

to the fact that $K_1(N)$ satisfies (3.187), i.e., there exists an integer N_6 , when $N > N_6$,

(3.267) is true. Let us define matrix \mathbf{A} to be

$$\mathbf{A} = \begin{bmatrix} \alpha_{k,1} & \alpha_{k,2} & \cdots & \alpha_{k,\varsigma} \\ \alpha_{k+1,1} & \alpha_{k+1,2} & \cdots & \alpha_{k+1,\varsigma} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k+\varsigma-1,1} & \alpha_{k+\varsigma-1,2} & \cdots & \alpha_{k+\varsigma-1,\varsigma} \end{bmatrix} \quad (3.268)$$

and define vectors \mathbf{b} , \mathbf{v} , \mathbf{u} to be

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_k^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \mathbf{b}_{k+1}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \vdots \\ \mathbf{b}_{k+\varsigma-1}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_k^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \mathbf{v}_{k+1}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \vdots \\ \mathbf{v}_{k+\varsigma-1}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_0^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \mathbf{u}_1^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \\ \vdots \\ \mathbf{u}_{\varsigma-1}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \end{bmatrix} \quad (3.269)$$

Then, by (3.257), we have

$$\mathbf{b} = \mathbf{A}\mathbf{u} + \mathbf{v} \quad (3.270)$$

In other words,

$$\mathbf{u} = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{v}) \quad (3.271)$$

thus, we have

$$\|\mathbf{u}\|^2 \leq \|\mathbf{A}^{-1}\|_2^2 (\|\mathbf{b}\| + \|\mathbf{v}\|)^2 \quad (3.272)$$

We start by evaluating $\|\mathbf{A}^{-1}\|_2^2$, which is equal to the inverse of the smallest eigenvalue of $\mathbf{A}^T \mathbf{A}$. From the definition of matrix \mathbf{A} in (3.268), we have

$$\mathbf{A}^T \mathbf{A} = \mathbf{D} + \mathbf{E} \quad (3.273)$$

where \mathbf{D} is an $\varsigma \times \varsigma$ diagonal matrix with the l -th diagonal element being $\sum_{i=0}^{\varsigma-1} (\alpha_{k+l-1,i})^2$, and \mathbf{E} is an $\varsigma \times \varsigma$ matrix with zero diagonals and (m, l) -th element being $\sum_{i=0}^{\varsigma-1} \alpha_{k+m-1,i} \alpha_{k+l-1,i}$, when $m \neq l$. The absolute difference between the smallest eigenvalue of $\mathbf{A}^T \mathbf{A}$ and \mathbf{D} is upper bounded by $\|\mathbf{E}\|_2$ [77]. The smallest eigenvalue of \mathbf{D} is

$$\min_{l \in \{0,1,\dots,\varsigma-1\}} \sum_{i=0}^{\varsigma-1} (\alpha_{k+l-1,i})^2 \geq \min_l \frac{(d_2 - 1) \|\mathbf{b}_{k+l}^{(N)}\|^2}{d_2} \quad (3.274)$$

$$\geq \frac{(d_2 - 1)(1 - B_5)}{d_2} \quad (3.275)$$

where (3.274) follows from (3.258), and (3.275) follows from (3.215) when $N > N_0$ since $k + \varsigma - 1 \leq K_1(N)$. We can upper bound the spectral norm of matrix \mathbf{E} , i.e., $\|\mathbf{E}\|_2$, by the Frobenius norm of \mathbf{E} , i.e.,

$$\|\mathbf{E}\|_2^2 \leq \sum_{m \neq l} \left(\sum_{i=0}^{\varsigma-1} \alpha_{k+m-1,i} \alpha_{k+l-1,i} \right)^2 \quad (3.276)$$

$$\leq \varsigma^2 \left(\frac{2(1 + B_5)}{d_2} \right)^2 \quad (3.277)$$

$$< \chi^2 \frac{4(1 + B_5)^2}{d_2^2} \quad (3.278)$$

$$\leq \bar{\chi}^2 \frac{4(1 + B_5)^2}{d_2^2} \quad (3.279)$$

where (3.277) follows from (3.267). Hence, we may conclude that

$$\|\mathbf{A}^{-1}\|_2^2 < \left(\frac{(d_2 - 1)(1 - B_5)}{d_2} - \frac{2\bar{\chi}(1 + B_5)}{d_2} \right)^{-1} \quad (3.280)$$

where the right hand side is a positive number, by the definition of d_2 . Next, we evaluate $\|\mathbf{v}\|^2$.

$$\|\mathbf{v}\|^2 = \sum_{i=0}^{\varsigma-1} \left(\mathbf{v}_{k+i}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \right)^2 \quad (3.281)$$

$$\leq \sum_{i=0}^{\varsigma-1} \left\| \mathbf{v}_{k+i}^{(N)T} \right\|^2 \left\| \mathbf{b}_{k+\varsigma}^{(N)} \right\|^2 \quad (3.282)$$

$$\leq \frac{\left\| \mathbf{b}_{k+\varsigma}^{(N)} \right\|^2}{d_2} \sum_{i=0}^{\varsigma-1} \left\| \mathbf{b}_{k+i}^{(N)} \right\|^2 \quad (3.283)$$

$$\leq \frac{\varsigma(1 + B_5)}{d_2} \left\| \mathbf{b}_{k+\varsigma}^{(N)} \right\|^2 \quad (3.284)$$

$$< \frac{\chi(1 + B_5)}{d_2} \left\| \mathbf{b}_{k+\varsigma}^{(N)} \right\|^2 \quad (3.285)$$

$$\leq \frac{\bar{\chi}(1 + B_5)}{d_2} \left\| \mathbf{b}_{k+\varsigma}^{(N)} \right\|^2 \quad (3.286)$$

where (3.283) follows from (3.260), and (3.284) follows from (3.215) when $N > N_0$

since $k + \varsigma - 1 \leq K_1(N)$. Finally, we evaluate $\|\mathbf{b}\|^2$.

$$\|\mathbf{b}\|^2 = \sum_{i=0}^{\varsigma-1} \left(\mathbf{b}_{k+i}^{(N)T} \mathbf{b}_{k+\varsigma}^{(N)} \right)^2 \leq \varsigma \left(\frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \right)^2 \quad (3.287)$$

$$< \chi \left(\frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \right)^2 \quad (3.288)$$

$$\leq \bar{\chi} \left(\frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \right)^2 \quad (3.289)$$

where (3.287) follows from (3.228) when $N > N_2$.

Following from (3.272), using (3.286), (3.289) and (3.280), we have

$$\|\mathbf{u}\|^2 \leq \|\mathbf{A}^{-1}\|_2^2 (\|\mathbf{b}\| + \|\mathbf{v}\|)^2 \quad (3.290)$$

$$\begin{aligned} &< \left(\frac{(d_2 - 1)(1 - B_5)}{d_2} - \frac{2\bar{\chi}(1 + B_5)}{d_2} \right)^{-1} \\ &\quad \left(\sqrt{\frac{\bar{\chi}(1 + B_5)}{d_2}} \|\mathbf{b}_{k+\varsigma}^{(N)}\| + \sqrt{\bar{\chi}} \left(\frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \right) \right)^2 \end{aligned} \quad (3.291)$$

$$\leq \left(\frac{(d_2 - 1)(1 - B_5)}{d_2} - \frac{2\bar{\chi}(1 + B_5)}{d_2} \right)^{-1} \left(2\sqrt{\frac{\bar{\chi}(1 + B_5)}{d_2}} \|\mathbf{b}_{k+\varsigma}^{(N)}\| \right)^2 \quad (3.292)$$

$$= \frac{d_2 - 1}{d_2} \|\mathbf{b}_{k+\varsigma}^{(N)}\|^2 \quad (3.293)$$

where (3.292) follows when N is large enough, due to the fact that $K_1(N)$ satisfies (3.187), i.e, there exists an integer N_5 , such that when $N > N_5$,

$$\sqrt{\bar{\chi}} \left(\frac{2B_3^2(K_1(N) + B_4)^{2\tau} U_0^{1+2\gamma}}{N^\gamma} \right) \leq \frac{\sqrt{\bar{\chi}(1 + B_5)}}{d_2} \sqrt{1 - B_5} \quad (3.294)$$

and (3.292) is true, and (3.293) follows from the definition of d_2 by (3.238). Hence, when N is large enough, more specifically, when $N > \max(N_0, N_2, N_3, N_4, N_5, N_6)$, we have a contradiction with (3.256). Therefore, we conclude that there must be at least χ eigenvalues of Σ'_N within distance (3.252) away from any of the clustered λ s, furthermore, from the definition of a cluster in (3.239), there must be at least χ

eigenvalues within distance

$$(2\chi + 1) \sqrt{d_2 d_0} \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta} \quad (3.295)$$

which is less than or equal to

$$(2\bar{\chi} + 1) \sqrt{d_2 d_0} \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta} \quad (3.296)$$

away from all of the clustered λ s. We pick χ eigenvalues of Σ'_N which are within distance (3.296) and arbitrarily pair each clustered λ with one of the eigenvalues. These eigenvalues will not be paired with any other λ because all other clusters of λ s are at least distance $2\sqrt{d_2 d_0} \frac{(k + \bar{\chi} + B_1)^\tau}{N^\beta}$ apart from this cluster.

Finally, by letting

$$d_1 = (2\bar{\chi} + 1) \sqrt{d_2 d_0}, \quad B_7 = \bar{\chi} + B_1 \quad (3.297)$$

we have the desired results when N is large enough. Note that B_7 is a positive integer and d_1 is a positive constant, independent of k and N .

Proof of Lemma 3.8

In the proof of Lemma 3.8, we will need results from Lemma 3.6 and 3.7. Thus, we will first prove that under the condition of Lemma 3.8, the results of Lemma 3.6 and

3.7 apply. Since

$$\lim_{N \rightarrow \infty} \vartheta_U^N = 0 \quad (3.298)$$

for any $0 < \kappa < 1$, when N is large enough, $\theta < \vartheta_U^N$ is small enough, which means that the result of Lemma 3.6 is valid. Now we show that the result of Lemma 3.7 is also true. Let $K_1(N) = \left(\frac{d_u}{\vartheta_L^N}\right)^{\frac{1}{x}} + c_u$. Because of

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{x\gamma}{2\tau}}} = 0, \quad \lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{x\beta}{x+1+\tau}}} = 0 \quad (3.299)$$

we have (3.187) and (3.188). Because of (3.298) and the fact that $\vartheta_L^N \leq \vartheta_U^N$, we have (3.186).

Hence, for any $0 \leq k \leq \left[\left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u\right]$, result of Lemma 3.7 applies because

$$k \leq \left[\left(\frac{d_u}{\vartheta_L^N}\right)^{\frac{1}{x}} + c_u\right] \leq K_1(N) \quad (3.300)$$

and N is large enough.

Now, we will use the result of Lemma 3.6 and 3.7 to prove Lemma 3.8. From the properties of the Karhunen-Loeve expansion, we know that

$$\sum_{k=0}^{\infty} \lambda_k = \int_0^{U_0} K(u, u) du < \infty \quad (3.301)$$

Thus, for any constant $0 < \kappa < 1$, we have

$$\sum_{k=0}^{\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \lambda_k = \sum_{k=0}^{\infty} \lambda_k - \sum_{k=\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{\infty} \lambda_k \quad (3.302)$$

$$\geq \int_0^{U_0} K(u, u) du - \sum_{k=\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{\infty} \lambda_k'' \quad (3.303)$$

$$\geq \int_0^{U_0} K(u, u) du - \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa} \theta^{1-\frac{1}{x}} \quad (3.304)$$

where we have used (3.77) in Lemma 3.6 to obtain (3.304).

