THESIS REPORT
Master's Degree

SYSTEMS
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Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), the University of Maryland, Harvard University, and Industry

Optimal Estimation of Domains of Attraction for Nonlinear Dynamical Systems

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M.S. 92-14

Optimal Estimation of Domains of Attraction for Nonlinear Dynamical Systems

by

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Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Science
1992

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Abstract

Title of Thesis: Optimal Estimation of Domains of Attraction for Nonlinear Dynamical Systems

Name of degree candidate: Stefano Paolo Coraluppi

Degree and Year: Master of Science, 1992

Thesis directed by: Professor André Tits and
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This thesis implements a recently proposed algebraic methodology for optimal domain of attraction estimation, and extends the method to include optimal estimation of the largest inscribed ball. In addition, a numerical optimal estimation methodology is proposed. The thesis addresses the important issues of Liapunov function construction and the optimal choice of parameters in the family of Liapunov functions. Several examples are included, including a detailed discussion of the classical inverted pendulum. Finally, the thesis addresses the importance of including a measure of the size of the domain of attraction as part of a generalized objective function in optimization-based controller design.

Dedication

to

Nonna Annamaria

with love

Acknowledgements

The primary ingredient that has made this thesis possible is the love and support of my parents, Luisa and Giorgio Coraluppi. I also want to recognize the support of my sisters Sabina, Monica, and Chiara.

I would like to thank Dr. André Tits and Dr. Eyad Abed for guiding my research and for many helpful discussions. They have unselfishly shared with me their time and their vast knowledge of control theory.

I am grateful to my friends Gísli Hjaltason and Nazi Maleki for their generous sharing of their time and talents. Gisli helped me in using *Mathematica* and allowed me to use his computer for the typing of this thesis, and assisted me in the use of *Word* 5.0. Nazi helped me in getting familiarized with the *Simnon* and *kaos* simulation software. They have saved me a lot of time and frustration and are sincerely thanked.

Finally, I thank the Graduate School of the University of Maryland for its financial support.

Contents

Chapter I. Introduction

The nature of control system design has undergone a rapid evolution as a result of developments in computer technology. Today's designer is no longer limited to the use of simple graphical schemes for controller design. It is now possible to develop high order as well as nonlinear models of the open loop system and to develop complex and optimized control schemes. Thus the modern setting for controller design is one in which the designer does not seek ad hoc designs or closed form, analytical solutions arising from simple design specifications and objectives. Rather, the designer deals with general constraints and seeks to maximize a combination of objectives.

The topic of this thesis is motivated by this design framework. In the design of controllers for many nonlinear systems, an important consideration is the size of the domain of attraction of an equilibrium point. A measure of the size is given by the radius of the largest ball in $\Re^n$ contained in the domain of attraction. This work presents a methodology for the optimal estimation of this size. This estimate can then be used in combination with other objectives as part of a generalized objective function.

An understanding of the estimation methodology presented in this thesis requires knowledge of mathematical tools from diverse fields. Chapter 2 presents a summary of these tools from nonlinear controls theory and abstract algebra. Many approaches have been proposed in the controls literature to perform domain of attraction estimates. These are presented in chapter 3. The methods are classified as non-Liapunov and Liapunov-based. The latter are characterized by their reliance on a suitably constructed Liapunov function. The available Liapunov based methods are classified as Zubov methods and LaSalle methods.

The methodology pursued in this thesis is a LaSalle method. As such, the issue of having a suitable Liapunov function must be addressed. The construction of Liapunov functions is a difficult task which has not progressed substancially since the days of Liapunov, though advances have recently been attained for the class of systems that are critically stable. The construction of families of Liapunov functions is addressed in chapter 4.

Given a function in the family of Liapunov functions, the next task is to find the critical level set value for that Liapunov function. In chapter 5 we approach this task in two ways. The first approach is through a scheme that exploits knowledge of the boundary of the domain of attraction. This reduces the problem to the solution of a set of equations with polynomial nonlinearities. The method is made viable by the application of the theory of Gröbner bases to the solution of this system of equations. This method has been proposed recently, and it is implemented in this work. The proposed method is extended by also applying the theory of Gröbner bases to the optimal estimation of the largest ball contained within the (optimal) estimate of the domain of attraction.

Although the algebraic approach in principle produces the optimal estimate, we find experimentally that it is limited to systems with simple dynamics and simple Liapunov functions. Thus in this thesis a second methodology is developed. This numerical methodology also results in the optimal estimation of the domain of attraction. Though less elegant than the algebraic approach, it is more feasible in practice with currently available tools.

Each choice of Liapunov function from a family of such functions results in a different estimate of the domain of attraction and of the radius of the inscribed sphere. Thus, there is a need for an intelligent selection of parameter values to obtain the best possible estimate. Chapter 6 addresses the task of optimizing the parameter values of the family of Liapunov functions.

There are several examples contained in the thesis as the optimal estimation methodology is developed. In addition, the methodology is applied to the inverted pendulum with linear stabilization. This example is developed in detail in chapter 7. Chapter 8 summarizes the goals and results of this research and proposes further directions of study. This includes research directed at improving the optimal estimation methodology, as well as on the use of this estimate in optimization-based controller design.

The appendices contain the computer code used to implement the algebraic estimation methodology. The code is written in *Mathematica,* a powerful high level language useful for numerical, symbolic, and graphical computation. The simulation of system dynamics to verify the computational results obtained is done with the use of the *Simnon* and *kaos* software packages.

Chapter II. Mathematical preliminaries

This chapter is intended to summarize the key mathematical tools used in this work. For a more detailed and complete presentation, sources in the bibliography or equivalent sources should be consulted.

i. Liapunov theory

Liapunov's second method, or direct method, is a key tool for the qualitative study of nonlinear differential equations. It was first presented by the Russian mathematician Aleksandr Liapunov in his doctoral dissertation in 1892. The following is the version of his theorem that will be useful for our purposes.

Theorem 1 (Liapunov). Let D be an open subset of $\Re^n$ and let $0 \in D$ be an equilibrium point of the autonomous system of equations

$$\dot{x} = f(x), \tag{1}$$

where $f{:}D \to \Re^n$ is a locally Lipschitz mapping from D into $\Re^n$. Let $V{:}D \to \Re$ be a continuously differentiable function such that $V(0) = 0$, $V(x) > 0$ for $x \in D \setminus \{0\}$, and $\dot{V}(x) \le 0$ for $x \in D$.

Then $x = 0$ is stable.

If in addition we have $\dot{V}(x) < 0$ for $x \in D \setminus \{0\}$, then $x = 0$ is asymptotically stable.

The significance of this result is that we can study the stability of an equilibrium point of a system without having to solve the differential equations describing the system. This is also possible through Liapunov's first or indirect method, i.e. through linearization about an operating point. However, linearization is inconclusive if it results in at least one eigenvalue with zero real part and none with positive part. In such a case, it has been proven that the system is not exponentially stable, but it may or may not be stable, and in fact it may still be asymptotically stable. The following example illustrates these cases.

Example 1. The scalar differential equations $\dot{x} = 0$, $\dot{x} = -x^3$, and $\dot{x} = x^3$ all have the same linearization about the origin, namely $\dot{x} = 0$. Note that the first is stable, the second asymptotically stable, and the third unstable.

A second advantage of the direct method is that it also provides global stability information. Specifically, the Liapunov function can be used to provide (conservative) estimates of the domain of attraction of the equilibrium point. The set $\Omega_c = \{x \in \Re^n \,/\, V(x) \le c\}$ is contained in the domain of attraction provided that Ω_c is bounded, connected, and contained in D.

There is a significant difficulty associated with the use of Liapunov's direct method, for though we are spared from the task of solving nonlinear differential equations, there is to date no general methodology for the construction of high order Liapunov functions.

The construction of quadratic Liapunov functions for an exponentially stable equilibrium point can be performed by using the linearized system. Consider the autonomous system (1). Let $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$. Since the equilibrium point $x = 0$ is exponentially stable, the eigenvalues of A satisfy $\mathrm{Re}[\lambda_i(A)] < 0 \forall i, 1 \le i \le n$. Then $V(x) = x^T P x$ is a Liapunov function for (1), where

P is chosen as follows. Choose a symmetric positive definite $Q \in \Re^{n\times n}$, and consider the Liapunov equation:

$$PA + A^T P = -Q \qquad (2)$$

Then the unique solution P of (2) is positive definite, and $V(x) = x^T P x$ is a Liapunov function for (1).

The Liapunov equation can be solved by using the Kronecker product, defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}.$$

We express P and Q in column form, with $p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$, where p_i and q_i are the ith columns of P and Q, respectively. It turns out that $\mathcal{A}p = -q$, where $\mathcal{A} = \left(A^T \otimes I\right) + \left(I \otimes A^T\right)$. Thus $p = -\mathcal{A}^{-1}q$. Note that $\mathcal{A}$ will always be nonsingular for an exponentially stable system, because $\mathrm{Re}[\lambda_i(\mathcal{A})] < 0 \forall i$, where $1 \le i \le n^2$. This follows from the fact that the set of eigenvalues of $\mathcal{A}$ is $\left\{\lambda_i(A) + \lambda_j(A), 1 \le i \le n, 1 \le j \le n\right\}$, and $\mathrm{Re}[\lambda_i(A)] < 0 \forall i, 1 \le i \le n$ since the linearization is exponentially stable.

The matrix $\mathcal{A}$ will be singular if the equilibrium point is aymptotically, but not exponentially, stable. In such a case, Liapunov's function cannot be utilized to construct a Liapunov function. A theorem to this effect can be found in Vidyasagar [27]:

Theorem 2. Let $A \in \Re^{n\times n}$ and let $\{\lambda_i, 1 \le i \le n\}$ be the (not necessarily distinct) eigenvalues of A. Then (14) has a unique solution P corresponding to every $Q \in \Re^{n \times n}$ if and only if $\lambda_i + \lambda_j^* \neq 0 \forall i, j$, where $*$ denotes complex conjugation.

The following is an example of the construction of a quadratic Liapunov function for an exponentially stable equilibrium point.

Example 2. Consider the system given by

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned} \qquad (3)$$

Linearizing this system about the equilibrium point (0,0), we obtain $\dot{x} = Ax$, where $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Set $Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

We have

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -2 \end{bmatrix} \text{ and } q = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so

$$p = -\mathcal{A}^{-1} q = -\begin{bmatrix} -1 & -0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0.5 & 0.5 & -0.5 & 0 \\ -0.5 & 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1 \\ -1 \\ 1.5 \end{bmatrix}.$$

Thus, a quadratic Liapunov function for (2) is given by $V(x) = x^T \begin{bmatrix} 2.5 & -1 \\ -1 & 1.5 \end{bmatrix} x$. This constructive method can be improved by exploiting the fact that P and Q are symmetric.

ii. Abstract Algebra and Gröbner Bases

To lead up to the key algebraic result that will be used in this work, several algebraic structures and definitions need to be briefly reviewed.

Definition 1. A *group* (G,+) is a set together with a binary operation for which the following properties hold:

1. Closure under (+).
2. Associativity under (+).
3. Existence of identity element in G.
4. Existence of inverses in G.

A group is said to be abelian (or commutative) if $\forall a,b \in G$, $a+b=b+a$.

Definition 2. A *ring* $(R,+,\times)$ is a set together with two binary operations for which the following properties hold:

1. (R,+) is an abelian group.
2. Closure and associativity under $(\times)$.
3. Multiplication is distributive over addition.

A ring is said to be commutative if $\forall a,b \in R$, $a \times b = b \times a$. In general, there need not be an element $1 \in R$ such that $a \times 1 = 1 \times a = a$ $\forall a \in R$. If there is, then $(R,+,\times)$ is called a ring with unit element. For our purposes, we will be interested in commutative rings with unit element. For simplicity, from this point on we refer to these simply as rings. Examples of these rings are the set of integers and the set of rationals, each with the usual addition and multiplication.

Definition 3. An *ideal* I of a ring R is a subset of R such that for all a,b∈I and for all r∈R, r×a∈I and a+b∈I.

Given a set P⊂R, (P) denotes the smallest ideal of R containing P. P is the generating set for (P).

We will be using ideals and rings in the context of polynomials. To introduce polynomial rings, we need to first introduce the notion of a field.

Definition 4. A ring for which the elements different from 0 form an abelian group under multiplication is called a *field*.

Definition 5. A *polynomial ring*, denoted by $F[x_1,x_2,\dots x_n]$, is the set of polynomials in the variables $x_1,x_2,\dots x_n$ with the usual addition and multiplication, where the coefficients are from the field F .

