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Asymptotic Stabilization of Low Dimensional Systems

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Asymptotic Stabilization of Low Dimensional Systems

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Abstract

This paper studies the asymptotic stabilization of two and three dimensional nonlinear control systems. In the two dimensional case we review some of our recent work and in the three dimensional case we give some new sufficient conditions and necessary conditions.

1 Introduction

We consider the single input system,

$$\dot{x} = f(x) + g(x)u \tag{1.1}$$

where $x \in \Re^n$, u is a scalar input, and f, g are C^1 vector fields. It is assumed that $f(0) = 0, g(0) \neq 0$. The system is said to be C^k feedback stabilizable at the origin of $\Re^n$ if there exists a real valued C^k function $\alpha(x)$ defined on some small neighborhood of the origin in $\Re^n$ such that $\dot{x} = f(x) + g(x)\alpha(x)$ is locally asymptotically stable at 0.

There has been much work done in the recent past on this problem. Prominent among them are the techniques based on center manifold theory, pioneered by Ayels [Ay1] and used effectively by Kokotovic and coauthors among others, the idea of zero dynamics introduced by Byrnes and Isidori [BI1,BI2] etc., and the topological obstructions derived by Brockett [Br1], Krosnosel'skii and Zabreiko [Kr1], the work on continuous feedback stabilization by Sontag and Sussmann [SS1], Kawski [Ka1] etc.

An extremely important observation on asymptotic stabilization was made by R. Brockett [Br1]. For the moment let us consider (1.1) with arbitrary state space dimension n and arbitrary number of inputs m. Brockett proved that the following are necessary for stabilization of (1.1) with a C^1 feedback function.

(B1:) The uncontrollable eigenvalues of the linearized system should be in the closed left half of the complex plane.

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(B2:) (1.1) is locally asymptotically controllable to the origin i.e. For an arbitrary open neighborhood W of the origin there exist a neighborhood W of the origin and control $u(\cdot)$ such that for all $x^o \epsilon W$ the solution $t \mapsto x(t, x^o, u(t))$ of (1.1) stays in U for all $t > o$ and converges to the origin as $t \mapsto \infty$,

(B3:) The function $(x, u) \mapsto \tilde{f}(x) + \tilde{g}(x)u : \Re^n \times \Re^m \to \Re^n$ is locally onto at (0,0).

The key condition here is (B3), which shows that very interesting pathologies are possible. This condition follows from a theorem due to M. A. Krosnosel'skii and P. P. Zabreiko [Kr1], which states that the index of a continous vector field in $\Re^n$ at a locally asymptotically stable equilibrium point is equal to $(-1)^n$. The focus of much of the research work on low dimensional cases has been on finding further necessary conditions and on finding rather strong sufficient conditions.

In section 2. of this paper we wil review our recent work on the two dimensional stabilization problem for real anlytic systems. In particular it will follow that (B3) is necessary and sufficient for C^0 stabilization. We will give some sufficient conditions for C^1 stabilizability and C^∞ stabilizability. In section three we will derive some necessary conditions and some sufficient conditions for the asymptotic stabilizability of homogeneous polynomial systems i.e. $f(x)$ is a homogeneous polynomial vector field and $g(x)$ is a constant vector.

2 Stabilization of two dimensional systems

In this section we will review some of our recent work on the stabilization problem for two dimensional systems. Throughout we will assume that the system is real analytic.

Since $g(0) \neq 0$ in (1.1) we may assume without any loss of generality that the system has the form,

$$\dot{x} = f(x_1, x_2) \quad (2.2)$$

$$\dot{x}_2 = u, \quad (2.3)$$

where $f(0) = 0; x_1, x_2 \in \Re, u \in \Re$ and f is real analytic.

The following theorem was proved in [DMK].

Theorem 2.1 *Consider the system (2.1). The following conditions are equivalent.*

(i) The system (hence (1.1)) is locally asymptotically stabilizable by C^0 feedback.

(ii) The Brockett condition (B3) is satisfied.

(iii) For all $\epsilon > 0$ there exist $p \epsilon B_\epsilon(0) \cap \Re^2_+$ and $q \epsilon B_\epsilon(0) \cap \Re^2_-$ such that $f(p) < 0$ and $f(q) > 0$. (Here $\Re^2_+ = \{(x_1, x_2) | x_1 > 0\}$ and $\Re^2_- =$

$\{(x_1, x_2) | x_1 < 0\}$ and $B_\epsilon(0)$ denotes the Euclidean ball of radius ϵ around the origin.