From the definition of matrix Σ_N , we have

$$\sum_{k=0}^{N-1} \mu_k^{(N)'} = \frac{U_0}{N} \text{tr}(\Sigma_N) = \frac{U_0}{N} \sum_{i=1}^N K\left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0\right) \quad (3.305)$$

Thus,

$$\begin{aligned} & \sum_{k=\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{N-1} \mu_k^{(N)'} \\ &= \sum_{k=0}^{N-1} \mu_k^{(N)'} - \sum_{k=0}^{\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \mu_k^{(N)'} \end{aligned} \quad (3.306)$$

$$\leq \frac{U_0}{N} \sum_{i=1}^N K\left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0\right) - \sum_{k=0}^{\left\lfloor \left(\frac{du}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \left(\lambda_k - d_1 \frac{(k + B_7)^\tau}{N^\beta} \right) \quad (3.307)$$

$$\begin{aligned}
&\leq \frac{U_0}{N} \sum_{i=1}^N K\left(\frac{i-1}{N}U_0, \frac{i-1}{N}U_0\right) - \int_0^{U_0} K(u, u)du \\
&\quad + \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa} \theta^{1-\frac{1}{x}} + \frac{d_1}{N^\beta} \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} (k + B_7)^\tau
\end{aligned} \tag{3.308}$$

$$\begin{aligned}
&\leq \frac{BU_0^{1+\alpha} 2^{\frac{\alpha}{2}}}{N^\alpha} + \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa} \theta^{1-\frac{1}{x}} \\
&\quad + \frac{d_1 \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1 \right) \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + B_7 \right)^\tau}{N^\beta}
\end{aligned} \tag{3.309}$$

$$\leq \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa^2} \theta^{1-\frac{1}{x}} \tag{3.310}$$

where (3.307) follows by Lemma 3.7. We have used (3.304) to obtain (3.308), and condition 2 in Section 3.2 to obtain (3.309), (3.310) follows because

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{\alpha x}{x-1}}} = 0 \tag{3.311}$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{\beta x}{1+x+\tau}}} = 0 \Rightarrow \lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{\beta x}{x+\tau}}} = 0 \tag{3.312}$$

and when N large enough, i.e., there exists a $N_5(\kappa) > 0$ such that when $N > N_5(\kappa)$,

we have

$$\frac{\frac{BU_0^{1+\alpha} 2^{\frac{\alpha}{2}} + U_0 K(0,0)}{N^\alpha}}{\frac{d_u^{\frac{1}{x}}}{(x-1)} \theta^{1-\frac{1}{x}}} \leq \frac{\frac{BU_0^{1+\alpha} 2^{\frac{\alpha}{2}} + U_0 K(0,0)}{N^\alpha}}{\frac{d_u^{\frac{1}{x}}}{(x-1)} (\vartheta_L^N)^{1-\frac{1}{x}}} \leq \frac{1}{2} \left(\frac{1}{\kappa^2} - \frac{1}{\kappa} \right) \quad (3.313)$$

$$\frac{\frac{2d_1 d_u^{\frac{\tau+1}{x}}}{(\tau+1)\theta^{\frac{\tau+1}{x}} N^\beta}}{\frac{d_u^{\frac{1}{x}}}{(x-1)} \theta^{1-\frac{1}{x}}} \leq \frac{\frac{2d_1 d_u^{\frac{\tau+1}{x}}}{(\tau+1)(\vartheta_L^N)^{\frac{\tau+1}{x}} N^\beta}}{\frac{d_u^{\frac{1}{x}}}{(x-1)} (\vartheta_L^N)^{1-\frac{1}{x}}} \leq \frac{1}{2} \left(\frac{1}{\kappa^2} - \frac{1}{\kappa} \right) \quad (3.314)$$

Therefore, for any $0 < \kappa < 1$, (3.190) holds for $\theta \in [\vartheta_L^N, \vartheta_U^N]$ when N is large enough.

3.8.8 Proof of Lemma 3.4

Since the condition of Lemma 3.4 is the same as Lemma 3.8, the results of Lemma 3.6, 3.7 and 3.8 hold. By the same argument as Lemma 3.6, Lemma 3.5 holds as well.

We first prove (3.54). Since ϑ_L^N satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^{\frac{\beta x}{x+\tau+1}}} = 0 \Rightarrow \lim_{N \rightarrow \infty} \frac{1}{\vartheta_L^N N^x} = 0 \quad (3.315)$$

when N is large enough such that

$$\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor + 1 \leq \left\lfloor \left(\frac{d_u}{\vartheta_L^N} \right)^{\frac{1}{x}} + c_u \right\rfloor + 1 < N - 1 \quad (3.316)$$

we can provide an upper bound on $R_a^N(\theta)$ by splitting the sum of N variables into

two parts,

$$R_a^N(\theta) = \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) + \sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \quad (3.317)$$

For any $0 < \kappa < 1$, we start with the first term in (3.317).

$$\begin{aligned} & \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \\ & \leq \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\lambda_k}{\theta} + d_1 \frac{(k + B_7)^\tau}{\theta N^\beta} \right) \end{aligned} \quad (3.318)$$

$$\leq \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\lambda_k''}{\theta} \right) + \frac{d_1}{2\theta N^\beta} \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor} (k + B_7)^\tau \quad (3.319)$$

$$\leq \left(\frac{\log 2 + x}{2\kappa} \right) d_u^{\frac{1}{x}} \theta^{-\frac{1}{x}} + \frac{d_1 \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + 1 \right) \left(\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}} + c_u \right\rfloor + B_7 \right)^\tau}{N^\beta} \quad (3.320)$$

$$\leq \left(\frac{\log 2 + x}{2\kappa^2} \right) d_u^{\frac{1}{x}} \theta^{-\frac{1}{x}} \quad (3.321)$$

where (3.318) follows from Lemma 3.7, (3.319) follows because the derivative of the function $\frac{1}{2} \log(1+x)$ is bounded by $\frac{1}{2}$ for $x \geq 0$ and the observation in (3.14), (3.320) follows because of (3.78) in Lemma 3.6, and (3.321) follows because of (3.312), and when N is large enough, more specifically, there exists an $N_6(\kappa) > 0$ such that when $N > N_6(\kappa)$, we have

$$\frac{\frac{d_1 d_u^{\frac{\tau+1}{x}}}{(\tau+1)\theta^{\frac{\tau+1+x}{x}} N^\beta}}{\left(\frac{\log 2 + x}{2} \right) d_u^{\frac{1}{x}} \theta^{-\frac{1}{x}}} \leq \frac{\frac{d_1 d_u^{\frac{\tau+1}{x}}}{(\tau+1)(\vartheta_L^N)^{\frac{\tau+1+x}{x}} N^\beta}}{\left(\frac{\log 2 + x}{2} \right) d_u^{\frac{1}{x}} (\vartheta_L^N)^{-\frac{1}{x}}} < \left(\frac{1}{\kappa^2} - \frac{1}{\kappa} \right) \quad (3.322)$$

Now, we will study the second term of (3.317).

$$\sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}}+c_u \right\rfloor+1}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \leq \frac{1}{2} \sum_{k=\left\lfloor \left(\frac{d_u}{\theta}\right)^{\frac{1}{x}}+c_u \right\rfloor+1}^{N-1} \mu_k^{(N)'} \quad (3.323)$$

$$\leq \frac{d_u^{\frac{1}{x}}}{2(x-1)\kappa^2} \theta^{-\frac{1}{x}} \quad (3.324)$$

where in obtaining (3.323) and (3.324), we have used the fact that $\log(1+x) \leq x$ and (3.190) in Lemma 3.8, respectively.

We combine the results of (3.321) and (3.324) and obtain

$$R_a^N(\theta) \leq \frac{d_u^{\frac{1}{x}} (x^2 - (1 - \log 2)x + (1 - \log 2))}{2(x-1)\kappa^2} \theta^{-\frac{1}{x}} \quad (3.325)$$

Using similar methods, we may also lower bound $R_a^N(\theta)$. We write

$$R_a^N(\theta) = \sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}}-c_l \right\rfloor} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) + \sum_{k=\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}}-c_l \right\rfloor+1}^{N-1} \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \quad (3.326)$$

We start with the first term of (3.326),

$$\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor \frac{1}{2} \log \left(1 + \frac{\mu_k^{(N)'}}{\theta} \right) \geq \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor \frac{1}{2} \log \left(1 + \frac{\lambda_k}{\theta} - d_1 \frac{(k + B_7)^\tau}{\theta N^\beta} \right) \quad (3.327)$$

$$\geq \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor \frac{1}{2} \log \left(\frac{\lambda'_k}{\theta} \right) - \frac{d_1}{2\theta N^\beta} \left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor \sum_{k=0}^{\left\lfloor \left(\frac{d_l}{\theta}\right)^{\frac{1}{x}} - c_l \right\rfloor} (k + B_7)^\tau \quad (3.328)$$

$$\geq \frac{\kappa x d_l^{\frac{1}{x}}}{4} \theta^{-\frac{1}{x}} \quad (3.329)$$

where (3.327) follows when applying the result of Lemma 3.7, (3.328) follows because the function $\frac{1}{2} \log(1 + x)$ has derivative bounded by $\frac{1}{2}$, and (3.329) follows because of (3.76) in Lemma 3.5 and (3.312), and when N is large enough, we have

$$\frac{\frac{d_1 d_l^{\frac{\tau+1}{x}}}{(\tau+1)\theta^{\frac{\tau+1+x}{x}} N^\beta}}{\frac{x d_l^{\frac{1}{x}}}{4} \theta^{-\frac{1}{x}}} \leq \frac{\frac{d_1 d_l^{\frac{\tau+1}{x}}}{(\tau+1)(\vartheta_L^N)^{\frac{\tau+1+x}{x}} N^\beta}}{\frac{x d_l^{\frac{1}{x}}}{4} (\vartheta_L^N)^{-\frac{1}{x}}} \leq \frac{1}{8} \quad (3.330)$$

A lower bound on the second term of (3.326) is zero. Hence, we can conclude that

$$R_a^N(\theta) \geq \frac{\kappa x d_l^{\frac{1}{x}}}{4} \theta^{-\frac{1}{x}} \quad (3.331)$$

Now we evaluate $D_b^N(\theta)$ for large enough N and $\theta \in [\vartheta_L^N, \vartheta_U^N]$, and prove (3.55).

$$D_b^N(\theta) = U_0^{-1} \sum_{k=0}^{N-1} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} \quad (3.332)$$

$$= U_0^{-1} \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} + U_0^{-1} \sum_{k=\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{N-1} \left(\frac{1}{\theta} + \frac{1}{\mu_k^{(N)'}} \right)^{-1} \quad (3.333)$$

$$\leq U_0^{-1} \sum_{k=0}^{\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor} \theta + U_0^{-1} \sum_{k=\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor + 1}^{N-1} \mu_k^{(N)'} \quad (3.334)$$

$$\leq U_0^{-1} \theta \left(\left\lfloor \left(\frac{d_u}{\theta} \right)^{\frac{1}{x}} + c_u \right\rfloor + 1 \right) + U_0^{-1} \frac{d_u^{\frac{1}{x}}}{(x-1)\kappa^2} \theta^{1-\frac{1}{x}} \quad (3.335)$$

$$\leq \frac{d_u^{\frac{1}{x}} (1 + \kappa^2(x-1))}{\kappa^3(x-1)U_0} \theta^{1-\frac{1}{x}} \quad (3.336)$$

where (3.333) follows because of the same reason as (3.317), and (3.334) follows because of the fact that for $a, b \geq 0$, $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} \leq \min(a, b)$, and (3.335) follows from (3.190) of Lemma 3.8, (3.336) follows because $\theta < \vartheta_U^N$, and ϑ_U^N goes to zero as N goes to infinity.

Therefore, for any $0 < \kappa < 1$, (3.54) and (3.55) are true for $\theta \in [\vartheta_L^N, \vartheta_U^N]$ when N is large enough.

