ABSTRACT

Title of dissertation:	Involutions of Shifts of Finite Type: Fixed Point Shifts, Orbit Quotients, and the Dimension Representation
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This thesis addresses several questions in symbolic dynamics. These involve the image of the dimension representation of a shift of finite type (SFT), the fixed point shifts of involutions of SFTs, and the conjugacy classes of orbit quotients of involutions of SFTs.

We present the first class of examples of mixing SFTs for which the dimension representation is surjective necessarily using nonelementary conjugacies.

Given a mixing shift of finite type X, we consider what subshifts of finite type $Y \subset X$ can be realized as the fixed point shift of an inert involution. We establish a condition on the periodic points of X and Y that is necessary for Y to be the fixed point shift of an inert involution of X. If X is the 2-shift, we show that this condition is sufficient to realize Y as the fixed point shift of an involution, up to shift equivalence on X. Given an involution f on X, we characterize what f-invariant subshifts can be realized as the fixed point shift of an involution.

Given a prime p, we classify the conjugacy classes of quotients of 1-sided mixing

SFTs which admit free $\mathbb{Z}/p$ actions. Finally, given p prime, and X_A a 1-sided mixing SFT, we classify the topological dynamical systems which arise as the orbit quotient systems for a free $\mathbb{Z}/p$ action on X_A.

Involutions of Shifts of Finite Type:
Fixed Point Shifts, Orbit Quotients, and
the Dimension Representation

by

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Dedication

I would like to dedicate this work to my son Nixon and my wife Jane.

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List of Abbreviations

SFT	Shift of Finite Type
MSFT	Mixing Shift of Finite Type
$\mathbb{Z}_+$	$\{0, 1, 2, ...\}$
$\mathbb{N}$	$\{1, 2, 3, ...\}$

Chapter 0

Organization and Summary of Results

Let $Aut(\sigma_A)$ be the group of homeomorphisms of a shift of finite type X_A that commute with the shift map σ_A. In Chapter 1 we describe the dimension representation of a SFT, ρ_A, from the mysterious $Aut(\sigma_{X_A})$ to the more tractable group of automorphisms of the dimension module, $Aut(\hat{A})$. An automorphism is inert if it is in the kernel of the dimension representation.

Let ϕ be an automorphism of a SFT X_A and let $fix_\phi(X_A)$ denote the set of points fixed by ϕ. It is well known that with dynamics given by the restriction of the shift, $fix_\phi(X_A)$, (a subshift of X_A) is a shift of finite type. We refer to $fix_\phi(X_A)$ as the fixed point shift of ϕ on X_A. The first question we consider is:

Question 0.0.1. *What can be the fixed point shift of an inert involution of a mixing shift of finite type?*

This is a generalization of the following question posed by John Smillie with motivation from complex dynamics: What are the fixed point shifts of involutions on the 2-shift? In fact, every involution of the 2-shift is inert and the inert case is still the fundamental case to understand even when noninert involutions exist. Apart from complex dynamics, Question 0.0.1 is natural from the viewpoint of symbolic dynamics, where a great deal of what is understood (and what is not understood) about the automorphism group of a SFT involves in a fundamental way the involutions. The following result shows how subshifts that are invariant under an inert automorphism can be realized as fixed point shifts.

Theorem 0.0.2. *Let f be an inert automorphism of a mixing shift of finite type X, with $fix_f(X) \subseteq Y$ where $Y \neq X$ and Y is a f-invariant subshift of finite type in X. Suppose $n \geq 2$ and n is the smallest possible integer such that $f^n = Id$. If the restriction of f to Y is inert, then Y can be realized as the fixed point shift of a finite order automorphism, ϕ on X, where $\phi^n = id$ and n is the minimal positive integer k such that $\phi^k = id$.*

For example, in Theorem 0.0.2 X could be the 2-shift, f could be the flip involution (which exchanges the two symbols), and Y could be any flip invariant subshift of finite type (since $fix_f(X) = \emptyset$ for f the flip). As Example 2.4.6 shows, Theorem 0.0.2 does not resolve Question 0.0.1 in general. Proposition 2.3.4 gives the necessary condition that if a shift of finite type Y is the fixed point shift of an inert involution on a mixing shift of finite type X, then $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades (as defined in Section 3.3.1). This raises the question:

Question 0.0.3. *Let Y be a SFT in a mixing shift of finite type X such that $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades. Can Y be realized as the fixed point shift of an inert involution on X?*

While Theorem 0.0.2 answers this question for certain special cases, our main result shows that the answer to Question 0.0.3 is yes up to shift equivalence when X is the full 2-shift.

Theorem 0.0.4. *For a shift of finite type Y, contained in the full 2-shift, X, the following are equivalent:*

1. *$Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades.*

2. *Y is the fixed point shift of an involution on a mixing shift of finite type which is SE to X.*

It is still unknown if a shift that is shift equivalent over $\mathbb{Z}^+$to the 2-shift is strong shift equivalent over $\mathbb{Z}^+$to the 2-shift. We also show that the answer to Question 0.0.3 is yes for a larger class of mixing shifts of finite type. We also give a (rather technical) proof that there is a finite decision procedure for checking condition (1) of Theorem 0.0.4.

An important part of our understanding of the action of inert automorphisms is the relationship between a shift of finite type X with finite order automorphism, U, and the quotient space X/U. We say that U is a strictly order n automorphism if every point lies in a U-orbit of cardinality n (i.e. U generates a free $\mathbb{Z}/n$ action on X). Kim and Roush asked the following question:

Question 0.0.5. *For p prime, when does a mixing SFT X have a strictly order n automorphism U such that X is conjugate to X/U?*

In the strongest result to date, when p is prime, Kim and Roush [KR3] showed that for a mixing shift of finite type, X, there exists X' shift equivalent to X with a strictly order p automorphism, f, such that X'/f is conjugate to X iff the periodic points of X are the disjoint union of p-cascades. For 1-sided mixing SFTs, the following result gives a complete answer to Question 0.0.5.

Theorem 0.0.6. *Let A be a totally out-amalgamated square matrix over $\mathbb{Z}^+$and let p be a prime integer. The following are equivalent:*

1. *The 1-sided shift of finite type, X_A^+ has a strictly order p automorphism, U, with X_A^+ conjugate to X_A^+/U*

2. *The matrix A^{red} is nilpotent where*

$$A_{ij}^{red} = \begin{cases} 0 & \text{if } A_{ij} \text{ is a multiple of } p \\ A_{ij} & \text{otherwise} \end{cases}$$

Here nilpotence of A^{red} refers to nilpotence as a matrix over $\mathbb{Z}^+$, and depends only on the zero-plus pattern of A^{red}. Question 0.0.5 is a specific case of the following question:

Question 0.0.7. *For a prime p and a mixing shift of finite type X, what are the conjugacy classes of X/U when U is a strictly order p automorphism?*

For an adjacency matrix A, let A' denote the matrix which is the total out-amalgamation of A (as described in Section 5.3). For a 1-sided mixing shift of finite type X_A, the following result characterizes the conjugacy classes of X/U in terms of the total out-amalgamation A'.

Theorem 0.0.8. *Let A be a totally out-amalgamated square matrix over $\mathbb{Z}^+$and let p be a prime integer. The 1-sided shift of finite type, X_A^+ has a strictly order p automorphism, U, with X_B^+ conjugate to $X_A^+/U \iff G_B$ is the quotient graph of an order p or order 1 graph automorphism ψ of G_A satisfying the following.*

Let C be the principal submatrix of A such that G_C is the maximal subgraph of G_A that has vertices fixed by ψ. The matrix C^{red} is nilpotent where

$$C_{ij}^{red} = \begin{cases} 0 & \text{if } C_{ij} \text{ is a multiple of } p \\ C_{ij} & \text{otherwise} \end{cases}$$

The dimension representation has been of fundamental importance in studying the structure of shifts of finite type. There is a known complete characterization of the actions of inert automorphisms on finite subsystems of shifts of finite type. An essential (and to a large extent sufficient) part of understanding how non-inert automorphisms can act on finite subsystems would be simply to know the image of the dimension representation. Additionally, given a classification of irreducible SFTs, Kim and Roush [KR6] describe how the classification of (reducible) SFTs can be found if and only if the range of the dimension representation is known.

The last question we address is:

Question 0.0.9. *Given A, a primitive matrix , what is the image of the dimension representation,* $\rho_A : Aut(\sigma_A) \to Aut(\hat{A})$*?*

Our contribution to addressing Question 0.0.9, though meaningful, is so far modest. Proposition 5.2.4 shows that the only general constructions to date, which are compositions of conjugates of elementary automorphisms, cannot construct certain candidate images of ρ_A. In Proposition 5.4.3 we examine a certain class of mixing shifts of finite type for which it is impossible by Proposition 5.2.4 to show that ρ_A is surjective using only elementary strong shift equivalences. For this class, we construct suitable nonelementary strong shift equivalences to show that the dimension representation is surjective. While this construction is complicated and not fully understood, it is the first class of essentially nonelementary examples constructed and will hopefully lead to further insight.

Chapter 1

Definitions and Background

1.1 Definitions of Shift Spaces

A discrete dynamical system is a topological space, X, equipped with a homeomorphism, f, from X to itself and is denoted by the pair (X, f). Let $\mathcal{A}$ be a finite set of symbols, called an alphabet, and let $\mathcal{A}^{\mathbb{Z}}$ denote the set of bi-infinite sequences $x = \{x_i\}$ where $x_i \in \mathcal{A}$ and $i \in \mathbb{Z}$. There is a natural map, σ, called the shift map that moves a sequence one step left, $\sigma(x)_i = x_{i+1}$. $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the full shift on the alphabet $\mathcal{A}$. When $\mathcal{A}$ has n symbols, the pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the full shift on n symbols or the full n-shift and is denoted by (X_n, σ_n). Unless otherwise indicated, $\mathcal{A} = \{0, 1, ..., n-1\}$. If $\mathcal{A}$ is given the discrete topology, then X_n has topology given by the product topology from $\mathcal{A}$ and is topologically a Cantor set. A compact, shift invariant subset of a full shift gives rise to a subspace with induced map given by the restriction of the shift. We refer to the subspace together with the restriction of the shift map as a subshift or as a shift space. A block is a finite sequence $[b_1 b_2 ... b_n]$ where each symbol $b_i \in \mathcal{A}$.

A continuous shift commuting map, ϕ, from a shift space X to a shift space Y is a block map or block code, meaning that there is a $k \in \mathbb{Z}^+$ and a function Φ such that for all $x \in X$, $\phi(x)_i = \Phi([x_{i-k} ... x_{i+k}])$. A 1-block code is a block map with $k = 0$. Dynamical systems (X, f) and (Y, g) are topologically conjugate if there exists a homeomorphism ϕ from X to Y such that $\phi \circ f = g \circ \phi$. In particular, shift spaces X and Y are conjugate if there exists a 1-1 and onto block code from X to Y.

A subshift of finite type X is defined by fixing a finite list of blocks, F, and excluding from X_n all sequences that contain a block from F. Equivalently, a shift of finite type X is the set of sequences $\{x \in X_n | x_{[i,i+m-1]} = b, b \in M\}$ where M is a fixed list of blocks of length m. Shifts of finite type or SFTs are a very rich and important class of shift spaces and are useful in applications to hyperbolic dynamical systems. See [LM] for an introduction to symbolic dynamics.

A SFT can be presented interchangeably by a directed graph and its adjacency matrix, a square matrix with entries in the semi-ring of the non-negative integers, $\mathbb{Z}^+ = \{0,1,...\}$. Let G be a finite directed graph with n ordered vertices and a finite edge set E. G is defined by its adjacency matrix, A, which is a $n \times n$ non-negative integral matrix with $A_{ij} =$ the number of edges from vertex i to vertex j. Let $t(e)$ and $i(e)$ denote the terminal and initial vertices of the edge $e \in E$. The shift of finite type X_G, or X_A, is the subshift of $E^{\mathbb{Z}}$ given by $\{x = (x_i)_{i \in \mathbb{Z}} \in E^{\mathbb{Z}} : t(e_i) = i(e_{i+1})\text{for all } i \in \mathbb{Z}\}$. We say that a square, non-negative integral matrix A is an edge presentation or simply presents the shift of finite type (X_A, σ_A).

Standing Convention 1.1.1. For simplicity, we will denote the shift dynamical system (X, σ) by the space X since the shift map is understood to be the underlying map, and we refer to σ specifically when we are talking about the dynamical map.

1.2 Conjugacy Invariants of SFTs

Dynamical systems (X, f) and (Y, g) are conjugate if there exists a homeomorphism, $\phi : X \to Y$, such that $\phi \circ f = g \circ \phi$. In general, conjugate systems have the same dynamical properties and a classification of conjugate SFTs would be especially useful. We will discuss several properties of SFTs that are invariant under conjugacy, and in the next section we will discuss the state of the classification problem for SFTs.

A SFT is mixing if there exists a $N \in \mathbb{N}$ such that for each pair of allowed blocks, u and v, and for each $n \geq N$, there is a block w of length n such that uwv is an allowed block.

A matrix, B, is primitive if its entries are nonnegative integers and there is some $n \in \mathbb{N}$ such that $(B^n)_{ij} > 0$ for all ij. If all rows and columns of a square matrix A over $\mathbb{Z}^+$are nonzero, then A is primitive iff X_A is a mixing shift of finite type. The class of mixing shifts of finite type (MSFTs) are the fundamental class of SFTs and many problems of involving SFTs can be reduced to the case of MSFTs. A SFT is irreducible if for each pair of allowed blocks, u and v, there is a block w, with uwv an allowed block. A SFT is reducible if it is not irreducible.

For a dynamical system, (X, f), let $Per(X, n)$ denote the set of points of X such that $f^n(x) = x$, and let $Per(X) = \cup_{n \in \mathbb{Z}^+} Per(X, n)$ be the collection of all periodic points. The length of an orbit is the number of points in the shift orbit.

When $Per(X, n)$ is finite for all $n \in \mathbb{Z}^+$, the periodic point counts of a dynamical system (X, f) are encoded by its Artin-Mazur zeta function,

$$\zeta_X(t) = \exp(\Sigma_{n=0}^{\infty} \frac{|Per(X, n)|}{n} t^n)$$

The zeta function for a SFT, X_A, can be computed as

$$\zeta_{X_A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{det(Id - tA)} \tag{1.1}$$

where $\chi_A(t)$ is the characteristic polynomial of the $r \times r$ matrix A. The non-zero spectrum of a matrix is the set of non-zero eigenvalues with corresponding multiplicity. The zeta function of a SFT, X_A, is determined by the nonzero spectrum of A and vice versa.

The entropy of a shift space is defined by $h(X) = \frac{lim}{n \to \infty} \frac{1}{n} log|B_n(X)|$, where $B_n(X)$ is the set of allowed blocks in X of length n. The entropy of a shift space measures the exponential rate at which the number of allowed words increases. The Spectral Radius Theorem and Perron-Frobenius theory imply that for a MSFT X_A, the entropy of X_A is the eigenvalue of A with largest modulus, which we will call λ_A, and that there is an eigenvector of λ_A which is positive.

1.3 The Conjugacy Problem for SFTs

Let (X_A, σ_A) or simply X_A denote the shift of finite type defined by the non-negative integral matrix A. For A and B matrices over $\mathbb{Z}^+$, it is natural to ask under what conditions do A and B present topologically conjugate shifts of finite type. Any two conjugate SFTs will have the same zeta function and entropy, thus if A and B present conjugate shifts of finite type, then A and B have the same non-zero spectrum. The non-zero spectrum is not enough to guarantee conjugacy, and in 1973 R. Williams gave an algebraic framework with which to study conjugacy classes of shifts of finite type.

1.3.1 Strong Shift Equivalence

Given matrices A and B over a unital semiring $\mathcal{S}$, A is elementary strong shift equivalent (ESSE) to B (over $\mathcal{S}$) if there exist matrices R and S over $\mathcal{S}$ with $A = RS$, $B = SR$. An ESSE, (R,S), has direction from A to B for $A = RS$ and $B = SR$, whereas the ESSE (S,R) has direction from B to A. An elementary conjugacy is one that arises from an elementary strong shift equivalence.

For matrices A and B over $\mathbb{Z}^+$, A is strong shift equivalent (SSE) to B over $\mathcal{S}$ if there is a chain of ESSE (over $\mathcal{S}$) between A and B. SSE is an algebraic equivalence relation whereas ESSE is not because ESSE is not a transitive relation.

Theorem 1.3.1. *[Wil] For A and B matrices over $\mathbb{Z}^+$, (X_A, σ_A) is conjugate to (X_B, σ_B) iff A is SSE to B over $\mathbb{Z}^+$.*

SSE over $\mathbb{Z}^+$is an algebraic equivalence relation whose equivalence classes correspond to conjugacy classes of shifts of finite type. This characterization of conjugacy does not solve the conjugacy problem because there is no known finite procedure for deciding when two non-negative integral matrices are SSE over $\mathbb{Z}^+$.

1.3.2 Shift Equivalence

Williams also defined the very tractable equivalence relation of shift equivalence. For matrices A and B over a unital semiring $\mathcal{S}$, A is shift equivalent (SE) to B over $\mathcal{S}$ if there exist matrices R and S over $\mathcal{S}$ and $l \in \mathbb{N}$ such that

$$RA = BR \qquad AS = SB \qquad A^l = RS \qquad B^l = SR.$$

The integer l is referred to as the lag of the shift equivalence given by (R,S,l). The advantage of using SE rather than SSE is that SE over $\mathbb{Z}$ and $\mathbb{Z}^+$are well understood. For example, matrices over $\mathbb{Z}$ are SE (over $\mathbb{Z}$) to a non-singular matrix. Further, two integral matrices are SE over $\mathbb{Z}$ iff they are SSE over $\mathbb{Z}$. Most importantly, SE over $\mathbb{Z}^+$is decidable. In various important special cases, SE over $\mathbb{Z}^+$is classified by well understood invariants. For example, all matrices over $\mathbb{Z}^+$with the same single non-zero eigenvalue, $\lambda > 0$, are SE over $\mathbb{Z}^+$. It is not known whether they must also be SSE over $\mathbb{Z}^+$.

The relation of shift equivalence can be given more concretely, as we present now. If A is an $n \times n$ matrix over $\mathbb{Z}^+$, then the eventual range of A, R_A, is given by $A^k\mathbb{Q}^n$, for large enough k such that A is an isomorphism from $A^k\mathbb{Q}^n$ to $A^{k+1}\mathbb{Q}^n$. By convention, the action of A is on row vectors. The dimension group of A, G_A, and its positive set G_A^+, are defined as

$$G_A = \{v \in R_A : vA^k \in \mathbb{Z}^n \text{for some } k \geq 0\} \tag{1.2}$$

$$G_A^+ = \{v \in R_A : vA^k \in (\mathbb{Z}^+)^n \text{for some } k \geq 0\} \tag{1.3}$$

$$\tag{1.4}$$

$(G_A, G_A^+, \hat{A})$ is called the dimension module or dimension triple. Dimension modules $(G_A, G_A^+, \hat{A})$ and $(G_B, G_B^+, \hat{B})$ are isomorphic if there exists an isomorphism, $\psi : G_A \to G_B$ that takes the positive set G_A^+ to G_B^+ and $\psi \circ \hat{A} = \hat{B} \circ \psi$.

Theorem 1.3.2. *[K2] Let A and B be matrices over $\mathbb{Z}^+$, then A is SE to B over $\mathbb{Z}$ iff $(G_A, \hat{A}) \cong (G_B, \hat{B})$, and A is SE to B over $\mathbb{Z}^+$iff $(G_A, G_A^+, \hat{A}) \cong (G_B, G_B^+, \hat{B})$.*

The dimension module has an important presentation in terms of polynomials. For a ring R, let $L(R)$ denote the Laurent ring of polynomials in $t^{\pm 1}$ with coefficients in R, and let $L(R)^{\mathbb{N}}$ be the $L(R)$-module of (countably infinite) column vectors with all but a finite number of entries zero. Let $cok(Id-tA)$ be the cokernel $L(\mathbb{Z})$-module given by $L(\mathbb{Z})^{\mathbb{N}}/(Id-tA)L(\mathbb{Z})^{\mathbb{N}}$. As above, matrices act from the right on row vectors. Let ϕ : $G_A \to cok(Id-tA)$ be defined by $v \to t^k v A^k$ for k such that $vA^k \in \mathbb{Z}^n$. ϕ is an isomorphism from G_A to $cok(Id-tA)$ such that $L(\mathbb{Z}^+)^{\mathbb{N}} \cap \{L(\mathbb{Z})^{\mathbb{N}}/(Id-tA)L(\mathbb{Z})^{\mathbb{N}}\}$ is isomorphic to the positive set G_A^+. The isomorphism of G_A given by $\hat{A}$ corresponds to multiplication by t^{-1} on $cok(Id-tA)$. So by Theorem 1.3.2, for A and B matrices over $\mathbb{Z}$, A is SE to B over $\mathbb{Z}$ iff $cok(I-tA)$ and $cok(I-tB)$ are isomorphic as $L(\mathbb{Z})$-modules, and A is SE to B over $\mathbb{Z}^+$iff $cok(I-tA)$ and $cok(I-tB)$ are isomorphic as ordered $L(\mathbb{Z})$-modules.

Example 1.3.3. If $A=[2]$, then A presents the full 2-shift. G_A is the ring $\mathbb{Z}[1/2]$ since $\mathbb{Z}[1/2]$ are the elements of $\mathbb{Q}$ that will be eventually mapped into $\mathbb{Z}$ by multiplication by 2. G_A^+ will be $\mathbb{Z}^+[1/2]$ and $\hat{A}$ will be the isomorphism of $\mathbb{Z}[1/2]$ given by multiplication by 2.

Proposition 1.3.4. *Let A, B, and C be integral matrices with B nilpotent. Then* $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ *and A are shift equivalent over* $\mathbb{Z}$ *.*

Suppose A is a $n \times n$ matrix over $\mathbb{Z}$ and det(A) = ± 1. Then $G_A = \mathbb{Z}^n$ and $\hat{A} = A$, since A is invertible over $\mathbb{Z}$. For B a $n \times n$ matrix over $\mathbb{Z}$, A will be SE to B over $\mathbb{Z}$ iff A and B are conjugate in the matrix group $Gl_n(\mathbb{Z})$.