Remark 2.1 : *The stabilizing feedback can be found to be Holder continous.*

Remark 2.2 : *Prior to our work M. Kawski has shown that (see [Ka1]) that small time local controllability is a sufficient condition for C^0 stabilization. Theorem 2.1 strengthens this result.*

The C^1 and C^∞ feedback stabilizability are much more subtle even in the two dimensional case. We derived some sufficient conditions in [DMK]. We first define two indices.

Since multiplication of f by a strictly positive function and coordinate changes do not affect stabilizability of (2.1), we may assume without any loss of generality that f is a Weierstrass polynomial, $x_1^m + a_1(x_2)x_1^{m-1} + \ldots + a_m(x_2)$ and $a_i(0) = 0$, $1 \le i \le m$. It is well known that the zero set of a Weierstrass polynomial can be written locally as the finite union of graphs of convergent rational power series $x_2 = \phi(x_1)$ where $x_1 \in [0, \epsilon)$ or $x_1 \in (-\epsilon, 0]$. Let us denote the positive rationals by Q_+ and define,

$$
\begin{aligned}
A^+ &= \{\gamma \in Q_+ \mid f(x_1, \phi(x_1)) < 0 \text{ for all } x_1 \in (0, \epsilon), \text{for some } \epsilon > 0. \\
&\quad \text{and for some convergent rational power series } \phi(x_1) \text{ with leading} \\
&\quad \text{exponent equal to } \frac{1}{\gamma}\} \\
A^- &= \{\gamma \in Q_+ \mid f(-x_1, \phi(x_1) > 0 \text{ for all } x_1 \in (0, \epsilon), \text{for some} \epsilon > 0 \\
&\quad \text{and for some convergent rational power series } \phi(x_1) \text{ with leading} \\
&\quad \text{exponent equal to } \frac{1}{\gamma}\}.
\end{aligned}
$$

Definition 2.1 *The index of stabilizability of f is* $\max\{\inf_{\gamma \in A^+}\{\gamma\}, \inf_{\gamma \in A^-}\{\gamma\}\}$.

Definition 2.2 *The fundamental stabilizability degree of f is the order of the zero of $a_m(x_2)$ at $x_2 = 0$. The secondary stabilizability degree of f is the order of the zero of $a_{m-1}(x_2)$ at $x_2 = 0$.*

Notation:

$$
\begin{aligned}
I &:= \text{Index of stabilizability of } f \\
s_1 &:= \text{Fundamental stabilizability degree of } f \\
s_2 &:= \text{Secondary stabilizability degree of } f.
\end{aligned}
$$

Theorem 2.2 *The system (2.2) and hence (1.1)) is C^1-stabilizable if $s_1 > 2I - 1$*

If $s_1 \leq 1 + 2s_2$ and s_1 is odd, then (2.1) is C^ω stabilizable.

If $s_1 < 1 + 2s_2$, then (2.1) is not C^∞ stabilizable.

3 Stabilization of homogeneous systems

In this section we consider a single input homogeneous system,

$$\dot{x} = f(x) + bu \tag{3.4}$$

where $x \in \Re^n$, $u \in \Re$, b is a real vector and f is a homogeneous polynomial vector field of some degree p i.e. $f(\lambda x) = \lambda^p f(x)$ for all $x\Re^n$ and $\lambda > 0$.. For the most part we will be seeking to find a feedback function $u = \alpha(x)$ which is homogeneous of degree p along rays from the origin i.e. $\alpha(\lambda x) = \lambda^p \alpha(x)$. For the sake of clarity henceforth we will use the term, *positively homogeneous* , to describe such functions. We remark that for this class of feedback the local and global stabilization are equivalent. Unless specified otherwise we will assume that f is C^1.

The following theortem is due to Andreini, Bacciotti and Stefani [ABS].
Theorem 3.1 *Consider the system,*

$$\begin{aligned} \dot{x_1} &= F(x_1, x_2) \\ \dot{x_2} &= u \end{aligned} \tag{3.5}$$

where $(x_1, x_2) \in \Re^p \times \Re^m, u \in \Re^m, F$ is homogeneous of some odd degree p. The system is asymptotically stabilizable by homogeneous feedback of degree p if $\dot{x_1} = F(x_1, 0)$ is asymptotically stable.

