ABSTRACT

Title of dissertation: ON STRING COMPACTIFICATIONS TO TWO (AND THREE) DIMENSIONS: CRITICAL AND NONCRITICAL

Ram Sriharsha
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Dissertation directed by: Professor Sylvester James Gates Jr.
Department of Physics

This thesis is devoted to certain aspects of string compactifications to two and three dimensions. We construct vacua that describe the propagation of noncritical strings on a singular Spin(7) space. We also examine compactifications which preserve six supercharges, and obtain the classical and quantum moduli space of such theories. We also consider compactifications on $\mathbb{T}^8/\mathbb{Z}_2$ and point out the relevance of these flux compactifications to understanding the mathematical result of Guan and Beauville on the Hodge numbers of hyper-Kähler four-folds.

ON STRING COMPACTIFICATIONS TO TWO (AND THREE) DIMENSIONS: CRITICAL AND NONCRITICAL

by

Ram Sriharsha

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Advisory Committee:
Professor Sylvester James Gates Jr., Chair/Advisor
Professor Markus A. Luty
Professor Theodore Jacobson
Professor Rabindranath Mohapatra
Professor Niranjan Ramachandran

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CONTENTS

1. INTRODUCTION

One of the fundamental problems of theoretical physics is that gravity is perturbatively non-renormalizable, and the quantization of gravity seems to be sensible only as an effective field theory. It is logically possible that the Einstein-Hilbert action is non-perturbatively finite (or renormalizable at least) even though it is not renormalizable in perturbation theory, but there is no evidence yet to back up this speculation. Having noted that quantum mechanics and Einstein's general relativity are fundamentally incompatible, where does string theory fit in?

String theory is based upon the idea that the elementary excitations are modes of closed strings, rather than point like objects that are collective excitations of a local quantum field theory. This effort to introduce structure to point-like objects turns out to have dramatic implications. String theory contains gravity, and indeed a massless spin-2 excitation is automatically present in such a theory. Furthermore, the theory is perturbatively finite, and is the first example of a perturbatively finite theory of gravity.

Scattering amplitudes in string theory are UV finite essentially due to the extended nature of strings, which smears out interactions in a manner still consistent with Lorentz invariance. Thus, in one fell swoop the problem of non-renormalizability of gravity is solved, making string theory the leading candidate for the quantum theory of gravity.

Closed string theory is still best understood only in the first quantized form, as a theory of maps of cylindrical worldsheets on to a target background space-time. This is analogous to thinking of quantum field theory as the theory of maps of the worldline of a point particle into space-time. In the case of string theory, due to the extended nature of strings, only splitting and joining interactions are allowed, and further more we can ask only on-shell questions. Probes external to the theory are not allowed. All this leads to a very rigid structure that is part of the appeal of string theory.

As a quantum theory of gravity, string theory becomes the right arena in which to seek solutions to some of the problems of semi-classical gravity. Indeed, string theory has been very successful in accounting for the microstates of extremal and near extremal black holes [40]. String theory may lead to a unitary description of black hole evaporation, allowing us to resolve the black hole information paradox. Perhaps the most important result to emerge from string theory so far is the realization that non-abelian gauge theories have a dual description in terms of a quantum theory of gravity [39], in the sense made precise by the AdS/CFT correspondence and its generalization to non conformal field theories.

String theory has also been a fertile ground for exploration of deep mathematics. The use of topological string theory to count curves in certain Calabi-Yau three-folds is just one example of the impact of string theory on mathematics [5]. The partition function of topological strings is a generating function for curve counting in the context of Calabi-Yau three-folds and Mirror symmetry allows a very efficient computation of these invariants.

It is fair to say that string theory has had a great impact on our understanding of particle physics, gravity, and modern mathematics. There is however still a long way to go before string theory can be connected up with the real world. Indeed, for all its success so far, string theory is still in its infancy. Given that much of what we know about string theory is based on perturbation theory, we discuss some of the issues we face in applying perturbative string theory to describe our world.

One of the features of string theory is that the theory has to live in ten (or eleven) space-time dimensions, in order for gravitational physics to emerge. Since our universe is four dimensional, how can this be compatible with observations? It has been well known since Kaluza and Klein that four dimensional physics can emerge from a higher dimensional theory upon compactification. That is, if the internal space is curled up (or compact) then the resulting four-dimensional theory has a finite collection of massless modes and an infinite tower of massive states, whose mass is related inversely to the compactification radius R. In this scenario, the massive states cannot be excited unless an energy of order $\frac{1}{R}$ is supplied, and in string theory one takes R of order Planck scale, which means that at all accessible energies the massive modes can be taken to be in their ground states, so the theory approximates a four-dimensional theory rather well.

Another feature of string theory (at least in critical dimension) is that the target space-time is supersymmetric. Indeed, perturbative string theory requires space-time supersymmetry in order to be tachyon free[1].

[1] To be precise, it is in perturbative critical string theory in NS-NS background that we require

One route to compactification is to start with critical string theory in ten dimensions and compactify to lower dimensions, preserving a fraction of the supersymmetry. Even before we address the issue of breaking supersymmetry, we have to contend with the fact that there is a large moduli space of vacua. In other words, there is a large number of massless scalars that arise by dimensional reduction, the number of such scalars being related to the betti numbers of the internal manifold.

Since the scalars couple with gravitational strength, these light scalars would have been ruled out by observations. A central issue is then how to lift the moduli, that is provide a mass for the moduli fields in such a manner as to render the resulting theory phenomenologically viable. In string theory, this is usually done by turning on fluxes. Fluxes typically provide a potential for moduli and lift some (or most) moduli. The remaining moduli are typically lifted by non-perturbative effects that are calculable. In doing so, one ends up with an embarrassment of riches. There is not one, but several such discrete vacua forming a so called landscape (see [41] for a review of flux compactifications and the landscape problem of string theory). The presence of this discrete set of vacua leads to a lack of predictability.

This issue exists even for non-supersymmetric string vacua, so in order to obtain predictions about our world from string theory, we need to confront the landscape problem.

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space-time supersymmetry to eliminate the tachyon. In non-critical string theory, the situation is not too different, and it turns out we require asymptotic supersymmetry in the sense of Kutasov and Seiberg [4]. Space-time supersymmetry is however not necessarily a requirement for string theory in a Ramond-Ramond background.

Given the lack of understanding of non-perturbative string theory, we are forced to use effective field theory arguments in trying to understand compactifications with fluxes. That is, we typically start with Calabi-Yau compactifications (which are known to be good string vacua, at least perturbatively, and in some cases non-perturbatively) and then break supersymmetry partially or completely by turning on fluxes. In the supergravity approximation, we can check the stability of such flux compactifications, and if these examples are consistent with all that we know about string theory so far, it gives us confidence to believe in the existence of such vacua.

It is however important to note that the only real consistency conditions we have at our disposal (for string theory in flux backgrounds) are the constraints of low energy effective field theory. It is therefore important to know if there are perhaps hidden constraints in a theory of gravity that may not be evident in effective field theory. Alternatively, we need to understand the non-perturbative dynamics of flux compactifications better before we can evaluate the landscape problem.

Can there be fundamental constraints in a theory of gravity, that are missed entirely by effective field theory as we know it?

There is a reason to entertain this thought, as we illustrate using an example [25]. Consider $\mathcal{N} = 4$ supergravity coupled to matter in four dimensions. With $\mathcal{N} = 4$ supersymmetry, the matter content is fixed to be in $\mathcal{N} = 4$ vector multiplets that contain 6 scalars, a gauge field and four Weyl fermions. Low energy supergravity allows any number n of such vector multiplets to be coupled to $\mathcal{N} = 4$ supergravity, but only $n \leq 22$ arise from string theory. These arise via compactification on $T^2 \times K3$

and freely acting orbifolds thereof.

Indeed there does not seem to be any way of obtaining $n > 22$. This seems to imply that only vacua with $n \leq 22$ can be embedded into a UV complete theory like string theory. If true, this strongly suggests that low energy effective field theory has to be supplanted with a heretofore unknown consistency condition. This difficulty arises specifically in theories with gravity, and it is easy enough to obtain $n > 22$ once we decouple gravity. The observation that there are very many more consistent looking effective field theories than the ones that can be embedded into string theory is due to Vafa [24].

With the above as a brief introduction to string theory and some current issues, we proceed to discuss broad outline of the thesis. The thesis naturally divides into two chapters which can be read independently, and are described in the two sections below.

1.1 Three dimensional compactifications of $\mathcal{M}$-theory

Some of the issues we have been discussing so far can be raised in the context of three dimensional supergravities also. There is no propagating graviton in three dimensions, so the issue of renormalizability of gravity is irrelevant. The landscape problem still remains, so does the issue of fixing moduli.

In the first part of the thesis, we focus essentially on three dimensional compactifications. We moreover restrict ourselves to supersymmetric compactifications. Supersymmetric theories are typically stable by virtue of the fact that the Hamiltonian is the square of the supercharge, ensuring energy positivity. Furthermore, supersymmetry allows us a better control of non-perturbative and strong coupling effects. We will try to understand the moduli space of supersymmetric compactifications with fluxes and the effect of fluxes on moduli in the context of $\mathcal{N} = 3$ supersymmetric compactifications. These compactifications preserve six supercharges in three dimensional sense and arise typically by choosing the internal manifold to be a hyper-Kähler four-fold. As $\mathcal{N} = 3$ supersymmetry is fairly exotic, we should first understand why such compactifications are interesting.

There are actually several distinct reasons to be interested in these compactifications.

Firstly, with $\mathcal{N} = 3$ it will turn out that we have enough control over perturbative and non-perturbative aspects of the compactification for us to study the moduli space of flux compactifications. It allows us to be able to precisely describe the vacua that arise by turning on fluxes, and it is possible to turn on fluxes and break $\mathcal{N} = 3$ supersymmetry to $\mathcal{N} = 2$ or even $\mathcal{N} = 1$. We will be able to provide a simple example of a flux compactification with four supercharges and all but one modulus fixed.

Secondly, study of $\mathcal{N} = 3$ supersymmetric string compactifications gives physical insight into the moduli space of hyper-Kähler four-folds, as these are the internal spaces in string theory that yield $\mathcal{N} = 3$ space-time supersymmetry in $3d$.

The theory of higher dimensional hyper-Kähler manifolds is still in its infancy. In particular, to date only two examples of compact hyper-Kähler four-folds are known: Beauville's example [13], and the Hilbert Scheme of two points on K3

(Hilb$^2(K3)$, or $K3^{[2]}$). In comparison, the theory of hyper-Kähler two-folds is very well understood, as there is only one such complex surface, up to diffeomorphisms and is called the K3 surface. The moduli space of K3 is remarkably well understood, and from a physical standpoint this simplicity is related to the string-string duality that relates type IIA compactifications on K3 surfaces and Heterotic compactifications on $\mathbb{T}^4$. Among the few results known for higher dimensional hyper-Kähler manifolds is that the Hodge numbers of four-folds are tightly constrained [15], [13]. This fact leads to the observation that the dimension of the moduli space of such compactifications is bounded and finite, as we show in the thesis.

In an effort to understand the origin of this bound from a physical perspective, we examine $\mathbb{T}^8/\mathbb{Z}_2$ compactifications of $\mathcal{M}$-theory. In the absence of G-flux, $\mathcal{M}$-theory on the orbifold $\mathbb{T}^8/\mathbb{Z}_2$ bears no relation to compactification on hyper-Kähler four-folds. However, in $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$ it turns out that in order to avoid a membrane anomaly, G-flux has to be turned on, and a naive compactification on the orbifold in the absence of such a flux is not a solution to $\mathcal{M}$-theory at the quantum level. It turns out that by switching on appropriate G-flux, we can solve the anomaly constraints, and preserve six supercharges. It would be very interesting to enumerate the various vacua that preserve $\mathcal{N} = 3$ supersymmetry this way, for they are expected to be in the same moduli space as that of hyper-Kähler four-fold compactifications, and may provide a physical understanding of the bounds on Hodge numbers of hyper-Kähler four-folds derived in [15]. This is an issue that is currently under investigation [32]. We discuss compactification on $\mathbb{T}^8/\mathbb{Z}_2$ and set up the conditions for fluxes to preserve $\mathcal{N} = 3$ supersymmetry. The analysis of these conditions, and the comparison to the bounds on Hodge numbers of hyper-Kähler four-folds will be performed in [32].

In his analysis of hyper-Kähler four-folds, Verbitsky [6] found an action of $SO(4, b_2 - 2)$ on the total cohomology $H^*(X; \mathbb{Z})$. Using string theory we provide a simple reason for this observation: the T-duality group acting on the moduli space of hyper-Kähler four-folds is $so(4, b_2 - 2)$, and T-duality acts on D-branes in string theory and in particular acts on $H^*(X; \mathbb{Z})$ which is the D-brane charge lattice.

Viewed from the standpoint of a low energy supergravity theory, compactifications of $\mathcal{M}$-theory on hyper-Kähler four-folds lead to a $\mathcal{N} = 3$ supergravity coupled to matter. The matter content is in the form of vector and hyper-multiplets. From this point of view alone, there is no obstruction to having an arbitrary number of matter multiplets. However, the above results strongly indicate that the matter content of string theories that yield $\mathcal{N} = 3$ supersymmetry in three dimensions is rather limited. This observation is nothing but the swampland interpretation suggested by Vafa. There seem to be very many consistent looking low energy Lagrangians of which only a few are realized in string theory. If this disparity can be understood better, it may help us understand the additional consistency conditions that a low energy theorist should impose in order to obtain a consistent quantum theory of gravity.

Finally, we point out a very interesting subtlety associated with compactifications on $\mathrm{Hilb}^2(K3)$. One can think of $\mathrm{Hilb}^2(K3)$ as the smooth resolution of the diagonal of the symmetric product $S^2(K3)$. The blow up replaces the singularity by an exceptional divisor which is a $\mathbb{P}^1$. Classically, one can start at any point on the moduli space of the Hilbert scheme and by tuning moduli reach the symmetric product point in moduli space. In the quantum theory G-flux has to be turned on, to prevent a membrane anomaly. It is possible to turn on $\mathcal{N} = 3$ supersymmetry preserving flux and solve the membrane anomaly constraint. It also seems that one can still tune moduli and reach the symmetric product point preserving $\mathcal{N} = 3$ supersymmetry. This however cannot occur, for as shown by Sethi, Rajesh and Dasgupta there is no way of turning on G-flux consistent with supersymmetry for $S^2(K3)$. We argue that the resolution of this puzzle lies in the quantum corrections to the moduli space in the presence of fluxes.

The analysis of the moduli space presented in the thesis is a weak coupling analysis. There will be important corrections to the moduli space, arising from non-perturbative effects, some of which arise via instantons obtained by wrapping M5-branes on divisors inside hyper-Kähler four-folds. We will show that such instantons do not contribute to the Kähler potential of the $\mathcal{N} = 3$ effective action.

Chapter two covers the first part of the thesis. In section 1 we collect some information on the mathematical aspects of hyper-Kähler four-folds. In sections 2 through 4 we describe compactifications of type IIA/B and $\mathcal{M}$-theory on hyper-Kähler four-folds at tree level, and obtain the classical moduli space. T-duality of the worldsheet conformal field theory is used in section 3 to provide a physical proof of Verbitsky's result on the action of $SO(4, b_2 - 2)$ on $H^*(X; \mathbb{Z})$. In section 6, we obtain the quantum moduli space in the $\mathcal{M}$-theory language. Here we show that there are vacua which preserve $\mathcal{N} = 2$ supersymmetry and give an example of such a vacuum with all but one modulus fixed. We also obtain vacua with $\mathcal{N} = 3$

supersymmetry and point out the subtlety involving $S^2(K3)$.

In section 8 we examine $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$ and show that for certain non-generic G-flux we obtain vacua with $\mathcal{N} = 3$ supersymmetry. We comment on the relevance of this observation to providing a physical proof of a result of Beauville [13] and Guan [15] on the bound on Hodge numbers of hyper-Kähler four-folds. Section 9 synthesizes the results of chapter 2.

1.2 Non-critical strings and Spin(7) compactifications

As we saw so far, compactifications on hyper-Kähler four-folds yields a theory with $\mathcal{N} = 3$ supersymmetry in three dimensions, or a theory with $\mathcal{N} = (3, 3)$ supersymmetry in two dimensions. The smallest amount of supersymmetry in two dimensions is $\mathcal{N} = (1, 1)$ and it is achieved by compactification on a Spin(7) holonomy manifold. These compactifications are the subject of the rest of the thesis. In the large radius limit, it was shown by Shatashvili and Vafa [8] that the worldsheet super conformal field theory (SCFT) governing string propagation on a compact Spin(7) manifold possessed an enhanced super conformal algebra (SCA) denoted in literature as $\mathcal{SW}(\frac{3}{2}, 2)$. Using techniques of CFT, we show that this algebra is present in Spin 7 compactifications even away from the large radius limit.

In our discussion so far, we have focused on critical string theory, which lives in ten dimensions and has a graviton in the spectrum of excitations. It turns out that there are consistent string theories that live in less than ten dimensions. These string theories do not have a massless graviton.

In an important paper, Kutasov and Seiberg [27] constructed tachyon free, space-time supersymmetric string theories that intrinsically live in even dimensions less that eight.

At this point it is natural to ask what if any, is the relationship between these non-critical string theories and the critical string theories that live in ten dimensions. It turns out that non-critical superstring theories arise in the moduli space of critical superstring compactifications, and they are effective descriptions of physics near singularities of Calabi-Yau space-times [9].

In the latter part of the thesis, we construct singular CFTs for degenerations of Spin 7 manifolds, generalizing the construction of Kutasov and Seiberg. Upon adding fundamental strings, all these vacua are connected to AdS_3 vacua with $\mathcal{N} = 1$ supersymmetry (SUSY), thus allowing us to classify AdS_3 vacua with $\mathcal{N} = 1$ SUSY. This completes the classification of AdS_3 vacua with NS-flux, the other cases being covered by the analysis of Giveon and Pakman [49].

Chapter 3 presents our results on Spin(7) compactifications, with section 1 extending the derivation of Shatashvili and Vafa to all orders in α' and section 3 derives the worldsheet SCFT governing singular Spin(7) compactifications. Section 2 is predominantly a review of the connection between non-critical strings and singularities of Calabi-Yau manifolds, while section 4 ends with conclusions and overlook.

2. HYPER-KÄHLER FOUR-FOLDS IN STRING AND $\mathcal{M}$-THEORY

In this chapter we determine the classical, and quantum corrected moduli space of hyper-Kähler four-fold compactifications in string and $\mathcal{M}$-theory. After collecting some information on the cohomology of hyper-Kähler four-folds in section 1, we proceed to perform the dimensional reduction in sections 2 to 4 retaining only the effects at tree level in string theory. In section 6 we examine how the results are modified by 1-loop corrections which typically require a non-vanishing background flux to solve the loop corrected equations of motion. This leads to a modification of the classical moduli space that is tractable, and its effects are captured in section 6. In section 8 we turn to compactification of $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$ which superficially differs from hyper-Kähler four-fold compactifications. It is however shown that upon turning on fluxes, both the orbifold as well as the compact hyper-Kähler four-folds belong to the same moduli space. This allows us to obtain an independent constraint on the dimension of the moduli space of $\mathcal{N} = 3$ supersymmetric compactifications in three dimensions, and may provide physical insight into an elegant result of Guan and Beauville.

In section 9, we close this chapter with a discussion of the swampland scenario of Vafa in the context of theories with six supercharges. Most of the results in this chapter are published in [33]. Compactifications on $\mathbb{T}^8/\mathbb{Z}_2$ and the relation to hyper-kähler four-folds is treated in the forthcoming preprint [32].

2.1 Some mathematical facts on hyper-Kähler four-folds

A hyper-Kähler 4-fold is a Kähler manifold with a nowhere vanishing non degenerate holomorphic 2-form ω. Then ω^2 trivializes the canonical line bundle, so by Yau's proof of Calabi conjecture, there is a unique Ricci-flat metric that respects the hyper-Kähler structure. The cohomology of a general Kähler manifold can be decomposed via Hodge decomposition. For a hyper-Kähler 4-fold, the non trivial Hodge numbers are $h^{1,1}, h^{2,1}, h^{3,1}$ and $h^{2,2}$. However, not all of them are independent. Given any type $(1,1)$-form we can create a $(3,1)$ form by wedging with ω, so that $h^{3,1} = h^{1,1}$. Also $h^{1,1} \geq 1$ as the space is Kähler, so we can write $h^{1,1} = 1 + p$ for some p in $\mathbb{Z}^+$. Furthermore, just as for a Calabi-Yau 4-fold, $h^{2,2}$ is not independent. The quickest way to note this is to consider the index of the Dolbeault operator $\bar{\partial}_{E_2}$ acting on the bundle E_2 of holomorphic type (2,0) forms[1]. This index is given by:

$$\mathrm{Ind}\bar{\partial}_{E_2} = \sum\nolimits_{q=0} (-1)^q h^{2,q} \tag{2.1}$$

However, the index also has a purely topological character, and can be expressed via the Atiyah-Singer Index theorem as:

$$\mathrm{Ind}(\bar{\partial}_{E_2}) = \int \mathrm{Todd}(X)\mathrm{Ch}(\Omega^{2,0}) \tag{2.2}$$

[1] Computation of characteristic classes and cohomology relations in this section are standard, a good introduction to which can be found in Bott and Tu [37].