In this thesis, the polynomial ring of interest is $\Re[x_1,x_2,\dots x_n]$.

The following notation is used to bijectively relate monomials of $\Re[x_1,x_2,\dots x_n]$ to vectors in $\mathcal{N}^n$: For $\alpha \in \mathcal{N}^n, x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$, and $\deg x^\alpha = \alpha$. This notion of degree is useful in the context of Gröbner bases but is somewhat unintuitive as it is unlike the usual notion of degree. For example, we typically think of $\deg(x_1^2x_2)$ as 3, whereas with the notion of degree just introduced it is $\begin{bmatrix}2\\1\end{bmatrix}$.

The degree of a polynomial is then defined as the highest degree of the monomials that it contains. For the notion of highest degree to be meaningful, we need an ordering relation on the vector-valued degree values.

Definition 6. A *term ordering* $\prec$ is a well-ordering on $\mathcal{N}^n$ satisfying the following: $\forall \alpha, \beta, \gamma \in \mathcal{N}^n : \alpha \prec \beta \Rightarrow \alpha + \gamma \prec \beta + \gamma$.

The specific term ordering we will be using is known as the strict lexicographic term ordering, represented by $\prec_L$. It is defined by the following: $\alpha \prec_L \beta \Leftrightarrow \exists j$ such that $\alpha_j < \beta_j$ and $\forall i < j : \alpha_i = \beta_i$.

Example 3. Consider the strict lexicographic term ordering on $\mathcal{N}^3$. We have $\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \prec_L \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. Equivalently, we say that $\deg(x_1^2 x_2^3 x_3^7) \prec_L \deg(x_1^2 x_2^4 x_3)$. Also, since $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \prec_L \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, we have $\deg(x_1^2 x_2^3 x_3^7 + x_1 x_3^2) \prec_L \deg(x_1^2 x_2^4 x_3)$.

It is clear that the strict lexicographic term ordering depends on the ordering that we impose on the polynomial variables. In the discussion above we have implicitly worked with the variable order $(x_1, x_2, \ldots, x_n)$ for n variables. For a general set of variables this order needs to be specified.

Example 4. Consider the strict lexicographic term ordering on $\mathcal{N}^2$, and consider polynomials in the variables x, y. With the variable order (x, y), we have $\deg(xy^2) \prec_L \deg(x^2 y)$. With the variable order (y, x), we have $\deg(x^2 y) \prec_L \deg(xy^2)$.

Now we present the main concept of this section. Let A be an ideal in $\Re[x_1, x_2, \ldots x_n]$.

Definition 7. A subset $G \subset A$ is a *Gröbner base* for A with respect to the term ordering $\prec_L$ if

i. $(G)=A$ and

ii. $\forall g \in G,\ f \in A \setminus \{0\}$, we have

$$\deg(f)=\deg(g) \text{ or } \deg(g) \prec_L \deg(f)$$

The significance of a Gröbner base of an ideal of a polynomial ring is captured by the following result:

Theorem 3 (Buchberger). Consider the Gröbner base G for the ideal $A \subset \Re[x_1, x_2, \ldots x_n]$ with respect to the strict lexicographic term ordering. Then G contains a polynomial $p \in \Re[x_n]$.

This is an interesting theoretical result which is useful in solving systems of polynomial equations in several unknowns. A complete discussion of our use of Gröbner bases is contained in chapter 5.

Example 5. The ideal $(\{xy-1, y-x\})$ has the Gröbner base $\{y-x, x^2-1\}$ with respect to the variable ordering (y,x). Notice that the Gröbner basis contains a polynomial in x alone.

Buchberger's algorithm is the most commonly employed to calculate Gröbner bases. The computational expenditure is considerable both in terms of the degree of the polynomials and the number of variables.

Chapter III. Survey of estimation methods

Several methodologies for estimating domains of attraction have been proposed in the controls literature. This chapter provides an overview of some of these methods, both Non-Liapunov and Liapunov-based. In addition, motivation is given for the methodology pursued in this thesis. Before discussing these methods, let us recall the definition of domain of attraction.

Definition 8. The *domain of attraction* R_A of an asymptotically stable equilibrium point x^* of (1) is the set of points $x \in \Re^n$ such that $\lim_{t\to\infty} \Phi(t,x) = x^*$, where $\Phi(t,x)$ is the solution to (1) starting at x at t=0.

i. Non-Liapunov Methodologies

This class of methods does not explicitly use Liapunov functions. Some attention has been given to the tracking function method and the describing function method. More recently, work has been done on what is known as the trajectory reversing method.

The trajectory reversing method involves using numerical methods to expand an initial estimate of the domain of attraction. This is done by doing backward integration of (1). Alternatively, this can be viewed as forward integration of the system

$$\dot{x} = -f(x). \tag{4}$$

Assume that we have an initial estimate W_0 of the domain of attraction R_A, so that $W_0 \subset R_A$. Define the backward mapping $F{:}\Re^n \to \Re^n$ by

$$F(x) = \Phi(-t_1, x), x \in W_0, \qquad (5)$$

where $\Phi(t,x)$ is the solution of (1) that starts at x at time t=0, and where $t_1 \in (0,T)$, T>0. T is chosen such that $\Phi(-t,x)$ is defined on [0,T] for all $x \in W_0$.

Let $W_1 = F(W_0)$. Then $W_0 \subset W_1 \subset R_A$. W_0 is a subset of W_1 because W_0 is positively invariant for (1). We define this notion below.

Definition 9. A set $M \in \Re^n$ is said to be *positively invariant* if $x \in M$ implies $\Phi(t,x) \in M, t > 0$.

In fact, it can be proved that W_0 is a proper subset of W_1 provided $W_0 \neq R_A$. Thus, W_1 is a better estimate of R_A than W_0. This argument can be extended as follows. Define

$$W_i = F_i(W_0), F_i(x) = \Phi(-t_i, x). \qquad (6)$$

For $t_{i+1} > t_i, i > 0$, we have $W_i \subset W_{i+1} \subset R_A$. Thus, the sequence $W_1, W_2, \ldots$ provides increasingly better estimates of R_A.

This method is generally implemented by obtaining approximations of the sets W_i through computer simulation. The method has proven to be effective, especially for low-order systems. Nevertheless, it does not generate analytical estimates of the domain of attraction.

ii. Liapunov-based methodologies

There are two broad categories of estimation methodologies involving the use of a suitable Liapunov function. These are typically referred to as Zubov methods and LaSalle methods.

The Zubov methods are based on Zubov's theorem, which gives necessary and sufficient conditions for a given region to be the domain of attraction of an equilibrium point.

Theorem 4 (Zubov). Consider the system of differential equations

$$\dot{x} = f(x), x \in \Re^n \tag{7}$$

where $f: D \rightarrow \Re^n$ is a locally Lipschitz mapping with domain $D \subset \Re^n$, $0 \in D$. Assume that there are two functions $V: D \rightarrow \Re$ and $\phi: \Re^n \rightarrow \Re$ which satisfy the following:

(i) V is continuously differentiable, $0 \leq V(x) \leq 1 \forall x \in D$, and $V^{-1}([0,b]) \cap D$ is bounded for b<1.

(ii) ϕ is continuous on $\Re^n$, $\phi(x) \geq 0$, and $\phi(x) = 0 \Rightarrow x = 0$.

(iii) For all $x \in D$, $V(x)$ and $\phi(x)$ satisfy

$$\dot{V}(x) = -\phi(x)[1 - V(x)]. \tag{8}$$

(iv) $\lim_{x \to \partial D} V(x) = 1$.

Then D is exactly the region of asymptotic stability R_A for the origin of (8).

To find the domain of attraction, we must find a function $V(x)$ which satisfies the conditions of the theorem for a chosen $\phi(x)$. Several methods are based on finding an approximate solution to (8), which does not admit a closed form solution. Other Zubov methods involve solving a different partial differential equation known as the Zubov generalized equation.

The problem with the Zubov methods is that approximating solutions to (8) involves considerable computation and, more significantly, partial sums in the series solution to $V(x)$ do not converge uniformly to the actual $V(x)$. As a result, with currently available techniques, approximate solutions to (8) are not related in any known way to the domain of attraction. Research in this area is currently directed at circumventing these numerical difficulties using novel numerical approaches.

Recently the "Crystal Growth Algorithm" has been proposed by Hilton [15]. This approach has the advantage that as the solution to (8) is constructed on an increasing number of points on a grid in phase space, a less conservative estimate of the domain of attraction is obtained. This is more in the spirit of the LaSalle estimation methodologies. Let us briefly examine Hilton's method.

The method involves numerically approximating the solution of (8) through a two-step process. First, values of $V(x)$ at grid points around the origin are determined by using the approximation $V(x) \approx 0$ near $x = 0$. Thus, the value of $V(x)$ at these points can be found by solving the simplified equation

$$\dot{V}(x) = -\phi(x), \qquad (9)$$

where $\phi(x)$ is any function satisfying (ii) from Theorem 4.

Once the $V(x)$ values at grid points around the origin have been determined, the second part of the numerical scheme is applied. This involves using a finite difference scheme to determine $V(x)$ at grid points adjacent to the starting grid. As additional $V(x)$ values at grid points are determined, those with $V(x)<1$ are included in the domain of attraction estimate. Those with $V(x) \geq 1$ are rejected. The process continues until the estimated region is surrounded by rejected grid points.

Currently Hilton's method is limited to systems with three state variables. This work is promising and performs well for low order systems and simple dynamics. For more complex systems, issues relating to memory allocation need to be further addressed. In addition, though a dense grid of points reduces numerical errors, these errors accumulate as the estimate is extended because each approximate value is based on previous approximate values. Thus, the values at grid points far from the origin become increasingly less accurate.

Hilton's method has suggested the numerical estimation methodology introduced in this thesis. Though this also involves the use of a grid of points in state space, the methodology is entirely different. Indeed, the methodology introduced here is a LaSalle method (to be discussed below), and does not involve the use of Zubov's equation. Despite the recent advances by Hilton, it is acknowledged in the controls literature [8] that the development of effective Zubov methods is a difficult task.

LaSalle methods are based on the use of a Liapunov function defined on a region D as stated in the asymptotic stability version of Theorem 1. In particular, on the set D the Liapunov function $V(x)$ is positive definite, it has a negative definite time derivative, and $V(0)=0$. LaSalle methods involve finding the largest positive invariant set of a certain type that is contained in D. D itself is not in general a positively invariant set. Several methods involve constructing sets that are not of the form $\Omega_c = \left\{x \in \Re^n \mathbin{/} V(x) \le c\right\}$. Such methods typically are based on the theory of dynamical systems and involve characterizing the boundaries of positively invariant sets. Often the following result characterizing the boundary of the domain of attraction is employed:

Theorem 5. Let $x=0$ be an asymptotically stable equilibrium point for (1).

Then its domain of attraction R_A is an open, invariant set. In addition, the boundary of R_A is a union of trajectories.

In this thesis, we work exclusively with sets of the form Ω_c. The key issues associated with the determination of a good estimate of this form include the choice of an optimal Liapunov function from a family of Liapunov functions and a choice of $c \in \Re$ such that Ω_c is as large as possible while remaining connected, bounded, and contained in D. One method to construct Ω_c is the following. First construct a quadratic Liapunov function as discussed in chapter 2. Then determine the largest circular region $C \subset \Re^n$ for which the Liapunov function is known to satisfy the conditions of Theorem 1. Finally, the largest positive invariant set of the form Ω_c contained in C is determined. This is then the estimate of the domain of attraction. Let us illustrate this method with the following example, which follows Khalil [18].

Example 6. Consider the following system:

$$\begin{aligned} x_1 &= -2x_1 + x_1x_2 \\ x_2 &= -x_2 + x_1x_2 \end{aligned} \qquad (10)$$

Linearizing the system about the equilibrium point at (0,0), we find that the system is exponentially stable at the origin, with $A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$.

Solving the Liapunov equation with $Q=I$, we obtain $P = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

We now must determine the largest circular region C satisfying the conditions of Theorem 1. Note first that V(x) is positive on all of $\Re^n \setminus \{0\}$.

