ABSTRACT

Title of Dissertation: THE UNCERTAINTY PRINCIPLE IN HARMONIC
                            ANALYSIS AND BOURGAIN'S THEOREM.

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We investigate the uncertainty principle in harmonic analysis and how it constrains the uniform localization properties of orthonormal bases. Our main result generalizes a theorem of Bourgain to construct orthonormal bases which are uniformly well-localized in time and frequency with respect to certain generalized variances. In a related result, we calculate generalized variances of orthonormalized Gabor systems. We also answer some interesting cases of a question of H. S. Shapiro on the distribution of time and frequency means and variances for orthonormal bases.

THE UNCERTAINTY PRINCIPLE IN HARMONIC

ANALYSIS AND BOURGAIN'S THEOREM.

by

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DEDICATION

To My Parents

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Chapter 1

Introduction

Given a function $f \in L^2(\mathbb{R})$, the Fourier transform of f, denoted $\widehat{f}$, is formally defined by

$$\widehat{f}(\gamma) = \int f(t)e^{-2\pi it\gamma}dt,$$

where the integral is over $\mathbb{R}$. If one views f as a signal which is a function of time, then $\widehat{f}$ describes how f is built up from different frequency components.

The uncertainty principle (UP) in harmonic analysis is a class of theorems which state that a nontrivial function, f, and its Fourier transform, $\widehat{f}$, can not both be simultaneously too well localized. One, of course, needs to be precise about what "localization" means. Roughly speaking, a function is well localized if it decays to zero quickly at $\pm\infty$, or if it is highly concentrated on a compact set. We shall state some concrete definitions of localization later. Likewise, once a measure of localization is specified, one needs to be precise about what it means to be "too well" localized. Again, there is a great deal of flexibility, and theorems range from giving highly technical and quantitative statements to more general and qualitive interpretations.

Heisenberg's uncertainty principle in quantum mechanics is the prototype of all uncertainty principles. His uncertainty principle deals with the inability

to precisely determine both position and momentum of a particle. While our focus and motivation here will be purely mathematical, Heiseberg's uncertainty principle will nonetheless play an important role for us. Later on, we shall discuss a relevant mathematical formulation of it as an $L^2(\mathbb{R})$ norm inequality. For some historical background on the uncertainty principle and for more information on its physical meaning, [20] and [16] both give a nice mathematical overview. The masterpiece [27] is perhaps the most comprehensive mathematical text on the subject, whereas [20] is possibly the most complete survey article on the topic.

Let us begin with an elementary, non-technical example of the uncertainty principle. Given $f \in L^2(\mathbb{R})$ and $\lambda > 0$, define the dilation, $f_\lambda(t)$, by $f_\lambda(t) = \lambda f(\lambda t)$. For fixed f, it is visually clear that f_λ becomes increasingly more concentrated about the origin as $\lambda \to \infty$. However, since $\widehat{f_\lambda}(\gamma) = \left(\frac{1}{\lambda}\right) \widehat{f}\left(\frac{\gamma}{\lambda}\right)$, one sees that $\widehat{f_\lambda}$ becomes increasingly more spread out as $\lambda \to \infty$. This example shows that when one dilates a function to make it more localized, the Fourier transform becomes less localized. While simple, this illustrates the uncertainty principle's main theme, namely the incompatibility of having both f and $\widehat{f}$ too sharply localized.

The limiting case (as $\lambda \to \infty$) in the above example gives rise to the Dirac delta measure, δ. Since the distributional support of δ is $\{0\}$, δ is about as well localized as possible. On the other hand, the distributional Fourier transform of δ is the constant function $\widehat{\delta} \equiv 1$, which is indeed very poorly localized. For this reason, the statement $\widehat{\delta} = 1$ may be viewed as a manifestation of the uncertainty principle, [5].

1.1 Qualitative uncertainty principles

Given a function f, define the support of f to be the closure of the set $\{t \in \mathbb{R} : f(t) \neq 0\}$, namely,

$$\text{supp}(f) = \overline{\{t \in \mathbb{R} : f(t) \neq 0\}}.$$

In view of our intuitive definition of localization, the notion of support gives a natural way to measure if a function is well localized. In particular, if a function has compact support then it fits our intuitive requirements for being well localized.

Using support as our measure of localization, we observe the following elementary uncertainty principle. Suppose $f \in L^2(\mathbb{R})$, and that f and $\widehat{f}$ both have compact support. Then the Paley-Wiener theorem, [33], states that f is the restriction to $\mathbb{R}$ of an entire function. Since an entire function can not vanish on any interval, it follows that no nontrivial $f \in L^2(\mathbb{R})$ can have both $\text{supp}(f)$ and $\text{supp}(\widehat{f})$ compact.

Benedicks, [10], gave the following extension of this result.

Theorem 1.1 (Benedicks). *If $f \in L^2(\mathbb{R})$ and the sets $\text{supp}(f)$ and $\text{supp}(\widehat{f})$ both have finite Lebesgue measure, then $f \equiv 0$.*

While this is an appealing result which illustrates the uncertainty principle nicely, its hypotheses are very strong and it does not give very precise insight into matters. A more advanced result along these lines is given by Hardy, [25]. Although the result dates back to 1933, there has been a recent upsurge of interest in it, see [24], [29].

Theorem 1.2 (Hardy). *Let $f \in L^2(\mathbb{R})$ and suppose that*

$$|f(t)| \leq Ce^{-\pi a t^2} \quad and \quad |\widehat{f}(\gamma)| \leq Ce^{-\pi b \gamma^2},$$

for some constant C and constants $a, b > 0$. Then

$$ab > 1 \implies f(t) \equiv 0$$

and

$$ab = 1 \implies f(t) = ce^{-a\pi t^2},$$

where $c \in \mathbb{C}$ is a constant.

Hardy's proof relies critically on complex analysis and the Phragmen-Lindelöf theorem. Motivated by Hardy's theorem, Ingham, [31], proved the following version for functions with compact support.

Theorem 1.3 (Ingham). *Let $\nu(t)$ be a positive function which monotonically approaches zero as $t \to \infty$. Suppose $f \in L^2(\mathbb{R})$ and that f is zero outside of the interval $[-l, l]$. Such an f can satisfy*

$$f(t) = O(e^{-|t|\nu(|t|)}), \quad |t| \to \infty$$

if and only if

$$\int_1^\infty \frac{\nu(t)}{t} dt$$

is convergent.

One direction of the proof of Ingham's theorem depends on the theory of quasi-analytic functions and the Carleman-Denjoy theorem, [30]. The other direction employs standard constructive methods.

In the case where $a = b$, Hardy's theorem deals with the symmetric weights $e^{-a\pi t^2}$ and $e^{-a\pi \gamma^2}$. Ingham's result replaces the $e^{-a\pi t^2}$ decay condition on f by the most extreme decay possible, namely that f is compactly supported, and replaces the other decay condition by one weaker than the original. Morgan considers the

problem for combinations of weights which lie "in between" those considered by Hardy and Ingham. While Morgan actually proved several theorems in this direction,[38], the following gives a typical flavor of his results.

Theorem 1.4 (Morgan). *Suppose $\epsilon > 0$, $f \in L^2(\mathbb{R})$ and that $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 2$. If*

$$f(t) = O(e^{-|t|^p}), \quad |t| \to \infty$$

and

$$\widehat{f}(\gamma) = O(e^{-|\gamma|^{q+\epsilon}}), \quad |\gamma| \to \infty,$$

then $f \equiv 0$.

Morgan's result makes use of the Phragmen-Lindelhöf theorem and saddle point methods.

The trio of theorems due to Hardy, Ingham and Morgan, respectively, shows how the uncertainty principle can be meaningfully refined by using different pairings of weights to measure localization. It is worth pointing out that although the above results are similar in appearance, they have different methods of proof. Understanding the role which different combinations of weights play in the uncertainty principle will be an important theme for us. The weights t^2 and γ^2 are especially important, and make their first appearance in the Heisenberg uncertainty principle.

1.2 The Heisenberg uncertainty principle

The Heisenberg uncertainty principle alluded to earlier may be stated as follows.

Theorem 1.5 (Heisenberg Uncertainty Principle). *For every $f \in L^2(\mathbb{R})$ and any $a, b \in \mathbb{R}$*

$$||(t-a)f||_{L^2(\mathbb{R})}||(\gamma - b)\widehat{f}||_{L^2(\mathbb{R})} \geq \frac{1}{4\pi}||f||_{L^2(\mathbb{R})}^2. \tag{1.1}$$

Moreover, equality holds in (1.1) if and only if $f(t) = Ce^{2\pi i b t}e^{-c(t-a)^2}$ for some constants $C \in \mathbb{C}$ and $c > 0$.

Rewriting (1.1) we have:

$$\left(\int |t-a|^2|f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int |\gamma - b|^2|\widehat{f}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \geq \frac{1}{4\pi}||f||_{L^2(\mathbb{R})}^2. \tag{1.2}$$

In this form, we see that Heisenberg's uncertainty principle measures localization using the t^2 and γ^2 weights. If a function is well localized in the sense of decaying quickly to zero at $\pm\infty$, then the integral

$$\int |t|^2|f(t)|^2 dt \tag{1.3}$$

will be finite. The relevant type of decay here is not a pointwise decay, but is instead an $L^2(\mathbb{R})$ decay. Moreover, the size of (1.3) tells us how spread out f is. For example, the functions $f_1(t) = \frac{1}{20}\chi_{[-10,10]}(t)$ and $f_2(t) = \frac{1}{2}\chi_{[-1,1]}(t)$ have both been normalized and it is visually clear that the first function is more spread out than the second. This is reflected by the fact that the integral (1.3) is larger for f_1 than f_2. Thus, the use of the t^2 and γ^2 weights to measure localization is both intuitively attractive and also allows one to make more quantitative statements about localization than in the previous section. The utility of the t^2 weight in measuring localization motivates the following definition.

Definition 1.6. *Given $f \in L^2(\mathbb{R})$ satisfying $||f||_{L^2(\mathbb{R})} = 1$, we define the mean of f by*

$$\mu(f) = \int t|f(t)|^2 dt \tag{1.4}$$

and the variance of f by

$$\Delta^2(f) = \int |t - \mu(f)|^2 |f(t)|^2 dt. \tag{1.5}$$

We shall often find it convenient to work with the square root of the variance

$$\Delta(f) = \left(\int |t - \mu(f)|^2 |f(t)|^2 dt \right)^{\frac{1}{2}}. \tag{1.6}$$

This quantity is usually refered to as the standard deviation or dispersion of f

We collect some interesting facts on means and variances in the following lemma.

Lemma 1.7. *Let $f \in L^2(\mathbb{R})$ and suppose $||f||_{L^2(\mathbb{R})} = 1$. If*

$$I(a) = \left(\int |t - a|^2 |f(t)|^2 dt \right)^{\frac{1}{2}} \tag{1.7}$$

is finite for a single value $a = a_0 \in \mathbb{R}$, then it is finite for every $a \in \mathbb{R}$. Moreover,

$$\Delta(f) = \inf_{a \in \mathbb{R}} \left(\int |t - a|^2 |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

In other words, $a = \mu(f)$ minimizes $I(a)$.

We may rewrite the Heisenberg uncertainty principle in terms of variances.

Theorem 1.8. *If $f \in L^2(\mathbb{R})$ satisfies $||f||_{L^2(\mathbb{R})} = 1$ then*

$$\Delta(f)\Delta(\widehat{f}) \geq \frac{1}{4\pi}. \tag{1.8}$$

There is an extensive literature on extensions and generalizations of Heisenberg's inequality, for example see [20]. We shall mention one particularly interesting example. In the next definition, we use the notation $\widehat{\mathbb{R}} = \mathbb{R}$ to denote the dual group of $\mathbb{R}$. We shall do this whenever we wish to emphasize that we are dealing with the frequency domain. For example, if $f \in L^2(\mathbb{R})$ then $\widehat{f} \in L^2(\widehat{\mathbb{R}})$.

Definition 1.9. *Let u and v be nonnegative Borel measurable functions on $\widehat{\mathbb{R}}$ and $\mathbb{R}$, respectively. If $1 < p, q < \infty$ and if there is a constant $K > 0$ such that*

$$\sup_{s>0} \left(\int_0^{1/s} u(\gamma) d\gamma \right)^{1/q} \left(\int_0^s v(t)^{-p'/p} dt \right)^{1/p'} = K$$

then we say $(u, v) \in F(p, q)$.

Definition 1.10. *Given a nonnegative Borel measurable function v, $L_v^P(\mathbb{R})$ is the set of all Borel measurable functions f for which*

$$\|f\|_{p,v} \equiv \left(\int |f(t)|^p v(t) dt \right)^{1/p} < \infty.$$

The following weighted uncertainty principle [4] is an elegant example of how the classical uncertainty principle can be generalized and strengthened.

Theorem 1.11 (Benedetto, Heinig). *Let $1 < p \le q < \infty$, and let u and v be even weights on $\widehat{\mathbb{R}}$ and $\mathbb{R}$ for which $(u, v) \in F(p, q)$ with constant K (as above). Assume $1/u$ and v are increasing on $(0, \infty)$. Then there is a constant $C(K)$ such that*

$$\|f\|_2^2 \le 4\pi C(K) \|t f(t)\|_{p,v} \|\gamma \hat{f}(\gamma)\|_{q', u^{-q'/q}}$$

for all f in the Schwartz class $\mathcal{S}(\mathbb{R})$.

This result once again illustrates the theme of how different pairings of weights translate into uncertainty principles.

1.3 Preview

The results surveyed in the introduction are all uncertainty principles for a single function. The main topic of this thesis is to examine how the uncertainty principle behaves for whole collections of functions, such as orthonormal bases. In

particular, we shall be interested in what sort of uniform localization the elements of an orthonormal basis for $L^2(\mathbb{R})$ can have.

Chapter 2 briefly presents some necessary background on frame theory. Our investigation of uncertainty principles for bases begins in chapter 3. There, we discuss Gabor systems and the Balian-Low theorem. The Balian-Low theorem is an uncertainty principle for Gabor orthonormal bases. We also discuss our proof of the fact that the Balian-Low theorem is sharp. In chapter 4, we examine a theorem of Bourgain which constructs orthonormal bases which are optimal, in a certain sense, with respect to the uncertainty principle. We generalize Bourgain's theorem to different weighted measures of localization. We devote chapter 5 to examining an orthonormalization calculation for the Gaussian coherent states. This sheds some light on the proof of Bourgain's theorem. In chapter 6 we examine a question due to Shapiro on means and variances of orthonormal bases and we answer some interesting cases of his question.

Chapter 2

Frame theoretic background

We shall briefly collect some background on frame theory in this chapter. Although frames will not be a main topic of investigation in this thesis, we shall make use of frame theoretic terminology and definitions at various places.

Frames are a generalization of orthonormal bases. A frame is a sequence of elements in a separable Hilbert space which can be used to give stable decompositions of all the elements in the Hilbert space. Unlike orthonormal bases, frames need not be orthonormal and may contain some redundancy.

Definition 2.1 (Frame). *Let H be a separable Hilbert space. A sequence of elements $\{x_n\} \subseteq H$ is a frame for H if there exist constants $0 < A \leq B < \infty$ such that*

$$\forall x \in H, \quad A||x||_H^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B||x||_H^2. \tag{2.1}$$

The constants A, B are called the frame constants of $\{x_n\}$.

Definition 2.2 (Frame Operator). *Let $\{x_n\}$ be a frame for the separable Hilbert space H. The associated frame operator*

$$S : H \to H$$

is defined by

$$Sx = \sum_n \langle x, x_n \rangle x_n.$$

Example 2.3. *If $\{x_n\}$ is an orthonormal basis for a separable Hilbert space, H, then $\{x_n\}$ is a frame for H with frame constants $A = B = 1$, and the associated frame operator is the identity.*

Theorem 2.4. *Let H be a separable Hilbert space. If $\{x_n\}$ is a frame for H with frame bounds $0 < A \le B < \infty$ then $||x_n||_H^2 \le B$ for all n.*

Proof. Fix n. Using the frame inequality (2.1) we have

$$||x_n||_H^4 = |\langle x_n, x_n \rangle|^2 \le \sum_j |\langle x_n, x_j \rangle|^2 \le B||x_n||_H^2.$$

Therefore, $||x_n||_H^2 \le B$. □

As mentioned above, frames are studied for their ability to give stable decompostions of separable Hilbert spaces. The following theorem, [22], [9], gives the connection between definition 2.1 and the decomposition property of frames.

Theorem 2.5. *Let H be a separable Hilbert space. If $\{x_n\}$ is a frame for H with frame constants A and B, then the associated frame operator is a positive, invertible operator satisfying*

$$AI \le S \le BI.$$

Consequently, for every $x \in H$ one has the decompositions

$$x = SS^{-1}x = \sum_n \langle x, S^{-1}x_n \rangle x_n, \tag{2.2}$$

and

$$x = S^{-1}Sx = \sum_n \langle x, x_n \rangle S^{-1}x_n, \tag{2.3}$$

where the convergence is in H.

Thus, (2.2) and (2.3) give two different ways to decompose $x \in H$ in terms of the frame elements $\{x_n\}$. Equivalently, one may refer to (2.2) and (2.3) as giving frame expansions for x. The coefficients in the decomposition, (2.3), are $\{\langle x, x_n \rangle\}$. So, definition 2.1 says that the l^2 norm of these coefficients is equivalent to the H norm of the function being decomposed. This norm equivalence means the frame gives stable decompositions in the sense that a small change in coefficients gives a small change in the element x being expanded, and vice versa.

Example 2.3 shows that an orthonormal basis for a separable Hilbert space H is a frame for H. Orthonormal bases are minimal frames in the sense that removing any element from an orthonormal basis leaves a system which is no longer a frame for H. To see this, suppose $X = \{x_n\}$ is an orthonormal basis for the separable Hilbert space H. Let $X_N = \{x_n\}_{n \neq N}$. By orthonormality,

$$\sum_{n \neq N} |\langle x_N, x_n \rangle|^2 = 0$$

and

$$\|x_N\|_H = 1.$$

Examining definition 2.1 shows that X_N is not a frame. The fact that orthonormal bases are not the only frames with this minimality property motivates the following definition.

Definition 2.6 (Exact frame). *Let $\{x_n\}$ be a frame for the separable Hilbert space H. $\{x_n\}$ is an exact frame if it is no longer a frame upon the removal any element x_N.*

Independent of frame theory, it turns out that exact frames are well known objects and have long been studied under the equivalent guise of Riesz bases.

Definition 2.7 (Riesz basis). *Let H be a separable Hilbert space. A sequence $\{x_n\} \subset H$ is a Riesz basis for H if $\{x_n\}$ is a frame for H and there exist constants $A, B > 0$ such that*

$$A||c||_{l^2} \leq ||\sum c_n x_n||_H \leq B||c||_{l^2} \tag{2.4}$$

holds for all finite sequences $c = \{c_n\}$.

We say that a sequence $\{x_n\} \in H$ is a Riesz basis for its span if (2.4) holds. In this case, we do not require $\{x_n\}$ to be complete. For example, any orthonormal sequence is a Riesz basis for its span.

The following result shows that Riesz bases and exact frames are actually the same.

Theorem 2.8. *Let H be a separable Hilbert space. $\{x_n\} \subseteq H$ is an exact frame for H if and only if it is a Riesz basis for H.*

A portion of the proof is illustrated by the following result.

Theorem 2.9. *Suppose $\{x_n\}_{n=0}^{\infty} \subset L^2(\mathbb{R})$ is a Riesz basis for its span. One can not have*

$$x_N \in \overline{\text{span}} \ \{x_j : j \neq N\}$$

for any N.