The following example captures the spirit of this theorem.

Example 3.1 *Consider the system,*

$$\begin{aligned} \dot{x_1} &= {x_2}^p \\ \dot{x_2} &= {x_3}^p \\ &\vdots \\ \dot{x_n} &= u \end{aligned} \tag{3.6}$$

where p is an odd integer. We show that this system is asymptotically stabilizable . This is done by using an induction argument.

When $n = 1$, $u = -x_1^p$ is a stabilizing feedback law and $V(x) = \frac{1}{2}x_1^2$ is a Lyapounov function.

Suppose that for some $n \geq 1$ (3.6) admits a stabilizing feedback function $u(x) = -(l(x_1, \ldots, x_n)^p$, where l is a linear function, and admits a quadratic Lyapounov function $V(x) = \frac{1}{2}x^T Q x$. Let us consider the $n + 1$

dimensional case. First let us change coordinates as, $y_i = x_i; i = 1, \ldots, n$ and $y_{n+1} = x_{n+1} + l(x_1, \ldots, x_n)$. By applying the Holder's inequality and by using the Lyapounov function $V(y_1, \ldots, y_n) + \frac{1}{2}y_{n+1}^2$ it is easily seen that for large enough $K, u = -K(y_{n+1}^p)$ is a stabilizing feedback function. This concludes the asymptotic stabilizability of (3.6).

For the rest of the section we will focus on the stabilization problem for three dimensional homogeneous systems. Necessary and sufficient conditions for the asymptotic stability of three dimensional homogeneous systems were derived by Coleman in [Co] (see [Ha1] also). Let us consider the system

$$\dot{x} = F(x) \tag{3.7}$$

where $x \in \Re^n$ and F is a positively homogeneous vector field (not necessarily polynomial) of degree p. One can derive an associated system on the $n-1$ dimensional sphere S^{n-1} by first writing an equation for $\frac{d}{dt}\left(\frac{x}{||x||}\right)$ as,

$$\frac{d}{dt}\left(\frac{x}{||x||}\right) = \frac{1}{||x||}F(x) - \frac{x^T F(x)}{||x||^3}x \tag{3.8}$$

and then changing the time scale, in an $||x||$ dependent way so that the equation depends only on $\frac{x}{||x||}$. Thus we obtain,

$$\frac{d}{dt}\left(\frac{x}{||x||}\right) = \frac{1}{||x||^p}F(x) - \frac{x^T F(x)}{||x||^{p+2}}x \tag{3.9}$$

Coleman's theorem states the following.

Theorem 3.2 *([Co]): Let A denote the union of all equilibrium points and periodic orbits of (3.8) on S^{n-1}. Let C denote the cone generated by C. Then the system (3.7) is asymptotically stable if and only if it is asymptotically stable when restricted to C.*

This can be used to generalize the theorem of Andreini, Bacciotti and Stefani [ABS] as follows in the three dimensional case. This theorem was proven independently by M. Kawski (see [Ka2]) also.

Theorem 3.3 *Consider the positively homogeneous control system*

$$\begin{aligned} \dot{y} &= h(y, z) \\ \dot{z} &= u \end{aligned} \tag{3.10}$$

where $y \in \Re^2$, $z \in \Re$, $u \in \Re$ and h is positively homogeneous of degree p i.e. $h(\alpha y, \alpha z) = \alpha^p h(y, z)$ for all $\alpha \in \Re$.

Suppose that there exist a Lipschitz continuous function $z = \phi(y) : \Re^2 \to \Re$ which is a positively homogeneous of degree 1 such that the system

$$\dot{y} = h(y, \phi(y)),$$

is asymptotically stable. Then there exists a Lipschitz continuous feedback function, $u = \alpha(y, z)$, which is homogeneous of degree p, such that the system,

$$\begin{aligned} \dot{y} &= h(y, z) \\ \dot{z} &= \alpha(y, z) \end{aligned} \tag{3.11}$$

is asymptotically stable.