Proposition 1.3.5. *[LM 7.3.6] For A and B primitive matrices, A is SE to B over* $\mathbb{Z}^+$*iff A is SE to B over* $\mathbb{Z}$ *.*

By Theorem 1.3.2 we can neglect the positive set when dealing with SE between primitive matrices.

Definition 1.3.6. X_A and X_B are eventually conjugate if there is an integer N such that (X_A, σ_A^n) and (X_B, σ_B^n) are topologically conjugate for all $n \geq N$.

Theorem 1.3.7. *[W2] For matrices A and B over $\mathbb{Z}^+$, X_A and X_B are eventually conjugate iff A and B are SE over $\mathbb{Z}^+$.*

Clearly if A is SSE over $\mathbb{Z}^+$to B, then A is SE over $\mathbb{Z}^+$to B, but when does A SE to B over $\mathbb{Z}^+$imply A is SSE to B over $\mathbb{Z}^+$? Williams [Wil] conjectured in 1974 that for matrices over $\mathbb{Z}^+$, SE over $\mathbb{Z}^+$implies SSE over $\mathbb{Z}^+$. This conjecture was refuted by Kim and Roush for the reducible case in 1992 [KR4] and for the irreducible and mixing cases in 1999 [KR1] but there remains much to be understood about the relation of SSE to SE. Essential to the counterexamples was a deeper understanding of the dimension representation of the automorphism group of a shift of finite type.

Standing Convention 1.3.8. For the rest of this paper, SE and SSE refer to SE over $\mathbb{Z}^+$and SSE over $\mathbb{Z}^+$unless otherwise stated.

1.4 The Dimension Representation

An automorphism of a shift space X is a shift commuting homeomorphism of X to itself. Let $Aut(\sigma_X)$ denote the group of automorphisms on a shift space X. Boyle, Lind, and Rudolph [BLR] showed that when a SFT, X, has non-zero entropy, the countably infinite group $Aut(\sigma_X)$ is not finitely generated and contains a copy of every finite group. $Aut(\sigma_X)$ is complicated and poorly understood.

Let $Aut(\hat{A})$ be the group of automorphisms of G_A that commute with $\hat{A}$. $Aut(\hat{A})$ is a much more tractable group to study and is typically finitely generated. For $A \in GL_n(\mathbb{Z})$, $G_A = \mathbb{Z}^n$ and $\hat{A} = A$ is the isomorphism given by multiplication by A, so $Aut(\hat{A})$ consists of invertible integral matrices that commute with A.

By Theorem 1.3.1, any $\phi \in Aut(\sigma_A)$ can be realized by some chain of ESSEs over $\mathbb{Z}^+$from A to A, $(R_1,S_1)(R_2,S_2)...(R_k,S_k)$. If (R,S) is an ESSE from A to B, then R induces an isomorphism from $(G_A, G_A^+, \hat{A})$ to $(G_B, G_B^+, \hat{B})$. For an automorphism ϕ and a corresponding SSE from A to A, $(R_1,S_1)(R_2,S_2)...(R_k,S_k)$, let $\hat{\phi}$ be the induced automorphism on $(G_A, G_A^+, \hat{A})$, where $\hat{\phi} = \prod(\hat{R}_i)^{\varepsilon_i}$ and ε_i is ± 1 according to the direction that the i-th ESSE is traversed. Since $\hat{\phi}$ does not depend on the choice of SSE representing ϕ, this gives a well defined map $\rho : Aut(\sigma_A) \to Aut(\hat{A})$ where $\rho(\phi) = \hat{\phi}$. ρ is called the dimension representation and elements in its kernel are called *inert* automorphisms. Krieger originally defined the dimension representation dynamically using a Grothendieck style construction on compact open subsets of unstable sets. We will use the algebraic definition given above because it is more convenient for our constructions which use chains of ESSEs. ρ depends explicitly on the presentation A, but for brevity we neglect A in the notation of the dimension representation.

Definition 1.4.1. A graph automorphism of $\mathcal{G}_A$ induces a 1-block map on X_A. The group of simple automorphisms is the subgroup of inert automorphisms generated by automorphisms conjugate to a block code induced by a graph automorphism that fixes all vertices.

In Chapter 3, we discuss at length the group of inert automorphisms, defined as the kernel of the dimension representation. In Section 3.3.2, we briefly discuss the known complete characterization of the actions of inert automorphisms on finite subsystems of shifts of finite type. In stark contrast, there has been little progress in describing how non-inert automorphisms can act on finite subsystems. An essential (and to a large extent sufficient) part of this understanding would be simply to know the image of the dimension representation. Additionally, given a classification of irreducible SFTs, Kim and Roush [KR6] describe how the classification of (reducible) SFTs can be found if Question 5.1.1 is answered.

Chapter 2

Fixed Point Shifts of Involutions

An involution of a shift of finite type, X, is an automorphism of X such that $U^2 = Id$. Recall from Section 1.4 that an automorphism of a shift of finite type is inert if it is in the kernel of the dimension representation. The question we consider in this chapter is:

Question 2.0.2. *What can be the fixed point shift of an inert involution of a mixing shift of finite type?*

For many shifts of finite type, such as full shifts, every involution is inert. Even when noninert involutions exist, the fundamental case to understand is the inert case. See Section 3.3 for further discussion. Question 2.0.2 is a natural generalization of a problem posed by John Smillie:

Question 2.0.3. *[Smillie, 2005] What are the fixed point shifts of involutions of the full 2-shift?*

In Section 3.1, we discuss the motivation of Smillie's question from complex dynamics and mention some motivation from symbolic dynamics. In Section 3.2, we recall background results from symbolic dynamics which will give context and be used in our later theorems. In Section 3.3, we discuss the class of inert automorphisms and conditions on periodic points that are necessary for the existence of inert automorphisms. In Section 3.4, we answer Question 2.0.2 in a special case and discuss the limitations of this result. In Section 3.5, we present a hierarchy of conditions involving cascades, zeta functions, and matrix traces, and establish a decision procedure for checking the necessary conditions of Question 2.4.7.

2.1 Application to Complex Dynamics

Smillie's Question (2.0.3) stems from a problem involving quadratic maps on $\mathbb{C}^2$. The Henoń family is a 2-parameter family of diffeomorphisms of $\mathbb{R}^2$ given by quadratic maps $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$, with $f_{a,b}(x,y) = (x^2 + a - by, x)$ and $a, b \in \mathbb{R}$ ($b \neq 0$ for a diffeomorphism). The Henoń family has been of interest for many years because of its relation to one-dimensional and complex dynamics. For $a \ll 0$, the restriction of $f_{a,b}$ to its chain recurrent set is hyperbolic and topologically conjugate to the full 2-shift, and when $a \gg 0$, the dynamics of $f_{a,b}$ are wandering [BS], but there are many open questions about what happens between these extremes. Cvitanovic conjectured that each map in the Henoń family can be described by horseshoe dynamics with collections of orbits removed; this conjecture has been supported by numerical evidence from Davis, MacKay, and Sannami [DMS].

Let $K_{a,b}$ be the set of bounded orbits of $f_{a,b}$. Let the real horseshoe locus, $H^{\mathbb{R}}$, be the set of $(a,b) \in \mathbb{R}^2$ such that the restriction of $f_{a,b}$ to $K_{a,b}$ is topologically conjugate to the full 2-shift, $(X_{[2]}, \sigma)$. Likewise, let the complex horseshoe locus, $H^{\mathbb{C}}$, be the set of $(a,b) \in \mathbb{C}^2$ such that the restriction of $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ to $K_{a,b}$ is topologically conjugate to the full 2-shift, $(X_{[2]}, \sigma)$. Bedford and Smillie [BS] describe how distinct connected components of $H^{\mathbb{R}}$ may be connected by paths in $H^{\mathbb{C}}$. Hubbard and Oberste-Vorth [Ob] show that $H^{\mathbb{C}}$ contains the set $HOV = \{(a,b) \in \mathbb{C}^2 : |a| > 2(|b|+1)^2, b \neq 0\}$.

For some $(a_0,b_0) \in H^{\mathbb{C}}$, pick ϕ_0, a conjugacy from $K_{(a_0,b_0)}$ to the full 2-shift. Now let $\gamma(t)$, $0 \leq t \leq 1$, be a closed loop in $H^{\mathbb{C}}$ with basepoint (a_0,b_0). Because real and complex horseshoes (represented here by the full 2-shift) are structurally stable, $\gamma(t)$ produces a homotopy of conjugacies h_t from $K_{(a_0,b_0)}$ to $K_{(a_t,b_t)}$.Thus, $\Theta(\gamma) = \phi_0 \circ h_1 \circ \phi_0^{-1}$ defines an automorphism of the 2-shift. Θ sends a loop in $H^{\mathbb{C}}$ to an automorphism of the full shift and depends only on the homotopy class of the loop, $[\gamma]$. So the map

$$\Theta : \pi_1(H^{\mathbb{C}},(a_0,b_0)) \to Aut(\sigma_{[2]})$$

given by $[\gamma(t)] \to \Theta(\gamma)$ is a well defined homomorphism. This homomorphism Θ provides a probe into the topological structure of connected components of $H^{\mathbb{C}}$.

Let $H^{\mathbb{C}}_{HOV}$ be the connected component of $H^{\mathbb{C}}$ that contains the connected set HOV. Hubbard [H] conjectured in 1986 that the image of $\pi_1(H^{\mathbb{C}}_{HOV})$ under Θ is isomorphic to the automorphism group of the full 2-shift. Recently, [BS] showed that the range of $\Theta(\pi_1(H^{\mathbb{C}}_{HOV}))$ is nontrivial: for γ a loop in HOV, $\Theta(\gamma)$ can be the automorphism defined by flipping the symbols 0 and 1. Even more recently, Arai's numerical work applying the theory of Bedford and Smillie, showed that $\Theta(\pi_1(H^{\mathbb{C}}_{HOV}))$ has an element of infinite order [A]. In contrast, the automorphism group of the 2-shift, is large and complicated. For example, it is countably infinite, residually finite, not finitely generated, it contains a copy of every finite group, the free group on infinitely many generators, and many other groups (but not any group with unsolvable word problem) [BLR].

Much more is understood in the analogous one-sided setting. Blanchard, Devaney, and Keen considered S_d, the space of monic polynomials of degree d on the complex plane such that the restriction of the polynomial to its bounded orbits is conjugate to the *one-sided* full d-shift, $X^+_{[d]}$. They defined $\Theta_d : \pi_1(S_d) \to Aut(\sigma_{X^+_{[d]}})$ as above. [BDK] exploited the interactions between the dynamical space and the parameter space to show that the map $\Theta_d : \pi_1(S_d) \to Aut(X^+_{[d]})$ is surjective.

In contrast to the two sided case, the automorphism group of the one-sided 2-shift contains only two elements. So if true, Hubbard's conjecture would show that the parameter space of the complex Henoń family is quite different than the set of monic quadratic maps on the complex plane and would give a geometric description of the still quite mysterious automorphism group of the two-sided 2-shift. Apart from complex dynamics, Question 2.0.2 is natural from the viewpoint of symbolic dynamics, where a great deal of what is understood (and what is not understood) about the automorphism group of a SFT involves in a fundamental way the involutions [F, BF, BLR, KRW1].

2.2 Embedding Theorems and Nasu's Masking Lemma

A map, g, from a shift of finite type X to a shift of finite type Y is an embedding if g is a continuous shift-commuting, one-to-one map. The following theorem of Krieger is a fundamental result of symbolic dynamics.

Theorem 2.2.1. *Let X be a shift space and Y a mixing shift of finite type. The following are equivalent:*

1. *$h(X) < h(Y)$ and there exists a shift commuting injection, $\phi : Per(X) \hookrightarrow Per(Y)$.*

2. *There exists an embedding of X into Y as a proper subshift.*

Theorem 2.2.1 in particular shows the very rich nature of subsystems of a SFT with positive entropy.

The following result of Nasu is a very useful tool which brings dynamical embeddings to the level of matrix presentations.

Theorem 2.2.2 (Nasu's Masking Lemma). *Let A be a matrix presentation of shift of finite type X. If X embeds into a shift of finite type Y, then there exists a matrix presentation, B, of Y such that A is a principal submatrix of B.*

(See [LM] for proofs and discussion of Theorems 2.2.1 and 2.2.2)

Let U be an automorphism of a shift of finite type X. Then let $fix_U(X)$ be the set of points of X that are not moved by U. Since U is a shift-commuting map, σ_X will move points fixed by U to points fixed by U, and therefore $fix_U(X)$ is a shift space. Additionally, $fix_U(X)$ will be a SFT because $fix_U(X)$ is the set of all bi-infinite sequences which can be built from the finite list of blocks of X, $\{b \in B_{2n+1}(X) | x_{[-n,n]} = b, U(x)_0 = x_0\}$, where U has radius n and $B_m(X)$ is the set of allowed words of length n in X.

It is a natural question to ask when a shift of finite type with a shift commuting finite group action can be embedded into another shift of finite type with a shift commuting finite group action. It is notable that the existence of embeddings is again characterized by entropy and periodic point structure.

Theorem 2.2.3. *[L] Let X and Y be mixing shifts of finite type with involutions U and V. Suppose the following hold:*

1. $h(X) < h(Y)$

2. *There exists a shift commuting injection* $\psi : Per(X) \hookrightarrow Per(Y)$ *such that* $\psi \circ U = V \circ \psi$

3. *There exists an embedding of* $fix_U(X)$ *into* $fix_V(Y)$.

Then there exists an embedding $\phi : X \hookrightarrow Y$ *with* $\phi \circ U = V \circ \phi$.

In an unpublished work, Lightwood [L] proved a generalization of this theorem as a tool for a construction for embedding $\mathbb{Z}^2$ subshifts into certain $\mathbb{Z}^2$ shifts of finite type. We will use this theorem to compare involutions of a shift of finite type to involutions of its subshifts.

Let us examine condition 2 of Theorem 2.2.3 with U and V involutions of mixing shifts of finite type X and Y. If $x \in Per(X)$ of least period n and $x \neq U(x)$, then U will map x to either $\sigma^{n/2}(x)$ or to another periodic point of least period n not in the shift orbit of x. A periodic point, x, is type 1 if U moves x to another periodic point in the σ-orbit of x. A periodic point is type 2 if U sends x to a periodic point that is not in the σ-orbit of x. A periodic point is called type 0 if it is fixed by U.

Standing Convention 2.2.4. Let the following be a standing convention for the rest of the paper: A symbolic block of length n, $b = b_0b_1...b_{n-1}$, will represent a shift orbit consisting of periodic points $\sigma^i((b)^\infty)$ for $0 \geq i \geq n-1$ where $(b)^\infty$ refers to x, the point of period n with $x_{[0,n-1]} = b$.

Example 2.2.5. Let X be the full shift on symbols $\{0,1,2,3\}$, and let U be the involution defined by switching the symbols 0 and 1 and fixing 2 and 3. Then $(0110)^\infty$ is mapped to $(1001)^\infty = \sigma^2((0110)^\infty)$, so $(0110)^\infty$ is a type 1 periodic point. The point $(0111)^\infty$

is mapped to $(1000)^\infty$, so $(0111)^\infty$ and $(1000)^\infty$ are type 2 periodic points. The point $(2332)^\infty$ is mapped to $(2332)^\infty$, so $(2332)^\infty$ is a periodic point of type 0.

Let $a_n^i(U)$ be the number of points of least shift period n $(\in \mathbb{N})$ of type i $(\in \{0,1,2\})$ with respect to the involution U.

Proposition 2.2.6. *If U and V are involutions of shifts of finite type X and Y, then there exists a shift commuting embedding $\psi : Per(X) \hookrightarrow Per(Y)$ with $\psi \circ U = V \circ \psi$ iff for all $n \in \mathbb{N}$ and $i \in \{0,1,2\}$, $a_n^i(V) \geq a_n^i(U)$.*

This proposition is immediately apparent and shows how the embedding of a shift commuting $\mathbb{Z}/2$ action on the periodic points of a shift of finite type is a set theoretic property of having enough periodic points of each type in the range SFT.

2.3 Inert Automorphisms

An automorphism, ϕ, of a shift of finite type, X, defines an equivalence relation on the points of X given by: if $x, y \in X$, then $x \sim_\phi y$ if x and y are in the same ϕ orbit. X/ϕ is the quotient space of X by the relation $\sim_\phi$. Let π be the projection of X onto the orbit space X/ϕ that takes a point $x \in X$ to its ϕ-orbit, $[x] = \{y \in X | x \sim_\phi y\}$. The shift on X induces a bijection, $\sigma_{X/\phi}$, from X/ϕ to X/ϕ which will define $(X/\phi, \sigma_{X/\phi})$ as a dynamical system. It is well known that X/ϕ will not be conjugate to a shift space unless for some $n \in \mathbb{N}$ every ϕ-orbit has cardinality n, i.e. ϕ is a strictly order n automorphism. Recall from Section 1.4, that an automorphism on a shift of finite type is inert if it is in the kernel of the dimension representation. Fiebig [F] gives a useful characterization of inertness in terms of zeta functions and orbit spaces.

Theorem 2.3.1. *[F] If ϕ is a finite order automorphism on a shift of finite type X, then $\zeta^{-1}_{X/\phi}(t) = \zeta^{-1}_X(t)$ iff ϕ is inert.*

Example 2.3.2. Let X be the full 2-shift, and let f be the automorphism that switches 0 and 1. Let g be the 2-to-1 sliding block code defined by $g(x)_i = x\_i + x_{i+1}$ mod 2. For $x, y \in X$, x is in the f-orbit of y iff $g(x) = g(y)$. Since X/f is topologically conjugate to $g(X)$ and the image of g is the full 2-shift, then by Theorem 3.3.1, f is inert.

Let X be a shift of finite type and ϕ be a finite order automorphism on X. Formula 1.1 shows that the reciprocal zeta function of a shift of finite type is a polynomial. Fiebig shows that the reciprocal zeta function of the orbit space, $\zeta^{-1}_{X/\phi}(t)$, is a polynomial factor of the reciprocal zeta function of X [F]. If a shift of finite type, X, has an irreducible reciprocal zeta function, then all finite order automorphisms of X are inert since $\zeta^{-1}_X(t)$ will not have polynomial factors, and thus $\zeta^{-1}_{X/\phi}(t) = \zeta^{-1}_X(t)$.

Example 2.3.3. Let $A = [2]$ be the matrix representation of the full 2-shift. Since $\zeta^{-1}_{X_A}(t) = 1 - 2t$ is irreducible, all finite order automorphisms on X_A are inert. In fact (see Example 5.4.1), $Aut(\sigma_{X_A}) = \mathbb{Z} \bigoplus Inert(\sigma_{X_A})$.

2.3.1 Cascades

A $(2,n)$-cascade is the union of two length n shift orbits and one shift orbit of length $2^i n$ for each $i \in \mathbb{N} = \{1, 2, \ldots\}$. The base of a $(2,n)$-cascade consists of the two least period n orbits and the tail of a $(2,n)$-cascade consists of its shift orbits of length $2n, 4n, \ldots, 2^i n, \ldots$. A 2-cascade is a $(2,n)$-cascade for some n. If U is an involution of a SFT X, then a $(2,n)$-U cascade is a $(2,n)$-cascade with a base of two type 2 length n shift orbits and a tail of one type 1 shift orbit of length $2^i n$ for each $i \in \mathbb{N} = \{1,2,\ldots\}$. A 2-$U$ cascade is a $(2,n)$-U cascade for some n. Note here that 2-U cascades are 2-cascades, so any condition involving 2-cascades will be true for 2-U cascades, but as is shown in Example 2.3.5, conditions involving 2-U cascades can not necessarily be weakened to 2-cascades.

Proposition 2.3.4. *Suppose U is an involution of a mixing shift of finite type X, and Y is the fixed point shift of U. Then the following are equivalent:*

1. *U is inert.*

2. $\zeta_{X/U}^{-1} = \zeta_X^{-1}$

3. *$Per(X) \backslash Per(Y)$ is a disjoint union of 2-U cascades.*

Proof:

(1) $\Leftrightarrow$ (2) from Theorem 2.3.1.

(2) $\Rightarrow$ (3): Let c_n be the number of type 2 shift orbits of length n, d_n be the number of type 1 shift orbits of length n, and f_n be the number of type 0 shift orbits of length n. Let P_n be the number of length n shift orbits in X and let Q_n be the number of length n shift orbits in X/U. Clearly $P_n = c_n + d_n + f_n$ and $Q_n = c_n/2 + d_{2n} + f_n$.

Since $\zeta_{X/U}^{-1} = \zeta_X^{-1}$, we have $P_n = Q_n$ for each $n \in \mathbb{N}$, so $d_{2n} = c_n/2 + d_n$. Let $n = 2^r q$ with q odd and $r \in \mathbb{Z}^+$. Since $d_q = 0$ for q odd, we have by induction on r that

$$d_n = \frac{1}{2}\sum_{i=0}^{r-1} c_k$$

Therefore type 1 length n shift orbits can be put in bijective correspondence with pairs of type 2 shift orbits of shorter length k such that $n/k = 2^i$ for $i > 0$. It follows that

$Per(X)\backslash Per(Y)$ is a disjoint union of 2-U cascades.

(3) $\Rightarrow$ (2): Let $Per(X)\backslash Per(Y)$ be a disjoint union of 2-U cascades, and let c_n, d_n, and f_n be defined as above. Then by the cascade decomposition of $Per(X)\backslash Per(Y)$, there are exactly as many type 1 length shift orbits as there are lower cascades, i.e. for each $n \in \mathbb{N}$, $d_n = \frac{1}{2}\sum_k c_k$ where $n/k = 2^i$ with $i > 0$. Note that this implies $d_{2n} = d_n + \frac{c_n}{2}$. So

$$Q_n = c_n/2 + d_{2n} + f_n = c_n/2 + (c_n/2 + d_n) + f_n = c_n + d_n + f_n = P_n$$

and thus $\zeta_{X/U}^{-1} = \zeta_X^{-1}$. $\square$

Example 2.3.5. If $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$, then $\zeta_{X_A}^{-1}(t) = (1-6t)(1+2t)$. By Theorem 2.3.4, $Per(X_A)$ is the disjoint union of 2-cascades because X_A has a fixed point free simple (inert) involution. X_A also has a fixed point free involution, ϕ, given by switching the vertices of the graph G_A. ϕ will not be inert since $\zeta_{X_A/\phi}^{-1}(t) = 1-6t \neq \zeta_{X_A}^{-1}(t) = (1-6t)(1+2t)$. This example shows that condition (3) of Proposition 2.3.4 can not be weakened to $Per(X)\backslash Per(Y)$ is a disjoint union of 2-cascades and displays the difference between 2-cascades and 2-U cascades.