The time derivative of $V(x) = x^T P x$ is given by

$$\dot{V}(x) = -(x_1^2 + x_2^2) + (\frac{1}{2}x_1^2 x_2 + x_1 x_2^2). \quad (11)$$

Now use the change of variables

$$x_1 = \rho\cos\vartheta, x_2 = \rho\sin\vartheta. \quad (12)$$

We then have

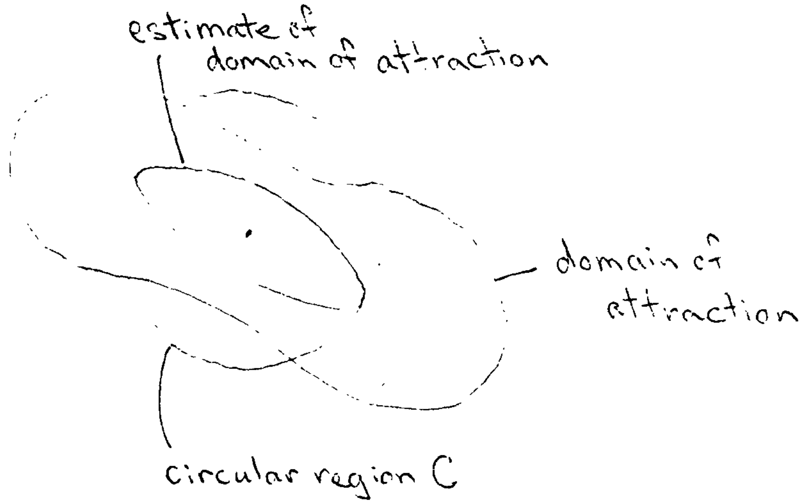
$$\dot{V}(x) = -\rho^2 + \rho^3 \cos\vartheta \sin\vartheta(\sin\vartheta + \frac{1}{2}\cos\vartheta)$$
$$\leq -\rho^2 + \frac{\sqrt{5}}{4}\rho^3 < 0, \text{ for } \rho < \frac{4}{\sqrt{5}}. \quad (13)$$

To determine the largest value of c such that Ω_c is contained in the circle of radius $\frac{4}{\sqrt{5}}$ centered at (0,0), we compute $\lambda_{min}(P) \times r^2 = \frac{1}{4} \times (\frac{4}{\sqrt{5}})^2 = 0.8$. Thus Ω_c with c=0.79 is an estimate of the domain of attraction.

In general, this method yields very conservative estimates of the domain of attraction of an equilibrium point. This is due to the fact that the region of state space in which we seek the largest acceptable level set value of the Liapunov function is limited to a circular region on which $\dot{V}(x)$ is negative definite. This limitation is shown schematically in figure 1.

Thus, the task before us is to improve upon this methodology. There are three key steps to effecting this improvement. The first step is to generate a higher order, more useful family of Liapunov functions than can be generated by Liapunov's equation. The second step is to use a more effective methodology to determine the critical level set value of the Liapunov function. Finally, the third step is to implement an optimization methodology to select the parameter values in our family of Liapunov functions.

Figure 1.



Chapter IV. Construction of Liapunov Functions

The construction of a parametrized family of Liapunov functions for asymptotically stable equilibria is the first step to implementing the proposed estimation methodology. We consider separately equilibrium points that are exponentially stable and those that are critically stable. The notion of critically stable is defined below.

i. Exponentially stable equilibria

The construction of quadratic Liapunov functions for exponentially stable equilibria has been examined in chapter 2. The justification for the method rests on the fact that an equilibrium point of a nonlinear system is exponentially stable if and only if the linearization is exponentially stable. This result does not hold for the more general notion of asymptotic stability, as can be seen from example 1.

The theory of quadratic Liapunov functions can be extended to include piecewise linear Liapunov functions. Quadratic and piecewise linear Liapunov functions can then be viewed as special cases of Liapunov functions constructed through the use of vector norms. This generalization is discussed by Kiendl, Adamy, and Stelzner [19]. There may be some hope of generalizing this notion further to include piecewise quadratic Liapunov functions, which can then be used as approximations to higher order Liapunov functions. Indeed, every higher order function can be approximately locally be a quadratic function.

A second extension to the theory of quadratic Liapunov functions constructed with Liapunov's equation,

$$PA + A^T P = -Q, \tag{14}$$

involves the definiteness of Q. It turns out that Q need only be positive semidefinite. This is discussed further by Ingwerson [16].

Besides the constructive methods based on Liapunov's equation, a second line of research is based on analytical methodologies involving the use of line integral techniques. Much of this work dates to the 1960's. Actually, the first result dates back to Liapunov, who gave the following result to prove his second (or direct) method. Assume that the origin is an exponentially stable equilibrium point for (1). If the Jacobian of $f(x)$ with respect to the state vector evaluated at the origin is symmetric, a Liapunov function for (1) is given by

$$V(x) = -\int_0^x f(x) \cdot dx. \tag{15}$$

Note that the integral is path independent. Further work includes the variable gradient method, which involves choosing a vector function $g(x)$ as a candidate gradient of a positive definite function $V(x)$, and such that $\dot{V}(x)$ is negative definite. Other related results are presented by Ingwerson [16] and by Reiss and Geiss [23], among others.

The disadvantage of these methodologies in the present context is that they are not strictly formal methods that can be implemented for an arbitrary system. As Ingwerson [16] puts it, "a certain amount of ingenuity is required". These methods can be useful in the analysis of specific classes of systems.

Additional work involves finding a Liapunov function based on the linearization of the system about the origin without resorting to Liapunov's equation. Chen and Chu [4] show how to convert a matrix A from controller canonical form to a form known as the Schwarz form. A Liapunov function can then be determined from the entries of the Schwarz form.

At present, there is no available methodology for the construction of high order Liapunov functions for exponentially stable equilibria. Of course, the addition of any cubic or higher order terms to a quadratic Liapunov function is still a valid Liapunov function, since the quadratic terms dominate near the origin. Yet it is not clear how these terms should be chosen to construct functions useful in improving domain of attraction estimates. Thus, in this thesis we will use the family of quadratic Liapunov functions based on the linearization of the system, as discussed in chapter 2. The construction of these functions has been implemented in *Mathematica*. The computer code can be found in Appendix A.

ii. Critically stable equilibria

We define the notion of critical stability as follows:

Definition 10. An equilibrium point for the autonomous system of equations (1) is said to be *critically stable* if it is stable but not exponentially stable.

In terms of the Jacobian of the system evaluated at the equilibrium point, there are no eigenvalues in the open right half plane, at least one simple eigenvalue on the imaginary axis, and no repeated eigenvalues on the imaginary axis.

The construction of high order Liapunov functions for critically stable equilibria is developed by Fu and Abed [7]. Specifically, they present an explicit construction of quartic and quintic Liapunov functions. These correspond to the case where the Jacobian has a simple zero eigenvalue, and the case where there is a pair of complex conjugate poles on the imaginary axis, respectively. Thus the estimation methodology for critical cases can make use of these functions.

These explicit constructions of Liapunov functions in critical cases are of more significance than one might realize. The explicit construction of quadratic Liapunov functions presented earlier assumes that the equilibrium point is exponentially stable. There is no methodology for the construction of quadratic Liapunov function for a critically stable equilibrium point. Indeed, there is no such function. We see this in the example below, due to Fu and Abed [7].

Example 7. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1^3 \\ \dot{x}_2 &= x_2 - 2x_1^2 \end{aligned} \tag{16}$$

The Jacobian for this system evaluated at the origin is given by $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

This corresponds to a critically stable equilibrium. It can be shown that there is no quadratic Liapunov function for the origin.

Chapter V. Determination of Critical Level Set and Radius of Inscribed Sphere

The principal difficulty with the determination of the critical level set as presented in chapter 3 is that we are restricted to a ball in $\Re^n$ in which to determine the largest set of the form Ω_c. The reason why this was done was the resulting ease in obtaining a domain of attraction estimate. However, to obtain less conservative estimates we must not restrict ourselves to a region of $\Re^n$ of any particular shape. Rather, we must work directly with the boundary of the domain of attraction and of the level sets of the form Ω_c. We need to determine the smallest $c \in \Re$ such that the boundary of Ω_c and the boundary of R_A intersect. This c will then be the critical level set value.

One method to determine c is based on the use of results due to Zaborsky [29]. The method relies on three key results. Before stating the results, the following assumptions are made:

(A1) The equilibrium points on the boundary of R_A, which we denote by ∂R_A, are finite in number and hyperbolic. (Recall that an equilibrium point is said to be *hyperbolic* if its Jacobian has no eigenvalues with zero real part).

(A2) The stable and unstable manifolds of equilibrium points on ∂R_A satisfy a technical condition known as the transversality condition.

For a hyperbolic equilibrium point σ of (1), its stable and unstable manifolds, $W^s(\sigma)$ and $W^u(\sigma)$ respectively, are defined as follows:

Definition 11.

$$\begin{aligned} W^s(\sigma) &= \left\{ x \in \Re^n \,/\, \lim_{t\to\infty} \Phi(t,x) = \sigma \right\} \\ W^u(\sigma) &= \left\{ x \in \Re^n \,/\, \lim_{t\to-\infty} \Phi(t,x) = \sigma \right\} \end{aligned} \qquad (18)$$

(A3) Every trajectory on ∂R_A approaches one of the equilibrium points as $t \to \infty$.

The strongest of these assumptions is (A3), which rules out systems where the boundary of the domain of attraction contains a limit cycle. With these assumptions, the following results are obtained. The first of these gives a more explicit characterization of the boundary of the domain of attraction than given by Theorem 5.

Theorem 6. (see [29]) Consider a nonlinear dynamical system described by equation (1) which satisfies assumptions (A1) to (A3). Let $\sigma_i, i = 1,2,\ldots,$ be the equilibrium points on the stability boundary ∂R_A of the asymptotically stable equilibrium point at the origin. Then

$$\partial R_A = \bigcup_{\sigma_i \in \partial R_A} W^s(\sigma_i). \qquad (19)$$

The next result presents conditions for an equilibrium point to be on the boundary of the domain of attraction.

Theorem 7. (see [29]) Let R_A be the domain of attraction of the origin, an asymptotically stable equilibrium point of the nonlinear dynamical system (1).

Let $\sigma \neq 0$ be a hyperbolic equilibrium point of (1). Then

i. $\sigma \in \partial R_A$ if and only if $W^u(\sigma) \cap R_A \neq \emptyset$.

ii. $\sigma \in \partial R_A$ if and only if $W^s(\sigma) \subset \partial R_A$.

We also need the following result concerning the values that a Liapunov function for (1) takes on the boundary of the domain of attraction.

Theorem 8. (see [29]) Consider (1) and assume that the origin is an asymptotically stable equilibrium point with domain of attraction R_A. Let $V(x)$ be a Liapunov function for this system. Then a point with the minimum value of the Liapunov function over the stability boundary ∂R_A exists, and it must be a type-1 equilibrium point. (An equilibrium point is said to be *type-1* if its Jacobian has exactly one eigenvalue with positive real part).

Based on these results, the following methodology can be used to determine the critical level set value:

1. Find all the type-1 equilibrium points.
2. Order those equilibrium points whose corresponding Liapunov function values $V(\cdot)$ are greater than 0. Let the one with lowest value of $V(\cdot)$ be $\hat{x}$.
3. Check whether $\hat{x}$ is on ∂R_A using the conditions of Theorem 7. If it is, then $V(\hat{x})$ is the critical value of the Liapunov function. If not, proceed to the point with the next lowest value of the Liapunov function and check whether this point is on ∂R_A. Continue in this manner until the critical value is determined.

This method in principle determines the critical level set value of the Liapunov function. The problem involves having a viable computational implementation of the method. It will in general be difficult to determine all the equilibrium points of a nonlinear system. In addition, implementing either of the conditions of Theorem 7 to determine whether an equilibrium point is on ∂R_A is a difficult task. Indeed, no computational implementation of this method has been proposed. Finally, it is difficult to verify that a given system satisfies the assumptions (A1)-(A3). If the system does not satisfy the assumptions and the method is used, we may obtain erroneous results. As an example, if the domain of attraction is bounded by a limit cycle, the methodology will not detect any equilibrium points on ∂R_A, and will fail to produce an estimate of the domain of attraction.