Proof: After a small perturbation of ϕ, we may assume that the function $\psi = \phi|_{S^1} : S^1 \to \Re$ is C^∞. (Here S^1 — denotes the standard unit circle in $\Re^2$). Now let M denote the intersection of the positive cone $\hat{C} \overset{\text{def}}{=} \{(y, z) \mid z = \phi(y),\ y \in \Re^2\}$ and S^2. Let $\sigma : S^2 \to S^2$ be a smooth diffeomorphism which preserves poles and moves points longitudinally such that $\sigma_0 \psi(S^1)$ is the equator of S^2.

Now let,

$$\dot{\theta} = a(\theta) + b(\theta) u \tag{3.12}$$

be the associated system on S^2, obtained by (3.10), as described in the introduction. Let q_n and q_s denote the north and the south poles of S^2 and let D be a band around the equator bounded by two latitudes and such that the inverse image of D under σ contains the equator. Now first transform (3.11) by σ to obtain,

$$\begin{aligned} \dot{\beta} &= \left(\sigma_* a \sigma^{-1}\right)(\beta) + \left(\sigma_* b \sigma^{-1}\right)(\beta) u \\ &= c(\beta) + d(\beta) u. \end{aligned} \tag{3.13}$$

Now find a smooth function $\gamma : S^2 \to \Re$ such that it has the following properties.

(p_1) $\gamma < 0$ above D and $\gamma > 0$ below D

(p_2) For all $\beta \in D$, the positive limit set $\omega(\beta)$ of the solution of

$$\dot{\beta} = c(\beta) + d(\beta)\gamma(\beta)$$

is contained in the equator. (In particular the equator is positively invariant).

Now consider the feedback function,

$$u = \alpha(y, z) = \|(y, z)\|^p \gamma\left(\sigma\left(\frac{(y, z)}{\|(y, z)\|}\right)\right).$$

Then it follows at once that $\hat{C}$ is an invariant cone of

$$\begin{aligned} \dot{y} &= h(y,z) \\ \dot{z} &= \alpha(y,z) \end{aligned} \tag{3.14}$$

and that the system is asymptotically stable on $\hat{C}$. Moreover all other invariant one or two dimensional cones meet S^2 outside of $\sigma^{-1} \circ D$. Since $z\alpha(y,z) < 0$ outside of the cone generated by $\sigma^{-1} \circ D$ it follows that the system is asymptotically stable on all such invariant cones. Hence by Coleman's theorem the asymptotic stability of (3.14) follows.

Q.E.D.

In view of this lemma, one can use known results on the stability of two dimensional homogeneous systems in order to derive sufficient conditions for asymptotic stabilization of three dimensional systems. The following theorem is of interest to us.

Theorem 3.4 *([Ha1]): Consider the two dimensional system,*

$$[\dot{x}_1, \dot{x}_2]^T = [f_1(x),\ f_2(x)]^T \tag{3.15}$$

where $f = [f_1, f_2]^T$ is Lipschitz continuous and is positively homogeneous of degree p. The system is asymptotically stable if and only if one of the following is satisfied:

(i) The system does not have any one dimensional invariant subspaces and

$$\int_0^{2\pi} \frac{\cos\theta f_1(\cos\theta,\sin\theta) + \sin\theta f_2(\cos\theta,\sin\theta)}{\cos\theta f_2(\cos\theta,\sin\theta) - \sin\theta f_1(\cos\theta,\sin\theta)} d\theta < 0$$

or

(ii) The restriction of the system to each of its one dimensional invariant subspaces is asymptotically stable.

As an application of theorems 3.2 and 3.3, let us consider the problem of stabilization of the angular velocity of a rigid body when only one of the control torques is available. This system has the structure,

$$\begin{aligned} \dot{x}_1 &= a_1 x_1 x_2 + b_1 u \\ \dot{x}_2 &= a_2 x_1 x_3 + b_2 u \\ \dot{x}_3 &= a_3 x_1 x_2 + b_3 u. \end{aligned}$$

D. Ayels and M. Szafranski have shown in [AS] that this system is locally asymptotically stabilizable when no two principal moment of inertia are equal. The case when two of the principal moment of inertia are equal (equivalently $a_1 = -a_2$) was the topic of study of the recent paper [SS2] by E. Sontag and H. J. Sussmann . They have shown that if none of the b_i's are equal to zero, then indeed the system is globally stabilizable by smooth feedback. Below we show that the system is globally stabilizable by Lipschitz continous, positively homogeneous feedback.