Using the standard expression for the Todd genus and Chern character, we compute:

$$\mathrm{Ind}(\bar{\partial}_{E_2}) = \frac{1}{120} \int (3{c_2}^2 + 79c_4) \tag{2.3}$$

where we used the fact that $c_1 = 0$. Now, the Todd genus of a hyper-Kähler 4-fold is precisely 3, and this implies a relation between ${c_2}^2$ and c_4 (incidentally, $\int c_4$ is the arithmetic genus or Euler characteristic of the hyper-Kähler 4-fold X). Specifically:

$$\mathrm{Todd}(X) = \frac{1}{720}(3{c_2}^2 - c_4) \tag{2.4}$$

so that

$$\int_X {c_2}^2 = 720 + \frac{\chi}{3} \tag{2.5}$$

Using (2.5) in (2.2) we get a relation between the various Hodge numbers. Denoting $h^{2,1} = 2q$ [2] this relation is:

$$h^{2,2} = 72 + 8p - 4q \tag{2.6}$$

So the hyper-Kähler 4-folds are characterized by two non negative integers (p, q). As we will learn in later sections, not all values of p and q are allowed.

[2] Here we used the fact that b_3 is divisible by 4, for a hyper-Kähler four-fold. Incidentally, this also implies χ is divisible by 12, which is a stronger result than the one for Calabi-Yau four-folds. The Hilbert scheme of two points on K3 gives us an example where χ is divisible by 12, and not by 24, so this is the strongest result we can get. In our notation $\frac{\chi}{24} = \frac{1}{2}(7 + p - q)$.

2.2 Compactification of type IIA on hyper-Kähler four-folds

In this section we will describe the compactification of type IIA string theory on a hyper-Kähler four-fold X. In the large volume limit these compactifications can be discussed by dimensionally reducing type IIA supergravity on hyper-Kähler four-folds.

The bosonic content of type IIA supergravity in ten dimensions is the metric g_{MN}, an antisymmetric two-form B_{MN} and dilaton ϕ from the NS-NS sector. The R-R sector gives rise to the one-form gauge field A_M and three form C_{MNP}. The bosonic action in string frame is of the form:

$$L = \int d^{10}x\sqrt{-g}[e^{-2\phi}(R^{10} + 4(\nabla\phi)^2 - \frac{1}{12}H^2) - \frac{1}{4}F^2 - \frac{1}{48}G^2] \tag{2.7}$$

Where:

$$F = dA \quad H = dB \quad G = dC + A \wedge H \tag{2.8}$$

are the gauge invariant field strengths. The action (2.7) is of course the tree level action for type IIA string theory in ten dimensions. There are higher order terms in the effective action that are not captured in (2.7). For the most part their structure is not known. There is however an important term of the form $B \wedge X_8$ where X_8 is a particular contraction of four powers of the Riemann tensor. This term was shown to be present in type IIA by considering scattering amplitudes in type II string theory [1]. This term leads to a tadpole for the B-field which has to be cancelled in type IIA by turning on G-flux and/ or adding N F1-strings such that:

$$N = \frac{\chi}{24} - \frac{1}{2(2\pi)^2}\int_X G \wedge G \tag{2.9}$$

If the Euler number of X is not divisible by 24, then the tadpole cannot be canceled by simply adding F-strings and we must turn on RR-flux G also. Of course, turning on G-flux we will typically end up breaking supersymmetry unless the G-flux happens to be primitive with respect to the $\mathbb{P}^1$ of complex structures on X. For the moment we will ignore these subtleties and address them in section 3. The action for type IIA supergravity is invariant with respect to 32 supercharges, 16 of which are left-chiral and 16 right-chiral with respect to the chirality operator in 10d. Upon compactifying on X, the resulting action in two dimensions possesses residual supersymmetry only if X admits a covariantly constant spinor. In the case of hyper-Kähler four-folds the holonomy group of X is $sp(2)$. A generic eight dimensional spinor is in one of the two inequivalent spinor representations of spin(8) say $\mathbf{8}_+$. Under $sp(2)$ we have the decomposition:

$$\mathbf{8}_+ = \mathbf{5} + \mathbf{1} + \mathbf{1} + \mathbf{1} \quad \mathbf{8}_- = \mathbf{4} + \mathbf{4} \tag{2.10}$$

so that there is a three-dimensional space of covariantly constant spinors on X. Via the decomposition:

$$\mathbf{16} = (8_+, +) + (8_-, -), \quad \mathbf{16}' = (8_+, -) + (8_-, +), \tag{2.11}$$

corresponding to $SO(1,9) \to SO(8) \times SO(1,1)$ we end up with a non-chiral two dimensional supergravity theory[3] with $\mathcal{N} = (3,3)$ supersymmetry upon compactifying type IIA on X.

[3] In (2.11) the $\mathbf{16}$ and $\mathbf{16}'$ refer to the ten dimensional Majorana-Weyl spinors of opposite chirality associated to type IIA, whereas the spinor representations of $SO(1,1)$ are labeled by their charges under $spin(1,1)$.

To determine the spectrum of the resulting two dimensional theory one performs Kaluza-Klein reduction of the various fields of type IIA. As the resulting two dimensional theory is non-chiral the fermions simply arise as $\mathcal{N} = (3,3)$ superpartners and it is enough to count the massless bosonic degrees. These are associated to the harmonics of the various bosonic fields of type IIA. Denoting the holomorphic 2-form on X by ω, one can expand the B_{MN} zero modes as:

$$B = \sum_i b^i \omega_i^{1,1} + b\omega \tag{2.12}$$

where:

$$\omega^{1,1} \in H^{1,1}(X) \quad b \in \mathbf{C} \quad b^i \in \mathbf{R} \tag{2.13}$$

leading to $h^{1,1} + 2$ scalars. The C_{MNP} zero modes lead to $2h^{2,1}$ scalars and $h^{1,1} + 2$ vectors via:

$$C = \sum_j c^j \omega_j^{2,1} + \sum_n C_\mu{}^n \omega_n^{1,1} + C_\mu \omega \quad \omega^{2,1} \in H^{2,1}(X) \quad c^j \in \mathbf{C} \tag{2.14}$$

The metric deformations lead to $3h^{1,1} - 2$ scalars g^k as follows: The zero modes of the graviton satisfy the Lichnerowicz equation which in a suitable gauge can be written as:

$$D_k D^k h_{ij} - R_{isjt} h^{st} = 0 \tag{2.15}$$

It is easy to see that the metric variations of the form δh_{ab} and $\delta h_{a\bar{b}}$ do not mix in (2.15) so they can be considered separately. For every element $\omega^{1,1}$ one obtains a variation of the form $\delta h_{a\bar{b}}$ so that the number of such deformations is $h^{1,1}$. Similarly, given $\omega^{1,1} \in H^{1,1}(X)$ one can construct a variation of type δh_{ab} as:

$$\delta h_{ab} = \omega^{\bar{c}}{}_{(a} \omega^{1,1}_{b)\bar{c}} \tag{2.16}$$

However if $\omega^{1,1}$ is proportional to the Kähler form then (2.16) vanishes, so that there are only $2h^{1,1} - 2$ deformations of type δh_{ab} so that the space of sp(2) holonomy metrics on a hyper-Kähler four-fold has dimension $3h^{1,1} - 2$.

Collecting all the matter content together we end up with $h^{1,1} = (p+1)$ $\mathcal{N} = (4,4)$ vector multiplets containing g^k, b_i as the scalar components, together with q $\mathcal{N} = (4,4)$ hyper multiplets containing the $4q$ scalars c^j. Even though we have only $\mathcal{N} = (3,3)$ supersymmetry, the matter sector arranges itself into $\mathcal{N} = (4,4)$ multiplets, which is a familiar fact given that any supersymmetric sigma model with $\mathcal{N} = 3$ supersymmetry is automatically $\mathcal{N} = 4$ supersymmetric also. Of course the higher order terms in the effective action will only be $\mathcal{N} = (3,3)$ supersymmetric.

The supergravity sector contains the graviton, three abelian gauge fields and a scalar, along with three gravitini and three Majorana fermions. The dilaton sits in the supergravity multiplet.

The low energy effective action for the vector and hyper-multiplet moduli will in general be given by a $\mathcal{N} = (4,4)$ supersymmetric sigma model. In the case of the vector multiplets with rigid supersymmetry this sigma model is based on a target space that is hyper-Kähler with torsion (HKT), so we expect upon coupling to supergravity that the target space is quaternionic Kähler with torsion (QKT). The hyper multiplet moduli space is similarly a hyper-Kähler or Quaternionic Kähler manifold. As the two multiplets carry scalars with different R-symmetries the moduli space factorizes just as in $\mathcal{N} = 2$ supergravity coupled to matter in four dimensions.

Denoting the moduli space $\mathcal{M}$ of type IIA on a hyper-Kähler four-fold as:

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H \tag{2.17}$$

what can be said about $\mathcal{M}_V$ and $\mathcal{M}_H$?

The worldsheet description of any $\mathcal{N} = (3,3)$ supersymmetric compactification to two dimensions is in the form of a $\mathcal{N} = 4$ SCFT with small $\mathcal{N} = 4$ SCA and $c = 12$. The space-time moduli that sit in the $(p+1)$ vector multiplets are all $\mathcal{N} = 4$ chiral primary operators of this internal $\mathcal{N} = 4$ SCA. Since any $\mathcal{N} = 4$ SCA has a $SU(2)_L \times SU(2)_R$ R-symmetry this implies[4] that the moduli space $\mathcal{M}_V$ has a $SO(4)$ isometry. It turns out due to a theorem of Berger and Simons (see [25] for a nice discussion on the Berger-Simons result) that the smooth manifolds with $SO(4)$ holonomy and dimension greater than 4 are only the symmetric spaces, the so called Grassmann manifolds. This leads us to identify:

$$\mathcal{M}_V = \frac{O(4, p+1)}{O(4) \times O(p+1)} \tag{2.18}$$

There is a natural $O(4, p+1; \mathbb{Z})$ symmetry of the moduli space which we can quotient by maintaining the Hausdorff property of the the resulting space. It is natural to conjecture that the U-duality group for this theory is $O(4, p+1; \mathbb{Z})$.

In type IIA the dilaton ϕ is in the $\mathcal{N} = (3,3)$ supergravity multiplet. This implies that the form of the moduli space is completely independent of string coupling $g_s = e^{\phi}$. For large g_s, type IIA goes over to 11d supergravity which is the low energy limit of $\mathcal{M}$-theory. This means the $\mathcal{M}$-theory moduli space is also given by

[4] Details of this standard argument are provided in Appendix B. This argument was first applied for determining the moduli space of $\mathcal{N} = 4$ SCFTs by Cecotti [38].

(2.18). By the same argument, the metric on the moduli space is independent of string coupling.

Given the moduli space of the form (2.18), we can take the large radius limit. The large radius limit can be determined by examining the Dynkin diagram of $O(4, p+1)$, and it turns out that the structure of the moduli space in the large radius limit is given by:

$$\mathcal{M} = \frac{O(3,p)}{O(3) \times O(p)} \times \mathbb{R}_+ \times \mathbb{R}^{p+3} \tag{2.19}$$

This is what we expect in the large radius limit. In this limit we expect the metric deformations to be characterized by the moduli space of $sp(2)$ holonomy metrics of fixed volume of a hyper-Kähler four-fold, which is the $O(3,p)$ factor, the $\mathbb{R}_+$ factor corresponds to the trivial radial mode. The $\mathbb{R}^{p+3}$ factor corresponds to the scalars arising from dimensional reduction of the NSNS 2-form. This provides a non-trivial consistency check.

The $\mathcal{N} = (3,3)$ supergravity coupled to matter has not been constructed in literature. There is however the case of $\mathcal{N} = (4,4)$ supergravity coupled to matter which has been analysed [2]. This theory has a gauged $SU(2) \in SO(4)$ R-symmetry and it has been shown that the target space parameterized by the scalars in this theory can be hyper-Kähler or Quaternionic Kähler. We expect a similar result to hold even in the case of $\mathcal{N} = (3,3)$ supergravity coupled to matter. That is, with $\mathcal{N} = (3,3)$ supersymmetry, the form of the moduli space remains non-trivial in general. This raises the puzzle as to how the CFT analysis was able to determine the local form of the moduli space. We will resolve this puzzle in the next section.

One subtlety that has to be pointed out is that there is a difference between the $K3$ case and the case of general hyper-Kähler manifolds which affects our understanding of the moduli space. For $K3$ surfaces the global Torelli theorem holds, so that the moduli space of complex structures is determined by the space of periods. It is the space of periods that the supergravity analysis is sensitive to, and so is the chiral primary ring of the $\mathcal{N} = (4, 4)$ worldsheet theory. It is not known whether a version of the global Torelli theorem holds for the higher dimensional cases. If it does not, then the choice of periods does not determine the complex structure fully. What will be lacking is some discrete data. It is known that all hyper-Kähler manifolds are deformations of a projective variety so they all have $\pi_1 = 0$. So it is not possible to have discrete torsion[3] in the worldsheet SCFT. I do not know what extra data the SCFT can have in this case that is not captured by the chiral primary ring. So the analysis of the moduli space in this thesis is carried out modulo the discrete ambiguity arising from lack of a global Torelli like theorem.

2.3 Compactification of type IIB on hyper-Kähler four-folds

The compactification of type IIB string theory on a hyper-Kähler four-fold X leads to a two dimensional $\mathcal{N} = (0, 6)$ supersymmetri theory in the non-compact directions. Its low energy limit is $\mathcal{N} = (0, 6)$ supergravity coupled to matter. In this section we determine the matter content of this theory and the moduli space. In the large volume liimit type IIB string theory in ten dimensions is well approximated by type IIB supergravity. The bosonic content of type IIB supergravity is the graviton

g_{MN}, the anti-symmetric two form B_{MN}, the dilaton ϕ, the RR axion C, along with the RR two form A_{MN} and the self-dual four-form G_{MNPQ}. Type IIB in ten dimensions has a $sl_2(\mathbb{Z})$ action where the two forms A, B form a doublet of $sl_2(\mathbb{Z})$ and the axio-dilaton can be combined as:

$$\tau = c + ie^{-\phi} \tag{2.20}$$

and transforms under $sl_2(\mathbf{Z})$ as:

$$\tau \rightarrow \frac{(a\tau + b)}{(c\tau + d)} \quad a, b, c, d \in \mathbf{Z} \quad ad - bc = 1 \tag{2.21}$$

As the five form field strength:

$$F = dG + \frac{3}{4} B \wedge dB \tag{2.22}$$

is self dual, there is no covariant action whose equation of motion yields the self-duality constraint. Agreeing to impose this constraint by hand, we can write down a lagrangian for type IIB supergravity. As in the type IIA case, we need to determine the massless spectrum of particles in the 2d theory. The NS-NS sector modes g, B and ϕ give rise to the same zero modes for both type IIA and IIB. So we end up with $4h^{1,1} + 1$ scalars from the NS-NS sector.

In type IIA we argued that the dilaton went into the supergravity multiplet. In type IIB it is the fluctuation of the radial mode of the metric that goes into the $\mathcal{N} = (0, 6)$ supergravity multiplet.

From the RR sector, the axion gives rise to a real scalar. The RR two form A gives rise to $h^{1,1} + 2$ scalars exactly as the B-field. The expansion of the self-dual five-form F is more complicated. It can be expanded as follows:

$$F = \sum dC^i {\omega^4}_{-i} + \sum dC'^j {\omega^4}_{+j} \tag{2.23}$$

where ω^4_- refers to the space of anti self-dual four-forms on X, while ω^4_+ refers to the space of self-dual four-forms. The self-duality of F implies that C^i are anti self-dual scalars, while C'^j are self-dual. This means the scalars C^i are left-moving while the scalars C'^j are right moving. In this notation the supersymmetries of the 2d theory are purely right-moving.

Therefore, in the purely right-moving matter sector we have $b^4_- + 5h^{1,1} + 3$ scalars.

The middle dimensional cohomology of X decomposes into self-dual and anti self-dual pieces by Poincare duality. The signature σ of X is nothing but $\sigma = b^4_+ - b^4_-$. The Hirzebruch signature theorem relates σ to the Euler character of X:

$$\sigma = \frac{1}{45}\int (7p_2 - p_1^2) = 48 + \frac{\chi}{3} \tag{2.24}$$

Furthermore, the Euler formula gives:

$$2(b^0 + b^2 + b^3) + b^4_+ + b^4_- = \chi \tag{2.25}$$

Using (2.24) and (2.25) together with the relation $\chi = 12(7 + p - q)$ we can easily determine:

$$b^4_- = 3(h^{1,1} - 1) = 3p \tag{2.26}$$

That is, we end up with $n = 8h^{1,1} = 8(p+1)$ right moving scalars which by $\mathcal{N} = (0, 6)$ supersymmetry have $8(p+1)$ right moving Majorana fermions as superpartners. Again this is consistent with the fact that in the rigid supersymmetry limit the dimension of the target space of the right-moving moduli must be a multiple of 8.

Of course, to complete the spectrum we need to compute the left moving fields as well, but since they will play no part in the rest of the discussion we will not explicitly count the left-movers. Suffice it to say that they ensure that the resulting two dimensional theory is free from gravitational anomalies.

The $\mathcal{N} = (0, 6)$ supergravity coupled to $8n$ matter multiplets has not been constructed in literature. The important point about this theory is that the target space for the scalars is completely fixed, even though the theory has only six supercharges, it behaves more like the case of $\mathcal{N} = 4$ supergravity with 16 supercharges in four dimensions. There is a simple argument to see why the target space for the right-moving moduli is fixed by n, in the case of $\mathcal{N} = (0, 6)$ supergravity . It starts out with the observation that in the case of rigid supersymmetry, any sigma model with $\mathcal{N} = (0, 6)$ supersymmetry is based on a flat target space (up to orbifolding by a discrete group). The reason for this is simple: with $\mathcal{N} = (0, 6)$ supersymmetry and beyond, the only super multiplet possible with this much supersymmetry has scalars transforming non trivially under the R-symmetry that rotates the supercharges. In the $\mathcal{N} = (0, 8)$ case for example the scalars form a $\mathbf{8}_v$ of the spin(8) R-symmetry, whereas in the $\mathcal{N} = (0, 6)$ case the scalars form a $\mathbf{4}$ of $SU(4) \sim SO(6)$. Every such sigma model if it were non trivial would give rise to a conformally invariant theory in the IR, with $\mathcal{N} = 6$ SCA and above. However there is no superconformal extension of the $\mathcal{N} = 6$ supersymmetry algebra. This means the IR theory must be scale invariant without being conformally invariant and therefore every such sigma model should actually correspond to a free theory[5].

Indeed in the $\mathcal{N} = (0,6)$ case the scalars transform in the $\mathbf{4}$ of $SU(4)$ so there are actually $8n$ scalar fields rather than a multiple of 6 which would have required the scalars to transform in the fundamental of $SO(6)$. As we have argued above, in the case of rigid $\mathcal{N} = (0,6)$ supersymmetry the target space parameterized by the right-moving scalars is actually flat and is simply $\mathbb{R}^{8n}$ locally (in our case $n = p+1$). This means that any non-trivial moduli space arises for these scalars precisely by coupling to $\mathcal{N} = (0,6)$ supergravity. Upon coupling to $\mathcal{N} = (0,6)$ supergravity there is a mass parameter κ that essentially plays the role of the gravitational Newton's constant in 4d (the 2d gravitational coupling is dimensionless). When $\kappa \to 0$ the target space becomes flat $\mathbb{R}^{8n}$ and for non-zero κ the target space for the right-moving moduli must have a curvature proportional to κ. All of this is analogous to what happens for $\mathcal{N} = 2$ supergravity coupled to matter in four dimensions. In this case in the rigid supersymmetry limit the target space for the scalars must be hyper-Kähler, whereas local supersymmetry requires the target space to be quaternionic Kähler with negative curvature proportional to the 4d Newton's constant. The only difference is that for $\mathcal{N} = (0,6)$ supergravity in 2d, the rigid supersymmetry limit is trivial and this we expect will put severe constraints on the moduli space arising out of local supersymmetry. In particular, this moduli space can be exactly determined. The actual construction of $\mathcal{N} = (0,6)$ supergravity coupled to matter

[5] In the non compact case one can have scale invariance without conformal invariance essentially by turning on the dilaton, but this is not possible in the compact case.

will be explored in a forthcoming paper.

This allows us to resolve the puzzle raised at the end of the previous section. Even though supergravity does not drastically constrain the moduli space of type IIA compactifications to three dimensions (with six supercharges), it turns out that supergravity does constrain the moduli space of type IIB compactification. Since type IIA and IIB are related to each other upon compactifying one dimension further (and T-dualizing) this provides us with an understanding of why the type II moduli space for hyper-Kähler four-fold compactifications can be determined locally.