Proof. Without loss of generality suppose $N = 0$, and that

$$x_0 \in \overline{\text{span}} \ \{x_j : j \neq 0\}.$$

Let $0 < A \leq B < \infty$ be the constants in (2.4). By our assumption there exists a sequence $\{c_j\}_{j=1}^{\infty}$ such that

$$\left\|x_0 - \sum_{j=1}^{\infty} c_j x_j\right\|_{L^2(\mathbb{R})} < \frac{A}{2}.$$

Combining this with (2.4) and the fact that $\{x_j\}$ is a Riesz basis for its span gives

$$A \le A \left(1 + \sum_{j=1}^{\infty} |c_j|^2 \right)^{\frac{1}{2}} \le \left\| x_0 - \sum_{j=1}^{\infty} c_j x_j \right\|_{L^2(\mathbb{R})} < \frac{A}{2}.$$

This is a contradiction, since $A > 0$. $\qquad\square$

A well known alternative definition of Riesz bases in terms of Grammian matrices appears in [22].

Theorem 2.10. *Suppose $X = \{x_n\}_{n \in \mathbb{Z}}$ is a frame for the separable Hilbert space H. X is a Riesz basis for H if and only if the Grammian matrix $G_{j,k} = \langle x_j, x_k \rangle$ defines a positive invertible operator on $l^2(\mathbb{Z})$.*

In finite dimensions, Riesz basis are particularly simple.

Theorem 2.11. *If $X = \{x_n\}_{n=1}^{N}$ is a finite, linearly independent subset of a Hilbert space, H, then X is a Riesz basis for its span.*

Proof. Since X is finite dimensional and linearly independent, it is easily verified that (2.4) holds. $\qquad\square$

Chapter 3

Gabor systems

We gave several examples of uncertainty principles in the introduction. The types of results we surveyed all dealt with uncertainty for an individual function and its Fourier transform. For example, the qualitative uncertainty principle, theorem 1.1, implies that a nontrivial function $f \in L^2(\mathbb{R})$ and its Fourier transform, $\widehat{f}$, can not both have compact support.

One of our goals is to understand how the uncertainty principle applies to certain collections of functions, as opposed to how it applies to individual functions. For us, the collection of functions under consideration will usually be an orthonormal basis. We want to know what sort of uniform localization, in time and frequency, the elements of an orthonormal basis can have. This chapter will focus on this question for Gabor orthonormal bases, but will also briefly address wavelet orthonormal bases.

Gabor systems and wavelet systems are both examples of coherent systems. A system of functions is *coherent* if it generated by a single function under the action of a group. For example, let $f \in L^2(\mathbb{R})$, and define $f_n(t) = f(t - n)$. It is clear that $\{f_n\}$ is a coherent system of functions, generated by the action of $\mathbb{Z}$ on f. It is well known, e.g., [12], that this particular system of functions can

not form an orthonormal basis, or even a frame, for $L^2(\mathbb{R})$. Gabor systems and wavelets are the simplest coherent systems for which one can obtain orthormal bases for $L^2(\mathbb{R})$. Although it will be not play a direct role in our work, let us mention that the respective groups associated with Gabor systems and wavelets are the Heisenberg group and affine group.

3.1 Gabor systems

Definition 3.1. *Given a function $f \in L^2(\mathbb{R})$ and constants $a, b > 0$, the Gabor system, $\mathcal{G}(f, a, b) = \{f_{m,n}\}_{m,n \in \mathbb{Z}}$ is defined by*

$$f_{m,n}(t) = e^{-2\pi i b m t} f(t - an).$$

Thus, a Gabor system consists of translates and modulates of a fixed function. Gabor systems have been widely studied because they can be used to give effective decompositions of functions. One of the main questions in Gabor analysis is to determine for which functions f and constants a, b the Gabor system, $\mathcal{G}(f, a, b)$, is an orthonormal basis, Riesz basis, or frame.

The following example is often called the trivial Gabor basis.

Example 3.2. *Let $f(t) = \chi_{[0,1]}(t)$. Using standard results on Fourier series it is easy to see that $\mathcal{G}(f, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$.*

The following result shows that the localization of a Gabor system is inherently uniform with respect to variances.

Theorem 3.3. *Suppose that $f \in L^2(\mathbb{R})$ and that the variances $\Delta(f)$ and $\Delta(\widehat{f})$ are both finite. Let $\{f_{m,n}\} = \mathcal{G}(f, a, b)$ for some $a, b > 0$. A direct calculation, [4], shows that*

$$\forall m, n \in \mathbb{Z}, \quad \Delta(f_{m,n}) = \Delta(f)$$

and

$$\forall m, n \in \mathbb{Z}, \quad \Delta(\widehat{f_{m,n}}) = \Delta(\widehat{f}).$$

This shows that the elements of a Gabor system have uniform localization with respect to time and frequency variances. The following example shows a Gabor frame which has excellent localization in time and frequency.

Example 3.4. *If $g(t) = e^{-\pi t^2}$ and $ab < 1$, then $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$. See [22] for further details. Moreover, a direct calculation combined with theorem 3.3 shows that*

$$\forall m, n \in \mathbb{Z}, \quad \Delta(g_{m,n}) = \Delta(g) = \frac{1}{2\sqrt{\pi}}$$

and

$$\forall m, n \in \mathbb{Z}, \quad \Delta(\widehat{g_{m,n}}) = \Delta(\widehat{g}) = \frac{1}{2\sqrt{\pi}}.$$

The problem with theorem 3.3 is that the uniform localization need not be a "good" localization when the Gabor system under consideration is an orthonormal basis. This is illustrated by example 3.2, where $f(t) = \chi_{[0,1]}(t)$ generates an orthonormal basis. Note that $\Delta(\widehat{f}) = \infty$, so that all elements of the trivial Gabor basis have uniformly poor localization in frequency. We shall see that this behavior is typical for Gabor orthonormal bases.

3.2 Density and duality

The difference between examples 3.2 and 3.4 is actually quite illuminating. In example 3.2, one has a poorly localized orthonormal basis with $a = b = 1$. In example 3.4, one has a well localized frame but one is required to take a "denser" set of translates and modulates (i.e., $ab < 1$).

The next theorem is a first step towards explaining the relationship between the value of (a, b) and localization and basis/frame properties of Gabor systems.

Theorem 3.5. *If $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ then $ab \leq 1$. If $\mathcal{G}(g, a, b)$ is a Riesz basis for $L^2(\mathbb{R})$ then $ab = 1$.*

The following closely related result gives further insight.

Theorem 3.6 (Ron-Shen Duality). *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz basis for its closed linear span.*

3.3   Linear independence

An interesting and useful result on Gabor systems is that any finite subset of a Gabor system is linearly independent. The case $a = b = 1$ was proven in [28] by Heil, Ramanathan, and Topiwala and the general case was shown by Linnell in [36].

Theorem 3.7. *Let $f \in L^2(\mathbb{R})$ be nontrivial, and $a, b > 0$. Any finite subset of $\mathcal{G}(f, a, b)$ is linearly independent.*

Heil, Ramanathan, and Topiwala conjectured, in [28], that this result still holds for irregular Gabor systems (i.e., those not defined on a lattice). While they have shown that this conjecture is true for certain interesting cases, the general case is still open.

3.4   The Zak transform

Definition 3.8. *The Zak transform, Zf, of a function $f \in L^2(\mathbb{R})$ is formally defined to be*

$$Zf(t, \gamma) = \sum_{n \in \mathbb{Z}} f(t - n) e^{2\pi i n \gamma}.$$

Note that the Zak transform is quasiperiodic, [22]. In other words, it satisfies the two equations

$$Zf(t + 1, \gamma) = e^{2\pi i \gamma} Zf(t, \gamma), \tag{3.1}$$

and

$$Zf(t, \gamma + 1) = Zf(t, \gamma). \tag{3.2}$$

For this reason, Zf is fully determined by its values on $Q = [0, 1)^2$. The next result gives a precise statement on the range and domain of the Zak transform.

Theorem 3.9. *The Zak transform is a unitary operator from $L^2(\mathbb{R})$ to $L^2(Q)$.*

The following result, [9], shows why the Zak transform is especially useful for studying Gabor systems on the $\mathbb{Z} \times \mathbb{Z}$ lattice.

Theorem 3.10. *Let $g \in L^2(\mathbb{R})$.*

1. *If $Zg \neq 0$ a.e. in Q then $\mathcal{G}(g, 1, 1)$ is complete in $L^2(\mathbb{R})$.*

2. *If $1/Zg$ is in $L^2(Q)$ then $\mathcal{G}(g, 1, 1)$ is complete and minimal in $L^2(\mathbb{R})$. A sequence, $\{x_n\}$, is minimal if $\forall j$, $x_j \notin \overline{\mathrm{span}}\{x_n : n \neq j\}$.*

3. *If there exist $0 < A \leq B < \infty$ such that $A \leq |Zg| \leq B$ a.e. in Q then $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$.*

4. *If $|Zg| = 1$ a.e. in Q then $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$.*

3.5 The Balian-Low theorem

The Balian-Low theorem is the classical uncertainty principle for Gabor systems. It shows that the localization behavior of the trivial Gabor orthonormal basis is typical of all Gabor orthonormal bases. The Balian-Low theorem traces its origins back to [1], [37], but there have been numerous corrections and simplifications of the original proof, e.g., [8], [2].

Theorem 3.11 (Balian-Low). *Let $g \in L^2(\mathbb{R})$. If the Gabor system $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ then either*

$$\int |t|^2 |g(t)|^2 dt = \infty$$

or

$$\int |\gamma|^2 |\widehat{g}(\gamma)|^2 d\gamma = \infty.$$

This is the simplest version of the Balian-Low theorem. It is actually true in much greater generality. For example, [8], the result still holds if "orthonormal basis" is replaced by "Riesz basis". The theorem above applies to Gabor systems on the lattice $\mathbb{Z} \times \mathbb{Z}$ and holds in one dimension. Gröchenig, Han, Heil, and Kutyniok have given extensions to symplectic lattices in higher dimensions, [23]. Benedetto, Czaja, and Maltsev have investigated the Balian-Low theorem for the symplectic form in higher dimensions, [7].

Rewriting the Balian-Low theorem in terms of variances gives

Theorem 3.12. *Let $g \in L^2(\mathbb{R})$. If $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ then either*

$$\Delta^2(g) = \infty \quad or \quad \Delta^2(\widehat{g}) = \infty.$$

3.5.1 Sharpness in the Balian-Low theorem

The Balian-Low theorem says there are no Gabor orthonormal bases localized with respect to both the t^2 and γ^2 weights. It is natural to ask to what extent this result is sharp. Namely, by how much can one weaken the t^2 and γ^2 weights so that the Balian-Low theorem no longer holds? We have proven the following result, [6], which shows that the Balian-Low theorem is essentially sharp.

Theorem 3.13 (Benedetto, Czaja, Gadziński, Powell). *Let $d > 2$. There exists a function $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and*

$$\int \frac{1 + |t|^2}{\log^d(|t| + 2)} |g(t)|^2 dt < \infty,$$

and

$$\int \frac{1 + |\gamma|^2}{\log^d(|\gamma| + 2)} |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

In particular, if one weakens the t^2 and γ^2 by the logarithmic terms in the theorem, then the Balian-Low theorem no longer holds.

3.6   A (p, q) Balian-Low theorem

While the t^2 and γ^2 weights associated with the Balian-Low theorem are natural and useful, it is also interesting to see what happens for other combinations of weights. A result in this direction follows from the work of Feichtinger and Gröchenig.

Theorem 3.14 (Feichtinger, Gröchenig). *Suppose $\epsilon > 0$ and that $\frac{1}{p} + \frac{1}{q} = 1$. If $g \in L^2(\mathbb{R})$ and $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ then either*

$$\int |t|^{p+\epsilon} |g(t)|^2 dt = \infty$$

or

$$\int |\gamma|^{q+\epsilon} |\widehat{g}(\gamma)|^2 d\gamma = \infty.$$

The proof of this makes use of the following results on Gabor systems and modulation space embeddings.

Definition 3.15. *Given $f, g \in L^2(\mathbb{R})$ the short time Fourier transform of f with respect to g is formally defined by*

$$\mathcal{S}_g[f](t, \gamma) = \int f(t)\overline{g(x-t)}e^{-2\pi i x \gamma} dx.$$

Definition 3.16. *Let g be a fixed Schwartz class function. The modulation space $M_{1,1}$ is defined to be the set of all measurable functions for which*

$$||f||_{1,1} = \int\int |\mathcal{S}_g f(x, y)| dx dy < \infty.$$

$M_{1,1}$ *is independent of the choice of $g \in \mathcal{S}(\mathbb{R})$ in the sense that different choices yield equivalent norms, [22].*

The following result appears in [18]. It may be viewed as a Balian-Low theorem for modulation spaces.

Theorem 3.17. *If $f \in M_{1,1}$, then $\mathcal{G}(f, 1, 1)$ is not an orthormal basis for $L^2(\mathbb{R})$.*

The following modulation space embedding appears in [21].

Theorem 3.18. *Fix $\epsilon > 0$ and assume $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p, q < \infty$. There exists a constant C such that*

$$||f||_{1,1} \le C \left(\left(\int |t|^{p+\epsilon} |f(t)|^2 dt \right)^{\frac{1}{2}} + \left(\int |\gamma|^{q+\epsilon} |\widehat{f}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} \right)$$

holds for all $f \in M_{1,1}$.

Thus, we see that theorem 3.14 follows from theorems 3.17 and 3.18. As with the standard Balian-Low theorem, we have the following "sharpness" result.

Theorem 3.19 (Benedetto, Czaja, Gadziński, Powell). *Let $d > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p, q < \infty$. There exists a function $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$, and*

$$\int \frac{1 + |t|^p}{\log^d(|t| + 2)} |g(t)|^2 dt < \infty$$

and

$$\int \frac{1 + |\gamma|^q}{\log^d(|\gamma| + 2)} |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

3.7   Wavelets

The past several sections dealt with Gabor systems, which are coherent systems associated to the Heisenberg group. Another popular class of coherent systems are the wavelet systems. Wavelets systems are generated by the action of the affine group.

Definition 3.20. *Given $\psi \in L^2(\mathbb{R})$, the associated wavelet system, $\mathcal{W}(\psi) = \{\psi_{m,n}\}_{m,n \in \mathbb{Z}}$, is defined by*

$$\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n).$$

As with Gabor systems, a fundamental question is to find $\psi \in L^2(\mathbb{R})$ for which the wavelet system, $\mathcal{W}(\psi)$, is an orthonormal basis for $L^2(\mathbb{R})$. In the following example, ψ is called the Haar wavelet.

Example 3.21 (Haar wavelet). *Let*

$$\psi(t) = \begin{cases} 1, & if \quad t \in [0, 1/2), \\ -1, & if \quad t \in [1/2, 1), \\ 0, & otherwise. \end{cases} \tag{3.3}$$

It is well known, e.g., see [13], that $\{\psi_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

This example is similar to the trivial Gabor basis (example 3.2), in that it has jump discontinuities and a poorly localized Fourier transform. The following example, due to Meyer, shows that one can do better than the Haar wavelet.

Example 3.22 (Meyer wavelet). *There exists $\psi \in L^2(\mathbb{R})$ such that $\psi \in \mathcal{S}(\mathbb{R})$, supp $\widehat{\psi}$ is compact, and $\{\psi_{m,n}\}$ is an othonormal basis for $L^2(\mathbb{R})$.*

This example shows that there are wavelet orthonormal bases whose generator is well localized in both time and frequency. By the Balian-Low theorem, this stands in contrast to the situation for Gabor bases. However, the following result shows that wavelet systems do not have the uniform localization found in Gabor systems by theorem 3.3.

Theorem 3.23. *Let $\psi \in L^2(\mathbb{R})$ and suppose that $\Delta(\psi)$ and $\Delta(\widehat{\psi})$ are both finite. A direct calculation, [4], shows that*

$$\forall m, n \in \mathbb{Z}, \quad \Delta(\psi_{m,n}) = 2^{-m}\Delta(\psi)$$

and

$$\forall m, n \in \mathbb{Z}, \quad \Delta(\widehat{\psi_{m,n}}) = 2^m\Delta(\widehat{\psi}).$$

In particular this shows that

$$\sup_{m,n}\Delta(\psi_{m,n}) = \infty \quad and \quad \sup_{m,n}\Delta(\widehat{\psi_{m,n}}) = \infty.$$

Thus, the elements of a wavelet system do not have uniform localization.

3.8   Battle's theorem

In the previous section we saw that wavelet systems lack the uniform localization of Gabor systems (compare theorem 3.3 with theorem 3.23). However, the generator of a wavelet orthonormal basis can have much better time-frequency localization than the generator of a Gabor orthonormal basis (compare the Meyer wavelet with the Balian-Low theorem). In this section, we present results which show how well localized the generator of a wavelet orthonormal basis can be.

The following result, [3], may be viewed as a version of the Balian-Low theorem for wavelet bases.

Theorem 3.24 (Battle). *Let $\psi \in L^2(\mathbb{R})$. If*

$$|\psi(t)| \leq Ce^{-|t|} \quad and \quad |\widehat{\psi}(\gamma)| \leq Ce^{-|\gamma|}$$

then the wavelet system $\{\psi_{m,n}\}$ can not be an orthonormal basis for $L^2(\mathbb{R})$.

As with the Balian-Low theorem, it is natural to ask if Battle's theorem is sharp. The following result, [17], shows that the hypotheses of Battle's theorem can not be significantly weakened.

Theorem 3.25 (Dziubański, Hernández). *For every $0 < \epsilon < 1$ there exists $\psi \in L^2(\mathbb{R})$ such that the wavelet system $\{\psi_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ satisfying*

$$|\widehat{\psi}(\gamma)| \leq C_\epsilon e^{-|t|^{1-\epsilon}} \quad and \quad \mathrm{supp}(\psi) \ \text{is compact,}$$

where C_ϵ is a constant.

Chapter 4

Bourgain's Theorem

In the previous chapter we saw how the Balian-Low theorem, theorem 3.11, imposes localization restrictions on Gabor orthonormal bases. It states that there are no functions $g \in L^2(\mathbb{R})$ which generate a Gabor orthonormal basis and satisfy both

$$\int |t|^2 |g(t)|^2 dt < \infty$$

and

$$\int |\gamma|^2 |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

The weights t^2 and γ^2 play a crucial role here. In contrast to the Balian-Low theorem, our sharpness result, theorem 3.13, states that if slightly weaker weights are used, then one can have Gabor orthonormal bases. Specifically, there are functions $g \in L^2(\mathbb{R})$ which generate Gabor orthonormal bases and satisfy

$$\int \frac{1 + |t|^2}{\log^d(|t| + 2)} |g(t)|^2 dt < \infty$$

and

$$\int \frac{1 + |\gamma|^2}{\log^d(|\gamma| + 2)} |\widehat{g}(\gamma)|^2 d\gamma < \infty,$$

where $d > 2$. In view of these two results, it is natural to ask what happens for general (i.e., non-Gabor) orthonormal bases, namely, what sort of "uniform

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localization" can a general orthonormal basis for $L^2(\mathbb{R})$ have with respect to the weights t^2 and γ^2. This question was posed by Balian, [1], and answered by Bourgain, [11].

Theorem 4.1 (Bourgain). *Let $\epsilon > 0$. There exists an orthonormal basis, $\{b_n\}$, for $L^2(\mathbb{R})$ such that*

$$\Delta(b_n) \leq \frac{1}{2\sqrt{\pi}} + \epsilon, \quad \forall n$$

and

$$\Delta(\widehat{b_n}) \leq \frac{1}{2\sqrt{\pi}} + \epsilon, \quad \forall n.$$

This result uses variances (see definition 1.5) to measure localization; the uniform boundedness of the variances reflects a type of uniform localization of the basis with respect to the t^2 and γ^2 weights.