It is easily seen that (see [SS2]) the problem can de reduced to the stabilization of,

$$\begin{aligned} \dot{x_1} &= x_2x_3 \\ \dot{x_2} &= -x_3x_1 - bx_3^2 \\ \dot{x_3} &= u \end{aligned}$$

where b is a nonzero constant. By theorem 3.2, if we can show that there is a Lipschitz continous function $x_3 = \phi(x_1, x_3)$ which is positively homogeneous of degree 1, which stabilizes,

$$\begin{aligned} \dot{x_1} &= x_2x_3 \\ \dot{x_2} &= -x_1x_3 - bx_3^2 \end{aligned} \tag{3.16}$$

then the desired conclusion follows.

Without any loss of generality we assume that $b > 0$. Since the stability is preserved under multiplication of the vector field by strictly positive functions we will first consider,

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 - bx_3 \end{aligned} \tag{3.17}$$

and seek to find a strictly positive stabilizing Lipschitz continous feedback function $x_3 = x_3(x_1, x_2)$ which is positively homogeneous of degree one . Since asymptotic stability of a positively homogeneous system is robust under small purtubations by functions of the same degree of homogeneity, we can relax the requirement of strictly positiveness to positiveness. It is seen at once by using the Lyapounov function $x_1^2 + x_2^2$ that,

$$x_3 = \begin{cases} 0; x_2 < 0 \\ x_2; x_2 \geq 0 \end{cases}$$

satisfies the requirements. This concludes the proof that the system is asymptotically stabilizable by globally Lipschitz continous feedback which is positively homogeneous of degree one.

Theorems 3.2 and 3.3 can be used to generate further sufficient conditions for the asymptotic stabilizability of positively homogeneous systems.

Let us consider the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, x_3)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ \dot{x}_3 &= u\end{aligned} \tag{3.18}$$

where $(x_1, x_2, x_3) \mapsto (f_1, f_2)(x_1, x_2, x_3) : \Re^3 \rightarrow \Re^2$ is a positively homogeneous function of some degree p.

Theorem 3.5 *:Suppose that there exists a smooth function $\varphi : S^1 \rightarrow \Re$ such that at least at one $\theta_0 \in S^1$, the vector $(f_1, f_2)^T(\cos\theta_0, \sin\theta_0, \varphi(\theta_0))$ points radially inwards and at no points $\theta_0 \in S^1$, the vector field $(f_1, f_2)^T(\cos\theta, \sin\theta, \varphi(\theta))$ points radially outwards. Then the system is asymptotically stabilizable.*

Proof: By (ii) of theorem (3.3) the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_1\\ f_2\end{bmatrix}(x_1, x_2)||(x_1, x_2)||\phi\left(||(x_1, x_2)||(x_1, x_2)\right)$$

is asymptotically stable. Now the theorem follows from theorems 3.2 and 3.3. **Q.E.D.** The sufficient condition given in theorem 3.4 can be tested

quite easily by using the locus of zeros of a certain function. Note that the crucial properties in the theorem are satisfied by the roots of the equation

$$f_1(x_1, x_2, x_3) - x_3 f_2(x_1, x_2, x_3) = 0. \tag{3.19}$$

Using homogeneity we rewrite (3.19) as

$$\text{since } f_1(\cos\theta, \sin\theta, x_3) - x_3 f_2(\cos\theta, \sin\theta, x_3) = 0. \tag{3.20}$$

One can now draw the locus of the zeros of (3.20) against $\theta \in [0, 2\pi]$ in a graph and decide at once the existence or nonexistence of a function φ as desired.

Our next sufficient condition is applicable to to homogeneous polynomial systems of odd degree and relates to (i) of theorem (3.3).

Now we consider the generic case and rewrite (ii) in the form,

$$\begin{aligned}\dot{x}_1 &= x_2^p + g_1(x_1, x_2, x_3)\\ \dot{x}_2 &= -x_3^p + g_2(x_1, c_2, x_3)\\ \dot{x}_3 &= u\end{aligned} \tag{3.21}$$

where g_1 and g_2 are homogeneous polynomials of odd degree p; g_1 does not contain x_2^p terms and g_1 and g_2 do not contain x_3^p terms. A generic system can be written in this form after a suitable linear change of coordinates.