3.8.9 Proof of Theorem 3.4

Note that (3.54) implies that

$$\frac{\kappa^x x^x d_l}{4^x R^x} \leq \theta_a^N(R) \leq \left(\frac{d_u^{\frac{1}{x}} (x^2 - (1 - \log 2)x + (1 - \log 2))}{2(x-1)\kappa^2} \right)^x R^{-x} \quad (3.337)$$

for large enough N and R in the interval of

$$\left[\frac{d_u^{\frac{1}{x}} (x^2 - (1 - \log 2)x + (1 - \log 2))}{2(x-1)\kappa^2} (\vartheta_U^N)^{-\frac{1}{x}}, \frac{\kappa x d_l^{\frac{1}{x}}}{4} (\vartheta_L^N)^{-\frac{1}{x}} \right] \quad (3.338)$$

From the definition of $D_a(R)$ in (3.49), we have

$$D_a(R) = D_a^N(\theta_a^N(R)) \quad (3.339)$$

$$\leq 2A^{(N)} + B^{(N)} + D_b^N(\theta_a^N(R)) \quad (3.340)$$

$$\leq 2A^{(N)} + B^{(N)} + \frac{d_u^{\frac{1}{x}} (1 + \kappa^2(x-1))}{\kappa^3(x-1)U_0} (\theta_a^N(R))^{1-\frac{1}{x}} \quad (3.341)$$

$$\begin{aligned} &\leq O(N^{-\alpha}) + O(N^{1/2-\alpha}) \\ &\quad + \frac{d_u(1 + \kappa^2(x-1)) (x^2 - (1 - \log 2)x + (1 - \log 2))^{x-1}}{U_0 \kappa^{2x+1} 2^{x-1} (x-1)^x} R^{1-x} \end{aligned} \quad (3.342)$$

where (3.340) follows from (3.45), (3.341) follows because of (3.55), (3.342) follows from (3.51), (3.52), (3.337) and the fact that R in (3.58) implies that R is in (3.338), and when R is in (3.338), $\theta_a^N(R)$ is in $[\vartheta_L^N, \vartheta_U^N]$. When R is in (3.58), we have that the third term in (3.342) is much larger than the sum of the first and second terms

when N is large enough due to the fact that

$$\lim_{N \rightarrow \infty} \frac{1}{\vartheta_{LL}^N N^{\frac{(\alpha-1/2)x}{x-1}}} = 0 \Rightarrow \lim_{N \rightarrow \infty} \frac{1}{\vartheta_{LL}^N N^{\frac{\alpha x}{x-1}}} = 0 \quad (3.343)$$

i.e., there exists an $N_9(\kappa) > 0$ such that when $N > N_9(\kappa)$, we have

$$\begin{aligned} & \frac{O(N^{-\alpha}) + O(N^{1/2-\alpha})}{\frac{d(1+\kappa^2(x-1))(x^2-(1-\log 2)x+(1-\log 2))^{x-1}}{U_0 2^{x-1}(x-1)^x} R^{1-x}} \\ & \leq \frac{O(N^{-\alpha}) + O(N^{1/2-\alpha})}{\frac{d(1+\kappa^2(x-1))(x^2-(1-\log 2)x+(1-\log 2))^{x-1}}{U_0 2^{x-1}(x-1)^x} \left(\frac{x d^{\frac{1}{x}}}{8} (\vartheta_{LL}^N)^{-\frac{1}{x}}\right)^{1-x}} \end{aligned} \quad (3.344)$$

$$\leq \frac{1}{\kappa^{2x+2}} - \frac{1}{\kappa^{2x+1}} \quad (3.345)$$

Therefore, for $0 < \kappa < 1$, (3.57) is true for R in the interval of (3.58) when N is large enough.

3.8.10 Proof of Theorem 3.5

Pick the sequences ϑ_{LL}^N and ϑ_U^N as

$$\vartheta_{LL}^N = \left(\frac{\nu}{\frac{x d^{\frac{1}{x}}}{8}} \log NP(N) \right)^{-x}, \quad \vartheta_U^N = \left(\frac{\nu}{\frac{\frac{1}{d_u} (x^2 - (1-\log 2)x + (1-\log 2))}{2(x-1)\kappa^2}} \log NP(N) \right)^{-x/2} \quad (3.346)$$

Then, because $P(N)$ satisfies (3.37) and (3.59), ϑ_{LL}^N satisfies (3.56) and ϑ_U^N satisfies (3.53). According to (3.38), we have the achievable rate, C_a^N , in the interval of (3.58), and thus, when N is large enough, Theorem 3.4 applies. Hence, an upper bound on

the minimum achievable expected distortion, or equivalently, the achievable rate in the separation-based scheme is

$$D_u^N = D_a(C_a^N) \tag{3.347}$$

$$\leq \frac{d_u(1 + \kappa^2(x - 1))(x^2 - (1 - \log 2)x + (1 - \log 2))^{x-1}}{U_0 \kappa^{3x+1} 2^{x-1} (x - 1)^x \nu^{x-1}} \left(\frac{1}{\log(NP(N))} \right)^{x-1} \tag{3.348}$$

Therefore, when $P(N)$ satisfies (3.37) and (3.59), for any $0 < \kappa < 1$, (3.61) holds when N is large enough.

Chapter 4

The Capacity Region of a Class of Discrete Degraded

Interference Channels

In wireless communications, where multiple transmitter and receiver pairs share the same medium, interference is unavoidable. How to best manage interference coming from other users and how not to cause too much interference to other users while maintaining the quality of communication is a challenging question and of a great deal of practical interest.

To be able to understand the effect of interference on communications better, interference channel (IC) was introduced in [74]. The IC is a simple network consisting of two pairs of transmitters and receivers. Each pair wishes to communicate at a certain rate with negligible probability of error. However, the two communications interfere with each other. To best understand the management of interference, we need to find the capacity region of the IC. However, the problem of finding the capacity region of the IC is essentially open except in some special cases, e.g., a class of deterministic ICs [28], discrete additive degraded interference channels (DADICs) [5], strong ICs [18, 70], ICs with statistically equivalent outputs [1, 14, 69].

In this chapter, we consider a class of discrete degraded interference channels (DDICs). In a DDIC, only the “bad” receiver faces interference, while the “good” receiver has the ability to decode both messages and thus, behaves like the receiver of a multiple access channel. It is this fact that makes the DDIC easier to analyze as compared to the IC, where both receivers are faced with interference.

We provide a single-letter characterization for the capacity region of a class of DDICs. The class of DDICs includes the DADICs studied by Benzel [5]. We show that for the class of DDICs studied, encoder cooperation does not increase the capacity region, and therefore, the capacity region of the class of DDICs is the same as the capacity region of the corresponding degraded broadcast channel, which is known.

4.1 System Model

A discrete memoryless IC consists of two transmitters and two receivers. Transmitter 1 has message W_1 , which is uniformly distributed in the set $\{1, 2, \dots, 2^{nR_1}\}$, to send to receiver 1. Transmitter 2 has message W_2 , which is uniformly distributed in the set $\{1, 2, \dots, 2^{nR_2}\}$, to send to receiver 2. Messages W_1 and W_2 are independent. The channel consists of two input alphabets, \mathcal{X}_1 and \mathcal{X}_2 , and two output alphabets, \mathcal{Y}_1 and \mathcal{Y}_2 . The channel transition probability is $p(y_1, y_2|x_1, x_2)$.

In this chapter, our definition of degradedness is in the stochastic sense, i.e., we say that an IC is DDIC if there exists a probability distribution $p'(y_2|y_1)$ such that

$$p(y_2|x_1, x_2) = \sum_{y_1 \in \mathcal{Y}_1} p(y_1|x_1, x_2)p'(y_2|y_1) \quad (4.1)$$

for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$ and $y_2 \in \mathcal{Y}_2$. However, we note that for any DDIC, we can form another DDIC (physically degraded) by

$$p(y_1, y_2 | x_1, x_2) = p(y_1 | x_1, x_2) p'(y_2 | y_1) \quad (4.2)$$

which has the same marginals, $p(y_1 | x_1, x_2)$ and $p(y_2 | x_1, x_2)$, as the original DDIC. Since the receivers do not cooperate in an IC, similar to the case of the broadcast channel [22, Problem 14.10], the capacity region is only a function of the marginals, $p(y_1 | x_1, x_2)$ and $p(y_2 | x_1, x_2)$, and the rate pairs in the capacity region can be achieved by the same achievability scheme for different ICs with the same marginals. Hence, the capacity results that we obtain for DDICs which satisfy (4.2) will be valid for any DDIC that has the same marginals, $p(y_1 | x_1, x_2)$ and $p(y_2 | x_1, x_2)$. Thus, without loss of generality, from now on, we may restrict ourselves to studying DDICs that satisfy (4.2).

A DDIC is characterized by two transition probabilities, $p'(y_2 | y_1)$ and $p(y_1 | x_1, x_2)$. For notational convenience, let T' denote the $|\mathcal{Y}_2| \times |\mathcal{Y}_1|$ matrix of transition probabilities $p'(y_2 | y_1)$, and $T_{\bar{x}_2}$ denote the $|\mathcal{Y}_1| \times |\mathcal{X}_1|$ matrix of transition probabilities $p(y_1 | x_1, \bar{x}_2)$, for all $\bar{x}_2 \in \mathcal{X}_2$.

Throughout the chapter, Δ_n will denote the probability simplex

$$\left\{ (p_1, p_2, \dots, p_n) \left| \sum_{i=1}^n p_i = 1, \quad p_i \geq 0, i = 1, 2, \dots, n \right. \right\} \quad (4.3)$$

and \mathcal{J}_n will denote the representation of the symmetric group of permutations of n

objects by the $n \times n$ permutation matrices.

The class of DDICs we consider in this chapter satisfies the following conditions:

1. T' is input symmetric. Let the input symmetry group be \mathcal{G} .
2. For any $x'_2, x''_2 \in \mathcal{X}_2$, there exists a permutation matrix $G \in \mathcal{G}$, such that

$$T_{x'_2} = GT_{x''_2} \quad (4.4)$$

3. $H(Y_1|X_1 = x_1, X_2 = x_2) = \eta$, independent of x_1, x_2 .
4. $p(y_1|x_1, x_2)$ satisfies

$$\sum_{x_2} p(y_1|x_1, x_2) = \frac{|\mathcal{X}_2|}{|\mathcal{Y}_1|}, \quad x_1 \in \mathcal{X}_1, y_1 \in \mathcal{Y}_1 \quad (4.5)$$

5. Let \mathbf{p}_{x_1, x_2} be the $|\mathcal{Y}_1|$ dimensional vector of probabilities $p(y_1|x_1, x_2)$ for a given x_1, x_2 . Then, there exists an $\tilde{x}_2 \in \mathcal{X}_2$, such that

$$\begin{aligned} & \left\{ \sum_{x_1, x_2} a_{x_1, x_2} \mathbf{p}_{x_1, x_2} : \sum_{x_1, x_2} a_{x_1, x_2} = 1, a_{x_1, x_2} \geq 0 \right\} \\ & \subseteq \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G \in \mathcal{G} \right\} \end{aligned} \quad (4.6)$$

The definition of an input symmetric channel is given in [91, Section II.D]. For completeness, we repeat it here. For an $m \times n$ stochastic matrix T' (an n input, m

output channel), the input symmetry group \mathcal{G} is defined as

$$\mathcal{G} = \{G \in \mathcal{J}_n : \exists \Pi \in \mathcal{J}_m, \quad T'G = \Pi T'\} \quad (4.7)$$

i.e., \mathcal{G} is the set of permutation matrices G such that the column permutations of T' with G may be achieved with corresponding row permutations. T' is input symmetric, if \mathcal{G} is transitive, i.e., any element of $\{1, 2, \dots, n\}$ can be mapped to every other element of $\{1, 2, \dots, n\}$ by some member of \mathcal{G} . \mathcal{G} being a transitive subgroup means that the output entropy of channel T' is maximized when the input distribution is chosen to be the uniform distribution, i.e.,

$$\max_{\mathbf{p} \in \Delta_n} H(T'\mathbf{p}) = H(T'\mathbf{u}) \quad (4.8)$$

where \mathbf{u} denotes the uniform distribution in Δ_n . This is because, for any $\mathbf{p} \in \Delta_n$, if we let $\mathbf{q} = |\mathcal{G}|^{-1} \sum_{G \in \mathcal{G}} G\mathbf{p}$, then we have

$$H(T'\mathbf{q}) = H\left(|\mathcal{G}|^{-1} \sum_{G \in \mathcal{G}} T'G\mathbf{p}\right) \quad (4.9)$$

$$= H\left(|\mathcal{G}|^{-1} \sum_{G \in \mathcal{G}} \Pi_G T'\mathbf{p}\right) \quad (4.10)$$

$$\geq |\mathcal{G}|^{-1} \sum_{G \in \mathcal{G}} H(\Pi_G T'\mathbf{p}) \quad (4.11)$$

$$= H(T'\mathbf{p}) \quad (4.12)$$

where (4.10) follows from the fact that $G \in \mathcal{G}$, and (4.11) follows from the concavity

of the entropy function. Note that for any $G' \in \mathcal{G}$,

$$G'\mathbf{q} = \mathbf{q} \tag{4.13}$$

by the fact that \mathcal{G} is a group. Since \mathcal{G} is also transitive, $\mathbf{q} = \mathbf{u}$.

Condition 2 implies that for any $p(x_1)$, $H(Y_1|X_2 = x_2)$ does not depend on x_2 . Combined with condition 1, condition 2 further implies that $H(Y_2|X_2 = x_2)$ does not depend on x_2 either. These two facts will be proved and utilized in other proofs later.