2.4 SSE classes of Fixed Point Sets

First we present a useful lemma from [BFK]:

Lemma 2.4.1. *Let ϕ be a finite order automorphism of a shift of finite type X_A. Then there exists a B such that X_A is conjugate to X_B and ϕ is defined by a graph automorphism of G_B.*

Proof: Let P_A be the partition of X_A by the symbol in the zero coordinate, and let $P' = \vee_{i\in\mathbb{Z}}\phi^i(P_A)$. P' is a finite clopen partition of X_A and if $P_i \in P'$, then $\phi(P_i) = P_j$ for some j. Each $x \in X$ corresponds to a point $x' \in (P')^{\mathbb{Z}}$ where $(x')_n = P_i$ for $\sigma^n(x) \in P_i$. Clearly, (X_A, σ_A) and (X', σ) are conjugate. Let X_B be a higher block presentation of X' such that X_B is a one-step shift of finite type. Then ϕ will act on X_B as a graph automorphism. □

We now present and discuss the following result addressing Question 2.0.2.

Theorem 2.4.2. *Let f be an inert automorphism of a mixing shift of finite type X, with $fix_f(X) \subseteq Y$ where Y is a f-invariant subshift of finite type in X. Suppose $f^n = Id$, with $n \geq 2$ and n minimal. If the restriction of f to Y is inert, then Y can be realized as a fixed point shift of a finite order automorphism, ϕ on X, where $\phi^n = id$ and n is minimal.*

Proof of Theorem 2.4.2:

By Lemma 2.4.1, we may assume that X has a graph presentation, G_X, such that f is a one block map defined by a graph automorphism of G_X, which we will also refer to as f. Let Y be defined by F, a finite set of forbidden length k blocks from X. Let $X^{[k]}$ be the k-block presentation of X and note that f will still act as a graph automorphism of $G_{X^{[k]}}$. Y will be presented by G_Y, a subgraph of $G_{X^{[k]}}$ that does not contain vertices defined by word in F and f will act on Y as a graph automorphism of G_Y. Let the image under f of an edge a in G_Y be denoted as $\bar{a}$, and the image of a vertex i be denoted $\bar{i}$.

Since f is inert on Y, we fix $N \in \mathbb{N}$ such that for i and j, vertices of G_Y, there are the same number of paths of length N in G_Y from j to i as there are paths of length N from j to $\bar{i}$ in G_Y. Let $g_{j,i}$ be a bijection from the set of paths of length N in G_Y from j to i to the set of paths of length N in G_Y from j to $\bar{i}$. Similarly, let $h_{j,i}$ be a bijection from the set of paths of length N in G_Y from j to i to the set of paths of length N in G_Y from $\bar{j}$ to i. We choose these bijections such that if $i_1,\ldots,i_k$ is a simple cycle of vertices under the action of U, then for all j $g_{ji_k} \circ \ldots \circ g_{ji_0} = id$ and $h_{i_k j} \circ \ldots \circ h_{i_0 j} = id$.

We define ϕ on X by the following rules:

1. If $x_{[i-N,i+N]}$ is a path in G_Y, then $\phi(x)_i = x_i$.

2. If $x_{[i-N,i+N-1]}$ is a path in G_Y and x_{i+N} is an edge not in G_Y, then $\phi(x)_{[i,i+N-1]} = g_{j,k}(x_{[i,i+N-1]})$, for j the initial vertex and k the terminal vertex of $x_{[i,i+N-1]}$.

3. If $x_{[i-N-1,i+N]}$ is a path in G_Y and x_{i-N} is an edge not in G_Y, then $\phi(x)_{[i-N+1,i]} = h_{j,k}(x_{[i-N+1,i]})$, for j the initial vertex and k the terminal vertex of $x_{[i-N+1,i]}$.

4. Otherwise, $\phi(x)_i = f(x)_i$.

ϕ is well defined by the preceding rules since each rule applies to a different disjoint set of paths in G_X. Note that $x_{[i,j]}$ is a G_Y path iff $\phi(x)_{[i,j]}$ is a G_Y path and $\phi(x) = x \Longleftrightarrow$X$\in Y$ since paths in G_Y are the only paths fixed by ϕ. Consequently, $\phi^m = id$, and ϕ is an automorphism of X with fixed point shift Y. $\square$

Corollary 2.4.3. *Let f be the flip map on the full 2-shift, X, that switches the symbols 0 and 1. If f is inert on a f-invariant SFT Y in X, then Y can be realized as the fixed point set of an involution of X.*

Corollary 2.4.3 raises two questions:

Question 2.4.4. *If Y is the fixed point shift of an inert involution of X, the 2-shift, is Y conjugate to a subshift of finite type in X on which the flip map, f, is inert?*

Question 2.4.5. *For Y a subshift of finite type of X, and $Per(X)\backslash Per(Y)$ a disjoint union of 2-cascades, when does there exist an inert involution, g, of X such that $g(Y) = Y$?*

We will show in Example 2.4.6 that the answer to Question 2.4.4 is no. In particular, this shows that Corollary 2.4.3 is not enough to characterize the fixed point shifts of inert involutions of the 2-shift. The main result of Chapter 4 shows that the answer to Question 2.4.5 is yes up to SE.

Example 2.4.6. There exists a fixed point shift, Y, of an inert involution on the 2-shift such that the flip map is not inert on any subshift conjugate to Y.

Note that the flip map on the 2-shift has an empty fixed point shift. There are 240 points of least period 8 in the full 2-shift which correspond to 30 length 8 shift orbits. Choose some pairing of these length 8 orbits, and choose higher length orbits such that the 30 length 8 shift orbits are the bases of 15 (2,8)-cascades. For each (2,8)-cascade there exists an inert involution on the points in the cascade which moves all points in the cascade. If we consider the disjoint union of the 15 (2,8)-cascades each with an inert fixed point free involution and the identity map on the points $(0)^\infty$ and $(1)^\infty$, then we have an inert involution on the subsystem of the 2-shift which contains only the 15 (2,8)-cascades and the points $(0)^\infty$ and $(1)^\infty$. The results of [BF] will give an inert involution of the 2-shift, g, which moves all points in the 15 (2,8)-cascades and fixes the points $(0)^\infty$ and $(1)^\infty$. If Y is the fixed point shift of g, then Y contains the point $(0)^\infty$ and contains no orbits of length 8. Thus $(0)^\infty$ can not be in a (2,1)-cascade, and f will not be inert on Y by Theorem 2.3.4. $\square$

Note that the last example shows that if $Per(X)$ and $Per(X)\backslash Per(Y)$ are the disjoint unions of 2-cascades, this does not mean that $Per(Y)$ is the disjoint union of 2-cascades.

In the absence of an involution, the following question arises from Theorem 2.3.4:

Question 2.4.7. *Let Y be a SFT in a mixing shift of finite type X such that $Per(X) \setminus Per(Y)$ is the disjoint union of 2-cascades. Can Y be realized as the fixed point shift of an inert involution on X?*

By Proposition 2.3.4, the cascade condition of Question 2.4.7 is necessary for Y to be the fixed point shift of an involution on X. We will comment more on the central nature of the cascade condition in the latter part of Section 3.4.2. In section 3.3, we saw the answer to Question 2.4.7 is yes for certain subshifts of the full 2-shift. In Chapter 4, we show that the cascade condition of Question 2.4.7 is sufficient to realize Y as the fixed point shift of an involution of X', where X' is shift equivalent to the 2-shift.

2.4.1 Inert Automorphism Constructions

An important tool in the manipulation of inert automorphisms has been the Inert Extension Theorem of Kim and Roush [KR2]. We will use the following special case.

Theorem 2.4.8. *[KR3] Let X and Y be shifts of finite type with Y a subshift of X. If U is an inert automorphism of Y such that $U^m = id$, then U can be extended to an inert automorphism V on X such that $V^m = id$.*

Proof: By Lemma 2.4.1, we may assume that Y has a graph presentation, G_Y, with adjacency matrix A such that U is a one block map defined by a graph automorphism of G_Y, which we will also refer to as U. Nasu's Masking Lemma (Lemma 3.1.2) gives a matrix presentation for X, and thus a graph presentation of X, G_X, such that G_Y appears as a subgraph of G_X. Let the image under U of an edge x and vertex i (of G_Y) be denoted by $\overline{x}$ and $\overline{i}$, respectively. Since U is inert on Y, we may fix $N \in \mathbb{N}$ such that for i and j, vertices of G_Y, there are the same number of paths of length N in G_Y from j to i as there are paths of length N from j to $\overline{i}$ in G_Y. Let g_{ji} be a bijection from the set of paths of length N in G_Y from j to i to the set of paths of length N in G_Y from j to $\overline{i}$. Similarly, let h_{ji} be a bijection from the set of paths of length N in G_Y from j to i to the set of paths of length N in G_Y from $\overline{j}$ to i. We choose these bijections such that if $i_1,...,i_k$ is a simple cycle of vertices under the action of U, then for all j $g_{ji_k} \circ ... \circ g_{ji_0} = id$ and $h_{i_k j} \circ ... \circ h_{i_0 j} = id$.

We define V on X as the extension of U by the following rules:

1. If $x_{[i-N,i+N]}$ is a path in G_Y, then $V(x)_i = U(x)_i$.

2. If $x_{[i-N,i+N-1]}$ is a path in G_Y and x_{i+N} is an edge not in G_Y, then $V(x)_{[i,i+N-1]} = g_{j,k}(x_{[i,i+N-1]})$, for j the initial vertex and k the terminal vertex of $x_{[i,i+N-1]}$.

3. If $x_{[i-N-1,i+N]}$ is a path in G_Y and x_{i-N} is an edge not in G_Y, then $V(x)_{[i-N+1,i]} = h_{j,k}(x_{[i-N+1,i]})$, for j the initial vertex and k the terminal vertex of $x_{[i-N+1,i]}$.

4. Otherwise, $V(x)_i = \mathrm{X}_i$.

V is well defined by the preceding rules since each rule applies to a different disjoint set of paths in G_X. Note that $x_{[i,j]}$ is a G_Y path iff $V(x)_{[i,j]}$ is a G_Y path. Consequently, the assumption $U^m = id$ and the cycle conditions on the choices of g_{ij} and h_{ij} imply that $V^m = id$. Clearly V is an automorphism of X which is an extension of U on Y. $\square$

In Section 3.3, we used a similar argument to realize some subshifts as a fixed point shift of finite order inert automorphisms. Note here that the fixed point shift of V will usually be larger than the the fixed point set of U.

We pause now to give some background on the role of cascade conditions in the construction and extension of finite order inert automorphisms.

Boyle and Fiebig [BF] characterize when automorphisms on finite subsystems of a shift of finite type, X, can be extended to a product of inert, finite order automorphisms on X. The complete characterization is quite complicated, but for automorphisms with order a power of a prime p, this extension is predicated on the existence of p-cascades. Boyle and Fiebig create a set of model systems with inert automorphisms that mimic the action of (p,n)-cascades and use Krieger's Embedding Theorem (Theorem 2.2.1) to show the existence of a subshift with the given action on the finite subsystem of (p,n)-cascades. The Inert Extension Theorem (2.4.8) is then used to extend the inert automorphism on the embedded model system to X.

Kim, Roush, and Wagoner [KRW1, KRW2] later gave a complete description of the action of inert automorphisms on finite subsystems of a mixing shift of finite type. KRW used the strategy of BF, except that their extremely complicated construction of model subsystems involved the "positive K-theory" method of polynomial matrix operations discussed in Section 4.1. The actions of compositions of finite order inert automorphisms on finite subsystems of a mixing SFT X realize the actions of all inert automorphisms on these finite subsystems, up to finitely many obstructions arising from low order periodic points.

2.5 Computability of 2-Cascade Condition

In this section we will discuss the related conditions of cascades, zeta functions, and matrix traces. In Proposition 2.5.1, we give a hierarchy of conditions involving cascade decompositions, zeta functions, and the traces of presenting matrices. In Proposition 2.5.3, we give a criterion for when $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades, and Proposition 2.5.5 shows that if $\zeta_X^{-1}(t) = 1 \bmod 2$ then the procedure given in Procedure 2.5.4 is decidable in a finite number of steps.

Consider the following four conditions on a $n \times n$ non-negative integral matrix A:

1. $Per(X_A)$ is the disjoint union of 2-cascades
2. $\det(Id - tA) = 1 \bmod 2$
3. A is nilpotent mod 2
4. $operatorname tr A^n = 0 \bmod 2 \ \forall n$

Note that condition 2 is the same as saying $\zeta_{X_A}(t) = 1 \bmod 2$ by Formula 1.1.

Proposition 2.5.1. *The conditions above satisfy the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) and (2) $\not\Rightarrow$ (1), (4) $\not\Rightarrow$ (3)*

Proof: (2) $\Leftrightarrow$ (3): Suppose A is a $k \times k$ matrix. Then $\det(Id - tA) = t^k \chi_A(t^{-1})$, where $\chi_A(t)$ is the characteristic polynomial of A. The matrix A, considered with its mod 2 entries lying in the field $\mathbb{Z}/2$, has $\chi_A(t) = t^k$ iff A is nilpotent.

(1) $\Rightarrow$ (2): Suppose $Per(X_A)$ is the disjoint union of 2-cascades. $\zeta_X^{-1}(t) = \prod_\gamma (1 - t^{|\gamma|})$, where the product is taken over all finite shift orbits in X and $|\gamma|$ denotes the length

of the shift orbit γ. The product of terms in a $(2,n)$-cascade is given by $(1-t^n)^2(1-t^{2n})(1-t^{4n})\cdots$, which is 1 mod 2. Since $Per(X_A)$ is the disjoint union of 2-cascades, the zeta function of X_A will be 1 mod 2.

(3) $\Rightarrow$ (4): If a $k \times k$ matrix, A, is nilpotent mod 2, then all of the coefficients, except for the t^k term, of the characteristic polynomial of A are 0 mod 2. The trace of A is the coefficient of the $k-1$ degree term of the characteristic polynomial, and so if A is nilpotent mod 2 then the trace of A is 0 mod 2. Also if A is nilpotent mod 2, then all powers of A are nilpotent mod 2, and thus all powers of A have trace that is 0 mod 2.

(2) $\not\Rightarrow$ (1): Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, then $det(Id - tA) = -4t^3 + 6t^2 - 4t + 1 = 1$ mod 2 but X_A has 4 points of least period 1 and no points of least period 2, so $Per(X_A)$ cannot be the disjoint union of 2-cascades.

(4) $\not\Rightarrow$ (3): If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then for all $n \in \mathbb{N}$, $tr(A^n) = 2 = 0$ mod 2, but A is not nilpotent mod 2. $\square$

Proposition 2.5.1 shows that the decomposition of periodic points into 2-cascades is a stronger condition than the mod 2 zeta function can capture. We devote the rest of this section to deciding (in the case we need) when a collection of periodic points is the disjoint union of 2-cascades.

Lemma 2.5.2. *Let Y be a SFT in SFT X. If $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades, then $\zeta_X^{-1}(t) = \zeta_Y^{-1}(t)$ mod 2.*

Proof of Lemma 2.5.2: $\zeta_X^{-1}(t) = \prod_\gamma (1-t^{|\gamma|})$, where the product is taken over all finite shift orbits in X and $|\gamma|$ denotes the length of the shift orbit γ. $\zeta_X^{-1}(t) = \prod_{\gamma \in Per(Y)}(1-t^{|\gamma|})\prod_{\gamma \in Per(X)\backslash Per(Y)}(1-t^{|\gamma|}) = \zeta_Y^{-1}(t)\prod_{\gamma \in Per(X)\backslash Per(Y)}(1-t^{|\gamma|})$. If $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades, then $\prod_{\gamma \in Per(X)\backslash Per(Y)}(1-t^{|\gamma|})$ is the product of series of the form $(1-t^n)^2(1-t^{2n})(1-t^{4n})\cdots$ which correspond to $(2,n)$-cascades. Since $(1-t^n)^2(1-t^{2n})(1-t^{4n})\cdots = 1 \bmod 2$, then $\prod_{\gamma \in Per(X)\backslash Per(Y)}(1-t^{|\gamma|}) = 1 \bmod 2$ and $\zeta_X^{-1}(t) = \zeta_Y^{-1}(t) \bmod 2$ □

2.5.1 Decision Procedure

Let X be a mixing SFT with subshift of finite type Y, such that $\zeta_X^{-1}(t) = \zeta_Y^{-1}(t)$ mod 2. Let P_n be the number of points of least period n in $Per(X)\backslash Per(Y)$. We define D_n with $n \in \mathbb{N} = \{1,2,\ldots\}$, recursively according to the following rules:

1. $D_q = 0$ for all q odd.

2. For n even, $D_n = D_{n/2} + P_{n/2}$.

Note that for $n = 2^r q$ with q odd and $r \geq 1$, it follows by induction on r that

$$D_n = \sum_{i=0}^{r-1} P_{2^i q}$$

Proposition 2.5.3. *Let P_n and D_n be as in the previous paragraph. Define $C_n = P_n - D_n$. Then $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades $\Leftrightarrow \forall n \in \mathbb{N}$, the following conditions hold:*

1. *(Parity condition) C_n is divisible by 2^{r+1} for $n = 2^r q$ with q odd.*

2. *(Quantity condition) C_n is non-negative.*

Moreover, $C_n = P_n - \sum_{i=0}^{r-1} P_{2^i q}$ *for* $n = 2^r q$ *with* q *odd.*

Proof: $\Rightarrow$: Assume $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades and let a_n be the number of $(2,n)$-cascades in $Per(X)\backslash Per(Y)$. For $n = 2^r q$ with q odd, let $b_n = \sum_{i=0}^{r-1} a_{2^i q}$ and note that b_n is the number of length n shift orbits in $(2,k)$-cascades where $n/k = 2^i$ for $i > 0$. Also note that $b_{2n} = b_n + a_n$ and for q odd, $b_q = 0$. By the assumption, $P_n = 2na_n + nb_n$.

We would like to show that $\forall n \in \mathbb{N}$, $C_n = 2na_n$ and $D_n = nb_n$. For n odd, $D_n = 0 = nb_n$ and $P_n = 2na_n = C_n$. Assume that for all $m \leq n$ that $C_m = 2ma_m$ and $D_m = mb_m$. Then $D_{2n} = D_n + P_n = 2D_n + C_n = 2nb_n + 2na_n = 2n(a_n + b_n) = 2nb_{2n}$ and $C_{2n} = P_{2n} - D_{2n} = 2(2n)a_{2n} + (2n)b_{2n} - 2nb_{2n} = 2(2n)a_{2n}$. So by induction, $\forall n \in \mathbb{N}$, $C_n = 2na_n$ and $D_n = nb_n$. The Parity and Quantity conditions are satisfied because a_n is a non-negative integer for all $n \in \mathbb{N}$ and $C_n = 2na_n$.

$\Leftarrow$: Assume that the Parity and Quantity conditions hold $\forall n \in \mathbb{N}$, and let P_n be the number of least period n points in $Per(X)\backslash Per(Y)$. $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades iff there exists non-negative integers, a_i such that for $n = 2^r q$ with q odd, $P_n = 2na_n + n * \sum_{i=0}^{r-1} a_{2^i q}$. If we let $a_n = \frac{C_n}{2n}$, then by the Parity and Quantity conditions, a_n will be a non-negative integer. It remains to show that for $n = 2^r q$ with q odd, $P_n = 2na_n + n * \sum_{i=0}^{r-1} a_{2^i q}$, which we will prove by induction on r.

For n odd, $P_n = C_n = 2na_n$. Assume that for $n = 2^r q$ with q odd, $P_n = 2na_n + n * \sum_{i=0}^{r-1} a_{2^i q}$. Then for

$P_{2n} = C_{2n} + D_{2n} = C_{2n} + D_n + P_n = C_{2n} + 2D_n + C_n =$

$2(2n)a_{2n} + (2n) * \sum_{i=0}^{r-1} a_{2^i q} + 2na_n = 2(2n)a_{2n} + (2n)\sum_{i=0}^{r} a_{2^i q}$

And so by induction on r, for $n = 2^r q$ with q odd, $P_n = 2na_n + n\sum_{i=0}^{r-1} a_{2^i q}$. $\square$

Proposition 2.5.3 gives criterion but not yet a finite procedure to decide if $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades.

Procedure 2.5.4. Procedure for deciding when $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades:

1. If $\zeta_X^{-1}(t) \neq \zeta_Y^{-1}(t)$ mod 2 then $Per(X)\backslash Per(Y)$ is not the disjoint union of 2-cascades.

2. Compute C_n for all $n \in \mathbb{N}$ recursively using the formula $C_n = P_n - \sum_{i=0}^{r-1} P_{2^i q}$ for $n = 2^r q$ with q odd.

3. If C_n satisfies the Parity and Quantity conditions of Proposition 2.5.3 for all $n \in \mathbb{N}$, then $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades.

Proposition 2.5.5. *Let X be a mixing shift of finite type such that X has positive entropy and $\zeta_X^{-1}(t) = 1$ mod 2. Given Y, a proper subshift of finite type in X, the procedure given by Procedure 2.5.4 will determine if $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades in a finite number of steps.*

If $\zeta_X^{-1}(t) = 1$ mod 2, then Lemma 2.5.2 shows that if Y is a SFT in X and $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades, then $\zeta_Y^{-1}(t) = 1$ mod 2.

