Bayesian Sequential Hypothesis Testing

by

D.C. MacEnany

BAYESIAN

SEQUENTIAL HYPOTHESIS TESTING

by

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[3] C. Gomez, A. Sulem, J.P. Quadrat, G.L. Blankenship, P. Kumar, A. LaVigna, D.C. Mac Enany, K. Paul, and I. Yan, "A Expert System for Stochastic Control and Signal Processing", *Proceesings IEEE Conference on Decision and Control*, Las Vegas, December 1984, pp. 716–723.

[4] D.C. Mac Enany, R.D. Short, "Multi-Source Target Tracking Estimation",*Sperry-Reston Corp. Technology Center, Internal Report* Rep 3410-RR-83-01, October 1983.

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ABSTRACT

Title of Thesis: Bayesian Sequential Hypothesis Testing

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In this thesis, optimality results are presented for Bayesian problems of sequential hypothesis testing. Conditions are given which are sufficient to demonstrate the existence and optimality of threshold policies and others are given which help characterize these policies. The general results are applied to solve four specific problems where the observations respectively arise from a time-homogeneous diffusion, a progressive semimartingale observed through a diffusion, a time-homogeneous Poisson process, and a predictable semimartingale observed through a point process. It is shown that threshold policies are optimal in all four cases. Exact formulas for the Bayesian costs in the point process cases will be presented for the first time.

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Introduction

This thesis addresses a class of problems in the theory of **optimal stopping rules**, or **sequential analysis**, which are called Bayesian sequential hypothesis testing problems. In the classical methods of mathematical statistics one supposes that an observation of a stochastic process is available for analysis, but that the length of the observation interval is fixed in advance. Thus in the classical binary hypothesis testing problem, one fixes a time say T, and then observes some process say $\{y_t : 0 \leq t \leq T\}$, whose statistics under each hypothesis are known. A decision is then made as to which hypothesis is the *best* to choose. In contrast to this approach wherein the length of time that the process is observed is fixed in advance, the approach in sequential analysis is to terminate the observations at a *random* stopping time which is chosen by the observer and based upon the information contained in the data observed. In the Bayesian version of this problem, one assumes knowledge of the prior statistics for each hypothesis and chooses that hypothesis which *best* minimizes a chosen cost criterion. The advantage of the sequential approach in Bayesian problems is that one can prove that they yield a cost smaller than any other method which uses a fixed observation time. This advantage notwithstanding, in some problems a fixed observation time is inappropriate due to the nature of the data. For instance, many situations demand that a cost be levied according to the amount of time it takes to reach a decision. In particular, this is usually true whenever data collection is expensive or in situations where it becomes increasingly risky to continue

to observe the data. In these situations one is compelled to use a sequential method.

The principal goal of this thesis is to demonstrate the optimality of certain statistical procedures, called threshold policies, as applied to problems of Bayesian sequential hypothesis testing. The importance of Bayesian sequential hypothesis testing is due not only to its applicability to many practical signal detection problems, but also because it provides a theoretical foundation with which to prove optimality results for other sequential statistical procedures.

The mathematical theory available to formulate and obtain optimality results in sequential analysis roughly divides into two main categories, the first of which can be called the 'dynamic programming' approach, and the second which might be called the 'excessive function' approach. Naturally, both approaches share many similarities but are sufficiently different to permit this broad classification. The methodology adopted in this thesis is best described as falling into the second category, but represents a significant modification to the usual development [DYNKIN], [SHIRYAYEV 77], [THOMPSON].

Loosely speaking, the excessive function theory approach extends well-known results for sequences of independent and identically distributed random variables to continuous-time (see also [KEL'BERT]). In this thesis, a more direct development is taken, one which yields a rich interplay between analytical and probabilistic ideas. While it is true that that the excessive function theory is more general and hence more powerful, it is also true that the approach contained herein requires considerably less sophisticated mathematical machinery. Thus one can argue that it renders important optimality results more accessible to a wider audience. Moreover, although it is less generally applicable, it is nonetheless powerful enough that with it one can obtain the solutions to four important applications, two of which, to this author's knowledge, have never before appeared.

The organization of the thesis by chapter is from the general to the specific, and

is given as follows.

In Chapter I is given a general, detailed treatment of certain types of semimartingales, special cases of which will be needed later in the applications chapter. Also given, is a proof of the existence and uniqueness of the solution to a functional differential equation. It was a tough decision to include this theorem in the first chapter, but it was felt that this type of differential equation is so intrinsic to the problems involving Poisson-type processes, that relegating the result to an unread appendix would be cheating the reader out of valuable insight into the nature of these types of problems, and their associated difficulties. Indeed, so intrinsic and problematic are these equations that they and their more complicated integro-differential brethren are deemed by Davis to be the 'bugbear of the subject' [DAVIS]. An interesting historical note is that one of the earliest (post-Euler) references to equations of the type considered herein is due to Poisson himself [POISSON].

In Chapter II, the general Bayesian sequential hypothesis testing problem is discussed and formulated. A principal goal of the chapter is to show how one can greatly reduce the complexity of the search for the optimal test. Also given are initial characterizations of the optimal Bayesian cost function.

In Chapter III, the principal optimality result is given, the essence of which is contained in Lemma 3.1.1. Also included are theorems whose importance is paramount in the discovery of the unique test which is optimal in a particular application.

In Chapter IV, the results of the previous chapters are applied to solve four Bayesian sequential detection problems, the first involving a fully-observed time-homogeneous diffusion, and the second involving a partially-observed progressively measurable semimartingale process. Partial optimality results for the latter have been previously presented only for a special case [LaVIGNA]. In the third problem, the Bayesian sequential hypothesis testing problem based on observations of a Poisson process is solved. Explicit formulae for the risk and thresholds are presented. These

results are then extended to solve the last sequential detection problem wherein the observations arise from a point process whose underlying rate is one of two predictable semimartingales.

After reading the thesis, one should be able to appreciate the difficulty in handling non-diffusion type processes in sequential analysis, especially in view of the formulae obtained in the point process cases. It seems safe to say that future work in sequential analysis will deal with ever more exotic processes as demanded by applications, and that therefore the research in this area will of necessity focus more on approximate and asymptotic methods. It appears that a good starting point for such research is with the relatively recent functional central limit theory [HELLAND]. In this respect, the explicit formulas obtained herein give one a numerical yardstick with which to judge candidate approximation schemes.

Chapter I
Prerequisite Considerations

1.0 Introduction

This chapter will establish some important results which will be used later on. Each theorem will be presented with slightly more generality than will be necessary in the hope that the essence of each is more clearly understood.

Theorem 1.1.1 demonstrates that special types of semimartingales always escape intervals in finite time. This result will be used to show that particular hypothesis testing strategies—threshold policies based on these semimartingales—in effect bifurcate the sample space Ω, in the sense that they terminate in finite time almost surely under either hypothesis.

Theorem 1.1.2 establishes a weak form of the differential rule for functions of locally finite variation semimartingales with piecewise monotone sample paths. The usual generalized Itô differential rule is stated for twice continuously differentiable functions of general semimartingales. This is sometimes specialized to semimartingales driven by discontinuous martingale processes where the smoothness requirement is weakened to 'once continuously differentiable'. It will become necessary in later chapters to consider stochastic differentials of functions even less smooth, specifically functions which are piecewise right continuous and have piecewise right continuous derivatives. In fact, functions of this type are intrinsic to the theory of discontinuous

semimartingales. The reason is that the formal application of the usual differential rule to discontinuous semimartingales very often leads to functional differential equations [e.g. HIBEY, SEGALL] of a type whose solutions are generically not continuously differentiable, and which do not, therefore, satisfy the assumptions which led to them.

Theorem 1.2.1 establishes the existence and uniqueness of the solution to a certain functional advance differential equation. Its solution yields an important quantity arising in the sequential Bayesian hypothesis testing problem involving point processes.

1.1 Semimartingale Prerequisites

Consider a probability triple $(\Omega, \mathcal{F}, P)$ equipped with a right-continuous **filtration** $\mathcal{F}_t, t \geq 0$. In this thesis, the following definition of semimartingale will be used.

Definition 1.1.1 *A* $(P, \mathcal{F}_t)$**-semimartingale** *is a random process* X *which has the decomposition,*

$$X_t = X_0 + M_t + A_t \quad t \geq 0 \quad P\text{-a.s.} \tag{1.1.1}$$

where X_0 *is a* $(P, \mathcal{F}_t)$*-r.v.,* $|X_0| < \infty$ *P-a.s.,* M *is a* **corlol** $(P, \mathcal{F}_t)$**-local martingale,** *with* $M_0 = 0$*, and* A *is a* **right-continuous,** $\mathcal{F}_t$**-adapted** *process, initially zero and of* **locally integrable variation** [ELLIOTT]. *The value set for* X *will be denoted as* $E \subseteq \mathbb{R}$.

Without loss of generality (wlog) then, take the **predictable** version of A and note that the random variable A_∞ exists [ELLIOTT]. In most of this thesis, the full generality of the definition will not be needed. However, as stated previously, the results given in this chapter are somewhat more general than will be necessary in later chapters but that this is done for clarity and because the general results were no harder to obtain.

Since a principal theme of the thesis is to consider threshold policies and hence first exit times of processes from intervals, one is naturally interested in the properties that should be satisfied by a process upon which one intends to base a threshold policy. One obvious requirement is that such a process should eventually exit the threshold interval. The next theorem gives three conditions which are sufficient to guarantee such behavior for a semimartingale. It is an adaptation and generalization of Lemma 17.7 in [LIPTSER & SHIRYAYEV 78].

Theorem 1.1.1 *Let* X *be a real* $(P, \mathcal{F}_t)$*-semimartingale as in definition 1.1.1, and suppose* $P\{|A_\infty| = \infty\} = 1$*, while* $P\{|A_t| = \infty\} = 0$, $\forall 0 \leq t < \infty$. *Furthermore,*

suppose $|E[A_{\tilde{\tau}}]| = E[|A_{\tilde{\tau}}|]$ *for any* $\mathcal{F}_t$ **-stopping time** $\tilde{\tau}$. *Then,* X *will escape any open interval containing* X_0 *in finite time, P-a.s.*

Proof:

Choose $X_0 = 0$ P-a.s. wlog and choose a, b to satisfy $-\infty < a < 0 < b < \infty$, a, b otherwise arbitrary. Define $\tau = \inf\{t \geq 0 : X_t \notin (a,b)\}$ and $\sigma_n = \inf\{t \geq 0 : \int_0^t |dA_s| \geq n\}, n \geq 0$. Note that τ and σ_n are both $\mathcal{F}_t$-stopping times and thus so is $\tau \wedge \sigma_n$. Moreover, since A is of locally integrable variation, $\sigma_n < \infty$ P-a.s. $\forall n \geq 0$, and $\sigma_n \leq \sigma_{n+1} \uparrow \infty$ P-a.s.

Next, assume that for all $n \geq 0$ it is true that,

$$a \leq X_{\tau \wedge \sigma_n} \leq b. \tag{1.1.2}$$

Otherwise, there exists an $n \geq 0$ satisfying $\tau \wedge \sigma_n = \tau < \sigma_n < \infty$ P-a.s., and such that $X_{\tau \wedge \sigma_n} = X_\tau \notin (a,b)$, from which the theorem follows. So assuming 1.1.2 holds for all $n \geq 0$, the remainder of the proof will proceed by *reductio ad absurdum.* From 1.1.2 then, using the fact that M is a $(P, \mathcal{F}_t)$-local martingale one obtains,

$$a - b < a \leq E[A_{\tau \wedge \sigma_n}] \leq b < b - a, \tag{1.1.3}$$

since clearly $a - b < a$, and $b < b - a$. Rewriting 1.1.3 yields,

$$|E[A_{\tau \wedge \sigma_n}]| < b - a < \infty, \tag{1.1.4}$$

and since $\sigma_n \uparrow \infty$ P-a.s., then $A_{\tau \wedge \sigma_n} \to A_\tau$ P-a.s, and invoking the bounded convergence theorem yields $|E[A_\tau]| < b - a$. From this it follows that $E[|A_\tau|] < b - a$ by hypothesis, and hence,

$$E[|A_\tau|] < \infty. \tag{1.1.5}$$

But then,

$$\begin{aligned} \infty > E[|A_\tau|] &= E[1_{\{\tau = \infty\}} |A_\tau|] + E[1_{\{\tau < \infty\}} |A_\tau|] \\ &\geq E[1_{\{\tau = \infty\}} |A_\tau|] = E[1_{\{\tau = \infty\}} |A_\infty|], \end{aligned} \tag{1.1.6}$$

and since $|A_\infty| = \infty$ P-a.s., it must be true that $\tau < \infty$ P-a.s. That is to say, X will exit (a, b) almost surely in finite time. Since a and b were arbitrary modulo X_0, the result is shown. ∎

Note that the result can be extended to a process whose **compensator** almost surely takes on one of two not necessarily infinite values at infinity, as long as one can exhibit a suitably regular, bijective function which maps those values into plus and minus infinity. The theorem then holds for the semimartingale resulting from the function acting on the original process, and hence for the original process by inversion.

Perhaps it should also be stressed that the theorem states only a sufficiency result. For instance, it is clear that no martingale can satisfy its hypotheses, because martingales lack (nontrivial) compensators. On the other hand, using the same notion as in the preceding paragraph, a bijective function of a martingale might yield a process with a nontrivial compensator which indeed satisfies the theorem. In fact, in Chapter IV there will be a need to guarantee that the exit times of a certain martingale (an *a posteriori* probability) are almost surely finite. The approach taken to accomplish this has just been outlined and utilizes Theorem 1.1.1.

When a semimartingale as in definition 1.1.1 escapes an open interval, questions naturally arise as to its whereabouts at the time of escape. The next definition provides a handle with which to phrase such questions.

Definition 1.1.2 *Let X be a real $(P, \mathcal{F}_t)$-semimartingale, taking values in $E \subseteq \mathbb{R}$, and let $X_0 \in I_0, P$-a.s., where $I_0 \subset E$, is some nonempty open interval. Define the sets,*

$$\Omega_0 = \{\omega \in \Omega : X_0(\omega) \in I_0\},$$

$$\Omega_\infty = \{\omega \in \Omega : X_\infty(\omega) \in I_0\},$$

and suppose τ is given as,

$$\tau = \inf\{t \geq 0 : X_t \notin I_0\}.$$

Then the X-**boundary** *of* I_0 *is defined and denoted as,*

$$\partial_x I_0 = \{x \in E : x = X_{\tau(\omega)}(\omega) \text{ for some } \omega \in \Omega_0 \setminus \Omega_\infty\}.$$

In addition, the X-**closure** *of* I_0 *is defined and denoted as,*

$$[I_0]_x = I_0 \cup \partial_x I_0.$$

It will also be convenient to define the upper and lower X*-boundary of* I_0 *as* $\partial_x^+ I_0$ *and* $\partial_x^- I_0$*, respectively, i.e.,* $\partial_x I_0 = \partial_x^- I_0 \cup \partial_x^+ I_0$*, and* $x' \in \partial_x^+ I_0$ *implies* $x' > x$ *for all* $x \in \partial_x^- I_0$*, and vice versa.*

Note that $\partial_x I_0$ is P-a.s. nonempty if X satisfies the hypotheses of Theorem 1.1.1, and often $\partial I_0 \subseteq \partial_x I_0$ for any such X, where ∂I_0 is the usual boundary of I_0 consisting of its endpoints. In fact, if X is a continuous semimartingale satisfying Theorem 1.1.1, then $\partial I_0 = \partial_x I_0$ P-a.s. Also note that neither $\partial_x I_0$ nor $[I_0]_x$ are necessarily closed sets. For instance, if X is a real right continuous semimartingale driven by a discontinuous martingale whose jumps are positive, then $\partial_x^+ I_0$ is generically a semi-open interval, closed on the left and open on the right.

As it was just indicated, the notion of the boundary of an interval with respect to a jump-type process is not trivial as it is for continuous processes. The simplest case is where the jumps of the process are always positive (negative) while the sample paths are strictly nonincreasing (nondecreasing) in between the jumps. Fortunately, this is the situation in the binary hypothesis testing of counting processes. It is also fortunate because a weaker form of the generalized Itô rule can be given for a process which has such sample paths, and because one soon discovers that the differential rule, as it is usually stated, is inadequate to handle the types of functions which arise in the hypothesis testing of such processes. Specifically, the Itô rule for jump processes is usually given for functions which are once continuously differentiable, whereas functions which are less smooth need to be considered. The next definition gives a general description of the type of functions which arise.

Definition 1.1.3 *Let $F : E \to \mathbb{R}$, with $E \subseteq \mathbb{R}$, and suppose that for all but an isolated set of real numbers, $D \subseteq \mathbb{R}$, F is n-times continuously differentiable, $n \geq 0$. Further, at each point $x_0 \in D$ assume that $\lim_{x \downarrow x_0} F^{(k)}(x)$ exists for $k = 0, 1, ..., n$, and define this limit to be the kth derivative of F at $x_0 \in D$. Also suppose that $\lim_{x \uparrow x_0} F^{(k)}(x)$ exists and is finite for all $x_0 \in D$, $k = 0, 1, ..., n$. Then, F is said to be a $\mathcal{C}^{n+}(E)$ function or one writes $F \in \mathcal{C}^{n+}(E)$.*

Loosely then, a $\mathcal{C}^{n+}(E)$ function is a function which is n-times, piecewise right-continuously differentiable, with left-hand limits, and only a countable (isolated) set of points prevents it from being n-times continuously differentiable. This set of points is called the set of **breakpoints** of the function.