To put this in perspective, note that there are $\psi \in \mathcal{S}(\mathbb{R})$ which generate wavelet orthonormal bases, $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$, for $L^2(\mathbb{R})$. Since $\psi \in \mathcal{S}(\mathbb{R})$, each $\Delta(\psi_{m,n})$ and $\Delta(\widehat{\psi_{m,n}})$ is finite. However we have already seen (fact 3.23) that for any wavelet system these variances are not uniformly bounded, [4].

Theorems 3.11, 3.13, and 4.1 form a trio of results which give insight into the boundaries of uncertainty for the t^2 and γ^2 weights. In this chapter we shall consider the situation for the weights t^p and t^q, where $\frac{1}{p}+\frac{1}{q} = 1$. This investigation is motivated by the (p,q) Balian-Low theorem of Feichtinger and Gröchenig, theorem 3.14, which says that if $g \in L^2(\mathbb{R})$ generates a Gabor orthonormal basis then one can not have both

$$\int |t|^{p+\epsilon}|g(t)|^2 dt < \infty$$

and

$$\int |\gamma|^{q+\epsilon}|g(\gamma)|^2 d\gamma < \infty.$$

As in the case $(p, q) = (2, 2)$, we proved a sharpness result, theorem 3.19, which says that Gabor bases are possible if the weights are weakened slightly. In particular, there are $g \in L^2(\mathbb{R})$ which generate Gabor orthonormal bases and satisfy

$$\int \frac{1 + |t|^p}{\log^d(|t| + 2)} |g(t)|^2 dt < \infty$$

and

$$\int \frac{1 + |\gamma|^q}{\log^d(|\gamma| + 2)} |\widehat{g}(\gamma)|^2 d\gamma < \infty,$$

where $d > 2$.

We shall consider what sort of localization a general orthonormal basis can have with respect to the t^p and γ^q weights. Our main result will generalize theorem 4.1. We use the following "generalized variances" to measure localization.

Definition 4.2. *Given $f \in L^2(\mathbb{R})$ and $\lambda > 0$ we define the generalized variance of f by*

$$\Delta_\lambda^2(f) = \inf_{a \in \mathbb{R}} \int |t - a|^\lambda |f(t)|^2 dt.$$

As with the standard definition of variance, it will often be convenient to work with the square root of the generalized variance

$$\Delta_\lambda(f) = \inf_{a \in \mathbb{R}} \left(\int |t - a|^\lambda |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

We refer to this as the generalized standard deviation or dispersion of f.

In terms of this definition, our main result is

Theorem 4.3. *Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Assume $q \in 2\mathbb{N}$. There exists an orthonormal basis, $\{b_n\}_{n \in \mathbb{N}}$, for $L^2(\mathbb{R})$, and a constant $C = C(p, q)$ such that*

$$\Delta_p(b_n) \leq C, \quad \forall n$$

and

$$\Delta_q(\widehat{b_n}) \leq C, \quad \forall n.$$

While we do not give an explicit value for the constant C, as Bourgain did, we can estimate possible values of C. However, we shall not comment any more on this here. Theorems 3.14, 3.19, and 4.3 comprise a trio of results which give insight into the role of the weights t^p and γ^q in uncertainty principles for orthonormal bases, and which extend the original $(p, q) = (2, 2)$ results.

4.1 Preliminary lemmas

In this section we shall state several lemmas which will be needed to prove theorem 4.3.

4.1.1 Decay rates of inverses of matrices

The following results relate the off-diagonal decay of an invertible matrix to the off-diagonal decay of its inverse. The results are due to Jaffard, [32], and have been further studied and simplified by Strohmer in [43]. We also note that Bourgain made use of similar results in [11] a few years prior to Jaffard's work. For example, see the transition between equations (2.11) and (2.12) in [11].

The following definition appears in [43].

Definition 4.4. *Let $A = (A_{m,n})_{m,n \in \mathcal{I}}$ be a matrix, where the index set is $\mathcal{I} = \mathbb{Z}, \mathbb{N},$ or $\{0, \cdots, N - 1\}$. Given $s > 1$, we say A belongs to $\mathcal{Q}_s$ if the coefficients $A_{m,n}$ satisfy*

$$|A_{m,n}| < \frac{C}{(1 + |m - n|)^s}$$

for some constant $C > 0$. We say that A belongs to $\mathcal{E}_s$ if

$$|A_{m,n}| < Ce^{-s|m-n|}.$$

The next result says that if A is in one of the two decay classes defined above, then A^{-1} has a similar kind of decay.

Theorem 4.5 (Jaffard). *Let $A : l^2(\mathcal{I}) \to l^2(\mathcal{I})$ be an invertible matrix, where $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or $\{0, \cdots, N-1\}$. Then*

$$A \in \mathcal{Q}_s \implies A^{-1} \in \mathcal{Q}_s$$

and

$$A \in \mathcal{E}_s \implies A^{-1} \in \mathcal{E}_{s'},$$

for some $0 < s' \leq s$.

The case $\mathcal{I} = \{0, 1, 2, \cdots, N-1\}$ should be interpreted as follows. We quote from [43]: "View the $n \times n$ matrix A_n as a finite section of an infinite dimensional matrix A. If we increase the dimension of A_n (and thus consequently the dimension of $(A_n)^{-1}$) we can find uniform constants independent of n such that the corresponding decay properties hold."

Let us next comment on the constants which arise in Jaffard's theorem. We restrict ourselves to the case $\mathcal{I} = \{0, 1, \cdots, N-1\}$. Suppose that A_N are sections of the infinite matrix A and that

$$|A_N(j,k)| \leq \frac{C}{1 + |j-k|^s}, \quad \forall \, j, k \in \mathcal{I}$$

holds for all N. Suppose for simplicity that there is a fixed $0 < r < 1$ such that

$$A_N = I_N - B_N \text{ with } ||B_N|| \leq r < 1$$

holds for each N. Jaffard's theorem then says there exists C' such that

$$|A_N^{-1}(j,k)| \leq \frac{C'}{1 + |j-k|^s}, \quad \forall \, j, k \in \mathcal{I}$$

holds for each N. The constant C' depends only on r, s, and C. One may see this by examining Jaffard's proofs, [32].

4.1.2 Bilinear form estimates

Next, we state some estimates on bilinear forms. Since the results are simple we include the proofs. Other results of this type appear in [26].

Lemma 4.6. *Given $\lambda > 1$, there exists a constant C_λ such that for every $\{a_n\} \in l^2(\mathbb{N})$, $N \in \mathbb{N}$*

$$\sum_{j=0}^{N} \sum_{k=0}^{N} \frac{|a_n||a_m|}{1 + |j - k|^\lambda} \leq C_\lambda \sum_{j=0}^{N} |a_j|^2.$$

Proof. The idea is to sum along the positively sloped diagonals of the finite grid

$$\{(j, k) \in \mathbb{Z}^2 : 0 \leq j, k \leq N\}.$$

Following this idea and applying Hölder's inequality at the appropriate place yields

$$
\begin{aligned}
\sum_{j=0}^{N} \sum_{k=0}^{N} \frac{|a_j||a_k|}{1 + |j - k|^\lambda} &= \sum_{j=0}^{N} |a_j|^2 + 2 \sum_{d=1}^{N} \sum_{j=0}^{N-d} \frac{|a_{d+j}||a_j|}{1 + d^\lambda} \\
&= \sum_{j=0}^{N} |a_j|^2 + 2 \sum_{d=1}^{N} \frac{1}{1 + d^\lambda} \sum_{j=0}^{N-d} |a_{d+j}||a_j| \\
&\leq \sum_{j=0}^{N} |a_j|^2 + 2 \left(\sum_{k=0}^{N} |a_k|^2 \right) \sum_{d=1}^{N} \frac{1}{1 + d^\lambda} \\
&\leq C_\lambda \sum_{j=0}^{N} |a_j|^2.
\end{aligned}
$$

$\square$

Lemma 4.7. *Let $\lambda > 2$. There exists C_λ such that for every $\{a_{m,n}\} \in l^2(\mathbb{N}^2)$*

$$\sum_{m,n=0}^{N} \sum_{j,k=0}^{N} \frac{|a_{m,n}||a_{j,k}|}{1 + |m - j|^\lambda + |n - k|^\lambda} \leq C_\lambda \left(\sum_{m,n=0}^{N} |a_{m,n}|^2 \right).$$

Proof.

$$\sum_{m,n=0}^{N}\sum_{j,k=0}^{N}\frac{|a_{m,n}||a_{j,k}|}{1+|m-j|^\lambda+|n-k|^\lambda} \le 2\sum_{c=0}^{N}\sum_{d=0}^{N}\sum_{m=0}^{N-c}\sum_{n=0}^{N-d}\frac{|a_{m,n}||a_{m+c,n+d}|}{1+|c|^\lambda+|d|^\lambda}$$

$$\le \sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}}\left(\sum_{m,n=0}^{N}|a_{m,n}|^2\right)\frac{1}{1+|c|^\lambda+|d|^\lambda} \le C_\lambda\left(\sum_{m,n=0}^{N}|a_{m,n}|^2\right).$$

$\square$

Consequently, one also has

Lemma 4.8. *Given* $\lambda > 1$, *there exists a constant* C_λ *such that for every* $\{a_{m,k}\} \in l^2(\mathbb{N}^2)$ *and* $N \in \mathbb{N}$

$$\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{j=0}^{N}\frac{|a_{m,k}||a_{j,k}|}{1+|j-m|^\lambda} \le C_\lambda\sum_{j,k=0}^{N}|a_{j,k}|^2.$$

4.1.3 Phase space localization

We begin by recalling the following definition.

Definition 4.9. *Given* $f, g \in L^2(\mathbb{R})$ *the short time Fourier transform of* f *with respect to* g *is formally defined by*

$$\mathcal{S}_g[f](t,\gamma) = \int f(t)\overline{g(x-t)}e^{-2\pi i x\gamma}dx.$$

The following lemma is theorem 11.2.5 in [22].

Lemma 4.10. *If* $f, g \in \mathcal{S}(\mathbb{R})$ *then* $\mathcal{S}_g[f] \in \mathcal{S}(\mathbb{R}^2)$.

Lemma 4.11. *Suppose* $\varphi \in L^2(\mathbb{R})$ *and define* $\varphi_j(t) = e^{-2\pi i j t}\varphi(t)$. *If*

$$|\widehat{\varphi}(\gamma)| \le \frac{C}{1+|\gamma|^N},$$

for some constant $C > 0$, *then*

$$|\langle\varphi_j,\varphi_k\rangle| \le \frac{C_1}{1+|j-k|^N}, \tag{4.1}$$

where C_1 *is a constant which may depend on* N.

Proof. Without loss of generality assume $n > 0$ and note that

$$|\langle \varphi_0, \varphi_n \rangle| \leq \int |\widehat{\varphi}(\gamma)||\widehat{\varphi}(\gamma - n)|d\gamma = I_1 + I_2,$$

where

$$
\begin{aligned}
I_1 &= \int_{\gamma \leq (n/2)} |\widehat{\varphi}(\gamma)||\widehat{\varphi}(\gamma - n)|d\gamma \\
&\leq \int_{\gamma \leq (n/2)} \frac{C^2}{(1 + |\gamma|^N)(1 + |\gamma - n|^N)} d\gamma \\
&\leq \frac{1}{1 + |(n/2)|^N} \int_{\gamma \leq (n/2)} \frac{C^2}{1 + |\gamma|^N} d\gamma \\
&\leq \frac{C_1}{1 + |n|^N}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_{\gamma > (n/2)} |\widehat{\varphi}(\gamma)||\widehat{\varphi}(\gamma - n)|d\gamma \\
&\leq \int_{\gamma > (n/2)} \frac{C^2}{(1 + |\gamma|^N)(1 + |\gamma - n|^N)} d\gamma \\
&\leq \frac{1}{1 + |(n/2)|^N} \int_{\gamma > n/2} \frac{C^2}{1 + |\gamma - n|^N} d\gamma \\
&\leq \frac{C_1}{1 + |n|^N}.
\end{aligned}
$$

Combining the estimates for I_1 and I_2, (4.1) follows. $\qquad\square$

4.2   Finite, orthonormal, well localized systems

Lemma 4.12. *Assume $\frac{1}{p} + \frac{1}{q} = 1$ and $q \in \mathbb{N}$. There exists a constant $C = C(p,q)$ and a constant $K_0 > 0$ so that for each $T \in \mathbb{N}$ and each integer $K > K_0$ there exists a finite orthonormal set $S_0 = S_0(T, K) = \{s_n\}_{n=0}^{T-1}$ of cardinality T satisfying*

$$\text{supp } s_n \subseteq [-1, 1] \tag{4.2}$$

and

$$\left(\int |t|^p |s_n(t)|^2 dt\right)^{\frac{1}{2}} \leq C \tag{4.3}$$

and

$$\left(\int |\gamma - nK|^q |\widehat{s}_n(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C, \tag{4.4}$$

for $n = 0, 1, \cdots, T - 1.$

Proof. Throughout the proof, C will denote various constants which are independent of T and K. C may depend on (p, q), φ, and N, all of which are fixed throughout the proof.

I. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a function of $L^2(\mathbb{R})$ norm one satisfying

$$\text{supp } \varphi \subseteq [-1, 1] \tag{4.5}$$

and

$$|\widehat{\varphi}(\gamma)| \leq \frac{C}{|\gamma|^N + 1}, \tag{4.6}$$

where $N > 4q, N \in \mathbb{N}$. Now define

$$\varphi_j(t) = e^{2\pi i j K t} \varphi(t), \quad j = 0, 1, \cdots, T - 1,$$

where $K > K_0$ are integers and K_0 will be defined later. Next, define

$$h_0(t) = \varphi_0(t) \tag{4.7}$$

and

$$h_n(t) = \varphi_n(t) - \sum_{j=0}^{n-1} a_{n,j} \varphi_j(t), \quad 1 \leq n \leq T - 1 \tag{4.8}$$

where the $a_{n,j}$ are chosen to make h_n orthogonal to $\{\varphi_j\}_{j=0}^{n-1}$. This choice of $a_{n,j}$ implies that for all $0 \leq l \leq n - 1$

$$\langle \varphi_n, \varphi_l \rangle = \sum_{j=0}^{n-1} a_{n,j} \langle \varphi_j, \varphi_l \rangle.$$

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Rewriting this in matrix form, we have

$$Ga = g,$$

where $G = \begin{pmatrix} \langle\varphi_{n-1},\varphi_{n-1}\rangle & \langle\varphi_{n-2},\varphi_{n-1}\rangle & \cdots & \langle\varphi_0,\varphi_{n-1}\rangle \\ \langle\varphi_{n-1},\varphi_{n-2}\rangle & \langle\varphi_{n-2},\varphi_{n-2}\rangle & \cdots & \langle\varphi_0,\varphi_{n-2}\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\varphi_{n-1},\varphi_0\rangle & \langle\varphi_{n-2},\varphi_0\rangle & \cdots & \langle\varphi_0,\varphi_0\rangle \end{pmatrix}$$

$$a = \begin{pmatrix} a_{n,n-1} \\ a_{n,n-2} \\ \cdots \\ a_{n,0} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} \langle\varphi_n,\varphi_{n-1}\rangle \\ \langle\varphi_n,\varphi_{n-2}\rangle \\ \cdots \\ \langle\varphi_n,\varphi_0\rangle \end{pmatrix}.$$

Note that these matrices all depend on n, but we shall usually suppress this for economy of notation. When we wish to emphasize the dependence on n, we shall write $G = G_n$ for example.

II. First of all, observe that G is an invertible matrix. To see this, note that the $\{\varphi_j\}_{j=0}^{n-1}$ are linearly independent by theorem 3.7. Hence, by theorem 2.11 $\{\varphi_j\}_{j=0}^{n-1}$ is a Riesz basis for its span. Thus, by theorem 2.10, $G = G_n$ is invertible for each n (recall that G depends on n). In particular, the $\{a_j^n\}_{j=0}^{n-1}$ are unique.

To apply Jaffard's lemma, we also need to know that the spectrum of $G = G_n$ stays uniformly bounded away from 0 independent of n. Note that the matrix G is a Toeplitz matrix, and by (4.6) has polynomial decay of order N off the main diagonal, in fact,

$$|G(j,k)| \leq \frac{C}{1 + K^N|j-k|^N} \leq \frac{C}{1 + |j-k|^N}. \tag{4.9}$$

For K large enough, the first inequality of (4.9) implies $G = G_n$ is diagonally

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dominant and has spectrum uniformly bounded away from 0. By our choice of $K > K_0$, this will be the case.

III. By the result of Jaffard, G^{-1} has the same type of decay off its main diagonal as G, namely,

$$|G^{-1}(j,k)| \leq \frac{C}{1 + |j - k|^N}.$$

Also, note that the comments after the statement of Jaffard's theorem ensure that C is independent of K.

Therefore, noting that $a_{n,n-j}$ is the j-th element of the vector a,

$$
\begin{aligned}
|a_{n,n-j}| &\leq \sum_{l=0}^{n-1} |G^{-1}(j,l)||g_l| = \sum_{l=0}^{n-1} |G^{-1}(j,l)||\langle \varphi_n, \varphi_{n-l-1} \rangle| \\
&\leq \sum_{l=0}^{n-1} \left(\frac{C}{1 + |j - l|^N} \right) \left(\frac{C}{1 + K^N |l + 1|^N} \right) \\
&\leq \sum_{l=0}^{n-1} \frac{C}{1 + |j - l|^N} \left(\frac{C}{K^N (l + 1)^N} \right) \\
&\leq \frac{C}{K^N} \sum_{l=0}^{n-1} \frac{1}{(1 + |j - l|^N)} \frac{1}{|l + 1|^N} \\
&\leq \frac{C}{K^N} \sum_{l=1}^{\infty} \frac{1}{(1 + |(j + 1) - l|^N)} \frac{1}{|l|^N} \\
&\leq \left(\frac{1}{K^N} \right) \frac{C}{|j + 1|^N}.
\end{aligned}
$$

To see the last step, note that

$$\sum_{1 \leq l \leq \frac{j+1}{2}} \frac{1}{|l|^N (1 + |j + 1 - l|^N)} \leq \frac{1}{(1 + |\frac{j+1}{2}|^N)} \sum_{l=1}^{\infty} \frac{1}{l^N}$$

combined with a similar estimate for the remaining range of summation gives the desired inequality.