We claim that the type IIB moduli space $\mathcal{M}$ is given by:

$$\mathcal{M} = \mathrm{T}^* \frac{O(4, p+1)}{O(4) \times O(p+1)} \tag{2.27}$$

That is locally $\mathcal{M}$ is a bundle over the Grassmannian with fibers $\mathbb{R}^{4p+4}$. Globally of course, the fibers are compact, and are actually tori (which is what makes the moduli space compact, assuming the T-duality group acts on the base). The form of the moduli space we expect from the large radius limit is:

$$\mathcal{M} = \mathbb{R}^{5p+6} \times \frac{SL_2(\mathbb{R})}{U(1)} \times \frac{O(3,p)}{O(3) \times O(p)} \tag{2.28}$$

We have been schematic in writing (2.28) and it should be thought of as a warped product of the individual factors. Since the radial mode goes into the $\mathcal{N} = (0, 6)$ supergravity multiplet, we expect the large radius limit to be exact.

The moduli space at weak coupling (the CFT moduli space) is of the form:

$$\mathcal{M} = \mathbb{R}^{4p+4} \times \frac{O(4, p+1)}{O(4) \times O(p+1)} \tag{2.29}$$

which agrees with the topology of (2.27). One way to show (2.27) is to simply dimensionally reduce type IIB on hyper-Kähler four-folds in a manner similar to and observe that the moduli space is of the form of a cotangent bundle over the space parameterized by the metric and B-field deformations. The cotangent bundle structure follows exactly as in the analysis of Gates, Gukov and Witten . This structure of course arises rather straightforwardly upon dimensional reduction, but our claim is that this form of the moduli space is fixed by the $\mathcal{N} = (0, 6)$ supergravity of the 2d theory.

2.3.1 T-duality

The space (2.27) has a base which can be thought of as the space of space-like four planes in $\mathbb{R}^{4,p+1}$. The group $O(4, p + 1)$ naturally acts on $\mathbb{R}^{4,p+1}$ into which we can embed an integral lattice $\Lambda^{4,p+1}$. The subgroup $O(3, p)$ of $O(4, p + 1)$ is the rotation of the integral lattice Λ^{3,b_2-3} of H^2. It was pointed out by Verbitsky [6] that there is a group action on the integral cohomology lattice of any hyper-Kähler $2n$-fold of the form $SO(4, b_2-2)$ which in particular holds for four-folds. The $O(4, p+1)$ factor can thus be identified with the symmetry of the integral cohomology lattice of the four-fold. This motivates the $O(4, p+1; \mathbf{Z})$ duality group of type IIA. In type IIB this has to be extended by the action of $sl_2(\mathbb{Z})$.

In the type II theories D-brane charges are vectors in the lattice $H^*(X; \mathbb{Z})$ and the action of $SO(4, b_2 - 2)$ that acts as an automorphism of this lattice rotates the D-brane charges the way T-duality is supposed to work. This leads us to suspect that $SO(4, p + 1; \mathbb{Z})$ is nothing but the T-duality group of the type IIA theory.

Indeed $SO(4,p+1;\mathbb{Z})$ is the T-duality group of the worldsheet SCFT corresponding to hyper-Kähler four folds. However as we will see soon some of these compactifications are destabilized by the 1-loop correction. In those cases the T-duality group may be strictly smaller. The fact that the classical T-duality group is $SO(4,p+1;\mathbb{Z})$ ties in neatly with the observation that $SO(4,p+1)$ acts on $H^*(X)$ via the fact that D-brane charges are Mukai vectors in the lattice $H^*(X;\mathbb{Z})$.

2.4 $\mathcal{M}$-theory on hyper-Kähler four-folds

The low energy limit of $\mathcal{M}$ theory is 11d supergravity whose bosonic content is a graviton and a 3-form potential A, with four-form flux G.

Dimensional reduction of 11d supergravity on a hyper-Kähler four-fold yields a three dimensional $\mathcal{N}=3$ supergravity coupled to matter. The matter multiplets are the vector multiplet (whose bosonic content is three scalars transforming as **3** of the SO(3) R-symmetry together a gauge field) and the hyper multiplet (which contains four scalars transforming as a complex doublet of the R-symmetry). Any action for the hyper multiplets is automatically $\mathcal{N}=4$ supersymmetric, so is the low energy effective action for the vector multiplets (in the absence of G-flux). Upon dimensional reduction, we end up with a $\mathcal{N}=3$ supergravity multiplet with a graviton, three gravitini. The matter sector consists $p+1$ vector multiplets (after dualizing some vectors into scalars) and q hyper multiplets. The moduli space factorizes as in the type IIA case. Upon dualizing the vectors into scalars, we expect the $\mathcal{M}$-theory moduli space to coincide with the type IIA case. The $\mathcal{M}$-

theory moduli space will be of the form:

$$\mathcal{M}_{11d} = \frac{O(4, p+1)}{O(4) \times O(p+1)} \otimes \mathcal{M}_H \tag{2.30}$$

2.4.1 Dimension of the $\mathcal{M}$-theory moduli space

The dimension of the moduli space of type IIA(or $\mathcal{M}$-theory) compactification on hyper-Kähler four-folds is set by two integers p and q. However it is easy to show that there is an upper bound on p for any hyper-Kähler four-fold.

In fact, Beauville [13] has shown that $b_2 \leq 23$ which implies $p \leq 20$. This restriction follows from the observation that $Sym^2(H^2) \hookrightarrow H^4$.

From this we note:

$$b_2(1 + b_2) \leq 2b_4 \tag{2.31}$$

Furthermore, by an Index theorem of Salamon [20]:

$$b_4 = -b_3 + 10b_2 + 46 \tag{2.32}$$

Using 2.31) and (2.32) we get:

$$(b_2 - 23)(b_2 + 4) \leq 0 \tag{2.33}$$

which implies $b_2 \leq 23$. In the case where $b_2 = 23$ the inclusion map $i : Sym^2(H^2) \hookrightarrow H^4$ is exact and gives the only non vanishing Hodge numbers leading to the Hodge diamond of the Hilbert scheme.

Rather non-trivially even the integer q is bounded from above for hyper-Kähler four-folds by a number that depends on p [15]. Indeed the analysis of [15] concludes that not every topological type of hyper-Kähler four-fold is possible. Either $b_2 = 23$ in which case the hyper-Kähler four-fold has the same Hodge diamond as $\mathrm{Hilb}^2(K3)$, or $b_2 \leq 8$. Furthermore, for all $b_2 \leq 8$, b_3 is bounded above by a number that depends on b_2.

It is clear that (2.31) and (2.32) also put a bound on q but the bounds derived in [15] are much stronger.

Including the moduli coming from the position of membranes, the entire moduli space is of bounded dimension. A similar situation arises for $\mathcal{N} = 4$ supersymmetric string compactifications in four dimensions as discussed in the introduction, but it is nice to see the moduli space of theories with six supercharges is bounded by a calculable finite number. We would like to find a more intuitive reason for these bounds. We will return to this question upon discussing flux compactifications on $\mathbb{T}^8/\mathbb{Z}_2$ in $\mathcal{M}$-theory.

2.5 Corrections in the absence of fluxes

In the next section we will generally conclude that all hyper-Kähler four-fold compactifications with $p - q$ even are unstable and G-flux has to be turned on to stabilize such compactifications. In the case where $p-q$ is odd, the compactification without flux is a stable solution at one-loop. In the absence of G-flux one can appeal to the standard RNS construction to note that all hyper-Kähler four-fold compactifications have $\mathcal{N} = 4$ supersymmetry on the worldsheet. This superconformal algebra cannot be broken by order α' or order g_s effects. That is, these compactifications are exact solutions to string theory. As the coset structure of the moduli space was deduced using only the $\mathcal{N} = 4$ SCA, this means that the moduli space is unaffected by higher order corrections.

2.6 The effect of fluxes

In this section, we return to the problem that we have briefly mentioned before, which is that compactifications on eight dimensional manifolds are typically destabilized at one-loop in string theory. In this section we study this in the context of $\mathcal{M}$-theory and find that one can turn on G-flux in $\mathcal{M}$-theory such that the 1-loop equations are satisfied, while at the same time preserving supersymmetry so that the compactifications remain stable solutions at one-loop. This however leads to a potential for moduli, lifting some moduli in a calculable manner.

2.6.1 Flux Quantization in $\mathcal{M}$-theory

$\mathcal{M}$-theory has a three form potential C with a four-form field strength G. The theory has membranes ($M2$-branes) that couple electrically to the flux G, and five-branes ($M5$-branes) that couple magnetically. By analogy with Dirac quantization, one might naively expect that G obeys the quantization condition:

$$\frac{G}{2\pi} \in H^4(X; \mathbb{Z}) \tag{2.34}$$

This is however incorrect in general, as pointed out by Witten [7]. Indeed, the shift in quantization condition of the $\mathcal{M}$-theory four-form is related to the K-theory classification of Ramond-Ramond flux. The correct quantization condition for G is given by [7]:

$$[\frac{G}{2\pi}] - \frac{1}{2}\lambda \in \mathbb{Z}, \tag{2.35}$$

where λ is one half of the first Pontryagin density:

$$\lambda = \frac{1}{2}p_1(X) \tag{2.36}$$

λ as defined in (2.36) is an integral class, if X is a complex manifold. It is however not necessarily an even class. If λ is even G obeys the Dirac quantization condition, whereas for λ not even, the quantization condition is modified.

There is another related condition in $\mathcal{M}$-theory called the tadpole cancelation condition. There is an important higher order correction to membrane charge, which acts as a tadpole for membrane charge in $\mathcal{M}$-theory. In order to cancel this tadpole, we need to solve:

$$d * G = \frac{1}{4\pi^2} G \wedge G - \frac{({p_1}^2 - 4p_2)}{192} \tag{2.37}$$

Integrating over the internal space, we obtain the constraint:

$$n + \frac{1}{2(2\pi)^2} \int_X G \wedge G = \frac{\chi}{24}, \tag{2.38}$$

where we used the relation:

$$(p_1^2 - 4p_2) = 8\chi, \tag{2.39}$$

that holds for any compact, complex eight dimensional manifold with $c_1 = 0$.

It is in the form (2.38) that the anomaly constraints are usually discussed in literature since most examples in literature are of complex manifolds with $c_1 = 0$. This will also be true in our case, as hyper-Kähler four-folds have $c_1 = 0$.

In the case where χ is divisible by 24 the anomaly constraints can be satisfied by adding $M2$-branes instead of turning on G-flux. Since the $M2$-branes can be freely placed at any point in the internal manifold, the effect of adding $M2$-branes is to increase the moduli space. However the moduli space cannot be increased arbitrarily, as $n < \frac{\chi}{24}$.

2.6.2 In the large radius limit

The shifted Dirac quantization condition for the G-flux in $\mathcal{M}$-theory forces introduction of flux backgrounds as long as λ is not even. This happens precisely when the Euler character χ is not divisible by 24. For hyper-Kähler four-folds this anomaly has values in $\mathbb{Z}_2$ and indeed when $p - q$ is even, we have to turn on G-flux to stabilize the background. Having worked hard to ensure the classical background was supersymmetric, we would like to be able to turn on G-flux in a manner compatible with supersymmetry, if this is possible. The conditions for turning on G-flux in a supersymmetric manner was first analyzed by Becker and Becker [11]. This is a supergravity analysis, and as such is valid only in the large radius limit. If we want to preserve at least $\mathcal{N} = 2$ supersymmetry, then the following equations should hold:

$$G \wedge J = 0, \quad G \wedge \Omega = 0, \quad G = *G \tag{2.40}$$

In (2.40), J is the Kähler form, and Ω is the holomorphic four-form of the Calabi-Yau four-fold. The interpretation of (2.41) is that turning on fluxes is equivalent to creating a potential for the moduli where the conditions (2.40) create a restriction on the Kähler and complex structure moduli of the Calabi-Yau four-fold. Indeed, as pointed out by [12], the conditions (2.40) follow from the following chiral and twisted-chiral superpotentials:

$$W = \int d^2\theta G \wedge J \wedge J, \quad \tilde{W} = \int d^2\tilde{\theta} G \wedge \Omega \tag{2.41}$$

For hyper-Kähler four-folds, $\Omega = \omega^2$, so the equations become:

$$G \wedge J = 0 \tag{2.42}$$

and

$$G \wedge \omega^2 = 0 \tag{2.43}$$

The condition (2.42) is the so called primitivity condition, which is essentially a restriction on the Kähler form J. Any G-flux satisfying (2.42) and (2.43) preserves four supercharges (or $\mathcal{N} = 2$ supersymmetry in the two dimensional sense). In order to preserve all six supercharges (or all of the original $\mathcal{N} = 3$ supersymmetry) we require:

$$G \wedge \omega = 0 \tag{2.44}$$

In order to explain this point better, we need to collect some relevant facts about Hyper-Kähler four-folds.

One definition of a hyper-Kähler manifold is that it is a compact, complex manifold with a triplet of complex structures J^A which satisfy the $SU(2)$ algebra:

$$J^A J^B = \epsilon^{ABC} J^C, \quad (J^A)^2 = -1, \quad A = 1, 2, 3 \tag{2.45}$$

Picking a complex structure J, the manifold is endowed with a Kähler form which we will also call J by abuse of notation. Then the remaining complex structures are of type $(2,0) \oplus (0,2)$ with respect to J and are denoted ω and $\bar{\omega}$ so that with respect to the complex structure J, a hyper-Kähler manifold has a nowhere vanishing holomorphic two-form ω. However, there is nothing special about the choice of complex structure. indeed, given J^A there is an entire $\mathbb{P}^1$ of complex structures possible for a hyper-Kähler manifold. Every J defined by :

$$J = aJ^1 + bJ^2 + CJ^3, \quad a^2 + b^2 + c^2 = 1, \tag{2.46}$$

is a point in the $\mathbb{P}^1$ of complex structures. Now suppose we turn on a G-flux that satisfies (2.43). Then the hyper-Kähler invariance requires (2.43) to hold upon rotating J into any of the $\mathbb{P}^1$ of complex structures so that $\mathcal{N} = 3$ supersymmetry is preserved. This is possible if and only if (2.44) is also satisfied.

We can recast these conditions into a form that will be better suited for analysis later. This requires us to introduce the lattice action on $H^2(X,\mathbb{Z})$ for an arbitrary hyper-Kähler manifold in analogy with the K3 surface. It is well known that the second cohomology of a K3 surface X can be thought of as an even integral, self-dual lattice. A K3 surface has $h^{1,1} = 20$ so $H^2(X,\mathbb{R})$ has dimension 22. Given elements α and β in $H^2(X,\mathbb{Z})$ we can define an inner product as follows:

$$\alpha.\beta = \int_X \alpha \wedge \beta, \quad \alpha, \beta \in H^2(X,\mathbb{Z}) \tag{2.47}$$

By virtue of Wu's formula[6], it follows that:

$$\alpha^2 = \int_X \alpha \wedge \alpha \in 2\mathbb{Z} \tag{2.48}$$

[6] Wu's formula [36] states $Q(x, w_2) = Q(x,x)\text{mod } 2$, where w_2 is the second Steiffel-Whitney

Furthermore, Poincare duality tells us that given $\alpha^i \in H^2(X,\mathbb{Z})$ we can find $\alpha^j* \in H^2(X,\mathbb{Z})$ such that:

$$\alpha^i.\alpha^j* = \delta^{ij} \tag{2.49}$$

so that $H^2(X,\mathbb{Z})$ has the structure of an abelian even, self-dual integral lattice Λ. The signature of Λ can be determined from the signature complex to be $\sigma = -16$ so that the lattice is of the form $\Lambda^{3,19}$. Classification of even self-dual lattices then tells us that $\Lambda^{3,19} = E_8 \oplus E_8 \oplus H^3$.

It turns out that this lattice notion generalizes to higher dimensional hyper-Kähler manifolds X, and the corresponding inner product is called the Beauville-Bogomolov form q_X.

The inner product q_X on $H^2(X,\mathbb{Z})$ was defined by Beauville [13] and the integrality of q_X was shown by Fujiki [14]. In the case of higher dimensional hyper-Kähler manifolds, the lattice Λ is not necessarily self-dual or even, though it turns out to be even for the two known cases of hyper-Kähler four-folds.

Now given a hyper-Kähler manifold X, the Beauville-Bogomolov form q_X has signature $3-p$, so that the lattice Λ is form the form $\Lambda^{3,p}$. A choice of complex structure for X is a choice of a space-like two plane $\mathcal{O}$ inside the lattice $\Lambda^{3,p}$. Indeed, given a holomorphic two-form ω, the Hodge-Riemann identities require:

$$\int_X \omega^2 = 0, \quad \int_X \omega \wedge \bar{\omega} > 0 \tag{2.50}$$

class of the complex surface and Q the intersection form. For a complex surface, w_2 is the mod 2 reduction of the first Chern class c_1 so w_2 vanishes for $K3$ leading to the observation that the intersection form on $K3$ is even.

Writing:

$$\omega = \alpha + i\beta, \quad \alpha, \beta \in H^2(X, \mathbb{Z}), \tag{2.51}$$

we can see that (2.50) implies:

$$q_X(\alpha, \alpha) = q_X(\beta, \beta) > 0, \quad q_X(\alpha, \beta) = 0 \tag{2.52}$$

That is, α and β span a space-like two-plane $\mathcal{O}$ in the lattice $\Lambda^{3,p}$.

For an abstract hyper-Kähler four-fold X, the G-flux is an element of $H^4(X, \mathbb{Z})$. We can split up $H^4(X, \mathbb{Z})$ into a part generated by $sym^2(H^2(X, \mathbb{Z}))$ and a part orthogonal to $sym^2(H^2(X, \mathbb{Z}))$. Let us focus on fluxes of the form:

$$G = \alpha \wedge \beta, \quad \alpha, \beta \in H^2(X, \mathbb{Z}) \tag{2.53}$$

The condition for such a G-flux to preserve $\mathcal{N} = 3$ supersymmetry is:

$$\alpha, \beta \perp \mathcal{O}, \quad \alpha \perp \beta \tag{2.54}$$

Before proceeding to analyze known examples of hyper-Kähler four-folds we should note the following puzzle. The conditions (2.42) and (2.43) remove at most three scalar moduli. However, compactifications that preserve $\mathcal{N} = 3$ supersymmetry must lift moduli in sets of four. This means there is an extra constraint on moduli that does not follow from (2.42) and (2.43) alone. This constraint is discussed below, as it is of interest even in other contexts of compactifications to three dimensions.

2.6.3 Extra constraints on moduli

In general, there are only three equations arising from the supergravity constraints (2.42) and (2.43). It appears therefore that only a triplet of moduli can be removed at each instance, whereas with $\mathcal{N} = 3$ supersymmetry we expect quaternionic dimensions to disappear. This means there is more moduli being removed than governed by (2.42) and (2.43). In the $\mathcal{M}$-theory setting this happens because certain modes of the 3-form C are constrained due to the Chern-Simons coupling $C \wedge G \wedge G$.

Indeed upon turning on G-flux this Chern-Simons coupling leads in a standard fashion to the $3d$ Chern-Simons action for the zero mode of C so that the G-flux appears to give topological mass to the vector field C_μ sitting in one of the p vector multiplets. Together with the mass terms for the triplet of scalars, this is enough to lift precisely one quaternionic dimension.

The 11d supergravity action is of the form:

$$S = \int d^{11}x\sqrt{-g}(R - \frac{1}{2}F \wedge *F) - \frac{1}{24\pi^2}C \wedge F \wedge F \tag{2.55}$$

Expanding the 3-form in harmonics:

$$C = \sum\nolimits_a \omega_a A^a_\mu + \sum\nolimits_i \omega_i^{2,1} A^i \quad \omega_a \in H^2(X; \mathbf{R}) \tag{2.56}$$

we end up with a 3d Chern-Simons action:

$$S = \frac{1}{4\pi}\int d^3x \lambda_{ab} A^a \wedge F^b, \quad \lambda_{ab} = \int_X \omega_a \wedge \omega_b \wedge \frac{G}{\pi} \tag{2.57}$$

In the case where λ is not even, $\frac{G}{\pi}$ is an integral class, so (2.57) as normalized is $\frac{1}{2}$ of the canonical Chern-Simons action in 3d. Furthermore, λ_{ab} as defined in (2.57) is integral. The Chern-Simons action on a 3-manifold W is defined by computing the Maxwell action:

$$S = \frac{1}{2\pi} \int_Z F \wedge F \tag{2.58}$$

for an arbitrary closed 4-manifold Z with boundary W by choosing an extension of the gauge field on Z. The action (2.58) is independent of the choice of Z modulo 2π. Suppose the 4-manifold Z is spin, then if L is a complex line bundle over Z with $c_1(L) = \frac{F}{2\pi}$ then $c_1^2(L)$ is divisible by 2 by Wu's formula[7] (as the second Steiffel-Whitney class w_2 vanishes). That is, given a 3-manifold with a chosen spin structure, the Chern-Simons action:

$$S = \frac{1}{4\pi} \int A \wedge F \tag{2.59}$$

is the basic action (the so called level-$\frac{1}{2}$ Chern-Simons action). This agrees with the normalization in (2.58).