A sufficient condition for condition 3 to hold is that the vectors $p(y_1|X_1 = x_1, X_2 = x_2)$ for all $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ are permutations of each other. This is true for instance when the channel from Y_1 to Y_2 is additive [5].

By condition 4, we can show that when X_2 takes the uniform distribution, Y_1 will also be uniformly distributed. Combined with condition 1, condition 4 implies that when X_2 takes the uniform distribution, $H(Y_2)$ is maximized, irrespective of $p(x_1)$.

In condition 5, the first line of (4.6) denotes the set of all convex combinations of vectors \mathbf{p}_{x_1, x_2} for all $x_1, x_2 \in \mathcal{X}_1 \times \mathcal{X}_2$, while the second line denotes all convex combinations, and their permutations with $G \in \mathcal{G}$, of vectors $\mathbf{p}_{x_1, \tilde{x}_2}$ for all $x_1 \in \mathcal{X}_1$, but for a fixed $\tilde{x}_2 \in \mathcal{X}_2$. Therefore, this condition means that all convex combinations of \mathbf{p}_{x_1, x_2} may be obtained by a combination of convex combinations of $\mathbf{p}_{x_1, \tilde{x}_2}$ for a fixed \tilde{x}_2 , and permutations in \mathcal{G} .

The DADICs considered in [5] satisfy conditions 1-5, as we will show in Section 4.5.1.

The aim of this chapter is to provide a single-letter characterization for the capac-

ity region of DDICs that satisfy conditions 1-5, and we will follow the proof technique of [5] with appropriate generalizations.

4.2 The Outer Bound (Converse)

When we assume that the encoders are able to fully cooperate, i.e., both encoders know both messages W_1 and W_2 , we get a corresponding degraded broadcast channel with input $x = (x_1, x_2)$. The capacity region of the corresponding degraded broadcast channel serves as an outer bound on the capacity region of the DDIC. The capacity region of the degraded broadcast channel is known [19,22,31], and thus, a single-letter outer bound on the capacity region of the DDIC is

$$\overline{\text{co}} \left[\bigcup_{p(u), p(x_1, x_2|u)} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq I(X_1, X_2; Y_1|U) \\ R_2 &\leq I(U; Y_2) \end{aligned} \right\} \right] \quad (4.14)$$

where $\overline{\text{co}}$ denotes the closure of the convex hull operation, and the auxiliary random variable U , which satisfies the Markov chain $U \longrightarrow (X_1, X_2) \longrightarrow Y_1 \longrightarrow Y_2$, has cardinality bounded by $|\mathcal{U}| \leq \min(|\mathcal{Y}_1|, |\mathcal{Y}_2|, |\mathcal{X}_1||\mathcal{X}_2|)$. More specifically, for DDICs that satisfy condition 3, (4.14) can be written as

$$\overline{\text{co}} \left[\bigcup_{p(u), p(x_1, x_2|u)} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq H(Y_1|U) - \eta \\ R_2 &\leq I(U; Y_2) \end{aligned} \right\} \right] \quad (4.15)$$

Let us define $T(c)$ as

$$\begin{aligned}
T(c) = & \max_{\substack{p(u)p(x_1, x_2|u) \\ H(Y_1|U) = c \\ |\mathcal{U}| \leq \min(|\mathcal{Y}_1|, |\mathcal{Y}_2|, |\mathcal{X}_1||\mathcal{X}_2|)}} I(U; Y_2) \quad (4.16)
\end{aligned}$$

where the entropies are calculated according to the distribution

$$p(u, x_1, x_2, y_1, y_2) = p(u)p(x_1, x_2|u)p(y_1|x_1, x_2)p'(y_2|y_1) \quad (4.17)$$

Using condition 3, we can show that $\eta \leq c \leq \log |\mathcal{Y}_1|$. $T(c)$ is concave in c $[3, 5]$, and therefore, (4.15) can also be written as

$$\bigcup_{\eta \leq c \leq \log |\mathcal{Y}_1|} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq c - \eta \\ R_2 &\leq T(c) \end{aligned} \right\} \quad (4.18)$$

4.3 An Achievable Region

Based on [69, Theorem 4], the following region is achievable,

$$\overline{\text{co}} \left[\bigcup_{p(x_1), p(x_2)} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq I(X_1; Y_1|X_2) \\ R_2 &\leq I(X_2; Y_2) \end{aligned} \right\} \right] \quad (4.19)$$

which corresponds to the achievability scheme that the “bad” receiver treats the signal for the “good” receiver as pure noise, and the “good” receiver decodes both messages as if it is the receiver in a multiple access channel.

For DDICs that satisfy condition 3, (4.19) reduces to

$$\overline{\text{co}} \left[\bigcup_{p(x_1), p(x_2)} \left\{ (R_1, R_2) : \begin{aligned} R_1 &\leq H(Y_1|X_2) - \eta \\ R_2 &\leq H(Y_2) - H(Y_2|X_2) \end{aligned} \right\} \right] \quad (4.20)$$

We note that (4.20) remains an achievable region if we choose $p(x_2)$ to be the uniform distribution. Furthermore, by choosing $p(x_2)$ as the uniform distribution, we have

$$p(y_1) = \sum_{x_1, x_2} p(y_1|x_1, x_2)p(x_1) \frac{1}{|\mathcal{X}_2|} \quad (4.21)$$

$$= \frac{1}{|\mathcal{X}_2|} \sum_{x_1} p(x_1) \sum_{x_2} p(y_1|x_1, x_2) \quad (4.22)$$

$$= \frac{1}{|\mathcal{Y}_1|} \quad (4.23)$$

where (4.23) uses condition 4. Thus, when $p(x_2)$ is chosen as the uniform distribution, $p(y_1)$ results in a uniform distribution as well. Let us define τ as

$$\tau = \max_{\mathbf{p} \in \Delta_{|\mathcal{Y}_1|}} H(T' \mathbf{p}) \quad (4.24)$$

Using the fact that the DDIC under consideration satisfies condition 1, i.e., it satisfies

(4.8), we have that when $p(x_2)$ is uniform, and consequently $p(y_1)$ is uniform,

$$H(Y_2) = \tau \quad (4.25)$$

Hence, choosing $p(x_2)$ to be the uniform distribution in (4.20), yields the following as an achievable region,

$$\overline{\text{co}} \left[\bigcup_{p(x_1)} \left\{ (R_1, R_2) : R_1 \leq \frac{1}{|\mathcal{X}_2|} \sum_{x_2} H(Y_1|X_2 = x_2) - \eta \right. \right. \\ \left. \left. R_2 \leq \tau - \frac{1}{|\mathcal{X}_2|} \sum_{x_2} H(Y_2|X_2 = x_2) \right\} \right] \quad (4.26)$$

Due to condition 2, for any $p(x_1) = \mathbf{p}$ and any $x'_2, x''_2 \in \mathcal{X}_2$, there exists a permutation matrix $G \in \mathcal{G}$ such that

$$H(Y_1|X_2 = x'_2) = H(T_{x'_2}\mathbf{p}) \quad (4.27)$$

$$= H(GT_{x'_2}\mathbf{p}) \quad (4.28)$$

$$= H(T_{x''_2}\mathbf{p}) \quad (4.29)$$

$$= H(Y_1|X_2 = x''_2) \quad (4.30)$$

which means that for any $p(x_1)$, $H(Y_1|X_2 = x_2)$ does not depend on x_2 . Furthermore, for any $p(x_1) = \mathbf{p}$ and any $x'_2, x''_2 \in \mathcal{X}_2$, there exist permutation matrices $G \in \mathcal{G}$ and

Π , of order $|\mathcal{Y}_1|$ and $|\mathcal{Y}_2|$ respectively, such that

$$H(Y_2|X_2 = x'_2) = H(T'T_{x'_2}\mathbf{p}) \quad (4.31)$$

$$= H(T'GT_{x''_2}\mathbf{p}) \quad (4.32)$$

$$= H(\Pi T'T_{x''_2}\mathbf{p}) \quad (4.33)$$

$$= H(T'T_{x''_2}\mathbf{p}) \quad (4.34)$$

$$= H(Y_2|X_2 = x''_2) \quad (4.35)$$

where (4.33) follows from the fact that $G \in \mathcal{G}$. (4.35) means that for any $p(x_1)$, $H(Y_2|X_2 = x_2)$ does not depend on x_2 either. Hence, the achievable region in (4.26) can further be written as

$$\overline{\text{co}} \left[\bigcup_{p(x_1)} \left\{ (R_1, R_2) : R_1 \leq H(Y_1|X_2 = x_2) - \eta \right. \right. \\ \left. \left. R_2 \leq \tau - H(Y_2|X_2 = x_2) \right\} \right] \quad (4.36)$$

for any $x_2 \in \mathcal{X}_2$. Since we will use condition 5 later, we choose to write the region of (4.36) as

$$\overline{\text{co}} \left[\bigcup_{p(x_1)} \left\{ (R_1, R_2) : R_1 \leq H(Y_1|X_2 = \tilde{x}_2) - \eta \right. \right. \\ \left. \left. R_2 \leq \tau - H(Y_2|X_2 = \tilde{x}_2) \right\} \right] \quad (4.37)$$

where \tilde{x}_2 is given in condition 5.

Let us define $F(c)$ as

$$F(c) = \min_{p(x_1)} H(Y_2|X_2 = \tilde{x}_2) \quad (4.38)$$

$$H(Y_1|X_2 = \tilde{x}_2) = c$$

where the entropies are calculated according to the distribution

$$p(y_1, y_2, x_1|\tilde{x}_2) = p(x_1)p(y_1|x_1, \tilde{x}_2)p'(y_2|y_1) \quad (4.39)$$

In (4.38), we can write \min instead of \inf by the same reasoning as in [90, Section I]. Note that $F(c)$ is not a function of \tilde{x}_2 because of (4.30) and (4.35). Again, by condition 3, we can show that $\eta \leq c \leq \log |\mathcal{Y}_1|$. Hence, the achievable region in (4.37) can be written as,

$$\overline{\text{co}} \left[\bigcup_{\eta \leq c \leq \log |\mathcal{Y}_1|} \left\{ (R_1, R_2) : R_1 \leq c - \eta \right. \right. \\ \left. \left. R_2 \leq \tau - F(c) \right\} \right] \quad (4.40)$$

which by [5, Facts 4 and 5], can further be written as

$$\bigcup_{\eta \leq c \leq \log |\mathcal{Y}_1|} \left\{ (R_1, R_2) : R_1 \leq c - \eta \right. \\ \left. R_2 \leq \tau - \underline{\text{env}} F(c) \right\} \quad (4.41)$$

where $\underline{\text{env}} F(\cdot)$ denotes the lower convex envelope of the function $F(\cdot)$.

4.4 The Capacity Region

In this section, we show that the achievable region in (4.41) contains the outer bound in (4.18), and thus, (4.18) and (4.41) are both, in fact, single-letter characterizations of the capacity region of DDICs satisfying conditions 1-5. To show this, it suffices to prove that

$$T(c) \leq \tau - \underline{\text{env}} F(c), \quad \eta \leq c \leq \log |\mathcal{Y}_1| \quad (4.42)$$

Let us fix a $c \in [\eta, \log |\mathcal{Y}_1|]$. Let $p^*(u), p^*(x_1, x_2|u)$ be the distributions that achieve the maximum in (4.16), i.e.,

$$H(Y_1|U) = c \quad (4.43)$$

$$I(U; Y_2) = T(c) \quad (4.44)$$

Using condition 5, for each $u \in \mathcal{U}$, there exists a $p^u(x_1) = \mathbf{p}^u$ and a permutation matrix $G^u \in \mathcal{G}$, such that

$$\sum_{x_1, x_2} p^*(x_1, x_2|U = u) \mathbf{p}_{x_1, x_2} = G^u T_{\tilde{x}_2} \mathbf{p}^u \quad (4.45)$$

Thus, we have

$$H(Y_1|U = u) = H(G^u T_{\tilde{x}_2} \mathbf{p}^u) = H(T_{\tilde{x}_2} \mathbf{p}^u) \quad (4.46)$$

(4.46) means that \mathbf{p}^u is in the feasible set of the optimization in (4.38) when $c = H(Y_1|U = u)$. Hence,

$$F(H(Y_1|U = u)) \leq H(T' T_{\tilde{x}_2} \mathbf{p}^u) \quad (4.47)$$

We have

$$H(Y_2|U = u) = H(T' G^u T_{\tilde{x}_2} \mathbf{p}^u) \quad (4.48)$$

$$= H(\Pi^u T' T_{\tilde{x}_2} \mathbf{p}^u) \quad (4.49)$$

$$= H(T' T_{\tilde{x}_2} \mathbf{p}^u) \quad (4.50)$$

$$\geq F(H(Y_1|U = u)) \quad (4.51)$$

where (4.48), (4.49) and (4.51) follow from (4.45), the fact that $G \in \mathcal{G}$, and (4.47), respectively. Thus,

$$H(Y_2|U) = \sum_u P(U = u) H(Y_2|U = u) \quad (4.52)$$

$$\geq \sum_u P(U = u) F(H(Y_1|U = u)) \quad (4.53)$$

$$\geq \sum_u P(U = u) \underline{\text{env}} F(H(Y_1|U = u)) \quad (4.54)$$

$$\geq \underline{\text{env}} F \left(\sum_u P(U = u) H(Y_1|U = u) \right) \quad (4.55)$$

$$= \underline{\text{env}} F(H(Y_1|U)) \quad (4.56)$$

$$= \underline{\text{env}} F(c) \quad (4.57)$$

where (4.53) follows from (4.51), (4.54) follows from the definition of $\underline{\text{env}}$, and (4.55) follows from convexity of $\underline{\text{env}}F(\cdot)$.