By the above, we have

$$|a_{n,j}| = |a_{n,n-(n-j)}| \leq \frac{C}{K^N |n - j + 1|^N}. \tag{4.10}$$

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IV. Note that

$$\sum_{j=0}^{n-1} |a_{n,j}|^2 \leq \frac{C^2}{K^{2N}} \sum_{j=0}^{n-1} \frac{1}{|n-j+1|^{2N}} \leq \frac{C^2}{K^{2N}} \sum_{j=2}^{n+1} \frac{1}{j^{2N}}$$

$$\leq \frac{C}{K^{2N}} \leq C.$$

Using (4.5), we can estimate the localization of the $h_n(t)$.

$$\int |t|^p |h_n(t)|^2 dt \leq ||h_n||^2_{L^2(\mathbb{R})}$$

$$= |\langle \varphi_n - \sum_{j=0}^{n-1} a_{n,j}\varphi_j, \varphi_n - \sum_{j=0}^{n-1} a_{n,j}\varphi_j \rangle|$$

$$\leq 1 + 2\sum_{j=0}^{n-1} |a_{n,j}||\langle \varphi_n, \varphi_j \rangle| + \sum_{j,k=0}^{n-1} |a_{n,j}||a_{n,k}||\langle \varphi_j, \varphi_k \rangle|$$

$$\leq 1 + 2\left(\sum_{j=0}^{n-1} |a_{n,j}|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} |\langle \varphi_j, \varphi_n \rangle|^2\right)^{\frac{1}{2}} + \sum_{j,k=0}^{n-1} |a_{n,j}||a_{n,k}|\frac{C}{1+K^N|j-k|^N}$$

$$\leq 1 + 2\left(\sum_{j=0}^{n-1} |a_{n,j}|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} \frac{C^2}{(1+K^N|j-n|^N)^2}\right)^{\frac{1}{2}} + C\sum_{j=0}^{n-1} |a_{n,j}|^2$$

$$\leq C,$$

where the penultimate inequality used lemma 4.8. Thus,

$$\int |t|^p |h_n(t)|^2 dt \leq C, \tag{4.11}$$

where C is independent of n, T, K.

V. We now estimate the localization of the $\widehat{h_n}(t)$. Using (4.10) we have

$$\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq \left(\int |\gamma - nK|^q |\widehat{\varphi}(\gamma - nK)|^2 d\gamma\right)^{\frac{1}{2}} + \left(\int |\gamma - nK|^q \left|\sum_{j=0}^{n-1} a_{n,j} \widehat{\varphi}(\gamma - jK)\right|^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \sum_{j=0}^{n-1} |a_{n,j}| \left(\int |\gamma - K(n-j)|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C \sum_{j=0}^{n-1} |a_{n,j}| \left(\sum_{l=0}^{q} |K(n-j)|^{(q-l)} \int |\gamma|^l |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C \sum_{j=0}^{n-1} |a_{n,j}| |n-j|^{q/2} \left(K^q \sup_{0 \leq l \leq q} \int |\gamma|^l |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C K^{q/2} \sum_{j=0}^{n-1} |a_{n,j}| |n-j|^{q/2}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C K^{q/2} \sum_{j=0}^{n-1} |a_{n,j}| |n-j+1|^{q/2}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C \frac{K^{q/2}}{K^N} \sum_{j=0}^{n-1} |n-j+1|^{-N} |n-j+1|^{q/2}$$

$$\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \frac{C}{K^{N-(q/2)}}$$

$$\leq C.$$

Thus,

$$\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C, \tag{4.12}$$

where C is independent of n, T, K.

VI. It remains to normalize the h_n. First note that the norms of the h_n are bounded away from 0, since by (4.8) and (4.10)

$$1 = ||\varphi_n||_{L^2(\mathbb{R})} \leq ||h_n||_{L^2(\mathbb{R})} + ||\varphi_n - h_n||_{L^2(\mathbb{R})}$$

$$\leq ||h_n||_{L^2(\mathbb{R})} + \sum_{j=0}^{n-1} |a_{n,j}|$$

$$\leq ||h_n||_{L^2(\mathbb{R})} + \frac{1}{K^N} \sum_{j=0}^{n-1} \frac{C}{|n-j+1|^N}$$

$$\leq ||h_n||_{L^2(\mathbb{R})} + \frac{C}{K^N} \sum_{j=2}^{\infty} \frac{1}{j^N}$$

$$\leq ||h_n||_{L^2(\mathbb{R})} + \frac{C}{K^N}.$$

Take K large enough so that $\frac{C}{K^N} < \frac{1}{2}$. From this it follows that

$$\frac{1}{2} \leq ||h_n||_{L^2(\mathbb{R})}. \tag{4.13}$$

In view of this and the discussion following (4.9), it is clear how to define the constant K_0 in the statement of the theorem.

Finally, let $s_n(t) = h_n(t)/||h_n||_{L^2(\mathbb{R})}$. By (4.11), (4.12), and (4.13) we see that (4.3) and (4.4) hold. Also, it is clear that (4.5) implies (4.2). □

Lemma 4.13. *Assume $\frac{1}{p} + \frac{1}{q} = 1$ and $q \in \mathbb{N}$. There exists a constant C and a constant K_0 such that for every $K > K_0$ and $T \in \mathbb{N}$ satisfying*

$$T^{2/p} \in \mathbb{N} \quad \text{and} \quad T^{2/q} \in \mathbb{N},$$

there exists a finite orthonormal set, $S = S(T, K) = \{s_{m,n}\}$, $(0 \leq m < T^{2/q}$ and $0 \leq n < T^{2/p})$ of cardinality T^2, satisfying

$$\text{supp } s_{m,n} \subseteq \left[\frac{1}{2}T^{2/p}, 2T^{2/p}K\right], \tag{4.14}$$

$$\left(\int |t - Kn - T^{(2/p)}K|^p |s_{m,n}(t)|^2 dt\right)^{\frac{1}{2}} \leq C, \tag{4.15}$$

and

$$\left(\int |\gamma - Km|^q |\widehat{s_{m,n}}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C. \tag{4.16}$$

Proof. **I.** Regarding the hypotheses of the lemma, note that there exist infinitely many $T \in \mathbb{N}$ for which

$$T^{2/p} \in \mathbb{N} \quad \text{and} \quad T^{2/q} \in \mathbb{N}.$$

To see this, note that $q \in \mathbb{N}$ implies $\frac{a}{b} = p \in \mathbb{Q}$ with $a, b \in \mathbb{N}$. Now, for every $N \in \mathbb{N}$, we can define $T_N = N^{aq} \in \mathbb{N}$, so that $T_N^{2/p} = N^{2bq} \in \mathbb{N}$ and $T_N^{2/q} = N^{2a} \in \mathbb{N}$.

II. Let $S_0(T^{q/2}, K) = \{s_m(t)\}_{m=0}^{T^{2/q}-1}$ be the system from the previous lemma. Define

$$s_{m,n}(t) = s_m(t - nK - T^{(2/p)}K) \quad \text{for} \quad 0 \leq m < T^{2/q} \quad \text{and} \quad 0 \leq n < T^{2/p}.$$

Now, (4.15) and (4.16) hold by the previous lemma. Also, note that by (4.2)

$$\text{supp } s_{m,n} \subseteq \left[nK + T^{2/p}K - 1, nK + T^{2/p}K + 1\right],$$

so that all the $s_{m,n}$ are supported in

$$\left[T^{2/p}K - 1, (T^{2/p} - 1)K + T^{2/p}K + 1\right] \subseteq \left[\frac{1}{2}T^{2/p}, 2KT^{2/p}\right].$$

$\square$

Lemma 4.14. *Let $\{\varphi_m\}$ be as in the proof of lemma 4.12 and let*

$$\varphi_{m,n}(t) = \varphi_m(t - nK - KT^{2/p}).$$

Then

$$\left|\int \gamma^l \widehat{\varphi_{m,n}}(\gamma)\overline{\widehat{\varphi_{j,k}}(\gamma)}d\gamma\right| \leq \frac{C|K(m+j)|^l}{|K(j-m)|^N + |K(k-n)|^N},$$

for all $j, m \geq 0$ satisfying $(j, m) \neq (0, 0)$.

Proof. Throughout the proof, C will denote a constant independent of T, K. Recall that $K \in \mathbb{N}$. So,

$$\left| \int \gamma^l \widehat{\varphi_{m,n}}(\gamma) \overline{\widehat{\varphi_{j,k}}(\gamma)} d\gamma \right|$$

$$= \left| \int \gamma^l e^{-2\pi i(nK + T^{2/p}K)\gamma} \widehat{\varphi}(\gamma - mK) e^{2\pi i(kK + T^{2/p}K)\gamma} \overline{\widehat{\varphi}(\gamma - jK)} d\gamma \right|$$

$$= \left| \int \gamma^l e^{-2\pi i(n-k)K\gamma} \widehat{\varphi}(\gamma - mK) \overline{\widehat{\varphi}(\gamma - jK)} d\gamma \right|$$

$$= \left| \int (\gamma + mK)^l e^{-2\pi i(n-k)K(\gamma + mK)} \widehat{\varphi}(\gamma) \overline{\widehat{\varphi}(\gamma - (j-m)K)} d\gamma \right|$$

$$= \left| \int (\gamma + mK)^l e^{-2\pi i(n-k)K\gamma} \widehat{\varphi}(\gamma) \overline{\widehat{\varphi}(\gamma - (j-m)K)} d\gamma \right|$$

$$\leq \sum_{d=0}^{l} |C_d| |mK|^{l-d} \left| \int \gamma^d \widehat{\varphi}(\gamma) \overline{\widehat{\varphi}(\gamma - (j-m)K)} e^{-2\pi i(n-k)K\gamma} d\gamma \right|$$

$$= \sum_{d=0}^{l} |C_d| |mK|^{l-d} \left| \mathcal{S}_{\widehat{\varphi}}[\gamma^d \widehat{\varphi}]((j-m)K, (n-k)K) \right|$$

$$\leq C |(m+j+1)K|^l \sum_{d=0}^{l} \left| \mathcal{S}_{\widehat{\varphi}}[\gamma^d \widehat{\varphi}]((j-m)K, (n-k)K) \right|$$

$$\leq \frac{C |(m+j+1)K|^l}{1 + |j-m|^N K^N + |n-k|^N K^N},$$

where we used lemma 4.11 and the fact that $\varphi \in \mathcal{S}(\mathbb{R})$ in the last inequality. $\quad\square$

Lemma 4.15. *Let $S = S(K, T)$ be the system from lemma 4.13. Also, assume $q \in \mathbb{N}$. For every $f \in \text{span } S$ one has*

$$\left| \int \gamma^l |\widehat{f}(\gamma)|^2 d\gamma \right| \leq CK^l T^{(2l)/q} ||f||_{L^2(\mathbb{R})}^2 \quad \text{for} \quad l = 0, 1, \cdots, q. \tag{4.17}$$

Proof. Throughout the proof, C will denote various constants which are independent of T, K. Assume $f \in \text{span } S$. Since the $\varphi_{m,n}$ are linearly independent and span $S(T, K)$, we have

$$f(t) = \sum_{m=0}^{(T^{(2/q)}-1)} \sum_{n=0}^{(T^{(2/p)}-1)} d_{m,n} \varphi_{m,n}(t),$$

for some finite sequence of constants $\{d_{m,n}\}$. The disjointness of the supports of $\varphi_{m,n}$ and $\varphi_{j,k}$ for $n \neq k$ implies

$$||f||^2_{L^2(\mathbb{R})} = \left| \sum_{m=0}^{T^{\frac{2}{q}}-1} \sum_{n=0}^{T^{\frac{2}{p}}-1} \sum_{j=0}^{T^{\frac{2}{q}}-1} \sum_{k=0}^{T^{\frac{2}{p}}-1} d_{m,n}\overline{d_{j,k}}\langle\varphi_{m,n},\varphi_{j,k}\rangle \right|$$

$$= \left| \sum_{m=0}^{T^{\frac{2}{q}}-1} \sum_{n=0}^{T^{\frac{2}{p}}-1} \sum_{j=0}^{T^{\frac{2}{q}}-1} d_{m,n}\overline{d_{j,n}}\langle\varphi_{m,n},\varphi_{j,n}\rangle \right|$$

$$= \left| \sum_{m,n} |d_{m,n}|^2 + \sum_{m\neq j}\sum_n d_{m,n}\overline{d_{j,n}}\langle\varphi_{m,n},\varphi_{j,n}\rangle \right|$$

$$\leq \sum_{m,n} |d_{m,n}|^2 + \sum_{m\neq j}\sum_n |d_{m,n}||d_{j,n}||\langle\varphi_{m,n},\varphi_{j,n}\rangle|.$$

That is,

$$||f||^2_{L^2(\mathbb{R})} \leq \sum_{m,n} |d_{m,n}|^2 + \sum_{m\neq j}\sum_n |d_{m,n}||d_{j,n}||\langle\varphi_{m,n},\varphi_{j,n}\rangle|. \tag{4.18}$$

I. Using lemma 4.8

$$\sum_{m\neq j}\sum_n |d_{m,n}||d_{j,n}||\langle\varphi_{m,n},\varphi_{j,n}\rangle| \leq \sum_k \sum_{m\neq j} |d_{m,k}||d_{j,k}|\frac{C}{1+K^N|j-m|^N}$$

$$\leq \sum_{m\neq j}\sum_n |d_{m,n}||d_{j,n}|\frac{C}{K^N|m-j|^N}$$

$$\leq \frac{C}{K^N}\sum_{m,n} |d_{m,n}|^2.$$

Therefore, applying the triangle inequality to (4.18) gives:

$$\sum_{j,k} |d_{j,k}|^2 \leq ||f||^2_{L^2(\mathbb{R})} + \sum_{m\neq j}\sum_n |d_{m,k}||d_{j,k}||\langle\varphi_{m,k},\varphi_{j,k}\rangle|$$

$$\leq ||f||^2_{L^2(\mathbb{R})} + \frac{C}{K^N}\sum_{m,n} |d_{m,n}|^2.$$

Take K large enough so that $\frac{C}{K^N} < \frac{1}{2}$. Thus

$$\sum_{m,n} |d_{m,n}|^2 \leq 2||f||^2_{L^2(\mathbb{R})}. \tag{4.19}$$

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II. Using lemma 4.14 and (4.19) we have

$$\left| \int \gamma^l |\widehat{f}(\gamma)|^2 d\gamma \right| = \left| \sum_{m,n} \sum_{j,k} d_{m,n} \overline{d_{j,k}} \int \gamma^l \widehat{\varphi_{m,n}}(\gamma) \overline{\widehat{\varphi_{j,k}}(\gamma)} d\gamma \right|$$

$$\leq \sum_{j=0}^{T^{2/q}-1} \sum_{m=0}^{T^{2/q}-1} \sum_{k=0}^{T^{2/p}-1} \sum_{n=0}^{T^{2/p}-1} |d_{m,n}||d_{j,k}| \left| \int \gamma^l \widehat{\varphi_{m,k}}(\gamma) \overline{\widehat{\varphi_{j,k}}(\gamma)} d\gamma \right|$$

$$\leq C \sum_{j=0}^{T^{2/q}-1} \sum_{m=0}^{T^{2/q}-1} \sum_{k=0}^{T^{2/p}-1} \sum_{n=0}^{T^{2/p}-1} |d_{m,k}||d_{j,k}| \frac{|K(j+m+1)|^l}{1 + |K(k-n)|^N + |K(j-m)|^N}$$

$$\leq C(2K)^l T^{(2l)/q} \sum_n \sum_j \sum_m \sum_k \frac{|d_{m,n}||d_{j,k}|}{1 + |k-n|^N + |j-m|^N}$$

$$\leq CK^l T^{(2l)/q} \left(\sum_{m,n} |d_{m,n}|^2 \right)$$

$$\leq CK^l T^{(2l)/q} ||f||_{L^2(\mathbb{R})}^2.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.3   A (p,q) version of Bourgain's theorem

We are now ready to prove our main result, theorem 4.3. The proof follows that of Bourgain, [11] which, in turn, is based on an idea of W. Rudin, [40], for constructing bounded bases for the Hardy space $H^2(\mathbb{C}^n)$.

Proof of Theorem 4.3. Throughout the proof C will denote various constants which are independent of n, T_n, K, and any indices.

Let $\{f_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be sequence which is dense in the unit sphere of $L^2(\mathbb{R})$. Fix $K > \max\{2, K_0\}$, where K_0 is the same as in lemma 4.13. The orthonormal basis we construct will be of the form $\bigcup_{n=1}^\infty B_n$ where B_n is a finite orthonormal set of C^∞ compactly supported functions. We shall construct the B_n inductively.

I. Suppose $B_1, \ldots, B_{n-1}$ are already defined such that B_j is a finite orthonormal set of C^∞ compactly supported functions and the elements of B_j and B_k are mutually orthonormal. Define $F_n = f_n - P_{[B_1, \ldots, B_{n-1}]} f_n$, where $P_{[B_1, \ldots, B_{n-1}]}$ is the orthonormal projection onto

$$[B_1, \ldots, B_{n-1}] \equiv \operatorname{span} \bigcup_{l=1}^{n-1} B_{n-1}.$$

For the base case of the induction we simply let $F_1 = f_1$. Using F_n, we now prepare the way to construct B_n.

i. Note that

$$||F_n||^2_{L^2(\mathbb{R})} \leq 1. \tag{4.20}$$

To see this, we shall first show $F_n \perp P_{[B_1, \cdots, B_{n-1}]} f_n$.

$$\begin{aligned}
\langle F_n, P_{[B_1, \cdots, B_{n-1}]} f_n \rangle &= \langle f_n - P_{[B_1, \cdots, B_{n-1}]} f_n, P_{[B_1, \cdots, B_{n-1}]} f_n \rangle \\
&= \langle f_n - P_{[B_1, \cdots, B_{n-1}]} f_n, P_{[B_1, \cdots, B_{n-1}]} (P_{[B_1, \cdots, B_{n-1}]} f_n) \rangle \\
&= \langle P_{[B_1, \cdots, B_{n-1}]} f_n - P_{[B_1, \cdots, B_{n-1}]} (P_{[B_1, \cdots, B_{n-1}]} f_n), P_{[B_1, \cdots, B_{n-1}]} f_n \rangle \\
&= \langle 0, P_{[B_1, \cdots, B_n]} f_n \rangle = 0.
\end{aligned}$$

Since $||f_n||^2_{L^2(\mathbb{R})} = 1$ it follows from the definition of F_n and the orthogonality proven above that

$$1 = ||F_n||^2_{L^2(\mathbb{R})} + ||P_{[B_1, \ldots, B_{n-1}]} f_n||^2_{L^2(\mathbb{R})}.$$

Thus $||F_n||_{L^2(\mathbb{R})} \leq 1$.

ii. Since f_n and all elements of the B_j are C^∞ and compactly supported it follows that F_n is also C^∞ and compactly supported.

Choose $T_n > 2$ large enough so that

$$\operatorname{supp} F_n \subseteq \left[-\frac{1}{2} T_n^{2/p}, \frac{1}{2} T_n^{2/p} \right], \tag{4.21}$$

$$\text{supp } b \subseteq [-\frac{1}{2}T_n^{2/p}, \frac{1}{2}T_n^{2/p}] \quad \text{for all} \quad b \in \bigcup_{j=1}^{n-1} B_j, \tag{4.22}$$

and

$$\int |\gamma|^l |\widehat{F_n}(\gamma)|^2 d\gamma \leq T_n^{2l/q}, \quad \text{for} \quad l = 0, 1, \ldots, q. \tag{4.23}$$

Note that we have no difficulties with the case $l = 0$ in (4.23), since $||\widehat{F_n}||_{L^2(\mathbb{R})} \leq 1$ by Parseval's theorem and (4.20).