In this work, we will pursue two methodologies to determine the critical level set value. The first is algebraic, the second numerical.

i. The Algebraic Approach

This approach to determine c is based on the use of Lagrange multiplier techniques. The critical level set value c corresponds to a local minimum of $V(x)$ subject to $\dot{V}(x)=0$. Assuming $\frac{\partial}{\partial x}\dot{V}(x)$ is full rank (i.e., nonzero), the first order necessary condition that a point must satisfy to be a local minimum is that there exist $\lambda \in \Re$ such that $\lambda \nabla \dot{V} - \nabla V = 0$. Thus, the value of c is given by solving the following system of nonlinear equations:

$$V - c = 0, \qquad (20a)$$

$$Q = 0, \qquad (20b)$$

$$\lambda \nabla Q - \nabla V = 0, \qquad (20c)$$

where we have used the abbreviation $Q = \dot{V} = \nabla V \cdot f$. If $\frac{\partial}{\partial x}\dot{V}(x)$=0, we must replace (20c) with $\frac{\partial}{\partial x}\dot{V}(x)$=0.

These equations can be interpreted as follows. The first simply sets the value of the Liapunov function at the point of intersection equal to the unknown c. The second states that the point on the boundary of the domain of attraction with the minimum value of the Liapunov function is an equilibrium point. This follows from Theorem 8. The third expresses the fact that at the point of tangency, the gradients of the level curves are parallel. Thus we have a system of n+2 equations (equation (20c) is a system of n equations), in n+2 unknowns (c, λ, and the n-dimensional state space vector x).

The problem now is to decide how to go about solving this sytems of equations. We focus on the case of polynomial-type nonlinearites. With an exclusively numerical approach, there is no certainty that all solutions will be generated, thus the approach is an unreliable one. We proceed instead with a combination of algebraic and numerical computation. This methodology has been investigated recently by Forsman [6].

Making use of Theorem 2, we can compute the Gröbner basis of the ideal $(\{V - c, Q, \lambda \nabla Q - \nabla V\})$ with the variable order $(\lambda, x_1, x_2, \ldots, x_n, c)$. This results in a set of polynomials, one of which is a polynomial in c alone. The roots to this polynomial can then be found numerically. Introducing a numerical procedure at this stage is less of a problem. Though roots of a polynomial are difficult to compute accurately, it is known that the number of roots equals the order of the polynomial. Thus there is no risk of not generating the solution of interest as was the case had the system of equations been solved directly with the use of numerical methods.

Both the algebraic and numerical aspects of this method can be programmed in *Mathematica*. The code used to implement the methodology can be found in Appendix B. The following example illustrates the method.

Example 8. Consider the same dynamical system considered in example 6:

$$\begin{aligned} x_1 &= -2x_1 + x_1x_2 \\ x_2 &= -x_2 + x_1x_2 \end{aligned} \tag{21}$$

We use the same quadratic Liapunov function for this system:

$$V(x) = \frac{1}{4}x_1^2 + \frac{1}{2}x_2^2. \tag{22}$$

Corresponding to this Liapunov function, we have

$$Q = -(x_1^2 + x_2^2) + (\frac{1}{2}x_1^2x_2 + x_1x_2^2) \tag{23}$$

The Gröbner basis for the ideal $(\{V - c, Q, \lambda\nabla Q - \nabla V\})$ with the variable order (λ, x_1, x_2, c) contains a polynomial p in c alone. The solutions to p=0 are 1.33333, 6.13921, 79.3745. The critical value of the Liapunov function is given by c=1.33333, the smallest positive solution, with corresponding state space variables $x_1 = 1.3333$ and $x_2 = 1.3333$. This is a significant improvement over the result obtained in example 6, which was c=0.79.

There is one difficulty that needs to be addressed. In general, the smallest positive root to the polynomial equation in c may or may not be the critical value of the Liapunov function. If it corresponds to a complex-valued state vector, the root should be discarded and the next smallest considered. To check that the state vector is real-valued, one might think to plug the chosen value of c into the system of equations, and solve, in turn, for each of the state variables by computing the Gröbner basis for the new system of equations. It turns out that, although this is conceptually correct, it is not implementable in finite precision arithmetic. A Gröbner basis for an ideal is not guaranteed to exist in general. If an inaccurate value for one unknown is put into the system of equations, there may be no values for the other variables corresponding to a solution to the system.

The difficulty can be resolved by discarding one of the n+2 equations when plugging in the approximate value for c. In our implementation, the equation Q=0 is discarded and candidate solutions for x_1 are determined. For each of these, x_2 is then determined. (Thus far only two-dimensional systems have been implemented). If a real valued state vector is found for which $Q \approx 0$, the critical value of the Liapunov function is accepted. If not, the value is disgarded and we proceed in the same manner with the next largest value, and so on, until the critical value is determined. This methodology is illustrated by the following example:

Example 9. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -2x_2 \end{aligned} \tag{24}$$

A Liapunov function for (24) is given by

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2). \tag{25}$$

With this choice of Liapunov function,

$$Q = x_1 x_2^2 - x_1^2 - 2x_2^2. \qquad (26)$$

The Gröbner basis for $(\{V - c, Q, \lambda\nabla Q - \nabla V\})$ with the variable ranking $\lambda \succ x_1 \succ x_2 \succ c$ contains the polynomial $p = 8c^3 - 71c^2 + 4c$. The positive roots of p=0 are 0.0567 and 8.8183.

First, we try plugging c=0.0567 into the system of equations and compute the Gröbner basis for $(\{V - 0.0567, \lambda\nabla Q - \nabla V\})$ with the variable order (λ, x_2, x_1). The possible values for x_1 are 0.33675 and 0.719224. However, neither of these results in feasible values for x_2. Thus, we disregard the root c=0.0567.

Next, we repeat the same procedure with c=8.8183. The possible values of x_1 are 0.33675, 0.719224, 2.78078, and 4.1996. With x_1=2.78078, we obtain a real value for x_2 of 3.14704. Thus the critical value for the Liapunov function is indeed c=8.8183.

The Lagrange multiplier/Gröbner basis methodology for determining the critical level set value of a Liapunov function is clearly a promising one. We see that it outperforms the methodology presented in chapter three, which does not make use of Lagrange multipliers and instead limits the region of interest of state space to a ball in $\Re^n$. The method is easier to implement than the manifold-based method discussed earlier, and does not involve the restrictive assumption (A3). The main drawback is the significant computational expenditure required to compute the Gröbner basis of an ideal. Indeed, for many systems of equations the *GroebnerBasis* subroutine in *Mathematica* is unable to determine a Gröbner basis. Also, note that the condition (20c) is a necessary condition for a local minimum. Thus, the solution c that we obtain may correspond to a local minimum which does not correspond to the critical level set value of the Liapunov function. In such an event, we obtain a more conservative estimate of the domain of attraction.

Essentially the same methodology used to determine the critical value of the Liapunov function can be used to determine the radius of the largest ball in $\Re^n$ contained in the optimal estimate of the domain of attraction. Recall that the motivation for determining this radius is to have a measure of the size of the domain of attraction that can be utilized in controller design.

In the special case of quadratic Liapunov functions, the radius can be evaluated easily by using $R = \sqrt{\dfrac{c}{\lambda_{\max}(P)}}$. In general, we seek the solution to the following system of equations:

$$V - c = 0 \tag{27a}$$

$$C - R^2 = 0 \tag{27b}$$

$$\nabla C - \lambda \nabla V = 0. \tag{27c}$$

In these equations we have used the abbreviation $C(x) = x^T x$, so that $C(x) = R^2$ represents a circle of radius R centered at the origin. We calculate the Gröbner basis for the ideal $(\{C - R^2, V - c, \nabla C - \lambda \nabla V\})$ with respect to the variable order $(x_1, x_2, \ldots, x_n, \lambda, R^2)$. Notice that c is not a variable, and has been assigned the critical value of the Liapunov function. The Gröbner basis contains a polynomial p in R^2 alone. We consider only positive roots to p=0, and determine the smallest root corresponding to a real-valued state vector. The square root of this solution is the radius of the largest ball contained in the estimate of the domain of attraction.

Example 10. Continuing with the system considered in example 9, we calculate the Gröbner basis for $(\{C - R^2, V - c, \nabla C - \lambda \nabla V\})$ and follow the method just described to determine R. We obtain R=4.199595.

Example 11. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_2(1 - x_1^2 - x_2^2) \end{aligned} \tag{28}$$

In addition to the exponentially stable equilibrium point at the origin, this system has the peculiar property that all points on the unit circle are (unstable) equilibrium points. Thus we know a priori that the domain of attraction is given by the open region bounded by the unit circle. Figure 2 illustrates the phase plane plot for this system, generated using *Simnon*.

A Liapunov function for this system is given by:

$$V(x) = x_1^2 + \frac{1}{2} x_2^2. \tag{29}$$

Determining the critical value of the Liapunov function, we obtain c=0.5. Calculating the radius of the largest ball contained in $\Omega_{0.5}$, we obtain R=0.707107.

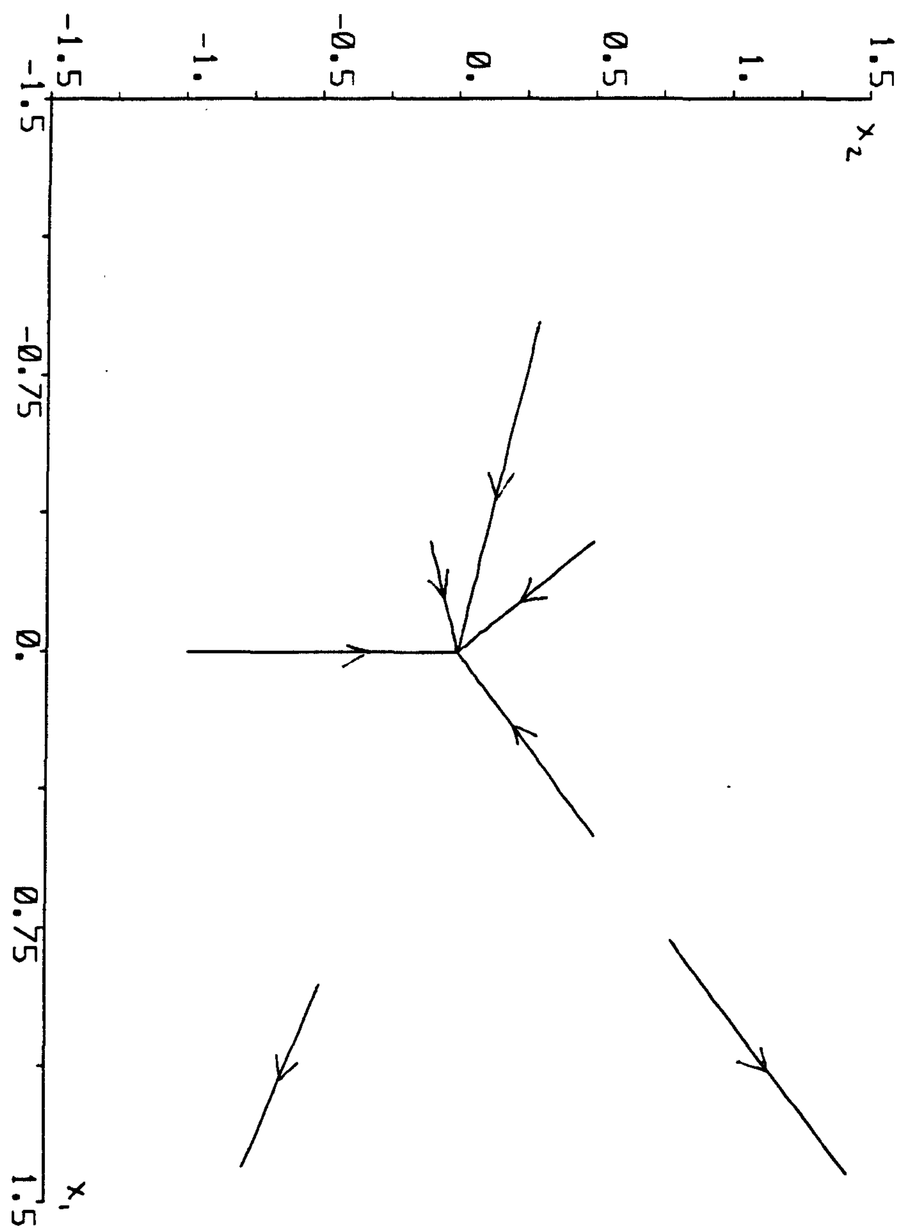
This last example points in an obvious way to the need for an optimal choice of the parameters in the Liapunov function for a system. If the Liapunov function were chosen as

$$V(x) = x_1^2 + x_2^2 \tag{30}$$

instead of (29), we would obtain R=1. The optimal choice of parameter values

Figure 2.