Theorem 3.6 *Suppose that the function*

$$\eta : x_3 \mapsto 1 + g_1(0, 1, x_3) : \Re \rightarrow \Re$$

takes either strictly positive values or strictly negative values. Then (3.21) is asymptotically stabilizable.

Proof: Let

$$f_1(x_1, x_2, x_3) = x_2^p + g_1(x_1, x_2, x_3)$$

and

$$f_2(x_1, x_2, x_3) = -x_3^p + g_2(x_1, x_2, x_3).$$

The objective here is to construct a "base" which is positively invariant and use it to establish the asymptotic stability. We will first consider the case when $Rng(\eta) \subset (0, \infty)$. Then the leading term of the polynomial $f_1(0, 1, x_3)$ is of even power. Now it follows at once that there exists a neighborhood $\mathcal{U} = [\pi/2 - \epsilon,\ \pi/2 + \epsilon]$ of $\pi/2$ such that,

$$f_1(\cos\theta, \sin\theta, x_3) > 0$$

for all $x_3 \in \Re$, and all $\theta \in \mathcal{U}$.

Similarly,

$$f_1(\cos\theta, \sin\theta, x_3) < 0 \text{ for all } \theta \in U + \{\pi\} \text{ and all } x_3 \in \Re.$$

Let

$$\lambda = \max\left\{ \frac{f_2(\cos\theta, \sin\theta, x_3)}{f_1(\cos\theta, \sin\theta, x_3)} \middle| \theta \in [\pi/2 - \epsilon, \pi/2],\ x_3 \in [0, \infty) \right\}$$

and

$$\mu = \max\left\{ \frac{-f_2(\cos\theta, \sin\theta, 0)}{f_1(\cos\theta, \sin\theta, 0)} \middle| \theta \in \left[\frac{3\pi}{2}, \frac{3\pi}{2} - \epsilon\right] \right\}.$$

Existence of μ is clear. Existence of λ follows since

$$\frac{f_2(\cos\theta, \sin\theta, x_3)}{f_1(\cos\theta, \sin\theta, x_3)} \text{ goes to } -\infty \text{ uniformly in } \theta \in \left[\frac{\pi}{2} - \epsilon, \frac{\pi}{2}\right]$$

as x_3 goes to infinity.

Now define the angle $\theta_0 \in (\pi/2 - \epsilon, \pi/2]$ via the following construction.

Let us define θ_0 by,

$$\theta_0 = \max\left\{\frac{\pi}{2} - \epsilon,\ \tan^{-1}\left(\frac{2\cos\epsilon + (m + 2\lambda)\sin\epsilon}{\sin\epsilon}\right)\right\}.$$

This choice of θ_0 can be explained via figure 1 rather easily.

Let us start with an arbitrary $\delta > 0$ and draw a line of slope λ through $(0, -\delta)$ until it meets the line with polar coordinate equal to $3\pi/2 + \epsilon$ at A. Now draw a line vertically upwards until it meets the line of slope m through $(0, 2\delta)$ at B. The polar coordinate of this point of intersection is equal to $\tan^{-1}\left(\frac{2\cos\epsilon + (m + 2\lambda)\sin\epsilon}{\sin\epsilon}\right)$. Of course one may need to decrease ϵ if necessary in order that the required intersection occur.