II. Let

$$S = S(T_n, K) = \{s_{j,k}^n : 0 \leq j < T_n^{(2/p)} \text{ and } 0 \leq k < T_n^{(2/q)}\}$$

be the system from lemma 4.13. We will switch from the double indexing to single indexing and enumerate the elements of the system as $\{s_{n,l}\}_{l=1}^{T_n^2}$. If l_1, l_2 are the indices for which $s_{n,l} = s_{l_1,l_2}^n$, let

$$x(s_{n,l}) = Kl_1 + T_n^{(2/p)}K \quad \text{and} \quad y(s_{n,l}) = Kl_2,$$

so that by lemma 4.13

$$\left(\int |t - x(s_{n,j})|^p |s_{n,j}(t)|^2 dt \right)^{\frac{1}{2}} \leq C \tag{4.24}$$

and

$$\left(\int |\gamma - y(s_{n,j})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} \leq C. \tag{4.25}$$

Note that

$$T_n^{(2/p)} K \leq x(s_{n,j}) \leq 2KT_n^{2/p} \quad \text{and} \quad 0 \leq y(s_{n,j}) \leq KT_n^{2/q}. \tag{4.26}$$

Let $0 < \Theta < \frac{1}{4}$ be fixed throughout the proof. Choosing Θ carefully (small enough) will allow one to estimate precise values of the constant C in theorem 4.3. We shall not consider this issue during the proof, and shall content ourselves with finding some, possibly large, constant C that works. Define

$$b_{n,1}(t) = \frac{\Theta}{T_n} F_n(t) + \alpha_{n,1} s_{1,n}(t)$$

$$b_{n,2}(t) = \frac{\Theta}{T_n} F_n(t) + \sigma_{n,1} s_{n,1}(t) + \alpha_{n,2} s_{n,2}(t)$$

$$\vdots \qquad\qquad \vdots$$

$$b_{n,T_n^2}(t) = \frac{\Theta}{T_n} F_n(t) + \sigma_{n,1} s_{n,1}(t) + \cdots + \sigma_{k,T_n^2-1} s_{n,T_n^2-1}(t) + \alpha_{n,T_n^2} s_{n,T_n^2}(t),$$

where the $\sigma_{n,j}$ and $\alpha_{n,j}$ are chosen to ensure that $\{b_{n,j}\}_{j=1}^{T_n^2}$ is orthonormal.

i. The choice of $\sigma_{n,j}$ and $\alpha_{n,j}$ implies that

$$|1 - \alpha_{n,j}| \le \frac{\Theta}{T_n} \quad \text{for} \quad j = 1, 2, \cdots, T_n^2, \tag{4.27}$$

and

$$|\sigma_{n,j}| \le \frac{\Theta}{T_n^2} \quad \text{for} \quad j = 1, 2, \cdots, T_n^2 - 1. \tag{4.28}$$

To see this, first note that $\{F_n\} \bigcup S(T_n, K)$ is an orthogonal set. Therefore, the assumption that $\{B_{n,j}\}_{j=1}^{T_n^2}$ is orthonormal implies that for $l = 1, 2, \cdots, T_n^2$ we have

$$0 = \frac{\Theta^2}{T_n^2} ||F_n||_{L^2(\mathbb{R})}^2 + \sigma_{n,1}^2 + \cdots + \sigma_{n,l-1}^2 + \sigma_{n,l} \alpha_{n,l} \tag{4.29}$$

and for $l = 1, 2, \cdots, T_n^2 - 1$

$$\alpha_{n,l}^2 = 1 - \frac{\Theta^2}{T_n^2} ||F_n||_{L^2(\mathbb{R})}^2 - \sigma_{n,1}^2 - \cdots - \sigma_{n,l-1}^2. \tag{4.30}$$

ii. Using (4.29) and (4.30) we shall now prove (4.27) and (4.28) by induction. The case $j = 1$ of (4.27) holds since (4.30) implies,

$$1 = \frac{\Theta^2}{T_n^2} ||F_n||_{L^2(\mathbb{R})}^2 + \alpha_{n,1}^2.$$

Since $2 < T_n$ and $\Theta < \frac{1}{4}$, we may choose $0 < \alpha_{n,1} \le 1$. So,

$$|1 - \alpha_{n,1}| \le |1 - \alpha_{n,1}^2| \le \frac{\Theta^2}{T_n^2} \le \frac{\Theta}{T_n}.$$

Using this, the case $j = 1$ of (4.28) now follows since by (4.29)

$$0 = \frac{\Theta^2}{T_n^2} ||F_n||_{L^2(\mathbb{R})}^2 + \alpha_{n,1} \sigma_{n,1}$$

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which implies

$$|\sigma_{n,1}| \leq \frac{\Theta^2}{T_n^2}\frac{1}{|\alpha_{n,1}|} \leq \frac{\Theta^2}{T_n^2}\frac{1}{(1-\Theta/T_n)} \leq \frac{\Theta}{T_n^2}.$$

The last inequality holds because $\Theta < \frac{1}{4}$ and $T_n > 2$.

iii. Next, assume $|\sigma_{n,j}| \leq \frac{\Theta}{T_n^2}$ holds for $j < l$. We may once again choose $0 < \alpha_{n,l} \leq 1$. Since the cardinality of $S(T_n, K)$ is T_n^2,

$$|1 - \alpha_{n,l}| \leq |1 - \alpha_{n,l}^2| \leq \frac{\Theta^2}{T_n^2} + \sum_{j=1}^{l-1}\sigma_{n,j}^2 \leq \frac{\Theta^2}{T_n^2} + T_n^2\frac{\Theta^2}{T_n^4} \leq 2\frac{\Theta^2}{T_n^2} \leq \frac{\Theta}{T_n},$$

and (4.27) follows by induction. For (4.28), assume that $|\sigma_{n,j}| \leq \frac{\Theta}{T_n^2}$ for $j < l$ and $|1 - \alpha_{n,l}| \leq \frac{\Theta}{T_n}$. Thus,

$$|\sigma_{n,l}| \leq \frac{1}{|\alpha_{n,l}|}\left(\frac{\Theta^2}{T_n^2}||F_n||_{L^2(\mathbb{R})}^2 + \sum_{j=1}^{l-1}\sigma_{n,j}^2\right) \leq \frac{1}{(1-\Theta/T_n)}\left(2\frac{\Theta^2}{T_n^2}\right) \leq \frac{\Theta}{T_n^2},$$

and (4.28) holds by induction.

III. By (4.27) and (4.28), we know that $\sigma_{n,j}$ is close to zero and $\alpha_{n,j}$ is close to one. Thus, we expect to have $b_{n,j}$ close to $s_{n,j}$. In fact,

$$||b_{n,j} - s_{n,j}||_{L^2(\mathbb{R})} \leq 3\frac{\Theta}{T_n}. \tag{4.31}$$

To see this, note that by (4.27) and (4.28)

$$\begin{aligned}
||b_{n,j} - s_{n,j}||_{L^2(\mathbb{R})} &\leq ||b_{n,j} - \alpha_{n,j}s_{n,j}||_{L^2(\mathbb{R})} + |1 - \alpha_{n,j}| \\
&\leq ||b_{n,j} - \alpha_{n,j}s_{n,j}||_{L^2(\mathbb{R})} + \frac{\Theta}{T_n} \\
&= \left(\frac{\Theta^2}{T_n^2}||F_n||_{L^2(\mathbb{R})}^2 + \sum_{k=1}^{j-1}|\sigma_{n,k}|^2\right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \\
&\leq \left(\frac{\Theta^2}{T_n^2} + \left(T_n^2\frac{\Theta^2}{T_n^4}\right)\right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \leq 3\frac{\Theta}{T_n}.
\end{aligned}$$

IV. Let us now prove that

$$\Delta_p(b_{n,j}) \leq \left(\int |t - x(s_{n,j})|^p|s_{n,j}(t)|^2dt\right)^{\frac{1}{2}} + CK^{p/2}\Theta. \tag{4.32}$$

Using (4.26),(4.31) and the fact that the $b_{n,j}$ are supported in $[-\frac{1}{2}T_n^{2/p}, 2T_n^{2/p}K]$ (since F_n and $s_{n,j}$ are), we have

$$\left(\int |t - x(s_j^n)|^p |b_{n,j}(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\leq \left(\int |t - x(s_j^n)|^p |b_{n,j} - s_{n,j}(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int |t - x(s_j^n)|^p |s_{n,j}(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\leq |2T_n^{2/p}K + 2KT_n^{2/p}|^{p/2}||b_{n,j} - s_j^n||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_{n,j}(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\leq CK^{p/2}T_n||b_{n,j} - s_{n,j}||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_{n,j}(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\leq CK^{p/2}\Theta + \left(\int |t - x(s_j^n)|^p |s_{n,j}(t)|^2 dt\right)^{\frac{1}{2}}.$$

Thus, by (4.15) and the definition of $x(s_j^n)$ we have

$$\Delta_p(b_{n,j}) \leq C + CK^{(p/2)}\Theta. \tag{4.33}$$

V. Here we show that

$$\Delta_q(\widehat{b_{n,j}}) \leq \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C\Theta K^{(q/2)}.$$

i. First we show that

$$\left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C\Theta K^{(q/2)}. \tag{4.34}$$

This follows from (4.23) and (4.26) since

$$\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{F_n}(\gamma)|^2 d\gamma \leq \int \sum_{k=0}^{q} c_k |\gamma|^k |y(\widehat{s_{n,j}})|^{q-k} |\widehat{F_n}(\gamma)|^2 d\gamma$$

$$\leq C \sum_{k=0}^{q} |y(\widehat{s_{n,j}})|^{q-k} \int |\gamma|^k |\widehat{F_n}(\gamma)|^2 d\gamma$$

$$\leq C \sum_{k=0}^{q} |y(\widehat{s_{n,j}})|^{q-k} K^k T_n^{(2k)/q}$$

$$\leq C \sum_{k=0}^{q} |3K T_n^{2/q}|^{q-k} K^k T_n^{(2k)/q}$$

$$\leq C(3K)^q \sum_{k=0}^{q} T_n^{(2-(2k)/q)} T_n^{(2k)/q}$$

$$= C K^q T_n^2.$$

ii. Next, we show that

$$\left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{b_{n,j}}(\gamma) - \widehat{s_{n,j}}(\gamma) - \frac{\Theta}{T_n} \widehat{F_n}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} \leq C \Theta K^{(q/2)}.$$

Let $\Psi(\gamma) = \widehat{b_{n,j}}(\gamma) - \widehat{s_{n,j}}(\gamma) - \frac{\Theta}{T_n} \widehat{F_n}(\gamma)$. Note that Ψ is in the span of $S(T_n, K)$.

Thus, using (4.26), lemma 4.15,(4.28) and that $q \in 2\mathbb{N}$

$$\int |\gamma - y(\widehat{s_{n,j}})|^q |\Psi(\gamma)|^2 d\gamma = \int (\gamma - y(\widehat{s_{n,j}}))^q |\Psi(\gamma)|^2 d\gamma$$

$$= \left| \sum_{k=0}^{q} c_k (y(\widehat{s_{n,j}}))^{q-k} \int \gamma^k |\Psi(\gamma)|^2 d\gamma \right|$$

$$\leq C3^q \sum_{k=0}^{q} K^{q-k} T_n^{(2-2k/q)} \left| \int \gamma^k |\Psi(\gamma)|^2 d\gamma \right|$$

$$\leq C \sum_{k=0}^{q} K^{q-k} T_n^{(2-2k/q)} K^k T_n^{2k/q} ||\Psi||^2_{L^2(\mathbb{R})}$$

$$\leq CK^q ||\Psi||^2_{L^2(\mathbb{R})} T_n^2$$

$$\leq CK^q \left(\sum_l |\sigma_{n,l}|^2 \right) T_n^2$$

$$\leq CK^q T_n^2 \left(T_n^2 \frac{\Theta^2}{T_n^4} \right)$$

$$\leq CK^q \Theta^2,$$

and the desired estimate follows. Note that this is the only step where we have made use of $q \in 2\mathbb{N}$ (as opposed to $q \in \mathbb{N}$).

iii. Combining the estimates from **i**. and **ii**. we have

$$\Delta_q(\widehat{b_{n,j}}) \le \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |b_{n,j}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}}$$

$$\le \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}}$$

$$+ \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{b_{n,j}}(\gamma) - \widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}}$$

$$\le \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} + \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\frac{\Theta}{T_n} \widehat{F_n}(\gamma)|^2 \gamma \right)^{\frac{1}{2}}$$

$$+ \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{b_{n,j}}(\gamma) - \widehat{s_{n,j}}(\gamma) - \frac{\Theta}{T_n} \widehat{F_n}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}}$$

$$\le \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} + \Theta C K^{(q/2)} + \Theta C K^{(q/2)}$$

$$= \left(\int |\gamma - y(\widehat{s_{n,j}})|^q |\widehat{s_{n,j}}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} + C \Theta K^{(q/2)}.$$

Thus,

$$\Delta_q(\widehat{b_{n,j}}) \le C + C K^{(q/2)} \Theta. \tag{4.35}$$

VI. Having shown that all the elements of $B = \bigcup_{j=1}^{\infty} b_{n,j}$ have the desired localization, it only remains to show that B is complete. To see this, note that

$$||P_{[B_1,\cdots,B_k]}f_k||^2_{L^2(\mathbb{R})} = ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}f_k||^2_{L^2(\mathbb{R})}$$

$$= ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}(F_k + P_{[B_1,\cdots,B_{k-1}]}f_k)||^2_{L^2(\mathbb{R})}$$

$$= 1 - ||F_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}F_k||^2_{L^2(\mathbb{R})}$$

$$= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \sum_{j=1}^{(T_k)^2} |\langle F_k, b_{k,j}\rangle|^2$$

$$= 1 - ||F_k||^2_{L^2(\mathbb{R})} + (T_k)^2 \left(\frac{\Theta}{T_k}||F_k||^2_{L^2(\mathbb{R})}\right)^2$$

$$= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \Theta^2||F_k||^4_{L^2(\mathbb{R})}$$

$$\geq \Theta^2,$$

To see the final inequality, let $h(t) = 1 - t^2 + a^2 t^4$ be defined on $[0,1]$, where $0 < a < \frac{1}{4}$ is fixed. It is easy to see that $h(t) \geq a^2$. Since $||F_n||_{L^2(\mathbb{R})} \leq 1$ and $\Theta < \frac{1}{4}$, the last step follows.

Now, suppose $y \in L^2(\mathbb{R})$ satisfies $\langle y, b\rangle = 0$ for all $b \in B$. If y is not identically zero, then $\tilde{y} = y/||y||_{L^2(\mathbb{R})}$ is in the unit sphere of $L^2(\mathbb{R})$ and there exists f_{n_k} such that $f_{n_k} \to \tilde{y}$ in $L^2(\mathbb{R})$ as $k \to \infty$. Thus,

$$0 < \Theta \leq ||P_{[B_1,\cdots,B_{n_k}]}f_{n_k}||_{L^2(\mathbb{R})} \leq ||P_{[B]}f_{n_k}||_{L^2(\mathbb{R})} \to ||P_{[B]}\tilde{y}||_{L^2(\mathbb{R})} = 0,$$

where the limit is as $k \to \infty$. This contradiction shows that B is complete and hence is an orthonormal basis. $\square$

Chapter 5

Orthonormalizing Coherent States

The proofs of Bourgain's theorem and our (p, q) generalization are both rather technical. Both proofs start by constructing finite, orthonormal, well-localized systems of functions in $L^2(\mathbb{R})$. Since these systems are not complete, one needs to carefully take linear combinations of these finite systems with a dense set of functions to obtain an orthonormal basis.

Question 5.1. *Why are the proofs of Bourgain's theorem and our (p, q) generalization necessarily so complicated?*

To answer this, let's begin by taking another look at Bourgain's theorem. A naive alternate idea for constructing a basis of the type in Bourgain's theorem is to start with a complete set of functions which already have the desired uniform localization and to orthonormalize them to obtain a basis.

For example, it follows from fact 3.3 that if $g(t) = 2^{1/4}e^{-\pi t^2}$, then the elements of the Gabor system $\mathcal{G}(g, 1, 1)$ all satisfy

$$\Delta(g_{m,n}) = \frac{1}{2\sqrt{\pi}} \quad \text{and} \quad \Delta(\widehat{g_{m,n}}) = \frac{1}{2\sqrt{\pi}}. \tag{5.1}$$

Moreover, $\mathcal{G}(g, 1, 1)$ is complete, but is not a Riesz basis for $L^2(\mathbb{R})$, or even a frame for $L^2(\mathbb{R})$. This follows (e.g., see [19], [22], [9]) from

Theorem 5.2. *Let* $g(t) = 2^{1/4}e^{-\pi t^2}$.

1. $\mathcal{G}(g, a, b)$ *is a frame for* $L^2(\mathbb{R})$, *if* $ab < 1$.

2. $\mathcal{G}(g, a, b)$ *is incomplete in* $L^2(\mathbb{R})$, *if* $ab > 1$.

3. $\mathcal{G}(g, a, b)$ *complete in* $L^2(\mathbb{R})$, *but not a frame, if* $ab = 1$. *Moreover,* $\mathcal{G}(g, a, b)$ *remains complete if any element is removed, but is no longer complete if two elements are removed.*

Thus, if one orthonormalizes $\mathcal{G}(g, 1, 1)$ with respect to some indexing of $\mathbb{Z} \times \mathbb{Z}$ then one obtains a new system, $\mathcal{O}(g, 1, 1)$, which is an orthonormal basis for $L^2(\mathbb{R})$. Since all the elements of $\mathcal{G}(g, 1, 1)$ satisfy (5.1) it is plausible that the elements of the orthonormalized system also have uniformly bounded time and frequency variances. However, it is difficult to estimate the variances here because $\mathcal{G}(g, 1, 1)$ is not a Riesz basis. The difficulties arise specifically because it is difficult to estimate the spectra of the Grammian matrices which arise in the Gram-Schmidt orthogonalization process. Theorem 2.10 sheds light on this.

On the other hand, if one chooses $ab > 1$ then it follows from the Ron-Shen duality theorem that $\mathcal{G}(g, a, b)$ is a Riesz basis for its span. By theorem 5.2, $\mathcal{G}(g, a, b)$ is incomplete. Thus, if one orthonormalizes $\mathcal{G}(g, a, b)$, the resulting system $\mathcal{O}(g, a, b)$ is not complete. However, since $\mathcal{G}(g, a, b)$ is a Riesz basis for its span, we show it is possible to estimate the variances of the elements of $\mathcal{O}(g, a, b)$.

In summary,

- $\mathcal{O}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$, but it is difficult to estimate the time and frequency variances of the elements in $\mathcal{O}(g, 1, 1)$.

- If $ab > 1$ then $\mathcal{O}(g, a, b)$ is not complete in $L^2(\mathbb{R})$, but one can derive variance estimates.

This explains why the proof of Bourgain's theorem proceeds as it does. Since the idea outlined in the first bullet is difficult to carry out, Bourgain's proof essentially uses the second idea and adds in completeness by cleverly taking linear combinations with a dense sequence.

Note that in Bourgain's theorem, one doesn't work with the Gaussian, but instead with a compactly supported function. The compact support makes the orthonormalization easier. In this chapter we shall focus on orthonormalizing $\mathcal{G}(g, 2, 2)$, where g is the Gaussian.

5.1 Orthonormalizing coherent states

Consider the indexing of $2\mathbb{Z} \times 2\mathbb{Z}$ which begins

$$(0,0), (2,2), (0,2), (-2,2), (-2,0), (-2,-2), (0,-2), (2,-2), (2,0),$$
$$(4,4), (0,4), (-4,4), (-4,0), (-4,-4), (0,-4), \cdots$$

and continues to spiral outwards in this manner. Let $\mathcal{O}(g, 2, 2)$ be the system which results when $\mathcal{G}(g, 2, 2)$ is orthonormalized in the above order. We examine the time and frequency localization of the elements in $\mathcal{O}(g, 2, 2)$. To simplify the exposition, we shall only derive estimates for those elements whose index is of the form $(n, n), n \in \mathbb{N}$.