Proof for Parity Condition:

Let Y be a subshift of finite type in X with $\zeta_Y^{-1}(t) = 1$ mod 2. Let A and B be matrices over $\mathbb{Z}^+$that present X and Y. By Proposition 2.5.1, A and B will be nilpotent mod 2. Let l be the minimum positive integer such that A^l and B^l have all entries divisible by 2, then $\mathrm{tr}(A^l)$ and $\mathrm{tr}(B^l)$ are divisible by 2. Let $\lfloor x \rfloor$ denote the largest integer that is less than or equal to $x \in \mathbb{R}$.

Clearly $2^{\lfloor n/l \rfloor}$ divides $\mathrm{tr}(A^n)$, and there exists $N \in \mathbb{N}$ such that for all $n > N$, $\lfloor n/l \rfloor > log_2(n) + 2$ since n/l is bounded below by a linear function of n and will eventually be larger than $log_2(n)+2$. So for all $n > N$, A^n and B^n are divisible by 2^{r+2} and thus $\mathrm{tr}(A^n)$ and $\mathrm{tr}(B^n)$ are divisible by 2^{r+2}, where $n = 2^r * q$ for q odd.

The number of least period n points in $Per(X) \backslash Per(Y)$ is equal to $\mathrm{tr}_n(A) - \mathrm{tr}_n(B)$, where the n-th net trace is given by $\mathrm{tr}_n(A) = \sum_{d|n} \mu(\frac{n}{d}) tr(A^d)$ and μ is the Mobius function,

$$\mu(m) = \begin{cases} (-1)^r & \text{if } m \text{ is the product of } r \text{ distinct primes} \\ 0 & \text{if } m \text{ contains a square factor} \\ 1 & \text{if } m = 1 \end{cases}.$$

Since $C_n = P_n - \sum_{i=0}^{r-1} P_{2^i q}$, it follows that

$$C_n = [\mathrm{tr}_n(A) - \sum_{i=0}^{r-1} \mathrm{tr}_{2^i q}(A)] - [\mathrm{tr}_n(B) - \sum_{i=0}^{r-1} \mathrm{tr}_{2^i q}(B)]$$

Applying the net trace formula and simplifying, for $n = 2^r * q$ with q odd, we get

$$C_n = \sum_{s|q} \mu(s) [\mathrm{tr}(A^{2^r q/s}) - 2\,\mathrm{tr}(A^{2^{r-1} q/s}) - \mathrm{tr}(B^{2^r q/s}) + 2\,\mathrm{tr}(B^{2^{r-1} q/s})] \tag{2.1}$$

Case 1: For $n = 2^r * q > N$ with q odd, if all non-zero terms in Formula 2.1 are $\mathrm{tr}(A^i)$ for $i > N$, then 2^{r+1} divides all terms and 2^{r+1} divides C_n.

Case 2: Let $q = p_1^{t_1} \ldots p_k^{t_k}$ with each p_i prime. If p_i is a prime greater than N with $t_i \geq 2$, then all terms in Formula 2.1 will have $\mathrm{tr}(A^i)$ for $i > N$ because either q/s is divisible by p_i or $\mu(s) = 0$.

Case 3: If $p_i > N$ and $t_i = 1$, then $n = 2^r p_i q'$ and

$C_n = \sum_{s|q'} \mu(s)[\mathrm{tr}(A^{2^r p_i q'/s}) - 2\,\mathrm{tr}(A^{2^{r-1} p_i q'/s}) - [\mathrm{tr}(A^{2^r q'/s}) + 2\,\mathrm{tr}(A^{2^{r-1} q'/s})]$

$- \mathrm{tr}(B^{2^r p_i q'/s}) - 2\,\mathrm{tr}(B^{2^{r-1} p_i q'/s}) - [\mathrm{tr}(B^{2^r q'/s}) + 2\,\mathrm{tr}(B^{2^{r-1} q'/s})]]$

All of the terms involving p_i will be $\mathrm{tr}(A^i)$ for $i > N$ and 2^{r+1} will divide those terms, so C_n will be divisible by 2^{r+1} iff the sum of the remaining terms will be divisible by 2^{r+1}. A careful examination of the terms that remains yields:

$$\sum_{s|q'} \mu(s)[-\mathrm{tr}(A^{2^r q'/s}) + 2\,\mathrm{tr}(A^{2^{r-1} q'/s}) + \mathrm{tr}(B^{2^r q'/s}) - 2\,\mathrm{tr}(B^{2^{r-1} q'/s})] = -C_{2^r q'}$$

By iterating the argument for Cases 2 and 3, we have reduced our problem to verifying C_n satisfies the Parity Condition when n contains only primes less than N. If α is the product of all primes less than N, then for $n > \alpha^2$ and n divisible only by primes less than N, all non-zero terms in Formula 2.1 will be $\mathrm{tr}(A^i)$ for $i > N$ because s will be at most α and $2^r q/s > \alpha > N$.

This shows that if the Parity condition is true up to $n = \alpha^2$, then the Parity condition will be satisfied for all $n \in \mathbb{N}$.

Quantity Condition:

D_n will grow as $n(\sqrt{\lambda_A}^n - \sqrt{\lambda_B}^n)$ whereas P_n grows as $\lambda_A^n - \lambda_b^n$. This means that at some finite M, for all $n > M$, P_n will be much larger than D_n, and thus the Quantity condition will be satisfied.

So, if L is the maximum of α^2 and M, then it only needs to be checked that C_n satisfies the Parity and Quantity conditions for $n < L$. $\square$.

Chapter 3

SE classes of Fixed Point Sets

The purpose of this chapter is to answer Smillie's Question (2.0.3) up to shift equivalence. The main result of this chapter is

Theorem 3.0.6. *For a shift of finite type Y, contained in the full 2-shift, X, the following are equivalent:*

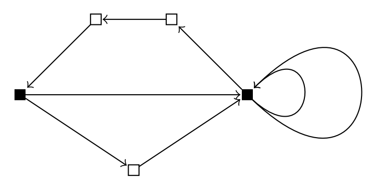
1. $Per(X) \backslash Per(Y)$ *is the disjoint union of 2-cascades.*

2. *Y is the fixed point shift of an involution on a mixing shift of finite type which is SE to X.*

Note that Condition (1) of Theorem 3.2.1 is decidable in a finite number of steps by Proposition 2.5.5. We also note that it is unknown (since 1974 [Wil]) whether a SFT which is SE to the 2-shift must be topologically conjugate to the 2-shift. The proof of our main result relies heavily on the use of polynomial matrix presentations of shifts of finite type and positive elementary matrix operations that produce presentations of conjugate SFTs as discussed in Section 4.1. Section 4.2 is dedicated to the proof of the main result and a discussion of its usefulness. In Section 4.2.2, we remark on some generalizations of the main result.

3.1 Path Presentations and Polynomial Matrices

Section 1.1 describes how shifts of finite type are presented as edge shifts by square matrices over $\mathbb{Z}^+$. Square matrices over $t\mathbb{Z}^+[t]$ can also present a shift of finite type, as can be understood from an example. Given $A = \begin{bmatrix} 0 & t^2+t \\ t^3 & 2t \end{bmatrix}$, we associate to A the following directed graph, $\mathcal{G}_A$.



The graph $\mathcal{G}_A$ is constructed as follows. Since A is 2 by 2, we begin with two vertices (the dark vertices of the above graph). These "essential"vertices will be the indices of the rows of A. For each monomial term, t^k, in the A_{ij} entry, we add a path of length k from i to j. For each path of length k, we add $k-1$ "nonessential" vertices to build the path. A "nonessential"vertex has exactly one incoming and exactly one outgoing edge. Let B be the 5 by 5 adjacency matrix of the graph $\mathcal{G}_A$. We regard A as a presentation of the SFT X_B. As can be seen from this example, matrices over $t\mathbb{Z}^+[t]$ and the corresponding path construction allow for a more compact presentations of graphs.

If B is a non-negative integer matrix, then $C = tB$ and B define the same directed graph. For a matrix A over $t\mathbb{Z}^+[t]$, the conversion from a path presentation to an edge presentation involves building the directed graph by the path construction and then creating the adjacency matrix of this graph. We can convert edge presentations to path presentations and vice versa as is convenient. Let X_A denote the shift of finite type defined by A, a matrix over either $t\mathbb{Z}^+[t]$ or $\mathbb{Z}^+$. For B a matrix over $t\mathbb{Z}^+[t]$, let $B^\sharp$ be the adjacency matrix of the graph $\mathcal{G}_B$ and note that X_B and $X_{B^\sharp}$ are the same SFT.

3.1.1 Constructions Using Polynomial Matrices

Several constructions using polynomial matrices and the path construction have been useful over the past 15 years. In this section, we will discuss how elementary matrix operations on polynomial matrices can be used to describe conjugacies between shifts of finite type, and how elementary positive operations can also be used to recode a polynomial matrix into convenient forms.

Let A be a nonnegative polynomial matrix that is indexed by $\{1,2,...\}$ and has finite support, i.e. there are finitely many non-zero entries. Constructing SFTs using matrices from this infinite setting allows us to use the following tools to compare polynomial matrices of different sizes. For polynomials x and y, we define $x \geq y$ to mean that $y - x \in \mathbb{Z}^+[t]$. Let $E_{ij}(x)$ be the matrix that is the identity matrix (also indexed over $\mathbb{N}$) except for the (i,j) $(i \neq j)$ entry which is a polynomial x over $\mathbb{Z}^+[t]$.

Standing Convention 3.1.1. When we refer to finite square polynomial matrices we mean that the matrix is actually embedded into the upper left corner of a matrix indexed by $\mathbb{N}$. In many cases we will be dealing with matrices of fixed size but in all generality these matrices will sit principally inside the infinite matrices described above.

Theorem 3.1.2. *[KRW, BW] For A,B square matrices over $t\mathbb{Z}^+[t]$, suppose that $Id - B = [E_{ij}(x)(Id - A)]$ or $Id - B = [(Id - A)E_{ij}(x)]$ with $x \in \mathbb{Z}^+[t]$ such that $x \leq A_{ij}$. Then B* *defines a polynomial matrix such that X_A is conjugate to X_B.*

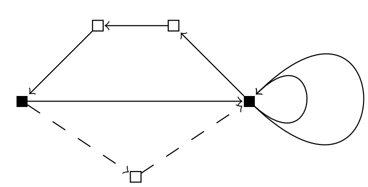
Multiplications by E_{ij} are called positive or elementary operations if they produce a presentation of a conjugate shift of finite type as in Theorem 3.1.2.

For example, if $A = \begin{bmatrix} 0 & t+t^2 \\ t^3 & 2t \end{bmatrix}$ and $x = t^2 < A_{1,2}$, then

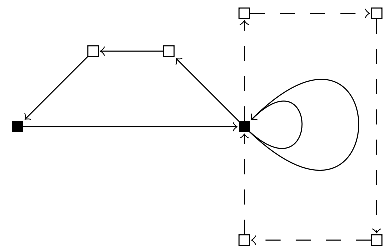
$$[Id - A]E_{2,1}(x) = \begin{bmatrix} 1 & -t-t^2 \\ -t^3 & 1-2t \end{bmatrix} \begin{bmatrix} 1 & t^2 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -t \\ -t^3 & 1-2t-t^5 \end{bmatrix} = [Id - B], \text{ where } B = \begin{bmatrix} 0 & t \\ t^3 & 2t+t^5 \end{bmatrix}.$$

So A and B present conjugate shifts of finite type by Theorem 3.1.2. A positive operation on a matrix A corresponds to deleting a path in the directed graph and adding paths that are the deleted path concatenated with either the predecessor or follower paths. In the example above, x corresponds to the dashed path in



The graph, $\mathcal{G}_B$ is created by deleting the dashed path and adding paths which are the concatenation of predecessor paths and the dotted path. In this example, we delete the length 2 dashed path and add a path of length 5 which is the concatenation of the length 3 path going from the second dark vertex to the first dark vertex and the length 2 dotted path from the previous graph. Thus $\mathcal{G}_B$ is



Let us note that it is possible to define a shift of finite type with a matrix A over $\mathbb{Z}^+[t]$, if A satisfies the No Zero Cycle (NZC) Condition. The NZC says that there are no closed loops in the corresponding directed graph that are traveled in zero time. This generality is not needed for the constructions used in Section 4.2, where we will only need polynomial matrices to be over $t\mathbb{Z}^+[t]$. The more general constructions involving NZC are necessary for the following theorems of Boyle and Wagoner (which we will not need but demonstrate the fundamental nature of positive operations).

Theorem 3.1.3 (Classification Theorem). *Suppose A and B are matrices over $\mathbb{Z}^+[t]$ satisfying the NZC, then the following are equivalent:*

1. *X_A and X_B are topologically conjugate*
2. *There is a sequence of positive row and column operations over $\mathbb{Z}^+[t]$ from $[Id - A]$ to $[Id - B]$*

Theorem 3.1.4 (Conjugacy Theorem). *Every topological conjugacy from (X_A, σ_A) to (X_B, σ_B) arises from some sequence of positive row and column operations over $\mathbb{Z}^+[t]$ from $[Id - A]$ to $[Id - B]$.*

Let us return to the example given above where $A = \begin{bmatrix} 0 & t+t^2 \\ t^3 & 2t \end{bmatrix}$ and $B =$

$\begin{bmatrix} 0 & t \\ t^3 & 2t+t^5 \end{bmatrix}$. Note that when we multiply $[Id - A]$ by an elementary matrix corresponding to a positive operation, $[Id - B]$ has a higher order term in the 2,2 position. The multiplication of the elementary matrices allows us to clear a low order off-diagonal term at the price of adding higher order terms.

A clearing process (or procedure) is a sequence of positive polynomial operations on a polynomial matrix such that all terms of degree less than some fixed d are cleared from all off-diagonal entries. Note here that after applying a clearing process to a matrix, all terms of degree less than d are removed from the off-diagonal entries, but there may be terms of degree less than d on the diagonal. For arbitrary d, it is impossible to remove all terms of degree less than d since periodic points of period less than d can only be built from such terms. Clearing processes enable us to deal with the structure of low order periodic points and higher length paths separately. This is a useful technique to exploit if we wish to extend some property from finite collections of periodic points to the entire shift of finite type. This technique is analogous to more traditional methods of coding between shift of finite types like the marker construction. For example, Kim and Roush used a clearing process to prove their p-fold covering theorem, for which the following theorem is a special case and will be used in proving Theorem 3.2.1.

Theorem 3.1.5. *[KR3] Let X be a mixing shift of finite type with $Per(X)$ a disjoint union of 2-cascades. Given a matrix tD over $t\mathbb{Z}^+[t]$ presenting X, there exist positive elementary operations from tD to $[tA_1 + tA_2]$, where tA_1 and tA_2 are matrices over $t\mathbb{Z}^+[t]$ and $[tA_1 - tA_2]$ is nilpotent.*

3.2 Fixed Point Shifts of Involutions up to SE

The following theorem will answer Smillie's Question 2.0.3 and Question 2.4.7 up to shift equivalence.

Theorem 3.2.1. *For a shift of finite type Y, contained in the full 2-shift, X, the following are equivalent:*

1. $Per(X) \backslash Per(Y)$ *is the disjoint union of 2-cascades.*

2. *Y is the fixed point shift of an involution on a mixing shift of finite type which is SE to X.*

The proof of Theorem 3.2.1 relies on Theorem 3.1.5 and the following lemma, which will be proven in the next section.

Lemma 3.2.2. *Let X be the 2-shift and let F be a non-negative integer matrix presentation of a subshift Y, where $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades. Then there exists a polynomial matrix A over $t\mathbb{Z}^+[t]$, where $A = \begin{bmatrix} tM & 2tB \\ tC & tF \end{bmatrix}$, $Per(X_{[tM]})$ is the disjoint union of 2-cascades, and X_A is conjugate to X.*

Proof of Theorem 3.2.1: Let X be the 2-shift and Y be a subshift of finite type in X with F a presentation of Y such that $Per(X)\backslash Per(Y)$ is the disjoint union of 2-cascades. Applying Lemma 3.2.2, we have a polynomial matrix $A = \begin{bmatrix} tM & 2tB \\ tC & tF \end{bmatrix}$, where F is a non-negative integer matrix presentation of the subshift Y, $Per(X_{[tM]})$ is the disjoint union of 2-cascades, and X_A is conjugate to the 2-shift. Applying Theorem 3.1.5 to $X_{[tM]}$, we get a sequence of positive polynomial operations from $[tM]$ to $[tA_1 + tA_2]$, where tA_1 and

tA_2 are matrices over $t\mathbb{Z}^+[t]$ and $tA_1 - tA_2$ is nilpotent, and these operations will also be positive operations from A to $\begin{bmatrix} t(A1+A2) & 2tB' \\ tC' & tF \end{bmatrix}$. These positive operations will not change the tF block or the even nature of the upper right block since they will correspond to adding a multiple of one of the first n rows or columns to another of the first n rows or columns, where tM is $n \times n$.

So by Theorem 3.1.2, $A = \begin{bmatrix} tM & 2tB \\ tC & tF \end{bmatrix}$ and $\begin{bmatrix} t(A1+A2) & 2tB' \\ tC' & tF \end{bmatrix}$ present conjugate SFTs.

If $D_1 = \begin{bmatrix} t(A1+A2) & 2tB' \\ tC' & tF \end{bmatrix}$, then let $\mathcal{G}_{D_1}$ be the directed graph defined by D_1. Let $\mathcal{G}_{D_2}$ be the graph created from $\mathcal{G}_{D_1}$ as follows. For each monomial term of the form at^k in A_{ij}, with A_{ij} from either of the upper blocks of D_1, we replace the corresponding a paths of length k from i to j with a single path of length $k-1$ from i to a nonessential vertex and a edges from this nonessential vertex to j. We let D_2 be the adjacency matrix of $\mathcal{G}_{D_2}$. X_A and X_{D_2} are conjugate shifts of finite type because there is an obvious bijective correspondence between bi-infinite paths in $\mathcal{G}_{D_1}$ and $\mathcal{G}_{D_2}$. The nonnegative integral matrix D_2 will also have the form $\begin{bmatrix} A_1^* + A_2^* & 2B^* \\ C^* & F \end{bmatrix}$, where $A_1^* - A_2^*$ is nilpotent.

D_2 will be SE over $\mathbb{Z}$ to $D_3 = \begin{bmatrix} A_1^* + A_2^* & 0 & 2B^* \\ -A_2^* & A_1^* - A_2^* & -B^* \\ C^* & 0 & F \end{bmatrix}$ because $A_1^* - A_2^*$ is nilpotent and thus for large enough l, D_3^l differ from D_2 by conjugation with a permutation matrix (Theorem 1.3.4).

Also, D_3 is SE over $\mathbb{Z}$ to $D_4 = \begin{bmatrix} A_1^* & A_2^* & B^* \\ A_2^* & A_1^* & B^* \\ C^* & C^* & F \end{bmatrix}$ because $D_4 = RS$ and $D_3 = SR$ for integral matrices

$$(R,S) = \left(\begin{bmatrix} Id & Id & 0 \\ 0 & -Id & 0 \\ 0 & 0 & Id \end{bmatrix}, \begin{bmatrix} A_1^* + A_2^* & 0 & 2B^* \\ -A_2^* & A_1^* - A_2^* & -B^* \\ C^* & 0 & F \end{bmatrix} \begin{bmatrix} Id & Id & 0 \\ 0 & -Id & 0 \\ 0 & 0 & Id \end{bmatrix} \right)$$

So D_2 is SE over $\mathbb{Z}$ to D_4, and by Theorem 1.3.5, D_2 is also SE over $\mathbb{Z}^+$to D_4 because they both present mixing shifts of finite type. Because $A^\sharp$ is SSE over $\mathbb{Z}^+$(thus SE over $\mathbb{Z}^+$) to D_2 and D_2 is SE over $\mathbb{Z}^+$to D_4, $A^\sharp$ and D_4 are SE over $\mathbb{Z}^+$. If we let D_4 present X' and A_i^* is $n \times n$, then X' is SE to X_A over $\mathbb{Z}^+$and X' has an obvious inert involution ϕ, defined by switching the first n vertices with the second n vertices. Clearly $X_F = Y$ will be the fixed point shift of ϕ. □

3.2.1 Proof of Lemma 3.2.2

We begin the proof of Lemma 3.2.2 with the following lemma.

Lemma 3.2.3. *Let X be the 2-shift and let F be a non-negative integer matrix presentation of a subshift Y, where $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades. Then for all sufficiently large m, there exists a polynomial matrix A over $t\mathbb{Z}^+[t]$, where $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$, such that X_A is conjugate to the full 2-shift and B' a non-negative integral matrix.*

Proof: Let F be some non-negative integer matrix presenting Y. By Lemma 2.5.2,

Y will have zeta function equal to 1 mod 2 and by Proposition 2.5.1, F will be nilpotent mod 2. Thus for large n, F^n will have entries a multiple of 2.

By Nasu's Masking Lemma 2.2.2, there exists a matrix over $\mathbb{Z}^+$, $A = \begin{bmatrix} M & B \\ C & F \end{bmatrix}$ such that X_A is conjugate to X. We will consider the polynomial presentation $[tA] = \begin{bmatrix} tM & tB \\ tC & tF \end{bmatrix}$.

By Theorem 3.1.2, if $E_{ij}(A_{ij})[Id - tA] = [Id - tA']$, then tA' presents a shift of finite type that is conjugate to X. If we multiply $[Id - tA]$ on the left by an elementary matrix for each entry in the upper right block, tB, then

For 1 $\leq$i$\leq$ n and $n+1 \leq j \leq n+k$ where M is a $n \times n$ matrix and F is a $k \times k$ matrix,

$$\prod_{ij} E_{(i,j)}(tA_{(i,j)})[Id - tA] =$$
$$\begin{bmatrix} Id & tB \\ 0 & Id \end{bmatrix} [Id - tA] =$$
$$\begin{bmatrix} Id & tB \\ 0 & Id \end{bmatrix} \begin{bmatrix} Id - tM & -tB \\ -tC & ID - tF \end{bmatrix} =$$
$$\begin{bmatrix} Id - tM - t^2BC & -t^2BF \\ -tC & Id - tF \end{bmatrix}.$$

The matrices $[tA]$ and $\begin{bmatrix} tM + t^2BC & t^2BF \\ tC & tF \end{bmatrix}$ present conjugate shifts of finite type by Theorem 3.1.2. We call multiplying on the left by $\begin{bmatrix} Id & * \\ 0 & Id \end{bmatrix}$ clearing the upper right block when * is the matrix in the upper right block. If we iterate clearing the upper right block of $[tA]$ m times, the resulting polynomial presentation is $\begin{bmatrix} tM' & t^{m+1}BF^m \\ tC & tF \end{bmatrix}$, where tM' contains mixed degree polynomial terms. But since F is nilpotent mod 2, we have for large enough m, $-t^{m+1}BF^m = -2t^{m+1}B'$ for some non-negative integer matrix B'. □

The remainder of this section is devoted to showing that for large enough$m$, the presentation from Lemma 3.2.3, $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$ presents a mixing shift of finite type which is conjugate to the 2-shift and for which $Per(X_{[tM]})$ is the disjoint union of 2-cascades.