Phase plane plot for the system of example 11. Notice that trajectories diverge, or converge to the origin, on a slope of $\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{x_2}{x_1}$. The domain of attraction is bounded by the unit circle.



for a family of Liapunov functions is addressed in the next chapter.

ii. Computational limitations of the Algebraic Approach

In the previous section it is mentioned that the key limitation in using the proposed algebraic methodology is the computational requirement in computing the Gröbner basis of an ideal. We now examine this issue in this section.

The computational cost for determining a Gröbner basis is very high as a function of both the number of variables in the system of polynomial equations as well as the highest degree present in the system of equations. Depending on some technical details, the complexity of computing a Gröbner basis for an ideal generated by polynomials of with highest degree d in n variables is $d^{O(n)}$, $d^{O(n^2)}$, or $d^{2^{O(n)}}$. In principle, it follows from Bezout's theorem that the $d^{O(n)}$ complexity is optimal (see Forsman [6]).

Unfortunately, there is very poor documentation of the *GroebnerBasis* function available in *Mathematica.* There is no warning when the input ideal is too complex for the function call to handle. In part this may be due to the fact that a relatively simple set of polynomials generating the ideal can result in extremely high-order polynomials in the Gröbner basis. This phenomenon is difficult to predict a priori.

In practice, we find experimentally that two-dimensional quadratic systems with quadratic Liapunov functions give rise to the most complex systems of equations that *Mathematica* can handle reliably. More complicated dynamics are handled on occasion, but more often than not the function call cannot be executed. This computational difficulty is a limitation to the use of the proposed algebraic methodology in practical settings, where the dimension of the state vector may be large. Note that only the performance of the *Mathematica* software has been tested. Other symbolic computing software, such as *Maple*, may prove to be more effective.

iii. The Numerical Approach

In this section we propose a new methodology to numerically determine the critical level set value of the Liapunov function for the asymptotically stable equilibrium point at the origin of (1). The motivation for the development of this second approach is the limited applicability of the algebraic methodology as discussed in the previous section.

In a practical setting, where we can expect to have many state variables and cubic or quartic nonlinearities, we would most likely use the Trajectory Reversing Method (discussed in chapter 3). A more recent entirely numerical, non-Liapunov estimation methodology is presented by Kadiyala [17]. This involves brute force forward-integration from a dense grid of starting points to develop a good picture of the domain of attraction.

The disadvantage of these numerical schemes is that the estimates obtained are not well-defined. By this we mean that we would require an infinitely dense grid of starting points to obtain sets that are known to be contained within the domain of attraction. This drawback is the fundamental motivation for pursuing a LaSalle method, where we obtain well-defined sets guaranteed to be contained within the domain of attraction.

Consider the autonomous system

$$\dot{x} = f(x), \tag{31}$$

where we assume $f{:}D \to \Re^n$ is a locally Lipschitz mapping with domain $D \subset \Re^n, 0 \in D$. We assume that the origin is an exponentially stable equilibrium point. Corresponding to this equilibrium point,we construct a Liapunov function $V(x)$ that is globally positive definite. This is not a necessary condition for a general Liapunov function, but it is assumed in this methodology. $V(x)$ may be constructed with a quadratic part based on the linearization of the system's dynamics, plus additional terms of even order.

The basic idea behind the method is to test the positive invariance of increasingly larger sets of the form $\Omega_c = \{x \in \Re^n, V(x) \le c\}$, where Ω_c is connected, bounded, and contains the origin. This can be done by checking the sign of $\dot{V}(x) = \nabla V(x) \cdot f(x)$ on a grid of points contained in Ω_c. With a quadratic Liapunov function on $\Re^2$ given by $V(x) = p_{11}x_1^{\ 2} + 2p_{12}x_1x_2 + p_{22}x_2^{\ 2}$, for some $p_{11}, p_{12}, p_{22} \in \Re$ satisfying $p_{11} > 0, p_{11}p_{22} > p_{12}^{\ 2}$, the set of grid points with spacing d contained within the set Ω_c is given by

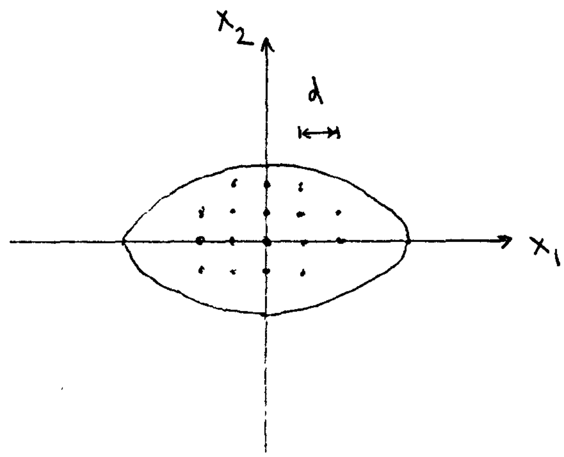
$$\{(x_1, x_2), -x_{1\_lower} \le x_1 \le x_{1\_lower}, -x_{2\_lower} \le x_2 \le x_{2\_lower}\} \quad (32)$$

where

$$\begin{aligned}
x_{1_lower} &= -d \cdot \left\lfloor \frac{\sqrt{\frac{c}{p_{11}}}}{d} \right\rfloor, x_{1_upper} = d \cdot \left\lfloor \frac{\sqrt{\frac{c}{p_{11}}}}{d} \right\rfloor, \\
x_{2_lower} &= d \cdot \left\lceil \frac{-2p_{12}x_1 - \sqrt{(2p_{12}x_1)^2 - 4p_{22}(p_{11}x_1^{\ 2} - c)}}{2p_{22}d} \right\rceil, \\
x_{2_upper} &= d \cdot \left\lfloor \frac{-2p_{12}x_1 + \sqrt{(2p_{12}x_1)^2 - 4p_{22}(p_{11}x_1^{\ 2} - c)}}{2p_{22}d} \right\rfloor.
\end{aligned} \quad (33)$$

Note that both x_{2_lower} and x_{2_upper} depend on x_1. This set of grid points corresponds closely to the actual set Ω_c, as shown in the figure below.

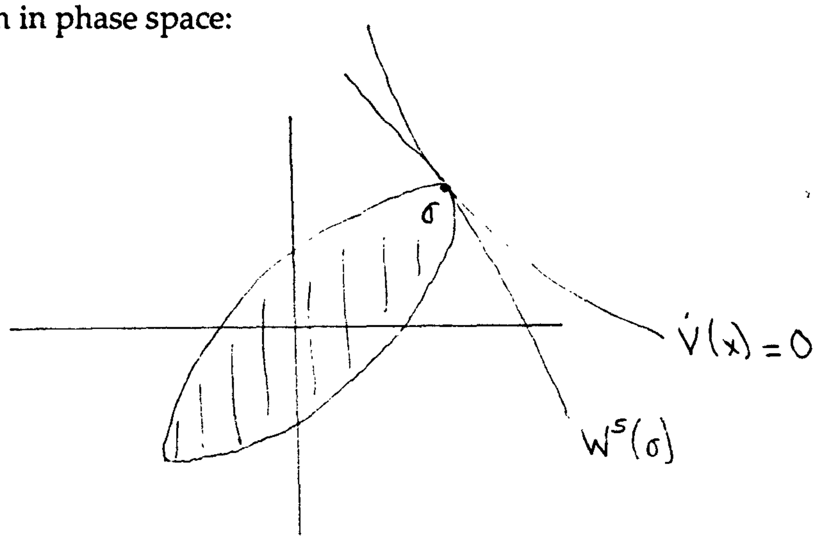
Figure 3.



Note that we must test grid points that are contained within sets of the form Ω_c. We cannot enlarge the estimate of the domain of attraction R_A by simply testing whether $\dot{V}(x) < 0$ at additional grid points. We must be certain to be testing points in a candidate positive invariant set. We continue to increase the value of c until we find a grid point x^* at which $\dot{V}(x^*) \geq 0$ or $\dot{V}(x^*) \approx 0$. At this point we stop, and the previous value of c is the estimate of the critical level set value.

To understand the theoretical basis for this methodology, consider the following sketch in phase space:

Figure 4.



The point σ corresponds to the point on the boundary of the domain of attraction on which $V(x)$ is minimized. We know from Theorem 8 that this will occur at a type-1 equilibrium point. Recall that this means that the Jacobian of (31) evaluated at σ has one right half plane pole, and no poles on the imaginary axis. The stable manifold of σ, $W^s(\sigma)$, is part of ∂R_A, the boundary of the domain of attraction of the origin.

Notice in figure 4 that the sets $\dot{V}(x)=0$ and $W^s(\sigma)$ are shown as being distinct sets whose intersection is $\{\sigma\}$. In general, these sets may coincide over portions of the phase plane, as there may be points in $W^s(\sigma)$ for which $\dot{V}(x)=0$. Notice also that points in region 1 are not in R_A yet they have $\dot{V}(x)<0$, emphasizing the importance of testing the negative definiteness of $\dot{V}(x)$ on sets of the form Ω_c.

The key concern with this method is that the grid of points be sufficiently dense such that there are points near σ. If this is the case, then based on the continuity of $\dot{V}(x)$ we conclude that there will be a grid point x^* with $\dot{V}(x^*)\geq 0$ or $\dot{V}(x^*)\approx 0$. This ensures that the methodology terminates when the critical level set value is reached. Denoting by c^* the previous value of c, the estimate of the domain of attraction is Ω_{c^*}.

Chapter VI. Optimization Methodology

There are some results in the controls literature concerning the determination of a Liapunov function that is optimal in some sense. We need to address the issue of determining the Liapunov function which results in the largest radius of the ball contained in the estimated domain of attraction. Before doing so, let us briefly introduce two interesting, different notions of optimal Liapunov functions.

Willson [28] and later Brayton and Tong [3] make use of a methodology to determine the best quadratic Liapunov function for a discrete time nonlinear system. In this context, the Liapunov function is used to predict ranges of parameter values within which a system is known to be asymptotically stable. The optimal Liapunov function is the one which results in the largest estimates of these ranges. The use of Liapunov functions allows one to avoid eigenvalue calculations.

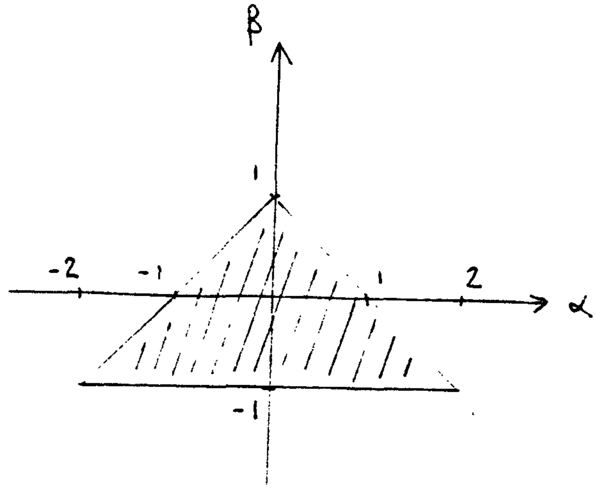
Example 12. Consider the discrete-time dynamical system in $\Re^2$ described by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix} x(k), k = 0,1,2... \qquad (34)$$

For this system to be asymptotically stable, we need its eigenvalues to be of magnitude less than one. This will be true for coefficient values in the region of parameter-space shown in figure 5.

Another notion of optimal Liapunov function involves maximizing the estimate of the allowable perturbation to a stable system, while insuring stability

Figure 5.



of the system. Consider the system

$$\dot{x} = Ax + g(t,x) \qquad (35)$$

where A is Hurwitz and $\|g(t,x)\|_2 \leq \gamma\|x\|_2 \forall t \geq 0$ and $\forall x \in \Re^n$. Let a symmetrix Q>0 be given, solve the Liapunov equation (2) for P, and let $V(x) = x^T Px$. $V(x)$ is then a Liapunov function for (35). Without perturbation, we have an exponentially stable linear system. The time derivative of $V(x)$ along the trajectories of the perturbed system is bounded as follows:

$$\dot{V}(x) \leq -\lambda_{\min}(Q)\|x\|_2{}^2 + 2\lambda_{\max}(P)\gamma\|x\|_2{}^2. \qquad (36)$$

The origin of the perturbed system will be globally exponentially stable if $\gamma < \lambda_{\min}(Q)/2\lambda_{\max}(P)$. This bound clearly depends on the choice of Q. It turns out that the bound is greatest when Q=I (see Khalil [19]). With this choice, the quadratic Liapunov function gives the least conservative estimate of the allowable magnitude of the perturbation term in (35).