Finally, for $\eta \leq c \leq \log |\mathcal{Y}_1|$, we have

$$T(c) = I(U; Y_2) \tag{4.58}$$

$$= H(Y_2) - H(Y_2|U) \tag{4.59}$$

$$\leq \tau - \underline{\text{env}}F(c) \tag{4.60}$$

where (4.60) follows from (4.57) and the definition of τ in (4.24).

Therefore, we conclude that the single-letter characterization of the capacity region of DDICs satisfying conditions 1-5 is (4.41), and also (4.18). To achieve point (R_1, R_2) on the boundary of the capacity region, if R_1 and R_2 are such that

$$R_1 = c - \eta, \quad R_2 = \tau - F(c) \tag{4.61}$$

for some $\eta \leq c \leq \log |\mathcal{Y}_1|$, transmitters 1 and 2 generate random codebooks according to $p^*(x_1)$, which is the minimizer of $F(R_1 + \eta)$, and $p^*(x_2)$, which is the uniform distribution, respectively, and transmit the codewords corresponding to the realizations of their own messages. Receiver 1 performs successive decoding, in the order of message 2, and then message 1. Receiver 2 decodes its own message treating interference from transmitter 1 as pure noise. To achieve point (R_1, R_2) on the capacity region, where R_1 and R_2 do not satisfy (4.61), time-sharing should be used. Furthermore, we note that for these DDICs, encoder cooperation cannot increase the capacity region.

4.5 Examples

In this section, we will provide three examples of DDICs for which conditions 1-5 are satisfied. The first example is the channel model adopted in [5], for which the capacity region is already known. In the second and third examples, the capacity regions are previously unknown, and using the results of this chapter, we are able to determine the capacity regions.

4.5.1 Example 1

A DADIC is defined as [5]

$$Y_1 = X_1 \oplus X_2 \oplus V_1 \tag{4.62}$$

$$Y_2 = X_1 \oplus X_2 \oplus V_1 \oplus V_2 \tag{4.63}$$

where

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{S} = \{0, 1, \dots, s-1\} \tag{4.64}$$

and \oplus denotes modulo- s sum, and V_1 and V_2 are independent noise random variables defined over \mathcal{S} with distributions

$$\mathbf{p}_i = (p_i(0), p_i(1), \dots, p_i(s-1)), \quad i = 1, 2 \tag{4.65}$$

Since $Y_2 = Y_1 \oplus V_2$, matrix T' is circulant, and thus input symmetric [91, Section II.D]. Hence, condition 1 is satisfied. It is straightforward to check that conditions 2-5 are also satisfied. For example, when $s = 3$, we have

$$T' = \begin{bmatrix} p_2(0) & p_2(2) & p_2(1) \\ p_2(1) & p_2(0) & p_2(2) \\ p_2(2) & p_2(1) & p_2(0) \end{bmatrix} \quad (4.66)$$

and the input symmetry group for T' is

$$\mathcal{G} = \left\{ G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\} \quad (4.67)$$

which is transitive, i.e., $1 \xrightarrow{G_2} 2, 1 \xrightarrow{G_1} 3, 2 \xrightarrow{G_1} 1, 2 \xrightarrow{G_2} 3, 3 \xrightarrow{G_2} 1, 3 \xrightarrow{G_1} 2$. From (4.62),

$$\begin{aligned} T_0 &= \begin{bmatrix} p_1(0) & p_1(2) & p_1(1) \\ p_1(1) & p_1(0) & p_1(2) \\ p_1(2) & p_1(1) & p_1(0) \end{bmatrix}, \quad T_1 = \begin{bmatrix} p_1(2) & p_1(1) & p_1(0) \\ p_1(0) & p_1(2) & p_1(1) \\ p_1(1) & p_1(0) & p_1(2) \end{bmatrix}, \\ T_2 &= \begin{bmatrix} p_1(1) & p_1(0) & p_1(2) \\ p_1(2) & p_1(1) & p_1(0) \\ p_1(0) & p_1(2) & p_1(1) \end{bmatrix} \end{aligned} \quad (4.68)$$

Conditions 2-4 are satisfied because

$$T_1 = G_1 T_0, \quad T_2 = G_2 T_0 \quad (4.69)$$

$$\eta = H(V_1) \quad (4.70)$$

$$\sum_{x_2} p(y_1 | x_1, x_2) = p_1(0) + p_1(1) + p_1(2) = 1 \quad (4.71)$$

Next, we check condition 5.

$$\left\{ \sum_{x_1, x_2} a_{x_1, x_2} \mathbf{p}_{x_1, x_2} : \sum_{x_1, x_2} a_{x_1, x_2} = 1, a_{x_1, x_2} \geq 0 \right\} \quad (4.72)$$

$$= \left\{ a \begin{pmatrix} p_1(0) \\ p_1(1) \\ p_1(2) \end{pmatrix} + b \begin{pmatrix} p_1(2) \\ p_1(0) \\ p_1(1) \end{pmatrix} + c \begin{pmatrix} p_1(1) \\ p_1(2) \\ p_1(0) \end{pmatrix} : a + b + c = 1, a, b, c \geq 0 \right\} \quad (4.73)$$

because even though (4.72) is a convex combination of 9 vectors, due to vectors repeating themselves in the columns of T_0 , T_1 and T_2 , the set, in fact, consists of convex combinations of only 3 vectors. On the other hand, for $\tilde{x}_2 = 0$,

$$\left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G = G_0 \right\} \quad (4.74)$$

$$= \left\{ a \begin{pmatrix} p_1(0) \\ p_1(1) \\ p_1(2) \end{pmatrix} + b \begin{pmatrix} p_1(2) \\ p_1(0) \\ p_1(1) \end{pmatrix} + c \begin{pmatrix} p_1(1) \\ p_1(2) \\ p_1(0) \end{pmatrix} : a + b + c = 1, a, b, c \geq 0 \right\} \quad (4.75)$$

because (4.74) is the convex combinations of the columns of T_0 , with the unitary permutation. Thus,

$$\begin{aligned} & \left\{ \sum_{x_1, x_2} a_{x_1, x_2} \mathbf{p}_{x_1, x_2} : \sum_{x_1, x_2} a_{x_1, x_2} = 1, a_{x_1, x_2} \geq 0 \right\} \\ &= \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G = G_0 \right\} \end{aligned} \quad (4.76)$$

$$\subseteq \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G \in \mathcal{G} \right\} \quad (4.77)$$

and condition 5 is satisfied.

4.5.2 Example 2

Next, we consider the following DDIC. We have $|\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{Y}_1| = 2, |\mathcal{Y}_2| = 3$, and $p(y_1|x_1, x_2)$ is characterized by

$$Y_1 = X_1 \oplus X_2 \oplus V_1 \quad (4.78)$$

where V_1 is Bernoulli with p . $p'(y_2|y_1)$ is an erasure channel with parameter $0 \leq \alpha \leq 1$,

i.e., the transition probability matrix is

$$T' = \begin{bmatrix} 1 - \alpha & 0 \\ \alpha & \alpha \\ 0 & 1 - \alpha \end{bmatrix} \quad (4.79)$$

Thus, the channel is such that the “bad” receiver cannot receive all the bits that the “good” receiver receives. More specifically, α proportion of the time, whether the bit is a 0 or 1 is unrecognizable, and thus denoted as an erasure e .

It is easy to see that T' is input symmetric because the input symmetry group

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \quad (4.80)$$

is transitive. Conditions 2-5 are satisfied because $p(y_1|x_1, x_2)$ is the same as in Example 1 in Section 4.5.1.

4.5.3 Example 3

Let a, b, c, d, e, f be non-negative numbers such that $a+b+c = 1$ and $d+e+f = 1/2$.

We have $|\mathcal{X}_1| = 4$, $|\mathcal{X}_2| = |\mathcal{Y}_1| = 3$, and $|\mathcal{Y}_2| = 6$. The DDIC is described as

$$T' = \begin{bmatrix} d & e & f \\ e & f & d \\ d & f & e \\ f & e & d \\ e & d & f \\ f & d & e \end{bmatrix}, \quad T_0 = \begin{bmatrix} a & b & c & c \\ b & c & a & b \\ c & a & b & a \end{bmatrix}, \quad T_1 = \begin{bmatrix} c & a & b & a \\ a & b & c & c \\ b & c & a & b \end{bmatrix}, \quad T_2 = \begin{bmatrix} b & c & a & b \\ c & a & b & a \\ a & b & c & c \end{bmatrix} \quad (4.81)$$

It is straightforward to see that T' is input symmetric because the input symmetry group

$$\mathcal{G} = \left\{ \begin{array}{l} G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\ G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \right\} \quad (4.82)$$

is transitive. Conditions 2-4 are satisfied because

$$T_1 = G_1 T_0, \quad T_2 = G_2 T_0 \quad (4.83)$$

$$\eta = -a \log a - b \log b - c \log c \quad (4.84)$$

$$\sum_{x_2} p(y_1 | x_1, x_2) = a + b + c = 1 \quad (4.85)$$

To show condition 5, we use Figure 4.1. The set on the first line of (4.6) in condition

5 is the convex combination of the following six points,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a \\ c \\ b \end{bmatrix}, \begin{bmatrix} c \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \\ c \end{bmatrix}, \begin{bmatrix} b \\ c \\ a \end{bmatrix}, \begin{bmatrix} c \\ b \\ a \end{bmatrix} \quad (4.86)$$

resulting in all the points within the hexagon in Figure 4.1. The three sets

$$\begin{aligned} & \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G = G_0 \right\} \\ &= \left\{ \mu_1 \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \mu_2 \begin{bmatrix} b \\ c \\ a \end{bmatrix} + \mu_3 \begin{bmatrix} c \\ a \\ b \end{bmatrix} + \mu_4 \begin{bmatrix} c \\ b \\ a \end{bmatrix} : \sum_{i=1}^4 \mu_i = 1, \mu_i \geq 0 \right\} \end{aligned} \quad (4.87)$$

and

$$\begin{aligned} & \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G = G_1 \right\} \\ &= \left\{ \mu_1 \begin{bmatrix} c \\ a \\ b \end{bmatrix} + \mu_2 \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \mu_3 \begin{bmatrix} b \\ c \\ a \end{bmatrix} + \mu_4 \begin{bmatrix} a \\ c \\ b \end{bmatrix} : \sum_{i=1}^4 \mu_i = 1, \mu_i \geq 0 \right\} \end{aligned} \quad (4.88)$$

and

$$\begin{aligned} & \left\{ G \left(\sum_{x_1} b_{x_1} \mathbf{p}_{x_1, \tilde{x}_2} \right) : \sum_{x_1} b_{x_1} = 1, b_{x_1} \geq 0, G = G_2 \right\} \\ &= \left\{ \mu_1 \begin{bmatrix} b \\ c \\ a \end{bmatrix} + \mu_2 \begin{bmatrix} c \\ a \\ b \end{bmatrix} + \mu_3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \mu_4 \begin{bmatrix} b \\ a \\ c \end{bmatrix} : \sum_{i=1}^4 \mu_i = 1, \mu_i \geq 0 \right\} \end{aligned} \quad (4.89)$$

correspond to the points in the three shaded areas, $[abc, cba, bca, cab]$, $[acb, abc, bca, cab]$,

and $[bac, cab, abc, bca]$, respectively. Since the three shaded areas cover the entire

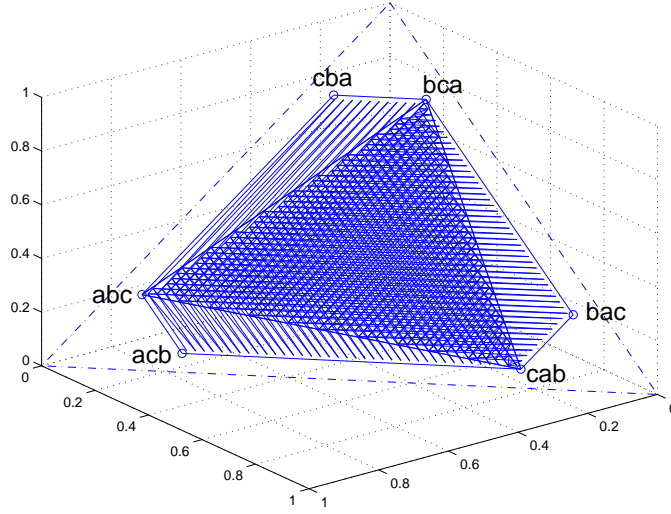


Figure 4.1: Explanation of condition 5 in example 3.

hexagon, and $\{G_0, G_1, G_2\} \subset \mathcal{G}$, condition 5 is satisfied.