Proposition 3.2.4. *If X_A is presented by polynomial matrix $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$ and $T = Per(X_A)\backslash\{Per(X_{[tM]}) \cup Per(X_F)\}$, then T is the disjoint union of 2-cascades.*

The set T is the subset of periodic points in the complement of $Per(X_F)$ that are not in $Per(X_{[tM]})$. $P = Per(X_A)\backslash Per(X_F)$ will be the disjoint union of 2-cascades by the hypothesis of the Lemma 3.2.2.

Proof: Recall from Definition 1.4.1, that the group of simple automorphisms is the subgroup of inert automorphisms that are generated by automorphisms of X_A which are conjugate to a graph automorphism that fixes the all vertices. Let ψ be a pairing of paths corresponding to terms in the upper right block of A, i.e. for each x, a path of length m from i to j that corresponds to a term in the upper right block, we associate to x another path of length m from i to j (which correspond to another term of the same power in the same entry of the upper right block). X_A has a simple involution defined by flipping paths according to ψ and T is exactly the set of periodic points moved by this involution. So by

Theorem 2.3.4, T must be the disjoint union of 2-cascades. $\square$

Proposition 3.2.5. *Let P_1 and P_2 be collections of periodic points of a SFT X such that P_1 and P_2 are the disjoint union of 2-cascades and $P_2 \subseteq P_1$. If c_n is the number of $(2,n)$-cascades in P_1, d_n is the number of $(2,n)$-cascades in P_2, and $c_n \geq d_n$ for all $n \in \mathbb{N}$, then $P_1 \backslash P_2$ is the disjoint union of 2-cascades.*

This proposition is immediately clear since $P_1 \backslash P_2$ will be the disjoint union of the remaining $c_n - d_n$ $(2-n)$-cascades for all $n \in \mathbb{N}$.

Lemma 3.2.6. *Let A be a polynomial matrix over $t\mathbb{Z}^+[t]$, where $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$, such that X_A is conjugate to the full 2-shift, $Per(X_A)\backslash Per(X_F)$ is the disjoint union of 2-cascades, and B' a non-negative integral matrix. Let $T = Per(X_A)\backslash\{Per(X_{[tM]}) \cup Per(X_F)\}$ and $P = Per(X_A)\backslash Per(X_F)$. There exists an $N \in \mathbb{N}$ such that for all $m \geq N$, $c_n \geq d_n$ for all $n \in \mathbb{N}$ where c_n is the number of $(2,n)$-cascades in P, and d_n is the number of $(2,n)$-cascades in T.*

Proof: Let p_n be the number of points of period n (not necessarily least period n) in P, a_n be the number of points of period n in P that are not least period n, then $p_n - a_n$ is the number of least period n points in P. If b_n is the number of least period n points in P that are in $(2,k)$-cascades for $n/k = 2^i$ with $i > 0$, then $n * c_n = p_n - a_n - b_n$ because each least period n point in P is either in a $(2,n)$-cascade or in a lower cascade. Let f_n be the number of points of least period n in X_F.

Given the presentation $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$, the Spectral Radius Theorem bounds the number of allowed blocks of length n in X_A between $C_1(\lambda_A)^n$ and $C_2(\lambda_A)^n$ where λ_A is the eigenvalue with largest modulus and C_1 and C_2 are positive constants. It is also possible to pick constants C_1 and C_2 in such a way that for large n, there are between $C_1(\lambda_A)^n$ and $C_2(\lambda_A)^n$ paths of length n between any 2 vertices.

If we apply this same argument to the graph defined by F, we can choose a constant C_3 such that there are less than $C_3(\lambda_F)^n$ paths between any two vertices. This implies that $p_n = 2^n - f_n > 2^n - C_3(\lambda_F)^n$.

Let $n = 2^r * q$ with q odd, then $b_n < \Sigma_{i=0}^{r-1} 2^{2^i * q}$ because the number of least period n points in the tail of cascades is clearly less than sum of the number of periodic points of order $2^i q$ for $0 \leq i < r$. Further, $b_n < \Sigma_{i=0}^{r-1} 2^{2^i * q} < (r) * 2^{n/2} < n(\sqrt{2})^n$ because the sum is less than the largest term times the number of terms. This shows that as n increases, b_n is bounded above by an exponential function with rate $\sqrt{2}$.

Similarly, the number of points of period but not least period n in P, a_n, can be bounded above by an exponential function with rate $\sqrt{2}$ because $a_n < \Sigma_{i|n, i \neq n} 2^i < n2^{n/2} = n(\sqrt{2})^n$.

We now need to find an upperbound on t_n, the number of points of least period n in T. A periodic point in T corresponds to a time m path from a term in the upper right block and a time $n-L$ path from G_A that together create a closed loop. This length $n-L$ path may have subpaths that correspond to terms in the upper right block, but we only care about overestimating the number of possible paths in G_A that will create a closed path. For large m, there are at least $C_4 \lambda_F^m$ paths that correspond to terms from the upper right block, where $\lambda_F (< 2)$ is the spectral radius of F and m is the power of t in the upper right block. So $t_n < C_2 2^{n-m} * C_4 \lambda_F^m$.

We now combine the estimates given above to compute $nc_n - nd_n$.

$$nc_n - nd_n = p_n - a_n - b_n - nd_n$$

$$> p_n - a_n - b_n - t - n$$

$$> 2^n - C_3(\lambda_F)^n - n*(\sqrt{2})^n - n(\sqrt{2})^n - C_4 2^{n-m}\lambda_F^m.$$

The only term that grows at the same exponential rate as the first term is the t_n term containing 2^{n-L}, but we can make the difference between $C_4(\lambda_F)^m$ and 2^m as large as we want by increasing m. So, there exists a large enough N, such that for all $m \geq N$, $nc_n - nd_n > 0$ for all $n \in N$. $\square$

Proof of Lemma 3.2.2:

Let X be the 2-shift and let F be a non-negative integer matrix presentation of a subshift Y, where $Per(X) \backslash Per(Y)$ is the disjoint union of 2-cascades. Then by Lemma 3.2.3, for all sufficiently large m, there exists a polynomial matrix A over $t\mathbb{Z}^+[t]$, where $A = \begin{bmatrix} tM & 2t^m B' \\ tC & tF \end{bmatrix}$, such that X_A is conjugate to the full 2-shift and B' a non-negative integral matrix. For $T = Per(X_A) \backslash \{Per(X_{[tM]}) \cup Per(X_F)\}$ and $P = Per(X_A) \backslash Per(X_F)$, Lemma 3.2.4 says that T is the disjoint union of 2-cascades and P is the disjoint union of 2-cascades by assumption. By Lemma 3.2.6, there exists an $N \in \mathbb{N}$ such that for all $m \geq N$, $c_n \geq d_n$ for all $n \in \mathbb{N}$ where c_n is the number of $(2,n)$-cascades in P, and d_n is the number of $(2,n)$-cascades in T. Proposition 3.2.5 says that for $m > N$ and $A = \begin{bmatrix} tM & 2t^m B \\ tC & tF \end{bmatrix}$, $Per(X_{[tM]})$ is the disjoint union of 2-cascades, and X_A is conjugate to the full 2-shift. $\square$

3.2.2 Generalizations of Theorem 3.2.1

In the proof of Lemma 3.2.3, we never used that X was conjugate to the full 2-shift, just that $\zeta_X(t) = 1 mod 2$. And the proof of Lemma 3.2.6, relies only on the entropy of X being larger than the entropy of the proper subshift Y. So the same proof above works for the following theorem:

Theorem 3.2.7. *If a mixing shift of finite type, W, has a zeta function that is 1 mod 2, then the following are equivalent:*

1. *Y is a subshift of finite type of W such that $Per(W) \backslash Per(Y)$ is the disjoint union of 2-cascades.*

2. *There is a mixing shift of finite type W' that is SE to W and W' has an inert involution with fixed point shift Y.*

Note that Condition (1) of Theorem 3.2.7 is decidable in a finite number of steps by Proposition 2.5.5.

3.2.3 Future work

There should be a straightforward generalization of Theorem 3.2.7 for strictly order n inert automorphisms.

If Lemma 4.2.2 could be proven relying on the cascade decomposition of $Per(X) \backslash Per(Y)$ rather than the zeta function of Y, then we could eliminate the assumption of Theorem 3.2.7 involving the zeta function of W. This might be accomplished by a different clearing procedure for the upper right block.

SE over $\mathbb{Z}^+$is a very strong equivalence relation on shifts of finite type (See section 1.4 for discussion), but still the use of SE in the statement of Theorem 3.2.1 reflects the mysterious gap between SE and SSE over $\mathbb{Z}^+$, which pervades the analysis of SFTs. For example, when X is the full 2-shift and X' is SE to X over $\mathbb{Z}^+$, it is not known if there is a fixed point free involution of X'.

Chapter 4

Strictly Order n Automorphisms of 1-sided SFTs

If ϕ is an automorphism of a shift of finite type X, then ϕ is called strictly order n if all ϕ orbits have cardinality n. Recall from Section 3.3 that X/ϕ is the quotient of X by the orbit relation of ϕ. The induced action of the shift map on X/ϕ, denoted $\sigma_{X/\phi}$, defines $(X/\phi, \sigma_{X/\phi})$ as a dynamical system. For X irreducible, it is well known that X/ϕ is conjugate to a SFT if ϕ is strictly order n, and $\sigma_{X/U}$ is not even expansive if ϕ does not have strict order n. For a shift of finite type X with finite order automorphism U, Fiebig showed that U is inert iff $\zeta_X(t) = \zeta_{X/U}(t)$ [F]. This result shows the relationship between the periodic point counts of the orbit space X/U (which is not usually even a shift space) and the inertness of U. Kim and Roush asked the following question:

Question 4.0.8. *When does a mixing SFT X have a strictly order n automorphism U such that X is conjugate to X/U?*

Note that by Fiebig's result, U must be inert for X/U to be conjugate to X. In the strongest result to date, Kim and Roush answered this question up to shift equivalence with the following theorem.

Theorem 4.0.9. *[KR3] For a mixing shift of finite type X and p prime, the following are equivalent:*

1. *There exists a mixing shift of finite type X' such that X' is SE over $\mathbb{Z}^+$ to X and X' has an inert strictly order p automorphism U with X'/U conjugate to X.*

2. *$Per(X)$ is the disjoint union of p-cascades.*

(See section 3.3.1 for a discussion of cascades.) We also note that it is still unknown if X being SE to the 2-shift implies that X is SSE to the 2-shift. In this chapter, we consider the more general question involving 1-sided SFTs of which Question 4.0.8 is a special case:

Question 4.0.10. *Given a 1-sided mixing shift of finite type X^+ and a prime p, what are the conjugacy classes of X^+/U for U a strictly order p automorphism of X^+?*

Our first result uses the structure theorem of Boyle, Franks, and Kitchens to completely describe the conjugacy classes of orbit quotient spaces of 1-sided mixing shifts of finite type by strictly order p automorphisms when p is prime.

Theorem 4.0.11. *Let A be a totally out-amalgamated square matrix over $\mathbb{Z}^+$ and let p be a prime integer. The 1-sided shift of finite type, X_A^+ has a strictly order p automorphism, U, with X_B^+ conjugate to $X_A^+/U \iff G_B$ is the quotient graph of an order p (or order 1) graph automorphism ψ of G_A satisfying the following condition:*

1. *Let C be the principal submatrix of A such that G_C is the maximal subgraph of G_A that has vertices fixed by ψ. The matrix C^{red} is nilpotent, where*

$$C_{ij}^{red} = \begin{cases} 0 & \text{if } C_{ij} \text{ is a multiple of } p \\ C_{ij} & \text{otherwise} \end{cases}$$

We also present the following result which shows that the orbit quotient of a mixing shift of finite type by a strictly order n automorphism is conjugate to the image of a particular kind of 1-block map defined by a graph homomorphism of the totally out-amalgamated graph.

Theorem 4.0.12. *Let A present a 1-sided shift of finite type, X_A^+, with a strictly order n automorphism U. Then X_A^+/U is conjugate to a 1-sided shift of finite type $X_{B'}$, such that there is a left resolving factor map $\delta' : X_{A'} \to X_{B'}$ where δ'_V, the vertex map of δ', is the quotient map of the vertex graph automorphism induced by U.*

While this result unlike Theorem 4.0.11 does not require U to have prime order, we do not have a way to determine which of the candidate image shifts will be the orbit quotient of a strictly order n automorphism of X_A. However, there are only finitely many possible candidates up to topological conjugacy. There is no analogous result known (or ruled out) for 2-sided SFTs.

In Section 5.1 we will introduce 1-sided shift spaces and present relevant properties including the solution to the conjugacy problem for 1-sided shifts of finite type. Section 5.2 is dedicated to proving Theorem 4.0.11. In Section 5.3, we give the proof of Theorem 4.0.12.

4.1 One-sided Shift Spaces

In the previous chapters, we considered bi-infinite symbol sequences and the corresponding bi-infinite walks in directed graphs as defining 2-sided shifts of finite type. For a shift space X, let X^+ be the set $\{x_{[0,\infty)} | x \in X\}$. We call X^+ a 1-sided shift space and X^+ is a 1-sided shift of finite type iff X is a shift of finite type. Finite directed graphs and their matrix presentations will also present 1-sided SFTs as the set of (forward) infinite walks through a directed graph.

Obviously, a block code on X with memory 0 will define a block code on X^+, and by considering a higher block presentation and shifting a code with memory $n > 0$ on X, it is possible to define a block code on X^+ from any block code on X. A block code from X to Y with zero memory, ϕ, will define an onto block code $\phi^+ : X^+ \to Y^+$ iff ϕ is an onto map. While Krieger's Embedding Theorem (2.2.1) characterizes the existence of proper embedded subshifts for mixing 2-sided shifts of finite type, there have been very limited results on when a 1-sided SFT can be embedded into another.

In general, it is much harder for a block map to be invertible at the 1-sided level because no memory is allowed. For example, the shift map is invertible on 2-sided shift spaces but the shift map will be one-to-one only on finite 1-sided shift spaces. For irreducible SFTs X and Y, a left resolving map $\psi : X \to Y$ is a 1-block code such that whenever $\phi(a) = b$ and $b'b$ is an allowed 2-block in Y, there exists exactly one symbol a' such that $a'a$ is an allowed 2-block in X and $\phi(a') = b'$. If ϕ is a 1-block conjugacy from X to Y, then ϕ^+ will be a conjugacy from X^+ to Y^+ iff ϕ is left resolving.

Let $Aut(\sigma_X^+)$ be the group of homeomorphisms of X^+ that commute with σ_X^+. For example, $Aut(\sigma_{X_{[2]}}^+)$ consists of only two elements. In contrast, recall that $Aut(\sigma_{X_{[2]}})$ is countably infinite, residually finite, and not finitely generated: it contains a copy of every finite group, the free group on infinitely many generators, and many other groups (but not any group with solvable word problem) [BLR]. Boyle, Franks, and Kitchens show that $Aut(\sigma_X^+)$ is generated by elements of finite order, and prove the following structure theorem for $Aut(\sigma_X^+)$.

Theorem 4.1.1. *[BFK] Let $Simp(X_A^+)$ be the group of simple automorphisms of a 1-sided shift space X_A^+ (as defined in 1.4.1). $Aut(\sigma_{X_A}^+)/Simp(X_A^+)$ is a finite group isomorphic to* *the group of permutation matrices that commute with the total out-amalgamation of A. Further,* $Aut(\sigma^+_{X_A})$ *is a semidirect product* $Simp(X^+_A) \ltimes Aut(\sigma^+_{X_A})/Simp(X^+_A)$.

Let A' be the total out-amalgamation of A. Fix an edge ordering of $G_{A'}$, and define the vertex graph automorphisms of $X_{A'}$ to be the set of graph automorphisms of $G_{A'}$ that preserve the edge ordering. We note that the vertex graph automorphisms are conjugate to the group of permutation matrices that commute with A'. A different choice of edge ordering would give a conjugate group of vertex graph automorphisms. Let the group of vertex graph automorphisms of a graph G be denoted, $Aut_V(G)$.

If $U \in Aut(\sigma^+_A)$, then by Theorem 4.1.1 $U = \phi \circ \psi$ where ϕ is a simple automorphism and ψ is $\varphi\psi'\varphi^{-1}$ for a vertex graph automorphism of $G_{A'}$, ψ' and $\varphi : X_A \to X_{A'}$ a conjugacy. In particular, if $U^n = Id$ then $\psi^n = Id$ because the following diagram will commute.

$$\begin{tikzcd} Aut(\sigma_A) \arrow[r, "U=\phi\circ\psi"] \arrow[d, "\pi"] & Aut(\sigma_A) \arrow[d, "\pi"] \\ Aut_V(G_{A'}) \arrow[r, "\psi'"] & Aut_V(G_{A'}) \end{tikzcd} \tag{4.1}$$

where π is the projection of $Aut(\sigma^+_A)$ onto $Aut(\sigma^+_{X_A})/Simp(X^+_A) \cong Aut_V(G_{A'})$.

Standing Convention 4.1.2. We will drop the $^+$ notation when referring to 1-sided shift spaces, and for the rest of this chapter we will assume a shift space is 1-sided unless otherwise noted.

In contrast to the 2-sided case, we know how to decide when nonnegative integral matrices A and B present topologically conjugate 1-sided SFTs.

Theorem 4.1.3. *[Wil] If A and B are nonnegative integer matrices, then* X_A *is conjugate to* X_B *iff the total out-amalgamations of A and B differ by conjugation with a permutation matrix.*

(We refer the reader to Section 5.3 for a description of state splitting and amalgamation on adjacency matrices including total out-amalgamation). Let G_A be the directed graph defined by adjacency matrix A. We will briefly describe 1-step total out-amalgamation, the graph operation which corresponds to the 1-step total out-amalgamation of an adjacency matrix. Let V_A and E_A be the vertices and edges of the graph G_A and let $E_A(u,v)$ be the number of edges from vertex u to vertex v in G_A. For $u_1, u_2 \in V_A$, we say that u_1 has the same incoming edge pattern or incoming edge structure as u_2 if for every $v \in V_A$, $E_A(v,u_1) = E_A(v,u_2)$. Let $[v]$ be the equivalence class of vertices with the same incoming edge pattern as v. The 1-step total out-amalgamation graph, G_B, is defined as follows.

- G_B has vertices given by the classes of vertices of V_A with the same incoming edge pattern.
- There are $\sum_{i \in [i]} E_A(i,j)$ edges in G_B from $[i]$ to $[j]$.

The total out-amalgamation of a graph G is the graph obtained by repeated total 1-step out-amalgamation until all vertices have unique incoming edge pattern. For a directed graph G, we denote the total 1-step out-amalgamation and the total out-amalgamation by G^* and G' respectively. Out-amalgamation of directed graphs correspond to out-amalgamations of adjacency matrices, so by Theorem 4.1.3, graphs G and H present conjugate SFTs X_G and X_H iff G' is graph isomorphic to H'.

Let p^* be the 1-block map from X_A to X_{A^*} that takes a vertex i to $[i]$ and an edge in E_A from i to j to an edge in E_{A^*} from $[i]$ to $[j]$. Similarly, let p' be the 1-block map from X_A to $X_{A'}$. The map p^* is described by a vertex map $p^*_V : V_A \to V_{A^*}$ and an edge map

$p^*_E : E_A \to E_{A^*}$. Similarly, the map p' is described by a vertex map $p'_V : V_A \to V_{A'}$ and an edge map $p'_E : E_A \to E_{A'}$.

Given a graph automorphism ψ on G, we define the quotient graph H as follows.

- H has vertices that are given by the ψ-vertex orbits.
- H has edges that are given by the ψ-edge orbits (if e is an edge in G from i to j, then $[e]$ is an edge from $[i]$ to $[j]$).

There is a canonical graph homomorphism from G to H that takes an edge $e \in E_G$ from i to j to an edge $[e] \in E_H$ from $[i]$ to $[j]$.

4.2 Quotients of Prime Order Automorphisms

Theorem 4.2.1. *Let A be a totally out-amalgamated square matrix over $\mathbb{Z}^+$and let p be a prime integer. The 1-sided shift of finite type, X_A^+ has a strictly order p automorphism, U, with X_B^+ conjugate to $X_A^+/U \iff G_B$ is the quotient graph of an order p (or order 1) graph automorphism ψ of G_A satisfying the following condition:*

1. *Let C be the principal submatrix of A such that G_C is the maximal subgraph of G_A that has vertices fixed by ψ. The matrix C^{red} is nilpotent, where*

$$C^{red}_{ij} = \begin{cases} 0 & \text{if } C_{ij} \text{ is a multiple of } p \\ C_{ij} & \text{otherwise} \end{cases}$$

Proof $\Rightarrow$: Let U be a strictly order p automorphism of X_A where A is totally out-amalgamated. By Theorem 4.1.1, $U = \phi \circ \psi$ where ϕ is a simple automorphism and ψ is a vertex automorphism of G_A. Further, by Equation 4.1, $\psi^p = Id$. Let G_B be the graph quotient of G_A by ψ, and let C be the principal submatrix of A such that G_C is the maximal subgraph of G_A that has vertices fixed by ψ.. Let C^{red} be a matrix defined by

$$C_{ij}^{red} = \begin{cases} 0 & \text{if } C_{ij} \text{ is a multiple of } p \\ C_{ij} & \text{otherwise} \end{cases}$$

Suppose C^{red} is not nilpotent. Let n be the lowest length such that there is a closed path γ of length n in $G_{C^{red}}$. Let k be the number of paths in $G_{C^{red}}$ that travel through the same vertices as γ and note that k will be the product of C_{ij}^{red} where γ has an edge from i to j. Then k will not be a multiple of p because p is prime. Let S be the set of k periodic points of X_A defined by the k closed paths of length n. U will map S into S and thus must partition S into length p U-orbits. This is a contradiction because p does not divide k, and thus C^{red} will be nilpotent.