Let $\varphi_{n,n}$ be the function obtained when $g_{n,n}$ is orthogonalized with respect to the previous elements of $\mathcal{G}(g, 2, 2)$, indexed as above, namely,

$$\varphi_{n,n}(t) = g_{n,n}(t) - \sum_{j=-(n-1)}^{n-1} \sum_{k=-(n-1)}^{n-1} a_{j,k}^n g_{j,k}(t), \tag{5.2}$$

where the $a_{j,k}^n$ are chosen to ensure that

$$\varphi_{n,n} \text{ is orthogonal to } \{g_{j,k} : -(n-1) \le j, k \le (n-1)\}. \tag{5.3}$$

We shall frequently suppress the dependence of $a^n_{j,k}$ on n and simply write $a_{j,k}$. Note that by theorems 3.7 and 2.11, the $a_{j,k}$ are unique. To normalize the $\varphi_{n,n}$, let

$$\psi_{n,n}(t) = \varphi_{n,n}(t)/||\varphi_{n,n}||_{L^2(\mathbb{R})}.$$

Thus, $\psi_{n,n}$ is the element of $\mathcal{O}(g,2,2)$ with index (n,n). We shall show

Theorem 5.3. *For every $p > 0$ there exists a constant C_p such that*

$$\Delta_p(\psi_{n,n}) \le C_p \quad and \quad \Delta_p(\widehat{\psi_{n,n}}) \le C_p$$

holds for all $n \in \mathbb{N}$.

The main part of the proof is devoted to estimating the $\{a^n_{j,k}\}$ in (5.2). This is the content of the next section.

5.2   Estimating the $\{a^n_{j,k}\}$

To estimate the $\{a^n_{j,k}\}$, we begin by using (5.3) to take inner products in (5.2). This yields

$$\langle g_{n,n}, g_{p,q}\rangle = \sum_{j=1-n}^{n-1} \sum_{k=1-n}^{n-1} a^n_{j,k}\langle g_{j,k}, g_{p,q}\rangle, \tag{5.4}$$

for all (p,q) satisfying $-(n-1) \le p, q \le n-1$. The following lemma gives explicit values for the inner products in (5.4).

Lemma 5.4. *Let $h(t) = 2^{1/4}e^{-\pi t^2}$. Fix $a, b > 0$ and let $\mathcal{G}(h, a, b) = \{h_{m,n}\}_{m,n \in \mathbb{Z}}$. Then*

$$\widehat{h} = h \quad and \quad ||h||_{L^2(\mathbb{R})} = 1 \tag{5.5}$$

and

$$\langle h_{m,n}, h_{k,l}\rangle = e^{-\frac{\pi}{2}b^2(n-l)^2}e^{-\frac{\pi}{2}a^2(m-k)^2}e^{-\pi iab(n+l)(m-k)}. \tag{5.6}$$

Proof. A standard calculation gives (5.5). To see (5.6) we calculate as follows:

$$\langle h_{m,n}, h_{k,l}\rangle = \int h(t-nb)\overline{h(t-lb)}e^{-2\pi i a t(m-k)}dt$$

$$= \sqrt{2}\int e^{-\pi(t-nb)^2}e^{-\pi(t-lb)^2}e^{-2\pi i a t(m-k)}dt$$

$$= \sqrt{2}\int e^{-\pi(t^2-2nbt+n^2b^2+t^2-2lbt+l^2b^2)}e^{-2\pi i a t(m-k)}dt$$

$$= \sqrt{2}e^{-\pi b^2(n^2+l^2)}\int e^{-2\pi(t^2-b(n+l)t)}e^{-2\pi i a t(m-k)}dt$$

$$= \sqrt{2}e^{-\pi b^2(n^2+l^2)}e^{2\pi(\frac{b}{2}(n+l))^2}\int e^{-2\pi(t^2-b(n+l)t+\frac{b^2}{4}(n+l)^2)}e^{-2\pi i a t(m-k)}dt$$

$$= \sqrt{2}e^{-\pi b^2(n^2+l^2-\frac{1}{2}(n+l)^2)}\int e^{-2\pi(t-\frac{b}{2}(n+l))^2}e^{-2\pi i a t(m-k)}dt$$

$$= e^{-\frac{\pi}{2}b^2(n-l)^2}e^{-\pi i a b(n+l)(m-k)}\mathcal{F}\left[e^{-2\pi t^2}\right](a(m-k))$$

$$= \sqrt{2}e^{-\frac{\pi}{2}b^2(n-l)^2}e^{-\pi i a b(n+l)(m-k)}\frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}a^2(m-k)^2}$$

$$= e^{-\frac{\pi}{2}b^2(n-l)^2}e^{\frac{\pi}{2}a^2(m-k)^2}e^{-\pi i a b(n+l)(m-k)},$$

where $\mathcal{F}$ denotes the Fourier transform. $\qquad\square$

Corollary 5.5. *Let* $g(t) = 2^{1/4}e^{-\pi t^2}$ *and let* $\mathcal{G}(g, 2, 2) = \{g_{m,n}\}_{m,n\in\mathbb{Z}}$. *Then*

$$\langle g_{m,n}, g_{k,l}\rangle = e^{-2\pi(n-l)^2}e^{-2\pi(m-k)^2}. \tag{5.7}$$

Using this corollary, we see that (5.4) is equivalent to

$$Ga = g, \tag{5.8}$$

where

$$g = \begin{pmatrix} e^{-2\pi(1)^2}v \\ e^{-2\pi(2)^2}v \\ \cdots \\ e^{-2\pi(2n-1)^2}v \end{pmatrix} \quad \text{with } v = \begin{pmatrix} e^{-2\pi(1)^2} \\ e^{-2\pi(2)^2} \\ \cdots \\ e^{-2\pi(2n-1)^2} \end{pmatrix}$$

and

$$a = \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ \cdots \\ y_{1-n} \end{pmatrix} \text{ with } y_j = \begin{pmatrix} a_{j,n-1} \\ a_{j,n-2} \\ \cdots \\ a_{j,1-n} \end{pmatrix}$$

and

$$G = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{2n-2} \\ B_1 & B_0 & B_1 & \cdots & B_{2n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{2n-2} & \cdots & B_2 & B_1 & B_0 \end{pmatrix}$$

where

$$B_j = e^{-2\pi j^2} C$$

and

$$C \equiv \begin{pmatrix} 1 & e^{-2\pi(1)^2} & e^{-2\pi(3)^2} & \cdots & e^{-2\pi(2n-2)^2} \\ e^{-2\pi(1)^2} & 1 & e^{-2\pi(1)^2} & \cdots & e^{-2\pi(2n-3)^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{-2\pi(2n-2)^2} & e^{-2\pi(2n-3)^2} & e^{-2\pi(2n-4)^2} & \cdots & 1 \end{pmatrix}.$$

Once again, we have suppressed the fact that all these matrices depend on n. We may use the following result to estimate the spectrum of G. It appears, among other places, in [39].

Theorem 5.6 (Gershgorin Theorem). *Let $A = (a_{i,j})$ be a $d \times d$ matrix with complex entries. The spectrum of A satisfies*

$$\sigma(A) \subseteq \bigcup_{i=1}^{d} R_i, \quad R_i = \{z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{j=1,j\neq i}^{d} |a_{i,j}|\}.$$

We now apply the Gershgorin theorem to our matrix G. Note that

$$\sup_i \sum_{j=1,j\neq i}^{2n-1} |G_{i,j}| = \left(\sum_{j=-n}^{n} e^{-2\pi j^2} \right)^2 - 1 \leq \left(\sum_{j\in\mathbb{Z}} e^{-2\pi j^2} \right)^2 - 1 \equiv r.$$

One may verify that $0 < r < 1$, and, in fact, numerically one has that $r \approx .0075$. Thus, $\sigma(G) \subseteq [1 - r, 1 + r]$ is bounded away from 0. Note that while G depends on n, this spectrum bound is independent of n. Likewise, if we let $R = I - G$, then we have $\sigma(R) \subseteq [-r, r]$. This allows us to derive the following block-version of Jaffard's lemma for our matrix G. Although the proof is essentially the same as Jaffard's,[32], we nonetheless include it for the sake of completeness.

Lemma 5.7. There exist $C, \delta > 0$ independent of n, such that

$$|G_{j,k}^{-1}| \leq C e^{-\delta|j'-k'|} e^{-\delta|J-K|},$$

where $j = Jn + j', k = Kn + k'$, and $0 \leq J, K, j', k' < n$. Recall G is an $(2n-1)^2 \times (2n-1)^2$ matrix.

Proof. Throughout the proof C will denote various constants independent of n. Let $R = I - G$. We have $\sigma(R) \subseteq [-r, r]$, where $0 < r < 1$ is as above.

Therefore,

$$|R^m(j,k)| \leq \left(\sum_{l=1}^{(2n-1)^2} |R^m(j,l)|^2 \right)^{\frac{1}{2}} = ||R^m e_j||_2 \leq ||R^m|| \leq r^m. \tag{5.9}$$

Here, $|| \cdot ||_2$ denotes the l^2 norm of a vector and $|| \cdot ||$ denotes the corresponding matrix norm it induces. Also, $\{e_j\}$ is the canonical basis for l^2. By the definitions of G and R we have

$$|R(j,k)| \leq c e^{-|J-K|} e^{-|j'-k'|}.$$

In particular,

$$\forall\, 0 < \delta < 1, \quad |R(j,k)| \leq c e^{-\delta|J-K|} e^{-\delta|j'-k'|},$$

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so that

$$|R^l(j,k)| \leq \sum_{m_1=1}^{(2n-1)^2} \cdots \sum_{m_{l-1}=1}^{(2n-1)^2} \sum_{m_l=1}^{(2n-1)^2} |R(j,m_1)| \cdots |R(m_{l-1},m_l)||R(m_l,k)|$$

$$\leq \sum_{M_1=1}^{2n-1}\sum_{m_1'=1}^{2n-1} \cdots \sum_{M_{l-1}=1}^{2n-1}\sum_{m_{l-1}'=1}^{2n-1} \sum_{M_l=1}^{2n-1}\sum_{m_l'=1}^{2n-1}$$

$$c^l e^{-|J-M_1|}e^{-|j'-m_1'|}\cdots e^{-|M_{l-1}-M_l|}e^{-|m_{l-1}'-m_l'|}e^{-\delta|K-M_l|}e^{-\delta|k'-m_l'|}$$

$$\leq \sum_{M_1=1}^{2n-1}\sum_{m_1'=1}^{2n-1} \cdots \sum_{M_{l-1}=1}^{2n-1}\sum_{m_{l-1}'=1}^{2n-1}$$

$$\tilde{c}c^l e^{-|J-M_1|}e^{-|j'-m_1'|}\cdots e^{-|M_{l-1}-M_l|}e^{-|m_{l-1}'-m_l'|}e^{-\delta|K-M_{l-1}|}e^{-\delta|k'-m_{l-1}'|}$$

$$\vdots$$

$$\leq C^l e^{-\delta|J-K|}e^{-\delta|j'-k'|}.$$

Combining this with (5.9) gives that for all $m \in \mathbb{N}$

$$|R^m(j,k)| \leq \min\{r^m, C^m e^{-\delta|J-K|}e^{-\delta|j'-k'|}\}. \qquad (5.10)$$

Take M large enough so that $\rho \equiv Cr^{M-1} < 1$. By (5.10)

$$|R^m(j,k)| \leq \left((r^m)^{M-1}C^m e^{-\delta|J-K|}e^{-\delta|j'-k'|}\right)^{\frac{1}{M}} = \left(\rho^m e^{-\delta|J-K|}e^{-\delta|j'-k'|}\right)^{\frac{1}{M}}.$$

Using $G^{-1} = \sum_{m=0}^{\infty} R^m$ and (5.10), we have

$$|G^{-1}(j,j)| \leq \sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$$

and

$$|G^{-1}(j,k)| \leq \delta_{j,k} + \sum_{n=1}^{\infty} (\rho^{\frac{1}{M}})^n e^{-\frac{\delta}{M}|j'-k'|}e^{-\frac{\delta}{M}|J-K|}$$

$$\leq \left(\frac{\rho^{1/M}}{1-\rho^{1/M}}\right) e^{-\frac{\delta}{M}|j'-k'|}e^{-\frac{\delta}{M}|J-K|}.$$

This completes the proof. $\qquad\qquad \square$

We shall use lemma 5.7 together with (5.8) and the definition of g to estimate the $\{a_{j,k}^n\}$. The following lemma will be useful.

Lemma 5.8. *There exist constants C and α, independent of n, such that*

$$\sum_{l=1}^{2n-1} e^{-\delta|n-j-l|} e^{-2\pi l^2} \le C e^{-\alpha|n-j|}$$

holds for all $n \in \mathbb{N}$ and $-(n-1) \le j \le n-1$.

Proof.

$$\sum_{l=1}^{2n-1} e^{-\delta|n-j-l|} e^{-2\pi l^2} \le \sum_{1 \le l \le \frac{1}{2}|n-j|} e^{-\delta|n-j-l|} e^{-2\pi l^2} + \sum_{l > \frac{1}{2}|n-j|} e^{-\delta|n-j-l|} e^{-2\pi l^2}$$

$$\le e^{-\frac{1}{2}\delta|n-j|} \sum_{1 \le l \le \frac{1}{2}|n-j|} e^{-2\pi l^2} + e^{-\frac{1}{2}\pi(n-j)^2} \sum_{l > \frac{1}{2}|n-j|} e^{-\delta|n-j-l|}$$

$$\le e^{-\frac{1}{2}\delta|n-j|} \sum_{l \in \mathbb{Z}} e^{-2\pi l^2} + e^{-\frac{1}{2}\pi(n-j)^2} \sum_{l \in \mathbb{Z}} e^{-\delta|n-j-l|}$$

$$\le C e^{-\alpha|n-j|}.$$

The second inequality holds because

$$1 \le l \le \frac{1}{2}|n-j| \implies |n-j-l| > \frac{1}{2}|n-j|.$$

$\square$

Using lemmas 5.7 and 5.8 we may now estimate the coefficants $\{a_{j,k}^n\}$.

Lemma 5.9. *There exist constants $C, \alpha > 0$ such that the coefficants $\{a_{j,k}^n\}$ satisfy*

$$|a_{j,k}^n| \le C e^{-\alpha|n-j|} e^{-\alpha|n-k|}.$$

Recall that $\{a_{j,k}^n\}$ depends on n. The above constants are independent of n.

Proof. Using (5.8), we have

$$a = G^{-1}g.$$

Using lemma 5.7 and the definitions of g and a, we have

$$
\begin{aligned}
|a_{n-j,n-k}| &\leq \left| \sum_{l=1}^{(2n-1)^2} G^{-1}(jn+k,l)g_l \right| \\
&= \left| \sum_{l'=1}^{2n-1} \sum_{L=1}^{2n-1} G^{-1}(jn+k,l'n+L)e^{-2\pi(l')^2}e^{-2\pi L^2} \right| \\
&\leq \sum_{l'=1}^{2n-1} \sum_{L=1}^{2n-1} e^{-\delta|j-l'|}e^{-\delta|k-L|}e^{-\delta|l'|}e^{-\delta|L|} \\
&\leq Ce^{-\alpha|j|}e^{-\alpha|k|}
\end{aligned}
$$

for each $-(n-1) \leq j,k \leq n-1$. Thus,

$$|a_{j,k}^n| \leq Ce^{-\alpha|n-j|}e^{-\alpha|n-k|}.$$

$\square$

5.3 Localization estimates

First, note that by theorem 5.2 the $L^2(\mathbb{R})$ norms of the $\{\varphi_{n,n}\}_{n\in\mathbb{N}}$ stay uniformly bounded away from 0.

Lemma 5.10. *There exists $\Delta > 0$ such that*

$$||\varphi_{n,n}||_{L^2(\mathbb{R})}^2 > \Delta$$

holds for all $n \in \mathbb{N}$.

Proof. We proceed by contradiction. Suppose there exists a subsequence of $\{\varphi_{n,n}\}_{n=0}^\infty$ satisfying

$$\lim_{m\to\infty} ||\varphi_{n_m,n_m}||_{L^2(\mathbb{R})}^2 = 0.$$

By (5.2),

$$g_{n_m,n_m} - \sum_{j=1-n_m}^{n_m-1} \sum_{k=1-n_m}^{n_m-1} a_{j,k} g_{j,k} \to 0, \quad \text{in } L^2(\mathbb{R}).$$

We may use the translation and modulation invariance of Gabor systems to convert this from a statement about the g_{n_m,n_m} to a statement about $g_{0,0}$. In particular, we have that for every $\epsilon > 0$ there exists $\{c_{j,k}\} \subset \mathbb{C}$ such that

$$\left\| g_{0,0} - \sum_{j<0} \sum_{k<0} c_{j,k} g_{j,k} \right\|_{L^2(\mathbb{R})} < \epsilon.$$

This implies that

$$g_{0,0} \in \overline{\text{span}} \, \{g_{j,k} : j, k < 0\}. \tag{5.11}$$

Recall that $\mathcal{G}(g, 2, 2)$ is a Riesz basis for its span by theorem 5.2 and the Ron-Shen duality theorem. Therefore, (5.11) is a contradiction by theorem 2.9. □

We need one final lemma before we can prove theorem 5.3.

Lemma 5.11. *For every $M \in \mathbb{N}$ there exists $C = C_M > 0$ such that*

$$\left| \int |t - n|^M g_{j,k}(t) \overline{g_{l,m}(t)} dt \right| \le C e^{-\frac{\pi}{2}(j-l)^2} \left| n - \frac{j+l}{2} \right|^M$$

holds for all $n \in \mathbb{N}$ and all $j, k, l, m \in \mathbb{Z}$.

Proof. Since $g_{j,0}(t) g_{l,0}(t) = \sqrt{2} e^{-\frac{\pi}{2}(j-k)^2} e^{-2\pi[t-(\frac{l+k}{2})]^2}$, it suffices to examine

$$\int |t - n|^p |g_{m,0}(t)|^2 dt.$$

Using

$$|t - n|^M \le K_M \sum_{k=0}^M |t|^k n^{M-k}$$

we have

$$\int |t - n|^p e^{-2\pi t^2} dt \le K_M \sum_{l=0}^M n^{M-l} \int |t|^l e^{-2\pi t^2} dt \le C_M n^M.$$

□

We are now in position to estimate the localization of the $\{\psi_{n,n}\}$.

Theorem 5.12. *Fix $p \in \mathbb{N}$. There exists a constant C_p such that*

$$\int |t - n|^p |\psi_{n,n}(t)|^2 dt \leq C_p$$

and

$$\int |\gamma - n|^p |\widehat{\psi_{n,n}}(\gamma)|^2 d\gamma \leq C_p$$

hold for all $n \in \mathbb{N}$.