Going back to (2.57), we notice that the effect of turning on background G-flux is to give topological mass to the gauge fields. By $\mathcal{N} = 3$ supersymmetry (2.58) is related to mass terms for the $\mathcal{N} = 3$ superpartners. The $\mathcal{N} = 3$ vector multiplet in 3d consists of a vector and three scalars. In $\mathcal{N} = 2$ notation we write the $\mathcal{N} = 2$ vector multiplet as Σ and the chiral multiplet as Φ. Σ contains a real scalar and a vector which together with the chiral multiplet form the content of a $\mathcal{N} = 3$ vector multiplet (the theory is parity invariant under $\lambda_{ab} \to -\lambda_{ab}$ so the multiplet is the same as a $\mathcal{N} = 4$ vector multiplet). In terms of this the superpotential can be

[7] For a spin four-manifold, w_2 vanishes. This implies the intersection form on the four-manifold is even, by Wu's formula.

schematically written as:

$$S = \int d^3x d^4\theta \lambda_{ab} \Sigma^a V^b - \int d^3x d^2\theta i \lambda_{ab} \Phi^a \Phi^b, \quad \Sigma = iD\bar{D}V \tag{2.60}$$

(2.60) is only schematic since we have ignored the coupling to gravity and as written (2.60) is simply the $\mathcal{N} = 3$ supersymmetric Chern-Simons action.

For non-zero G, parity invariance in 3d is generically broken by the Chern-Simons coupling (2.57). This is simply because the G-flux is odd under 11d parity, and any expectation value breaks parity in 11d, and upon compactification in the resulting 3d theory also. This does not however happen for the $\mathcal{N} = 3$ vacua that arise from hyper-Kähler four-fold compactifications. The reason is that in this case, the topological mass matrix is such that for every positive eigenvalue there is a negative eigenvalue of equal magnitude.

The analysis leading to (2.57) is really independent of the details of the internal manifold which are subsumed in λ_{ab}. Suppose we consider $K3 \times K3$. In this case the 3d theory has $\mathcal{N} = 4$ supersymmetry which prevents the appearance of a Chern-Simons term. However, as shown in ([22]) it is possible to turn on G-flux consistent with $\mathcal{N} = 4$ supersymmetry. Indeed as we saw above, the 11d Chern-Simons coupling gives rise to a topological mass for the gauge fields irrespective of the precise amount of supersymmetry, so we should expect this coupling to be present even for $K3 \times K3$. However, it is well known that there is no $\mathcal{N} = 4$ supersymmetric Chern-Simons action. There is in fact only one way to complete (2.57) in a manner consistent with $\mathcal{N} = 4$ supersymmetry. To explain this let us consider the dimensional reduction on $K3 \times K3$.

Upon dimensionally reducing (2.56) the gauge fields A_μ^a arise via reduction of C on $K3\times K3$. An equal number of such gauge fields arise via dimensional reduction on either $K3$. Instead of considering all those gauge fields together as $2(h^{1,1}+2)$ vector multiplets of the $\mathcal{N}=4$ supersymmetry, we can rather consider them as $h^{1,1}+2$ vector multiplets and $h^{1,1}+2$ twisted vector-multiplets. Doing so the 11d Chern-Simons coupling leads upon dimensional reduction to a BF type coupling between the vector and twisted vector-multiplets, lifting a pair of quaternionic dimensions at a time.

To be more precise, the $\mathcal{N}=4$ supersymmetry algebra in three dimensions has a $SU(2)_R\times SU(2)_N$ R-symmetry, the eight super charges being doublets under the two R-symmetry factors. The $\mathcal{N}=4$ vector multiplet has a vector, three real scalars transforming as **3** of $SU(2)_R$ as bosonic components. The $\mathcal{N}=4$ supersymmetry algebra admits an automorphism that exchanges the two $SU(2)$ factors and takes a vector multiplet into a so-called twisted vector-multiplet which has three scalars that transform as **3** of $SU(2)_N$. In the $\mathcal{N}=4$ supergravity that arises upon compactifying $\mathcal{M}$-theory on $K3\times K3$, the two $SU(2)$ factors can be related to the holonomies of the $K3$s. In fact, upon compactifying $\mathcal{M}$-theory on a product of four-manifolds $Y\times Y$, the holonomy group is $SO(4)\times SO(4)$ generically, leading to the absence of R-symmetries in the resulting 3d theory (which is not supersymmetric unless Y has reduced holonomy). Suppose Y is a $K3$ surface, then decomposing $SO(4)$ as $SO(4)=SU(2)\times SU(2)$ the holonomy of Y can be taken to be one of the two $SU(2)$ factors, and the other $SU(2)$ factor therefore becomes an R-symmetry. The same thing happens with the other factor of Y thus leading to a $SU(2)_R \times SU(2)_N$ R-symmetry as noted. The important point is that the two $SU(2)$ factors are associated with the two $K3$ surfaces.

With this identification, it is clear that the $2(h^{1,1}+2)$ vector multiplets that arise by dimensional reduction of the 3-form have to be treated as $(h^{1,1}+2)$ vector multiplets and $(h^{1,1}+2)$ twisted vector-multiplets as claimed, because the scalars in these multiplets transform under different R-symmetries. There is a unique renormalizable coupling that involves vector and twisted vector-multiplets and is called the BF coupling [21],[26]. It is precisely this coupling that arises via dimensional reduction of $\mathcal{M}$-theory to 3d.

Again schematically the $\mathcal{N}=4$ superpotential can be written as:

$$S = \int d^3x d^4\theta \lambda_{aa'} \Sigma^a \tilde{V}^{a'} - i\lambda_{aa'} \Phi^a \tilde{\Phi}^{a'} \tag{2.61}$$

where Σ, Φ form a $\mathcal{N}=4$ vector multiplet and $\tilde{\Sigma}$ and $\tilde{\Phi}$ form a $\mathcal{N}=4$ twisted-vector multiplet.

Now that we have identified the effect that allows us to lift moduli in $\mathcal{N}=3$ supersymmetric multiplets, we are ready to discuss how to turn on G-flux in a manner consistent with $\mathcal{N}=2$ or $\mathcal{N}=3$ supersymmetry.

2.6.4 $\mathcal{N}=2$ supersymmetry from fluxes

There is a simple way of obtaining $\mathcal{N}=2$ supersymmetry upon turning on G-flux. Suppose we set:

$$G = \mu(\omega \wedge \bar{\omega} + \nu J^2), \tag{2.62}$$

where μ and ν are constants, then the G-flux is of type $(2,2)$ with respect to the complex structure defined by J. Primitivity of G as defined in (2.62) requires:

$$\nu = \frac{1}{3}\frac{q_X(\omega, \bar{\omega})}{q_X(J, J)} \tag{2.63}$$

This means, turning on a G-flux of the form (2.62) preserves at least $\mathcal{N} = 2$ supersymmetry. The fact that it cannot preserve more than $\mathcal{N} = 2$ supersymmetry follows from the fact that G as defined in (2.62) does not satisfy (2.44).

Interestingly enough, turning on G-flux of the form (2.62) lifts all but one complex modulus, the volume modulus and its $\mathcal{N} = 2$ super-partner. The fact that the volume modulus is un-lifted follows from the fact that (2.62) is invariant under a scaling $\omega \to \lambda\omega$, $J \to \lambda J$ if we simultaneously let $\mu \to \lambda^{-2}\mu$. This reflects the fact that the supergravity constraints are in sensitive to the volume modulus.

To see that all other moduli are lifted, one can simply examine the superpotential to show that all other moduli gain tree level mass. A more intuitive proof however follows simply by observing that a choice of G-flux of the form (2.62) fixes a space-like 3-plane in $\Lambda^{3,19}$, thus lifting all moduli associated with the hyper-Kähler structure. (2.62) is not the only choice of G-flux that preserves $\mathcal{N} = 2$ supersymmetry. Indeed any choice of G-flux of the form:

$$G = \mu(\alpha \wedge J + \nu J^2), \quad \alpha \in H^2(X, \mathbb{Z}) \quad \alpha \perp \mathcal{O} \tag{2.64}$$

preserves $\mathcal{N} = 2$ supersymmetry for appropriate choice of μ and ν, leading to a vector multiplet moduli space that is locally:

$$\mathcal{M}_V = \frac{O(2,p)}{O(2) \times O(p)} \tag{2.65}$$

2.6.5 Example: the Hilbert Scheme of two points on $K3$

We will discuss one of the two known examples of compact hyper-Kähler fourfolds, known as the Hilbert scheme of two points on $K3$, and determine the moduli space of the resulting $\mathcal{N} = 3$ supersymmetric vacua. The Hilbert scheme of two points on $K3$, $\mathrm{Hilb}^2(K3)$ is defined as the smooth resolution of the diagonal in the singular symmetric product $S^2(K3)$. To understand the singularity of $S^2(K3)$ and its resolution it is convenient to replace $K3$ by $\mathbb{C}^2$ locally (think of $\mathbb{C}^2/\mathbb{Z}_2$ as a non-compact $K3$ surface). Then $S^2(\mathbb{C}^2)$ is nothing but $\frac{\mathbb{C}^2\times\mathbb{C}^2}{\mathbb{Z}_2}$ where the $\mathbb{Z}_2$ acts by an exchange of the two factors. By a change of variables we can think of the space as $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_2$. Now the singularity of $\mathbb{C}^2/\mathbb{Z}_2$ is due to a collapsing $\mathbb{P}^1$ and it admits a smooth resolution by blowing up the $\mathbb{P}^1$, into the total space of the bundle $\mathcal{O}(-2)_{|\mathbb{P}^1}$. Similarly, $S^2(K3)$ can be resolved by blowing up the exceptional divisor. The resulting space can be proven to be hyper-Kähler and is nothing but $\mathrm{Hilb}^2(K3)$. The cohomology of $\mathrm{Hilb}^2(K3)$ is closely related to the cohomology of $K3$ and indeed the second betti number of the Hilbert scheme is 23, while the third betti number vanishes. Using this we determine $p = 20$ and $q = 0$, so $\chi/24 = 27/2$, which means G-flux must be turned on for this background to solve the string equations.

For $\mathrm{Hilb}^2(K3)$ we can determine the Fujiki constant [14]:

$$\int_X \alpha^4 = 3{q_X}^2(\alpha) \quad \alpha \in H^2(X,\mathbb{R}) \tag{2.66}$$

To preserve $\mathcal{N} = 3$ supersymmetry we can turn on a G-flux of the form (2.53):

$$\frac{G}{\pi} = \mu\alpha \wedge \beta \quad \alpha,\beta \in H^2(X,\mathbb{Z}), \quad \mu \in \mathbb{Z}^+ \tag{2.67}$$

The anomaly cancelation condition can be written as:

$$n + \frac{3}{8} q_X(\alpha) q_X(\beta) = \frac{27}{2} \tag{2.68}$$

The lattice $\Lambda^{3,20}$ of $\mathrm{Hilb}^2(K3)$ is nothing but $E_8 \oplus E_8 \oplus H^3 \oplus 2\mathbb{Z}e$, which is an even integral lattice, so α^2 is even for $\alpha \in H^2(X, \mathbb{Z})$. This means (2.68) always has a solution with integral number n of $M2$-branes. It also has a solution with no $M2$-branes, where $n = 0$ and leads to a moduli space that is of the form:

$$\mathcal{M} = \frac{O(4,19)}{O(4) \times O(19)} \tag{2.69}$$

In (2.69) we have written down the full moduli space, which is the same as the vector multiplet moduli space, since $q = 0$.

To remove more moduli we can simply generalize (2.67) to allow for more complicated G-flux. This way we can obtain moduli spaces of the form:

$$\mathcal{M} = \frac{O(4, 21-2n)}{O(4) \times O(21-2n)} \quad n \leq 9 \tag{2.70}$$

Similarly, we can find $\mathcal{N} = 2$ supersymmetric solutions by turning on G-flux of the form:

$$\frac{G}{\pi} = \mu(\omega \wedge \bar{\omega} - \frac{1}{3} \frac{q_X(\omega, \bar{\omega})}{q_X(J, J)} J^2) \quad \mu \in \mathbb{R}^+ \tag{2.71}$$

The anomaly cancelation condition becomes:

$$n + \frac{5}{24} \mu^2 {q_X}^2(\omega, \bar{\omega}) = \frac{27}{2} \tag{2.72}$$

Interestingly enough, the only solution of (2.72) has $n = 6$, so there are always extra moduli arising from the location of membranes.

The analysis of flux compactifications on $\text{Hilb}^2(K3)$ that we have carried out led us to the conclusion that there are $\mathcal{N}=2$ and $\mathcal{N}=3$ supersymmetric vacua with G-flux. This is surprising, because an analysis of $S^2(K3)$ performed in [23] reveals that there are no supersymmetric vacua with G-flux, that obey the flux quantization condition in $\mathcal{M}$-theory. To pose this puzzle better, we must understand how to identify the symmetric product point $S^2(K3)$ in the hyper-Kähler moduli space of $\text{Hilb}^2(K3)$.

The moduli space of complex structures on $\text{Hilb}^2(K3)$ has dimension 20, which is one greater than the moduli space of complex structures of $K3$. This means at a generic point in moduli space the internal manifold is not of the form $\text{Hilb}^2(K3)$. Geometrically as we mentioned before, $\text{Hilb}^2(K3)$ is obtained from $S^2(K3)$ by blowing up the exceptional divisor e. In fact $q_X(e,e)=-2$ so e is a time-like vector in $\Lambda^{3,20}$. A choice of complex structure is the same as a choice of a positive 3-plane $\mathcal{O}$in $\Lambda^{3,20}$ and this induces a polarization of e as $e=e_+^{3,0}+e_-^{0,20}$ where the $\pm$ serve to indicate the projection into space-like and time-like parts. To reach the point in moduli space where we have $S^2(K3)$ we need to ensure that e is orthogonal to the 3-plane $\mathcal{O}$ spanned by (J,ω). In other words, one has to rotate the 3-plane spanned by (J,ω) such that it is orthogonal to e. This way, starting from a generic point in the moduli space of $\text{Hilb}^2(K3)$ we can blow-down the exceptional divisor e to reach the symmetric product $S^2(K3)$. Now, as we saw before, $\text{Hilb}^2(K3)$ admits $\mathcal{N}=3$ supersymmetric vacua with G-flux turned on. The resulting moduli space is a sub-space of the classical moduli space defined by the lattice $\Lambda^{4,21}$. Starting from a generic point in the classical moduli space, turning on $\mathcal{N}=3$ supersymmetric flux is equivalent to creating a potential for some moduli, which acquire tree level mass and can be integrated out. The remaining moduli do not acquire a potential either perturbatively or non-perturbatively, as it is forbidden by the $\mathcal{N} = 3$ supersymmetry. This means we can always tune the remaining moduli to reach a point in moduli space where $e^{3,0} = 0$ with G-flux turned on, but this would mean we can preserve $\mathcal{N} = 3$ supersymmetry on $S^2(K3)$ with suitable G-flux, in contradiction with the results of Sethi et.al [23]. The resolution of this puzzle has to be that at the point in moduli space where $e^{3,0} = 0$ the internal space is no longer $S^2(K3)$. This is indeed possible as one of the effects of turning on G-flux is to warp the internal space. Even though this resolves the puzzle qualitatively, it would be very interesting to determine the effect of quantum corrections in the presence of G-flux more precisely. As we will show below, instanton effects are absent, so the only corrections to the moduli space metric arise from radiative corrections and possible strong dynamics.

2.7 Effects of instantons

Upon turning on G-flux, the worldsheet theory is no longer an $\mathcal{N} = 4$ superconformal field theory, as the Ramond-Ramond vertex operators are non-local with respect to the worldsheet supersymmetry generators invalidating the RNS approach. This makes it plausible that the moduli space metric can acquire perturbative and (or) non-perturbative corrections. In this section we examine possible corrections to the moduli space picture we obtained at weak coupling.

Non-perturbative effects that occur in field theories at weak coupling are due to instantons. These are finite action solutions to the Euclidean field equations. In supersymmetric theories, these instantons can be BPS, or non BPS. BPS states in three dimensional $\mathcal{N}=3$ supersymmetric theories can be $\frac{1}{3}$ or $\frac{2}{3}$ BPS, preserving two or four supercharges respectively, but it turns out that only $\frac{1}{3}$ BPS instantons exist in the string theory. These instantons arise from wrapping $M5$-branes on divisors inside the hyper-Kähler four-fold X. Since a BPS instanton breaks four of the six charges, there are four such fermionic zero modes in the instanton background, and these zero modes have to be soaked up in order to provide a non trivial amplitude. This means such an instanton can potentially correct the Kähler metric of the target space.

The analysis of instantons in $\mathcal{M}$-theory was first carried out by Witten [16] and later generalized by Kallosh et.al [17] to include flux compactifications in $\mathcal{M}$-theory. We will review Witten's result below and its generalization to include fluxes. We will then be able to note that there are no instanton induced corrections to the metric in the $\mathcal{N}=3$ vacua we discussed in the context of $\mathrm{Hilb}^2(K3)$.

2.7.1 Non-perturbative superpotentials in $\mathcal{M}$-theory

The non-perturbative superpotential in Calabi-Yau four-fold compactifications of $\mathcal{M}$-theory to three dimensions can be effectively computed, following Witten [16]. Upon compactifying $\mathcal{M}$-theory on a Calabi-Yau four-fold, we obtain moduli which belong in chiral or linear multiplets, depending on whether the moduli arise from complex structure or Kähler deformations respectively. The Kähler moduli pair with periods of the C-field to form linear multiplets. One can dualize the linear multiplets into ordinary chiral multiplets in the absence of Chern-Simons terms. The gauge invariance of C translates into a perturbative shift symmetry for the dual scalar ϕ. Any superpotential in such a compactification must depend on Kähler moduli, as otherwise we can always scale the Kähler moduli to reach the large radius limit where the superpotential has to be absent. However, perturbatively there can be no dependence of the superpotential on the Kähler moduli, since any such dependence requires the superpotential to depend on ϕ thus breaking the perturbative shift invariance.

This argument implies that any superpotential in $\mathcal{M}$-theory compactification on Calabi-Yau four-folds must be non-perturbatively generated. The natural candidate for a non-perturbative superpotential is an instanton which in the $\mathcal{M}$-theory context is an $M5$-brane wrapping a divisor inside the Calabi-Yau four-fold. The shift invariance of ϕ is broken by $M5$-branes since the five-brane is a magnetic source for the C-field.

It is however a non-trivial problem to determine which divisors if any, lead to a non-perturbative superpotential. Witten determined a criterion that had to be satisfied in order for a non-perturbative superpotential to be generated: the holomorphic Euler characteristic of the divisor D has to equal one, that is $\chi_D = 1$. This is a very strong restriction on the possible divisors that generate a superpotential. The important observation due to Witten was that the superpotential vanishes due to an anomaly unless $\chi_D = 1$. Assuming the divisor D is smooth, the normal bundle to D in the four-fold X is a line bundle N. If z is a coordinate in the normal direction there is a $U(1)$ which acts by $z \to e^{i\theta} z$. This $U(1)$ plays the role of an

R-charge in the three dimensional effective theory and Witten related the R-charge to the number of zero modes of the Dirac equation on the five-brane (which is twisted by the spin-bundle associated to the normal bundle; the spinors take values in $S_+ \otimes (S'_+ \oplus S'_-)$ where S_+ is the positive chirality spin bundle associated to the tangent bundle of the divisor, and $S'_\pm$ are the respective chirality spin bundles associated to the normal bundle of D. Witten shows that this R-charge is given by χ_D, the arithmetic genus of the divisor D. In order for the instanton induced superpotential to be non vanishing by charge conservation we then require $\chi_D = 1$.

The analysis of Witten can be generalized to the case with background flux [17]. The main difference is that in the case with fluxes the solutions to Dirac's equation do not have definite chirality. It turns out that one can define an F-chirality (essentially a $\mathbb{Z}_2$ subgroup of the $U(1)$ normal bundle action together with a sign flip $G \to -G$ for the G-flux) and count the zero modes of different F-chirality. A straightforward generalization of Witten's analysis then reveals a relation between $\chi_D(F)$ and χ_D of the form:

$$\chi_D(F) = \chi_D - (h^{0,2} - n), \tag{2.73}$$

where n is the number of solutions to:

$$\mathcal{H}(G_{\bar{a}\bar{z}bc}\phi^{bc} d\bar{z}^{\bar{a}}) = 0, \tag{2.74}$$

where $\mathcal{H}$ is the projector on to the space of harmonic forms and ϕ^{ab} is a harmonic $(2,0)$ form.

We are interested in applying these results to hyper-Kähler four-folds, so $h^{0,2} = 1$. Also, if we are turning on G-flux preserving $\mathcal{N} = 3$ supersymmetry, then $n = 1$.