Proof $\Leftarrow$: Let G_B be the graph quotient of G_A by an order p vertex automorphism, ψ. Let C be the principal submatrix of A such that G_C is the maximal subgraph of G_A that has vertices fixed by ψ. Let C^{red} be a matrix defined by

$$C_{ij}^{red} = \begin{cases} 0 & \text{if } C_{ij} \text{ is a multiple of } p \\ C_{ij} & \text{otherwise} \end{cases}$$

Assume C^{red} is nilpotent. We define ϕ, a 1-block automorphism of X_A as follows. If $C(i,j)$ is nonzero and divisible by p, then let ϕ_{ij} be an order p permutation of the edges between i and j and define $\phi(x)_0 = \phi_{ij}(x_0)$ if x_0 is an edge from i to j with $C(i,j)$ divisible by p and $\phi(x)_0 = x_0$ otherwise. We define U to be the composition of ϕ with ψ. Clearly, $\phi \circ \psi = \psi \circ \phi$ and $U^p = id$. Because p is prime, U is strictly order p if U has no fixed points. Every point will be moved by U since points not moved by ψ will be infinite paths in G_C and paths that are in G_C for a long time will have an edge moved by ϕ because C^{red} is nilpotent. □

4.3 Strictly Order n Automorphisms of 1-sided SFTs

Before we will begin our proof of Theorem 4.0.12, we will need several lemmas.

Given a matrix B, we let B^* and B' denote the 1-step total out-amalgamation and total out-amalgamation of B. For γ a 1-block map from X_A to $X_{B'}$, we can ask whether the action of γ on vertices and edges factors through the maps p_V^* and p_E^* (as defined in Section 5.1).

Standing Convention 4.3.1. If γ is a 1-block map from X_A to X_B, then γ is also a graph homomorphism of G_A to G_B. We will refer to both the map from X_A to X_B and the map from G_A to G_B as γ. In particular, we will denote the vertex map and edge map of γ as γ_V and γ_E respectively.

Lemma 4.3.2. *Let γ be a 1-block left resolving onto map from X_A to $X_{B'}$. Let p^* denote the one block conjugacy from X_A to X_{A^*} with vertex map p_V^* and edge map p_E^*. There exists a vertex map $\gamma_V^* : V_{A^*} \to V_{B'}$ such that $\gamma_V = \gamma_V^* \circ p_V^*$ and also γ_V^* is the vertex map of a left resolving graph homomorphism $\gamma^* : G_{A^*} \to G_{B'}$. Moreover, there is a left resolving graph homomorphism $\delta : G_A \to G_{B'}$ such that $\delta = \gamma^* \circ p^*$.*

Proof: In order to show that γ_V factors through p_V^*, we must show that for any $u_1, u_2 \in V_A$, if $p_V^*(u_1) = p_V^*(u_2)$, then $\gamma_V(u_1) = \gamma_V(u_2)$. Let $\bar{u}$ represent the image vertex of u under p_V^* and $[u_1]$ be the image of a vertex u under γ_V. If $p_V^*(u_1) = p_V^*(u_2)$, then u_1 and u_2 must have the same incoming edge pattern. If u_1 and u_2 have the same incoming edge pattern, then $[u_1]$ and $[u_2]$ have the same incoming edge pattern because γ is left resolving. But since $G_{B'}$ is totally out-amalgamated, each vertex has unique incoming edge pattern and thus $[u_1] = [u_2]$. Therefore, γ_V factors through p_V^*, $\gamma_V = \gamma_V^* \circ p_V^*$.

Now for each $v \in V_{A^*}$, pick j in V_A such that $v = \bar{j}$. Let p_j^* denote the bijection from j-incoming edges to $[j]$-incoming edges. Then, because γ is left resolving, the map $\gamma \circ (p_j^*)^{-1}$ is a bijection from the $\bar{j}$-incoming edges to $\bar{j}$-incoming edges, and it is compatible with the vertex map γ_V^*. Therefore this edge map defines the required left resolving graph homomorphism $\gamma^* : G_{A^*} \to G_{B'}$. Define $\delta : G_A \to G_{B'}$ on vertices by $\delta_V = \gamma_V$ and on edges by $\delta_E = \gamma_E^* \circ p_E^*$. Now $\delta = \gamma^* \circ p^*$. $\square$

The following example shows how a left resolving map given by an orbit quotient of a strictly order n automorphism does not factor through a conjugacy $p^* : X_A \to X_{A^*}$.

Example 4.3.3. Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ present a labeled directed graph G_A with two vertices and edge from i to j labeled by the symbol in A_{ij}. The (1-step) total out-amalgamation of A is $[a+b]$. The map U on X_A is a fixed point free involution (strictly order 2) defined by the exchange of vertices in G_A and X_A/U is conjugate to X_B where $B = [a+b]$. Let γ be the 2-to-1 left resolving factor map from X_A onto X_B defined by the orbit quotient of U. The 1-block map γ will not factor through the conjugacy $p^* : X_A \to X_{A'}$ because the composition of p^* and any left resolving map from $X_{A'}$ to X_B will be one-to-one whereas γ will be 2-to-one.

Theorem 4.3.4. *Let $\gamma : X_A \to X_{B'}$ be a left resolving factor map. Let p' be a left resolving conjugacy from X_A to $X_{A'}$. Then there are 1-block left resolving factor maps $\delta : X_A \to X_{B'}$ and $\delta' : X_{A'} \to X_{B'}$ such that the vertex maps γ_V and δ_V are equal and $\delta = \delta' \circ p'$.*

Proof: The map p' is the composition of total 1-step out-amalgamation maps p^*. So by iteration of Lemma 4.0.12, we define the required maps δ, δ', and p'. □

Below by "n-to-one" we mean constant n-to-one.

Proposition 4.3.5. *Let X_A be a 1-sided MSFT presented by the block circulant matrix*

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n_1} \\ \vdots & & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}$$

where all of the A_i are $k \times k$ matrices. Let $U \in Aut(\sigma_A)$ be defined by a graph automorphism of G_A that sends the i-th vertex to the $(k+i)$-th vertex, such that $U^n = id$. For $B = A_1 + ... + A_n$, let G_B be the quotient graph of G_A by U and let $\pi : X_A \to X_B$ be the 1-block map defined by the corresponding graph homomorphism from G_A to G_B. Let $\gamma : X_A \to X_{B'}$ be defined by the 1-block map $\gamma = p'_B \circ \pi$, where $p'_B : X_B \to X_{B'}$ is a left resolving conjugacy. The map γ is left resolving, n-to-one, and onto.

Proof: If π is a left resolving, n-to-one, onto map, then the composition $\gamma = p'_B \circ \pi$ will be a left resolving, n-to-one, onto map because p' is a left resolving conjugacy. So it suffices now to consider π.

Each vertex of G_A is in a vertex orbit under U consisting of n distinct vertices. Since $U^n = id$, it follows for every $v \in V_A$ that no two incoming edges of v can be in the same U-orbit of edges. Therefore the map $G_A \to G_B$ sends incoming edges of v bijectively to incoming edges of $[v]$, and π is left resolving. The map π is n-to-one because two points of X_A are colllapsed by π if and only if they lie in the same U-orbit. Clearly π is onto.□

Proof of Theorem 4.0.12: Let A present a 1-sided shift of finite type, X_A, with a strictly order n automorphism U. We care only about the conjugacy class of X_A/U and not on the particular presentation for X_A or X_A/U or even the incarnation of U on X_A. So without loss of generality, by Theorem 2.4.1 we can assume that X_A is presented by the block circulant matrix

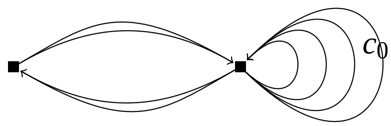
$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n_1} \\ \vdots & & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}$$

where all of the A_i are $k \times k$ matrices. Let $U \in Aut(\sigma_A)$ be defined by a graph automorphism of G_A that sends the i-th vertex to the $(k+i)$-th vertex, such that $U^n = id$. For $B = A_1 + \ldots + A_n$, let G_B be the quotient graph of G_A by U and let $\pi : X_A \to X_B$ be the 1-block map defined by the corresponding graph homomorphism from G_A to G_B.

By Lemma 4.3.5, we now have γ a left resolving, n-to-one factor map from X_A onto $X_{B'}$, where $X_{B'}$ is conjugate to X_A/U. Let $p' : X_A \to X_{A'}$ be a left resolving conjugacy. By Theorem 4.3.4, there is a left resolving map $\delta' : X_{A'} \to X_{B'}$, such that $\delta'_V \circ p'_V = \gamma_V$. $\square$

The following examples show how condition (1) in Theorem 4.0.11 is no longer necessary if U does not have prime order.

Example 4.3.6. If $A = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}$, then G_A is the following graph with names of its associated edges given.



Let ϕ be the 2-block automorphism defined by

$$\phi(x)_i = \begin{cases} a_{j+1} & \text{if } x_i = a_j \\ c_{j+1} & \text{if } x_i = c_j \\ b_{j+k+1} & \text{if } x_{[i,i+1]} = b_j a_k \end{cases}$$

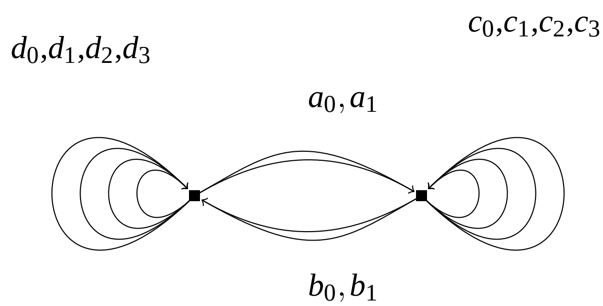
where the subscripts of a and b are taken mod 2 and the subscript of c is taken mod 4. The map ϕ is strictly order 4 since c and ba blocks are permuted with order 4 because

$$b_j a_k \xrightarrow{\phi} b_{j+k+1} a_{k+1} \xrightarrow{\phi} b_{j+1} a_k \xrightarrow{\phi} b_{j+k} a_{k+1} \xrightarrow{\phi} b_j a_k$$

Also, X_A/ϕ is conjugate to X_A because ϕ is simple. In particular, this example shows that condition (1) (involving A^{red}) of Theorem 4.0.11 is not necessary if ϕ is not of prime order because $A^{red} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ is not nilpotent.

This is an example like Example 4.3.6 with the additional property that G_A has a non-trivial vertex graph automorphism.

Example 4.3.7. If $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, then G_A is



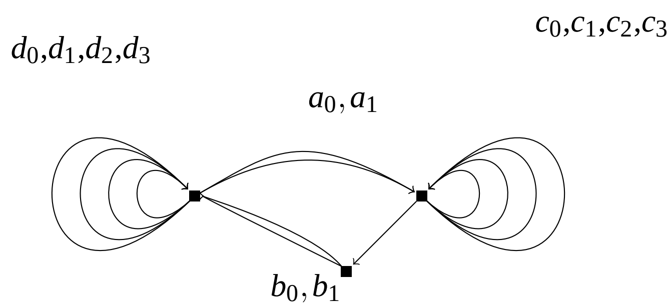
Let ϕ be the 2-block automorphism defined by

$$\phi(x)_i = \begin{cases} a_{j+1} & \text{if } x_i = a_j \\ c_{j+1} & \text{if } x_i = c_j \\ d_{j+1} & \text{if } x_i = d_j \\ b_{j+k+1} & \text{if } x_{[i,i+1]} = b_j a_k \text{ or } b_j d_k \end{cases}$$

where the subscripts of a and b are taken mod 2 and the subscripts of c and d are taken mod 4. The map ϕ is strictly order 4 since c, d, bd, and ba blocks are permuted with order 4. Also, X_A/ϕ is conjugate to $X_{[6]}$. This example shows that condition (1) (involving A^{red}) of Theorem 4.0.11 is not necessary if ϕ is not of prime order because $A^{red} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ is not nilpotent when ϕ has nontrivial vertex graph automorphism.

This is an example like Example 4.3.6 with the additional properties that G_A has a non-trivial vertex graph automorphism and A has relatively prime entries.

Example 4.3.8. If $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, then G_A is



Similarly to the previous examples, X_A will have a strictly order 4 automorphism and $A^{red} = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is not nilpotent. This example shows that the previous examples are not predicated on A being divisible by 2.

Chapter 5

Mixing Shifts of Finite Type with Surjective Dimension Representations

5.1 Importance of Dimension Representation

The fundamental question we consider is:

Question 5.1.1. *Given A, a primitive matrix , what is the image of the dimension representation,* $\rho : Aut(\sigma_A) \to Aut(\hat{A})$*?*

The significance of ρ_A was indicated in Chapter 1. Our contribution to addressing Question 5.1.1, though meaningful, is so far modest. We will show that the only general constructions to date, using elementary strong shift equivalences, cannot construct many candidate images of ρ_A (Proposition 5.2.4). Then we will give a construction of surjective dimension representations for a class of examples (Proposition 5.4.3), for which it is impossible to show that ρ_A is surjective using only conjugacies arising from ESSEs by Proposition 5.2.4. The construction itself is complicated and poorly understood. Nevertheless, it is the only such class which has been constructed, and we hope it will lead to further insight.

Recall from Section 1.4, that $Aut(\hat{A})$ is the group of automorphisms of G_A that commute with $\hat{A}$. Boyle, Lind, and Rudolph show that if A has simple non-zero spectrum (i.e. every nonzero eigenvalue is a simple root of the characteristic polynomial of A), then $Aut(\hat{A})$ is a finitely generated abelian group. However, $Aut(\hat{A})$ in general can be nonabelian and not finitely generated. There are just a few sophisticated examples of SFTs for which the dimension representation is shown to be non-surjective [KRW3, W3].

Let A be a primitive matrix and $Aut_+(\hat{A})$ be the positive automorphisms of the dimension group, i.e. automorphisms of G_A which multiply the Perron eigenvector by a positive constant. Now we regard ρ_A as a map from $Aut(\sigma_A) \to Aut(\hat{A})$ and say that ρ_A is surjective if its image is $Aut_+(\hat{A})$.

Question 5.1.2. *Under what conditions does ρ_A map $Aut(\sigma_{X_A})$ onto $Aut_+(\hat{A})$?*

In some easy cases (e.g. for full shifts) the dimension representation is known to be surjective. There is just one general positive result known for showing elements lie in the image of the dimension representation.

Theorem 5.1.3. *[BLR] Suppose $\Phi \in Aut(\hat{A})$, then for all sufficiently large n, there is a $\phi \in Aut(\sigma_A^n)$ with $\rho(\phi) = \Phi$, and moreover such that ϕ is presented as an elementary conjugacy of (X_{A^n}, σ_{A^n}), i.e. ϕ arises from some ESSE (R,S) from A^n to A^n.*

In Proposition 5.2.4, we will see an obstruction to generalizing the ESSE result of Theorem 5.1.3 to the case $n = 1$ (even after replacing A with some matrix SSE to A).

The main result of this chapter (Proposition 5.4.3) is the presentation of a nontrivial class of examples in which ρ is surjective even though the ESSE obstruction of Proposition 5.2.4 holds. In Section 2.2, we describe $Aut_+(\hat{A})$, the candidate range of the dimension representation and compute several relevant examples. In Section 2.3, we describe state splitting, an operation on matrices over $\mathbb{Z}^+$which is used in the constructions of Section 2.4. In Section 2.4, we give the promised examples of mixing shifts of finite type with surjective dimension representation.

5.2 $Aut_+(\hat{A})$

Recall from Section 1.4 that $Aut(\hat{A})$ is the group of automorphisms of G_A that commute with $\hat{A}$ and let $\rho : Aut(\sigma_A) \to Aut(\hat{A})$ be the dimension representation of the SFT X_A. Also note that $G_A = G_{A^n}$, $Aut(\hat{A}) \subset Aut(\hat{A^n})$, and typically (e.g. if all eigenvalues of $\hat{A}^n$ are simple roots of χ_{A^n}) $Aut(\hat{A}) = Aut(\widehat{A^n})$.

Recall from Section 1.4 that the eventual range of A, R_A, is given by $A^k\mathbb{Q}^k$, for large enough k such that A is an isomorphism from $A^k\mathbb{Q}^n$ to $A^{k+1}\mathbb{Q}^n$. Every element $\hat{\phi} \in Aut(\hat{A})$ is the restriction of a unique invertible real linear transformation $\tilde{\phi} : R_A \otimes \mathbb{R} \to R_A \otimes \mathbb{R}$. The use of $\hat{\phi}$ and $\tilde{\phi}$ is an abuse of notation since we do not in general have an associated automorphism of the shift, ϕ, but we use the hat notation simply to refer to an element of $Aut(\hat{A})$. Assume A is a primitive matrix with spectral radius λ_A. Let v_A be a positive row eigenvector of λ_A (a Perron eigenvector of A). In general, $\tilde{\phi}(v_A) = \alpha v_A$, where α depends only on $\tilde{\phi}$. We define

$$Aut_+(\hat{A}) = \{\hat{\phi} \in Aut(\hat{A}) : \tilde{\phi}(v_A) = \alpha v_A, \alpha > 0\}$$

It is well known that when A is primitive, $\rho_A(Aut(\sigma_{X_A})) \subseteq Aut_+(\hat{A})$. We say that the dimension representation ρ is surjective if $\rho_A(Aut(\sigma_{X_A})) = Aut_+(\hat{A})$.

5.2.1 Examples of $Aut(\hat{A})$ and $Aut_+(\hat{A})$

Example 5.2.1. Let $A = [n]$, so X_A is the full n-shift. G_A is the ring $\mathbb{Z}[1/n]$ since $\mathbb{Z}[1/n]$ are the elements of $\mathbb{Q}$ that will be eventually mapped into $\mathbb{Z}$ by multiplication by n. G_A^+ will be $\mathbb{Z}^+[1/n]$ and $\hat{A}$ will be the isomorphism of $\mathbb{Z}[1/n]$ given by multiplication by n. If $n = p_1^{r_1} ... p_k^{r_k}$ with each of the p_i distinct primes, then $Aut(\hat{A})$ consists of elements of the form $\hat{\phi}(x) = \pm p_1^{t_1} \ldots p_k^{t_k} x$ for $t_i \in \mathbb{Z}$ and $Aut_+(\hat{A})$ are the automorphisms of G_A of the form $\hat{\phi}(x) = p_1^{t_1} \ldots p_k^{t_k} x$ for $t_i \in \mathbb{Z}$, Here $Aut_+(\hat{A})$ is isomorphic to the finitely generated abelian group $\mathbb{Z}^k$.

Example 5.2.2. Suppose A is a $n \times n$ matrix over $\mathbb{Z}$ and $det(A) = \pm 1$. Then $G_A = \mathbb{Z}^n$ and $\hat{A} = A$, since A is invertible over $\mathbb{Z}$. $Aut(\hat{A})$ consists of the elements of $GL(n,\mathbb{Z})$ that commute with A. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we have $G_A = \mathbb{Z}^2$, $Aut(\hat{A}) = \{\pm A^m : m \in \mathbb{Z}\}$, and $Aut_+(\hat{A}) = \{A^m : m \in \mathbb{Z}\}$. Here the group $Aut_+(\hat{A})$ is isomorphic to $\mathbb{Z}$.

Example 5.2.3. Let $A = \begin{bmatrix} 8 & 5 \\ 5 & 8 \end{bmatrix}$, the matrix A has eigenvalues 13 and 3 with eigenvectors $u = [1,1]$ and $v = [1,-1]$. If $\hat{\phi} \in Aut(\hat{A})$, then $\tilde{\phi}$ sends u to $\alpha_\phi u$ and v to $\beta_\phi v$, where $\alpha_\phi = \pm 13^n$ for $n \in \mathbb{Z}$ and $\beta_\phi = \pm 3^m$ for $m \in \mathbb{Z}$, and the pair $(\alpha_\phi, \beta_\phi)$ determines $\hat{\phi}$. $Aut_+(\hat{A})$ consists of the automorphisms $\hat{\phi}$ such that $\alpha_\phi > 0$. Clearly for $\hat{\phi} \in Aut_+(\hat{A})$ we have

$$(\alpha_\phi, \beta_\phi) \in \{(13^n, (-1)^l 3^m : l, m, n \in \mathbb{Z}\}$$

Thus $L_A : \hat{\phi} \to (l,m,n)$ defines an embedding of the group $Aut_+(\hat{A})$ into

$$\{(l,m,n) \in \mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}\}$$

The integral matrices $\begin{bmatrix} 7 & 6 \\ 6 & 7 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ commute with A and thus define elements of $Aut_+(\hat{A})$ with $(\alpha_\phi, \beta_\phi)$ respectively being $(13,1)$, $(1,-1)$, and $(1,3)$. The associated images of (l,m,n) under L_A are respectively $(0,0,1)$, $(1,0,0)$, and $(0,1,0)$. Now it is clear for this A that the embedding L_A is an isomorphism from $Aut_+(\hat{A})$ onto $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$.

The automorphism of X_A, ψ, corresponding to the ESSE

$$(R,S) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

has $(\alpha_\psi, \beta_\psi) = (1, -1)$ and $L_A(\hat{\psi}) = (1,0,0)$. The shift map, σ_A, has $(\alpha_\sigma, \beta_\sigma) = (13,3)$ and $L_A(\hat{\sigma}) = (0,1,1)$. However it is not obvious whether ρ maps $Aut(\sigma_A)$ onto $Aut_+(\hat{A})$.