The notion of monotonic sample paths introduced above is useful when considering $\mathcal{C}^{1+}$ functions of jump semimartingales. For instance, if one applies a right-discontinuous function, such as the greatest integer function, to a right-continuous jump semimartingale with piecewise nondecreasing sample paths, then the resulting process will be right-continuous. However, applying the same function to right-continuous semimartingale with piecewise nonincreasing sample paths yields a left-continuous process.

The next result is an extension of the usual differential rule to include $\mathcal{C}^{1+}$ functions of **locally finite variation** semimartingales which have piecewise nondecreasing sample paths.

Theorem 1.1.2 *Let X be a $(P, \mathcal{F}_t)$-semimartingale of locally finite variation with piecewise nondecreasing sample paths. Suppose a function F is given such that $F \in \mathcal{C}^{1+}(E)$. Then $F(X)$ is a semimartingale, and with equality denoting P-*indistinguishability [BREMAUD],

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-})\, dX_s + \sum_{0<s\leq t} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\}, \quad t \geq 0 \quad (1.1.7)$$

where ΔX_t denotes $X_t - X_{t-}$ for all $t > 0$, any process X.

Proof:

First note that the right continuity and left-hand limits of F are necessary in order to yield a corlol process. To see this, consider an arbitrary $\omega \in \Omega$. Let $t \geq 0$, then, because X is piecewise monotone increasing, there exist $\{t_n\}_{n=0}^{\infty}$, $t_n \downarrow t$ such that $X_{t_n}(\omega) \downarrow X_t(\omega)$. But then $F(X_{t_n}(\omega)) \to F(X_t(\omega))$ because F is right-continuous, and hence $F(X)$ is continuous on the right. For emphasis note that in general,

$$\lim_{x \uparrow c} F(x) = F(c-) \neq F(c) = \lim_{x \downarrow c} F(x) = F(c+) \tag{1.1.8}$$

It is clear that $F(X_t)$ has limits on the left. Similarly, the right continuity of F' and implied left-hand limits yield $F'(X_{s-})$ locally bounded, and so the integral in 1.1.7 makes sense.

Let $\mathcal{D}$ be the set of disjoint, consecutive open intervals constructed from the set of breakpoints of F, say $\{d_k\}$, so that $\mathcal{D} = \{..., (d_{-1}, d_0), (d_0, d_1), (d_1, d_2), ...\}$. Write $D_i \in \mathcal{D}$ for $D_i = (d_i, d_{i+1})$. Without loss of generality, suppose $X_0 \in D_{i_0}$ for some $D_{i_0} \in \mathcal{D}$. Define τ_0 as the first time after $t = 0$ that X escapes D_{i_0} i.e., $\tau_0 = \inf\{t \geq 0 : X_t \notin (d_{i_0}, d_{i_0+1})\}$. Now, as long as $t \in [0, \tau_0)$, the usual differential rule [SNYDER] for jump processes holds for $F(X_t)$ since F is assumed strictly continuously differentiable on D_{i_0}. Hence for $t \in [0, \tau_0)$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-})\, dX_s + \sum_{0<s\leq t} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\} \tag{1.1.9}$$

Since $F(X_t)$ is corlol, then,

$$F(X_{\tau_0-}) = F(X_0) + \int_0^{\tau_0-} F'(X_{s-})\, dX_s + \sum_{0<s\leq \tau_0-} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\} \tag{1.1.10}$$

Decomposing dX_s as $dX_s = dX_s^c + \Delta X_s$ [WONG] one obtains,

$$\int_0^{\tau_0-} F'(X_{s-})\, dX_s = \int_0^{\tau_0} F'(X_{s-})\, dX_s^c + \sum_{0<s\leq \tau_0-} F'(X_{s-})\, \Delta X_s \tag{1.1.11}$$

Combining 1.1.10 and 1.1.11 yields,

$$F(X_{\tau_0-}) = F(X_0) + \int_0^{\tau_0} F'(X_{s-})\, dX_s^c + \sum_{0<s\leq\tau_0-} \Delta F(X_s) \tag{1.1.12}$$

Now the possibilities as to whether X jumps a $t = \tau_0$ and/or F has discontinuities at X_{τ_0-} and/or X_{τ_0} reduce to the cases $\Delta F(X_{\tau_0}) = 0$ or $\Delta F(X_{\tau_0})$ is nonzero but bounded (implicitly F is locally bounded). Either way, 1.1.12 yields,

$$F(X_{\tau_0-}) + \Delta F(X_{\tau_0}) = F(X_0) + \int_0^{\tau_0} F'(X_{s-})\, dX_s^c + \sum_{0<s\leq\tau_0-} \Delta F(X_s) + \Delta F(X_{\tau_0}) \tag{1.1.13}$$

and so,

$$F(X_{\tau_0}) = F(X_0) + \int_0^{\tau_0} F'(X_s)\, dX_s^c + \sum_{0<s\leq\tau_0} \Delta F(X_s) \tag{1.1.14}$$

Since $X_{\tau_0} \in D_{i_1}$ for some $D_{i_1} \in \mathcal{D}$ one can define $\tau_1 = \inf\{t \geq \tau_0 : X_t \notin D_{i_1}\}$. Then, if $t \in [\tau_0, \tau_1)$, $X_t \in D_{i_1}$, and F is continuously differentiable there so, arguing as before it follows that,

$$F(X_{\tau_1-}) = F(X_{\tau_0}) + \int_{\tau_1}^{\tau_1-} F'(X_{s-})\, dX_s^c + \sum_{\tau_1<s\leq\tau_2-} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\} \tag{1.1.15}$$

and from this,

$$F(X_{\tau_1}) = F(X_{\tau_0}) + \int_{\tau_0}^{\tau_1} F'(X_{s-})\, dX_s^c + \sum_{\tau_0<s\leq\tau_1} \Delta F(X_s) \tag{1.1.16}$$

By induction, for any $n \in \mathbb{N}$,

$$F(X_{\tau_n}) = F(X_{\tau_{n-1}}) + \int_{\tau_{n-1}}^{\tau_n} F'(X_{s-})\, dX_s^c + \sum_{\tau_{n-1}<s\leq\tau_n} \Delta F(X_s) \tag{1.1.17}$$

where $\tau_n = \inf\{t \geq \tau_{n-1} : X_t \notin D_{i_n}\}$, and where $D_{i_n} \ni X_{\tau_{n-1}}$. Performing the recursion indicated in 1.1.17 one obtains,

$$F(X_{\tau_n}) = F(X_0) + \int_0^{\tau_n} F'(X_{s-})\, dX_s^c + \sum_{0<s\leq\tau_n} \Delta F(X_s) \tag{1.1.18}$$

Note that $\tau_n \leq \tau_{n+1}$, and that since X is of locally finite variation, X can only cross $\partial\mathcal{D}$ a finite number of times in finite time, P-a.s. So if τ is any stopping time which is P-a.s.-finite, there exists $n_0 \in \mathbb{N}$ such that $\tau \wedge \tau_{n_0} = \tau$ P-a.s., and 1.1.18 yields,

$$F(X_\tau) = F(X_0) + \int_0^\tau F'(X_{s-})\, dX_s^c + \sum_{0<s\leq\tau} \Delta F(X_s) \tag{1.1.19}$$

or,

$$F(X_\tau) = F(X_0) + \int_0^\tau F'(X_{s-})\, dX_s + \sum_{0<s\leq\tau} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\} \tag{1.1.20}$$

Since τ is any P-a.s.-finite stopping time, 1.1.7 follows from 1.1.20 *a fortiori.* ∎

Hence, the usual differential rule in the non-diffusion case holds under slightly weaker conditions on the function F than are customarily imposed, if one is willing to assume slightly stronger conditions on the sample paths of X. There are two key ingredients to the proof. The first is that F must respect the sample path properties of X such that $F(X)$ is corlol. For example, if X has nondecreasing sample paths, then F must be at least piecewise right-continuous. The second ingredient is that X must have locally finite variation. This yields a boundary behavior which is sufficiently simple to permit the argument to proceed as in the proof. Without this assumption, for instance in the diffusion case, this type of argument is confounded at the discontinuities, where such a process can cross the boundary infinitely often in an arbitrarily small time period.

This concludes the general semimartingale results which will be needed in the applications chapter. In the next section, a result is presented which will be necessary in the hypothesis testing problem dealing with point processes.

1.2 A Functional Differential Equation

The next theorem plays a key role in Bayesian hypothesis testing problem in the case of point process observations.

Theorem 1.2.1 *Let a, b, c, and u be given such that $0 < a < b < 1$, and both c and u are positive. Suppose a function $E(x)$ is given which is continuous for all $x \in (0,1)$. Consider the following functional-advance differential equation,*

$$UR(x) = -c \qquad x \in (0,b), \tag{1.2.1}$$

where the linear operator U is defined via,

$$UR(x) = -x(1-x)R'(x) + (u+x)\Big[R(\frac{u+1}{u+x}x) - R(x)\Big],$$

where the boundary conditions are given by,

$$\begin{aligned} R(a) &= E(a) \\ R(x) &= E(x) \qquad \forall x \in [b, \frac{u+1}{u+b}b) \end{aligned} \tag{1.2.2}$$

and with the functional requirement,

$$R(x) \quad \textit{is continuous at each} \quad x \in (0,b). \tag{1.2.3}$$

Then a unique solution exists for all $x \in (0,b)$.

Proof:

Before getting into the details of the proof a short informal discussion will be given, since problems of this type are somewhat peculiar (see [EL'SGOL'TS & NORKIN]). Consider the fact that as long as $x \in [b - \frac{b(1-b)}{u+b}, b)$ that,

$$R(\frac{u+1}{u+x}x) = E(\frac{u+1}{u+x}x), \tag{1.2.4}$$

and thus one may write,

$$R'(x) = F(x, R(x)) \qquad x \in J_1[b), \tag{1.2.5}$$

where $J_1[b)$ is the stated semi-open interval and where,

$$F(x,R) = G(x) - \frac{u+x}{x(1-x)}R, \tag{1.2.6}$$

with G(x) a known continuous function given by,

$$G(x) = \frac{c + (u+x)E(\frac{u+1}{u+x}x)}{x(1-x)}. \tag{1.2.7}$$

Hence, if one can prove the existence and uniqueness of a continuous solution to the ODE 1.2.5, with arbitrary initial condition, then the process may be repeated inductively to the left, each induction step involving a proof of the existence and uniqueness of an ODE with a different forcing function obtained from the previous induction step. The induction process is continued until the ODE under consideration lives on the interval containing $x = a$. On this interval, call it J_a, the ODE solution of interest is the one passing through $(a, E(a))$, which may or may not be an initial condition for that interval. In either case, assuming a unique trajectory exists passing through $(a, E(a))$, then this solution is extended to the right hand endpoint of J_a, yielding a value for R at this point. This value of R is then taken to be the initial condition for the solution obtained on the previous induction step. Note that matching the value at this endpoint is justified since the proof of the existence and uniqueness on the previous step was specified with an arbitrary initial condition. The process of matching endpoints is then continued to the right until the starting interval $J_1[b)$ is reached, at which time the matching procedure is terminated since one is not free to choose the value of $E(x)$ at $x = b$. However, after the induction step on J_a, and after the endpoint matching procedure, one will have proven the existence of a unique continuous solution to the following problem,

$$\begin{aligned} U\tilde{R}(x) &= -c \qquad x \in [A,b) \\ \tilde{R}(x) &= E(a) \\ \tilde{R}(x) &= E(x) \qquad \forall x \in J_0[b), \end{aligned} \tag{1.2.8}$$

where A is the left hand endpoint of J_a, and $J_0[b)$ is the semi-open interval in 1.2.2. The only difference between 1.2.8 and 1.2.1, is that the solution to 1.2.1 lives on all of $(0, b)$, whereas the solution to 1.2.8 does not. However, consider the problem,

$$\begin{aligned} U\hat{R}(x) &= -c \qquad x \in (0, A) \\ \hat{R}(x) &= \tilde{R}(A^+) \\ \hat{R}(x) &= \tilde{R}(x) \qquad \forall x \in J_a. \end{aligned} \tag{1.2.9}$$

If the existence and uniqueness of a continuous solution to 1.2.9 can be shown, then by combining this and the result for 1.2.8, it follows that,

$$R(x) = \hat{R}(x)1\{x < A\} + \tilde{R}(x)1\{x \geq A\}, \tag{1.2.10}$$

will solve the original problem.

The problem 1.2.8 will be handled first, dropping the tildes for notational convenience. To start, it is necessary to define the discretization intervals and towards this end, consider the advance Σ defined as,

$$\Sigma^0 x = x, \;\; \Sigma^1 x = \frac{u+1}{u+x} x, \;\; \Sigma^n x = \Sigma^{n-1}(\Sigma x). \tag{1.2.11}$$

If one asks that Σ^{-1} satisfy $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = \Sigma^0$, then this yields,

$$\Sigma^{-1} x = \frac{ux}{u+1-x}, \tag{1.2.12}$$

and Σ^{-1} is called the one step retardation. For any $x \in (0, 1)$, it follows easily that,

$$0 < \Sigma^{-n} x < x < \Sigma^n x < 1 \quad \forall n \geq 1, \tag{1.2.13}$$

and hence the significance of the terminology 'advance' and 'retardation'. With these notions, one can define,

$$J_n[b) = [\Sigma^{-n} b, \Sigma^{-(n-1)} b) \quad n \geq 0, \tag{1.2.14}$$

as the nth discretization interval; this agrees with the previous definition given for $J_0[b)$ given above. Note that there will be occasion to use the notation $J_n(x)$, with the obvious interpretation. Observe that $\Sigma^{-n}x$ ($\Sigma^n x$) is monotone decreasing (increasing) and bounded below (above). Hence, there exists $n = n_a$ such that $\Sigma^{-n_a}b \leq a$, and such that this is the smallest such number satisfying the inequality. In the previous informal notation then, $J_a = J_{n_a}[b)$, and $A = \Sigma^{-n_a}b$. Thus, the existence and uniqueness of a soution to 1.2.8 will follow if one can prove this for solutions of

$$\begin{aligned} R_n'(x) &= F_n(x, R_n(x)) \qquad x \in J_n[b) \\ R_n(\Sigma^{-n}b) &= R_n, \end{aligned} \tag{1.2.15}$$

for each $1 \leq n < n_a$, and in addition do the same for a solution of,

$$\begin{aligned} R_{n_a}'(x) &= F_{n_a}(x, R_{n_a}(x)) \quad x \in J_{n_a}(b) \\ R_{n_a} &= E(a), \end{aligned} \tag{1.2.16}$$

where in 1.2.15, R_n is an arbitrary constant and,

$$F_n(x, R) = G_n(x) - \frac{u + x}{x(1 - x)}R, \tag{17}$$

where,

$$G_n(x) = \frac{c + (u + x)R_{n-1}\left(\frac{u+1}{u+x}x\right)}{x(1 - x)}, \tag{1.2.18}$$

with,

$$R_0(x) = E(x) \qquad x \in J_0[b). \tag{1.2.19}$$

Let n=1, for the first induction step. Then it is a simple consequence of elementary ODE theory that a unique solution to 1.2.15 exists passing through any point interior to the set $(J_1[b), \mathbb{R})$. To prove a unique solution exists starting from the boundary point $(\Sigma^{-1}b, \mathbb{R})$, one can choose $\epsilon_1 > 0$ such that that $\Sigma^{-1}b - \epsilon_1 > 0$, and then define $\bar{J}_1(b) = (\Sigma^{-1}b - \epsilon_1, b)$. Using these open intervals, consider the auxiliary ODE,

$$\begin{aligned} \bar{R}'(x) &= F_1(x, \bar{R}(x)) \qquad x \in \bar{J}_1(b) \\ \bar{R}(\Sigma^{-1}b) &= R_1. \end{aligned} \tag{1.2.20}$$

Now apply ODE theory to 1.2.20. Specifically, $F_1(x,R)$ is continuous for all (x,R) in the open set $(\bar{J}_1(b),\mathbb{R})$. In view of 1.2.19, F_1 has a continuous first partial derivative in R for all $(x,R) \in (\bar{J}_1(b),\mathbb{R})$. So, applying the standard results (see HALE, Theorem 3.1), $\bar{R}$ satisfying 1.2.20 exists and is unique. Therefore, if one defines the restriction,

$$R_1(x) = \bar{R}(x) \qquad x \in J_1[b), \tag{1.2.21}$$

then $R_n(x)$ uniquely exists satisfying 1.1.15 for $n = 1$. Notice from 1.2.15 that $R_1'(x)$ is continuously differentiable for all $x \in J_n(b)$. Therefore, $R_1(x)$ and thence also $G_2(x)$ is continuous there. The proof for any $n < n_a$ is the same as for $n = 1$ with the following observations. First, realize that one can always find an ϵ_n, $n \geq 1$, satisfying $\Sigma^{-n}b - \epsilon_n > 0$, in view of 1.2.13. Thus for any $n \geq 1$, $\bar{J}_n(b)$ is well-defined. Second, from 1.2.17, it is clear that $F_n(x,R)$ has a continuous first partial in R for all $(x,R) \in ((0,1),\mathbb{R})$, and hence this is true for all $(x,R) \in (\bar{J}_n(b),\mathbb{R})$. Lastly, it should be clear that $F_n(x,R)$ is continuous for all $(x,R) \in (\bar{J}_n(b),\mathbb{R})$. It is clearly true for n=1, has been shown to be true for $n = 2$, and is true by induction in general. In summary, a unique solution to 1.2.15 with arbitrary initial condition has been shown to exist for each n, $1 \leq n \leq n_a$. The proof of the existence of a unique solution to 1.2.16 is the analogous. If $a = \Sigma^{-n_a}b$, then the proof is exactly the same. If $a > \Sigma^{-n_a}b$, then there is no need to consider the auxiliary problem since then 'a' is interior to $J_{n_a}[b)$, and the proof follows directly.