Now BPS instantons preserve two supercharges, so there are four zero modes, giving rise to an R-charge violation of two units. This means $\chi_D(F) = 2$ in order for instantons to generate a correction to the moduli space metric. This in turn requires:

$$\chi_D = 2 \tag{2.75}$$

It remains to compute the arithmetic genus of a divisor D inside a hyper-Kähler four-fold and check if (2.75) is satisfied. The arithmetic genus of the Hilbert scheme $\mathrm{Hilb}^2(K3)$ can be expressed in terms of the cohomology class α that is Poincare dual to the divisor D as [34]:

$$\chi_D = \frac{1}{2}(\frac{1}{2}q_X(\alpha) + 2)(\frac{1}{2}q_X(\alpha) + 3) \tag{2.76}$$

It is easy enough to verify that there is no divisor D that satisfies (2.75). Thus there are no instanton corrections to the moduli space metric in the case of the Hilbert scheme $\mathrm{Hilb}^2(K3)$. Incidentally, for vacua with $\mathcal{N} = 2$ supersymmetry, the conditions for an instanton generated superpotential also require the divisor D to satisfy (2.75). This means there is no instanton induced superpotential in the $\mathcal{N} = 2$ vacua that arise from turning on G-flux on $\mathrm{Hilb}^2(K3)$.

2.8 $\mathcal{M}$-theory on the orbifold $\mathbb{T}^8/\mathbb{Z}_2$

We have so far treated $\mathcal{N} = 3$ supersymmetric $\mathcal{M}$-theory compactifications to three dimensions as if they arose from hyper-Kähler four-folds alone. In this section, we will show that there is one other class of $\mathcal{M}$-theory compactifications which also gives us $\mathcal{N} = 3$ supersymmetry in $d = 3$.

Consider compactification of $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$. The resulting three dimensional theory has $\mathcal{N}=8$ supersymmetry, and a moduli space that is of the form:

$$\mathcal{M} = \frac{O(8,8)}{O(8)\times O(8)} \tag{2.77}$$

Indeed, this $\mathcal{M}$-theory compactification is related by S^1 reduction to type IIA on $\mathbb{T}^8/\mathbb{Z}_2$. By $\mathcal{N}=(8,8)$ supersymmetry, the moduli space is constrained to be of the form:

$$\mathcal{M} = \frac{O(8,n)}{O(8)\times O(n)}, \tag{2.78}$$

so that the moduli space has dimension $8n$. The twisted sectors of the orbifold $\mathbb{T}^8/\mathbb{Z}_2$ do not contribute any extra massless modes, so that all massless scalars arise from the $\mathbb{Z}_2$ projection of type IIA on $\mathbb{T}^8$, leading to a moduli space of the form (2.77). Interestingly, by choosing the $\mathcal{M}$-theory circle to be embedded in $\mathbb{T}^8/\mathbb{Z}_2$ we can connect $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$ to type IIA on $\mathbb{T}^7/\mathbb{Z}_2$ orientifold.

String theory on this orbifold makes sense, even though the orbifold cannot be blown-up. As far as geometry is concerned though, this orbifold cannot be connected to a smooth manifold by turning on blow-up modes. As we are dealing with a compactification on an eight dimensional manifold, we once again face the fact that we may have to turn on G-flux to solve (2.38). Standard orbifold techniques allow us to compute the Euler number χ of the orbifold to be $\chi = 24\times 16$ so that we need to solve:

$$n + \frac{1}{4}\int_{\mathbb{T}^8}(\frac{G}{2\pi})^2 = 16 \tag{2.79}$$

Turning on fluxes of course breaks the $\mathcal{N}=8$ supersymmetry. We like to consider the question of whether there are any solutions that preserve $\mathcal{N} = 3$ supersymmetry.

Before we turn to some examples, there is one important point about the orbifold $\mathbb{T}^8/\mathbb{Z}_2$ that will be important to understand. In the usual orbifold construction, a discrete sub-group of space-time symmetry is gauged. In string theory this can be generalized to define the notion of an orientifold by gauging in addition worldsheet parity. Indeed the type I superstring in ten dimensions can be obtained as the orientifold of type IIB. Analogously, orientifolds can be defined for type IIA upon compactification on odd dimensional tori: the product of space reflection and worldsheet parity can be gauged. For example, type IIA compactified on $\mathbb{T}^7/\mathbb{Z}_2$ admits orientifold 2-planes denoted as $O2^-$, $O2^+$ and $\tilde{O}2$. The $O2^-$ orientifold carries $-\frac{1}{8}$ units of $D2$-brane charge, while the $O2^+$ and $\tilde{O}2$ planes carry $\frac{1}{8}$ and $\frac{3}{8}$ units of $D2$-brane charge respectively. As we noted above, the $\mathbb{T}^7/\mathbb{Z}_2$ orientifold lifts to the $\mathbb{T}^8/\mathbb{Z}_2$ orbifold in $\mathcal{M}$-theory. The $O2^-$ lifts to the so called $OM2^-$-plane which carries $-\frac{1}{16}$ units of $M2$-brane charge.

Indeed the space $\mathbb{T}^8/\mathbb{Z}_2$ has $M2$-brane charge $-\frac{\chi}{24}$ which is -16. There are a total of 2^8 fixed points so each fixed point carries $-\frac{1}{16}$ units of $M2$-brane charge and define the $OM2^-$-plane. What is the $\mathcal{M}$-theory interpretation of the $O2^+$ and $\tilde{O}2$-planes? Near each fixed point of $\mathbb{T}^8/\mathbb{Z}_2$ the space is locally of the form $\mathbb{R}^8/\mathbb{Z}_2$. Deleting a neighborhood around the origin, the transverse space is a copy of $\mathbb{R}P^7$. The number of distinct choice of G-flux (or discrete torsion as it is called) in $\mathcal{M}$-theory on the orbifold $\mathbb{R}^8/\mathbb{Z}_2$ is given by $H^4(\mathbb{R}P^7, \mathbb{Z})$ which is $\mathbb{Z}_2$. That is, there are precisely two possible choices for G-flux. The first possibility with trivial torsion corresponds to the $\mathcal{M}$-theory lift of the $O2^-$-plane, so both the $O2$ and $\tilde{O}2$-planes must flow to the same $\mathcal{M}$-theory object. This object carries $\frac{3}{16}$ units of $M2$-brane charge [35].

2.8.1 An $\mathcal{N} = 6$ example

In order to obtain $\mathcal{N} = 6$ supersymmetric flux compactifications, let us choose a complex structure on $\mathbb{T}^8$ by defining complex coordinates w^i, $i = 1...4$. The Kähler form is taken to be:

$$J = \sum_i dw^i d\bar{w}^i \tag{2.80}$$

Suppose we choose G to be of type $(4, 0) \oplus (0, 4)$ with respect to this complex structure. Then G is primitive. G is of type $(2, 2)$ with respect to a different complex structure where $(z^1, z^2, z^3, z^4) = (w^1, \bar{w}^2, w^3, \bar{w}^4)$. Indeed there are six complex structures with respect to which G is primitive and type $(2, 2)$. This means any flux of type $(4, 0) \oplus (0, 4)$ with respect to the complex structure defined by w preserves $\mathcal{N} = 6$ supersymmetry in $d = 3$.

Suppose we consider compactifications without $OM2^+$ and $\tilde{O}M2$ planes. Then the flux quantization condition becomes:

$$\frac{G}{2\pi} = (\mu dw^1 \wedge dw^2 \wedge dw^3 \wedge dw^4 + \text{hc}) \quad \text{Re}\mu, \text{Im}\mu \in \mathbb{Z} \tag{2.81}$$

The anomaly cancelation condition (2.79) becomes:

$$n + 8(\text{Re}\mu)^2 + 8(\text{Im}\mu)^2 = 16 \tag{2.82}$$

whose solutions are:

$$n = 0 \quad \mu = 1 + i, \quad n = 8 \quad \mu = 1, i \tag{2.83}$$

To determine the moduli that are lifted, we need to examine the couplings of the G-flux to the moduli. Turning on a G-flux of the form $G = \mu G^{4,0} + \mathrm{hc}$ fixes the complex structure of the four-fold entirely. Before orbifolding, $\mathbb{T}^8$ has 16 Kähler moduli, while the complex structure is specified by 16 complex parameters. Unlike a Calabi-Yau manifold however, the complex parameters do not uniquely specify the Ricci-flat metric, that is, not all of these complex parameters are moduli in the effective action. Since the metric deformations give rise to 36 scalars, it is clear that there is a six complex dimensional space of parameters that preserve the Ricci-flat metric. Therefore, a G-flux of the form (2.81) lifts 20 real moduli corresponding to complex structure deformations that change the Ricci-flat metric. This leaves 16 real moduli coming from the metric. The C-field gives rise to 28 scalars which can acquire mass only through the Chern-Simons term (2.57). It is easy to see that 12 of the C-field moduli are thus lifted, leaving a scalar moduli space of dimension 32. In the absence of $M2$-branes we expect the moduli space to be of the form:

$$\mathcal{M} = \frac{O(8,4)}{O(8) \times O(4)} \tag{2.84}$$

If there are n additional membranes, the moduli space takes the form:

$$\mathcal{M} = \frac{O(8,4+n)}{O(8) \times O(4+n)} \tag{2.85}$$

In the presence of orientifolds, the flux quantization condition can be different. As we have already seen, flux quantization in $\mathcal{M}$-theory requires the 4-form G to satisfy (2.35). For $\mathbb{T}^8$ the class λ vanishes. This means the G-flux satisfies the naive quantization condition:

$$[\frac{G}{2\pi}] \in H^4(\mathbb{R}P^7, \mathbb{Z}) \tag{2.86}$$

For the orbifold $\mathbb{R}^8/\mathbb{Z}_2$ a similar statement can be made [7]. The flux quantization condition can be written as:

$$[\frac{G}{2\pi}] - \frac{1}{2}w_4 \in H^4(X, \mathbb{Z}), \tag{2.87}$$

where w_4 is the fourth Steiffel-Whitney class of the four-fold X. For $\mathbb{R}^8/\mathbb{Z}_2$, deleting a neighborhood of the origin, we can consider X to be $\tilde{\mathbb{R}}^8/\mathbb{Z}_2$. Let D be a four-cycle surrounding the origin in X. Then, the tangent bundle of X is a sum of eight copies of the unoriented real bundle ϵ with $w_1(\epsilon) = x$. The total Steiffel-Whitney class of X is:

$$(1+x)^8 = 1, \tag{2.88}$$

so that w_4 vanishes. Thus the flux quantization condition becomes:

$$\int_D \frac{G}{2\pi} = \mathbb{Z} \tag{2.89}$$

So, on $\mathbb{R}^8/\mathbb{Z}_2$, the G-flux of $\mathcal{M}$-theory obeys the naive quantization condition. There is however a small puzzle here. Let us choose coordinates x^i, $i = 1, 2, 3, 4$ and y^i to describe a square $\mathbb{T}^8$, with the identification $x^i \to x^i + 1$, $y^i \to y^i + 1$. Then we can define a four-cycle:

$$\gamma : 0 \leq x^1, x^2, x^3, x^4 \leq 1, \quad y^i = 0 \tag{2.90}$$

Let us assume the flux through g is an integer n_γ. We are also allowed to consider four-cycles γ' in $\mathbb{T}^8/\mathbb{Z}_2$ which lift to half-cycles on $\mathbb{T}^8$:

$$\gamma' : 0 \leq x^1, x^2, x^3 \leq 1, \quad 0 \leq x^4 \leq \frac{1}{2}, \quad y^i = 0 \tag{2.91}$$

It is easy to note that the flux on γ' obeys integral quantization only if n_γ is even. Does this imply that the G-flux on $\mathbb{T}^8/\mathbb{Z}_2$ has to be even? As shown by Frey and Polchinski [28], the orientifold is consistent as long as flux quantization is obeyed on the covering space, which in our case is $\mathbb{T}^8$. If the G-flux is not even, then there is discrete G-flux trapped at certain fixed points that allow for the flux quantization in covering space to be recovered. These fixed points with discrete G-flux are precisely the orientifold planes $OM2^+$ and $\tilde{O}M2$. In writing (2.82) we imposed that the G-flux be even, a requirement that allowed us to discuss vacua without exotic orientifold planes. To describe the full set of $\mathcal{N} = 6$ supersymmetric vacua, we have to include exotic orientifold planes. In this case the equivalent of the condition (2.82) becomes:

$$n + \frac{m}{4} + 8((\mathrm{Re}\mu)^2 + (\mathrm{Im}\mu)^2) = 16 \quad m, n \in \mathbb{Z}, \quad \mathrm{Re}\mu, \mathrm{Im}\mu \in \frac{1}{2}\mathbb{Z} \tag{2.92}$$

More precisely, m is the number of orientifold planes of type $OM2^+$ or $\tilde{O}M2$ and as long as $m \neq 0$ at least one of $\mathrm{Re}\mu$ or $\mathrm{Im}\mu$ must be half-integral. Again we find a finite set of solutions to (2.92). These are all possible $\mathcal{N} = 6$ vacua that arise from $\mathbb{T}^8/\mathbb{Z}_2$ by turning on G-flux and are in one to one correspondence with the dimensional reduction of $\mathcal{N} = 3$ vacua in $d = 4$ found in [28]. This is not surprising, as we know that $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$ is dual to type IIA on the $\mathbb{T}^7/\mathbb{Z}_2$ orientifold, which itself is T-dual to type IIB on $S^1 \times \mathbb{T}^6/\mathbb{Z}_2$ orientifold studied in [28].

2.8.2 An $\mathcal{N}=4$ example

We can also turn on G-flux on $\mathbb{T}^8/\mathbb{Z}_2$ in such a manner as to preserve eight supercharges, leading to $\mathcal{N}=4$ supersymmetry in $d=3$. We will briefly discuss this example, as it will set the stage for the analysis of $\mathcal{N}=3$ supersymmetric vacua. Let us choose complex coordinates w^i, $i=1,2,3,4$ to describe $\mathbb{T}^8$. The $\mathbb{Z}_2$ acts by $w^i \to -w^i$. The most general G-flux that is of type (2,2) and primitive with respect to the Kähler form (2.80) is given by:

$$\frac{G}{2\pi} = Ad\bar{w}^1dw^2d\bar{w}^3dw^4 + Bd\bar{w}^1dw^2dw^3d\bar{w}^4 + Cd\bar{w}^1d\bar{w}^2dw^3dw^4 + \mathrm{hc} \qquad (2.93)$$

For $B=C=0$, we can transform (2.93) to the form (2.81) by a change of variables. This means that for $B=C=0$ a G-flux of the form (2.93) can preserve $\mathcal{N}=6$ supersymmetry. We will now show that for $C=0$, a G-flux of the form (2.93) preserves $\mathcal{N}=4$ supersymmetry.

One way to observe that (2.93) preserves $\mathcal{N}=4$ supersymmetry for $C=0$ (by duality we could equally well set $A=0$ or $B=0$) is to note that when $C=0$ there are in total four complex structures with respect to which (2.93) is type (2,2) and primitive. This argument is quick, but fails to generalize to vacua with $\mathcal{N}=3$ supersymmetry and forces us to come up with a better proof.

Consider compactification of $\mathcal{M}$-theory on a Calabi-Yau four-fold X. Four forms on an eight dimensional manifold transform in the $[\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8}]_A$ representation, which is the $\mathbf{70}$ of $SO(8)$. On a complex eight dimensional manifold there is a natural action of $U(4)$ and since X is Calabi-Yau, there is actually an action of $SU(4)$, which is nothing but the holonomy group of a Calabi-Yau four-fold. Now on any Kähler manifold X of complex dimension n, there is a natural action of $SU(2)$ on forms G of type $(p, n-p)$ generated by:

$$G \to G \wedge J, \quad G \to (n-p)G, \quad G \to i_J G \tag{2.94}$$

The contraction with the Kähler form J is the adjoint operation of $SU(2)$, while the multiplication by $n-p$ is the action of the diagonal J_3 of $SU(2)$. For middle dimensional forms, primitivity is equivalent to carrying zero charge under the diagonal $U(1)$. We are interested in G-flux that is primitive and type (2,2). Forms of type (2,2) transform in the $\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$ of $SU(4)$ with the $\mathbf{20}$ carrying charge zero under J_3. This means that for a flux to be of type (2,2) and primitive, it must transform in the $\mathbf{20}$ dimensional representation of $SU(4)$.

In order to preserve $\mathcal{N} = 4$ supersymmetry, we need to have an unbroken $SU(2) \times SU(2)$ R-symmetry. The torus $\mathbb{T}^8$ has a $SO(8)$ global symmetry which must embed the $SO(4)_R \sim SU(2) \times SU(2)$. Under the decomposition $SO(4)_H \times SO(4)_R \in SO(8)$, the $\mathbf{8}$ of $SO(8)$ transforms as:

$$\mathbf{8} = (\mathbf{2}, \mathbf{2}; \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2}) \tag{2.95}$$

The $\mathbf{20}$ of $SU(4)$ decomposes under $SO(4)_H$ as:

$$\mathbf{20} = (\mathbf{3}, \mathbf{3}) \oplus 2(\mathbf{2}, \mathbf{2}) \oplus 3(\mathbf{1}, \mathbf{1}) \tag{2.96}$$

That is, we can turn on G-flux in one of the three representations $(\mathbf{3}, \mathbf{3})$, $(\mathbf{2}, \mathbf{2})$ or $(\mathbf{1}, \mathbf{1})$ of $SO(4)_H$, and still preserve $\mathcal{N} = 2$ supersymmetry. We will show below that in order to preserve $\mathcal{N} = 4$ supersymmetry we can only turn on G-flux in the $(\mathbf{3}, \mathbf{3})$ representation of $SO(4)_H$. Using (2.95) we can compute the decomposition of **70** of $SO(8)$ under $SO(4)_H \times SO(4)_R$. Under this decomposition, a G-flux will preserve $\mathcal{N} = 3$ supersymmetry if and only if it is a singlet with respect to $SO(4)_R$. It is straightforward to check that the $(\mathbf{3}, \mathbf{3})$ appears with a singlet in $SO(4)_R$ leading to our assertion that only a G-flux of type $(\mathbf{3}, \mathbf{3})$ preserves $\mathcal{N} = 4$ supersymmetry. Going back to coordinate language, a G-flux of the form:

$$\frac{G}{2\pi} = (Ad\bar{w}^1 dw^2 + A^* dw^1 d\bar{w}^2)(Bd\bar{w}^3 dw^4 + B^* dw^3 d\bar{w}^4) \tag{2.97}$$

transforms in $(\mathbf{3}, \mathbf{3})$ representation under the $SO(4)_H$ generated by the two $SU(2)$ rotations that act on (w^1, w^2) and (w^3, w^4) respectively. Now it is easy enough to see that up to a redefinition of constants (2.97) is precisely (2.93) with $C = 0$ as advertised.

As usual, we must supplement (2.97) with the anomaly cancelation conditions. In the absence of exotic orientifold planes the G-flux must solve:

$$n + 8(|A|^2 + |B|^2) = 16, \quad (\mathrm{Re}A + \mathrm{Re}B) \in \mathbb{Z}, \quad (\mathrm{Im}A + \mathrm{Im}B) \in \mathbb{Z} \tag{2.98}$$

2.8.3 $\mathcal{N} = 3$ supersymmetry, first steps

For a G-flux to preserve $\mathcal{N} = 2$ supersymmetry, it has to transform in the **20** of $SU(4)$. The holonomy of a hyper-Kähler manifold is in $sp(2)$ (sometimes denoted as USp(4)). Under $sp(2)$, the **6** of $SU(4)$ decomposes as:

$$\mathbf{6} = \mathbf{5} \oplus \mathbf{1}, \tag{2.99}$$

so that the **20** of $SU(4)$ decomposes as:

$$\mathbf{20} = \mathbf{1} \oplus \mathbf{5} \oplus \mathbf{14} \tag{2.100}$$

The easiest way to obtain (2.100) is to note the Lie algebra isomorphisms:

$$sp(2) \simeq Spin(5), \quad SU(4) \simeq Spin(6), \tag{2.101}$$

and note that **20** of $SU(4)$ is the same as the trace-free, symmetric rank-2 tensor representation of $SO(5)$.

Although we can preserve $\mathcal{N} = 2$ supersymmetry by turning on G-flux in any of the three representations of $sp(2)$ in (2.100), the criterion for preserving $\mathcal{N} = 3$ supersymmetry is more stringent. In order to preserve $\mathcal{N} = 3$ supersymmetry, the G-flux has to transform in the **14** of $SO(5)$.

Turning on such G-flux, we can again find solutions to the anomaly cancelation condition and determine the space of $\mathcal{N} = 3$ supersymmetric vacua. Under the assumption that the $\mathcal{N} = 3$ moduli space is connected, these vacua will be related to hyper-Kähler four-fold compactifications. This allows us to enumerate such vacua and infer results about hyper-Kähler four-folds [32].