Let us first consider if we can create a generating set of $Aut_+(\hat{A})$ using the image of ESSE under the dimension representation. If (R,S) is an ESSE from A to A, then R (and S) commute with A and thus R (and S) have eigenvectors $[1,1]$ and $[1,-1]$. This means that R (and S) will have fixed column sum of either 13 or 1 and column difference of either 1 or 3. If R has column sum of 13 and column difference of 3, then $R = A$ or $\begin{bmatrix} 5 & 8 \\ 8 & 5 \end{bmatrix}$, and if R has column sum of 1 and column difference of 1, then $R = Id$ or $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The only other possibility is that either R or S has column sum of 1 and column difference of 3, which would imply that either R or S contains negative entries, which is a contradiction of the assumption that (R,S) is an ESSE over $\mathbb{Z}^+$. So $(1,0,0)$, $(1,1,1)$, and $(0,1,1)$ are the only possible coordinates in $L_A(Aut_+(\hat{A}))$ that can be the image of an ESSE.

Using our construction from Section 2.4, Appendix A explicitly gives γ, a chain of 4 ESSEs from A to A with $(\alpha_\gamma, \beta_\gamma) = (13,1)$ and $L(\hat{\gamma}) = (0,1,0)$. The three automorphisms of X_A given by ψ, γ, and σ_A will map to a generating set of $Aut_+(\hat{A})$ given by their L_A coordinates of $(1,0,0)$, $(0,1,0)$, and $(0,1,1)$, and thus ρ will be surjective. $\square$

The construction of the embedding L_A is in no way particular to the preceding example. Let A be a primitive matrix with simple integer eigenvalues $\lambda_1, \ldots, \lambda_n$ where λ_1

has the largest modulus. If λ_i is divisible by m_i primes, then the map L_A is an embedding of $Aut_+(\hat{A})$ into $(\mathbb{Z}/2)^{n-1} \times \mathbb{Z}^{m_1} \times ... \times \mathbb{Z}^{m_n}$ given by

$$(p_1^{i_1}...p_{m_1}^{i_{m_1}}, (-1)^{l_2} q_1^{j_1}...q_{m_2}^{j_{m_2}}, ..., (-1)^{l_n} r_1^{k_1}...r_{m_n}^{k_{m_n}}) \to (l_2, ..., l_n, i_1, ..., i_{m_1}, ..., k_1, ..., k_{m_n})$$

One could try to build the image of ρ_A by finding primitive matrices C which are SSE to A, finding ESSEs of C ($C = RS = SR$) with R a non-trivial action on G_C, and pulling back to G_A. The following proposition shows this approach cannot succeed in general.

Proposition 5.2.4. *Suppose $C = RS = SR$ with C a primitive matrix with its eigenvalue of largest modulus being a prime integer p. Let ϕ be the conjugacy associated to the ESSE (R, S). Then there is a $\hat{\psi} \in Aut(\hat{C})$ and $k \in \mathbb{Z}^+$ such that $\hat{\psi}^k = id$ and $\hat{\phi}\hat{\psi} = \hat{C}$ or $\hat{\phi} = \hat{\psi}$.*

Proof: The matrices R and S commute with C. Let v_C be the positive eigenvector of C. Because λ_C is a simple eigenvalue of C, there are constants α, $\beta > 0$ such that $v_C = \alpha v_C$, $v_C S = \beta v_C$. Now $\alpha\beta = p$, so either $\alpha = 1$ or $\beta = 1$. Suppose $\beta = 1$. Because $v_C > 0$ and $S_{ij} \geq 0$ and $v_C \beta = v_C S$, we have that β is the spectral radius of S by the Spectral Radius Theorem. If $\hat{\psi} = S$, then for some $k \in \mathbb{Z}^+$, $\hat{\psi}^k = id$ since S will have eigenvalues of largest modulus that are k-th roots of unity. This would imply that $\hat{\phi}\hat{\psi} = \hat{R}\hat{S} = \hat{C}$.

Suppose $\beta = p$ and $\alpha = 1$. The same argument above shows that for $\hat{\psi} = R$, there is some $k \in \mathbb{Z}^+$ such that $\hat{\psi}^k = id$. □

Examples 5.4.1 and 5.4.2 do not satisfy the hypothesis of Proposition 5.2.4 and a generating set of $Aut_+(\hat{A})$ can be made by image under ρ_A of ESSEs. However, the matrices presented in Proposition 5.4.3 are subject to the obstruction of Proposition 5.2.4, but ρ_A is still surjective. So not only are the examples in Proposition 5.4.3 nontrivial, but they demonstrate that we are not missing some miraculous obstruction.

Lastly, as an obstruction to another proof strategy, we give a cautionary example where the image of $Aut_+(\hat{A})$ under the embedding L_A constructed above need not be all of the lattice $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$. This does not preclude A from having a surjective dimension representation, but it shows that one cannot find a general proof which simply realizes automorphisms whose L_A images are arbitrary elements of the lattice.

Example 5.2.5. Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. A has eigenvalues of 5 and 2 with eigenvectors $u = [2,1]$ and $v = [1,-1]$. In order to compute the image of L_A, we need to examine matrices that commute with A and have non-zero spectrum 5^{p_1} and $\pm 2^{p_2}$ with corresponding eigenvectors u and v. The unique matrix that has eigenvalues 5 and 1 with eigenvectors u and v is $C = \frac{1}{3}\begin{bmatrix} 11 & 4 \\ 8 & 7 \end{bmatrix}$. C corresponds to the elementary vector $(0,0,1) \in \mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$, but $C \notin Aut(\hat{A})$, because for all $n \in \mathbb{N}$, $[1,0]CA^n \notin \mathbb{Z}^2$. Therefore $(0,0,1) \notin L_A(Aut_+(\hat{A}))$. In fact, $(0,n,m) \notin L_A(Aut_+(\hat{A}))$ if $n+m$ is odd.

5.3 State splittings

State splitting is an important type of ESSE between matrices over $\mathbb{Z}^+$. Any SSE between shifts of finite type can be decomposed into state splittings and the inverse operations of state amalgamations. State splittings will be used to generate the SSEs used in Proposition 5.4.3.

Let A be a $n \times n$ matrix over $\mathbb{Z}^+$. An in-splitting of A is given by some splitting of the rows of A, i.e. the i-th row of A, a_i, is split into k_i rows over $\mathbb{Z}^+$, $b_1,\ldots,b_{k_i}$, such that $\sum_{j=1}^{k_i} b_j = a_i$. Let $k = \sum_{i=1}^{n} k_i$. The in-splitting matrix of A is the k by n matrix, R, of the split rows of A, i.e. that the first k_1 rows of R are the rows split from a_1, the k_1+1 to k_1+k_2 rows of R are the split rows of a_2, and so on. The split matrix, B, is created by taking R and copying the i-th column of R k_i times. Let S be the $n\times k$ matrix such that $S_{ij} = 1$ if the j-th row of R is split from the i-th row of A and $S_{ij} = 0$ otherwise. Then $A = SR$, $B = RS$ and (R,S) is an ESSE from B to A. S is a so called subdivision matrix in which every row has exactly one entry equal to 1 and every column has at least one entry equal to 1. $A = RS$ since S will sum the columns of R that are split from the same column of A. $B = SR$ since S will copy the the rows of R according to how the columns of R were split from the columns of A. The matrix A is called an out-amalgamation of B if B can be made from a finite sequence of in-splittings of A.

Example 5.3.1. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and let the first row, $[3,1]$, be split into $[1,1]$ and $[2,0]$ and the second row, $[2,4]$ be split into $[1,1]$, $[1,2]$, and $[0,1]$.

Then $R = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$, so $B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

There is an analogous procedure for the out-splitting of a matrix A. For example, if we split the first column of A, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, into $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the second column, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, into $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$R = \begin{bmatrix} 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

We say that B is a out-splitting of A.

A matrix B is a in-amalgamation of A if B can be obtained by a finite sequence of out-splittings of A. A matrix B is a 1-step splitting of a matrix A if B can be obtained as a single splitting of A, i.e. if A and B are ESSE by some (R,S), given by a splitting. R is called the in-/out-splitting matrix (or the in-/out-amalgamation matrix) for the out-/in-splitting of A to B. S is called the subdivision matrix for the splitting of A to B (or the amalgamation matrix for the amalgamation of B to A).

The total 1-step in-amalgamation of A is defined as follows. If A is n by n and A has $k(\leq n)$ distinct rows, then let R be the k by n matrix made up of the distinct rows of A. R is unique up to some permutation of its rows. For a fixed choice of the rows of R, S is given by a unique subdivision matrix such that $A = SR$. If $B = RS$, then B is called the total 1-step in-amalgamation of A and is uniquely determined by A up to conjugation by a permutation matrix. The total 1-step column amalgamation is defined similarly. The total in-/out-amalgamation of a matrix A is the matrix arrived at by performing total 1-step in-/out-amalgamations until every row/column is distinct.

Example 5.3.2. Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $A = [2]$. The total 1-step in-amalgamation of C is B and the total in-amalgamation of C is A.

Theorem 5.3.3 (LM 7.1.2). *Let ϕ be a conjugacy from X_A to X_B. Then ϕ is a composition of conjugacies given by ESSEs from splittings and amalgamations.*

Furthermore, it is possible to decompose an automorphism of X_A, ϕ, into the composition of k conjugacies arising from in-splittings and k conjugacies arising from in-amalgamations.

5.4 Examples of Surjective Dimension Representations

Example 5.4.1. For $n \in \mathbb{N}$, the dimension representation of the full n-shift is surjective.

Let $A = [n]$, so X_A is the full n-shift. G_A is the ring $\mathbb{Z}[1/n]$, G_A^+ will be $\mathbb{Z}^+[1/n]$, and $\hat{A}$ will be the isomorphism of $\mathbb{Z}[1/n]$ given by multiplication by n. If $n = p_1^{r_1} \ldots p_k^{r_k}$ for primes $p_1, \ldots, p_k$, then $Aut(\hat{A})$ consists of elements of the form $\hat{\phi}(x) = \pm p_1^{t_1} \ldots p_k^{t_k} x$ for $t_i \in \mathbb{Z}$ and $Aut_+(\hat{A}) = \{\hat{\phi} : \hat{\phi}(x) = p_1^{t_1} \ldots p_k^{t_k} x\}$. Clearly, $Aut(\hat{A}) \cong \mathbb{Z}/2 \times \mathbb{Z}^k$ and $Aut_+(\hat{A}) \cong \mathbb{Z}^k$. Consider γ_i, the ESSE from A to A given by $([p_i], [n/p_i])$. $\rho_A(\gamma_i) = [p_i]$ and $L_A(\hat{\gamma}_i) = e_i$, where e_i is the i-th elementary row vector. So $\gamma_1, \ldots, \gamma_k$ get mapped by ρ_A to a generating set of $Aut_+(\hat{A})$, and thus the dimension representation of A is surjective.

Example 5.4.2. Let $B = nA$ where A is primitive symmetric matrix with eigenvalues n and 1, both of multiplicity 1. If n is prime, then the dimension representation of B is surjective.

If A has integer eigenvectors u and v for eigenvalues n and 1, then B has eigenvectors u and v for eigenvalues n^2 and n (by Perron-Frobenius theory, we assume u is positive). $\hat{B}$ will be given by multiplication by B on G_B.

$Aut(\hat{B})$ consists of matrices over $\mathbb{Q}$ that are automorphisms of G_A and commute with B (thus must have the same eigenspaces). $Aut_+(\hat{B})$ will consist of the matrices that have eigenvalue n^j on u and $\pm n^k$ on v for $j,k \in \mathbb{Z}$. We will show that L_A will map $Aut_+(\hat{A})$ isomorphically onto $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$ by giving elements of $Aut_+(\hat{B})$ whose images under L_A generate all of $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$. Since A is symmetric, B will be symmetric, and ψ is an ESSE $[D, BD]$ from B to B where D is the permutation matrix such that conjugation by D gives the transpose of a matrix. In this case, $(\alpha_\psi, \beta_\psi) = (1, -1)$ and $L_B(\hat{\psi}) = (1,0,0)$. Also note that $\rho(\sigma_B) = \hat{B}$ and $L_B(\hat{B}) = (0,2,1)$. If γ is the ESSE from B to B given by (A, nId), then $L_B(\hat{\gamma}) = (0,1,0)$. Since $(1,0,0)$, $(0,2,1)$, and $(0,1,0)$ will generate all of $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z} \cong Aut_+(\hat{B})$, the dimension representation of B is surjective.

Alternatively, it is possible to view X_B as a product shift of $X_{[n]} \times X_A$. A point in X_B is a point in the full n-shift cross a point in X_A and γ corresponds to the automorphism of the product shift given by $\sigma_{X_A} \times id_{[n]}$.

Recall that a conjugacy arising from an ESSE is called an elementary automorphism.

Theorem 5.4.3. *Let n and k be prime odd integers such that $n > 1$ and $0 < k^2 < n$. Let* $A = \begin{bmatrix} \frac{n+k}{2} & \frac{n-k}{2} \\ \frac{n-k}{2} & \frac{n+k}{2} \end{bmatrix}$*, then the following are true:*

1. *The dimension representation, ρ, of A is surjective.*

2. *The restriction of ρ to the subgroup of $Aut(\sigma_A)$ generated by conjugates of elemen-*

tary automorphisms in not surjective.

Remark: Note that the previous example shows for the case $n = k^2$, A will have surjective dimension representation.

Proof of (2): The matrix A has simple spectrum of n and k, with eigenvectors of $u = [1,1]$ and $v = [1,-1]$ respectively. By Proposition 5.2.4, the restriction of ρ to the subgroup of $Aut(\sigma_A)$ generated by conjugates of elementary automorphisms is a subgroup that is generated by $\hat{A}$ and finite order elements. As show below, $Aut_+(\hat{A}) \cong \mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$, which is clearly larger than the subgroup generated by $\hat{A}$ and finite order elements. $\square$

Proof of (1): The matrix A has simple spectrum of n and k, with eigenvectors of $u = [1,1]$ and $v = [1,-1]$ respectively. For $\hat{\phi} \in Aut_+(\hat{A})$ with $(\alpha_\phi, \beta_\phi) = (n^t, (-1)^l k^s))$, L_A maps $\hat{\phi}$ to $(l,s,t) \in \mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$. Further, L_A will map $Aut_+(\hat{A})$ onto $\mathbb{Z}/2 \times \mathbb{Z} \times \mathbb{Z}$ because $\begin{bmatrix} \frac{n^t+(-1)^l k^s}{2} & \frac{n^t-(-1)^l k^s}{2} \\ \frac{n^t-(-1)^l k^s}{2} & \frac{n^t+(-1)^l k^s}{2} \end{bmatrix}$ will be an integral matrix that commutes with A for any $(l,s,t) \in \mathbb{Z}/2 \times \mathbb{Z}^+ \times \mathbb{Z}^+$, which generates the group $mathbbZ/2 \times \mathbb{Z} \times \mathbb{Z}$.

Since A is symmetric, there exists ψ, an ESSE $[D, AD]$ from A to A with $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $L_B(\hat{\psi}) = (1,0,0)$. Another generator of $Aut_+(\hat{A})$ is given by $\rho_A(\sigma_A) = \hat{A}$ with $L_A(\hat{A}) = (0,1,1)$. In the construction below, we produce γ, such that $(\alpha_\gamma, \beta_\gamma) = (n,1)$ and ρ composed with L_A maps γ to the $(0,1,0)$ element of $\mathbb{Z}/2 \times \mathbb{Z}^2 \cong Aut_+(\hat{A})$. The construction of γ does not require n or k to be prime: it does use $n = k \bmod 2$, which is required for the entries of A to be integers. The three automorphisms of X_A given by ψ, γ, and σ_A will map to a generating set of $Aut_+(\hat{A})$ given by their L_A coordinates of $(1,0,0)$, $(0,1,0)$, and $(0,1,1)$, and thus ρ_A will be surjective.

The automorphism γ will be given by a chain of ESSEs $A \xrightarrow{(D_1,S_1)} A_1 \xrightarrow{(D_2,S_2)} A_2 \xrightarrow{(D_3,S_3)} A_3 \xrightarrow{(D_4,S_4)} A$ where D_1 and D_2 are subdivision matrices for in-splittings and D_3 and D_4 are amalgamation matrices for in-amalgamations. Further, we will show that $[1,1]*D_1*D_2*D_3*D_4=[n,n]$ and $[1,-1]*D_1*D_2*D_3*D_4=[1,-1]$, which implies $(\alpha_\gamma,\beta_\gamma)=(n,1)$ and ρ_A composed with L_A maps γ to the $(0,1,0)$ element of $\mathbb{Z}/2\times\mathbb{Z}^2\cong Aut_+(\hat{A})$.

We will now briefly describe the general procedure for the splittings (D_1,S_1), (D_2,S_2), (D_3,S_3), and (D_4,S_4).

The splitting (D_1,S_1):

(D_1,S_1) will be a row splitting of the two rows of A. The first row, $[\frac{n+k}{2},\frac{n-k}{2}]$ will be split into $\frac{k+1}{2}$ rows of the form $[k,0]$, $\frac{k-1}{2}$ rows of the form $[0,k]$, and one row of the form $[\frac{n-k^2}{2},\frac{n-k^2}{2}]$. This is a valid splitting because $n>k^2$ and

$$\frac{k+1}{2}[k,0]+\frac{k-1}{2}[0,k]+[\frac{n-k^2}{2},\frac{n-k^2}{2}]=[\frac{n+k}{2},\frac{n-k}{2}]$$

The second row , $[\frac{n-k}{2},\frac{n+k}{2}]$, will be split into $\frac{k-1}{2}$ rows of the form $[k,0]$, $\frac{k+1}{2}$ rows of the form $[0,k]$, and one row of the form $[\frac{n-k^2}{2},\frac{n-k^2}{2}]$. This is a valid splitting because $n>k^2$ and

$$\frac{k-1}{2}[k,0]+\frac{k+1}{2}[0,k]+[\frac{n-k^2}{2},\frac{n-k^2}{2}]=[\frac{n-k}{2},\frac{n+k}{2}]$$

The matrix S_1 will have 2 columns and $2k+2$ rows because both rows of A are split $k+1$ times.

(For presentation purposes, we write out S_1 transpose.)

$$\begin{pmatrix} \overbrace{k,\cdots,k,}^{(k+1)/2} & \overbrace{0,\cdots,0,}^{(k-1)/2} & \overset{1}{\frac{n-k^2}{2},} & \overbrace{k,\cdots,k,}^{(k-1)/2} & \overbrace{0,\cdots,0,}^{(k+1)/2} & \overset{1}{\frac{n-k^2}{2}} \\ 0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & 0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2} \end{pmatrix}$$

$$\begin{array}{ll} \text{\# of cols=} & \begin{array}{cc} k+1 & k+1 \end{array} \\ D_1 = & \begin{pmatrix} \overbrace{1,\cdots,1,} & \overbrace{0,\cdots,0} \\ 0,\cdots,0, & 1,\cdots,1 \end{pmatrix} \end{array}$$

$A_1 = S_1 D_1$ and will be $k+1$ copies of the first column of S_1 and $k+1$ copies of the second column of S_1 because the first row of A was split $k+1$ times and the second row of A was split $k+1$ times.

$$
\begin{array}{cccl}
\text{\# of cols=} & \overbrace{}^{k+1} & \overbrace{}^{k+1} & , \\
A_1 = & \left(\begin{array}{c} K \\[2em] 0 \\[2em] \frac{n-k^2}{2} \\[2em] K \\[2em] 0 \\[2em] \frac{n-k^2}{2} \end{array}\right. & \left.\begin{array}{c} 0 \\[2em] K \\[2em] \frac{n-k^2}{2} \\[2em] 0 \\[2em] K \\[2em] \frac{n-k^2}{2} \end{array}\right) & \begin{array}{l} (k+1)/2 \text{ rows} \\[2em] (k-1)/2 \text{ rows} \\[2em] 1 \text{ row} \\[2em] (k-1)/2 \text{ rows} \\[2em] (k+1)/2 \text{ rows} \\[2em] 1 \text{ row} \end{array}
\end{array}
$$

with K denoting a matrix with all entries equal to k.

The splitting (D_2, S_2):

A_1 has 3 different rows, $[k, \cdots, k, 0, \cdots, 0]$, $[0, \cdots, 0, k, \cdots, k]$, and $[\frac{n-k^2}{2}, \cdots, \frac{n-k^2}{2}]$.

Each of the k rows of A_1 with the form

$$
\text{\# of cols} = \quad [\overbrace{k, \cdots, k,}^{k+1} \quad \overbrace{0, \cdots, 0}^{k+1}]
$$

should be split into k rows

$$
\begin{array}{lccc}
\text{\# of cols =} & k & & k+1 \\
& [\overbrace{k,0,\cdots,0,} & 1, & \overbrace{0,\cdots,0}] \\
& [0,k,\cdots,0, & 1, & 0,\cdots,0] \\
& \vdots & \vdots & \vdots \\
& [0,0,\cdots,k, & 1, & 0,\cdots,0]
\end{array}
$$

For $1 \leq i \leq k$, we will call the i-th row above a type (1,i) row.

Each of the k rows of A_1 with the form

$$
\begin{array}{lcc}
\text{\# of cols =} & k+1 & k+1 \\
& [\overbrace{0,\cdots,0,} & \overbrace{k,\cdots,k}]
\end{array}
$$

should be split into k rows

$$
\begin{array}{lccc}
\text{\# of cols =} & k+1 & k & 1 \\
& [\overbrace{0,\cdots,0,} & \overbrace{k,0,\cdots,0,} & 1] \\
& [0,\cdots,0, & 0,k,\cdots,0 & 1] \\
& \vdots & \vdots & \vdots \\
& [0,\cdots,0, & 0,\cdots,k, & 1]
\end{array}
$$

For $1 \leq i \leq k$, we will call the i-th row above a type $(2,i)$ row.

The two rows of A_1 of the form $[\frac{n-k^2}{2},\cdots,\frac{n-k^2}{2}]$ should be split into $\frac{n-k^2}{2}$ pairs of rows with each pair summing to $[1,...,1]$ and such that the first row of the pair has ones in the first $\frac{k+1}{2}$ entries and from the $k+1$ entry to the $\frac{3k+1}{2}$ entry, and zeros otherwise. This pair is chosen such that the transpose will match the resulting columns that show up in A_2. Each pair of rows will look like

$$\begin{pmatrix} \overbrace{1\ldots1}^{\frac{k+1}{2}} & \overbrace{0\cdots0}^{\frac{k-1}{2}} & \overbrace{1}^{1} & \overbrace{1\cdots1}^{\frac{k-1}{2}} & \overbrace{0\cdots0}^{\frac{k+1}{2}} & \overbrace{0}^{1} \\ 0\ldots0 & 1\cdots1 & 0 & 0\cdots0 & 1\cdots1 & 1 \end{pmatrix}$$

We will refer to this pair of rows as complementary rows.