In the present context, the notion of optimality is yet another. The optimal Liapunov function in a family of Liapunov functions will result in the largest radius of the sphere contained in the estimated domain of attraction. Each Liapunov function will result in a different estimate of the domain of attraction, and a different radius value. Note that we assume here, as throughout the thesis, that the equilibrium point of interest is at the origin. The assumption is made without loss of generality as it is sufficient to make a suitable choice of state variables.

Unfortunately, unlike the other notions of optimality, we do not have necessary and sufficient conditions for optimal parameters of the Liapunov function. Also, we lack conditions on the matrix Q that results in an optimal quadratic Liapunov function. Thus, we must intelligently search the parameter space to determine the optimal vector of parameter values.

Consider the following framework for an optimazion scheme in a vector space. Let X denote the set of vectors of parameter values that correspond to Liapunov functions, for a given family of Liapunov functions. Also, let $R(x)$ denote the radius of the largest ball contained in the estimate of the domain of attraction, where the estimate is based on the Liapunov function determined by x. Starting with an initial vector $x_0 \in X$, successive iterates are generated with a mapping $A: \mathfrak{R}^n \to \mathfrak{R}^n$, so that

$$x_{k+1} = A(x_k), k \geq 0. \qquad (37)$$

The mapping $A: \mathfrak{R}^n \to \mathfrak{R}^n$ has the following properties:

i. $A(X) \subset X$. This ensures that a feasible vector generates another feasible vector.

ii. $\forall x \in X \setminus X^*, R(A(x)) > R(x)$, where X^* is the set of local maxima of $R(x)$. For all points $x \in X^*$, we have $R(A(x)) = R(x)$. Note that $R(x)$ is a complicated function of a family of Liapunov functions for the system as well as the system dynamics.

The mapping A involves a two step algorithm. The first step involves choosing an ascent direction in parameter space. The second step involves choosing how far to move along the chosen direction to arrive at the next iterate. For instance, in Newton's method the direction chosen is given by the gradient of R evaluated at the current iterate value. Thus at step k of the iteration, the direction is $\nabla R(x_{k-1})$. The choice of the next iterate along the chosen direction is known as a line search. There are several iterative schemes available for its determination.

The difficulty with implementing an iterative optimization scheme in the present context is the complicated dependence of the function $R(x)$ on the vector of parameter values, x. In particular, we cannot straightforwardly compute the gradient of $R(x)$ with respect to x. Thus, we cannot utilize the standard optimization schemes (Newton, Conjugate Gradient, etc.).

Some results are available for n^{th}-order quadratic systems. Genesio and Tesi [9] have studied the problem of estimating domains of attraction for this restricted class of systems. Their proposed methodology depends on a set of parameter values, and the optimal values can be determined by utilizing linear programming techniques. In addition, the methodology depends on an additional parameter to be chosen from a specified range. Genesio and Tesi [9] do not have any methodology to fix this additional parameter other than sweeping through admissible values.

The situation in the general case is more difficult, since we have more than one parameter to be fixed. The approach that we use is the following. We search the parameter space by evaluating $R(x)$ at a set of feasible vector values. This includes vector values where all parameters are of equal magnitude, and where, in turn, a parameter is made much larger than the others.

In the case of quadratic Lyapunov functions, we can proceed similarly by successively weighing the entries of a diagonal, symmetric, positive definite matrix Q. This matrix is then used to generate the function $V(x)$ as usual. The choice of Q resulting in the largest value of $R(x)$ corresponds to the pseudo-optimal Liapunov function. The following examples illustrate the improved estimates that can be obtained in this manner. Both examples employ the algebraic approach of section V.i. Example 13 continues with the system of Example 8, and example 14 examines the Van der Pol oscillator in reverse time.

Example 13. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2 + x_1 x_2 \end{aligned} \qquad (38)$$

We have already obtained an estimate the largest ball in $\Re^2$ contained in the domain of attraction of (38) by making use of a Liapunov functions generated with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The Liapunov function was $V(x) = \frac{1}{4}x_1^2 + \frac{1}{2}x_2^2$, and the critical level set value was found to be c=1.3333. The resulting radius of the largest inscribed sphere is 1.63299. (With the method presented by Khalil [19], the radius is 1.257). We now try $Q = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$ and attempt to improve the estimate. We find that with $Q = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$ and the corresponding Liapunov function $V(x) = 2x_1^2 + \frac{1}{2}x_2^2$, we obtain a critical level set value of c=5.86316. The resulting radius of the largest inscribed sphere is found to be 1.71219, an improvement over the previous estimate of 1.63299.

For two-dimensional systems as the one above we could easily try many more choices for Q, including nondiagonal matrices. Nevertheless, in the general setting of n-dimensional systems this would result in a methodology that is very expensive computationally.

Example 14. Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 \end{aligned} \tag{39}$$

This is the general form of Van der Pol's equation; we consider the system with $\varepsilon = 1$. Linearizing the system at the origin, we find that the Jacobian is given by $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, with eigenvalues $\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. Thus the origin is an unstable focus. Using the Poincaré-Bendixon Theorem with an appropriately chosen annular region such as $M = \{x \in \Re \,/\, 1 \le (x_1^2 + x_2^2)^{\frac{1}{2}} \le 4\}$, it can be proven that there exists a stable limit cycle encircling the origin and contained in M. All trajectories in the plane (not including the origin) are drawn to the limit cycle. Figure 6 shows a phase plane plot for the system, generated with *Simnon*.

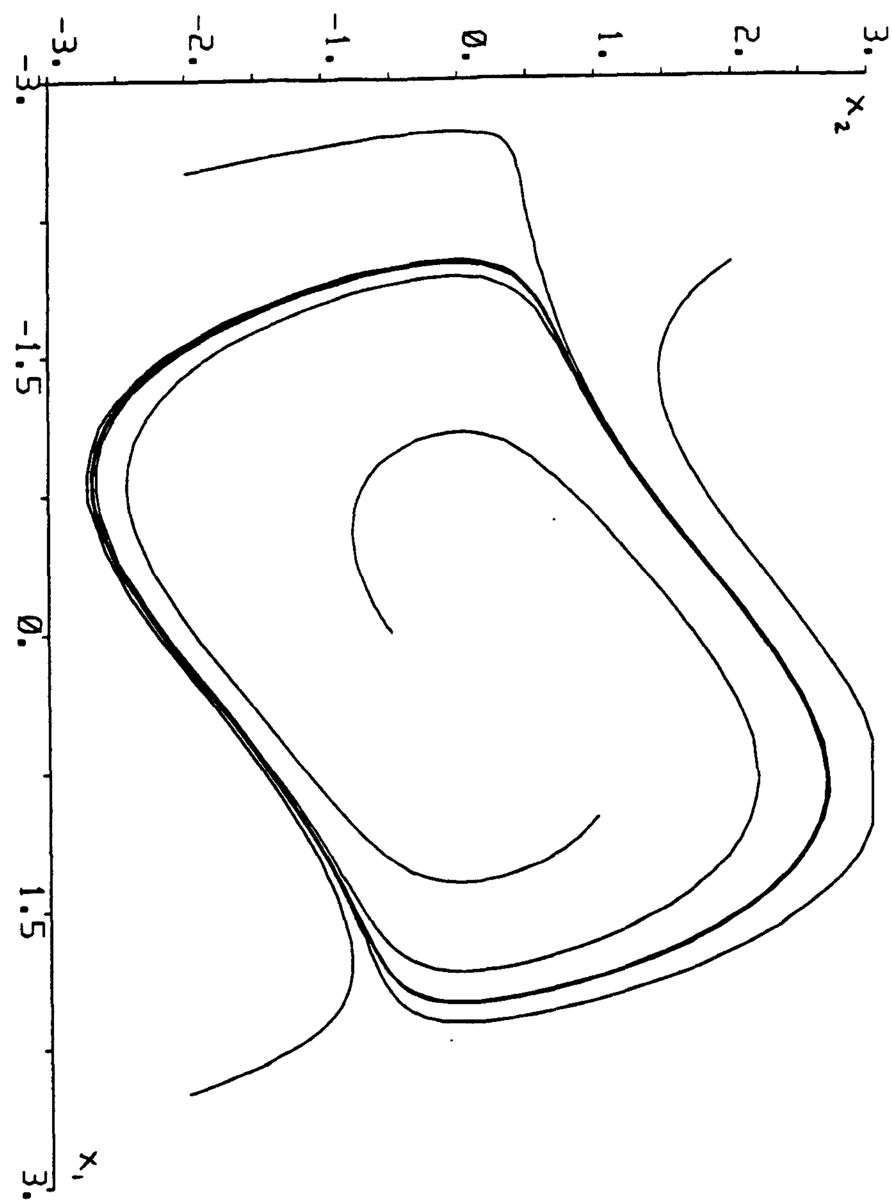
If we now consider the Van der Pol equation (with $\varepsilon = 1$) in reverse time, we have

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - (1 - x_1^2)x_2 \end{aligned} \tag{40}$$

We now have a stable focus at the origin and a domain of attraction for the origin that is bounded by an unstable limit cycle. In the class of quadratic Liapunov functions, a near-optimal Liapunov function is given by

Figure 6.

Phase plane plot for the Van der Pol oscillator. Notice that all trajectories in the plane with the exception of the origin, are drawn to the stable limit cycle.



$$V(x) = \frac{13}{12}x_1^{\ 2} - \frac{1}{2}x_1x_2 + \frac{13}{12}x_2^{\ 2}. \qquad (41)$$

Corresponding to this choice of $V(x)$, the critical level set value is approximately 1.0. Thus, the optimal estimate of the domain of attraction is given by $\Omega_{1.0}$.

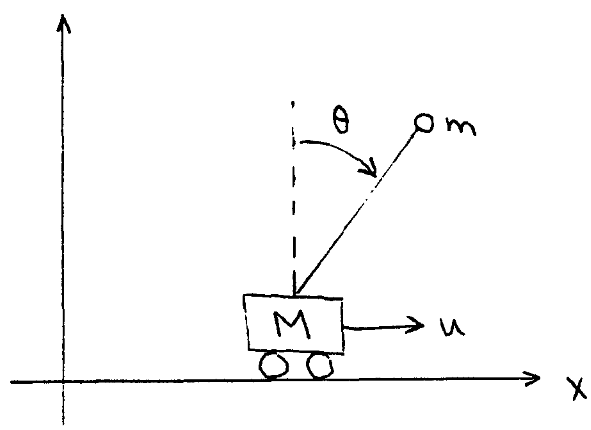
Chapter VII. An Example: The inverted pendulum

In this chapter we present a further example of the proposed estimation methodology. This example, developed in detail, is of an inverted pendulum with linear stabilization. A similar development of the dynamics for the uncontrolled system can be found in Henders and Soudack [15].

i. Dynamics of the inverted pendulum

The system that we propose to study consists of a massless rod with a ball of mass m attatched to one end. The other end of the rod is mounted on a cart of mass M. We assume that the attatchment to the cart is such that the rod can swing freely through any angle, with its motion restricted to a plane spanned by the direction of motion of the cart and the vertical direction. We further assume that the cart moves on a frictionless surface, and that the system can be controlled through a force u in the direction of motion of the cart. The setup is illustrated below.

Figure 7.