Proof.

$$\int |t - n|^p |\psi_{n,n}(t)|^2 dt = \frac{1}{||\varphi_{n,n}||^2_{L^2(\mathbb{R})}} \int |t - n|^p |\varphi_{n,n}(t)|^2 dt$$

$$= \frac{1}{||\varphi_{n,n}||^2_{L^2(\mathbb{R})}} \left(\int |t - n|^p \left| g_{n,n}(t) - \sum_{j=-(n-1)}^{n-1} \sum_{k=-(n-1)}^{n-1} a_{j,k} g_{j,k}(t) \right|^2 dt \right)$$

$$\leq \frac{1}{\Delta} (S_1 + S_2 + S_3),$$

where

$$S_1 = \int |t - n|^p |g_{n,n}(t)|^2 dt = \int |t|^p |g(t)|^2 dt,$$

$$S_2 = \sum_{j=-(n-1)}^{n-1} \sum_{k=-(n-1)}^{n-1} a_{j,k} \int |t - n|^p g_{j,k}(t) \overline{g_{n,n}(t)} dt$$

$$+ \sum_{j=-(n-1)}^{n-1} \sum_{k=-(n-1)}^{n-1} \overline{a_{j,k}} \int |t - n|^p \overline{g_{j,k}(t)} g_{n,n}(t) dt,$$

$$S_3 = \sum_{j=-(n-1)}^{n-1} \sum_{k=-(n-1)}^{n-1} \sum_{l=-(n-1)}^{n-1} \sum_{m=-(n-1)}^{n-1} a_{j,k} \overline{a_{l,m}} \int |t - n|^p g_{j,k}(t) \overline{g_{l,m}(t)} dt.$$

It is clear that S_1 is uniformly bounded in n. To see that S_2 is bounded indepen-

dantly of n we use lemmas 5.9 and 5.11

$$|S_2| \leq 2C \sum_{j=1-n}^{n-1} \sum_{k=1-n}^{n-1} \left(e^{-\alpha|n-j|} e^{-\alpha|n-k|} \right) \left(e^{-\frac{\pi}{2}(j-n)^2} \left| n - \frac{(n+j)}{2} \right|^p \right)$$

$$= 2C \left(\sum_{j=1-n}^{n-1} e^{-\alpha|n-k|} \right) \left(\sum_{k=1-n}^{n-1} e^{-\alpha|n-j|} e^{-\frac{\pi}{2}(n-j)^2} \left| \frac{n-j}{2} \right|^p \right)$$

$$\leq 2C \left(\sum_{k=1-n}^{n-1} e^{-\alpha|n-k|} \right) \left(\sum_{j=1-n}^{n-1} e^{-\alpha'|n-j|} \right)$$

$$\leq 2C \left(\sum_{k=1}^{\infty} e^{-\alpha|k|} \right) \left(\sum_{j=1}^{\infty} e^{-\alpha'|j|} \right).$$

Likewise, for S_3 we have

$$|S_3| \leq C \sum_{j,k=1-n}^{n-1} \sum_{l,m=1-n}^{n-1} e^{-\alpha|j-n|} e^{-\alpha|k-n|} e^{-\alpha|l-n|} e^{-\alpha|m-n|} e^{-\frac{\pi}{2}(j-l)^2} \left| n - \frac{(j+l)}{2} \right|$$

$$= C \sum_{j,k=1}^{2n-1} \sum_{l,m=1}^{2n-1} e^{-\alpha|j|} e^{-\alpha|k|} e^{-\alpha|l|} e^{-\alpha|m|} e^{-\frac{\pi}{2}(j-l)^2} \left| \frac{j+l}{2} \right|$$

$$\leq C \left(\sum_{j,l=1}^{\infty} e^{-\alpha|j|} e^{-\alpha|l|} |j+l| \right) \left(\sum_{k,m=1}^{\infty} e^{-\alpha|k|} e^{-\alpha|m|} \right).$$

Thus, we see that for any $p > 0$ there is C_p, independent of n, such that

$$\int |t - n|^p |\psi_{n,n}(t)|^2 dt < C_p, \quad \forall n \in \mathbb{N}.$$

For the other inequality, note that

$$\widehat{g_{m,n}}(\gamma) = e^{-2\pi i n b(\gamma + am)} \widehat{g}(\gamma + am) = (\widehat{g})_{n,-m} = g_{n,-m}.$$

Therefore, the same calculations as above yield the uniform boundedness of

$$\int |\gamma - n|^p |\widehat{\psi_{n,n}}(\gamma)|^2 d\gamma.$$

$\square$

We conclude the section with the following question.

Question 5.13. *Let* $g(t) = 2^{1/4}e^{-\pi t^2}$. *If one orthonormalizes* $\mathcal{G}(g, 1, 1)$ *does there exist a constant* C *such that the resulting system* $\mathcal{O}(g, 1, 1) = \{o_{m,n}\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *which satisfies*

$$\Delta(o_{m,n}) \leq C \quad and \quad \Delta(\widehat{o_{m,n}}) \leq C$$

for all $m, n \in \mathbb{Z}$?

Chapter 6

Shapiro's Question

We have already seen how means and variances convey information about where a function is "located" in the time-frequency plane. The Balian-Low theorem and Bourgain's theorem both address the question of whether or not the sequences of time and frequency variances of an orthonormal basis can be bounded. Recall that Bourgain's theorem constructs an orthonormal basis, $\{b_n\}$, for $L^2(\mathbb{R})$ whose variance sequences, $\{\Delta^2(b_n)\}$ and $\{\Delta^2(\widehat{b_n})\}$, are both bounded. On the other hand, the Balian-Low theorem shows that no Gabor orthonormal basis can have both of these variance sequences bounded. A better understanding of how orthonormal bases "cover" the time-frequency plane requires one also examine the mean sequences.

In 1991 H. Shapiro posed the following question, [41]

Question 6.1 (Shapiro). *Given four sequences of real numbers,*

$$\{a_n\},\ \{b_n\},\ \{c_n\},\ \{d_n\},$$

does there exist an orthonormal basis $\{\varphi_n\}$ for $L^2(\mathbb{R})$ such that

$$\mu(\varphi_n) = a_n,\ \ \mu(\widehat{\varphi_n}) = b_n,\ \ \Delta^2(\varphi_n) = c_n,\ \ \Delta^2(\widehat{\varphi_n}) = d_n$$

holds for all n?

The following theorem will serve as a starting point for our investigation.

Theorem 6.2. *There does not exist an infinite orthonormal sequence $\{f_n\} \in L^2(\mathbb{R})$ such that all four of the mean and variance sequences are bounded.*

Shapiro, [41], gives an elegant elementary proof of theorem 6.2 which relies on a compactness result of Kolmogorov. The result also follows from the theory of prolate spheroidal wavefunctions, [35]. We shall discuss prolate spheroidal wavefunctions later.

Motivated by theorem 6.2, we consider the following question.

Question 6.3. *If $\{\varphi_n\}$ is an orthonormal basis for $L^2(\mathbb{R})$, how many of the sequences $\{\mu(\varphi_n)\}, \{\mu(\widehat{\varphi_n})\}, \{\Delta^2(\varphi_n)\}, \{\Delta^2(\widehat{\varphi_n})\}$ can be bounded? Which combinations of these sequences can be bounded?*

6.1 Examples

Let us consider some examples.

Example 6.4 (Wavelet Basis). *Let $\psi \in L^2(\mathbb{R})$ be such that the wavelet system $\mathcal{W}(\psi) = \{\psi_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. A direct calculation, [4], shows that for wavelet systems the three sequences*

$$\{\mu(\psi_{m,n})\}, \{\Delta^2(\psi_{m,n})\}, \{\Delta^2(\widehat{\psi_{m,n}})\}$$

are unbounded.

Example 6.5 (Gabor basis). *Let g be any function such that the corresponding Gabor system $\mathcal{G}(g,1,1) = \{g_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. A direct*

computation shows that both $\{\mu(g_{m,n})\}$ and $\{\mu(\widehat{g_{m,n}})\}$ are unbounded sequences. Moreover, by the Balian-Low theorem, at least one of the two variance sequences, $\{\Delta^2(g_{m,n})\}$ and $\{\Delta^2(\widehat{g_{m,n}})\}$ must be unbounded (in fact constantly equal to ∞).

Example 6.6 (Hermite basis). *Let $\{h_n\}$ be the Hermite functions defined by*

$$h_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(\frac{-1}{2\sqrt{\pi}} \right)^k e^{\pi t^2} \frac{d^k}{dt^k} (e^{-2\pi t^2}).$$

We follow the notation of [20]. The Hermite functions are eigenfunctions of the Fourier transform, form an orthonormal basis for $L^2(\mathbb{R})$, and satisfy

$$2\sqrt{\pi} \ t h_k(t) = \sqrt{k+1} h_{k+1}(t) + \sqrt{k} h_{k-1}(t). \tag{6.1}$$

By taking the inner product of (6.1) with h_k and using the orthonormality of the Hermite functions, it follows that $\mu(h_k) = 0$ for all k. Since each h_n is an eigenfunction of the Fourier transform, we also have $\mu(\widehat{h_k}) = 0$. In particular, both mean sequences are bounded. Using (6.1) again, one can show that $\Delta(h_k) = \Delta(\widehat{h_k}) = \frac{\sqrt{2k+1}}{2\sqrt{\pi}}$, so that both variance sequences are unbounded.

Example 6.7 (Bourgain basis). *Let $\epsilon > 0$. In [11], Bourgain constructs an orthonormal basis, $\{f_n\}$ for $L^2(\mathbb{R})$ satisfying $\Delta^2(f_n) \leq \left(\frac{1}{2\pi} + \epsilon \right)^2$ and $\Delta^2(\widehat{f_n}) \leq \left(\frac{1}{2\pi} + \epsilon \right)^2$ for all n. However, the mean sequences are both unbounded.*

Example 6.8 (Wilson Basis). *Let $g \in L^2(\mathbb{R})$ and define the associated Wilson system, $\{\psi_{l,k}\}_{l \leq 0, k \in \mathbb{Z}}$, by*

$$\psi_{0,k}(t) = g(t - k), \qquad\qquad\qquad k = 0$$

$$\psi_{l,k}(t) = \sqrt{2} g(t - k/2) \cos(2\pi l t), \qquad l \neq 0, k + l \ \text{even},$$

$$\psi_{l,k}(t) = \sqrt{2} g(t - k/2) \sin(2\pi l t), \qquad l \neq 0, k + l \ \text{odd}.$$

See [22] for background on Wilson bases. For any g one can verify that

$$\{\mu(\psi_{l,m})\} \text{ and } \{\Delta^2(\widehat{\psi_{l,k}})\} \text{ are unbounded sequences.}$$

For examples of Wilson bases with exponential localization in time and frequency see, [15], [14].

We shall prove two theorems which answer question 6.3. The first shows that theorem 6.2 holds for orthonormal bases even if the hypotheses are weakened to allow one of the mean sequences to be unbounded. Namely, there does not exist an orthonormal basis $\{f_n\}$ for $L^2(\mathbb{R})$ with $\{\mu(\widehat{f_n})\}, \{\Delta^2(f_n)\}$ and $\{\Delta^2(\widehat{f_n})\}$ being bounded sequences.

Theorem 6.9. *There does not exist an orthonormal basis $\{f_n\}$ for $L^2(\mathbb{R})$ such that $\{\Delta^2(f_n)\}, \{\Delta^2(\widehat{f_n})\}$ and $\{\mu(\widehat{f_n})\}$ are all bounded sequences.*

The second result shows that theorem 6.2 does not hold for orthonormal bases if the hypotheses are weakened to allow one of the variance sequences to be unbounded. Namely, there are orthonormal bases $\{f_n\}$ for $L^2(\mathbb{R})$ such that $\{\Delta^2(f_n)\}, \{\mu(f_n)\}$ and $\{\mu(\widehat{f_n})\}$ are bounded sequences.

Theorem 6.10. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ there exists an othonormal basis, $\{f_n\}$, for $L^2(\mathbb{R})$ satisfying $|\mu(f_n)| \le \epsilon, |\mu(\widehat{f_n})| \le \epsilon$ and $\Delta^2(f_n) \le C$ for all n.*

6.2 Two variances and one mean

In this section we shall prove theorem 6.9. We shall first need some background on the prolate spheroidal wavefunctions.

6.2.1 Prolate spheroidal wavefunctions

Our brief discussion of prolate spheroidal wavefunctions will follow that of [42], [34], [35]. These papers, written by combinations of Landau, Slepian, and Pollak, are the original and authoritative references on prolate spheroidal wavefunctions.

Definition 6.11. *Given $\Omega > 0$, the Paley Wiener space, PW_Ω is defined by*

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq [-\Omega, \Omega]\}.$$

Theorem 6.12 (Slepian, Pollak). *Given any $T > 0$ and $\Omega > 0$, there exists a sequence $\{\psi_n\}_{n=0}^\infty \subset L^2(\mathbb{R})$, called the prolate spheroidal wave functions, and a monotone decreasing sequence of positive numbers, $\{\lambda_n\}$, such that:*

1. *The ψ_n are complete and orthonormal in PW_Ω,*

2. *The ψ_n are complete and orthogonal in $L^2([-\frac{T}{2}, \frac{T}{2}])$ with*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \psi_n(t)\psi_m(t)dt = \lambda_n \delta(m, n),$$

3.

$$\lambda_n \psi_n(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{sin(\Omega(t-s))}{\pi(t-s)} \psi_n(s)ds.$$

Definition 6.13. *Given constants $\epsilon, \eta, \Omega > 0$, we define*

$$S = S_{T,\Omega,\epsilon,\eta} = \{f \in L^2(\mathbb{R}) : \int_{|t| \geq T} |f(t)|^2 dt \leq \epsilon^2 \text{ and } \int_{|\gamma| \geq \Omega} |\widehat{f}(\gamma)|^2 d\gamma \leq \eta^2\}.$$

Landau and Pollak showed that

Theorem 6.14 (Landau, Pollak). *If $f \in S_{T,\Omega,\epsilon,\eta}$ with $\|f\|_{L^2(\mathbb{R})} = 1$ then*

$$\|f - \sum_{n=0}^{[2T\Omega]} a_n \psi_n\|_{L^2(\mathbb{R})}^2 \leq 12(\epsilon + \eta)^2 + \eta^2,$$

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where the coefficants, $a_n = a_n(f)$ are defined by

$$Pf = \sum_{n=0}^{\infty} a_n \psi_n$$

and P is the projection onto PW_Ω.

The next result follows directly from Landau and Pollak's result. It is well known, but we include a proof for the sake of completeness.

Theorem 6.15. *Let $S = S_{T,\Omega,\epsilon,\eta}$. Suppose ϵ and η are small enough. There exists $N \in \mathbb{N}$ such that S contains no orthonormal subset containing more than N elements.*

Proof. Suppose $\{f_l\}_{l=1}^{N} \subseteq S$ is orthonormal, where N is some fixed integer. Let $\{\psi_n\}_{n=0}^{\infty}$ be the prolate spheroidal wavefunctions for $[-T,T] \times [\Omega,\Omega] \subset \mathbb{R} \times \hat{\mathbb{R}}$.

So, by Landau and Pollak's theorem, for each $l = 1,\ldots,N$ there exists $\{a_{n,l}\}_{n=0}^{\infty}$ such that if $f_l = Pf_l + h_l$, where P is the projection onto PW_Ω, then

$$Pf_l = \sum_{n=0}^{\infty} a_{n,l} \psi_n,$$

$$\|h_l\|_{L^2(\mathbb{R})}^2 = \|f_l - Pf_l\|_{L^2(\mathbb{R})}^2 = \eta^2,$$

and

$$\|f_l - \sum_{n=0}^{[2T\Omega]} a_{n,l}\psi_n\|_{L^2(\mathbb{R})}^2 \leq 12(\epsilon + \eta)^2 + \eta^2.$$

First, note that

$$\|\sum_{n=[2T\Omega]+1}^{\infty} a_{n,l}\psi_n\|_{L^2(\mathbb{R})} = \|f_l - h_l - \sum_{n=0}^{[2T\Omega]} a_{n,l}\psi_n\|_{L^2(\mathbb{R})}$$

$$\leq \|f_l - \sum_{n=0}^{[2T\Omega]} a_{n,l}\psi_n\|_{L^2(\mathbb{R})} + \|h_l\|_{L^2(\mathbb{R})}$$

$$\leq \sqrt{12(\epsilon + \eta)^2 + \eta^2} + \eta.$$

Since $h_l \perp Pf_j$ for all j and l, it follows from orthonormality that $0 = \langle f_l, f_j \rangle = \langle h_l, h_j \rangle + \sum_{n=0}^{\infty} a_{n,l} \overline{a_{n,j}}$ for $j \neq l$. Thus,

$$\left| \sum_{n=0}^{[2T\Omega]} a_{n,l} \overline{a_{n,j}} \right| \leq |\langle h_l, h_j \rangle| + \left(\sum_{n=[2T\Omega]+1}^{\infty} |a_{n,l}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=[2T\Omega]+1}^{\infty} |a_{n,j}|^2 \right)^{\frac{1}{2}}$$

$$\leq \eta^2 + (\sqrt{12(\epsilon + \eta)^2 + \eta^2} + \eta)^2 \qquad \text{for } j \neq l.$$

Next, note that we also have

$$\left(\sum_{n=0}^{[2T\Omega]} |a_{n,l}|^2 \right)^{\frac{1}{2}} = \| \sum_{n=0}^{[2T\Omega]} a_{n,l} \psi_n \|_{L^2(\mathbb{R})} \geq \|f_l\|_{L^2(\mathbb{R})} - \|f_l - \sum_{n=0}^{[2T\Omega]} a_{n,l} \psi_n \|_{L^2(\mathbb{R})}$$

$$\geq 1 - \sqrt{12(\epsilon + \eta)^2 + \eta^2}.$$

Thus, defining $v_l = (a_{0,l}, a_{1,l}, \ldots, a_{[2T\Omega],l}) \in \mathbb{R}^{[2T\Omega]+1}$ for $l = 1, 2, \ldots, N$, we have

$$1 \geq |v_l| \geq 1 - \sqrt{12(\epsilon + \eta)^2 + \eta^2} \tag{6.2}$$

and

$$|\langle v_l, v_j \rangle| \leq \eta^2 + \left(\sqrt{12(\epsilon + \eta)^2 + \eta^2} + \eta \right)^2 \text{ for } l \neq j. \tag{6.3}$$

If N is too large, (6.2) and (6.3) yield a contradiction. Recall ϵ and η are assumed to be sufficiently small. $\qquad \square$

6.2.2 Preliminary lemmas

Lemma 6.16. *Suppose* $g \in L^2(\mathbb{R}), \|g\|_{L^2(\mathbb{R})} = 1$, *satisfies*

$$|\mu(g_n)| < A, \ |\mu(\hat{g}_n)| < B \ , \Delta(g_n) < J \ , \Delta(\hat{g}_n) < K.$$

Fix $\epsilon > 0$. *For any* $R > \max\{\frac{J^2}{\epsilon^2}, \frac{K^2}{\epsilon^2}\}$ *we have*

$$g \in S_{A+R,B+R,\epsilon,\epsilon}.$$

Proof. Since $R > \frac{J^2}{\epsilon^2}$,

$$\int_{|t|\geq A+R} |g(t)|^2 dt \leq \int_{|t|\geq|\mu(g)|+R} |g(t)|^2 dt \leq \int_{|t-\mu(g)|\geq R} |g(t)|^2 dt$$
$$\leq \frac{1}{R}\int_{\mathbb{R}} |t-\mu(g)|^2 |g(t)|^2 dt \leq \frac{J^2}{R} < \epsilon^2.$$

Likewise,

$$\int_{|t|\geq B+R} |\widehat{g}(\gamma)|^2 d\gamma \leq \frac{K^2}{R} < \epsilon^2.$$

$\square$

Lemma 6.17. *Suppose* $f, g \in L^2(\mathbb{R}), ||f||_{L^2(\mathbb{R})} = ||g||_{L^2(\mathbb{R})} = 1,$ *and that the means and variances*

$$\mu(f), \mu(\hat{f}), \mu(g), \mu(\hat{g}), \Delta^2(f), \Delta^2(\hat{f}), \Delta^2(g), \Delta^2(\hat{g})$$

are all finite. Then,

$$|\langle f, g\rangle| \leq 2\frac{\Delta(f) + \Delta(\hat{f}) + \Delta(g) + \Delta(\hat{g})}{|\mu(f) - \mu(g)| + |\mu(\hat{f}) - \mu(\hat{g})|}.$$

Proof. Let

$$S_1 = \{t : |t - \mu(f)| \geq \frac{1}{2}|\mu(f) - \mu(g)|\}$$

and

$$S_2 = \{t : |t - \mu(g)| \geq \frac{1}{2}|\mu(f) - \mu(g)|\}.$$

So,

$$|\langle f, g \rangle| \leq \int |f(t)||g(t)|dt \leq \int_{S_1} |f(t)||g(t)|dt + \int_{S_2} |f(t)||g(t)|dt$$

$$\leq \frac{2}{|\mu(f) - \mu(g)|} \int |t - \mu(f)||f(t)||g(t)|dt$$

$$+ \frac{2}{|\mu(f) - \mu(g)|} \int |t - \mu(g)||f(t)||g(t)|dt$$

$$\leq \frac{2\Delta(f)}{|\mu(f) - \mu(g)|} + \frac{2\Delta(g)}{|\mu(f) - \mu(g)|}$$

$$= \frac{2\left(\Delta(f) + \Delta(g)\right)}{|\mu(f) - \mu(g)|}.$$

Likewise,

$$|\langle f, g \rangle| = |\langle \hat{f}, \hat{g} \rangle| \leq 2 \frac{\Delta(\hat{f}) + \Delta(\hat{g})}{|\mu(\hat{f}) - \mu(\hat{g})|}.$$

Now, combining the previous two inequalities gives

$$|\langle f, g \rangle| \leq 2 \frac{\Delta(f) + \Delta(\hat{f}) + \Delta(g) + \Delta(\hat{g})}{|\mu(f) - \mu(g)| + |\mu(\hat{f}) - \mu(\hat{g})|},$$

as desired. $\qquad \square$

6.2.3 Two variances and one mean: the proof

We can now prove theorem 6.9. We restate the theorem here.