2.9 *Synthesis*

As we have seen in this chapter, vacua with $\mathcal{N} = 3$ supersymmetry in $d = 3$ can be constructed from $\mathcal{M}$-theory on hyper-Kähler four-folds. We have also motivated a relationship between these vacua and vacua arising from $\mathcal{N} = 3$ supersymmetric flux compactifications of $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$. This relationship may allow us to qualitatively understand the existence of bounds on the Hodge numbers of hyper-Kähler four-folds. From a physical perspective, we have tried to emphasize the existence of this bound on the moduli space of $\mathcal{N} = 3$ vacua arising from hyper-

Kähler four-folds as a bound on all string vacua with $\mathcal{N} = 3$ supersymmetry in $d = 3$. This is a rather strong statement which if true supports the argument that there are very many consistent looking effectve supergravity theories that are not embedded in string theory. It would be very important to understand if this statement is true and if so the reason, as this will provide insight into quantum gravity itself. Unfortunately, at this time not enough is known about string theory to conclusively rule out other vacua with $\mathcal{N} = 3$ supersymmetry that may arise via string theory. It is indeed possible that there are vacua which preserve $\mathcal{N} = 3$ supersymmetry in $d = 3$ that are intrinsically non-perturbative in the sense that the string coupling g_s is forced to be of order unity in these vacua. We have already encountered one such example in the context of type IIA on the orientifold $\mathbb{T}^7/\mathbb{Z}_2$. In this case a linear combination of the dilaton and the volume modulus is fixed (the orientifold couples to the dilaton). It is certainly possible for g_s itself to be fixed at order one leading to non-perturbative vacua. We believe that such vacua exist, but they are all related to the $\mathcal{N} = 3$ supersymmetric vacua that arise from hyper-Kähler four-folds [33]. One reason to believe this may be true is that all attempts to construct perturbative $\mathcal{N} = 3$ supersymmetric vacua lead to the possibilities already considered in this thesis [33]. Furthermore, unlike the case of $\mathcal{N} = 4$ supergravity in $d = 4$, flux vacua with different dimensions of moduli space can be connected in three (and two) dimensions. Indeed, we exhibited a simple example where by turning on G-flux inside $\mathrm{Hilb}^2(K3)$ we could obtain various solutions with different n (2.70). Let us pause for a bit, to explain why these different solutions are really connected.

The explanation is more general than just the special case of $\mathrm{Hilb}^2(K3)$. In specifying an $\mathcal{M}$-theory vacuum, we pick a choice of G-flux and a choice of internal manifold X. Supersymmetry provides a constraint on X and G, so does the anomaly cancelation condition, and the flux quantization condition. Suppose we have two different choices of G-flux, say G_1 and G_2 that solve both constraints for a chosen X. Then, they do not lead to two different theories, rather they lead to different vacua in the same theory. Indeed, one can exhibit a soliton that interpolates between these vacua: an $M5$-brane wrapping the divisor Poincare dual to the four-form $\frac{(G_1-G_2)}{2\pi}$ leads to such a soliton [12].

This is a precise argument relating models with different G-flux on the same X. It is also possible that models with different X are also connected in the landscape of flux vacua. This has already been shown in the context of $K3\times K3$ compactifications in three dimensions [22], and in four dimensions it is a well known fact [42] that the Calabi-Yau moduli space is connected by conifold transitions. This leads us to believe that a similar story holds for $\mathcal{N}=3$ vacua in three dimensions.

Indeed one approach to investigate this issue in the context of hyper-Kähler folds proceeds as follows [32]. Suppose we wish to argue that $\mathbb{T}^8/\mathbb{Z}_2$ upon deformation by G-flux leads to vacua that are smoothly connected to the hyper-Kähler moduli space. Then one approach to showing this would be to consider the theory on the world volume of a stack of N D2-branes probing the $\mathbb{R}^7/\mathbb{Z}_2$ orientifold in type IIA. The theory one the world volume of this stack is an $\mathcal{N}=8$ supersymmetric gauge theory with gauge group SO(2N), Sp(N) or SO(2N+1) (depending on whether the orientifold plane is $O2^-$, $O2^+$ or $\tilde{O}2$). Upon turning on G-flux, this gauge theory is deformed into a $\mathcal{N} = 3$ supersymmetric gauge theory. The IR limit of this three dimensional GLSM will be a $\mathcal{N} = 3$ SCFT. If $\mathbb{T}^8/\mathbb{Z}_2$ is related by turning on G-flux to a compact hyper-Kähler four-fold, then the IR SCFT will be identical to one obtained from M2-branes at a suitable singularity of hyper-Kähler four-folds. This approach is currently under investigation, and may be a promising avenue for exploring dualities between flux vacua [32].

3. SINGULAR CFTS FOR SPIN(7) SPACES

In this chapter, we will consider string compactifications to two space-time dimensions. These vacua preserve $\mathcal{N} = (1,1)$ or $\mathcal{N} = (0,2)$ supersymmetry and arise via compactification of type II A (or IIB) on so called Spin(7) holonomy manifolds. We will be essentially interested in the worldsheet aspects of such compactifications. As reviewed in appendix A, the worldsheet description of superstrings propagating on a Calabi-Yau n-fold is in terms of a $\mathcal{N} = 2$ SCFT with central charge $c = 3n$. It was shown by Shatashvili and Vafa [8] that the worldsheet CFT governing superstring propagation on compact Spin(7) manifolds has an enhanced SCA which is denoted as $\mathcal{SW}(\frac{3}{2}, 2)$. This enhanced algebra plays the same role for the Spin(7) CFT as the $\mathcal{N} = 2$ SCA plays for the Calabi-Yau spaces. In section 1, we re-derive the algebra of Shatashvili and Vafa. In doing so, we will be extending the derivation to small radius, and confirming that the extended algebra exists away from the large radius limit.

Compact Calabi-Yau spaces can develop degenerations that occur at finite distance in the Calabi-Yau moduli space (as measured by the Weil-Peterson metric for example). It is an interesting question to determine the effective string theory that describes the local physics in the neighborhood of the singularity. From the space-time point of view, near the singularity gravitational physics de-couples, so we require a string theory that does not have a massless spin-2 particle in the spectrum. Such a string is called a non critical superstring, for reasons that we will explain in section 2. For Calabi-Yau manifolds, there is a nice prescription for determining the non-critical string given the non compact Calabi-Yau, essentially due to Giveon, Kutasov and Pelc [9].

Indeed it is possible to provide a simple connection between singularities of Calabi-Yau manifolds and the occurrence of non-critical strings, in the context of gauged linear sigma models, as we show in section 2 (this is done for the sake of completeness as this result seems to have not been explicitly derived in literature, though it is implicit in the work of Silverstein and Witten [30]). It is natural to ask if singularities of other manifolds can be described by a suitable non-critical string. This is the question we ask in the context of Spin(7) manifolds, and find [18] that singularities of Spin(7) manifolds are captured by a non-critical string whose worldsheet SCFT has an enlarged symmetry algebra denoted in literature as $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$. We provide evidence for this relationship between Spin(7) singularities and the appearance of the enhanced symmetry algebra in section 3.

With the exception of section 2, which is largely a review, the rest of chapter 3 follows from work done in [33].

3.1 Worldsheet aspects of Spin(7) compactifications

As reviewed in Appendix A, the worldsheet description of superstring propagation is in the form of a two dimensional $\mathcal{N} = 1$ SCFT with $c = 15$. In order to describe a compactification of the form $\mathbb{R}^2 \times X$, we simply tensor the free conformal field theory of two bosons and two Majorana fermions, along with the $\mathcal{N} = 1$ SCFT describing the internal manifold X. In order to obtain $c = 15$, this internal SCFT must have $c = 12$. However, any internal SCFT with $c = 12$ will not do: we need to eliminate the tachyon, and as we saw this required space-time supersymmetry. It is therefore necessary to impose space-time supersymmetry as a sensible requirement for the string theory. An important fact in string theory is that local symmetries in space-time appear as global symmetries on the worldsheet. In the presence of gravity the space-time supersymmetry is gauged, so we expect there exists a global charge on the worldsheet corresponding to the gauged supersymmetry in the target space-time. This global charge is nothing but the zero momentum left (or right)-moving part of the gravitino vertex operator. In a standard choice of picture this charge is written as:

$$Q = \int dz e^{-\varphi/2} e^{iH} \Sigma \tag{3.1}$$

In (3.1) the spin-field Σ is the operator that creates the R-vacuum of the internal SCFT of X from the NS-vacuum, and we have written the global charge for a theory with $\mathcal{N} = 1$ space-time supersymmetry in $2d$, which happens to be the case relevant for Spin(7) compactifications. From a space-time point of view, the existence of $\mathcal{N} = (1, 1)$ supersymmetry upon compactification on X is non-trivial and leads to the observation that the internal manifold X must possess a covariantly constant spinor of definite $SO(8)$ chirality, and hence X must be a manifold of Spin(7) holonomy. The un-compactified type IIA on $\mathbb{R}^{10}$ preserves 32 supercharges and admits two constant Majorana-Weyl spinors in $\mathbf{16}$ and $\mathbf{16'}$ representations of $SO(1,9)$. Upon compactifying on X the spinors can be decomposed as $\mathbf{16} = (\mathbf{8}^+, +) \oplus (\mathbf{8}^-, -)$ and $\mathbf{16'} = (\mathbf{8}^+, -) \oplus (\mathbf{8}^-, +)$. Precisely when the internal manifold X has Spin(7) holonomy, $\mathbf{8}^+ = \mathbf{7} \oplus \mathbf{1}$ leading to a covariantly constant spinor of definite $SO(8)$ chirality. This gives rise to $\mathcal{N} = (1,1)$ space-time supersymmetry in type IIA and $\mathcal{N} = (0,2)$ supersymmetry in type IIB. From a worldsheet point of view, we can similarly surmise that the existence of the holomorphic spin-field Σ as required in (3.1) will lead to severe restrictions on the possible internal SCFT describing the internal manifold X. Is it possible to use this to constrain the internal SCFT? It is indeed possible to determine the internal SCFT this way, and what we would have deduced in this manner would be the worldsheet SCFT that governs string propagation on manifolds with Spin(7) holonomy. The SCFT governing Spin(7) compactifications was first described by [8]. The approach we use here has the advantage that it is valid even for small radius compactifications, and indeed we shall find that all Spin(7) compactifications have a $\mathcal{SW}(\frac{3}{2}, 2)$ algebra on the worldsheet.

First of all, the existence of a holomorphic spin-field that gives rise to the space-time supercharge as in (3.1) requires that the Σ is a free Majorana fermion. The simplest way to see this is as follows: The supercharges Q must square to P^+ up to picture changing. This requires Σ to have the following OPEs:

$$\Sigma(z)\Sigma(w) = \frac{1}{z-w} \tag{3.2}$$

These are precisely the OPEs of a free Majorana fermion. Indeed by a classic result of Federbush and Johnson [10], we can identify Σ with a free Majorana fermion. BRST invariance of the gravitino vertex requires the Σ, G OPE to be:

$$\Sigma(z)G(w) \sim \frac{A(w)}{(z-w)^{\frac{1}{2}}} + (z-w)^{\frac{1}{2}}(W + \partial A), \tag{3.3}$$

where we introduced two holomorphic fields A and W of spin $\frac{3}{2}$ and $\frac{5}{2}$ respectively. The worldsheet SCFT has $\mathcal{N} = 1$ supersymmetry with stress tensor T and supercurrent G. If A is not a multiple of G, then we will have two spin $\frac{3}{2}$ currents. The existence of two such spin $\frac{3}{2}$ currents either implies the underlying CFT has $\mathcal{N} = 2$ supersymmetry (which is possible only for Calabi-Yau four-folds and not Spin(7) manifolds) or the underlying manifold is a circle fibration over a G_2 holonomy manifold. This means, for a Spin(7) manifold the only possibility is $A = G$. There are altogether four holomorphic fields T,G,W and $U =: \Sigma\partial\Sigma :$. It is straightforward to verify that there exists precisely one closed superconformal algebra containing these fields, the $\mathcal{SW}(\frac{3}{2}, 2)$ algebra at $c = 12$ [18]. This confirms the result of Shatashvili and Vafa. In the proof we did not use any aspect of the non-linear sigma model representation of the Spin(7) CFT, nor the large radius limit. Indeed this enhanced SCA is expected to hold regardless of the size of the internal manifold.

3.2 Non-critical Strings and Calabi-Yau manifolds

As reviewed in Appendix A, the Liouville mode ϕ becomes a dynamical field on the worldsheet of a non-critical string. Including this mode we may view the spacetime as a copy of $\mathbb{R}^n \times \mathbb{R}_\phi$ where $SO(1, n+1)$ Lorentz invariance is broken to $SO(1, n)$ by the Liouville field. Inclusion of this Liouville field restores conformal invariance on the worldsheet. For the superstring, one can view non-critical superstrings as coupling of matter SCFT to $\mathcal{N} = 2$ super-Liouville. The $\mathcal{N} = 2$ super-Liouville action is written in terms of a chiral superfield Φ as:

$$S = \int d^2 z d^4 \theta \bar{\Phi} \Phi + \int d^2 z d^2 \theta \mu e^{\gamma \Phi} \tag{3.4}$$

The scalar component of Φ is denoted as $\phi_l + i\theta$ where ϕ_l is the Liouville field with background charge Q, and θ is the $\mathcal{N} = 2$ super=partner of ϕ_l. The central charge of $\mathcal{N} = 2$ super-Liouville is given by:

$$c_L = 3 + 3Q^2 \tag{3.5}$$

The action (3.4) is conformally invariant for:

$$\gamma = \frac{1}{Q} \tag{3.6}$$

Note that, for $\gamma = \frac{1}{Q}$ the field θ lives on a circle of radius Q.

$\mathcal{N} = 2$ super-Liouville can be tensored with a $\mathcal{N} = 2$ SCFT $\mathcal{N}$ to form a non-critical string background, the non-compact flat directions being $\mathbb{R}^{10-2n}$. We require:

$$n = \hat{c}_{\mathcal{N}} + \frac{1}{2} + Q^2 \tag{3.7}$$

In this case, we can perform GSO projection in a manner that preserves supersymmetry, as shown in [27],[9].

As a simple example, let us consider a non-critical string vacuum of the form $\mathbb{R}^4 \times \mathbb{R}_\phi \times \mathcal{N}$. Assuming ϕ is a Liouville field with background charge Q, and $\mathcal{N}$ is a $\mathcal{N} = 1$ worldsheet SCFT with central charge c we would like to determine the conditions under which eight supercharges are preserved in the field theory on

$\mathbb{R}^4$[1]. As shown by [9], if $\mathcal{N}$ possesses a $U(1)$ affine current J with respect to which $\mathcal{N}/U(1)$ has $\mathcal{N}=2$ SCA on the worldsheet, then it is possible to obtain a GSO projection that preserves eight supercharges. First, we can write J in terms of a chiral boson Y as:

$$J = i\partial Y, \quad Y(z)Y(w) = -\log(z-w) \tag{3.8}$$

Now $\mathcal{N}/U(1)$ is supposed to be a $\mathcal{N}=2$ SCFT with central charge :

$$\hat{c}_{\mathcal{N}/U(1)} = \hat{c}_{\mathcal{N}} - \frac{1}{2} = n - 1 - Q^2 \tag{3.9}$$

This implies the $U(1)_R$ current $J_{\mathcal{N}/U(1)}$ satisfies:

$$J_{\mathcal{N}/U(1)} = i\sqrt{n-1-Q^2}\,\partial H \tag{3.10}$$

Now, we can write down the FMS vertex:

$$Q_A = \int dz e^{-\frac{1}{2}\varphi} S_A e^{\pm\frac{i}{2}\sqrt{n-1-Q^2}H} e^{\pm\frac{i}{2}\sqrt{1+Q^2}Y} \tag{3.11}$$

We have written (3.11) in $-\frac{1}{2}$ picture, introducing spin-fields S_A for the flat $\mathbb{R}^{10-2n}$. These spin-fields can be explicitly described by bosonization. We can pairwise bosonize the $(10-2n)$ Majorana fermions by introducing $5-n$ chiral bosons H_i, $i=1,...,5-n$. In terms of the H_i it is easy to describe the spin-fields S_A:

$$S_A = e^{\frac{i}{2}r_i H_i} \quad r_i = \pm\frac{1}{2}, \quad i = 1,..,5-n \tag{3.12}$$

[1] Note that the field theory is five dimensional, though the supersymmetries, as well as the particle content are more naturally arranged in four dimensional multiplets. Indeed the four dimensional field theory is holographically dual to string theory on $\mathbb{R}^4 \times \mathbb{R}_\phi \times \mathcal{N}$.

There are in total 32 supercharges arising from (3.11) and another 32 from the right movers. It is however a simple matter to check that only 2^{6-n} supercharges are mutually local. In other words, we obtain a string vacuum with precisely the same supersymmetry as in a critical string compactification on Calabi-Yau n-folds.

That this is more than just a numeric coincidence of supersymmetries and that there is a simple connection between Calabi-Yau n-folds and non-critical strings was first noted by Witten [29] (see also [30]), and later made more precise by [9]. The approach of [29] has the advantage that the connection between critical and non-critical strings is made in an elegant manner, and we will briefly describe the approach followed in [29] with an example. Consider the worldsheet CFT governing string propagation on a Calabi-Yau three-fold. Calabi-Yau three-folds acquire conifold singularities, that is points in Calabi-Yau moduli space where the metric looks locally like that of a conifold. A conifold can be thought of as a complex cone in $\mathbb{C}^4$:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \quad z_i \in \mathbb{C}^4 \tag{3.13}$$

By a change of variables we can write (3.13) as:

$$xy - uv = 0, \quad x, y, u, v \in \mathbb{C}^4 \tag{3.14}$$

In the form (3.13) it is clear that the conifold singularity has a collapsing $S^2 \times S^3$. The conifold admits a small resolution obtained by blowing up the S^2 (thought of as $\mathbb{P}^1$) into the total space of the line bundle $\mathcal{O}(-1) + \mathcal{O}(-1)_{\mathbb{P}^1}$.

Now consider the worldsheet SCFT corresponding to the non-compact Calabi-Yau three-fold defined by $\mathcal{O}(-1) + \mathcal{O}(-1)_{\mathbb{P}^1}$.

In order to describe this CFT we are going to follow a short-cut which is remarkably efficient, and due to Witten [31]. Instead of describing the $\mathcal{N} = 2$ SCFT that corresponds to the sigma model on the resolved conifold, we will find it easier to describe a $\mathcal{N} = 2$ supersymmetric abelian gauge theory whose D-flatness conditions parameterize the resolved conifold (3.14). Then, we can argue that the IR limit of this gauged linear sigma model (GLSM) is the nonlinear sigma model that describes the $\mathcal{N} = 2$ SCFT corresponding to the resolved conifold.

$\mathcal{N} = (2,2)$ supersymmetry in two dimensions can be conveniently thought of as arising via dimensional reduction of $\mathcal{N} = 1$ supersymmetry in four dimensions. $\mathcal{N} = 1$ supersymmetry in $d = 4$ has a $U(1)$ R-symmetry, and another $U(1)$ R-symmetry arises upon dimensional reduction, leading to a global symmetry $SO(1,1) \times U(1)_V \times U(1)_A$. There are two types of multiplets of interest to us: the chiral and twisted-chiral multiplets. The chiral multiplet Φ has a complex scalar ϕ as bosonic content and is related to the four dimensional chiral multiplet by reduction. The four dimensional vector multiplet, upon dimensional reduction gives rise to a two dimensional vector multiplet V, with a complex scalar σ, and gauge field A_μ as its bosonic content. The field strength of the vector field lies in a twisted-chiral multiplet Σ. Introducing (2,2) superspace notation [31], we can write:

$$D_\pm = \partial_{\theta_\pm} - i\bar{\theta}^\pm \partial_\pm, \quad \bar{D}_\pm = -\partial_{\bar{\theta}^\pm} + i\theta^\pm \partial_\pm, \quad \partial_\pm = \partial_{x_0} \pm \partial_{x_1} \tag{3.15}$$

The supercharges $Q_\pm$ and $\bar{Q}_\pm$ are defined by:

$$Q_\pm = \partial_{\theta_\pm} + i\bar{\theta}^\pm \partial_\pm, \quad \bar{Q}_\pm = -\partial_{\bar{\theta}^\pm} - i\theta^\pm \partial_\pm, \tag{3.16}$$

and they obey the supersymmetry algebra:

$$\{Q_\pm, \bar{Q}_\pm\} = P_\pm, \quad {Q_\pm}^2 = 0, \quad \{Q_+, Q_-\} = 0 \tag{3.17}$$

The chiral superfield Φ is defined by:

$$\bar{D}_\pm \Phi = 0, \tag{3.18}$$

and can be expanded as:

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + 2\theta^+\theta^- F + ... \tag{3.19}$$

The twisted-chiral multiplet satisfies:

$$\bar{D}_+\Sigma = D_-\Sigma = 0, \tag{3.20}$$

and can be expanded as:

$$\Sigma = \sigma + i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\bar{\theta}^-\lambda_- + 2\theta^+\bar{\theta}^-(D - iF_{01}) + ... \tag{3.21}$$

The vector multiplet V is defined as:

$$\begin{aligned} V = {} & \theta^-\bar{\theta}^-(A_0 - A_1) + \theta^+\bar{\theta}^+(A_0 + A_1) \\ & -\theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} + \sqrt{2}i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) \\ & +\sqrt{2}i\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + 2\theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D \end{aligned} \tag{3.22}$$

Consider a $\mathcal{N} = 2$ supersymmetric gauge theory with an abelian vector multiplet V and four chiral superfields Φ^i, and $\tilde{\Phi}^i$, $i = 1, 2$. The fields Φ^i carry charge +1 under the $U(1)$ gauge field, while $\tilde{\Phi}^i$ carry charge -1. The action in (2,2) superspace is given by:

$$S = \int d^4\theta(\bar{\Phi}^i e^V \Phi^i + \bar{\tilde{\Phi}}^i e^{-V}\tilde{\Phi}^i - \frac{1}{2e^2}\bar{\Sigma}\Sigma) - \frac{1}{2}\int d\bar{\theta}^+ d\theta^- \xi\Sigma, \quad \xi = r - i\theta \tag{3.23}$$

In (3.23), ξ is the complexified Fayet-Iliopuolos (FI) parameter and θ is the so called theta angle of the gauge theory. The moduli space of vacua correspond to the space of solutions to the D-flatness conditions:

$$D = \sum_i (|\phi^i|^2 - |\tilde{\phi}^i|^2) = r \tag{3.24}$$

modulo the gauge symmetry that acts on ϕ^i and $\tilde{\phi}^i$. For $r = 0$, the space of D-flatness conditions modulo gauge invariance is parameterized by the gauge invariant composite super-fields $X = \Phi^1\tilde{\Phi}^1$, $Y = \Phi^2\tilde{\Phi}^2$, $U = \Phi^1\tilde{\Phi}^2$ and $V = \Phi^2\tilde{\Phi}^1$ subject to the relation:

$$xy - uv = 0, \quad x, y, u, v \in \mathbb{C}^4 \tag{3.25}$$

Turning on the FI parameter r turns the moduli space of solutions to the D-flatness into the bundle $\mathcal{O}(-1) + \mathcal{O}(-1)_{\mathbb{P}^1}$. At a generic point on the Higgs branch of the $\mathcal{N} = 2$ supersymmetric gauge theory defined in (3.23) the entire gauge multiplet acquires mass by the super-Higgs mechanism. The light modes that survive are tangent to the space of solutions to (3.24) and we expect that the IR limit of this GLSM is nothing but a sigma model on the resolved conifold. The FI parameter ξ is an exactly marginal deformation of the SCFT describing the conifold and r is related to the size of the resolved $\mathbb{P}^1$, while θ is the integral of the NS-NS B-field on the $\mathbb{P}^1$.