There are two basic approaches to modeling the dynamics of a mechanical system. The first is through the use of Newton's equations of motion. The second approach is through the use of Lagrange's equations. These equations have a different formulation but are equivalent to Newton's equations. They come from the application of the calculus of variations to Hamilton's Principle. This principle states that of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies. Thus the motion is such that $\int_{t_1}^{t_2}(K-P)dt$ is minimized, where K represents kinetic energy and P represents potential energy, and the system is considered from time t_1 to t_2. From the calculus of variations, it is known that for a functional of the form

$$J=\int_{x_1}^{x_2} f(y_1(x),\dot{y}_1(x),y_2(x),\dot{y}_2(x),\ldots)dx \qquad (42)$$

to achieve a minimum, it is necessary that the function f satisfy

$$\frac{\partial f}{\partial y_i}-\frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i}=0, i=1,2,\ldots \qquad (43)$$

Defining the Lagrangian as $L=K-P$ and the state vector as $v=\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$, and applying the above result from the calculus of variations, we have

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{v}_i}-\frac{\partial L}{\partial v_i}=u, i=1,2,\ldots \qquad (44)$$

The kinetic energy for our system includes a term for the cart and one for the pendulum. In particular, we have

$$
\begin{aligned}
K &= \frac{M}{2}x^2 + \frac{m}{2}\{\frac{d}{dt}[x + r\sin(\theta)]\}^2 + \frac{m}{2}\{\frac{d}{dt}[r\cos(\theta)]\}^2 \\
&= \frac{M}{2}x^2 + \frac{m}{2}[\dot{x}^2 + r^2\dot{\theta}^2 + 2r\dot{x}\dot{\theta}\cos(\theta)]. \quad (45)
\end{aligned}
$$

The potential energy with respect to the height of the cart is given by

$$
P = mgr\cos(\theta). \qquad (46)
$$

Applying (44) and simplifying, we obtain the following dynamical equations for the system:

$$
\ddot{x} = \frac{u - mg\sin(\theta)\cos(\theta) + mr\sin(\theta)\dot{\theta}^2}{M + m\sin^2(\theta)} \quad (47)
$$

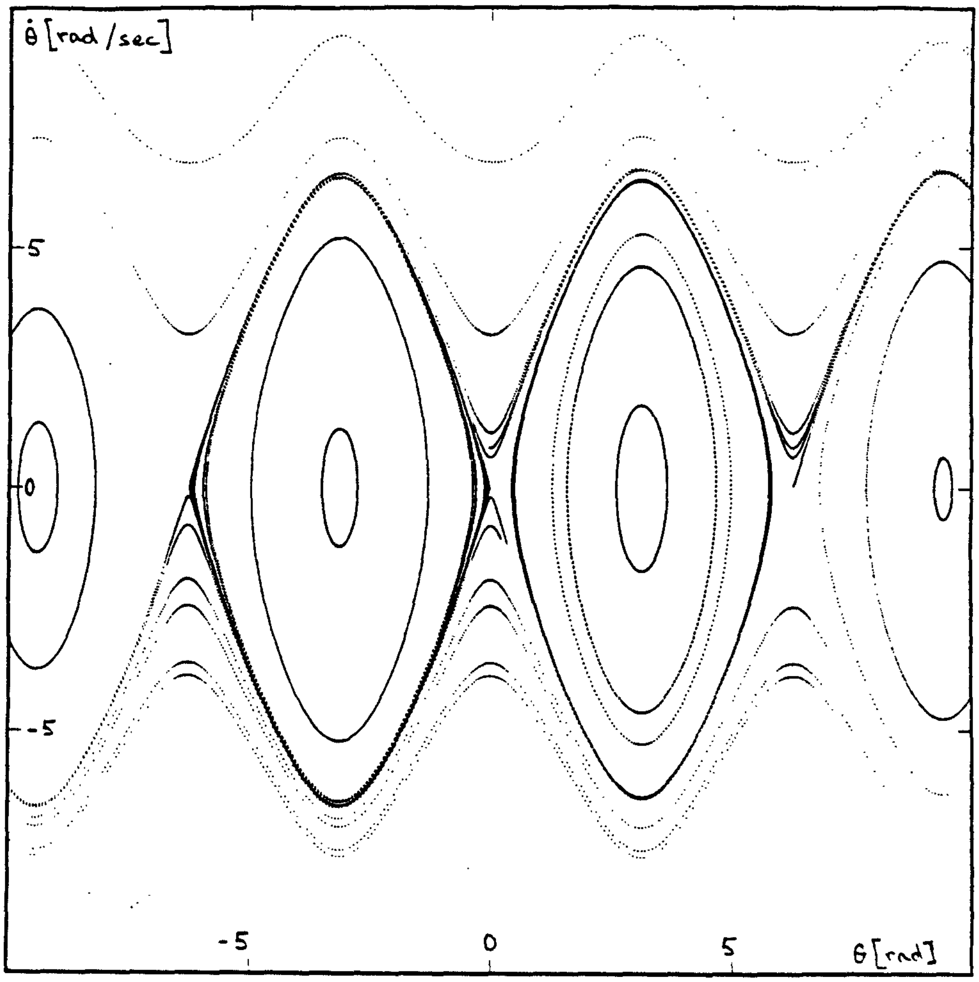
$$
\ddot{\theta} = \frac{-mr\sin(\theta)\cos(\theta)\dot{\theta}^2 + (m+M)g\sin(\theta) - u\cos(\theta)}{r[M + m\sin^2(\theta)]} \qquad (48)
$$

The global dynamics of the uncontrolled system are illustrated in figure 8, generated using *kaos*. We see, as expected, that the origin is an unstable equilibrium. In particular, the origin is a saddle point with eigenvalues given by $\pm\sqrt{\frac{(m+M)g}{rM}}$. The parameter values used to generate the phase plane plot are those given in (55).

We are interested in studying the global behavior of the pendulum with linear state feedback stabilization. Thus we begin by linearizing the system at the origin. The state equations are given by

Figure 8.

Phase plane plot for the uncontrolled inverted pendulum. Notice that the origin is a saddle point.



$$\dot{v} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(m+M)g}{rM} & 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{rM} \end{bmatrix} u. \quad (49)$$

For our purposes, we will be interested in stabilizing the dynamics of the pendulum without concerning ourselves with the dynamics of the cart. Note that we can do this since the dynamics of the pendulum do not depend on those of the cart. Thus we consider the reduced system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(m+M)g}{rM} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-1}{rM} \end{bmatrix} u, \quad (50)$$

where $x_1 = \theta$ and $x_2 = \dot{\theta}$.

We denote this as

$$\dot{z} = Az + bu. \quad (51)$$

We now implement linear state feedback with $u = f^T z$, for some vector f of feedback gains. We then have

$$\dot{z} = [A + bf^T]z. \quad (52)$$

Our system is already in controller canonical form, so we see by inspection that the system is controllable. Thus, we are free to choose pole locations; we choose a damping ratio $\zeta = 0.7071$ and a natural frequency $\omega_n = 1$.

Setting $A + bf^T = \begin{bmatrix} 0 & 1 \\ -1 & -1.4142 \end{bmatrix}$, we find that our design is implemented with the feedback gain vector $f^T = \left[(m+M)g + rM \quad 1.4142rM\right]$.

ii. Domain of attraction estimation for the inverted pendulum

The above design results in a system for which to this point we have only local information about the (stable) origin in state space. Before implementing our domain of attraction estimation methodology, let us develop some intuition as to the global behavior of the system. The stabilized system is given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-mr\sin(x_1)\cos(x_1)x_2{}^2 + (m+M)g\sin(x_1) - f^T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cos(x_1)}{r[M + m\sin^2(x_1)]} \tag{53}$$

To determine the equilibrium points of the system, we set $\dot{x}_1 = \dot{x}_2 = 0$. We then have $x_2 = 0$ and $(m+M)g\sin(x_1) = [(m+M)g + rM]x_1\cos(x_1)$. Simplifying, we conclude that the equilibria are given by the solutions to the following:

$$\frac{(m+M)g}{[(m+M)g + rM]}\tan(x_1) = x_1$$
$$x_2 = 0 \tag{54}$$

In order to work out this example explicitly, we choose the following set of parameter values:

$$\begin{aligned} m &= 0.1kg \\ M &= 1.0kg \\ r &= 1.0m \\ g &= 9.81m/s^2 \end{aligned} \qquad (55)$$

Solving (54) numerically, we find that the following are the equilibrium points of the system: $(-28.6^\circ,0),(0,0),(28.6^\circ,0)$. We consider the linearization of the system at these equilibria to determine their nature. At both $(-28.6^\circ,0)$ and $(28.6^\circ,0)$, the Jacobian of the system is given by $\begin{bmatrix} 0 & 1 \\ 1.5655 & -1.2132 \end{bmatrix}$. The eigenvalues of this matrix are +0.784 and -2.00. Thus, these equilibria correspond to saddle points.

Figure 9 contains a phase plane plot that illustratrates the global behavior of the system, generated using *kaos*. We see that the origin is a stable focus, as expected from our controller design. As expected from the calculations above, we find that there are saddle points at $(-28.6^\circ,0)$ and $(28.6^\circ,0)$. Also, the stable manifolds of these saddle points form the boundary of the domain of attraction of the equilibrium point, as we expect from Theorem 6.

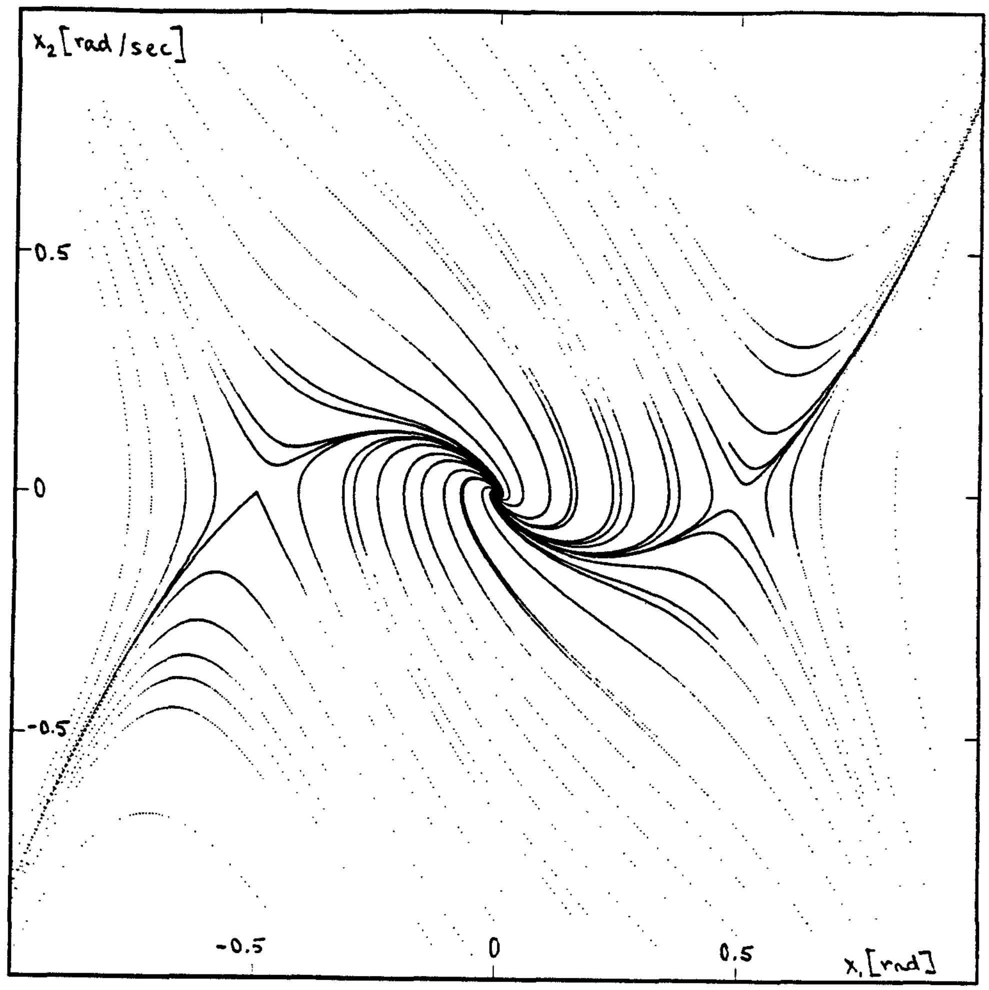
Now that we have a good understanding of the global dynamics of the pendulum with linear feedback, let us implement the algebraic optimal estimation methodology. In the class of quadratic Liapunov functions, a near-optimal Liapunov function is given by

$$V(x) = \frac{5}{4}x_1^2 + \frac{5}{6}\sqrt{3}x_1x_2 + \frac{13}{12}x_2^2. \qquad (56)$$

Corresponding to this choice of $V(x)$, the critical level set value is approximately 1.0 Thus, the optimal estimate of the domain of attraction is given by $\Omega_{1.0}$.

Figure 9.

Phase plane plot for the inverted pendulum with linear state feedback stabilization. The domain of attraction for the origin is bounded by the stable manifolds of the saddle point equilibria at (–28.6°,0) and (28.6°,0).



Chapter VIII. Conclusion and Directions for Future Work

Obtaining a sufficiently large domain of attraction for an equilibrium point of a dynamical system is an objective which must be explicitly considered in the design of controllers for many nonlinear systems. A good example of its importance is illustrated by Khalil [18] in the context of repair time after failure for power systems. A good knowledge of the domain of attraction of a system allows for the determination of the time available to repair the system, ensuring that it will then return to its proper equilibrium. The larger the domain of attraction, the more time is available for repair. Knowlege of the domain of attraction also allows one to predict in general what level of disturbance is tolerable for a given system.