Theorem 6.18. *There does not exist an orthonormal basis $\{g_n\}$ for $L^2(\mathbb{R})$ such that $\{\Delta^2(g_n)\}$, $\{\Delta^2(\widehat{g_n})\}$ and $\{\mu(\widehat{g_n})\}$ are all bounded sequences.*

Proof. We proceed by contradiction. Suppose such a basis, $\{g_n\}_{n \in \mathbb{Z}}$, exists and that

$$|\mu(\hat{g}_n)| < B, \ \Delta(g_n) \leq K, \ \Delta(\widehat{g_n}) \leq K$$

holds for each $n \in \mathbb{Z}$. Let $I_n \subset \mathbb{R} \times \hat{\mathbb{R}}$ be the rectangle $[n, n+1] \times [-B, B]$. So, $\{(\mu(g_n), \mu(\hat{g}_n))\}_{n \in \mathbb{Z}}$ is contained in the disjoint union $\bigcup_{n \in \mathbb{Z}} I_n$. By lemma 6.16 and theorem 6.15

$$\exists N \text{ such that } \forall n, \quad \text{card}\left(I_n \bigcap \{(\mu(g_j), \mu(\hat{g}_j))\}_j\right) \leq N. \tag{6.4}$$

Let $f(t) = e^{-2\pi i t \mu(\hat{g}_0)} g_0(t + \mu(g_0))$ and $f_M(t) = e^{2\pi i M t} f(t)$. So, $\|f_M\|_{L^2(\mathbb{R})} = 1$ and

$$\mu(f_M) = 0, \ \mu(\hat{f}_M) = M, \ \Delta(f_M) < K, \ \Delta(\hat{f}_M) < K.$$

Let $S_n = \{j : (\mu(g_j), \mu(\hat{g}_j)) \in I_n\}$. Thus, by Parseval's theorem, lemma 6.17, and (6.4) we have

$$
\begin{aligned}
1 = \sum_n |\langle f_M, g_n \rangle|^2 &\leq \sum_n \left| \frac{2(4K)}{|0 - \mu(g_n)| + |M - \mu(\hat{g}_n)|} \right|^2 \\
&= 64K^2 \sum_n \frac{1}{(|\mu(g_n)| + |M - \mu(\hat{g}_n)|)^2} \\
&= 64K^2 \sum_n \sum_{k \in S_n} \frac{1}{(|\mu(g_k)| + |M - \mu(\hat{g}_k)|)^2} \\
&\leq 128NK^2 \sum_{n=0}^{\infty} \frac{1}{(|n| + |M - B|)^2}.
\end{aligned}
$$

Since the right hand side of this inequality approaches 0 as $M \to \infty$, we have a contradiction. $\qquad\square$

6.3   Two means and one variance

We prove theorem 6.10 in this section. Let us restate the theorem here.

Theorem 6.19. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ there exists an orthonormal basis, $\{f_n\}$, for $L^2(\mathbb{R})$ satisfying*

$$|\mu(b_j)| \leq \epsilon, \ |\mu(\hat{b}_j)| \leq \epsilon, \ \text{and} \ \Delta^2(b_j) \leq C$$

for all j.

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Proof. We begin by defining a system of functions $\mathcal{G}(T, N)$, which we shall need for the proof.

I. Let $g \in \mathcal{S}(\mathbb{R})$ be a function satisfying

- $||g||_{L^2(\mathbb{R})} = 1$ and $\widehat{g} \in C_c^\infty(\mathbb{R})$

- supp $\widehat{g} \subseteq [-1/2, 1/2]$

- g is real and even

- $\mu(g) = \mu(\widehat{g}) = 0$ and $\Delta(g) \equiv \delta < \infty$.

Regarding the third and fourth bullets, note that g is real and even if and only if $\widehat{g}$ is real and even. Also, the mean of an even function is 0. Now define $g_n(t) = \sqrt{2} \cos(2\pi n t) g(t)$. The functions $\{g_n\}_{n=1}^\infty$ have the following properties which are easily verified:

- $\widehat{g_n}(\gamma) = \frac{\sqrt{2}}{2}(\widehat{g}(t - n) + \widehat{g}(t + n))$

- $\langle g_n, g_m \rangle = \delta_{n,m}$.

- $\mu(g_n) = 0 = \mu(\widehat{g_n})$

- $\Delta(g_n) \leq (\sqrt{2})\delta$

- supp $\widehat{g_n} \subseteq [-n - \frac{1}{2}, -n + \frac{1}{2}] \bigcup [n - \frac{1}{2}, n + \frac{1}{2}]$

Given $T, N \in \mathbb{N}$, we define the orthonormal system $\mathcal{G}(T, N) = \{g_n\}_{n=N}^{N+T}$.

II. Let $\{\varphi_n\}_{n=1}^\infty \subseteq \mathcal{S}(\mathbb{R})$ be dense in the unit sphere of $L^2(\mathbb{R})$ and satisfy

$$||\varphi_n||_{L^2(\mathbb{R})} = 1 \text{ and } \widehat{\varphi_n} \in C_c^\infty(\mathbb{R}).$$

The basis shall be of the form $\bigcup_{j=1}^{\infty} B_j$, where each B_j is a finite set of C^{∞} functions whose Fourier transforms are compactly supported. We shall construct the B_j inductively.

Suppose we have already constructed $B_1, \ldots, B_{n-1}$. Let

$$\Phi_n = \varphi_n - P_{[B_1,\ldots,B_{n-1}]}\varphi_n,$$

where $[B]$ is the notation in [11] which denotes the span of the set of functions B. For the base case of the induction let $\Phi_1 = \varphi_1$. Observe that $||\Phi_n||_{L^2(\mathbb{R})} \leq 1$ and Φ_n is orthogonal to the elements of B_j for each $j < n$. Note that $\widehat{\Phi_n} \in C_c^{\infty}(\mathbb{R})$ since φ_n and the elements of $\bigcup_{j=1}^{n-1} B_j$ also satisfy this property.

Take N_n large enough so that $[-N_n + 1, N_n - 1]$ contains the support of $\widehat{\Phi_n}$ and the supports of the Fourier transforms of the functions in $\bigcup_{j=1}^{n-1} B_j$. Take T_n large enough so that:

$$\int |t|^2 |\Phi_n(t)|^2 dt \leq T_n^2 \tag{6.5}$$

and

$$\left| \int t|\Phi_n(t)|^2 dt \right| \leq \epsilon T_n^2 \quad \text{and} \quad \left| \int \gamma |\widehat{\Phi_n}(\gamma)|^2 d\gamma \right| \leq \epsilon T_n^2. \tag{6.6}$$

Enumerate the elements of $\mathcal{G}(T_n, N_n)$ as $\{g_{j,n}\}_{j=0}^{T_n^2}$. The support properties of $\mathcal{G}(T_n, N_n)$ ensure that the elements of $\mathcal{G}(T_n, N_n)$ are orthogonal to Φ_n and the elements of $\bigcup_{j=1}^{n-1} B_j$. We now define the elements of B_n as

$$b_{1,n}(t) = \frac{\Theta}{T_n}\Phi_n(t) + \alpha_{1,n}g_{1,n}(t)$$

$$b_{2,n}(t) = \frac{\Theta}{T_n}\Phi_n(t) + \beta_{1,n}g_{1,n}(t) + \alpha_{2,n}g_{2,n}(t)$$

$$\vdots$$

$$b_{T_n^2,n}(t) = \frac{\Theta}{T_n}\Phi_n(t) + \beta_{1,n}g_{1,n}(t) + \cdots + \beta_{T_n^2-1,n}g_{T_n^2-1,n}(t) + \alpha_{1,T_n^2}g_{1,T_n^2}(t)$$

where $0 < \Theta < \frac{1}{4}$ is a fixed constant and the $\alpha_{j,n}$ and $\beta_{j,n}$ are choosen to ensure the $b_{j,n}$ are orthonormal. As in Bourgain's theorem, we have

$$|\beta_{j,n}| \le \frac{\Theta}{T_n^2} \le \frac{1}{T_n^2} \tag{6.7}$$

and

$$|1 - \alpha_{j,n}| \le \frac{\Theta}{T_n} \le \frac{1}{T_n}. \tag{6.8}$$

III. Let us now prove estimates for $\mu(b_{j,n})$. Using the fact that $\widehat{\Phi}_n$ and the $\widehat{g}_{j,n}$ all have disjoint support we have

$$
\begin{aligned}
\mu(b_{j,n}) &= \int t|b_{j,n}(t)|^2 dt = \langle (\widehat{b_{j,n}})', \widehat{b_{j,n}} \rangle \\
&= \frac{\Theta^2}{T_n^2} \langle (\widehat{\Phi}_n)', \widehat{\Phi}_n \rangle + \sum_{k=1}^{j-1} |\beta_{k,n}|^2 \langle (\widehat{g_{k,n}})', \widehat{g_{k,n}} \rangle + |\alpha_{j,n}|^2 \langle (\widehat{g_{j,n}})', \widehat{g_{j,n}} \rangle \\
&= \frac{\Theta^2}{T_n^2} \int t|\Phi_n(t)|^2 dt + \sum_{k=1}^{j-1} |\beta_{k,n}|^2 \mu(g_{k,n}) + |\alpha_{j,n}|^2 \mu(g_{j,n}) \\
&= \frac{\Theta^2}{T_n^2} \int t|\Phi_n(t)|^2 dt.
\end{aligned}
$$

Thus,

$$|\mu(b_{j,n})| \le \frac{1}{T_n^2} \left| \int t|\Phi_n(t)|^2 dt \right| \le \epsilon.$$

IV. Next we estimate $|\mu(\widehat{b_{j,n}})|$. Using the support properties of the functions, we have

$$
\begin{aligned}
\mu(\widehat{b_{j,n}}) &= \int \gamma |\widehat{b_{j,n}}(\gamma)|^2 d\gamma \\
&= \frac{\Theta^2}{T_n^2} \int \gamma |\widehat{\Phi}_n(\gamma)|^2 d\gamma + \sum_{k=1}^{j-1} |\beta_{k,n}|^2 \mu(\widehat{g_{k,n}}) + |\alpha_{j,n}|^2 \mu(\widehat{g_{j,n}}) \\
&= \frac{\Theta^2}{T_n^2} \int \gamma |\widehat{\Phi}_n(\gamma)|^2 d\gamma.
\end{aligned}
$$

Thus,

$$|\mu(\widehat{b_{j,n}})| \le \frac{1}{T_n^2} \left| \int \gamma |\widehat{\Phi}_n(\gamma)|^2 d\gamma \right| \le \epsilon.$$

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V. Now we estimate $\Delta^2(b_{j,n})$. Once again, using the disjointness of supports and proceeding as in part **III**, we have

$$
\begin{aligned}
\Delta^2(b_{j,n}) &\leq \int |t|^2 |b_{j,n}(t)|^2 dt \\
&= \int |t|^2 |\frac{1}{T_n} \Phi_n(t)|^2 dt + \sum_{k=1}^{j-1} \int |t|^2 |\beta_{j,n} g_{j,n}(t)|^2 \\
&\quad + \int |t|^2 |\alpha_{j,n} g_{j,n}(t)|^2 dt \\
&= \frac{1}{T_n^2} \int |t|^2 |\Phi_n(t)|^2 dt + \sum_{k=1}^{j-1} |\beta_{k,n}|^2 \Delta^2(g_{k,n}) + |\alpha_{j,n}|^2 \Delta^2(g_{j,n}) \\
&\leq \frac{1}{T_n^2} \int |t|^2 |\Phi_n(t)|^2 dt + \sum_{k=1}^{j-1} \frac{1}{T_n^4} 2\delta^2 + |\alpha_{j,n}|^2 2\delta^2 \\
&\leq \frac{1}{T_n^2} \int |t|^2 |\Phi_n(t)|^2 dt + T_n^2 \left(\frac{2\delta^2}{T_n^4} \right) + (1)2\delta^2 \\
&\leq \frac{1}{T_n^2} \int |t|^2 |\Phi_n(t)|^2 dt + 4\delta^2 \leq 1 + 4\delta^2.
\end{aligned}
$$

The second step above follows by proceeding as in part III.

VI. It only remains to show that $\bigcup_{j=1}^{\infty} B_j$ is complete. The verification is identical to that in [11] and our chapter 4, but we repeat the details here for the sake of

completeness.

$$||P_{[B_1,\cdots,B_k]}\varphi_k||^2_{L^2(\mathbb{R})} = ||P_{[B_1,\cdots,B_{k-1}]}\varphi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}\varphi_k||^2_{L^2(\mathbb{R})}$$

$$= ||P_{[B_1,\cdots,B_{k-1}]}\varphi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}(\Phi_k + P_{[B_1,\cdots,B_{k-1}]}\varphi_k)||^2_{L^2(\mathbb{R})}$$

$$= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}\Phi_k||^2_{L^2(\mathbb{R})}$$

$$= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + \sum_{j=1}^{(T_k)^2} |\langle \Phi_k, b_{k,j}\rangle|^2$$

$$= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + (T_k)^2 \left(\frac{\Theta}{T_k}||\Phi_k||^2_{L^2(\mathbb{R})}\right)^2$$

$$= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + \Theta^2||\Phi_k||^4_{L^2(\mathbb{R})}$$

$$\geq \Theta^2.$$

To see the final inequality, let $h(t) = 1 - t^2 + a^2t^4$ be defined on $[0,1]$, where $0 < a < \frac{1}{4}$ is fixed. It is easy to see that $h(t) \geq a^2$. Since $||\Phi_k||_{L^2(\mathbb{R})} \leq 1$ and $\Theta < \frac{1}{4}$, the last step follows.

Now, suppose $y \in L^2(\mathbb{R})$ satisfies $\langle y, b\rangle = 0$ for all $b \in B$. If y is not identically zero, then $\tilde{y} = y/||y||_{L^2(\mathbb{R})}$ is in the unit sphere of $L^2(\mathbb{R})$ and there exists φ_{n_k} such that $\varphi_{n_k} \to \tilde{y}$ in $L^2(\mathbb{R})$ as $k \to \infty$. Thus,

$$0 < \Theta \leq ||P_{[B_1,\cdots,B_{n_k}]}\varphi_{n_k}||_{L^2(\mathbb{R})} \leq ||P_{[B]}\varphi_{n_k}||_{L^2(\mathbb{R})} \to ||P_{[B]}\tilde{y}||_{L^2(\mathbb{R})} = 0,$$

where the limit is as $k \to \infty$. This contradiction shows that B is complete and hence is an orthonormal basis. $\square$

6.4   Means of Bourgain bases

Theorem 6.9 shows that an orthonormal basis, $\{b_n\}$, for which

$$\{\Delta(b_n)\} \quad \text{and} \quad \{\Delta(\widehat{b_n})\} \quad \text{are bounded sequences} \tag{6.9}$$

can not have either of its mean sequences bounded. If an orthonormal basis for $L^2(\mathbb{R})$ satisfies (6.9) we shall refer to it as a Bourgain basis, in view of theorem 4.1. In this section we reexamine theorem 6.9 and look for more precise contraints on the time and frequency mean sequences.

We begin with the following proposition.

Proposition 6.20. *Let $\{b_n\}$ be the basis in Bourgain's theorem, theorem 4.1. The sequence $\{(\mu(b_n), \mu(\widehat{b_n}))\}_n$ lies in a quarter-plane of the form*

$$W_{a,b} = \{(x,y) \in \mathbb{R}^2 : a \le x \ and \ b \le y\}.$$

This proposition follows by examining the proof of Bourgain's theorem. The following result says a bit more about the mean sequences of a Bourgain basis.

Theorem 6.21. *Suppose $\{b_n\}$ is an orthonormal basis for $L^2(\mathbb{R})$ with*

$$\Delta(b_n) \le K \quad and \quad \Delta(\widehat{b_n}) \le K \quad \forall n.$$

If

$$\{(\mu(b_n), \mu(\widehat{b_n}))\}_n \subset W \equiv \{(x,y) \in \mathbb{R}^2 : x \le 0 \ or \ y \le 0\} \tag{6.10}$$

then

$$\sum_n \frac{1}{(1 + |\mu(b_n)| + |\mu(\widehat{b_n})|)^2} = \infty. \tag{6.11}$$

Proof. The proof is essentially the same as that of theorem 6.9. We proceed by contradiction and begin by assuming such a basis exists and satisfies

$$\sum_n \frac{1}{(1 + |\mu(b_n)| + |\mu(\widehat{b_n})|)^2} < \infty. \tag{6.12}$$

Let $f = b_0$ and assume without loss of generality that $\mu(f) = \mu(\widehat{f}) = 0$. Let $f_N(t) = e^{2\pi i N t} f(t - N)$, so that $\mu(f_N) = \mu(\widehat{f_N}) = N$. As in the proof of theorem

6.9 we have

$$1 \leq \sum_j \frac{CK^2}{(1 + |\mu(b_j) - N| + |\mu(\widehat{b}_j) - N|)^2} \tag{6.13}$$

for some constant C. If we let

$$S_1 = \{j : \mu(\widehat{b}_j) \leq 0\} \text{ and } S_2 = \{j : \mu(b_j) \leq 0\}$$

then we can overestimate the above sum by two sums, one over S_1 and one over S_2. If $j \in S_1$ then for $N > 0$

$$|\mu(b_j)| + |\mu(\widehat{b}_j)| + N \leq |\mu(b_j) - N| + |\mu(\widehat{b}_j) - N| + 3N$$
$$\leq |\mu(b_j) - N| + 4|\mu(\widehat{b}_j) - N|$$
$$\leq 4|\mu(b_j) - N| + 4|\mu(\widehat{b}_j) - N|.$$

Therefore,

$$\sum_{j \in S_1} \frac{CK^2}{(1 + |\mu(b_j) - N| + |\mu(\widehat{b}_j) - N|)^2} \leq \sum_{j \in S_1} \frac{C_0 K^2}{(1 + |\mu(b_j)| + |\mu(\widehat{b}_j)| + N)^2}$$

for some constant C_0. By (6.12), the left side of this inequality goes to 0 as $N \to \infty$. Combining this with a similar estimate for the sum over S_2 shows that the right side of (6.13) goes to 0 as $N \to \infty$. Thus, we have a contradiction. $\square$

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