The analysis that leads to the conclusion that the IR limit of the GLSM of (3.23) is the sigma model on the resolved conifold, depends on the fact that $|r| >> 0$ (or if $r = 0$ then $\theta = \pi$. Indeed, we can integrate the gauge multiplet V out classically, only when $|r| >> 0$. The GLSM of (3.23) has two branches classically:

the Coulomb branch and the Higgs branch. The IR limit of the gauge theory on the Higgs branch is what we found to be the nonlinear sigma model associated with the resolved conifold. The Coulomb branch is parameterized by the VEV of σ and exists only when the VEVs of the matter multiplets vanish. Unlike in higher dimensions, the Coulomb and Higgs branches are not distinct for finite gauge coupling e. Indeed, the scalar fields cannot be localized in two dimensions due to strong IR divergences and at finite e, the Coulomb and Higgs branches are not really distinct. However as argued by Witten [29], the two branches become distinct in the limit $e \to \infty$ which is precisely the IR limit of the GLSM. Now classically the Coulomb and Higgs branches meet at a point, so how can they be separate in the IR limit? This is possible if a throat develops on the Higgs and Coulomb branches in the limit $\xi \to 0$ such that the distance between the branches goes to ∞ in the limit $e \to \infty$. This is precisely what happens, as we will verify shortly. What is the implication of this result? The worldsheet SCFT corresponding to the nonlinear sigma model on the resolved conifold develops a singularity in the limit $\xi \to 0$ and the effective string description near this singularity is obtained by considering the throat metric. We will show below that the SCFT describing the throat is precisely the $\mathcal{N} = 2$ super-Liouville CFT at $\hat{c} = 3$ (or $Q = \sqrt{3}$) providing a concrete relationship between non-critical superstrings and the singularities of Calabi-Yau manifolds.

Indeed, going back to the action (3.23) we see that for $\xi \to 0$, the wave-function can spread on to the Coulomb branch, meaning the light degree of freedom is the gauge multiplet Σ. For a non-zero VEV for σ we can integrate out all the chiral super-fields (they acquire a tree-level mass), and the resulting 1-loop effective action is an action that depends only on the twisted-chiral super-field Σ. However, the central charge of the field theory has to be $\hat{c} = 3$ whereas the theory of a free chiral super-field has $\hat{c} = 1$. The only way to make $\hat{c} > 1$ is to endow one of the scalars with a background charge. This means Σ, rather than being a free chiral super-field is actually the $\mathcal{N} = 2$ linear dilaton. Including the FI term, we obtain the $\mathcal{N} = 2$ super-Liouville action with $Q = \sqrt{3}$.

This argument provides an elegant relation between non-critical strings and singularities of Calabi-Yau spaces. In the next section, we describe non-critical strings that describe singularities of Spin(7) manifolds [18].

3.3 Noncritical strings and Spin(7) manifolds

As we have seen so far, compact Spin(7) manifolds are characterized on the worldsheet by the appearance of an enhanced SCA of the form $\mathcal{SW}(\frac{3}{2}, 2)$ with $c = 12$. It is a very interesting problem to determine the singularities of Spin(7) manifolds, in particular those singularities that occur at finite distance in Spin(7) moduli space. This is a daunting task from the mathematical point of view, since Spin(7) geometry is real geometry unlike Calabi-Yau geometry which is complex geometry and singularities of complex manifolds are much tamer than real singularities. From the string theory point of view the elegant relationship between gauged linear sigma models and Calabi-Yau manifolds (or equivalently the relation between catastrophe theory and conformal field theory [43], [44]) is what is responsible for the relative ease of description of singularities of Calabi-Yau manifolds. This simplicity in turn reflects the fact that certain quantities are protected in a $\mathcal{N} = 2$ supersymmetric field theory and in particular the super-potential is one such protected quantity that effectively parameterizes the IR fixed point SCFT. Equivalently, the fact that the GLSM flows to a Calabi-Yau SCFT hinges crucially on the $\mathcal{N} = 2$ worldsheet supersymmetry. In the case of Spin(7) manifolds, if we attempt to write down a GLSM, there is no guarantee that it flows to a non trivial Spin(7) CFT. Similarly, the $\mathcal{N} = 1$ super-potential receives uncontrollable radiative corrections, and does not parameterize the IR fixed point CFT. For these reasons, a direct attack on the problem of determining singular CFTs corresponding to Spin(7) spaces is not feasible at the present juncture. However, knowing that singular CFTs are effectively described by noncritical strings, we will show that the IR SCFT corresponding to singular Spin(7) spaces can be determined. This may help us understand the nature of singularities in the Spin(7) moduli space.

The worldsheet description of $\mathcal{N} = 1$ non-critical strings contains a $\mathcal{N} = 1$ linear dilaton factor. The $\mathcal{N} = 1$ linear dilaton CFT has a scalar ϕ with background charge Q and a free Majorana fermion ψ_ϕ so that the central charge is:

$$c = \frac{3}{2} + 3Q^2 \tag{3.26}$$

A naive attempt to generalize the construction of [9] would involve simply tensoring $\mathcal{N} = 1$ super-Liouville together with some $\mathcal{N} = 1$ internal SCFT $\mathcal{M}$ and look for GSO projection that preserves two supercharges in $d = 2$. This would then be the non-critical string vacuum for singular Spin(7) spaces in a manner generalizing the result for Calabi-Yau spaces [9]. This does not work however, as there turns out to be no GSO projection that preserves supersymmetry. In order to describe a superstring propagating on a Spin(7) manifold we expect that the internal SCFT has a $\mathcal{SW}(\frac{3}{2},2)$ algebra at $c = 12$. The fact that the string theory is non-critical means that this internal SCFT contains a $\mathcal{N} = 1$ linear dilaton factor. We shall argue that de-coupling the $\mathcal{N} = 1$ linear dilaton multiplet from the Spin(7) SCA leads to an enhanced SCA that is denoted as $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$, at central charge:

$$c = \frac{21}{2} - 3Q^2 \tag{3.27}$$

The $\mathcal{N} = 1$ $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ algebra contains two spin-$\frac{3}{2}$ multiplets and a spin-2 multiplet. One of the two spin-$\frac{3}{2}$ multiplets is the super-current multiplet containing (T, G) where T is the stress tensor and G is the super-current. The other spin-$\frac{3}{2}$ multiplet is denoted (U, H), while the spin-2 multiplet will be denoted as (V, W). For more details on the representation theory of this SCA, and the OPEs that define the algebra the reader is urged to refer to [45] and [46]. For $Q = 0$ the $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ algebra becomes the G_2 SCA found in [8].

The straightforward way to show this claim is to simply de-couple the $\mathcal{N} = 1$ linear dilaton factor from the Spin(7) SCA and obtain the resulting commutation relations that define the $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA. Interestingly, this is a simple extension of what has been observed in [45] under an entirely different context. Gepner and Noyvert were interested in studying representations of the Spin(7) SCA, and in particular were interested in the different extensions of the Spin(7) SCA, that is other enhanced SCAs that contained the Spin(7) SCA. One of the results found in [45] is that the $\mathcal{SW}(\frac{3}{2},2)$ SCA at central charge c admitted an extension by a dimension-$\frac{1}{2}$ multiplet such that the resulting enhanced SCA was the $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA at $\tilde{c}=c-\frac{3}{2}$. What was essentially done in [45] was that the $h=\frac{1}{2}$ multiplet was de-coupled from the $\mathcal{SW}(\frac{3}{2},2)$ SCA and what one thus obtained was the $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA. There is a simple explanation for this result, which we will be able to explain later in this section, but it is already clear that our problem can now be solved. Indeed, what we want to do is simply to de-couple a $h=\frac{1}{2}$ multiplet from the $\mathcal{SW}(\frac{3}{2},2)$ algebra at $c=12$, with the only difference being that the $h=\frac{1}{2}$ multiplet contains a scalar ϕ with background charge Q. It is straightforward to generalize the result of [45] to this case, and we find that the result of de-coupling the $\mathcal{N}=1$ linear dilaton from the Spin(7) SCA yields a $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA at $c=\frac{21}{2}-3Q^2$.

Phrased differently, a non-critical string background for Spin(7) spaces can be obtained by tensoring the $\mathcal{N}=1$ linear dilaton with an internal SCFT on $\mathcal{M}$ that has a $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA at central charge $c=\frac{21}{2}-3Q^2$. The GSO projection that yields a string vacuum with two supercharges is obtained by using the FMS vertex with the internal spin-field taken to be the Majorana fermion of the Spin(7) SCA.

We will provide alternative arguments that lead to the same conclusion as stated above. These different arguments serve to reinforce the conclusion as well as to provide more intuitive understanding of the result. First of all, suppose we knew nothing about Spin(7) compactifications and the underlying SCFT, but all we had was our understanding of Calabi-Yau SCFTS, we could have obtained information about the Spin(7) SCFTs as follows. A compact Calabi-Yau four-fold is an eight dimensional Kähler manifold with a nowhere vanishing holomorphic four-form Ω. By a result of Bochner, this four-form is in fact covariantly constant and defines a calibration called the SLAG calibration. A Spin(7) manifold can be thought of as an eight dimensional manifold with a nowhere vanishing real four-form. Now given a Calabi-Yau manifold with a $\mathbb{Z}_2$ involution that acts on the Kähler form J by $J \to -J$ while preserving the holomorphic four-form Ω, we can quotient by this involution to produce a Spin(7) manifold[2].

From the worldsheet point of view, the Calabi-Yau four-folds are described by a $\mathcal{N} = 2$ SCFT with $c = 12$. Such a theory contains a $U(1)$ R-current J which can be written in terms of a chiral boson ϕ as :

$$J = 2i\partial\phi, \quad \phi(z)\phi(w) = -\log(z-w) \tag{3.28}$$

There is also a spectral flow operator U given by:

$$U =: e^{2i\phi} : \tag{3.29}$$

It is easy to see that $V = U + \bar{U}$ is a holomorphic spin-2 field. The $\mathbb{Z}_2$ involution acts on the Calabi-Yau CFT by $J \to -J$ and leaves V invariant. It also leaves the $\mathcal{N} = 1$ supercurrent G and the stress tensor T invariant. Now including T, G, v and W the $\mathcal{N} = 1$ superpartner of V, we have the right field content to generate the Spin(7) SCA, that is the $\mathcal{SW}(\frac{3}{2}, 2)$ SCA at $c = 12$.

One can do an analogous computation to relate singular Calabi-Yau CFTs and singular Spin(7) CFTs. Non-critical string vacua that describe singular Calabi-Yau

[2] Technically the holonomy group of such a quotient X is in $SU(4)\times\mathbb{Z}_2$ so in particular $\pi_1(X) = \mathbb{Z}_2$ and is not a Spin(7) manifold which has $\pi_1(X) = 0$ and holonomy strictly Spin(7). This distinction fades however upon deformation of the quotient, and string theory on such spaces is identical to string theory on Spin(7) manifolds.

four-folds are obtained by tensoring the $\mathcal{N}=2$ linear dilaton CFT with an internal CFT $\mathcal{N}$ and performing a suitable GSO projection. The key requirement is that $\mathcal{N}$ must give rise to a $\mathcal{N}=2$ SCFT on the worldsheet. Again, we can perform a $\mathbb{Z}_2$ involution on this CFT to obtain a Spin(7) CFT. This $\mathbb{Z}_2$ acts as the orbifold $\frac{S^1\times\mathcal{N}}{\mathbb{Z}_2}$ where $\mathcal{N}$ is a $\mathcal{N}=2$ SCFT. Our claim then amounts to the statement that $\frac{S^1\times\mathcal{N}}{\mathbb{Z}_2}$ admits a $\mathcal{SW}(\frac{3}{2},\frac{3}{2},2)$ SCA precisely when $\mathcal{N}$ yields a $\mathcal{N}=2$ SCA.

This statement can indeed be viewed as the CFT generalization of the statement that we can obtain G_2 holonomy manifolds by considering the quotient $\frac{S^1\times CY_3}{\mathbb{Z}_2}$.

In the case of G_2 holonomy manifolds, one can exhibit the worldsheet equivalent of the relation between the orbifold $\frac{S^1\times CY_3}{\mathbb{Z}_2}$ and G_2 as follows.

The CY_3 CFT contains a $U(1)$ R-current that can be written in terms of a chiral boson ϕ as:

$$J=\frac{\sqrt{3}}{2}i\partial\phi \tag{3.30}$$

As far as the S^1 CFT is concerned, it is described by a free boson θ and a free Majorana fermion ψ_θ. The $\mathbb{Z}_2$ acts by $\theta\to-\theta$, $\phi\to-\phi$ such that upon tensoring the two CFTs, it is possible to identify as holomorphic spin-$\frac{3}{2}$ current H defined by:

$$H=:e^{i\sqrt{3}\phi}:+:e^{-i\sqrt{3}\phi}:+J\psi_\theta \tag{3.31}$$

such that H is invariant under the involution.Taking OPEs with $G=G^++G^-$ defines its spin-2 superpartner M while the other generators that fill out the G_2 SCA are obtained from the HM and MM OPEs.

Analogously, suppose we consider $\frac{S^1\times\mathcal{N}}{\mathbb{Z}_2}$. The CFT corresponding to $\mathcal{N}$ has a $U(1)$ R-current which can be written as $J=i\sqrt{\frac{c}{3}}\partial\phi$. The CFT corresponding to S^1

is again defined by θ and ψ_θ. The conserved spin-$\frac{3}{2}$ current can be now written as:

$$H =: e^{iQ\theta} e^{i\sqrt{\frac{c}{3}}\phi} : + : e^{-iQ\theta} e^{-i\sqrt{\frac{c}{3}}\phi} : + J\psi_\theta, \quad c = 9 - 3Q^2 \tag{3.32}$$

The nontrivial fact is that H as defined in (3.32) is local with respect to the space-time supercharges. Including the $\mathcal{N} = 1$ superpartner and the other generators forced by the HH and MM OPEs allows us to obtain the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA at $c = \frac{21}{2} - 3Q^2$.

An important point is that the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA is defined by two parameter, the central charge c and a parameter λ that is called the self-coupling. The central charge was fixed in terms of the background charge Q to be:

$$c = \frac{21}{2} - 3Q^2. \tag{3.33}$$

What fixes λ? Examining the representation theory of the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA presented in [46] we realize that the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA has a $\mathcal{N} = 1$ subalgebra isomorphic to the Tri-critical Ising model precisely when:

$$\lambda = 4\frac{(63 - 6c)(6c - 9)^2}{240(30c - 21)} \tag{3.34}$$

What is the significance of this observation? Precisely when the $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA contains a Tri-critical Ising piece, it is possible to obtain a GSO projection that preserves two supercharges upon tensoring with the $\mathcal{N} = 1$ linear dilaton [47].

We are now in a position to explain the observation of [45] that de-coupling the $h = \frac{1}{2}$ multiplet from the Spin(7) SCA yields the G_2 algebra. Indeed, starting with the G_2 algebra and tensoring with a free $h = \frac{1}{2}$ multiplet is equivalent to considering a compactification on $S^1 \times G_2$. We know that G_2 holonomy manifolds lead to three dimensional theories with four supercharges, so upon compactification on S^1 we expect to obtain a two dimensional theory with four supercharges. Two of these supercharges must arise from left movers, and two from right movers. That is, in RNS formalism the FMS vertex must be:

$$Q^1 = \int dz e^{-\frac{1}{2}\varphi} e^{\frac{i}{2}H} \Sigma, \quad Q^2 = \int dz e^{-\frac{1}{2}\varphi} e^{\frac{i}{2}H} \Sigma' \tag{3.35}$$

Then Σ defines the Ising field of the Spin(7) SCA and Σ' defines the spin-$\frac{1}{2}$ field that sits in the $h = \frac{1}{2}$ multiplet.

We have provided evidence for the fact that non-critical strings propagating on singular Spin(7) spaces are obtained by tensoring $\mathcal{N} = 1$ linear dilaton with an internal SCFT with $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA. In [18] we provide further evidence that supports this conclusion, which for reasons of space and time cannot be included here. The reader may refer to [18] for further details.

3.4 Conclusion

In chapter 3, we have obtained the criterion for a non-critical string background to preserve two supercharges in $d = 2$ and related such non-critical strings to singularities of Spin(7) spaces. We also provided a simple derivation of the Spin(7) SCA of [8] and argued that the results of [8] hold away from the large radius limit. We also examined some results of [45] regarding extension of the Spin(7) SCA and provided a physical reasoning for the observations in [45].

At this point we should note that the results of this chapter allow us to complete the classification of AdS_3 vacua with NS-flux. AdS_3 vacua with purely NS-flux admit an elegant description on the worldsheet. Using worldsheet techniques AdS_3 vacua that admitted $\mathcal{N} = 2, 3, 4$ supersymmetry were classified in [48] and [49]. Unfortunately, the techniques of [48] and [49] are not useful to classify vacua with $\mathcal{N} = 1$ supersymmetry. This case follows directly from our analysis of non-critical strings, since by adding F1-strings to vacua of the form $\mathbb{R}^2 \times \mathbb{R}_\phi$ we flow to AdS_3 vacua so our results imply that AdS_3 vacua with NS-flux preserve $\mathcal{N} = 1$ space-time supersymmetry if the internal SCFT has $\mathcal{SW}(\frac{3}{2}, \frac{3}{2}, 2)$ SCA with central charge (3.33) and self-coupling (3.34).

Having derived the worldsheet description of singular spin(7) CFTs, it would be very interesting to understand if there is a tractable relation between geometry and CFT for spin(7) spaces. Indeed all known examples of Spin(7) manifolds start with orbifolding Calabi-Yau four-folds and then blowing up the orbifold in a manner in which one preserves the real four-form that forms the Spin(7) calibration. This suggests that perhaps a relation between algebraic geometry and CFT exists wherein the real algebraic variety is obtained in some sense as a blow up of a quotient of a complex algebraic variety by an anti-holomorphic involution. It would be interesting to make this precise, as this will allow us to analyze Spin(7) CFTs by using super-renormalizable field theories in $2d$.

It is also possible to generalize our results to describe singular G_2 manifolds which yield three dimensional gauge theories which preserve four supercharges. In this case we expect holomorphy to allow better control of the singular CFT and it is certainly worth exploring further.

A. CONFORMAL FIELD THEORY AND COMPACTIFICATIONS

In this chapter, we review the basic principles of string theory, and introduce aspects of critical and non-critical strings that are useful to understand the thesis.

A.1 Basics of string theory

In the first quantized approach string theory is presented as the theory of maps of a cylindrical worldsheet on to a fixed background space-time in a manner analogous to considering quantum field theory as the worldline theory of a point particle. Unlike point particle theory of course, string theory is much more constrained. The worldsheet action is taken to be the Polyakov action[1]:

$$S = \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad \alpha = 0, 1 \tag{A.1}$$

In (A.1) we have introduced the auxiliary worldsheet metric h and σ^α are the worldsheet coordinates while X^μ are target space coordinates and $\eta_{\mu\nu}$ is the flat background metric with Minkowski signature (the choice of flat metric here is for simplicity, we will drop this requirement momentarily). Classically, the action has two dimensional diffeomorphism invariance, and Weyl invariance on the worldsheet.

[1] This is the action for the fundamental string, which is why terms involving the extrinsic curvature of the string worldsheet embedded in the target space are not added.