To begin the endpoint matching process, note that since F_n is so well behaved for all $(x,R) \in ((0,1),\mathbb{R})$, then the solution on $J_{n_a}[b)$ can be extended naturally to include the right-hand endpoint (see HALE, Lemma 2.1). Then define $R_{n_a-1}(\Sigma^{-(n_a-1)}b) = R_{n_a}(\Sigma^{-(n_a-1)}b)$. Continue this process of extension and matching up to and including $J_2[b)$. The unique solution to 1.2.8, for a, b as given, has now been demonstrated to exist.

To complete the proof, it is necessary to extend the solution continuously to the

left, i.e., solve the problem given in 1.2.9. First observe that the solution to 1.2.8 can be extended naturally to the left-hand endpoint of $J_{n_a}[b)$ (again, see HALE, Lemma 2.1). Thus solving 1.2.9 is equivalent to proving the existence and uniqueness of a solution to (hats removed for notational politeness),

$$\begin{aligned} R_n'(x) &= F_n(x, R_n(x)) \qquad x \in J_n[b) \\ R_n((\Sigma^{-(n-1)}b)^-) &= R_n, \end{aligned} \tag{1.2.22}$$

for all $n > n_a$, with $R_{n_a+1} = R_{n_a}(\Sigma^{-n_a}b)$. The proof in this case is also by induction on the analogous 'overbarred' problem (see 1.2.20), where now $\bar{J}_n(b) = (\Sigma^{-n}b, \Sigma^{-(n-1)}b+\epsilon_n)$, and again, $\epsilon_n > 0$ can always be chosen so that $\Sigma^{-(n-1)}b+\epsilon_n < 1$. This time, at each step of the induction, the solution is extended naturally to the left, providing an initial condition on the next induction step. Finally, since for any $x \in (0, b)$, there exists a smallest n_x such that $\Sigma^{-n_x} < x$, the solution to 1.2.9 is secured. Recalling 1.2.10, the theorem has been shown. ∎

Corollary *If in addition, $E(x)$ is n-times continuously differentiable on $J_0[b)$, $n \geq 1$, then the unique solution to the problem is also n-times continuously differentiable except possibly for $x \in \{\Sigma^{-k}b : k = 0, ...n\}$. Also, if $R(b^-) = R(b)$ for b given, then $R(x)$ is n-times continuously differentiable for all $n \geq 1$, and $x < b$.*

Proof: Follows directly from 1.2.1; solve for the derivative and use induction. ∎

This concludes the current section and chapter. In the next chapter, the problem of Bayesian sequential hypothesis testing is presented.

Chapter II
Bayesian Sequential Hypothesis Tests

2.0 Introduction

In this chapter, the Bayesian sequential hypothesis testing problem is presented. The first section establishes the probabilistic framework which will be used throughout the thesis. In addition, the class of possible solutions to the sequential hypothesis testing problem is defined. Within this class of so called *admissible policies*, is the class of *threshold policies* which is also defined. The second section gives the particulars of the general Bayesian problem, however, only the binary case is presented. The Bayes risk is defined and is shown to be concave.

In section 2.3, the first steps are taken to reduce the complexity of the minimization problem posed by the Bayes risk. Theorem 2.3.1 shows that for any policy $\tilde{u} = (\tilde{\tau}, \tilde{\delta}) \in \mathcal{U}$, one can replace $\tilde{\delta}$ with a certain $\mathcal{F}_{\tilde{\tau}}$-measurable random variable and by so doing obtain a new policy whose risk is not larger. The immediate consequence is that the infimum over all admissible policies is no smaller than the infimum over their first component. Further insight is gained in Theorem 2.3.2 which helps characterize the form of the risk. Next it is shown that threshold policies are unchanged by the replacement procedure of Theorem 2.3.1 — an obvious property to check before attempting to prove that such policies are optimal. The chapter ends by demonstrating that policies invariant under Theorem 2.3.1, and in particular threshold policies, have terminal errors whose sum is bounded by one.

2.1 Preliminaries

In this section, the problem of testing two statistical hypotheses will be presented using the Bayesian risk approach. The following probablistic framework forms the starting point of the investigation.

On a measurable space $(\Omega, \mathcal{F})$, there are given two additional probability measures P_i , i=0,1. One is given a filtration $\{\mathcal{O}_t : t \in T\}$, where $T \subseteq [0, \infty)$ is the time parameter set, either continuous or discrete. If $T = [0, \infty)$, then $\mathcal{O}_t$ is taken to be right continuous. The $\mathcal{O}_t$ filtration should be thought of as being the history of some observable process. There is also given Θ, the hypothesis parameter set. For instance, in the simplest case of binary hypotheses, $\Theta = \{\theta_0, \theta_1\}$.

The general hypothesis testing problem can be loosely described as discovering a plan, or policy, which dictates when to stop and what to decide, based upon the available information, and which also satisfies a given performance criterion. Any policy then, ought to be a pair consisting of a stopping time and a deciding rule, and an optimum policy is one which achieves a specified performance. In view of this, it is clear that in seeking optimum policies, the search is naturally limited to a class of candidate policies whose membership in the class, or *admissibility*, is defined by practical measurability requirements. That is to say, each component of an eligible policy pair should at least, *and at most*, be measurable with respect to some **σ-algebra** which contains all the information that might possibly be observed. A natural definition of admissible policy then, is given in terms of the observation filtration, $\mathcal{O}_t$.

Definition 2.1.1 *An* **admissible policy**, u, *is a pair,* $u = (\tau, \delta)$, *where* τ *is an* $\mathcal{O}_t$*-stopping time taking values in* T, *and* δ *is a* $(P, \mathcal{O}_\tau)$*-random variable taking values in* Θ. *The class of admissible policies will be denoted as* $\mathcal{U}$.

In the binary hypothesis testing problem, the choice of $u \in \mathcal{U}$ prescribes a policy which specifies, via τ, when the observation is to be terminated, and specifies via δ, which hypothesis is to be accepted. That is to say if $\delta = i$, then the hypothesis $\theta = \theta_i$

will be accepted, $i = 0, 1$.

An important subclass of admissible policies are the threshold policies. This is due both to their simple specification and remarkable optimality properties.

Definition 2.1.2 *Let X be an $\mathcal{O}_t$-adapted, $\mathbb{R}$-valued process. An* (X, I)**-threshold policy** $\bar{u} \in \mathcal{U}$ *is a pair,* $\bar{u} = (\bar{\tau}, \bar{\delta})$*, where,*

$$\bar{\tau} = \inf\{t \geq 0 : X_t \notin I\}$$

and,

$$\bar{\delta} = \begin{cases} 1 & X_{\bar{\tau}} \geq b \\ 0 & X_{\bar{\tau}} \leq a \end{cases}$$

where $a \leq b$ are the **thresholds,** *and $I = (a, b)$ is called the* **threshold,** *or* **continuation interval.** *The class of threshold policies will be denoted as $\bar{\mathcal{U}}$; note that, $\bar{\mathcal{U}} \subseteq \mathcal{U}$. If $a = b$, the threshold policy is said to be* **degenerate** *and one defines $\bar{\tau} \equiv 0$ and $\bar{\delta} = 1$ if $X_0 = b$.*

Observe that for a given filtration, there are as many possible threshold policies as there are ways to choose a pair (X, I) which satisfies the definition. For a given $\mathcal{O}_t$-adapted process X however, the hope is that one can find *a priori* a particular threshold pair such that the threshold policy so defined is optimal in some sense. For our purposes, X will be taken to be either a certain *a posteriori* probability or (log)likelihood ratio process, the important similarity being that they all satisfy Theorem 1.1.1. As a matter of style, the policies in Definition 2.1.2 will be called simply 'threshold policies' when there is no ambiguity in either the reference process or threshold interval.

Consider an arbitrary admissible policy $u \in \mathcal{U}$. For each $\omega \in \Omega$, the policy $u = (\tau, \delta)$ can be described very generally as incurring two kinds of losses, one due to the cost of waiting to decide, and the other due to making a wrong decision. A reasonably general characterization of the cost of the observation time, or *running*

cost, is given by $\int_0^\tau c_s\, ds$, where $\{c_t\,,\, t \geq 0\}$ is some $\mathcal{O}_t$-adapted process which serves as a suitable cost measure. For instance, $c_t = c \geq 0$, a positive constant, is often chosen with the intention of using an apparently simple running cost to capture the behavior: 'the longer it takes to decide, the more it costs'. Of course, this is not the only cost process which captures this behavior, nor is it necessarily simple, for the simplicity of a running cost is often best judged not by its specification, but by how easily it yields to mathematical analysis. Indeed, a fundamental tradeoff in the modeling aspects of a sequential testing problem is choosing a running cost which can be worked with mathematically, *and still captures a desired behavior.*

Next, consider that each admissible policy can make two types of incorrect decisions,

$$\begin{cases} \delta = 0 \quad \text{while} \quad \theta = \theta_1 \\ \delta = 1 \quad \text{while} \quad \theta = \theta_0. \end{cases}$$

Naturally, it will be desirable to include in the total cost structure a penalty for each of these events and then minimize in some way the total cost, and in this sense minimize the probabilities of these errors.

This concludes the section. In the next, the discussion will focus on the details of the Bayesian formulation.

2.2 Bayesian Formulation

Let θ be a $\{0,1\}$-valued, $\mathcal{F}$-measurable random variable, $\theta : \Omega \to \Theta$. Fix an arbitrary $\pi \in [0,1]$ and define a probability measure P_π on $(\Omega, \mathcal{F})$ via,

$$P_\pi\{\omega : \theta(\omega) = 1\} = \pi \quad , \quad P_\pi\{\omega : \theta(\omega) = 0\} = 1 - \pi, \tag{2.2.1}$$

such that for each set $A \in \mathcal{F}$ there holds,

$$P_\pi\{A\} = \pi P_1\{A\} + (1-\pi)P_0\{A\}. \tag{2.2.2}$$

In words, it is assumed that θ is a binary random variable taking on the values *zero* or *one* with the *a priori* probabilities $1-\pi$ and π, respectively. The *a posteriori* probability of the $\theta = 1$ hypothesis is then defined as $\pi_t = P_\pi\{\theta = 1 | \mathcal{O}_t\}$, $t \in T$, and since $\mathcal{O}_0 = \{\emptyset, \Omega\}$, it follows that $P_\pi\{\pi_0 = \pi\} = 1$.

Consider an arbitrary admissible policy $u \in \mathcal{U}$ as defined above. For each $\omega \in \Omega$, the policy $u = (\tau, \delta)$ can be described as incurring two kinds of losses, one due to the cost of waiting to decide, the other due to making wrong decisions. On average therefore, a natural definition for the average running cost is given by $E_\pi[\int_0^\tau c_s ds]$, where $\{c_t, t \geq 0\}$ is some nonnegative $\mathcal{O}_t$-adapted process which yields a desired cost behavior.

To describe the cost of making wrong decisions, define

$$w(\theta,\delta) = \begin{cases} c^0 & \text{if} \quad \theta = 1,\ \delta = 0; \quad c^0, > 0 \\ c^1 & \text{if} \quad \theta = 0,\ \delta = 1; \quad c^1, > 0 \\ 0 & \text{if} \quad \delta = \theta, \end{cases} \tag{2.2.3}$$

yielding the average cost of incorrect decisions as $E_\pi[w(\theta,\delta)]$. Observe that it is without loss of generality in the search for an optimal policy that there is no cost levied for correct decisions in 2.2.3. Now, one can easily show that

$$E_\pi[w(\theta,\delta)] = c^0 \pi P_1\{\delta = 0\} + c^1(1-\pi)P_0\{\delta = 1\}.$$

Putting the pieces together yields,

$$\rho(\pi, u) = E_\pi[\int_0^\tau c_s ds + w(\theta, \delta)] \tag{2.2.4}$$

as the total P_π-average Bayesian cost, or risk, of the policy $u = (\tau, \delta)$. By definition it is clear that $\rho(\pi, u)$ is nonnegative for all $\pi \in [0,1]$ and $u \in \mathcal{U}$, and therefore the **Bayes risk**, or **cost**, given by,

$$\rho(\pi) = \inf_{u \in \mathcal{U}} \rho(\pi, u) \tag{2.2.5}$$

is nonnegative and exists for all $\pi \in [0,1]$. For a particular $\pi \in [0,1]$, one would like to find a $u \in \mathcal{U}$ which minimizes the risk at that prior. From 2.2.5 it is easy to show that

$$\rho(0) = 0 = \rho(1), \tag{2.2.6}$$

and thus for $\pi \in \{0,1\}$, an optimal policy is $u = (\tau, \delta) = (0, \pi)$, i.e., stop immediately and decide π. Obviously this trivial policy is not optimal for other values of π, but any rival policy which is must behave as u for $\pi \in \{0,1\}$, i.e., it must stop immediately and decide π.

From 2.2.5 it also follows that $\rho(\pi)$ is concave. This important observation is recorded in the following theorem for future reference.

Theorem 2.2.1 *The Bayes risk (2.2.5) is concave in π on $[0,1]$.*

Proof:

Combine 2.2.1-2.2.4 to get,

$$\rho(\pi, u) = \left[E_1[\int_0^\tau c_s\, ds + c^0 \pi_\tau 1\{\delta = 0\}] - E_0[\int_0^\tau c_s\, ds + c^1(1-\pi_\tau)1\{\delta = 1\}]\right] \cdot \pi + E_0[\int_0^\tau c_s\, ds + c^1(1-\pi_\tau)1\{\delta = 1\}].$$

With this in mind, it follows that for any $\alpha \in (0,1)$, and π', $\pi'' \in [0,1]$,

$$\begin{aligned}\rho(\alpha\pi' + (1-\alpha)\pi'') &= \inf_{u \in \mathcal{U}} \rho(\alpha\pi' + (1-\alpha)\pi'', u) \\ &\geq \alpha \inf_{u \in \mathcal{U}} \rho(\pi', u) + (1-\alpha) \inf_{u \in \mathcal{U}} \rho(\pi'', u) \\ &= \alpha\rho(\pi') + (1-\alpha)\rho(\pi''),\end{aligned}$$

The desired result. ∎

The following definition serves to characterize the optimality of admissible policies for the Bayesian formulation.

Definition 2.2.1 *Let $I \subseteq [0,1]$. The policy $u^* \in \mathcal{U}$ is said to be* I-**Bayesian** *if*

$$\rho(\pi, u^*) = \rho(\pi) \qquad \forall \pi \in I \tag{2.2.7}$$

If $u^* \in \mathcal{U}$ is $[0,1]$-Bayesian, it is said to be **Bayesian.**

Consider that any reasonable, admissible policy will be at least $\{0,1\}$-Bayesian, and so for convenience of notation, I will be understood to denote $I \cup \{0,1\}$ unless noted otherwise. Given this definition, one of the principal goals of this thesis can be succinctly stated as demonstrating that there exists a threshold policy, say with thresholds a, b, $a \leq b$, which is at least (a,b)-Bayesian, and preferably Bayesian, for a given observation filtration. It is this goal of demonstrating the existence of such a threshold policy and thresholding *interval* which motivates the use of the suggestive 'I' notation in the definition.

For completeness, one technical assumption perhaps deserves mention. In 2.2.3 it is assumed that both costs for wrong decisions are strictly positive. This is done to avoid degenerate cases for which no Bayesian threshold policy exists or is unique. For instance, let $i \in \{0,1\}$ and suppose $c^i = 0$, while the other cost is nonzero. Then it is easy to show that the policy $u_i = (\tau, \delta_i) = (0, i)$ is ($[0,1]$-) Bayesian. It also follows that there exists no threshold policy which is Bayesian optimal (consider $\pi = |i-1|$). Likewise, if $c^0 = c^1 = 0$, then any policy with stopping time $\tau = 0$ P_π-a.s. $\forall \pi \in [0,1]$ is Bayesian, and this includes all the degenerate threshold policies.

This concludes the section. The next section begins by showing how the infimum over all admissible policies appearing in 2.2.5 can be replaced by a much simpler minimization. This result is the first to reveal why threshold policies enjoy their special status.

2.3 Optimal Stopping

Having considered basic notions of Bayesian hypothesis testing in the previous sections, the focus here is to lessen the complexity of the search for the optimal policy. Towards this end it is shown in Theorem 2.3.1 that finding a Bayesian policy can be reduced to a problem of optimal stopping, i.e., one can restrict the search for the optimal policy, an infimum operation over both policy components, to a search for an optimal stopping policy, an infimum operation over the stopping time component alone. This is accomplished by showing that any policy pair can be replaced by another pair whose risk is no larger. The new pair is obtained from the original by replacing its decision component with a new decision variable which has an explicit representation in terms of the *a posteriori* probability; the stopping time component is kept the same. This yields a simpler expression for the Bayes risk which is then exploited to discover more of its structure. Also, the question arises as to the effect of the replacement procedure in Theorem 2.3.1 on threshold policies, and it is shown that such policies remain unchanged—an obvious property to check before one tries to prove that threshold policies are Bayesian! Prior to this, the theorem is used to further characterize the risk as was begun in Theorem 2.2.1. The section ends by showing that the sum of the terminal errors for a threshold policy are bounded by one. The next theorem is a generalization of Lemma 4.1.1 in [SHIRYAYEV].