4.6 Chapter Summary and Conclusions

We provide a single-letter characterization for the capacity region of a class of DDICs, which is more general than the class of DADICs studied by Benzel [5]. We show that for the class of DDICs studied, encoder cooperation does not increase the capacity region, and the best way to manage the interference is through random codebook design and treating the signal for the “good” receiver as pure noise at the “bad” receiver.

The results of this chapter have been published in [58] and submitted for publication in [52].

Chapter 5

On the Capacity Region of the Gaussian Z-channel

In this chapter, we aim to find or bound the capacity region of a modified interference channel, the Gaussian Z-channel; see Figure 1.2. In [83], an achievable region for the Gaussian Z-channel is provided for the case of $\alpha > 1 + P_1$. In this chapter, we focus on the model of the Gaussian Z-channel where the cross-over link is weak, more specifically, $\alpha < 1$. We derive an achievable region and show that this region is almost equal to the capacity region by proving most of the converse. We also derive some lower and upper bounds on the capacity region. Finally, for the special case of $\alpha = 1$, we determine the capacity region exactly.

5.1 System Model

The Gaussian Z-channel has two transmitters and two receivers as shown in Figure 1.2.

The received signals at receivers R1 and R2 are given as,

$$Y_1 = X_1 + \sqrt{\alpha}X_2 + Z_1 \tag{5.1}$$

$$Y_2 = X_2 + Z_2 \tag{5.2}$$

where X_1 and X_2 are the signals transmitted by transmitters T1 and T2, and Z_1, Z_2 are independent Gaussian random variables with zero mean and unit variance and are independent of everything else. The transmitters T1 and T2 are subject to power constraints P_1 and P_2 , respectively. The received signals in (5.1) and (5.2) can equivalently be written as,

$$Y_1 = \frac{X_1}{\sqrt{\alpha}} + X_2 + \frac{Z_1}{\sqrt{\alpha}} \quad (5.3)$$

$$Y_2 = X_2 + Z_2 \quad (5.4)$$

since scaling does not affect the capacity region. For the rest of this chapter, we will be working with the channel model in (5.3) and (5.4).

Three independent messages are transmitted in a Z-channel: the message from transmitter T1 to receiver R1, denoted as W_{11} , the message from transmitter T2 to receiver R1, denoted as W_{21} , and the message from transmitter T2 to receiver R2, denoted as W_{22} . W_{11} , W_{21} and W_{22} are uniformly distributed on the sets $\{1, 2, \dots, 2^{nR_{11}}\}$, $\{1, 2, \dots, 2^{nR_{21}}\}$, and $\{1, 2, \dots, 2^{nR_{22}}\}$, respectively, and they are independent of each other. The capacity region of the Z-channel is a three dimensional volume, with axes R_{11} , R_{21} and R_{22} corresponding to the rates of messages W_{11} , W_{21} and W_{22} .

In this chapter, we mainly study the case of $\alpha < 1$. Reference [83] studied the case of $\alpha > 1 + P_1$. These two cases correspond to two different kinds of “degradedness” conditions on the channels from transmitter T2 to both receivers. In the absence of the link between transmitter T1 and receiver R1, the channels from transmitter T2

to both receivers constitute a traditional broadcast channel [22]. Given the existence of the link from transmitter T1 to receiver R1, the condition, $\alpha > 1 + P_1$ assumed in [83], corresponds to the case that the signal of transmitter T2 received at receiver R2 is a “degraded” version of the same signal received at receiver R1 (for Gaussian inputs). The condition, $\alpha < 1$, that we assume in this chapter, corresponds to the case that the signal of transmitter T2 received at receiver R1 is a “degraded” version of the same signal at receiver R2. The “degradedness” condition we have here is stronger than the one in [83], in that, it is valid for all distributions of X_1 .

In this chapter, we consider only deterministic encoders, which incur no loss in performance [89]. All logarithms are defined with respect to base e .

5.2 Achievable Region

Let us define four quantities:

$$c_{11}(\beta) = \frac{1}{2} \log \left(1 + \frac{P_1}{\alpha\beta P_2 + 1} \right) \quad (5.5)$$

$$c_{21}(\beta) = \frac{1}{2} \log \left(1 + \frac{\alpha(1 - \beta)P_2}{\alpha\beta P_2 + 1} \right) \quad (5.6)$$

$$c_{22}(\beta) = \frac{1}{2} \log (1 + \beta P_2) \quad (5.7)$$

$$c_1(\beta) = \frac{1}{2} \log \left(1 + \frac{P_1 + \alpha(1 - \beta)P_2}{\alpha\beta P_2 + 1} \right) \quad (5.8)$$

The following theorem states an achievable region for the Gaussian Z-channel when $\alpha < 1$.

Theorem 5.1 *If $\alpha < 1$, the following region is achievable in the Gaussian Z-channel:*

$$R_{11} \leq c_{11}(\beta) \tag{5.9}$$

$$R_{21} \leq c_{21}(\beta) \tag{5.10}$$

$$R_{22} \leq c_{22}(\beta) \tag{5.11}$$

$$R_{11} + R_{21} \leq c_1(\beta) \tag{5.12}$$

for any $0 \leq \beta \leq 1$.

A proof of Theorem 5.1 is given in Appendix 5.6.1.

We show an example of the achievable region in Figure 5.1, where $P_1 = 1$, $P_2 = 5$ and $\alpha = 0.5$. The boundary of the capacity region is traced as we change β from 0 to 1, and interpret inequalities in (5.9)-(5.12) as equalities. Each fixed β determines a pentagon on a plane parallel to the R_{11} - R_{21} plane as defined by inequalities (5.9), (5.10) and (5.12), and also a rate R_{22} as defined by inequality (5.11). Therefore, the achievable region is a concatenation of pentagons of varying sizes along the R_{22} axis.

We have established the achievability of the region defined by (5.9)-(5.12). Next, we will investigate the converse of this achievable region. We will show that, in most places, the achievable region is actually tight, i.e., it is equal to the capacity region of the channel.

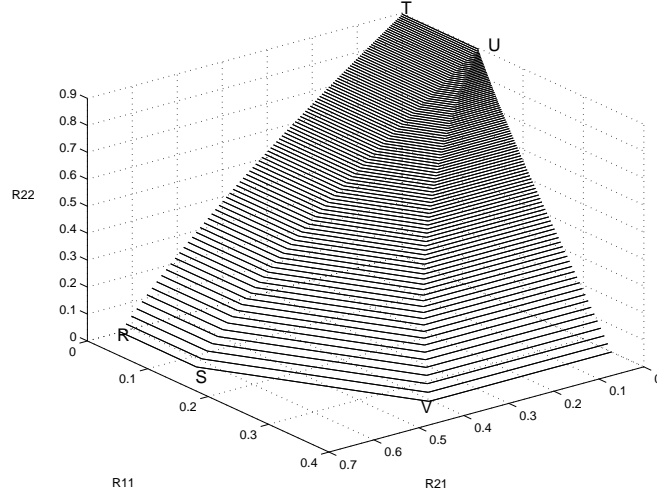


Figure 5.1: The achievable region.

5.3 The Converse

Theorem 5.2 *The achievable rate triplets (R_{11}, R_{21}, R_{22}) have to satisfy*

$$R_{21} \leq c_{21}(\beta) \quad (5.13)$$

$$R_{22} \leq c_{22}(\beta) \quad (5.14)$$

$$R_{11} + R_{21} \leq c_1(\beta) \quad (5.15)$$

for some $0 \leq \beta \leq 1$.

A proof of Theorem 5.2 is given in Appendix 5.6.2.

Referring back to Figure 5.1, this theorem states that, of the three surfaces that make up the achievability region, two of them, the surface defined by $TRSU$ and the surface defined by USV , are actually tight.

The converse that is missing is the part that describes R_{11} , when R_{21} is so small that $R_{11} + R_{21} < c_1(\beta)$. This will be addressed in the discussion section next by

developing some lower and upper bounds on the capacity region.

5.4 Discussion

As stated above, combining Theorems 5.1 and 5.2, we see that the achievable region in Theorem 5.1 for $R_{21}, R_{22}, R_{11} + R_{21}$ is in fact tight. The only unsureness comes from R_{11} .

As mentioned in [83], the Z-channel includes the multiple access channel, the broadcast channel and the Z-interference channel as special cases. By setting $\beta = 0$ in the achievable region in Theorem 5.1, we get

$$R_{11} \leq \frac{1}{2} \log(1 + P_1) \quad (5.16)$$

$$R_{21} \leq \frac{1}{2} \log(1 + \alpha P_2) \quad (5.17)$$

$$R_{11} + R_{21} \leq \frac{1}{2} \log(1 + P_1 + \alpha P_2) \quad (5.18)$$

$$R_{22} = 0 \quad (5.19)$$

which is exactly the capacity region for the Gaussian multiple access channel with link gains 1 and $\sqrt{\alpha}$, and noise variance 1 [22]. By setting $P_1 = 0$ in the achievable region in Theorem 5.1, we get

$$R_{22} \leq \frac{1}{2} \log(1 + \beta P_2) \quad (5.20)$$

$$R_{21} \leq \frac{1}{2} \log \left(1 + \frac{\alpha(1 - \beta)P_2}{\alpha\beta P_2 + 1} \right) \quad (5.21)$$

$$R_{11} = 0 \quad (5.22)$$

which is exactly the capacity region for the Gaussian broadcast channel with channel gains 1 and $\sqrt{\alpha}$, and noise variance 1 [22]. By setting $R_{21} = 0$ in the capacity region of the Gaussian Z-channel, we should get the capacity region of the Gaussian Z-interference channel [17], which is still an open problem.

5.4.1 Sum Capacity of the Gaussian Z-channel

Similar to the Z-interference channel case, the sum capacity of the Gaussian Z-channel is known for $\alpha < 1$ based on the achievable region of Theorem 5.1 and the converse theorem, Theorem 5.2. The sum capacity is

$$\max_{0 \leq \beta \leq 1} c_{22}(\beta) + c_1(\beta) \quad (5.23)$$

It can be easily verified that when $\beta = 1$, we attain the maximum and the sum capacity for the Gaussian Z-channel when $\alpha < 1$ is

$$\frac{1}{2} \log \left(\frac{(1 + P_2)(1 + P_1 + \alpha P_2)}{1 + \alpha P_2} \right) \quad (5.24)$$

The sum capacity is attained at point U in Figure 5.1.

5.4.2 Lower and Upper Bounds for the Capacity Region

Next, we will derive lower and upper bounds for the capacity region for the portion where a converse is missing. An obvious upper bound for the capacity region is

obtained by combining

$$R_{11} \leq \frac{1}{2} \log(1 + P_1) \quad (5.25)$$

with (5.10), (5.11) and (5.12) for any $0 \leq \beta \leq 1$. In Figure 5.2, the achievable region in Theorem 5.1 is shown in black and this upper bound is shown in yellow.

Two other achievable regions can be derived to close the gap between the lower and upper bounds on the capacity region.