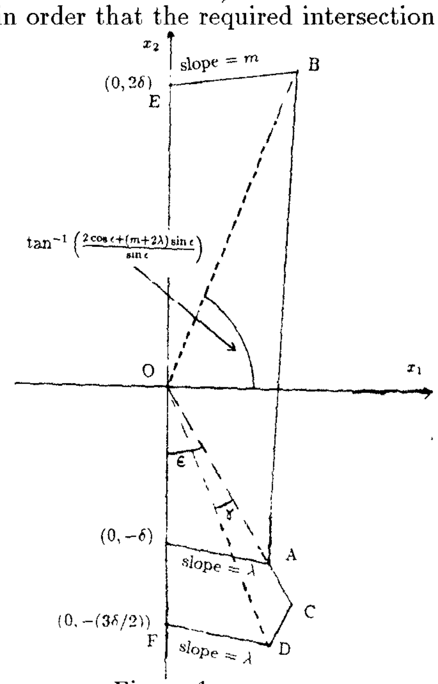

Figure 1

Now lets define a line segment ℓ_1 and an angle $\gamma \in (0, \epsilon)$ in the following way. Start from B and draw ℓ to be of very large negative slope until it hits the line $\theta = 3\pi/2 + \epsilon$ at C. Now draw a vertical line downwards until it hits the line $\theta = 3\pi/2 + \epsilon - \gamma$ at D. The choice of the slope of ℓ and γ is made such that the line of slope m through D meets the negative x_2-axis at $\left(0, \frac{-3\delta}{2}\right)$.

Let $E = (0, 2\delta)$ and $F = (0, -3\delta/2)$.

We will now define a Lipschitz continuous function $x_3 = \phi(x_1, x_2)$ which is homogeneous of degree 1 such that the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \varphi(x_1, x_2)) \\ \dot{x}_2 &= f_2(x_1, x_2, \varphi(x_1, x_2)) \end{aligned} \tag{3.22}$$

is asymptotically stable. Let us first consider the line ℓ. We fix φ to be a large positive constant L on ℓ such that $\left|\frac{f_2(x_1, x_2, \varphi(x_1, x_2))}{f_1(x_1, x_2, \varphi(x_1, x_2))}\right|$ is always greater than the magnitude of the slope of ℓ. This is obviously possible from the hypothesis on g_1 and g_2. Vary φ smoothly from L at C to zero at D along CD. Set $\varphi \equiv 0$ on FD. Increase φ from 0 at E to L smoothly along EB. Now use homogeneity to define φ on $\Re^2$. It is clear that one can construct a Lipschitz continuous function φ this way.

Now let us consider (3.22). It is clear that there aren't any one dimensional invariant unstable subspaces, for by our construction the vector field $[f_1, f_2]^T$ points into the region $EBCDF$ along the portion of the boundary which does not lie on the x_2-axis. Suppose that there aren't any one dimensional invariant stable subspaces either. Then the solution with initial condition $(0, 2\delta)$ enters into $EBCDF$ and cannot leave it on $EB \cup BC \cup CD \cup DF$ and hence has to cross OF. But by homogeneity this now implies asymptotic stability. Now by theorem 3.2 the stabilizability of (3.21) follows.

In the case when $Rng(\eta) \subset (-\infty, 0)$, one can do essentially the same construction in the left half plane instead of the right half plane as above.
Q.E.D.

Now we discuss some topological aspects of the stabilization problem for the homogeneous three dimensional systems(3.10). We focus on finding some stronger requirement of the Krosnosel'skii - Zabreiko theorem which cannot be captured by (B3).

For the sake of simplicity we will assume that $h(x)$ only has isolated zeroes on the unit sphere S^2. Let $u = \alpha(x)$ be a (not necessarily homogeneous) continous feedback function. Let $\phi(x) = [(h(x)^T, \alpha(x)]^T$. Let S^2_ϵ denotes a small enough ball in $\Re^3$ such that the origin is the only zero of ϕ on and inside S^2_ϵ. Let $Z = \{p \epsilon S^2_\epsilon | h(p) = 0\}$. Let deg(h,p,w) denotes the Brower degree of h with respect to $p \in S^2_\epsilon$ and $w \in \Re^2$.

Theorem 3.7 : *A necessary condition for the asymptotic stabilizability of 3.10 is that there exist $W \subset Z$ such that $\sigma_{p \in W} deg(h,p,0) = -1$.*

Proof: Let $\psi = \phi / \| \phi \|: S^2_\epsilon \to S^2_1$ and let $\deg(\psi, p, q)$ denotes the Brower degree of ψ with respect to $p \in S^2_\epsilon$ and $q \in S^2_1$. Then it is easily seen that $\deg(\psi, p, \psi(p)) = \text{sgn } \alpha(p) \deg(h, p, 0)$ for all $p \in Z$. Since a necessary condition for stabilizability is that index $\phi = \sum_{p \in S^2_\epsilon} \deg\ (\psi, p, [0,0,1]^T) = -1$, the conclusion follows. **Q.E.D.**

Some other necessary conditions which are similar in spirit appear in [Ka2] and [Cor].

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