Theorem 2.3.1 *Let $\tilde{u} = (\tilde{\tau}, \tilde{\delta}) \in \mathcal{U}$ be arbitrary, and define $u = (\tau, \delta) \in \mathcal{U}$ via, $\tau = \tilde{\tau}$ and,*

$$\delta = \begin{cases} 1 & c^0 \pi_\tau \geq c^1(1 - \pi_\tau) \\ 0 & c^0 \pi_\tau < c^1(1 - \pi_\tau). \end{cases} \tag{2.3.1}$$

Then, for all $\pi \in [0, 1]$,

$$\rho(\pi, \tilde{u}) \geq \rho(\pi, u) = E_\pi[\int_0^\tau c_s ds + e(\pi_\tau)] \tag{2.3.2}$$

where,

$$e(\pi) = \min\{c^0 \pi, c^1(1 - \pi)\}, \tag{2.3.3}$$

and $\{c_t : t \geq 0\}$ *is an* $\mathcal{O}_t$*-adapted, nonnegative process.*

Proof: If $\pi = 0$ or 1, then the assertions are easy to check, so suppose $\pi \in (0,1)$. From definition 2.2.1, it is clear that since $\{\tilde{\delta} = i\} \in \mathcal{O}_{\tilde{\tau}}$ for $i = 0, 1$, there holds,

$$\begin{aligned} E_\pi[1\{\tilde{\delta} = 0\}1\{\theta = 1\}] &= E_\pi[1\{\tilde{\delta} = 0\}E_\pi[1\{\theta = 1\}|\mathcal{O}_{\tilde{\tau}}]] \\ &= E_\pi[1\{\tilde{\delta} = 0\}\pi_{\tilde{\tau}}]. \end{aligned} \tag{2.3.4}$$

and similarly,

$$E_\pi[1\{\tilde{\delta} = 1\}1\{\theta = 0\}] = E_\pi[1\{\tilde{\delta} = 1\}(1 - \pi_{\tilde{\tau}})]. \tag{2.3.5}$$

So, combining 2.2.3, 2.2.4, 2.3.4, and 2.3.5 yields,

$$\begin{aligned} \rho(\pi, \tilde{u}) &= E_\pi[\int_0^{\tilde{\tau}} c_s ds + w(\theta, \tilde{\delta})] \\ &= E_\pi[\int_0^{\tilde{\tau}} c_s ds + c^0 1\{\tilde{\delta} = 0\}1\{\theta = 1\} + c^1 1\{\tilde{\delta} = 1\}1\{\theta = 0\}] \\ &= E_\pi[\int_0^{\tilde{\tau}} c_s ds + c^0(1 - \tilde{\delta})\pi_{\tilde{\tau}} + c^1\tilde{\delta}(1 - \pi_{\tilde{\tau}})] \\ &= E_\pi[\int_0^{\tau} c_s ds + c^0(1 - \tilde{\delta})\pi_\tau + c^1\tilde{\delta}(1 - \pi_\tau)] \\ &\geq E_\pi[\int_0^{\tau} c_s ds + e(\pi_\tau)] = \rho(\pi, u), \end{aligned} \tag{2.3.6}$$

and the theorem is shown. ∎

Observe that the theorem is very general. No assumptions were made regarding the continuity of either the time parameter or state-space; one has only to interpret the integrals in the proper way. The extension to multiple hypotheses is tedious but straightforward. In addition, notice from definition 2.1.2 that a threshold policy $(\bar{\tau}, \bar{\delta}) \in \bar{\mathcal{U}}$ is entirely specified by its first component (via the thresholds). That is to say, if $\bar{\tau}$ is given, $\bar{\delta}$ is completely determined, a characteristic not shared by admissible policies in general. The theorem however shows that in minimizing the risk, the infimum over all policies is no smaller than the infimum over all policies which share this characteristic. This is a welcome simplification of the problem for it

says that in the search for Bayesian decision policies, one need only consider policies $u = (\tau, \delta)$ with δ given by 2.3.1 and with the risk given by,

$$\rho(\pi) = \inf_{u \in \mathcal{U}} \rho(\pi, u) = \inf_{\tau \in \mathcal{T}} \rho(\pi, \tau),$$

where,

$$\rho(\pi, \tau) = E_\pi[\int_0^\tau c_s ds + e(\pi_\tau)], \tag{2.3.7}$$

and where $\mathcal{T}$ denotes the class of $\mathcal{O}_t$-stopping times. In fact, without loss of generality it is sufficient to take $\mathcal{T}$ to be the subclass of P_π-a.s. finite stopping times if the following technical condition holds,

$$\text{(T):} \qquad E_\pi[\int_0^\infty c_s ds] = \infty.$$

Intuitively, the condition (T) is entirely natural since it is unusual to impose a running cost which fails to force decisions in finite time in problems of sequential hypothesis testing. As a result, *it will be assumed throughout the remainder of this thesis that condition (T) holds.* As a sidenote the reader should be aware that the cost due to making wrong decisions is often called the 'terminal cost' in the Bayesian formulation. This usage is a direct consequence of the theorem, and equation 2.3.1 in particular.

To motivate the next theorem, consider the set defined via,

$$C_0 = \{\pi : \rho(\pi) < e(\pi)\}.$$

The set C_0 is called the **natural continuation set** for the obvious reason that one honestly expects to continue collecting data as long as the risk is less than the terminal cost. In addition to C_0, one can define the **natural stopping set** as,

$$S_0 = \{\pi : \rho(\pi) = e(\pi)\},$$

without loss of generality (see Theorem 2.3.2 below). It is easy to see that the two sets are disjoint and that their union is the whole interval, $[0, 1]$. Now because the

goal is to prove that threshold policies are optimal, it had better be the case that C_0 is an open and connected set since one believes intuitively that the stopping interval of a threshold policy ought to be identical to C_0, if the policy is Bayesian. This and more is shown to be true in the theorem below using the simpler form of the risk given in 2.3.7.

Theorem 2.3.2 *Let* $\pi_e = \frac{c^1}{c^0+c^1}$, *i.e., the (unique) point at which* $e(\pi)$ *attains its maximum value* $(e(\pi_e) = \frac{c^0 c^1}{c^0+c^1})$. *Then there exist two numbers* a_* *and* b_* *where*

$$0 \leq a_* \leq \pi_e \leq b_* \leq 1, \tag{2.3.8}$$

and such that the Bayes risk satisfies,

$$\rho(\pi) < e(\pi) \qquad \forall \pi \in (a_*, b_*)$$

$$\rho(\pi) = e(\pi) \qquad \forall \pi \in [0,1] \setminus (a_*, b_*).$$

In addition, $\rho(\pi)$ *is continuous on* $[0,1]$.

Proof:

In view of the fact that $e(\pi)$ is concave, for any $\pi \in [0,1]$ an easy consequence of Jensen's Inequality is,

$$\begin{aligned}
\rho(\pi) &= \inf_{\tau \in \mathcal{T}} E_\pi [\int_0^\tau c_s \, ds + e(\pi_\tau)] \\
&\leq \inf_{\tau \in \mathcal{T}} [E_\pi \int_0^\tau c_s \, ds + e(E_\pi[\pi_\tau])] \\
&= \inf_{\tau \in \mathcal{T}} E_\pi \int_0^\tau c_s \, ds + e(\pi) \\
&= e(\pi),
\end{aligned}$$

using the obvious fact that $\{\pi_t : t \geq 0\}$ is a **uniformly integrable** martingale. Thus,

$$\rho(\pi) \leq e(\pi) \qquad \forall \pi \in [0,1]. \tag{2.3.9}$$

Now since $e(0) = 0 = e(1)$, it is clear from 2.2.6, the concavity of $e(\pi)$, and 2.3.9 that $\rho(\pi)$ is continuous on $[0,1]$. This inequality also justifies the definition of S_0 given above.

It is now geometrically obvious that there exist the two numbers a_* and b_* as claimed, nevertheless one can argue as follows. If $S_0 = [0,1]$, then $C_0 = \emptyset$ and choosing $a_* = \pi_e = b_*$ yields the result. If $S_0 = \{0,1\}$, then $C_0 = (0,1)$ and choosing $a_* = 0$, $b_* = 1$ does the same. Otherwise, choose $\pi' \in S_0$, $0 < \pi' < 1$, and suppose $\pi' \leq \pi_e$. Then the following is true,

$$\rho(\pi') = e(\pi') \Longrightarrow \pi \in S_0 \quad \forall\, \pi \leq \pi' \leq \pi_e, \tag{2.3.10}$$

for if not, there exists $\tilde{\pi} < \pi'$ such that $\tilde{\pi} \in C_0$, implying $\rho(\tilde{\pi}) < e(\tilde{\pi})$. But this violates the concavity and continuity of $\rho(\pi)$ on $[0,1]$. To see this, choose $\alpha \in [0,1]$ such that

$$\tilde{\pi} = \alpha \cdot 0 + (1-\alpha)\pi'$$

and then,

$$\begin{aligned}
\rho(\tilde{\pi}) &= \rho(\alpha \cdot 0 + (1-\alpha)\pi') \\
&\geq \alpha\rho(0) + (1-\alpha)\rho(\pi') \\
&= (1-\alpha)e(\pi') \\
&= e(\tilde{\pi}) \\
&> \rho(\tilde{\pi}),
\end{aligned}$$

a contradiction. The same contradiction is derived if one supposes $\pi' \geq \pi_e$, and therefore the following is also true,

$$\rho(\pi') = e(\pi') \Longrightarrow \pi \in S_0 \quad \forall\, \pi \geq \pi' \geq \pi_e. \tag{2.3.11}$$

Having established 2.3.10 and 2.3.11, it is now simple to obtain the final case. It is clear that C_0 is an open set so choose π', $\pi'' \in C_0$, with $\pi' < \pi''$, and of course assume $C_0 \neq \emptyset$. Now if C_0 is not a connected set then there exists $\alpha \in (0,1)$ such that

$$\alpha\pi' + (1-\alpha)\pi'' \in S_0,$$

whence at least $\pi' \in S_0$ or $\pi'' \in S_0$ in view of 2.3.10 and 2.3.11. A contradiction results in either case. Hence, C_0 is an open interval and therefore take a_*, b_* to be its right- and left-hand endpoints, respectively. ∎

The preceding theorem has shown that the natural continuation set is an interval and is therefore the primary candidate upon which to base a threshold policy, but there are special cases to consider. If C_0 is an empty interval, then $\rho(\pi) = e(\pi)$ and the identically zero stopping time is clearly Bayesian optimal. Obviously the degenerate threshold policy with thresholds $a = \pi_e = b$ is also Bayesian optimal in this case. On the other hand if $C_0 = (0,1)$ then there is no optimal threshold policy. To see this, pick an arbitrary threshold policy $\bar{u} \in \bar{\mathcal{U}}$, with thresholds $0 < a < b < 1$. Choose $\pi \in \partial_\pi(a,b)$ and suppose $\pi \notin \{0,1\}$. Then,

$$E_\pi[\int_0^{\bar{\tau}} c_s\, ds + e(\pi_{\bar{\tau}})] = e(\pi) > \rho(\pi).$$

Thus $\bar{u}$ is not $\{\pi\}$-optimal and therefore cannot be Bayesian optimal. Alternatively, choose $(a,b) = (0,1)$ to be the thresholds. Then $\bar{\tau} = \infty$, P_π-a.s. $\forall\, \pi \in (a,b)$, and in view of (T),

$$E_\pi[\int_0^{\bar{\tau}} c_s\, ds + e(\pi_{\bar{\tau}})] = \infty,$$

which is clearly not an optimal policy. Similarly one can show that if one endpoint of C_0 is *zero* or *one*, then no optimal threshold policy exists. In the remainder of the thesis therefore, these special cases are of no consequence and it will be assumed that *the class of threshold policies $\bar{\mathcal{U}}$ is further restricted to those policies whose thresholds obey the following inequalities,*

$$\text{(I)} \qquad 0 < a \leq \pi_e \leq b < 1; \quad a < b,$$

where $\pi_e = \arg\max_{\pi \in [0,1]} e(\pi)$. Put another way, under condition (I), if $\bar{u} \in \bar{\mathcal{U}}$, then $0 \leq \bar{\tau} < \infty$, P_π-a.s. $\forall\, \pi \in (a,b)$. For the remainder of the thesis the notation I_0 will be used with the implicit understanding that $I_0 = (a,b)$ satisfies condition (I).

A short recapitulation is in order. Theorem 2.3.1 showed that any policy could be replaced by another which would do no worse and this lead to a simpler expression for the Bayes risk. Using this simpler form of the risk, the last theorem yielded more insight into the structure of the risk. But Theorem 2.3.1 raises another point. In light of that theorem, a natural question to ask is what policy results when a threshold policy $(\bar{\tau}, \bar{\delta})$ is exchanged for the policy $(\bar{\tau}, \delta)$, with δ as in 2.3.1. If $\bar{u}$ is a threshold policy based upon the *a posteriori* probability, the next theorem shows that $(\bar{\tau}, \bar{\delta})$ and $(\bar{\tau}, \delta)$ are identical.

Theorem 2.3.3 *Let $I_0 = (a, b)$ (satisfying condition (I) above), and consider the threshold policy $(\bar{\tau}, \bar{\delta})$ based on I_0 and $\{\pi_t : t \geq 0\}$ (see def. 2.1.2). Then,*

$$\bar{\delta}(\omega) = \delta(\omega) \qquad \forall \omega \in \Omega,$$

with δ as in Theorem 2.3.1.

Proof:

Suppose $\omega \in \Omega$ is such that, $\pi_{\bar{\tau}(\omega)}(\omega) \geq b$, and therefore $\bar{\delta}(\omega) = 1$. Then from 2.3.1,

$$\frac{\pi_{\bar{\tau}}}{1 - \pi_{\bar{\tau}}} \geq \frac{b}{1 - b}.$$

On the other hand, $b > c^1/(c^0 + c^1)$, and so,

$$\frac{b}{1 - b} > \frac{c^1}{c^0}.$$

Thus $\pi_{\bar{\tau}} \geq b$ implies $c^0 \pi_{\bar{\tau}} > c^1(1 - \pi_{\bar{\tau}})$, and thus $\delta(\omega) = 1$. Now let $\omega \in \Omega$ be such that $\delta(\omega) = 1$, implying,

$$\frac{\pi_{\bar{\tau}(\omega)}(\omega)}{1 - \pi_{\bar{\tau}(\omega)}(\omega)} \geq \frac{c^1}{c^0},$$

so that upon rearranging one obtains,

$$\pi_{\bar{\tau}(\omega)}(\omega) \geq \frac{c^1}{c^0 + c^1} > a,$$

and therefore $\bar{\delta}(\omega) = 1$. From this one concludes that,

$$\delta(\omega) = 1 \Longleftrightarrow \bar{\delta}(\omega) = 1 \qquad \forall \omega \in \Omega.$$

Similarly, one can show,

$$\delta(\omega) = 0 \Longleftrightarrow \bar{\delta}(\omega) = 0 \qquad \forall \omega \in \Omega.$$

This completes the argument. ∎

The fact that threshold policies are invariant with respect to Theorem 2.3.1 allows one to prove the next interesting result. Clearly, the proof below depends on this invariance property alone, but it will be stated only in terms of threshold policies for emphasis.

Theorem 2.3.4 *Let $\bar{u} \in \bar{U}$ be a threshold policy. Then,*

$$P_0\{\bar{\delta} = 1\} + P_1\{\bar{\delta} = 0\} \leq 1.$$

Proof:

From Theorem 2.3.3 there holds,

$$E_\pi[\int_0^{\bar{\tau}} c_s \, ds + w(\theta, \bar{\delta})] = E_\pi[\int_0^{\bar{\tau}} c_s \, ds + e(\pi_{\bar{\tau}})],$$

and therefore,

$$c^0 \pi P_1\{\bar{\delta} = 0\} + c^1(1-\pi)P_0\{\bar{\delta} = 1\} = E_\pi[e(\pi_{\bar{\tau}})] \leq e(\pi), \tag{2.3.12}$$

where the inequality follows as in 2.3.9. Suppose $\pi \leq \pi_e$ and thence $e(\pi) = c^0\pi$. Then 2.3.12 implies,

$$\begin{aligned} 0 &\leq c^0\pi(1 - P_1\{\bar{\delta} = 0\}) - c^1(1-\pi)P_0\{\bar{\delta} = 1\} \\ &\leq c^0\pi_e(1 - P_1\{\bar{\delta} = 0\}) - c^1(1-\pi_e)P_0\{\bar{\delta} = 1\} \\ &= e(\pi_e)[1 - (P_0\{\bar{\delta} = 1\} + P_1\{\bar{\delta} = 0\})]. \end{aligned}$$

The same inequality is obtained if one supposes that $\pi \geq \pi_e$. Thus except in the excluded cases ($c^0c^1 = 0$) it follows from the fact that $e(\pi_e) > 0$ that

$$P_0\{\bar{\delta} = 1\} + P_1\{\bar{\delta} = 0\} \leq 1,$$

and the theorem is shown. ∎

Corollary *Suppose $u = (\tau, \delta) \in \mathcal{U}$ is Bayesian and*

$$P_0\{\delta = 1\} + P_1\{\delta = 0\} = 1.$$

Then, the degenerate threshold policy with thresholds $a = b = \pi_e$ is Bayesian with risk $\rho(\pi) = e(\pi)$.