Larger Achievable Region 1: It is clear that the following three points given as triplets of (R_{11}, R_{21}, R_{22}) are achievable.

$$\text{Point } A: \left(\frac{1}{2} \log(1 + P_1), 0, \frac{1}{2} \log \left(1 + \frac{\alpha P_2}{P_1 + 1} \right) \right) \quad (5.26)$$

$$\text{Point } B: \left(\frac{1}{2} \log(1 + P_1), \frac{1}{2} \log \left(1 + \frac{\alpha P_2}{P_1 + 1} \right), 0 \right) \quad (5.27)$$

$$\text{Point } C: \left(\frac{1}{2} \log \left(1 + \frac{P_1}{\alpha P_2 + 1} \right), 0, \frac{1}{2} \log(1 + P_2) \right) \quad (5.28)$$

These three points are shown in Figure 5.2. Joining the lines between points A and B and points A and C , and the curve connecting points B and C , we can obtain a plane which is achievable by time sharing.

Larger Achievable Region 2: Using the technique of successive decoding [14], we can split X_2 into three parts:

$$X_2 = X_{21} + X_{comm} + X_{priv} \quad (5.29)$$

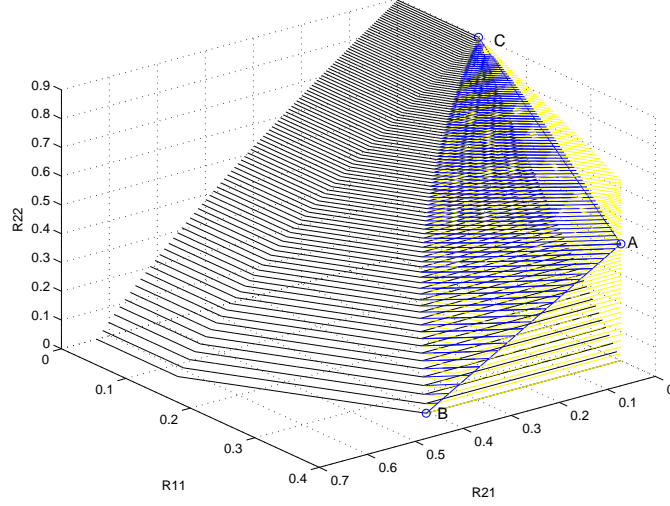


Figure 5.2: All achievable regions and the upper bound in (5.25).

where X_{21} is a function of message W_{21} and $X_{comm} + X_{priv}$ together carry message W_{22} . Let X_{comm} and X_{priv} be independent. X_{comm} is intended to be decoded by both receiver R1 and receiver R2, even though receiver R1 is not interested in decoding any part of message W_{22} . X_{priv} is decoded by receiver R2 only. Receiver R1 treats X_{priv} as noise. Transmitter T2 uses power $\bar{\gamma}P_2$ for X_{21} , power $\mu\gamma P_2$ for X_{comm} , and power $\bar{\mu}\gamma P_2$ for X_{priv} , where $\bar{\gamma} = 1 - \gamma$, $\bar{\mu} = 1 - \mu$ and γ and μ vary from 0 to 1.

Receiver R1 uses decoding order X_{21} , then X_{comm} and finally X_1 , and receiver R2 uses decoding order X_{21} , then X_{comm} and finally X_{priv} . Let $\mathcal{A}1(\mu, \gamma)$ be the set of R_{11} , R_{21} and R_{22} that satisfies the following inequalities:

$$R_{21} \leq \frac{1}{2} \log \left(1 + \frac{\bar{\gamma}P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \gamma P_2} \right) \quad (5.30)$$

$$R_{comm} \leq \frac{1}{2} \log \left(1 + \frac{\mu\gamma P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu}\gamma P_2} \right) \quad (5.31)$$

$$R_{11} \leq \frac{1}{2} \log \left(1 + \frac{\frac{P_1}{\alpha}}{\frac{1}{\alpha} + \bar{\mu}\gamma P_2} \right) \quad (5.32)$$

$$R_{priv} \leq \frac{1}{2} \log (1 + \bar{\mu} \gamma P_2) \quad (5.33)$$

$$R_{22} = R_{comm} + R_{priv} \quad (5.34)$$

Receiver R1 uses decoding order X_{comm} , then X_{21} and finally X_1 , and receiver R2 uses decoding order X_{comm} , then X_{21} and finally X_{priv} . Let $\mathcal{A}2(\mu, \gamma)$ be the set of R_{11} , R_{21} and R_{22} that satisfies the following inequalities:

$$R_{comm} \leq \frac{1}{2} \log \left(1 + \frac{\mu \gamma P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu} \gamma P_2 + \bar{\gamma} P_2} \right) \quad (5.35)$$

$$R_{21} \leq \frac{1}{2} \log \left(1 + \frac{\bar{\gamma} P_2}{\frac{P_1}{\alpha} + \frac{1}{\alpha} + \bar{\mu} \gamma P_2} \right) \quad (5.36)$$

$$R_{11} \leq \frac{1}{2} \log \left(1 + \frac{\frac{P_1}{\alpha}}{\frac{1}{\alpha} + \bar{\mu} \gamma P_2} \right) \quad (5.37)$$

$$R_{priv} \leq \frac{1}{2} \log (1 + \bar{\mu} \gamma P_2) \quad (5.38)$$

$$R_{22} = R_{comm} + R_{priv} \quad (5.39)$$

Then, an achievable region for the part where the converse is missing is the convex hull of

$$\left(\bigcup_{0 \leq \mu, \gamma \leq 1} \mathcal{A}1(\mu, \gamma) \right) \cup \left(\bigcup_{0 \leq \mu, \gamma \leq 1} \mathcal{A}2(\mu, \gamma) \right) \quad (5.40)$$

Figure 5.2 shows *Larger Achievable Region 2* and the lines AB and AC defined in *Larger Achievable Region 1* in blue. As we can see, there is still a gap between the lower and upper bounds, and additional research is needed to find the exact capacity region. We would like to mention here that using a coding scheme similar to [63], we

would get an even larger achievable region than *Larger Achievable Region 2*.

5.4.3 The Capacity Region when $\alpha = 1$

Finally, it is worth noting that, similar to [17], the Gaussian Z-channel with $\alpha \leq 1$ has the same capacity region as the channel in Figure 5.3 where Z and Z_2 are zero-mean Gaussian random variables with variance $\frac{1}{\alpha} - 1$ and 1, respectively. Even though there are three messages in the Gaussian Z-channel, as compared to two messages in the Gaussian Z-interference channel, the proof in [17, Appendix A] still follows straightforwardly. Noting that the two channels have the same capacity is useful, since the capacity region of the channel in Figure 5.3 might be easier to determine in some cases. For example, for $\alpha = 1$, Y_1 and Y_2 are statistically equivalent, thus both receiver R1 and receiver R2 are able to decode all three messages, W_{11} , W_{12} and W_{22} , similar to a multiple access channel. Thus, the capacity region of the Gaussian Z-channel with $\alpha = 1$ is

$$R_{11} \leq \frac{1}{2} \log(1 + P_1) \quad (5.41)$$

$$R_{21} + R_{22} \leq \frac{1}{2} \log(1 + P_2) \quad (5.42)$$

$$R_{11} + R_{21} + R_{22} \leq \frac{1}{2} \log(1 + P_1 + P_2) \quad (5.43)$$

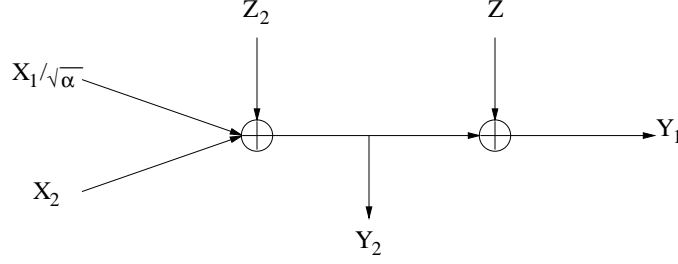


Figure 5.3: The equivalent channel.

5.5 Chapter Summary and Conclusions

In this chapter, we provide an achievable region for the recently proposed Gaussian Z-channel when $\alpha < 1$. We are able to prove most of the converse for this achievable region. We also provide an upper bound and two larger achievable regions to characterize the capacity region better. We determine the exact capacity region when $\alpha = 1$.

The results of this chapter have been published in [56].

5.6 Appendix

5.6.1 Proof of Theorem 5.1

For simplicity, we will not present probability of error calculations, but rather, we will describe a scheme the transmitters and receivers may use to achieve the region given in (5.9) to (5.12).

Fix a β between 0 and 1, it suffices to show that the two rate triplets: $(R_{11}, R_{21}, R_{22}) = (c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ and $(R_{11}, R_{21}, R_{22}) = (c_1(\beta) - c_{21}(\beta), c_{21}(\beta), c_{22}(\beta))$ are achievable. This is because, if these two triplets are achievable, then all other points

of the region can be achieved by the usual time-sharing technique.

First, we will show that $(R_{11}, R_{21}, R_{22}) = (c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ can be achieved. Transmitter T2 dedicates βP_2 power for transmitting message W_{22} using codebook C_{22} , and $(1 - \beta)P_2$ power for transmitting message W_{21} using codebook C_{21} . It transmits the sum of the two codewords. Transmitter T1 uses all its power P_1 for transmitting message W_{11} using codebook C_{11} .

Receiver R1 looks at codebook C_{21} only, treating everything else as noise, and therefore obtains a rate of

$$R_{21} = \frac{1}{2} \log \left(1 + \frac{(1 - \beta)P_2}{\frac{P_1}{\alpha} + \beta P_2 + \frac{1}{\alpha}} \right) = c_1(\beta) - c_{11}(\beta) \quad (5.44)$$

Then, it subtracts the effect of W_{21} off, looks at codebook C_{11} , treating everything else as noise, and obtains a rate of

$$R_{11} = \frac{1}{2} \log \left(1 + \frac{\frac{P_1}{\alpha}}{\beta P_2 + \frac{1}{\alpha}} \right) = c_{11}(\beta) \quad (5.45)$$

Together, this is a rate of $R_{11} + R_{21} = c_1(\beta)$.

Receiver R2, looks at codebook C_{21} only, treating everything else as noise, since

$$R_{21} = \frac{1}{2} \log \left(1 + \frac{(1 - \beta)P_2}{\frac{P_1}{\alpha} + \beta P_2 + \frac{1}{\alpha}} \right) \quad (5.46)$$

$$\leq \frac{1}{2} \log \left(1 + \frac{(1 - \beta)P_2}{\beta P_2 + 1} \right) \quad (5.47)$$

it can decode W_{21} without error. Subtracting the effect of W_{21} off, looking at codebook

C_{22} , receiver R2 gets a rate of

$$R_{22} = \frac{1}{2} \log(1 + \beta P_2) = c_{22}(\beta) \quad (5.48)$$

Thus, rate triplet $(c_{11}(\beta), c_1(\beta) - c_{11}(\beta), c_{22}(\beta))$ is achieved.

When both transmitters and receiver R2 operate in exactly the same way as explained above, and receiver R1 performs the successive decoding in the reverse order (i.e., it decodes W_{11} first and then W_{21}), the rate triplet $(R_{11}, R_{21}, R_{22}) = (c_1(\beta) - c_{21}(\beta), c_{21}(\beta), c_{22}(\beta))$ is achieved.

5.6.2 Proof of Theorem 5.2

We will prove this by using ideas similar to El Gamal's alternative proof [27] to Bergmans' proof [8].

Since there is no cooperation between the two receivers, the capacity region of this channel depends on the joint distribution $p(y_1, y_2 | x_1, x_2)$ only through the two marginals $p(y_1 | x_1, x_2)$ and $p(y_2 | x_1, x_2)$ [17]. Thus, we will concentrate on the following channel which will yield the same capacity region as our original channel (5.3)-(5.4),

$$Y_1 = \frac{X_1}{\sqrt{\alpha}} + Y_2 + \tilde{Z} \quad (5.49)$$

$$Y_2 = X_2 + Z_2 \quad (5.50)$$

where \tilde{Z} and Z_2 are Gaussian random variables with zero mean and variance $\frac{1}{\alpha} - 1$ and 1, respectively. Let rate triplets (R_{11}, R_{21}, R_{22}) be achievable. Then by Fano's

inequality [22], there exists an ϵ_n such that

$$H(W_{22}|Y_2^n) \leq n\epsilon_n \quad (5.51)$$

$$H(W_{21}, W_{11}|Y_1^n) \leq n\epsilon_n \quad (5.52)$$

and as $n \rightarrow \infty$, $\epsilon_n \rightarrow 0$.