In two dimensions the worldsheet metric h can always be put in the form $h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}$ locally. By Weyl invariance of the classical action ϕ naively disappears from the action. The resulting two dimensional action is then conformally invariant. In the quantum theory, there is a conformal anomaly which depends on the central charge c_m of the matter sector. So naively, until $c_m = 0$ conformal invariance cannot hold quantum mechanically for arbitrary worldsheets. This fortunately turns out to be too naive. In going to conformal gauge we have fixed diffeomorphism invariance. In the quantum theory this introduces bc ghosts, which contribute to the conformal anomaly in such a manner as to cancel the matter contribution if $c_m = 26$. For the free field theory in (A.1), this means the target space-time has dimension 26, leading to the statement that the bosonic string lives in 26 dimensions.

Strings can interact only by splitting and joining. Using conformal invariance it is possible to describe scattering of strings by insertion of suitable vertex operators on the worldsheet. In the bosonic string theory, conformal invariance requires vertex operators to be conformal primaries of dimension 1. For the free theory of 26 scalars, it is easy to enumerate the massless vertex operators, among which is the tachyon vertex operator:

$$V = e^{ik^{\mu}X_{\mu}} \tag{A.2}$$

The quantum dimension of V is $-\frac{1}{2}k^2$, so for $k^2 = -2$, V has conformal dimension 1. As $k^2 < 0$, this means such a perturbation is a space-time tachyon. This means the bosonic string vacuum is unstable in perturbation theory. Another problem with the bosonic string is that there are no space-time fermions in the perturbative spectrum. Both problems can be cured if we turn to superstrings. The worldsheet action for the superstring is a $\mathcal{N}=1$ SCFT with $c_m = 15$. The $\mathcal{N}=1$ worldsheet supersymmetry is gauged, leading to the bc and $\beta\gamma$ ghosts such that for $c_m = 15$ the conformal anomaly vanishes, and the resulting string action is Weyl invariant. As simple example of a $\mathcal{N}=1$ SCFT with $c = 15$ is the theory of ten free bosons and Majorana fermions. Thus superstring theory lives in ten dimensions.

In superstring theory, the worldsheet SCFT has a Ramond (R) sector and a Neveu-Schwarz (NS) sector. The worldsheet theory being a superconformal field theory has holomorphic currents T and T_F that are conserved. T is a spin-2 current and is nothing but the stress tensor, whereas T_F is a spin-$\frac{3}{2}$ superpartner called the supercurrent. The CFT can be divided into operators which have branch cuts with respect to T_F and belong to the so called Ramond sector, along with operators that are single valued with respect to the supercurrent and belong to the NS sector. Physically, the Ramond sector vertex operators are space-time fermions, while the NS sector operators are space-time bosons. The RNS formalism has worldsheet supersymmetry manifest, but space-time supersymmetry is realized indirectly. The existence of space-time supersymmetry requires the presence of a fermionic charge Q on the worldsheet that is constructed as shown below, following FMS:

$$Q_\alpha = \int dz e^{-\frac{1}{2}\varphi} S_\alpha \tag{A.3}$$

In (A.3) the Q_α are the space-time supercharges and S_α are the ten-dimensional spin fields. An explicit formula for the spin fields can be given via bosonization. To do so, start with the ten worldsheet fermions ψ^μ and pair them up as

$\frac{1}{2}(\psi^{2m+1} + i\psi^{2m}) = e^{iH_m}$ where $m = 0, ..., 4$ are five chiral bosons. Then:

$$\Sigma_\alpha = e^{i\sum_i r_i H_i} \quad i = 0, ..., 4 \quad r_i = \pm\frac{1}{2} \tag{A.4}$$

There are thirty two such spin fields half of which lead to mutually local charges. From the right moving sector we get sixteen more supercharges so that in total the superstring leads to a ten dimensional theory with 32 supercharges. There are two types of GSO projections that preserve 32 supercharges, and lead to the type IIA and IIB theories.

Type IIA preserves 32 supercharges in the **16** and **16**$'$ of $SO(1,9)$, whereas type IIB preserves **16** and **16** and leads to a chiral ten-dimensional supergravity.

A.2 compactifications and worldsheet CFT

Kaluza-Klein compactification has been vigorously pursued in literature as a means to relate the ten-dimensional string vacua to our four-dimensional world. A general compactification to $10 - 2n$ dimensions starts with a vacuum of the form $\mathbb{R}^{10-2n} \times \mathcal{N}$. Corresponding to $\mathcal{N}$ is a $\mathcal{N} = 1$ supersymmetric worldsheet CFT with central charge $c = 3n$. The flat directions give rise to $10 - 2n$ free scalars X^i, and $5 - n$ Weyl fermions ψ^i. In general, the worldsheet theory only preserves $\mathcal{N} = 1$ supersymmetry. This is not enough however to remove the tachyon and obtain a supersymmetric vacuum. For a supersymmetric vacuum there must exist worldsheet charges Q:

$$Q \sim \int dz e^{-\frac{1}{2}\varphi} e^{\sum_i r_i H_i} \Sigma \quad i = 1, ..., 5 - n \tag{A.5}$$

Σ is a holomorphic, spin-$\frac{n}{8}$ operator in the Ramond sector of the worldsheet SCFT corresponding to $\mathcal{N}$. Using this, and the fact that Q must square to the $10-2n$ dimensional momentum (upto picture changing), we can deduce the fact that the internal SCFT possesses $\mathcal{N}=2$ worldsheet supersymmetry. Indeed, there exists a $U(1)_R$ current J which can be written as $J = i\sqrt{n}\phi$ in terms of a chiral boson ϕ such that Σ becomes:

$$\Sigma = e^{\frac{i}{2}\sqrt{n}\phi}, \quad \phi(z)\phi(w) = -\log(z-w) \tag{A.6}$$

The current J, together with the stress tensor T and $\mathcal{N}=1$ supercurrent G force the $\mathcal{N}=2$ SCA at $c=3n$, and require the introduction of another spin-$\frac{3}{2}$ field $\tilde{G}$. We can write $G = G^+ + G^-$ and $\tilde{G} = G^+ - G^-$ so that $G^\pm$ carry charges ± 1 under $U(1)_R$. In order for GSO projection (A.5) to apply to all vertex operators in the theory, the $U(1)_R$ charges of the operators must be properly quantized. This means that vacua of the form $\mathbb{R}^{10-2n} \times \mathcal{N}$ lead to theories with 2^{6-n} supercharges if the internal SCFT corresponding to $\mathcal{N}$ has $\mathcal{N}=2$ SCA with $c=3n$, and obeys a charge integrality condition. Purely geometrically, we also know that a compactification on $\mathcal{N}$ preserves 2^{6-n} supercharges if $\mathcal{N}$ is a Calabi-Yau n-fold. What we have thus noted is that a Calabi-Yau n-fold leads to a $\mathcal{N}=2$ SCFT on the worldsheet with $c=3n$ and vice versa.

A.2.1 Non-critical strings and the Liouville mode

What we have discussed above is the case of critical string compactifications. Now we proceed to briefly discuss non-critical strings. The criterion for a consistent bosonic string background was a worldsheet CFT with central charge $c = 26$. What happens if $c \neq 26$? When $c \neq 26$, the CFT is not Weyl invariant on a curved worldsheet. This does not mean the string theory is inconsistent. We used two dimensional diffeomorphisms to gauge away two of the three degrees of freedom of the auxiliary metric h_{ab}, and Weyl invariance allowed us to remove the third degree of freedom, leaving us with X^i as the only worldsheet fields along with the bc ghosts. When $c \neq 26$, the Weyl degree of freedom cannot be removed this way and becomes a physical mode on the worldsheet and is called the Liouville mode ϕ.

As shown by Polyakov, the gauge fixed action depends on ϕ in the following fashion:

$$S_l = \int d^2 z(\sqrt{\hat{g}}\frac{1}{8\pi}(\hat{\nabla}\phi)^2 + \frac{Q}{8\pi}\phi\hat{R} + \frac{\mu}{8\pi\gamma^2}e^{\gamma\phi}) \tag{A.7}$$

In writing (A.7), one introduces the Liouville mode using a background metric $\hat{g}$ to write $h = e^{\gamma\phi}\hat{g}$. It is a non-trivial fact that the action (A.7) is conformally invariant if:

$$Q = \gamma + \frac{2}{\gamma}, \tag{A.8}$$

leading to a CFT with central charge:

$$c = 1 + 3Q^2 \tag{A.9}$$

This means, we can view the X^i along with ϕ as a $(d+1)$ dimensional spacetime with central charge:

$$c_{tot} = d + 1 + 3Q^2 \tag{A.10}$$

Now (A.11) makes it clear that we can form a critical string background by setting $c_{tot} = 26$ which leads to:

$$Q = \sqrt{\frac{25 - d}{3}} \tag{A.11}$$

The Liouville mode is space-like for $d < 25$ and time-like for $d \geq 25$. An important fact about Liouville theory is that even though the central charge in (A.9) is greater than 1, the Liouville mode has only one effective degree of freedom. This means that the effective string theory lives in $d + 1$ dimensions and is also the reason why it is called a non-critical string. There is an entirely analogous reason why the Liouville mode appears in superstring theory when $d \neq 10$.

B. WORLDSHEET ASPECTS OF HYPER-KÄHLER FOUR-FOLD COMPACTIFICATIONS

The worldsheet description of $\mathcal{N} = (3,3)$ and $\mathcal{N} = (0,6)$ supersymmetric string compactifications to two dimensions starts with an internal $\mathcal{N} = 4$ SCFT with $c = 12$. This fact is proven as follows: In the RNS formalism supersymmetries that are gauged in space-time must arise from global currents on the worldsheet. This in particular means there must exist global space-time fermionic currents of the following form:

$$Q_+{}^A = \int dz e^{-\varphi/2} e^{iH/2} \Sigma^A \quad A = 1,2,3 \tag{B.1}$$

(B.1) is the standard FMS vertex which is the holomorphic part of the gravitino vertex operator at zero momentum. As usual the free fermions ψ^0 and ψ^1 corresponding to the flat two dimensional space-time have been bosonized into a chiral boson H and (B.1) is in the standard $(-\frac{1}{2})$ picture so Σ^A must have dimension $\frac{1}{2}$. Furthermore the space-time supersymmetry algebra without central charges is of the form:

$$\{Q_+^A, Q_+^B\} = \delta^{AB} P_+ \tag{B.2}$$

which requires Σ^A to satisfy the following OPEs:

$$\Sigma^A(z)\Sigma^B(w) = \delta^{AB}\frac{1}{(z-w)} \tag{B.3}$$

which automatically identify Σ^A as free Majorana fermions [10]. One can choose a pair of fermions out of the three free Majorana fermions and bosonize the pair as:

$$\frac{1}{2}(\Sigma^1 + i\Sigma^2) = e^{i\phi} \tag{B.4}$$

This defines a $U(1)$ current:

$$J_3 = 2i\partial\phi \tag{B.5}$$

which is actually an R-current. Using the remaining free Majorana fermion Σ^3 we can define two more $U(1)$ generators:

$$J^{\pm} =: e^{\pm i\phi}\Sigma^3 : \tag{B.6}$$

$(J^{\pm}, J_3)$ together generate the $SU(2)$ Kac-Moody algebra at level $k = 2$. Therefore the internal SCFT turns out to have a small $\mathcal{N} = 4$ SCA with $c = 12$. This corresponds to the case of hyper-Kähler four-fold compactifications. In the large radius limit the worldsheet description of a hyper-Kähler four-fold compactification is via a $\mathcal{N} = 4$ supersymmetric sigma model which is also conformally invariant and leads to a SCFT with small $\mathcal{N} = 4$ SCA and $c = 12$. The moduli of the SCFT are the $\mathcal{N} = 4$ chiral primaries.

In order to deform the $\mathcal{N} = (4,4)$ worldsheet SCFT one adds operators of the form:

$$\delta S = \int d^2z\mathcal{O}(z,\bar{z}) \tag{B.7}$$

To preserve conformal invariance $\mathcal{O}$ must be a dimension $(1,1)$ operator. However in order to preserve $\mathcal{N}=(4,4)$ worldsheet supersymmetry we require more. A $\mathcal{N}=(4,4)$ SCA has four left-moving (and four right-moving) supercharges which can be denoted as $G^{\pm}$ and $\tilde{G}^{\pm}$. The $\pm$ indices indicate the $U(1)$ R-charge of these operators under J_3. In order to preserve $\mathcal{N}=(2,2)$ supersymmetry generated by $G^{\pm}$ and J_3 we require the operator $\mathcal{O}$ to be the top component of a chiral superfield whose bottom component is a chiral primary operator. That is, given a ϕ_i annihilated by $G^{+}_{-\frac{1}{2}}$ and carrying charge $+1$ and conformal dimension $\frac{1}{2}$ one deforms the action S into:

$$S' = S + \int d^2z(t^iG^-\phi_i + \bar{t}^iG^+\bar{\phi}_i) \tag{B.8}$$

where by $G^+\phi_i$ one means picking the z^{-1} pole of the $G^+\phi_i$ OPE. Under what circumstances will a deformation of the form (B.8) respect $\mathcal{N}=4$ superconformal invariance? For this the deformation (B.8) must be an $SU(2)$ singlet. It is obviously a singlet under $U(1)$ generated by J so we need to only check invariance under $J^{\pm}$. This requires:

$$\tilde{G}^+\phi_i = 0 \tag{B.9}$$

That is ϕ_i is a $\mathcal{N}=4$ primary with dimension $\frac{1}{2}$. This is the standard result that the CFT moduli arise from $\mathcal{N}=4$ primary operators with dimension $\frac{1}{2}$.

Clearly, from (B.8) the CFT moduli space admits an action of $SU(2)\times SU(2)$ (we have used open string vertex operators in (B.8) so only one $SU(2)$ factor is obvious).

BIBLIOGRAPHY

[1] C. Vafa and E. Witten, "A One Loop Test Of String Duality," Nucl. Phys. B **447**, 261 (1995) [arXiv:hep-th/9505053].

[2] M. Pernici and P. van Nieuwenhuizen, "A Covariant Action For The SU(2) Spinning String As A Hyperkahler Or Quaternionic Nonlinear Sigma Model," Phys. Lett. B **169**, 381 (1986).

[3] C. Vafa, "Modular Invariance And Discrete Torsion On Orbifolds," Nucl. Phys. B **273**, 592 (1986).

[4] D. Kutasov and N. Seiberg, "Number Of Degrees Of Freedom, Density Of States And Tachyons In String Nucl. Phys. B **358**, 600 (1991).

[5] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal Nucl. Phys. B **359**, 21 (1991).

[6] M. Verbitsky, "Cohomology of compact hyperkähler manifolds," [alg-geom/9501001].

[7] E. Witten, "On flux quantization in M-theory and the effective action," J. Geom. Phys. **22**, 1 (1997) [arXiv:hep-th/9609122].

[8] S. L. Shatashvili and C. Vafa, "Superstrings and manifold of exceptional holonomy," Selecta Math. **1**, 347 (1995) [arXiv:hep-th/9407025].

[9] A. Giveon, D. Kutasov and O. Pelc, "Holography for non-critical superstrings," JHEP **9910**, 035 (1999) [arXiv:hep-th/9907178].

[10] P. Federbush, K. Johnson, Phys. Rev. **120**, 1926 (1960)

[11] K. Becker and M. Becker, "M-Theory on Eight-Manifolds," Nucl. Phys. B **477**, 155 (1996) [arXiv:hep-th/9605053].

[12] S. Gukov, C. Vafa and E. Witten, "CFT's from Calabi-Yau four-folds,"

Nucl. Phys. B **584**, 69 (2000) [Erratum-ibid. B **608**, 477 (2001)] [arXiv:hep-th/9906070].

[13] A. Beauville, “Varietes Kähleriennes dont la primiere classe de Chern est nulle,” J. Diff. Geom **18**, 755 (1983).

[14] A. Fujiki, “On the de Rham cohomology of a compact Kähler symplectic manifold,” Adv. Stud. Pure Math. **10** 105, (1987).

[15] D. Guan, “On the betti numbers of Irreducible compact hyperKähler manifolds of complex dimension four,”Math. Research Letters 4(2001),663-669.

[16] E. Witten, “Non-Perturbative Superpotentials In String Theory,” Nucl. Phys. B **474**, 343 (1996) [arXiv:hep-th/9604030].

[17] R. Kallosh, A. K. Kashani-Poor and A. Tomasiello, “Counting fermionic zero modes on M5 with fluxes,” JHEP **0506**, 069 (2005) [arXiv:hep-th/0503138].

[18] R. Sriharsha, “Noncritical strings and W-algebras,” arXiv:hep-th/0611168.

[19] D. Kutasov and N. Seiberg, “Noncritical superstrings,” Phys. Lett. B **251**, 67 (1990).

[20] D. Huybrechts, “Compact hyperkähler manifolds: basic results,” preprint 1997.

[21] A. Kapustin and M. J. Strassler, “On mirror symmetry in three dimensional Abelian gauge theories,” JHEP **9904**, 021 (1999) [arXiv:hep-th/9902033].

[22] P. S. Aspinwall, “An analysis of fluxes by duality,” arXiv:hep-th/0504036.

[23] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux,” JHEP **9908**, 023 (1999) [arXiv:hep-th/9908088].

[24] C. Vafa, “The string landscape and the swampland,” arXiv:hep-th/0509212.

[25] P. S. Aspinwall, “K3 surfaces and string duality,” arXiv:hep-th/9611137.

[26] R. Brooks and S. J. J. Gates, “Extended supersymmetry and superBF gauge theories,” Nucl. Phys. B **432**, 205 (1994) [arXiv:hep-th/9407147].

[27] D. Kutasov and N. Seiberg, “Noncritical superstrings,” Phys. Lett. B **251**, 67 (1990).

[28] A. R. Frey and J. Polchinski, "N = 3 warped compactifications," Phys. Rev. D **65**, 126009 (2002) [arXiv:hep-th/0201029].

[29] E. Witten, "Some comments on string dynamics," arXiv:hep-th/9507121.

[30] E. Silverstein and E. Witten, "Criteria for conformal invariance of (0,2) models," Nucl. Phys. B **444**, 161 (1995) [arXiv:hep-th/9503212].

[31] E. Witten, "Phases of N = 2 theories in two dimensions," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[32] R. Sriharsha, "On $\mathcal{N} = 3$ supersymmetry from $\mathcal{M}$-theory on $\mathbb{T}^8/\mathbb{Z}_2$," in preparation.

[33] R. Sriharsha, "The moduli space of hyper-K'ahler four-fold compactifications," JHEP **0703**, 095 (2007) [arXiv:hep-th/0612023].

[34] G. Ellingsrud, L. Gottsche and M. Lehn, "On the cobordism class of the Hilbert scheme of a surface,"] [arXiv:math.AG/9904095].

[35] S. Sethi, "A relation between N = 8 gauge theories in three dimensions," JHEP **9811**, 003 (1998) [arXiv:hep-th/9809162].

[36] J. Milnor and J. Stasheff, "Characteristic Classes," Annals of Mathematics Studies, No.76, Princeton University Press, Princeton,NJ.

[37] R. Bott and L. Tu, "Differential forms in Algebraic Topology," Graduate Texts in Mathematics, Vol.82, Springer, 1982.

[38] S. Cecotti, "N=2 Landau-Ginzburg versus Calabi-Yau sigma models: Nonperturbative aspects," Int. J. Mod. Phys. A **6**, 1749 (1991).

[39] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[40] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," Phys. Lett. B **379**, 99 (1996) [arXiv:hep-th/9601029].

[41] M. R. Douglas and S. Kachru, "Flux compactification," arXiv:hep-th/0610102.

[42] B. R. Greene, D. R. Morrison and A. Strominger, "Black hole condensation

and the unification of string vacua," Nucl. Phys. B **451**, 109 (1995) [arXiv:hep-th/9504145].

[43] C. Vafa and N. P. Warner, "Catastrophes and the Classification of Conformal Theories," Phys. Lett. B **218**, 51 (1989).

[44] E. J. Martinec, "Algebraic Geometry and Effective Lagrangians," Phys. Lett. B **217**, 431 (1989).

[45] D. Gepner and B. Noyvert, "Unitary representations of SW(3/2,2) superconformal algebra," Nucl. Phys. B **610**, 545 (2001) [arXiv:hep-th/0101116].

[46] B. Noyvert, "Unitary minimal models of SW(3/2,3/2,2) superconformal algebra and manifolds of G(2) holonomy," JHEP **0203**, 030 (2002) [arXiv:hep-th/0201198].

[47] T. Eguchi, Y. Sugawara and S. Yamaguchi, "Supercoset CFT's for string theories on non-compact special holonomy manifolds," Nucl. Phys. B **657**, 3 (2003) [arXiv:hep-th/0301164].

[48] A. Giveon and M. Rocek, "Supersymmetric string vacua on AdS(3) x N," JHEP **9904**, 019 (1999) [arXiv:hep-th/9904024].

[49] A. Giveon and A. Pakman, "More on superstrings in AdS(3) x N," JHEP **0303**, 056 (2003) [arXiv:hep-th/0302217].