Proof:

Observe that,

$$E_\pi[\int_0^\tau c_s\, ds + c^0\pi P_1\{\delta = 0\} + c^1(1-\pi)P_0\{\delta = 1\}] = \rho(\pi) \leq e(\pi) \quad \forall\, \pi \in [0,1].$$

Thus,

$$0 \leq E_\pi \int_0^\tau c_s\, ds \leq e(\pi_e)[1 - (P_0\{\delta = 1\} + P_1\{\delta = 0\})],$$

arguing as above; and therefore,

$$E_\pi \int_0^\tau c_s\, ds \equiv 0.$$

As a result,

$$\rho(\pi) = c^0\pi P_1\{\delta = 0\} + c^1(1-\pi)P_0\{\delta = 1\},$$

so if $\pi \geq \pi_e$ one can show

$$\rho(\pi) \geq c^1(1-\pi) = e(\pi);$$

and similarly if $\pi \leq \pi_e$,

$$\rho(\pi) \geq c^0\pi = e(\pi).$$

Hence $\rho(\pi) = e(\pi)\ \forall \pi \in [0,1]$. This same risk is achieved with the stated degenerate threshold policy. This completes the proof. ∎

Obviously, it is necessary that any Bayesian optimal policy satisfy the last two theorems. In the general **excessive function** theory approach [SHIRYAYEV] to optimal stopping, one defines at the outset the Bayesian stopping time candidate in terms of the natural stopping set as,

$$\tau = \inf\{t \geq 0 : \pi_t \in S_0\}$$

with the intention of using Theorem 2.3.1 to define the decision component. Such a policy is seen to satisfy the consequents of the last two theorems. Assuming one can prove under given conditions that τ is Bayesian, then the task necessary to demonstrate that a threshold policy is optimal becomes precisely one of showing that C_0 is open and connected. These properties of C_0 have already been established and so τ as defined above is indeed the stopping time of a threshold policy. Now, to prove directly that this stopping time is Bayesian, one can use the excessive function approach, but this requires 'rather deep results from the general theory of Markov processes and martingales' [*loc. cit.* 113], in general. Alternatively, one might try to discover simple sufficient conditions which yield optimality and then show them to hold in a particular application. In the next chapter, such a set of conditions is set forth, one which is sufficient to prove the existence of a Bayesian threshold policy. This concludes the section and chapter.

Chapter III
Sufficient Conditions for Bayesian Optimality

3.0 Introduction

The purpose of this chapter is to identify the conditions under which threshold policies based on the *a posteriori* probability are Bayesian optimal, and indicate how to use them in a particular application to demonstrate optimality as well as compute the thresholds. The chapter is organized as follows.

Section 3.1 gives a set of three conditions which are sufficient to guarantee that threshold policies are Bayesian. Two lemmas are given which help characterize the risk and the continuation interval. The main theorem, which is based on these lemmas, then demonstrates the sufficiency of the conditions mentioned.

In a particular application, the first two sufficient conditions are obtained by construction as will be shown in Chapter IV. A direct verification of the third condition is difficult in general, so in section 3.2, auxiliary conditions are given which allow one to obtain the third condition from the other two. The chapter ends with a section discussing an alternative set of sufficient conditions.

3.1 Sufficient Conditions

The purpose of this section is to spell out a set of three conditions which are sufficient to prove that there exists a unique threshold policy which is Bayesian. From the last chapter, it is clear that the process of fundamental interest is the *a posteriori*

probability process $\{\pi_t : t \geq 0\}$. In the remainder of this chapter it will be assumed that $\{\pi_t : t \geq 0\}$ is in fact a **standard Markov process** [DYNKIN].

Since the goal of this thesis is to demonstrate the optimality of threshold policies, it is natural that the topic of first exit times of a process from an open interval should arise. In turn then, it might be expected that the notion of characteristic operator should appear. This is indeed the case.

Let U denote the **characteristic operator** [DYNKIN] for the *a posteriori* probability process $\{\pi_t : t \geq 0\}$ and $\mathcal{D}(U)$ denote its domain of definition. Now given that $\{\pi_t : t \geq 0\}$ is as above, suppose that $r : [0,1] \to \mathbb{R}$ satisfies the hypotheses of a generalized Itô rule for this particular $\{\pi_t : t \geq 0\}$, so that one can write

$$r(\pi_\tau) - r(\pi_0) = \int_0^\tau Ur(\pi_s)ds + M_\tau, \tag{3.1.1}$$

for all $(P_\pi, \mathcal{F}_t)$-stopping times satisfying $E_\pi[\tau] < \infty$, where M is some P_π-local martingale. Finally. let $I_0 \subset [0,1]$, strictly, be some nonempty, open interval and $r \in \mathcal{D}(U)$ be such a function. Consider the following set of conditions on the pair (r, I_0):

$$\begin{aligned}
&\text{(C1):} && Ur(\pi) = -c(\pi) && \forall \pi \in (0,1);\\
&\text{(C2):} && r(\pi) = e(\pi) && \forall \pi \in \partial_\pi I_0;\\
&\text{(C3):} && r(\pi) < e(\pi) && \forall \pi \notin \partial_\pi I_0.
\end{aligned}$$

In a particular application, the first two conditions pose a generalized **Stefan problem** [SHIRYAYEV] which does not have a unique solution in general. The third condition serves to constrain the problem so that its solution is essentially unique, as will be described below. Notice that the function $r(\pi)$ is like a Dynkin function in that it satisfies (C1), is similar to the risk function in as much as it satisfies (C1) and (C2), but differs from both in the additional requirement (C3). The remainder of this section is devoted to proving that the conditions (C1)-(C3) are sufficient to guarantee

that there exists a threshold policy which is Bayesian. The first lemma goes a long way towards showing this.

Lemma 3.1.1 *Suppose there exists a pair* (r, I_0) *satisfying the conditions (C1)-(C3). Then,*

$$\rho(\pi) = r(\pi) \qquad \forall \pi \in [I_0]_\pi, \tag{3.1.2}$$

where $\rho(\cdot)$ *is the Bayes risk.*

Proof:

First note that (C2) and (C3) together imply,

$$e(\pi) \geq r(\pi) \qquad \forall \pi \in [0,1]. \tag{3.1.3}$$

Next, in view of (C1), one obtains,

$$\begin{aligned} \rho(\pi) &= \inf_{\tau \in \mathcal{T}} \rho(\pi, \tau) \\ &= \inf_{\tau \in \mathcal{T}} E_\pi [\int_0^\tau c(\pi_s)\, ds + e(\pi_\tau)] \\ &= r(\pi) + \inf_{\tau \in \mathcal{T}} E_\pi [e(\pi_\tau) - r(\pi_\tau)] \qquad \forall \pi \in [0,1]. \end{aligned} \tag{3.1.4}$$

Hence, combining 3.1.4 and 3.1.5 yields,

$$\rho(\pi) \geq r(\pi) \qquad \forall \pi \in [0,1]. \tag{3.1.5}$$

On the other hand, if one defines the $(P_\pi, \mathcal{O}_t)$-stopping time $\bar{\tau}$ as,

$$\bar{\tau} = \inf\{t \geq 0 : \pi_t \notin I_0\}, \tag{3.1.6}$$

then, from (C1) and (C2) it follows that,

$$\rho(\pi, \bar{\tau}) = r(\pi) \qquad \forall \pi \in [I_0]_\pi. \tag{3.1.7}$$

Thus, 3.1.5 and 3.1.6 together yield,

$$\rho(\pi) = r(\pi) \qquad \forall \pi \in [I_0]_\pi, \tag{3.1.8}$$

and the lemma is shown. ∎

Because of 3.1.8 and (C3), it is convenient and suggestive to call $r(\cdot)$ the **subrisk.** Likewise, it is appropriate to call I_0 the continuation interval.

The next lemma shows that if there are two pairs, say (r, I_0) and (s, J_0), which satisfy the conditions (C1-C3), then $I_0 \equiv J_0$ and $r(\pi) = s(\pi)$, $\forall\, \pi \in [I_0]_\pi$. This is called **essential uniqueness** because, as it will be shown, the nonuniqueness of $r(\pi)$ off of $[I_0]_\pi$ is irrelevant to any questions concerning the risk. This is fortunate since $r(\pi)$ is not in general unique there.

Lemma 3.1.2 *Suppose there exists a pair (r, I_0) such that the conditions (C1-C3) hold. Then (r, I_0) is essentially unique and in particular, $I_0 \equiv C_0$.*

Proof:

To prove the theorem it is only necessary to show that $I_0 \equiv C_0$, in view of the previous lemma. Pick any $\pi \in I_0$, and observe that

$$\rho(\pi) = r(\pi) < e(\pi),$$

in view of Lemma 3.1.1 and (C3), and so $\pi \in C_0$. Thus $I_0 \subset C_0$, and claim $I_0 \equiv C_0$, for if not $\partial I_0 \subset C_0$. In this case, choose $\pi \in \partial I_0$ and note $\pi \in \partial_\pi I_0 \subset [I_0]_\pi$. From Lemma 3.1.1 and (C2) then

$$\rho(\pi) = r(\pi) = e(\pi).$$

But $\pi \in C_0$ also, implying $\rho(\pi) < e(\pi)$ which yields the contradiction. ∎

Corollary *Under the same assumptions,*

$$\rho(\pi) = e(\pi) \qquad \forall \pi \notin I_0 \tag{3.1.9}$$

Proof: Immmediate from the lemma and Theorem 2.3.2. ∎

The preceding lemmas are the main ingredients to the principal optimality result proved next. To summarize, Lemma 3.1.1 showed that the three conditions imply that

the risk and the subrisk are identical on the π-closure of the continuation interval. Hence, by itself the lemma suggests that threshold policies are at least $[I_0]_\pi$-Bayesian. Lemma 3.1.2 and its corollary showed that the risk equals the terminal cost outside of the continuation interval. Putting the two lemmas together yields the following.

Theorem 3.1.3 *Suppose there exists a pair (r, I_0) satisfying the conditions (C1-C3). Then the threshold policy based on I_0 is Bayesian.*

Proof:

Suppose (C1),(C2), and (C3) hold, and define $\tau^* = \bar{\tau}$ as in 3.1.6. Then 3.1.7 and 3.1.8 give,

$$\rho(\pi, \tau^*) = \rho(\pi) \qquad \forall \pi \in [I_0]_\pi, \tag{3.1.10}$$

i.e., τ^* solves the optimal stopping problem for all $\pi \in [I_0]_\pi$. Since τ^* is of threshold type, it follows that the policy $u^* = (\tau^*, \delta^*)$ exists and is $[I_0]_\pi$-Bayesian, where δ^* is given by,

$$\delta^* = \begin{cases} 1 & \text{if } \pi_{\tau^*} \geq b \\ 0 & \text{if } \pi_{\tau^*} \leq a, \end{cases}$$

for $I_0 = (a, b)$.

It follows from Lemma 3.1.2 that,

$$\rho(\pi) = e(\pi) \qquad \forall \pi \notin I_0. \tag{3.1.11}$$

On the other hand, $\tau^* = 0$ P_π-a.s. $\forall \pi \notin I_0$, and then,

$$\begin{aligned} \rho(\pi, \tau^*) &= E_\pi[\int_0^{\tau^*} c(\pi_s) ds + e(\pi_{\tau^*})] \\ &= E_\pi[0 + e(\pi)] \\ &= e(\pi). \end{aligned} \tag{3.1.12}$$

Hence τ^* solves the optimal stopping problem for all $\pi \notin I_0$. Combining this fact with the above argument for $\pi \in [I_0]_\pi$ shows that there exists a policy, in particular $u^* = (\tau^*, \delta^*)$, τ^* as in 3.1.6 and δ^* as above, such that u^* is Bayesian. ∎

Thus, in order to prove that there exists a Bayesian threshold policy, one need only exhibit a subrisk-interval pair satisfying the three conditions. In practice, one constructs a family of pairs whose members all satisfy the first two conditions. The remaining difficulty is to show that there exists within this family at least one pair which enjoys the third condition. An existence proof is all that is necessary here in view of the essential uniqueness property. In the next section, auxiliary conditions are given which allow one to obtain (C3) from (C1), (C2) and properties of the pair which are more readily verifiable.

3.2 Auxiliary Conditions

In the last section, the goal was to demonstrate the sufficiency of the conditions (C1-C3) in proving that threshold policies are Bayesian. The emphasis in this section is to replace condition (C3) with properties of the subrisk and the continuation interval which are more easily verified. Such auxiliary conditions play a primary role in applications, by providing characterizations which guide the search for the (essentially) unique subrisk and continuation interval pair obeying (C1-C3).

The next theorem is very important in applications in that it provides a way to demonstrate the concavity of the subrisk based on its smoothness properties, quite a bargain since a direct proof of concavity can be much more difficult than a verification of smoothness. Note then the intimate connection between this theorem and the condition (C3) in an applications setting.

Theorem 3.2.1 *Let $r(\pi)$ exist satisfying (C1), and suppose $r(\pi)$ is continuous on $[I_0]_\pi$. Then $r(\pi)$ is concave on $[I_0]_\pi$ if $r \in \mathcal{C}^{2+}([I_0]_\pi)$.*

Proof:

The proof will proceed by contradiction. Suppose $r(\pi)$ is not concave on $[I_0]_\pi$. Then it is strictly convex on some subinterval $J \subset [I_0]_\pi$ in view of the fact that $r \in (\mathcal{C}^0 \cap \mathcal{C}^{2+})([I_0]_\pi)$. Define,

$$\tau_J = \inf\{t > 0 : \pi_t \notin J\}, \tag{3.2.1}$$

and suppose $\pi_0 \in J$, P_π-a.s. Now, consider any stopping time $\sigma \in \mathcal{T}$, where $0 \leq \sigma \leq \tau_J$, P_π-a.s. Since $r(\pi)$ is strictly convex on J, one obtains,

$$E_\pi[r(\pi_\sigma)] \geq r(E_\pi[\pi_\sigma]) = r(\pi) \qquad \pi \in J, \tag{3.2.2}$$

where the inequality is an application of Jensen's Inequality, and the equality follows since $\{\pi_t : t \geq 0\}$ is a uniformly integrable $(P_\pi, \mathcal{O}_t)$-martingale. On the other hand,

$$Ur(\pi) < 0 \quad \Longrightarrow \quad E_\pi[r(\pi_\tau)] \leq r(\pi) \quad \forall \tau \in \mathcal{T},\ \pi \in [I_0]_\pi, \tag{3.2.3}$$

i.e., condition (C1) guarantees that $E_\pi[r(\pi_\tau)]$ is bounded above by its initial value (indeed, it can be shown that $r(\pi_t)$ is a $(P_\pi, \mathcal{O}_t)$-supermartingale). As a result, the strict convexity of r is a clear contradiction, and therefore $r(\pi)$ is concave on $[I_0]_\pi$. ∎

Corollary *If in addition $r(\pi)$ satisfies (C2), then $r(\pi)$ is continuous on $[I_0]_\pi$*

Proof: Immediate from the theorem. ∎

The next theorem is used in the applications to follow to obtain the condition (C3) from the conditions (C1) and (C2), and properties of the subrisk function.

Theorem 3.2.2 *Let there exist a pair (r, I_0) satisfying the conditions (C1) and (C2), and suppose $r \in (C^0 \cap C^{2+})([I_0]_\pi)$ is right continuously differentiable on $(0,1)$. With $I_0 = (a,b)$ assume that,*

$$\begin{aligned} r'(a) &\leq e'(a) \\ r'(b^-) &\geq e'(b). \end{aligned} \tag{3.2.4}$$

Then,

$$r(\pi) < e(\pi) \qquad \forall\, \pi \in I_0.$$

Proof:

From Theorem 3.2.1, $r(\pi)$ is concave and continuous on $[I_0]_\pi$. Thus,

$$r(\pi) \leq l_a(\pi) \qquad \forall\, \pi \in [I_0]_\pi$$

where,

$$l_a(\pi) = r(a) + r'(a)(\pi - a).$$

Likewise,

$$r(\pi) \leq l_b(\pi) \qquad \forall\, \pi \in [I_0]_\pi$$

where,

$$l_b(\pi) = r(b) + r'(b^-)(\pi - a).$$

From condition (C2),

$$l_a(\pi) = e(a) + r'(a)(\pi - a),$$

and so from 3.2.4 it follows that

$$r(\pi) \leq e(\pi) \qquad \forall\, \pi \in (a, \pi_e].$$

Similarly,

$$r(\pi) \leq e(\pi) \qquad \forall\, \pi \in [\pi_e, a),$$

and so the subrisk is upper bounded by the terminal cost on $[I_0]_\pi$. Now arguing as in Theorem 2.3.2 with the subrisk instead of the risk, one can show that there exist two numbers a_r, b_r such that,

$$0 < a \leq a_r \leq \pi_e \leq b_r \leq b < 1;$$

where,

$$r(\pi) < e(\pi), \qquad \forall\, \pi \in (a_r, b_r);$$

$$r(\pi) = e(\pi), \qquad \forall\, \pi \in (a, a_r] \cup [b_r, b).$$

Now if one can show that $a_r = a$ and $b_r = b$, then the proof will be complete. To this end suppose $b_r < b$ and choose $\pi \in [b_r, b) \subset I_0$. Then

$$U(r(\pi) - e(\pi)) \equiv 0,$$

and so,

$$Ue(\pi) < 0,$$

in view of condition (C1). But $e(\pi)$ is affine on $[b_r, 1]$ yielding the contradiction,

$$Ue(\pi) \equiv 0,$$

since the characteristic operator of a martingale clearly annihilates affine functions. The same contradiction is derived if $a_r > a$. ∎

3.3 Discussion

This chapter has set down and investigated a set of conditions sufficient to demonstrate Bayesian optimality. It should be clear that these conditions are not the most simple. Indeed, the simplest (nontrivial) sufficient conditions are given as follows. Suppose that for any $\tau \in \mathcal{T}$ one can find a threshold stopping time $\bar{\tau} \in \mathcal{T}$ satisfying,

$$e(\pi_\tau) \geq e(\pi_{\bar{\tau}}); \tag{3.3.1}$$

$$E_\pi \int_0^\tau c_s \, ds \geq E_\pi \int_0^{\bar{\tau}} c_s \, ds, \tag{3.3.2}$$

for all $\pi \in [0,1]$. Then clearly one need only consider threshold policies, for if a Bayesian policy exists, then one can by assumption find a threshold policy which is Bayesian optimal (see 2.3.7). This is the essence of the argument used in [LaVIGNA] to exhibit necessary and sufficient conditions for the existence of an (a,b)-Bayesian threshold policy when the observations arise from a diffusion observed through another diffusion and for a particular choice of the cost $\{c_t : t \geq 0\}$. The condition 3.3.1 is obtained therein *with equality* by choosing the thresholds so as to match the error probabilities of the threshold policy to any arbitrary (admissible) rival policy.† The matching is made possible by the fact that the error probabilities are related to the thresholds by continuous, invertible mappings. In fact, these mappings are simple and explicit (see equation 17.124 in [LIPTSER & SHIRYAYEV 78]. They are the well known Wald approximations [WALD] which are exact in the diffusion case where there is no overshoot.