We develop a series of bounds on R_{22} ,

$$nR_{22} = H(W_{22}) \quad (5.53)$$

$$= H(W_{22}|Y_2^n) + I(W_{22}; Y_2^n) \quad (5.54)$$

$$\leq H(W_{22}|Y_2^n) + I(W_{22}; Y_2^n|W_{21}) \quad (5.55)$$

$$= H(W_{22}|Y_2^n) + h(Y_2^n|W_{21}) - h(Y_2^n|W_{21}, W_{22}) \quad (5.56)$$

$$= H(W_{22}|Y_2^n) + h(Y_2^n|W_{21}) - h(Z_2^n) \quad (5.57)$$

$$\leq n\epsilon_n + h(Y_2^n|W_{21}) - \frac{n}{2} \log(2\pi e) \quad (5.58)$$

where (5.55) is obtained from (5.54) using the independence of messages W_{21} and W_{22} , (5.57) is obtained from (5.56) because we consider deterministic encoders, thus given W_{21} and W_{22} , we know X_2^n , and therefore the only remaining randomness is in Z_2^n . Finally, (5.58) follows from (5.51) and the fact that Z_2^n is an i.i.d. Gaussian sequence with unit variance.

Next, we develop a bound for R_{21} ,

$$nR_{21} = H(W_{21}) \quad (5.59)$$

$$= H(W_{21}|Y_1^n) + I(W_{21}; Y_1^n) \quad (5.60)$$

$$\leq H(W_{11}, W_{21}|Y_1^n) + I(W_{21}; Y_1^n|W_{11}) \quad (5.61)$$

$$= H(W_{11}, W_{21}|Y_1^n) + h(Y_1^n|W_{11}) - h(Y_1^n|W_{11}, W_{21}) \quad (5.62)$$

$$\leq n\epsilon_n + h(Y_1^n|W_{11}) - h(Y_1^n|W_{11}, W_{21}) \quad (5.63)$$

$$\leq n\epsilon_n + \frac{n}{2} \log(2\pi e) \left(P_2 + \frac{1}{\alpha} \right) - h(Y_1^n|W_{11}, W_{21}) \quad (5.64)$$

Finally, we develop a bound for $R_{11} + R_{21}$,

$$n(R_{11} + R_{21}) = H(W_{11}, W_{21}) \quad (5.65)$$

$$= H(W_{11}, W_{21}|Y_1^n) + I(W_{11}, W_{21}; Y_1^n) \quad (5.66)$$

$$\begin{aligned} &= H(W_{11}, W_{21}|Y_1^n) + h(Y_1^n) \\ &\quad - h(Y_1^n|W_{11}, W_{21}) \end{aligned} \quad (5.67)$$

$$\leq n\epsilon_n + h(Y_1^n) - h(Y_1^n|W_{11}, W_{21}) \quad (5.68)$$

$$\begin{aligned} &\leq n\epsilon_n + \frac{n}{2} \log(2\pi e) \left(\frac{P_1}{\alpha} + P_2 + \frac{1}{\alpha} \right) \\ &\quad - h(Y_1^n|W_{11}, W_{21}) \end{aligned} \quad (5.69)$$

where (5.64) and (5.69) follow from [26, Lemma 2].

Consider the following series of inequalities,

$$\frac{n}{2} \log(2\pi e) \left(\frac{1}{\alpha} \right) = h(Y_1^n | W_{11}, W_{21}, W_{22}) \quad (5.70)$$

$$\leq h(Y_1^n | W_{11}, W_{21}) \quad (5.71)$$

$$\leq h(Y_1^n | W_{11}) \quad (5.72)$$

$$\leq \frac{n}{2} \log(2\pi e) \left(P_2 + \frac{1}{\alpha} \right) \quad (5.73)$$

Thus, there exists a $\beta \in [0, 1]$, such that

$$h(Y_1^n | W_{11}, W_{21}) = \frac{n}{2} \log(2\pi e) \left(\beta P_2 + \frac{1}{\alpha} \right) \quad (5.74)$$

From (5.64), (5.69) and (5.74), we see that there exists a $\beta \in [0, 1]$ such that

$$nR_{21} \leq n\epsilon_n + nc_{21}(\beta) \quad (5.75)$$

$$n(R_{11} + R_{21}) \leq n\epsilon_n + nc_1(\beta) \quad (5.76)$$

Finally, for R_{22} , we argue as follows,

$$h(Y_1^n | W_{11}, W_{21}) = h\left(\frac{X_1^n}{\sqrt{\alpha}} + Y_2^n + \tilde{Z}^n | W_{11}, W_{21}\right) \quad (5.77)$$

$$= h(Y_2^n + \tilde{Z}^n | W_{11}, W_{21}) \quad (5.78)$$

$$= h(Y_2^n + \tilde{Z}^n | W_{21}) \quad (5.79)$$

where (5.78) follows because X_1^n is a deterministic function of W_{11} , and (5.79) follows

because Y_2^n and \tilde{Z}^n are independent of W_{11} .

Now, let us consider $h(Y_2^n + \tilde{Z}^n|W_{21})$. We know that

$$h(\tilde{Z}^n|W_{21}) = h(\tilde{Z}^n) = \frac{n}{2} \log(2\pi e) \left(\frac{1}{\alpha} - 1 \right) \quad (5.80)$$

Applying entropy power inequality [8, Lemma II], we have

$$h(Y_2^n + \tilde{Z}^n|W_{21}) \geq \frac{n}{2} \log(2\pi e) \left(\frac{e^{\frac{2}{n}h(Y_2^n|W_{21})}}{2\pi e} + \frac{1}{\alpha} - 1 \right) \quad (5.81)$$

Combining (5.81) with (5.74) and (5.79), we have

$$h(Y_2^n|W_{21}) \leq \frac{n}{2} \log(2\pi e)(\beta P_2 + 1) \quad (5.82)$$

Thus, from (5.58), we have

$$nR_{22} \leq n\epsilon_n + c_{22}(\beta) \quad (5.83)$$

Since $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, using (5.75), (5.76) and (5.83), we obtain the inequalities (5.13), (5.14) and (5.15), proving the theorem.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Wireless communications has gained great popularity over the past decades. The wireless medium has many unique characteristics, which create new challenges as well as new opportunities in the communication problem: interference, cooperation, correlation, diversity and feedback.

In this thesis we have addressed, from an information-theoretic point of view, some aspects of the fundamental issues arising in entirely wireless networks: correlation, cooperation and interference. The results in the thesis owe to the synthesis of several methods from information theory, estimation and detection theory, optimization theory, matrix analysis, probability and statistics. The main contributions of this thesis are as follows.

Capacity region and optimum power allocation for fading Gaussian multiple access channel with common data

Correlated data is an inherent part of wireless networks. Even in the simple multiple access channel, the optimum transmission of arbitrarily correlated data is an

extremely difficult and open problem. Thus, we investigated correlated data by considering a simplified model for the correlation, which is called common data. We first investigated the system where no fading is present and provided an explicit characterization for the capacity region and developed a simpler encoding/decoding scheme, that is specially tailored for the Gaussian channel. Next, we studied the system with fading, and obtained a characterization of the ergodic capacity region. We also characterized the optimum power allocation schemes that achieve the rate tuples on the boundary of the capacity region. In addition, we provided an iterative method for the numerical computation of the ergodic capacity region, and the optimum power control strategies.

This thesis provides a first look at the effect of fading on correlated data, and our results justify the intuition that the common message enjoys the beamforming gain, and is only transmitted when channels from both transmitters to the receiver are reasonably good. Furthermore, the received power of the common message comes from both transmitters. In fact, the amount of power each transmitter spends for the common message is proportional to its channel gain at that time instant.

Scaling laws for the Gaussian sensor networks and the order optimality of separation

In practical situations, correlated data manifests itself in more general forms. One practically interesting application is the sensor networks. The sensor network is a system where both correlation and cooperation play critical roles. This thesis studied the effects of correlation and cooperation in the many-to-one sensor network by characterizing the order optimal performance. Under some general conditions,

we determined an order-optimal achievability scheme, and identified the minimum achievable expected distortion at the collector node as a function of the number of nodes and the sum power constraint. Our order-optimal achievability scheme is separation-based. In multi-user information theory, generally speaking, separation principle does not hold. However, in our case, we found a scheme which is separation based, and is order-optimal.

The results of this work quantify the type of performance we may expect from sensor networks and provide guidelines for the design of sensor networks. The results also illustrate how we may exploit correlation in sensor data and cooperate among sensor nodes in an order-optimal fashion.

Capacity region of a class of discrete degraded interference channels

Interference is unavoidable in wireless networks with multiple source-destination pairs. Since all transmissions share the same wireless medium, the desired information co-exists with undesired information in the received signal. The capacity region of the interference channel is open except for some special cases. We provided sufficient conditions on degraded interference channels such that treating interference as noise is optimal. We provided a single-letter characterization for the capacity region of a class of degraded interference channels. The class includes the additive degraded interference channels studied by Benzel [5]. We showed that for the class of degraded interference channels studied, encoder cooperation does not increase the capacity region, and therefore, the capacity region of the class of degraded interference channels is the same as the capacity region of the corresponding degraded broadcast channel, which is known.

In this thesis, not only have we found the capacity region of a class of discrete degraded interference channels which was previously unknown, but we have also characterized conditions under which the optimal treatment of interference is treating it as noise. When looking for new achievability schemes that perform better than treating interference as noise, based on our results, one should focus on degraded interference channels that do not satisfy the conditions characterized in this thesis.

On the Gaussian Z-channel

Traditional interference channels are simple models for four isolated nodes; and the need to modify the interference channel, so that it represents a stage of a multi-hop wireless network, is clear. We followed the modified interference channel model proposed in [83], and studied the Gaussian Z-channel, when the cross-over link is weak. We derived an achievable region and showed that this region is almost equal to the capacity region by proving most of the converse. We also derived some additional lower and upper bounds on the capacity region.

This result improves our understanding of interference management in cases where each transmitter, in addition to the message intended for its own receiver, has messages for other receivers in the network.

6.2 Future Work

Despite the efforts made in this thesis, and recent progress made by many researchers in this field, the understanding of the fundamental performance limits of entirely wireless networks is far from satisfactory. There is much room for future work. We

list a few problems that the author would like to pursue in the future.

Gaussian sensor networks

We have characterized the order optimal performance when the eigenvalues of the underlying random process have a polynomial decrease rate and when the sum power constraint is not too small. In these cases, separation is order-optimal, i.e., it is order optimal to transmit in two stages, where in the first stage we compress the data to get rid of the correlation, and in the second stage we let sensors send the compressed and almost independent data using cooperation.

When one or both of these conditions are not satisfied, it is expected that non-separation based achievability schemes perform order better than separation-based schemes, i.e., compressing data first may be suboptimal, as correlation may facilitate cooperation in the second stage. It is left to future work that we investigate scenarios not covered in this thesis, and propose non-separation based schemes with better or even order optimal performance. The two-user non-cooperative multiple access channel with correlated Gaussian sources studied in [12, 46, 47] may be of help.

Correlation, cooperation and feedback

Through the study of the Gaussian sensor network in this thesis, we have realized that even though correlation, cooperation and feedback have mostly been studied separately, there are some connections between these three components. In a multiuser scenario, the main benefit brought by cooperation and feedback is the increase in correlation in the channel inputs. Therefore, it is preferable to understand these three phenomena within a unified framework. Interesting questions arise such as how much additional correlation can be obtained through cooperation and feedback; and what

amount of underlying correlation renders cooperation and/or feedback links useless.

As a part of our future work, we may conduct our study in the context of a two-user multiple access channel with cooperation, correlation and/or feedback. For example, we may study a multiple access channel with correlated data and feedback to understand how much more correlation can be gained from the feedback link. We may also study a multiple access channel with correlation and cooperation to understand the difference in performance between using correlation alone, cooperation alone and both correlation and cooperation. Eventually we may study a multiple access channel with all these three components and seek achievability and converse results in this general problem.

Interference management

The fundamental question on how to manage interference has been partially answered in this thesis, that is, we established the conditions on channels, under which the most efficient method of managing interference is to treat it as pure noise. Our future research will target the complete answer to this fundamental question. To this end, we will start by studying the simple degraded binary *non-symmetric* interference channel and investigate the structure of codes that enable better interference management than treating interference as pure noise. The results obtained may enrich the methods of interference management in wireless networks.

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