Some efforts have already been made to design controllers with this specific objective in mind. Recently Teel [26] has made use of linear output regulation theory to develop a controller to extend the domain of attraction. Saydy, Abed, and Tits [25] present sufficient conditions for the existence of a linear feedback stabilizing an equilibrium point of a given nonlinear system with the resulting domain of attraction containing a prespecified ball in $\Re^n$. They also explicitly determine this feedback law in the case of planar systems. In a more general setting, the availability of good estimates of the domain of attraction and of the largest ball contained therein provides a tool for use in optimization-based controller design, where a generalized objective function is used.

This thesis provides a contribution to the LaSalle class of estimation methodologies. It provides an implementation of the algebraic approach based on Gröbner bases and Lagrange multipliers, and extends the method to include optimal estimation of the inscribed ball in $\Re^n$. It also develops a novel numerical scheme for optimal estimation. In addition, the important questions of the construction of a family of Liapunov functions and the optimal choice of parameters in this function are addressed.

There is much work still to be done on these last two points. Research needs to be done on the construction of a high order family of Liapunov functions that is useful for estimating domains of attraction. The lower the degree of the Liapunov function, the more generic it is in some sense. It is a function which corresponds to a number of different systems with different dynamics, so that it will inevitably result in conservative domain of attraction estimates. Indeed, the family of quadratic Liapunov functions will be the same for all systems having the same linearization. Further work is needed on the determination of an optimal choice of parameter values when the objective depends in a complicated manner on the parameters, as in the present context. Perhaps the most promising approach is to try to extend the results of Genesio and Tesi [9], which are currently restricted to the class of quadratic systems.

An additional direction for future work is of a strictly algebraic nature. The available algorithms for the computation of the Gröbner basis of an ideal is extremely costly. An improved algorithm would extend the applicability of the algebraic methodology to systems with more complicated dynamics. In addition, it would provide a useful tool for use in other areas of control theory where problems can be reduced to the solution of a set of equations with polynomial nonlinearities. As an example, this includes the design of controllers with a precise specification of the set in which the Nyquist curve should lie. Another example is the computation of equilibrium points for nonlinear systems.

Finally, the natural extension to the topic of this thesis is the development of design techniques that explicitly consider specifications on the domain of attraction as part of a generalized objective function. Kokotovic and Marino [20] address the danger associated with traditional design techniques. By this we refer to designs based on achieving a faster step response, disturbance rejection, and insensitivity to parameter variations, while neglecting the system's nonlinearities.

These classical specifications are typically achieved through high gain feedback. However, in the presence of neglected nonlinearities, such designs may lead to extremely small domains of attraction, or even to instabilities. Consider the following example, due to Kokotovic and Marino [20].

Example 15. The origin of the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(t) + \frac{1}{3}x_2^{\,2}\end{aligned} \tag{57}$$

can be stabilized with the following linear state feedback control law:

$$u(t) = -\gamma^2 x_1 - \gamma x_2. \tag{58}$$

The eigenvalues of the linearization at the origin of the system with the feedback control law (58) are given by $-\frac{\gamma}{2} \pm \frac{3\gamma}{2}j$. Clearly, the system's response speeds up as $\gamma \to +\infty$. However, it can be shown that as $\gamma \to +\infty$, the domain of attraction of the origin shrinks to zero. This shows the danger of designs based on neglected nonlinearities and based solely on specifications on the linearized system.

In many practical settings, an entirely numerical, non-Liapunov estimation methodology like the Trajectory Reversing Method remains the best way to go to obtain rough domain of attraction estimates. Yet the estimates remain rough, for the method in principle requires that all points from an initial estimate be integrated in reverse time. The advantage of LaSalle methodologies over non-Liapunov methodologies, as well as over the Zubov class of Liapunov-based methodologies, is that we obtain well-defined sets that are known to be contained in the actual domain of attraction. With further research, the goal is for LaSalle methodologies to be a viable alternative in practical settings.

Appendix A. *Mathematica* code for Liapunov function construction

The following subroutine takes as input a positive scalar n and returns an n-dimensional identity matrix.

```
iden[n_] :=
      Block[{i,j,Id={}},
            For [i=1,i<=n,i++,
                  For [j=1,j<=n,j++,
                        If [i==j,AppendTo[Id,1],AppendTo[Id,0]]
                  ]
            ];
            Id=Partition[Id,n];
            Id
      ]
```

The following subroutine takes a scalar function of the state variables as well as the corresponding state vector, and returns the gradient of the function with respect to the state vector.

```
grad[V_,z_] :=
      Block [{f={},i,n=Length[z]},
            For [i=1,i<=n,i++,
                  AppendTo[f,D[V,z[[i]]]]
            ];
            f
      ]
```

The following subroutine takes a vector function of the state variables and the state vector and returns the Jacobian of the vector function. In the present context we will use this subroutine to determine the linearization of the system dynamics with respect to the state variables.

```
jacobian[f_,z_] :=
      Block [{j,J={}},
            For [j=1,j<=Length[z],j++,
                  AppendTo[J,grad[f[[j]],z]]
```

```
            ];
            J
      ]
```

The following subroutine determines the kronecker product of two n-dimensional matrices.

```
kronecker [M1_,M2_,n_] :=
      Block [{i,j,k,K={},row={}},
            For [i=1,i<=n,i++,
                  For [k=1,k<=n,k++,
                        row={};
                        For [j=1,j<=n,j++,
                              row=Join [row,M1[[i]][[j]]*M2[[k]]
                        ];
                        AppendTo[K,row]
                  ]
            ];
            K
      ]
```

The subroutines above are utilized in the main subroutine, `getliap[f,z,Q]`. It takes as input the vector of dynamics of the system, the state vector, and a choice for the matrix Q in Liapunov's equation. The subroutine returns the corresponding quadratic Liapunov function.

```
getliap [f_,z_,Q_] :=
      Block [{n,J={},A={},K={},q={},
                        Id={},p={},Id={},p={},P={},Kinv={},V}
            n=Length[z];
            (* calculate the Jacobian      *)
            J=jacobian [f,z]
            (* set A = J(x1=0,x2=0)        *)
            A=J /. {x1->0,x2->0}
            (* get q from Q                      *)
            q=Flatten[Q];
            (* get K from A                    *)
            K=kronecker[A,Q,n];
            Id=iden[n];
            K=kronecker[Id,Transpose[A],n]
                  +kronecker[Transpose[A],Id,n];
            (* compute p                         *)
            Kinv=Inverse [K];
            p=-Kinv . q;
            (* get V from p                    *)
            P=Partition [p,n];
```

```
        V = z . P . z
    ]
```

The example below shows the use of `getliap[f,z,Q]` in a *Mathematica* simulation:

In[1]:=

(* define dynamics and state vector *)

z = {x1,x2};

f = {x1^2-x2,x1-x2};

In[2]:=

(* fix Q and determine Liapunov function *)

Q = {{2,0},{0,1}};

V = getliap [f,z,Q]

Out[2]:=

$$x1(\frac{5x1}{2}-x2)+x2(-x1+\frac{3x2}{2})$$

In[3]:=

(* fix Q and determine Liapunov function *)

Q = {{2,1},{1,3}};

V = getliap [f,z,Q]

Out[3]:=

$$x1(\frac{9x1}{2}-x2)+x2(-x1+\frac{5x2}{2})$$

Appendix B. *Mathematica* code for determination of critical level set and radius of inscribed sphere

The following two subroutines are used to retain positive real and real elements from a list of complex numbers, respectively. They are utilized in the main subroutine `critvalue[V,f,z]`.

```
retainpositive[alist_] :=
      Block [{worklist={},j},
            For [j=1,j<=Length[alist],j++,
                  If [Re[alist[[j]]]==alist[[j]] &&
                                                Re[alist[[j]]]>0,
                        AppendTo[worklist,alist[[j]]]
                  ]
            ];
            worklist
      ]
```

```
retainreal[alist_] :=
      Block [{worklist={},j},
            For [j=1,j<=Length[alist],j++,
                  If [Re[alist[[j]]]==alist[[j]],
                        AppendTo[worklist,alist[[j]]]
                  ]
            ];
            worklist
      ]
```

The subroutine `critvalue[V,f,z]` is given below. It takes as input a Liapunov function for a system, the vector of system dynamics, and the state vector. It returns the critical level set value d of the Liapunov function, corresponding to the largest positive invariant set of the form Ω_d. The methodology employed follows the discussion in section V.i. In addition to the above subroutines, `critvalue[V,f,z]` also uses `grad[V,z]` given in Appendix A.

```
critvalue[V_,f_,z_] :=
      Block[{Q,G,L,P,P1,p,h,n,d,j,test,
                  dlist={},dnewlist={},x1list={},

x2list={},x1newlist={},x2newlist={},x2viable={},
```

```
                    const=0.000001},
        (* calculate the time derivative of the Liapunov function     *)
                Q = grad[V,z] . f;
        (* calculate the Lagrange multiplier equations                *)
                G = grad[Q-L*V,z];
        (* fing GB, solving for a polynomial in d                     *)
                P = GroebnerBasis [{Q,V-d,G},L,z,d];
                p = First [P];
        (* find roots of polynomial in d and put in a list            *)
                h = {ToRules [NRoots [p==0,d]]};
                n = Length [h];
                For [i=1,i<=n,i++,AppendTo [dlist,d /. h[[i]]]];
        (* retain only positive real roots in dnewlist                *)
                dnewlist = retainpositive[dlist];

        (* generate feasible (x1,x2) pairs for increasing d           *)
        (* until a value of d is found with a feasible pair           *)
                j = 1;
                test = False;
        (* find x1 values, retain real ones in x1newlist              *)
                While [test==False && j<=Length[dnewlist],
                      Clear[x1];
                      x1newlist = {};
                      workingd = dnewlist[[j]];
                      P = GroebnerBasis [{V-workingd,G},L,x2,x1];
                      p = First [P];
                      NRoots [p==0,x1];
                      h = {ToRules [NRoots [p==0,x1]]};
                      n = Length [h];
                      For [i=1,i<=n,i++,AppendTo[x1list,x1 /.
                                                      h[[i]]]];
                      x1newlist = retainreal[x1newlist];

        (* for fixed x1, determine real x2 for which Q is            *)
        (* close to zero and retain values in x2viable               *)
                      For [i=1,i<=Length[x1newlist] && Not[test],i++,
                            x1 = x1newlist[[i]];
                            h = {ToRules [NRoots [V-
workingd==0,x2]]};
                            n = Length [h];
                            x2list = {};
                            For [k=1,k<=n,k++,AppendTo [x2list, x2 /.

h[[k]]]];
                            x2newlist = retainreal [x2list];
                            x2viable = {};
                            For [k=1,k<=Length[x2newlist],k++,
                                  x2 = x2newlist[[k]];
                                  If [Abs[Q]<=const,
                                        AppendTo[x2viable,x2]];
                            ]:

        (* if x2viable is not empty, we have a solution              *)
                            If [Length[x2viable] != 0,test=True];
                            Clear[x2];
                      ];
                      j++;
                ];
```

```
(* return level set value                                           *)
              workingd
       ]
```

With the critical level set value determined, the next subroutine finds the radius of the largest inscribed ball. The inputs are the Liapunov function, the critical level set value, and the state vector.

```
findgenradius [V_,d_,z_] :=
       Block [{R2,C,G,L,P,p,h,n,R2list={},R2newlist={}},
(* determine system of equations                                    *)
              C = z . z;
              G = grad [C,z] - L * grad [V,z];
(* determine polynomial in R2, which represents R squared           *)
              P = GroebnerBasis [{C-R2,V-d,G},z,L,R2];
              p = First [P];
(* find roots of polynomial in R2 and put in a list                 *)
              h = {ToRules [NRoots [p==0,R2]]};
              n = Length[h];
              For [i=1,i<=n,i++,AppendTo [R2list,R2 /. h[[i]]]];
(* retain positive real roots in dnewlist and determine radius *)
              R2newlist = retainpositive [R2list];
              R2 = First [R2newlist];
              radius = R2^0.5;
              radius
              ]
```

The following subroutine determines the radius of the largest inscribed ball for the case where the Liapunov function is quadratic. The inputs are a positive definite matrix P corresponding to the Liapunov function $V(x) = x^T Px$, and the critical level set value.

```
findradius[P_,d_] :=
       Block[{r},
              r = (d / Max[Eigenvalues[P]])^0.5;
              r
       ]
```

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