S_2 will have the form of

$$\begin{bmatrix} (k+1)/2 \text{ blocks of type 1 rows} \\ (k-1)/2 \text{ blocks of type 2 rows} \\ (n-k^2)/2 \text{ pairs of complementary rows} \\ (k-1)/2 \text{ blocks of type 1 rows} \\ (k+1)/2 \text{ blocks of type 2 rows} \\ (n-k^2)/2 \text{ pairs of complementary rows} \end{bmatrix}$$

A_2 will have k copies of the first $(k+1)/2$ columns of R_2 because the first $(k+1)/2$ rows of A_1 are split k times. Then A_2 will have k copies of the $(k+1)/2+1$ to $(k+1)/2+(k-1)/2$ columns of R_2 because the $(k+1)/2+1$ to $(k+1)/2+(k-1)/2$ rows of A_1 are split k times, and so on.

$$
\begin{array}{ccl}
\text{\# of cols} = & \begin{array}{cccc} k^2 & n-k^2 & k^2 & n-k^2 \end{array} & \\
A_2 = & \left(\begin{array}{cccc}
DK & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
DK & 1 & 0 & 0 \\
0 & 0 & DK & 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & DK & 1 \\
P & P & P & P \\
DK & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
DK & 1 & 0 & 0 \\
0 & 0 & DK & 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & DK & 1 \\
P & P & P & P
\end{array}\right) &
\begin{array}{l}
\left.\begin{array}{c} \cdot \\ \\ \cdot \end{array}\right\} k(k+1)/2 \text{ rows} \\
\left.\begin{array}{c} \cdot \\ \\ \cdot \end{array}\right\} k(k-1)/2 \text{ rows} \\
\qquad n-k^2 \text{ rows} \\
\left.\begin{array}{c} \cdot \\ \\ \cdot \end{array}\right\} k(k-1)/2 \text{ rows} \\
\left.\begin{array}{c} \cdot \\ \\ \cdot \end{array}\right\} k(k+1)/2 \text{ rows} \\
\qquad n-k^2 \text{ rows}
\end{array}
\end{array}
$$

where 0 and 1 represent matrices filled with zeros and ones respectively, DK is the

k by k^2 matrix

$$DK = \begin{pmatrix} \overbrace{k,\cdots,k,}^{k} & \overbrace{0,\cdots,0,}^{k} & \cdots & \overbrace{0,\cdots,0}^{k} \\ 0,\cdots,0, & k,\cdots,k, & \cdots & 0,\cdots,0 \\ \vdots & \vdots & & \vdots \\ 0,\cdots,0, & 0,\cdots,0, & \cdots & k,\cdots,k \end{pmatrix}$$

(# of cols = k, k, k)

and the $n-k^2$ rows of P are given by repeating $(n-k^2)/2$ times the following pair of rows.

$$\begin{pmatrix} \overbrace{1,\cdots,1,}^{\frac{k(k+1)}{2}} & \overbrace{0,\cdots,0,}^{\frac{k(k-1)}{2}} & \overbrace{1,\cdots,1,}^{n-k^2} & \overbrace{1,\cdots,1,}^{\frac{k(k-1)}{2}} & \overbrace{0,\cdots,0,}^{\frac{k(k+1)}{2}} & \overbrace{0,\cdots,0}^{(n-k^2)} \\ 0,\cdots,0, & 1,\cdots,1, & 0,\cdots,0, & 0,\cdots,0, & 1,\cdots,1, & 1,\cdots,1 \end{pmatrix}$$

The amalgamation (D_3, S_3):

We now turn to the third ESSE, $A_2 \to A_3$. The matrix A_3 will be the total 1-step row amalgamation of A_2. The matrix A_2 has $2k+2$ distinct rows and S_3 is the $(2k+2)\times 2n$ matrix whose rows are the distinct rows of A_2. The matrix D_3 is the amalgamation matrix such that $A_2 = D_3 S_3$ and $A_3 = S_3 D_3$. Explicitly we choose the ordering of the rows in S_3

so that S_3 has the following form:

$$\begin{pmatrix}
\overbrace{k,\cdots,k}^{k},\dots & \overbrace{0,\cdots,0}^{k}, & \overbrace{1,\cdots,1}^{(n-k^2)}, & \overbrace{0,\cdots,0}^{k},\dots & \overbrace{0,\cdots,0}^{k}, & \overbrace{0,\cdots,0}^{(n-k^2)} \\
\vdots & \vdots & \vdots & & \vdots & \\
0,\cdots,0,\dots & k,\cdots,k, & 1,\cdots,1, & 0,\cdots,0,\dots & 0,\cdots,0, & 0,\cdots,0 \\
0,\cdots,0,\dots & 0,\cdots,0, & 0,\cdots,0, & k,\cdots,k,\dots & 0,\cdots,0, & 1,\cdots,1 \\
\vdots & \vdots & \vdots & & \vdots & \\
0,\cdots,0,\dots & 0,\cdots,0, & 0,\cdots,0, & 0,\cdots,0,\dots & k,\cdots,k, & 1,\cdots,1 \\
1,\cdots,1,\dots & 0,\cdots,0, & 1,\cdots,1, & 1,\cdots,1,\dots & 0,\cdots,0, & 0,\cdots,0 \\
0,\cdots,0,\dots & 1,\cdots,1, & 0,\cdots,0, & 0,\cdots,0,\dots & 1,\cdots,1, & 1,\cdots,1
\end{pmatrix}$$

A_3 can be computed from S_3 as follows:

- for $1 \leq i \leq k$, the i-th column of A_3 is the sum of the $i+jk$ columns of S_3 for $0 \leq j \leq \frac{k+1}{2}-1$ and the $n+i+jk$ columns of S_3 for $0 \leq j \leq \frac{k-1}{2}-1$.
- For $1 \leq i \leq k$, the $(k+i)$-th column of A_3 is the sum of the $\frac{k(k+1)}{2}+i+jk$ columns of S_3 for $0 \leq j \leq \frac{k-1}{2}-1$ and the $n+\frac{k(k-1)}{2}+i+jk$ columns of S_3 for $0 \leq j \leq \frac{k+1}{2}-1$.
- The 2k+1 column of A_3 will be the sum of the k^2+1 to n columns of S_3.
- The 2k+2 column of A_3 will be the sum of the $n+k^2+1$ to $2n$ columns of S_3.

$$
\begin{array}{rcccc}
\text{\# of cols} = & k & k & 1 & 1
\end{array}
$$

$$
A_3 = \left(\begin{array}{cccc}
\overbrace{k,\cdots,k,} & \overbrace{0,\cdots,0,} & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
\vdots & \vdots & \vdots & \vdots \\
k,\cdots,k, & 0,\cdots,0, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
\vdots & \vdots & \vdots & \vdots \\
0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
k,\cdots,k, & 0,\cdots,0, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
\vdots & \vdots & \vdots & \vdots \\
k,\cdots,k, & 0,\cdots,0, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
\vdots & \vdots & \vdots & \vdots \\
0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
k,\cdots,k, & 0,\cdots,0, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\
0,\cdots,0, & k,\cdots,k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2}
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \frac{k+1}{2} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \frac{k-1}{2} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \frac{k-1}{2} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \frac{k+1}{2} \\
\\
\\
\end{array}
$$

The amalgamation (D_4, S_4):

As shown above, A_3 will have only 2 different row patterns,

$[k, ..., k, 0, ..., 0, \frac{n-k^2}{2}, \frac{n-k^2}{2}]$ and $[0, ..., 0, k, ..., k, \frac{n-k^2}{2}, \frac{n-k^2}{2}]$. A_4 is the total 1-step row amalgamation of A_3 So,

# of cols = k k .

$$S_4 = \begin{pmatrix} \overbrace{k, \cdots, k,} & \overbrace{0, \cdots, 0,} & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \\ \\ 0, \cdots, 0, & k, \cdots, k, & \frac{n-k^2}{2}, & \frac{n-k^2}{2} \end{pmatrix}$$

- The first column of A_4 will be the sum of columns 1 to $(k+1)/2$, $k+1$ to $k+(k-1)/2$, and 2k+1 column of S_3.

- The second column of A_4 is the sum of columns $(k+1)/2+1$ to k, $k+(k-1)/2+1$ to $2k$, and $2k+2$ column of S_3.

A_4 ends up being $\begin{bmatrix} \frac{n+k}{2} & \frac{n-k}{2} \\ \frac{n-k}{2} & \frac{n+k}{2} \end{bmatrix} = A$.

All that remains is to show $[1,1]D_1D_2D_3D_4 = [n,n]$ and $[1,-1]D_1D_2D_3D_4 = [1,-1]$. D_1 and D_2 will copy columns according to how the rows of A and A_1 are split. D_3 and D_4 will sum columns according to how the rows of A_2 and A_3 are amalgamated. Because the first n rows of A_2 are split from the first row of A and the second n rows of A_2 are split from the second row of A, $[1,1]D_1D_2 = [\underbrace{1, \cdots, 1}_{n \text{ cols}} \quad \underbrace{1, \cdots, 1}_{n \text{ cols}}]$.

$$[1,1]D_1D_2D_3 = [1,\cdots,1]D_3 = [\overbrace{k,\cdots,k,}^{k}\ \overbrace{k,\cdots,k,}^{k}\ n-k^2,\ n-k^2]$$

# of cols =

because there are k copies of the first $2k$ rows of S_3 in A_2 and $n-k^2$ copies of the each of the last two rows of S_3 in A_2. $[1,1]D_1D_2D_3D_4 =$

$$[\overbrace{k,\cdots,k,}^{(k+1)/2}\ \overbrace{k,\cdots,k,}^{(k-1)/2}\ \overbrace{k,\cdots,k,}^{(k-1)/2}\ \overbrace{k,\cdots,k,}^{(k=1)/2}\ n-k^2,\ n-k^2]D_4$$

$= [n,n]$ because

- the first to $(k+1)/2$, $k+1$ to $k+(k-1)/2$, and $2k+1$ rows of A_3 are the same as the first row of S_4, so D_4 will sum these columns and $k*(k+1)/2+k*(k-1)/2+n-k^2 = n$.

- the $(k+1)/2+1$ to k,$k+(k-1)/2+1$ to $2k$, and $2k+2$ rows of A_3 are the same as the second row of S_4, so D_4 will sum these columns and $k*(k+1)/2+k*(k-1)/2+n-k^2 = n$.

Because the first n rows of A_2 are split from the first row of A and the second n rows of A_2 are split from the second row of A,

$$[1,-1]D_1D_2 = [\underbrace{1,\cdots,1}_{n \text{ cols}}\ \underbrace{-1,\cdots,-1}_{n \text{ cols}}]$$

Let $(S_3)_i$ be the i-th row of the matrix S_3. The i-th coordinate of $[1,-1]*D_1*D_2*D_3 =$

$[1,...1,-1,...-1]*D_3$ is the difference between the number of the first n rows of A_2 that equal $(S_3)_i$ and the number of the second n rows that are equal to $(S_3)_i$.

- There are $(k+1)/2$ copies of $(S_3)_1$ in the first n rows of A_3 and $(k-1)/2$ copies of $(S_3)_1$ in the second n rows of A_3, which means that the first coordinate of $[1,-1]D_1D_2D_3$ is 1.

- The same argument applies to the first k coordinates of $[1,-1]*D_1*D_2*D_3$.

- For $k+1 \leq i \leq 2k$, there are $(k-1)/2$ copies of $(S_3)_i$ in the first n rows of A_3 and $(k+1)/2$ copies of $(S_3)_i$ in the second n rows of A_3, so the i-th coordinate of $[1,-1]*D_1*D_2*D_3$ is -1.

- For $i = 2k+1, 2k+2$, there are $(n-k^2)/2$ copies of $(S_3)_i$ in the first n rows of A_3 and $(n-k^2)/2$ copies of $(S_3)_i$ in the second n rows of A_3, so the i-th coordinate of $[1,-1]*D_1*D_2*D_3$ is 0.

This means that

$$[1,-1]*D_1*D_2*D_3 = [\underbrace{1,\cdots,1}_{k \text{ cols}} \quad \underbrace{-1,\cdots,-1}_{k \text{ cols}} \quad 0 \quad 0] .$$

In order to compute

$$\begin{array}{lcccc} [1,-1]D_1D_2D_3D_4 = & [\underbrace{1,\cdots,1,} & \underbrace{-1,\cdots,-1,} & 0 & 0]D_4 \\ \text{\# of cols=} & k & k & & \end{array}$$

note that $(k+1)/2$ of the first k rows and $(k-1)/2$ of the second k rows of A_3 are equal to the first row of S_4, and $(k-1)/2$ of the first k rows and $(k+1)/2$ of the second k rows of A_3 are equal to the second row of S_4. This means that

$[1,-1]*D_1*D_2*D_3*D_4=[1,-1]$. This completes Example 5.4.3.

While the preceding example is not general, it is my hope that this example will lead to some insight for more general constructions.

Appendix A

Computations for Theorem 5.2.3

Let us consider the case when n = 13 and k = 3, and $A = \begin{bmatrix} 8 & 5 \\ 5 & 8 \end{bmatrix}$. In the construction below, we produce γ, such that $(\alpha_\gamma, \beta_\gamma) = (13,1)$ and ρ composed with L_A maps γ to the $(0,1,0)$ element of $\mathbb{Z}/2 \times \mathbb{Z}^2 \cong Aut_+(\hat{A})$. γ will be given by a chain of ESSEs $A \xrightarrow{(D_1,S_1)} A_1 \xrightarrow{(D_2,S_2)} A_2 \xrightarrow{(D_3,S_3)} A_3 \xrightarrow{(D_4,S_4)} A$ where D_1 and D_2 are subdivision matrices for row splittings and D_3 and D_4 are amalgamation matrices for row amalgamations. Further, we will show that $[1,1]*D_1*D_2*D_3*D_4 = [13,13]$ and $[1,-1]*D_1*D_2*D_3*D_4 = [1,-1]$, which implies $(\alpha_\gamma, \beta_\gamma) = (13,1)$ and ρ_A composed with L_A maps γ to the $(0,1,0)$ element of $\mathbb{Z}/2 \times \mathbb{Z}^2 \cong Aut_+(\hat{A})$.

Below is the Matlab code and comments that compute γ and show γ has the proper attributes.

```
A = [8 , 5;

 5 , 8]

% A has eigenvalues of 13 and 3.

x = [1,1]

% x is the Perron eigenvector of A.

y = [1,-1]

% y is the eigenvector of 3.

S1 = [3,0;
```

```
  3,0;
  0,3;
  1,1;
  1,1;
  3,0;
  0,3;
  0,3;
  1,1;
  1,1]

D1 = [1,1,1,1,1,0,0,0,0,0;
  0,0,0,0,0,1,1,1,1,1]
D1*S1
ans =
 8 5
 5 8
% This shows that D1*S1=A and below we define A1=S1*D1
A1 = S1*D1
A1 =
 3 3 3 3 3 0 0 0 0 0
 3 3 3 3 3 0 0 0 0 0
 0 0 0 0 0 3 3 3 3 3
 1 1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1 1
```

```
 3 3 3 3 3 0 0 0 0 0

 0 0 0 0 0 3 3 3 3 3

 0 0 0 0 0 3 3 3 3 3

 1 1 1 1 1 1 1 1 1 1

 1 1 1 1 1 1 1 1 1 1

% We define the row splitting matrix of A1 by

S2 = [3,0,0,1,1,0,0,0,0,0;

  0,3,0,1,1,0,0,0,0,0;

  0,0,3,1,1,0,0,0,0,0;

  3,0,0,1,1,0,0,0,0,0;

  0,3,0,1,1,0,0,0,0,0;

  0,0,3,1,1,0,0,0,0,0;

  0,0,0,0,0,3,0,0,1,1;

  0,0,0,0,0,0,3,0,1,1;

  0,0,0,0,0,0,0,3,1,1;

  1,1,0,1,1,1,0,0,0,0;

  0,0,1,0,0,0,1,1,1,1;

  1,1,0,1,1,1,0,0,0,0;

  0,0,1,0,0,0,1,1,1,1;

  3,0,0,1,1,0,0,0,0,0;

  0,3,0,1,1,0,0,0,0,0;

  0,0,3,1,1,0,0,0,0,0;

  0,0,0,0,0,3,0,0,1,1;

  0,0,0,0,0,0,3,0,1,1;
```

```
  0,0,0,0,0,0,0,3,1,1;
  0,0,0,0,0,3,0,0,1,1;
  0,0,0,0,0,0,3,0,1,1;
  0,0,0,0,0,0,0,3,1,1;
  1,1,0,1,1,1,0,0,0,0;
  0,0,1,0,0,0,1,1,1,1;
  1,1,0,1,1,1,0,0,0,0;
  0,0,1,0,0,0,1,1,1,1]

D2 = [1,0,0,0,0,0,0,0,0,0;
  1,0,0,0,0,0,0,0,0,0;
  1,0,0,0,0,0,0,0,0,0;
  0,1,0,0,0,0,0,0,0,0;
  0,1,0,0,0,0,0,0,0,0;
  0,1,0,0,0,0,0,0,0,0;
  0,0,1,0,0,0,0,0,0,0;
  0,0,1,0,0,0,0,0,0,0;
  0,0,1,0,0,0,0,0,0,0;
  0,0,0,1,0,0,0,0,0,0;
  0,0,0,1,0,0,0,0,0,0;
  0,0,0,0,1,0,0,0,0,0;
  0,0,0,0,1,0,0,0,0,0;
  0,0,0,0,0,1,0,0,0,0;
  0,0,0,0,0,1,0,0,0,0;
```

```
  0,0,0,0,0,1,0,0,0,0;
  0,0,0,0,0,0,1,0,0,0;
  0,0,0,0,0,0,1,0,0,0;
  0,0,0,0,0,0,1,0,0,0;
  0,0,0,0,0,0,0,1,0,0;
  0,0,0,0,0,0,0,1,0,0;
  0,0,0,0,0,0,0,1,0,0;
  0,0,0,0,0,0,0,0,1,0;
  0,0,0,0,0,0,0,0,1,0;
  0,0,0,0,0,0,0,0,0,1;
  0,0,0,0,0,0,0,0,0,1]'
A1 =
 3 3 3 3 3 0 0 0 0 0
 3 3 3 3 3 0 0 0 0 0
 0 0 0 0 0 3 3 3 3 3
 1 1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1 1
 3 3 3 3 3 0 0 0 0 0
 0 0 0 0 0 3 3 3 3 3
 0 0 0 0 0 3 3 3 3 3
 1 1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1 1
D2*S2
ans =
```

```
 3 3 3 3 3 0 0 0 0 0
 3 3 3 3 3 0 0 0 0 0
 0 0 0 0 0 3 3 3 3 3
 1 1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1 1
 3 3 3 3 3 0 0 0 0 0
 0 0 0 0 0 3 3 3 3 3
 0 0 0 0 0 3 3 3 3 3
 1 1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1 1
% A1 =D2*S2 and we define A2=S2*D2
A2 = S2*D2
A2 =

3 3 3 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 3 3 3 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 3 3 3 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
3 3 3 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 3 3 3 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 3 3 3 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 1 1 1 1
1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
```

```
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
3 3 3 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 3 3 3 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 3 3 3 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 1 1 1 1
1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1

% We define S3 to be the distinct rows of A2 which create
A3 as the 1-step total column amalgamation of A2.
S3 = [ A2(1,:) ; A2(2,:) ; A2(3,:); A2(7,:); A2(8,:);
A2(9,:); A2(10,:) ; A2(11,:)]
S3 =

3 3 3 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
```

0 0 0 3 3 3 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 3 3 3 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 0 0 0 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 0 0 0 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 3 3 1 1 1 1

1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1

D3 =

1 0 0 0 0 0 0 0

0 1 0 0 0 0 0 0

0 0 1 0 0 0 0 0

1 0 0 0 0 0 0 0

0 1 0 0 0 0 0 0

0 0 1 0 0 0 0 0

0 0 0 1 0 0 0 0

0 0 0 0 1 0 0 0

0 0 0 0 0 1 0 0

0 0 0 0 0 0 1 0

0 0 0 0 0 0 0 1

0 0 0 0 0 0 1 0

0 0 0 0 0 0 0 1

1 0 0 0 0 0 0 0

0 1 0 0 0 0 0 0

```
 0 0 1 0 0 0 0 0

 0 0 0 1 0 0 0 0

 0 0 0 0 1 0 0 0

 0 0 0 0 0 1 0 0

 0 0 0 1 0 0 0 0

 0 0 0 0 1 0 0 0

 0 0 0 0 0 1 0 0

 0 0 0 0 0 0 1 0

 0 0 0 0 0 0 0 1

 0 0 0 0 0 0 1 0

 0 0 0 0 0 0 0 1

max(max(A2-D3*S3)

ans = 0

%This computation shows that $A2 = D3*S3

A3 = S3*D3

A3 =

 3 3 3 0 0 0 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2
```

```
%S4 is the set of distinct rows of A3 which will define A4 as the

1-step total column amalgamation of A3.

S4 = [ A3(1,:) ; A3(3,:) ]

S4 =

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

D4 =

 1 0

 1 0

 0 1

 1 0

 0 1

 0 1

 1 0

 0 1

A3 =

 3 3 3 0 0 0 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2
```

```
D4*S4

ans =

 3 3 3 0 0 0 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

 0 0 0 3 3 3 2 2

 3 3 3 0 0 0 2 2

 0 0 0 3 3 3 2 2

%We see that A3 = D4*S4 and below that A4=A.

A4 = S4*D4

A4 =

 8 5

 5 8

A =

 8 5

 5 8


x*D1*D2*D3*D4

ans =

    13    13

y*D1*D2*D3*D4

ans =
```

```
 1    -1

% This shows that (\alpha_{\gamma}, \beta_{\gamma}) = (13,1).
```

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