The condition 3.3.2 is shown to be in effect as follows. First, one computes a lower bound for the left hand side of 3.3.2 for all $\tau \in \mathcal{T}$ using Jensen's Inequality (see [WALD], [LIPTSER & SHIRYAYEV 78]). In fact, this lower bound holds ir-

† Be aware that the hypothesis test there is based on the likelihood ratio. However, the likelihood ratio is related to the *a posteriori* probability through a bijection and thus the tests are equivalent, only the thresholds differ. See Appendix I.

paths. In essence, the Wald-Jensen bound, if you will, is simply not tight enough for processes with noncontinuous sample paths.

Given this state of affairs, the attempt was made to find an alternative set of sufficient conditions which might yield optimality for both diffusions and processes with noncontinuous sample paths (but continuous time parameter). The alternative set of conditions were given in section 3.1 and our attention now turns to using them in applications.

This concludes the chapter. In the next, the results of the previous chapters will be used to solve four problems arising in applications. The first two deal with the Bayesian problem when the observations come from a diffusion, and the second two when one is observing a point process.

Chapter IV
Applications in Sequential Testing

4.0 Introduction

In this chapter four problems in Bayesian sequential hypothesis testing will be formulated and solved. Given the framework established in the second chapter, formulation here means specifying the two hypothesized probability measures, and the observation σ-algebra. In section 4.1 two problems are considered where one observes a diffusion and seeks to minimize a given Bayes risk. In the first, it is assumed that one of two constant drifts is responsible for the observations. Actually, the detection version of the hypothesis testing problem is solved because it is mathematically equivalent while notationally simpler. A constant, positive cost-rate is chosen for this problem which is proportional to the the square of the nonzero drift. It is shown that a threshold policy based on the *a posteriori* probability is optimal, and the risk and thresholds are given. The second problem considered is the case where a diffusion with a drift which is assumed to be merely progressively measurable is the basis for the detection problem. It is shown that, if one chooses a constant, positive cost-rate which is proportional to the square of the stochastic drift, then again, a threshold policy is optimal. In fact, it is shown that the risk and thresholds are identical to those in the homogeneous case.

In section 4.2, two additional formulations are given. The first considers the case where under each hypothesis, it is assumed that the observations are due to a constant

rate Poisson process. Here the hypothesis testing problem is presented because the detection problem—a constant rate versus a rate of one—is no real notational bargain. A threshold policy is shown to be Bayesian, and explicit formulas for the risk and thresholds are computed. The last problem deals with the situation where a point process with one of two predictable rates is assumed responsible for the observations. It is shown that if one chooses a constant positive cost-rate which is proportional to the difference of the stochastic rates, then there exists a Bayesian threshold policy. Again, the risk and the thresholds are the same as those in the homogeneous case.

4.1 Diffusion Observations

This section will draw on the formulation and notation laid down in sections 2.1 and 2.2 on the Bayesian formulation to the sequential hypothesis testing problem. Following the briefest review, the problem will be specialized to the case where the filtration history arises from observations of a continous semimartingale driven by a standard Wiener process.

Let $\pi \in [0,1]$ be arbitrary throughout the remainder of the section, unless otherwise noted. This will save the continual trouble of having to restate the obvious, and this will not be done, except for emphasis in some of the numbered equations.

Recall that on the measurable space, $(\Omega, \mathcal{F})$ there is given a family of probability measures $\{P_\pi : 0 \leq \pi \leq 1\}$, satisfying 2.2.2, and a $\{\theta_0, \theta_1\}$-valued random variable, θ, with the *a priori* probability distribution given in 2.2.1. Based on the observation filtration, $\mathcal{O}_t$, one wants to choose a value for θ which minimizes the risk, defined in 2.2.4. Values for θ are chosen by a two-step procedure. First, the decision to terminate the observation procedure is made according to a $(P_\pi, \mathcal{O}_t)$-stopping time, say τ, and then the value for θ is chosen according to a $(P_\pi, \mathcal{O}_\tau)$-binary random variable, say δ (see definition 2.1.1). A particular application begins by specifying the nature of $\{\mathcal{O}_t : t \geq 0\}$, the σ-algebra generated by the observations.

It is assumed that the random variable θ is unobservable, but that one can observe an *initially zero* continuous random process $\{y_t : t \geq 0\}$ whose statistics under each of the hypotheses—$\theta = \theta_0$, $\theta = \theta_1$—are governed by the probability measures P_0 and P_1, respectively. To wit, for each $\omega \in \Omega$, the observation process has the stochastic differentials,

$$dy_t(\omega) = \begin{cases} h_t^0(\omega)dt + dw_t(\omega), & \text{if } \theta(\omega) = \theta_0; \\ h_t^1(\omega)dt + dw_t(\omega), & \text{if } \theta(\omega) = \theta_1, \end{cases}$$

where, $\{w_t : t \geq 0\}$ is a $(P_\pi, \mathcal{F}_t)$-**standard Wiener process** which is P_π-independent of θ, and where $\{h_t^i : t \geq 0\}$ is an $\mathcal{F}_t$-progressive process satisfying, $E_i[\int_0^t |h_s^i| ds] < \infty$,

$i = 0, 1$. Now the above hypothesis testing problem can be easily shown, without loss of generality, to be equivalent to a detection problem, and so, letting $\theta_0 = 0$, and $\theta_1 = 1$, the problem can be recast in the detection format, in which case $\{y_t : t \geq 0\}$ has the $(P_\pi, \mathcal{F}_t)$-stochastic differential

$$dy_t = \theta h_t dt + dw_t \qquad t \geq 0, \tag{4.1.1}$$

with the same requirements on $\{h_t : t \geq 0\}$, and $\{w_t : t \geq 0\}$ just mentioned.

From 4.1.1, one takes the observation filtration, $\mathcal{O}_t$, to be the σ-algebra generated by the $\{y_t : t \geq 0\}$ process. Having done this, recall from section 2.2 that the *a posteriori* probability is $\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\}$, $t \geq 0$. This permits the total average risk to be given as

$$\rho(\pi, \tau) = E_\pi[\int_0^\tau c(\pi_s)\, ds + e(\pi_\tau)], \tag{4.1.2}$$

where $c_t(\pi)$ is a given cost-rate function, and $e(\pi)$ is the terminal cost as in 2.3.3. Note that 4.1.2 is free of δ, as achieved in section 2.3.

Having defined the observation filtration, the immediate goal is to compute a stochastic differential for the $\{\pi_t : t \geq 0\}$ process. From 4.1.1 it is clear that $P_0 \ll P_1$, so take,

$$\Lambda_t(\omega) = \frac{dP_1}{dP_0}(\mathcal{O}_t)(\omega), \tag{4.1.3}$$

to be the likelihood ratio for the problem, i.e., it is the Radon-Nikodym derivative of the $\mathcal{O}_t$-restriction of the measure P_1 with respect to the $\mathcal{O}_t$-restriction of the measure P_0. If $\pi = 1$, then $P_\pi\{\pi_t = 1\} = 1$ with P_π probability one, for all $t \geq 0$. Suppose $\pi < 1$; in Appendix I, it is shown in this case that (see A.1.11),

$$\pi_t = \frac{\frac{\pi}{1-\pi}\Lambda_t}{1 + \frac{\pi}{1-\pi}\Lambda_t} = \phi(\Lambda_t), \tag{4.1.4}$$

and so, applying the **Itô differentiation rule** one obtains,

$$\pi_t - \pi_0 = \int_0^t \frac{\pi_s(1-\pi_s)}{\Lambda_s} d\Lambda_s - \int_0^t \frac{\pi_s^2(1-\pi_s)}{\Lambda_s^2} d[\Lambda, \Lambda]_s. \tag{4.1.5}$$

The likelihood ratio for this problem is well known and is given by,

$$\Lambda_t = \exp\{\int_0^t \hat{h}_s dy_s - \frac{1}{2}\int_0^t \hat{h}_s^2 ds\}, \tag{4.1.6}$$

where,

$$\hat{h}_t = E_1[h_t|\mathcal{O}_t]. \tag{4.1.7}$$

From 4.1.3 and 4.1.6, one can obtain,

$$\Lambda_t = 1 + \int_0^t \hat{h}_s \Lambda_s dy_s. \tag{4.1.8}$$

Now, if one observes that (see A.1.7),

$$E_\pi[\theta h_t|\mathcal{O}_t] = \pi_t \hat{h}_t, \tag{4.1.9}$$

then a straightforward calculation shows that there exists a $(P_\pi, \mathcal{O}_t)$-standard Wiener martingale, say $\{\bar{w}_t : t \geq 0\}$, such that $\{y_t : t \geq 0\}$ has stochastic differential,

$$dy_t = \pi_t \hat{h}_t dt + d\bar{w}_t. \tag{4.1.10}$$

Combining 4.1.5, 4.1.8, and 4.1.10 yields,

$$\begin{aligned} \pi_t - \pi_0 &= \int_0^t \pi_s(1-\pi_s)\hat{h}_s dy_s - \int_0^t \pi_s^2(1-\pi_s)\hat{h}_s^2 ds \\ &= \int_0^t \pi_s(1-\pi_s)\hat{h}_s d\bar{w}_s \qquad \forall t \geq 0, \end{aligned} \tag{4.1.11}$$

and so,

$$d\pi_t = \pi_s(1-\pi_s)\hat{h}_s d\bar{w}_s. \tag{4.1.12}$$

It is clear from 4.1.12 that since $\{\bar{w}_t : t \geq 0\}$ is a continuous martingale that $\partial_\pi I_0 = \partial I_0 = \{a, b\}$, if $I_0 = (a, b)$ is some interval such that $a < \pi_0 < b$, P_π-a.s., and $0 < a < b < 1$. It may not be clear that $\{\pi_t : t \geq 0\}$ can be guaranteed to satisfy Theorem 1.1.1, i.e., can be guaranteed to escape any such interval. To see that this is so, first note that $\pi_t = \Phi(L_t)$, where $\Phi : \mathbb{R} \to [0, 1]$ is the bijective function given

implicitly by 4.1.4 and the relation, $L_t = \log \Lambda_t$, i.e., $\Phi(L) = \phi(\log(L))$; L is called the log-likelihood ratio. Now, equations 4.1.6 and 4.1.10 show,

$$dL_t = (\pi_t - \frac{1}{2})\hat{h}_t^2 dt + \hat{h}_t d\bar{w}_t \qquad t \geq 0, \tag{4.1.13}$$

which yields the $(P_\pi, \mathcal{O}_t)$-compensator for L as,

$$A_t^L = (\pi_t - \frac{1}{2})\hat{h}_t^2 \qquad t \geq 0. \tag{4.1.14}$$

As prerequisites to Theorem 1.1.1, it is assumed that,

$$\text{(A1):} \qquad P_\pi\{|A_\infty^L| = \infty\} = 1 \quad \forall \pi \in [0,1];$$

$$\text{(A2):} \qquad P_\pi\{|A_t^L| = \infty\} = 0 \quad \forall \pi \in [0,1],\ \forall t < \infty.$$

These assumptions are necessary to ensure that $\{\pi_t : t \geq 0\}$ will escape I_0 in finite time P_π-a.s., and thence guarantee that the threshold policy with continuation interval I_0 will eventually terminate. Since $P_i \ll P_\pi$, $i = 0, 1$, it follows that,

$$P_\pi\{|A_\infty^L| = \infty\} = 1 \iff P_i\{|A_\infty^L| = \infty\} = 1 \qquad i = 0, 1. \tag{4.1.15}$$

In view of 4.1.15, assumptions equivalent to (A1) and (A2) can be recast in terms of the measures P_i, $i = 0, 1$ as,

$$\text{(A1)}' : \qquad P_i\{\int_0^\infty \hat{h}_s^2 ds = \infty\} = 1 \quad i = 0,1;$$

$$\text{(A2)}' : \qquad P_i\{\int_0^t \hat{h}_s^2 ds = \infty\} = 0 \quad i = 0,1,\ \forall t < \infty.$$

The third condition of Theorem 1.1.1 can be shown directly as,

$$\begin{aligned} |E_\pi[A_t^L]| &= |(1-\pi)E_0[A_t^L] + \pi E_1[A_t^L]| \\ &= |(\pi - 1)E_0[\int_0^t \frac{1}{2}\hat{h}_s^2 ds] + \pi E_1[\int_0^t \frac{1}{2}\hat{h}_s^2 ds]| \\ &= |(\pi - 1)E_0|A_t^L| + \pi E_1|A_t^L|| \\ &= |E_\pi|A_t^L| - \pi E_1|A_t^L| + \pi E_1|A_t^L|| \\ &= E_\pi|A_t^L| \end{aligned} \tag{4.1.18}$$

Hence, by Theorem 1.1.1, L is guaranteed to escape any open interval in finite time P_π-a.s., and since $\pi_t = \Phi(L_t)$ where Φ is bijective, then the same holds true for the *a posteriori* probability.

Next, let $r : \mathbb{R} \to [0,1]$ be some twice continuously differentiable function. An application of the Itô rule gives,

$$r(\pi_t) - r(\pi_0) = \int_0^t \pi_s^2(1-\pi_s)^2\hat{h}_s^2 r''(\pi_s)ds + \int_0^t \pi_s(1-\pi_s)\hat{h}_s r'(\pi_s)d\bar{w}_s, \quad (4.1.19)$$

using 4.1.12.

Suppose $h_t = h$, $t \geq 0$, where h is some deterministic, nonzero constant. Thus, $(A1)'$ and $(A2)'$ hold. In addition, suppose that the cost-rate function is given as $c(\pi) = ch^2$, with c a positive deterministic constant; this choice of cost-rate entails no loss of generality since $c > 0$ is otherwise arbitrary. In addition, it follows that condition (T) holds, thus $E_\pi[\int_0^\infty c_s ds] = \infty$. This is a classic Bayesian set-up: a constant running cost risk with a homogeneous diffusion. From 4.1.19, the characteristic operator of the *a posteriori* probability process is found to be,

$$Ur(\pi) = h^2\pi^2(1-\pi)^2 r''(\pi) \qquad \forall \pi \in (0,1). \quad (4.1.20)$$

The application of the results in Chapter III begins by setting up the problem specified by conditions (C1) and (C2), which is:

$$\begin{cases} Ur(\pi) = -ch^2 & \forall \pi \in (0,1); \\ r(\pi) = e(\pi) & \pi \in \{a,b\}. \end{cases}$$

Substituting 4.1.20 into the above and dividing by h^2 yields,

$$\begin{cases} \pi^2(1-\pi)^2 r''(\pi) = -c & \forall \pi \in (0,1); \\ r(\pi) = e(\pi) & \pi \in \{a,b\}, \end{cases} \quad (4.1.21)$$

which is a family of ODE's indexed by the boundary points $\{a,b\}$, $a < b$. Motivated by Theorem 3.2.1, it is required that a solution to 4.1.21 for a, b given at least satisfies,

$$r(\pi) \text{ is continuous } \forall \pi \in [a,b]. \quad (4.1.22)$$

Clearly, the solution to 4.1.21-22 exists, is unique, and is easily obtained in closed form as,

$$r(\pi) = \frac{b-\pi}{b-a}e(a) + \frac{\pi-a}{b-a}[e(b) - d(b) + d(a)] + d(\pi) - d(a), \tag{4.1.23}$$

where,

$$d(\pi) = c(1-2\pi)\log\frac{\pi}{1-\pi}.$$

Now it is clear that $r(\pi)$ is concave for all $\pi \in (0,1)$ directly from 4.1.21. However, it is also true that 4.1.21 shows that $r \in \mathcal{C}^{2+}(E)$, with $E = [a,b]$, and so invoking Theorem 3.2.1, it follows that $r(\pi)$ is concave for all $\pi \in [a,b]$. Obviously, this weaker concavity argument is trivial and unnecessary, but it is given for pedagogical reasons—to show at which step Theorem 3.2.1 is applied. It will turn out that the concavity of the subrisk is not a 'free gift' in the discontinuous martingale case considered in the next section.

Since $r(\pi)$ is concave for all $\pi \in (0,1)$, then condition (C3) is satisfied if one imposes,

$$r'(a) = e'(a), \qquad r'(b) = e'(b). \tag{4.1.24}$$

It is obvious that 4.1.24 guarantees (C3), nevertheless, an appeal to Theorem 3.2.2 can be made. The solution $\{a^*, b^*\}$ to 4.1.24 is easily shown to exist (see below). Its uniqueness is also easily shown, but follows *a fortiori* in view of Lemma 3.1.2 simply from its existence. Again, in the jump process case, this is a welcome device to proving uniqueness because the existence proof will be difficult enough.

In summary, the three sufficient conditions, (C1-C3), have been shown to be in force. Thus, the following theorem has been demonstrated to be true.

Theorem 4.2.1 *In the problem of sequential detection based on observations of the homogeneous diffusion process,*

$$y_t = \theta h t + w_t \qquad h \neq 0; \quad t \geq 0,$$

with running cost,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[ch^2\tau], \tag{4.1.25}$$

there exists a Bayesian rule, $u^* = (\tau^*, \delta^*)$, *which is a threshold policy given by,*

$$\begin{aligned} \tau^* &= \inf\{t \geq 0 : \pi_t \notin (a^*, b^*)\} \\ \delta^* &= \begin{cases} 1, & \text{if } \pi_{\tau^*} \geq b^*; \\ 0, & \text{if } \pi_{\tau^*} \leq a^*. \end{cases} \end{aligned} \tag{4.1.26}$$

The Bayes risk is given by,

$$\rho(\pi) = \begin{cases} e(\pi) & \pi \notin (a^*, b^*); \\ r^*(\pi) & \pi \in (a^*, b^*), \end{cases}$$

with subrisk,

$$r^*(\pi) = \frac{b^* - \pi}{b^* - a^*} e(a^*) + \frac{\pi - a^*}{b^* - a^*}\Big[e(b^*) - d(b^*) + d(a^*)\Big] + d(\pi) - d(a^*),$$

and,

$$d(\pi) = c(1 - 2\pi)\log\frac{\pi}{1-\pi}. \tag{4.1.27}$$

The constants a^* *and* b^* *are uniquely determined from the system of transcendental equations,*

$$\begin{aligned} e'(a^*) - d'(a^*) &= f(a^*, b^*); \\ e'(b^*) - d'(b^*) &= f(a^*, b^*), \end{aligned} \tag{4.1.28}$$

with,

$$f(a, b) = \frac{e(b) - e(a) - (d(b) - d(a))}{b - a}, \tag{4.1.29}$$

where a^*, b^* *satisfy condition (I), i.e.,*

$$0 < a^* \leq \pi_e \leq b^* < 1; \quad a^* < b^*.$$

Proof:

Given largely in the preceding discussion. Note that the conditions (T), (A1), and (A2) are satisfied trivially because $h \neq 0$ and because $c(\pi) = ch^2$, with $0 < c < \infty$.

Also note that $\tau^* < \infty$, $P_\pi - a.s.$, $\forall \pi \in [0,1]$, in view of Theorem 1.1.1. The system 4.1.28 to solve for the optimal thresholds a^*, b^*, is obtained from 4.1.23 and 4.1.24. In order to show that a^*, b^* exist satisfying 4.1.28, begin by transforming the system to the following form:

$$\begin{aligned} c^1 - d'(b) &= c^0 - d'(a)\,; \\ c^1 + bd'(b) - d(b) &= ad'(a) - d(a)\,. \end{aligned} \tag{4.1.30}$$

The origin of the first equality should be clear; obtain the second by eliminating c^0 from the system 4.1.28. To simplify the remaining computations let $A = \frac{a}{1-a}$, $B = \frac{b}{1-b}$, and thus the requirement $0 < a < b < 1$ becomes $0 < A < B < \infty$. Upon substituting these values into 4.1.30 and rearranging there holds,†

$$\begin{aligned} c^0 + c^1 + c[\frac{1}{B} - B - 2\log B] &= c[\frac{1}{A} - A - 2\log A]\,; \\ c^1 - c[B + \log B] &= -c[A + \log A]\,. \end{aligned} \tag{4.1.31}$$

Next, eliminate c^1 in the first equation, divide both sides of both the result and the second equation by $c > 0$, and rearrange to get,

$$\begin{aligned} \tilde{c}^0 + \frac{1}{B} + \log\frac{1}{B} &= \frac{1}{A} + \log\frac{1}{A}\,; \\ \tilde{c}^1 + A + \log A &= B + \log B\,, \end{aligned} \tag{4.1.32}$$

where $\tilde{c}^0 = c^0/c$ and $\tilde{c}^1 = c^1/c$. If one considers 4.1.32 as an implicit definition of the mappings $A \mapsto B_0(A)$ and $A \mapsto B_1(A)$, respectively, it follows that $B_0(A)$ and $B_1(A)$ have the same terminal behavior as $A \downarrow 0$ and as $A \uparrow \infty$. Such behavior is undesirable since our goal is the equivalent of finding a fixed point to $B_0(A) = B_1(A)$. With this in mind, rearrange 4.1.32 as,

$$\begin{aligned} \frac{\frac{B}{A} - 1}{B} + \log\frac{B}{A} &= \tilde{c}^0\,; \\ B\frac{\frac{B}{A} - 1}{\frac{B}{A}} + \log\frac{B}{A} &= \tilde{c}^1\,. \end{aligned} \tag{4.1.33}$$

† Compare 4.1.31 with equations 4.77 and 4.78 in [SHIRYAYEV].

Define $R = \frac{B}{A}$ and make the substitution,

$$\begin{aligned} \frac{R-1}{B} + \log R &= \tilde{c}^0; \\ B\frac{R-1}{R} + \log R &= \tilde{c}^1, \end{aligned} \tag{4.1.34}$$

with the requirements $0 < B < \infty$, $1 < R < \infty$. Consider the first equation in 4.1.34 and the implicit mapping $R \mapsto B_0(R)$. One can show via the Implicit Function Theorem that $B_0(R) \in C^1(1,\infty)$ where $B_0(1) = 0$ and,

$$B_0'(R) = \frac{B_0(R)}{R-1}\Big(1 + \frac{B_0(R)}{R}\Big) > 0.$$

On the other hand, it also follows that $B_1(R) \in C^1(1,\infty)$ where $B_1(1) = \infty$ and,

$$B_1'(R) = \frac{-1}{R-1}\Big(1 + \frac{B_1(R)}{R}\Big) < 0.$$

Consequently, there exists a unique R satisfying

$$0 < B_0(R) = B_1(R) < \infty; \quad 1 < R < \infty. \tag{4.1.35}$$

Chaining back through the argument, it has been shown that there is a unique solution (a^*, b^*) satisfying 4.1.28 and also $0 < a^* < b^* < 1$. From Lemma 3.1.2 and Lemma 2.3.2, it follows in addition that condition (I) is satisfied. This completes the proof. ∎

To facilitate the numerical calculation of the thresholds, begin by subtracting the first equation in 4.1.34 from the second to obtain,

$$B\frac{R-1}{R} - \frac{R-1}{B} = \tilde{c}^1 - \tilde{c}^0,$$

and thence,

$$B^2 - (\tilde{c}^1 - \tilde{c}^0)\Big(\frac{R}{R-1}\Big)B - R = 0,$$

yielding,

$$B = \frac{\tilde{c}^1 - \tilde{c}^0}{2}\Big(\frac{R}{R-1}\Big) + \sqrt{\Big[\frac{\tilde{c}^1 - \tilde{c}^0}{2}\Big(\frac{R}{R-1}\Big)\Big]^2 + R}. \tag{4.1.36}$$

Replacing B by 4.1.36 in the second equation of 4.1.34 and defining $S = \sqrt{R} > 1$, one can obtain the following compact expression,

$$\left(S - \frac{1}{S}\right) \exp\left[\sinh^{-1}\left(\frac{1}{2}\frac{\tilde{c}^1 - \tilde{c}^0}{S - S^{-1}}\right)\right] + 2\log S = \tilde{c}^1. \tag{4.1.37}$$

If one computes a solution to 4.1.37, then the optimal thresholds (a^*, b^*) are obtained from,

$$\begin{aligned} A &= \frac{1}{S}\exp\left[\sinh^{-1}\left(\frac{1}{2}\frac{\tilde{c}^1 - \tilde{c}^0}{S - S^{-1}}\right)\right]; \\ B &= S\exp\left[\sinh^{-1}\left(\frac{1}{2}\frac{\tilde{c}^1 - \tilde{c}^0}{S - S^{-1}}\right)\right]; \\ a^* &= \frac{A}{1+A}; \\ b^* &= \frac{B}{1+B}. \end{aligned} \tag{4.1.38}$$

In the special case where $c^0 = c^1 > 0$, it follows that $B = S = 1/A$ and B satisfies,

$$B - \frac{1}{B} + 2\log B = \tilde{c}^0 = \tilde{c}^1. \tag{4.1.39}$$

Note in this case that $b^* = 1 - a^*$, and $0 < a^* < \frac{1}{2} < b^* < 1$. As an example, if $c^0 = c^1$ where,

$$c^0 = (3 - \frac{1}{3} + 2\log(3)) \cdot c,$$

then,

$$a^* = \frac{1}{4}; \quad b^* = \frac{3}{4}.$$

In general, 4.1.39 can be used to obtain an initial guess for an iterative method involving 4.1.37, or can be used as an approximation when $c^0 \approx c^1$.

Notice that the drift coefficient has been effectively 'factored out' of the problem as a result of choosing the cost-rate as in 4.1.25. A particular choice of the cost-rate constant for an instance of the above problem is purely a design decision, modulo the positivity requirement. There is no choice which is uniformly good for all problems. However, the choice made herein, $c(\pi) = ch^2$, is intuitively satisfying since it not

only enjoys positivity, but moreover exhibits the appealing property of penalizing the detector more, or less, depending upon whether the magnitude of the drift is greater, or smaller than one, and does so according to the square of the magnitude. One might say that this choice of cost-rate is reasonable in that it reflects a designer's modest desire to expect better performance from the detector (faster decisions on average) in 'favorable' problems (large drifts to detect), and to allow worse performance (slower decisions on average) in 'hard' problems (drifts close to zero). Apparently, such a designer believes "not all filtrations are created equal", to put it colloquially.

As mentioned, a by-product of this choice of cost is that the drift is factored out of the risk, and therefore the risk is independent of any particular drift. This is a significant advantage because it implies immediately that the results of Theorem 4.1.1 can be extended to a more general case. To see this, first consider that the proper generalization of the cost choice discussed above is to take the running cost as,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[\int_0^\tau c\hat{h}_s^2 ds], \tag{4.1.40}$$

in the case where the drift of $\{y_t : t \geq 0\}$ is in general only known to be $(P_\pi, \mathcal{F}_t)$-progressive. For this choice of cost, it is clear that $c_t = c\hat{h}_t^2$, $c > 0$, $t \geq 0$, is a $(P_\pi, \mathcal{O}_t)$-progressive process. Having made this choice of running cost, one can prove the following.

Theorem 4.1.2 *Assume (A1) and (A2) hold. In the problem of sequential detection based on observations of the process (see 4.1.1),*

$$y_t = \theta \int_0^t h_s ds + w_t \qquad t \geq 0, \tag{4.1.41}$$

with running cost,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[\int_0^\tau c\hat{h}_s^2 ds] \qquad c > 0,\ t \geq 0, \tag{4.1.42}$$

the threshold policy, $u^* = (\tau^*, \delta^*)$, given in the homogeneous case, is Bayesian. Moreover, the risk and thresholds are exactly the same.

Proof:

First note that (T) follows from (A1).

The computation of the risk is given by,

$$\begin{aligned}\rho(\pi) &= \inf_{\tau\in\mathcal{T}} E_\pi[\int_0^\tau c_s\,ds + e(\pi_\tau)] \\ &= \inf_{\tau\in\mathcal{T}} E_\pi[\int_0^\tau c\hat{h}_s^2\,ds + e(\pi_\tau)] \\ &= \inf_{\tau\in\mathcal{T}} E_\pi[-\int_0^\tau \hat{h}_s^2\tilde{U}r^*(\pi_s)\,ds + e(\pi_\tau)],\end{aligned} \tag{4.1.43}$$

with r^* as Theorem 4.1.1 and where $\tilde{U}$ is the operator as in 4.1.21. Hence,

$$\rho(\pi) = r^*(\pi) + \inf_{\tau\in\mathcal{T}} E_\pi[e(\pi_\tau) - r^*(\pi_\tau)] = r^*(\pi), \tag{4.1.44}$$

but this is the same risk as in the homogeneous case, and so the result is shown. ∎

A special case of the above result is given in [LaVIGNA], using an argument based upon a theorem due to Shiryayev [LIPTSER & SHIRYAYEV 77] for the generalized problem of Wald sequential detection [WALD]. The theorem above was proven independently and in fact can be used to obtain the generalized Wald result simply and directly, using an argument which is formally equivalent to the method of LeCam used in the discrete-time case [LEHMANN]. The proof given here has another advantage in that it makes it clear how to generalize the Bayesian problem to include other cost-rates and thence also to extend the Wald results. As a final note, the reader should be aware that the formulas given here are different from those appearing elsewhere, but lead to the same results.

This concludes the section. In the next, the Bayesian problem is considered for the homogeneous Poisson case, which has interesting similarities with the homogeneous diffusion case. Lastly, a more general theorem is obtained from the homogeneous result by choosing the proper cost, in much the same way that the homogeneous result was extended in this section.

4.2 Point Process Observations

This section will consider the sequential hypothesis testing problem from the Bayesian viewpoint when the observations arise from a continuous-time point process. The notation and basic set-up are given in sections 2.1 and 2.2.

Recall that on a measurable space $(\Omega, \mathcal{F})$, there is given a family of probability measures $\{P_\pi : 0 \leq \pi \leq 1\}$, satisfying 2.2.2, and a binary valued random variable, θ, with the *a priori* probability distribution given in 2.2.1. Based on the observation filtration, $\mathcal{O}_t$, one wants to choose a value for θ which minimizes the risk, defined in 2.2.4. The manner in which values for θ are chosen has two parts. First, the decision to terminate the observation procedure is made according to a $(P_\pi, \mathcal{O}_t)$-stopping time, say τ, and second, a value for θ is chosen according to a $(P_\pi, \mathcal{O}_\tau)$-binary random variable, say δ (see definition 2.1.1). A particular application begins by specifying the nature of $\{\mathcal{O}_t : t \geq 0\}$, the σ-algebra generated by the observations.

It is assumed that the random variable θ is unobservable, but that one can observe a counting process $\{n_t : t \geq 0\}$ whose statistics under each of the hypotheses—$\theta = 0$, $\theta = 1$—are governed by the probability measures P_0 and P_1, respectively. Specifically, for each $\omega \in \Omega$, the observation process is an *initially zero* counting process with semimartingale representation,

$$n_t(\omega) = \begin{cases} \int_0^t \lambda_s^0(\omega)ds + m_t(\omega), & \text{if } \theta(\omega) = 0; \\ \int_0^t \lambda_s^1(\omega)ds + m_t(\omega), & \text{if } \theta(\omega) = 1, \end{cases}$$

where, $\{m_t : t \geq 0\}$ is a $(P_\pi, \mathcal{F}_t)$-martingale which is P_π-independent of θ, and where $\{\lambda_t^i : t \geq 0\}$ is an $\mathcal{F}_t$-predictable process satisfying $E_i \int_0^t |\lambda_s^i| ds < \infty$, $i = 0, 1$. Thus, it is assumed that the observation process is a $(P_\pi, \mathcal{F}_t)$-semimartingale with stochastic differential,

$$dn_t = \big(\theta\lambda_t^1 + (1-\theta)\lambda_t^0\big)dt + dm_t \qquad t \geq 0, \tag{4.2.1}$$

while the observation filtration is the σ-algebra generated by this process.

The first step is to compute a $(P_\pi, \mathcal{O}_t)$-semimartingale representation for the *a posteriori* probability, $\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\}$, $t \geq 0$, as a precursor to minimizing the total average risk given in 2.3.7. To begin, note that the likelihood ratio for the problem is well known and is given by,

$$\Lambda_t = \exp\Big[\int_0^t \log \frac{\hat{\lambda}_s^1}{\hat{\lambda}_s^0} dn_s - \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0) ds\Big] \quad t \geq 0, \tag{4.2.2}$$

where,

$$\hat{\lambda}_t^i = E_i[\lambda_t^i | \mathcal{O}_{t-}] \qquad t \geq 0. \tag{4.2.3}$$

From 4.2.2 it follows that,

$$d\Lambda_t = \Lambda_{t-} \frac{\hat{\lambda}_t^1 - \hat{\lambda}_t^0}{\hat{\lambda}_t^0} \big(dn_t - \hat{\lambda}_t^0 dt\big). \tag{4.2.4}$$

Now, using 4.1.4, one obtains,

$$\pi_t - \pi_0 = \int_0^t \frac{\pi_s(1-\pi_s)}{\Lambda_s} d\Lambda_s^c + \sum_{0<s\leq t} \Delta\phi(\Lambda_s), \tag{4.2.5}$$

and from 4.2.4

$$d\Lambda_t^c = -\Lambda_s(\hat{\lambda}_s^1 - \hat{\lambda}_s^0) ds. \tag{4.2.6}$$

The computation for $\Delta\phi(\Lambda_t)$ is given as,

$$\begin{aligned} \Delta\phi(\Lambda_t) &= \frac{\frac{\pi}{1-\pi}\Delta\Lambda_t}{(1+\frac{\pi}{1-\pi}\Lambda_t)(1+\frac{\pi}{1-\pi}\Lambda_{t-})} \\ &= \pi_{t-}(1-\pi_t)\frac{\Delta\Lambda_t}{\Lambda_{t-}} \\ &= \pi_{t-}(1-\pi_t)\frac{\hat{\lambda}_t^1 - \hat{\lambda}_t^0}{\hat{\lambda}_t^0} \\ &= \frac{\pi_{t-}(1-\pi_t)}{u_t}, \end{aligned} \tag{4.2.7}$$

where the last line also serves to define $\{u_t : t \geq 0\}$. In addition, note that,

$$\begin{aligned} 1-\pi_t &= \frac{1}{1+\frac{\pi}{1-\pi}\Lambda_t} \\ &= \frac{1}{1+\frac{\pi}{1-\pi}\Lambda_{t-}+\frac{\pi}{1-\pi}\frac{\Lambda_{t-}}{u_t}} \\ &= \frac{(1-\pi_{t-})u_t}{u_t+\pi_{t-}}. \end{aligned} \tag{4.2.8}$$

Combining 4.2.5, 4.2.6, 4.2.7, and 4.2.8 yields,

$$\begin{aligned}\pi_t - \pi_0 &= -\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\pi_s(1-\pi_s)ds + \sum_{0<s\leq t} \frac{\pi_{s-}(1-\pi_{s-})}{u_s + \pi_{s-}} \\ &= -\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\pi_s(1-\pi_s)ds + \int_0^t \frac{\pi_{s-}(1-\pi_{s-})}{u_s + \pi_{s-}} dn_s,\end{aligned} \tag{4.2.9}$$

where the second line follows since $\{m_t : t \geq 0\}$, and therefore $\{n_t : t \geq 0\}$, has unity jumps. Now, if one observes that (see A.1.7),

$$E_\pi[\theta\lambda_t^1 + (1-\theta)\lambda_t^0 | \mathcal{O}_{t-}] = \hat{\lambda}_t^1\pi_{t-} + \hat{\lambda}_t^0(1-\pi_{t-}), \tag{4.2.10}$$

then a straightforward calculation shows that there exists a $(P_\pi, \mathcal{O}_t)$-martingale, say $\{\bar{m}_t : t \geq 0\}$, such that $\{n_t : t \geq 0\}$ has the $(P_\pi, \mathcal{O}_t)$-stochastic differential,

$$dn_t = \big(\hat{\lambda}_t^1\pi_{t-} + \hat{\lambda}_t^0(1-\pi_{t-})\big)dt + d\bar{m}_t, \tag{4.2.11}$$

Substituting 4.2.11 into 4.2.9 gives,

$$\pi_t - \pi_o = \int_0^t \frac{\pi_{s-}(1-\pi_{s-})}{u_s + \pi_{s-}} d\bar{m}_s. \tag{4.2.12}$$

Having computed 4.2.12, the next step is to investigate under what conditions a process as in 4.2.12 can be guaranteed to escape an interval and hence serve as the basis of a threshold policy.

Consider an interval $I_0 = (a, b)$ such that $0 < a < b < 1$, and $a < \pi_0 < b$, P_π-a.s. From 4.2.2, 4.2.11, and the relation $L_t = \log \Lambda_t$, there holds,

$$dL_t = \Big[\big(\hat{\lambda}_t^1\pi_{t-} + \hat{\lambda}_t^0(1-\pi_{t-})\big)\log\frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0} - (\hat{\lambda}_t^1 - \hat{\lambda}_t^0)\Big]dt + \log\frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0} d\bar{m}_t, \tag{4.2.13}$$

yielding the $(P_\pi, \mathcal{O}_t)$-compensator for L as,

$$A_t^L = \big(\hat{\lambda}_t^1\pi_{t-} + \hat{\lambda}_t^0(1-\pi_{t-})\big)\log\frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0} - (\hat{\lambda}_t^1 - \hat{\lambda}_t^0). \tag{4.2.14}$$

As in section 4.1, the assumptions (A1) and (A2) must hold in order to apply Theorem 1.1.1 and thus ensure that L will escape I_0 in finite time P_π-a.s. In view of 4.1.5, the assumptions (A1) and (A2) can be recast in terms of the measures P_i, $i = 0, 1$, equivalently as,

$$(\text{A1})'' : \qquad P_i\{\int_0^\infty \varphi(\hat{\lambda}_s^{1-i}, \hat{\lambda}_s^i)ds = \infty\} = 1 \quad i = 0, 1;$$

$$(\text{A2})'' : \qquad P_i\{\int_0^t \varphi(\hat{\lambda}_s^{1-i}, \hat{\lambda}_s^i)ds = \infty\} = 0 \quad i = 0, 1, \ \forall t < \infty.$$

where,

$$\varphi(\lambda^1, \lambda^0) = \lambda^1 \log \frac{\lambda^1}{\lambda^0} - (\lambda^1 - \lambda^0). \tag{4.2.15}$$

Notice, for $\lambda^0, \ \lambda^1 > 0$,

$$\begin{aligned} \varphi(\lambda^1, \lambda^0) &= \lambda^1 \log \frac{\lambda^1}{\lambda^0} - (\lambda^1 - \lambda^0) \\ &\geq \lambda^1 \log \frac{\lambda^1}{\lambda^0} + \lambda^1 \log \frac{\lambda^1}{\lambda^0} = 0, \end{aligned} \tag{4.2.16}$$

and similarly,

$$\begin{aligned} \varphi(\lambda^0, \lambda^1) &= -\lambda^0 \log \frac{\lambda^1}{\lambda^0} - (\lambda^0 - \lambda^1) \\ &\geq -(\lambda^1 - \lambda^0) + (\lambda^1 - \lambda^0) = 0. \end{aligned} \tag{4.2.17}$$

The third condition of Theorem 1.1.1 is an easy consequence of 4.2.16 and 4.2.17 using the same steps as in 4.1.18. Thus by Theorem 1.1.1, L is guaranteed to escape I_0 in finite time P_π-a.s., and since $\pi_t = \Phi(L_t)$, where Φ is bijective, then the same holds true for the *a posteriori* probability. A difficulty however, is that the π-boundary of I_0 is not simply its endpoints as in the diffusion case, and worse, it is stochastic unless u_t is deterministic for all $t \geq 0$. Therefore, although conditions have been given under which $\{\pi_t : t \geq 0\}$ can be guaranteed to escape intervals, the π-boundary of the process is difficult to work with in the general case.

Avoiding this difficulty for the moment, it is desired to compute the characteristic operator for $\{\pi_t : t \geq 0\}$. So, assuming $r \in \mathcal{C}^{1+}(E)$, with $E = (0, 1)$, an application

of Theorem 1.1.2 gives,

$$r(\pi_t) - r(\pi_0) = \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\Big[-\pi_s(1-\pi_s)r'(\pi_s) + (u_s+\pi_s)[r\Big(\frac{u_s+1}{u_s+\pi_s}\pi_s\Big) - r(\pi_s)]\Big]ds + \int_0^t [\, r\Big(\frac{u_{s-}+1}{u_{s-}+\pi_{s-}}\pi_{s-}\Big) - r(\pi_{s-})]d\bar{m}_t, \tag{4.2.18}$$

using 4.2.11 and 4.2.12.

Now consider the case where $\lambda_t^1 = \lambda^1$, $\lambda_t^0 = \lambda^0$, $t \geq 0$, are deterministic constants. Assume that these constants satisfy $\lambda^1 > \lambda^0 > 0$, without loss of generality. Thus (A1)$''$ and (A2)$''$ hold. Also suppose that the cost-rate function is given as $c(\pi) = c(\lambda^1 - \lambda^0)$, a positive deterministic constant; thus $E_\pi[\int_0^\infty c(\pi_s)ds] = \infty$ and therefore condition (T) holds. The terminal cost is $e(\pi)$, as usual.

In view of 4.2.9 with

$$u_t = u = \frac{\lambda^0}{\lambda^1 - \lambda^0},$$

it is clear that $\partial_\pi^- I_0 = \{a\}$, i.e., $\{\pi_t : t \geq 0\}$ will exit I_0 continuously on the right. However, since $\{\pi_t : t \geq 0\}$ will exit I_0 on the left only by jumping, then it follows that $\partial_\pi^+ I_0 = [b, \frac{u+1}{u+b}b)$. To see this, consider that whenever $\pi_t \in [b - \frac{b(1-b)}{u+b}, b)$, it has the potential of getting into $\partial_\pi^+ I_0$ because its jump size is $\frac{\pi_{t-}(1-\pi_{t-})}{u+\pi_{t-}}$ at that time. Letting $\Sigma\pi_{t-}$ denote the new state arrived at by jumping from π_{t-}, i.e.,

$$\Sigma\pi_{t-} = \pi_{t-} + \frac{\pi_{t-}(1-\pi_{t-})}{u+\pi_{t-}} = \frac{u+1}{u+\pi_{t-}}\pi_{t-},$$

yields,

$$\partial_\pi I_0 = \{a, [b, \Sigma b)\},$$

as the π-boundary of I_0, for I_0 as defined above.

From 4.2.18, the characteristic operator of $\{\pi_t : t \geq 0\}$ is found to be,

$$Ur(\pi) = (\lambda^1 - \lambda^0)\big[-\pi(1-\pi)r'(\pi) + (u+\pi)\Big[r(\frac{u+1}{u+\pi}\pi) - r(\pi)\Big]\big] \quad \forall \pi \in (0,1). \tag{4.2.19}$$

The application of the results in sections 3.1 and 3.2 begins by setting up the problem specified by conditions (C1) and (C2), which is:

$$\begin{cases} Ur(\pi) = -c(\lambda^1 - \lambda^0) & \forall \pi \in (0,1); \\ r(\pi) = e(\pi) & \pi \in \{a, [b, \Sigma b)\}, \end{cases}$$

choosing a particular form for the cost rate constant without loss of generality (see diffusion case in section 4.1). Substituting 4.2.19 into the above and dividing by $(\lambda^1 - \lambda^0) > 0$ yields

$$\begin{aligned} -\pi(1-\pi)r'(\pi) + (u+\pi)\Big[r(\frac{u+1}{u+\pi}\pi) - r(\pi)\Big] = -c \quad \forall \pi \in (0,1) \\ r(\pi) = e(\pi) \quad \pi \in \{a, [b, \Sigma b)\}, \end{aligned} \tag{4.2.20}$$

which is a family of functional advance differential equations, indexed by the boundary points $\{a, b\}$. Motivated by Theorem 3.2.1, it is required that a solution to 4.2.20 be continuous for all $\pi \in [a, \Sigma b)$. Letting $\tilde{U}$ represent the operator in the equivalent problem, it was shown in Theorem 1.1.2 that for every $0 < a < b < 1$, that there exists a unique solution to,

$$\tilde{U}\, R(\pi) = -c \qquad \forall\, \pi \in (0, b)$$

$$R(\pi) = e(\pi) \quad \forall\, \pi \in \{a, [b, \Sigma b)\}$$

with $R(\pi)$ continuous for all $\pi \in (0, b)$. Also, in the theorem to follow it is shown that there exists a right–continuous function, $d(\pi)$, satisfying,

$$\tilde{U}\, d(\pi) = -c \qquad \forall\, \pi \in (0, 1)$$

$$d(\pi) = 0 \qquad \forall\, \pi \in [b, \Sigma b)$$

$$d(\pi) < 0 \qquad \forall\, \pi \geq \Sigma b,$$

for any $b \in (0, 1)$. So, take $D(\pi) = d(\pi) + e(\pi)$, $\pi \in [b, 1)$, and construct,

$$r(\pi) = R(\pi)\, 1_{\{\pi < b\}} + D(\pi)\, 1_{\{\pi \geq b\}}.$$

This gives a family of right–continuous functions, $\Re = \{r(\cdot\,; a, b) \,:\, 0 < a < b < 1\}$ satisfying,

$$\tilde{U}\, r(\pi) = -c \qquad \forall\, \pi \in (0, b)$$

$$r(\pi) = e(\pi) \quad \forall\, \pi \in \{a, [b, \Sigma b)\},$$

which are strictly continuous for all $\pi < b$, and all of which furnish the conditions (C1) and (C2). In addition, by the corollary to Theorem 1.2.1, every $r \in \Re$ also enjoys, $r \in C^{2+}(E)$, with $E = (0, b)$, and as a result, by the corollary to Theorem 3.2.2 it follows that every $r \in \Re$ is concave on E. Recall also from the corollary to Theorem 1.2.1 that, $r'(\pi)$ is continuous for all $\pi \in (0, b)$, with the possible exception of $\pi = \Sigma^{-1}b$. This implies,

$$r'(\pi^-) \geq r'(\pi) \qquad \pi = \Sigma^{-1}b, \ \forall r \in \Re, \tag{4.2.21}$$

since every $r \in \Re$ is concave. Solving for the derivative in 4.2.20,

$$r'(\pi) = \frac{c + (u + \pi)[r(\frac{u+1}{u+\pi}\pi) - r(\pi)]}{\pi(1 - \pi)}, \tag{4.2.22}$$

and evaluating it at $B^{\pm}$, with $B = \Sigma^{-1}b$, yields,

$$r'(B^{\pm}) = \frac{c + (u + B)[r(b^{\pm}) - r(B)]}{B(1 - B)}, \tag{4.2.23}$$

with the immediate consequence,

$$r(b^-) \geq r(b), \qquad \forall r \in \Re \tag{4.2.24}$$

in view of (4.2.21). Now if any $r \in \Re$ satisfies 4.2.24 with strict inequality, then obviously such a member violates condition (C3). As a result, one is only interested in the subfamily,

$$\Re_1 = \{r \in \Re : r(b^-; a, b) = e(b)\}. \tag{4.2.25}$$

Again, the corollary to Theorem 1.2.1 has something to say. If $r \in \Re_1$, then $r(\pi)$ is twice continuously differentiable at $\pi = a$ (unless $b = \frac{u+1}{u+a}a$) and therefore, in order to satisfy condition (C3), it is necessary that $r(\pi)$ satisfy,

$$r'(a) = e'(a) \qquad r \in \Re_1. \tag{4.2.26}$$

Thus, if there exists $r \in \Re_2 = \{r \in \Re_1 : r'(a;a,b) = e'(a)\}$, then r satisfies the conditions (C1), (C2), and (C3), and in addition, is concave on $[(a,b)]_\pi$. In this case, $r(\pi)$ is the subrisk being sought.

An interesting subtlety has arisen here however. Contrasted with the subrisk problem in the diffusion case, it appears that the problem here is 'overdetermined'. Consider the diffusion case. There, the two free constants of the *second order differential equation* were chosen to satisfy the boundary condition (C2). Then, the boundary points 'a' and 'b' were chosen as the solution to a system of two equations in two unknowns whose unique solution was guaranteed to match the derivatives at the boundary, and thence yield condition (C3). Here, the single free constant of the *first order differential equation* is chosen to satisfy the boundary condition at $\pi = a$, and this leaves only 'a' and 'b' with which to satisfy the boundary condition at $\pi = b$ *and* the condition on *both* derivatives to obtain condition (C3). The pathway out of this apparent difficulty is provided by the following lemma.

Lemma 4.2.1 *Let $r \in \Re$. Then $r'(b^-) > e'(b)$.*

Proof:

From 4.2.22,

$$\begin{aligned} r'(b^-) - \frac{c}{b(1-b)} &= \frac{u+b}{b(1-b)}\, e\Big(\frac{u+1}{u+b}b\Big) - \frac{u+b}{b(1-b)}\, r(b^-) \\ &= \frac{u\, e(b)}{b(1-b)} - \frac{u+b}{b(1-b)}\, r(b^-) \\ &= -\frac{b\, e(b)}{b(1-b)} + \frac{u+b}{b(1-b)}[e(b) - r(b^-)] \qquad (4.2.27) \\ &= -\frac{e(b)}{1-b} + \frac{u+b}{b(1-b)}[r(b) - r(b^-)] \\ &= e'(b) + \frac{u+b}{b(1-b)}[r(b) - r(b^-)]. \end{aligned}$$

Since $r \in \Re$, it is clear from the last line and 4.2.24, that

$$r'(b^-) \geq \frac{c}{b(1-b)} + e'(b) > e'(b) \qquad (4.2.28)$$

since $c > 0$ is assumed. ∎

Corollary: *If* $r \in \Re_1$, *then,*

$$r'(b^-) = \frac{c}{b(1-b)} + e'(b). \tag{4.2.29}$$

Proof: Immediate from 4.2.27. ∎

Hence, if there exists $r \in \Re_2$, then it indeed satisfies the conditions (C1-3). In summary then, the problem to solve in the search for the subrisk is to prove the existence of *any* pair a_*, b_* satisfying,

$$\begin{aligned} r(b_*^-; a_*, b_*) &= e(b_*); \\ r'(a_*; a_*, b_*) &= e'(a_*), \end{aligned} \tag{4.2.30}$$

or,

$$\begin{aligned} r'(b_*^-; a_*, b_*) &= e'(b_*) + \frac{c}{b_*(1-b_*)}; \\ r'(a_*; a_*, b_*) &= e'(a_*), \end{aligned}$$

where $r \in \Re$. Recall that the uniqueness of the pair will follow from its existence in view of Theorem 3.1.4. Be aware that 4.2.30 is a well–posed problem because it has been shown that $\Re$ is not empty, i.e., for any $0 < a < b < 1$, there exists a solution to 4.2.20. Proving that a_*, b_* exist satisfying 4.2.30 is somewhat involved, and so is given in Appendix II. Based upon their existence, the following theorem can be shown.

Theorem 4.2.1 *In the problem of sequential hypothesis testing, based on observations of the homogeneous Poisson process,*

$$n_t = (\theta\lambda^1 + (1-\theta)\lambda^0)\, t + m_t \qquad \lambda^1 > \lambda^0,\ t \geq 0,$$

with running cost,

$$E_\pi[\int_0^\tau c(\pi_s)\, ds] = E_\pi[\, c(\lambda^1 - \lambda^0)\tau],$$

there exists a Bayesian rule, $u_* = (\tau_*, \delta_*)$, *which is a threshold policy given by,*

$$\tau_* = \inf\{\, t \geq 0 : \pi_t \notin (a_*, b_*)\,\}$$

$$\delta_* = \begin{cases} 1 & \text{if } \pi_\tau \geq b_* \\ 0 & \text{if } \pi_\tau \leq a_*. \end{cases}$$

The Bayes risk is given by,

$$\rho(\pi) = \begin{cases} e(\pi) & \pi \notin (a_*, b_*); \\ r_*(\pi) & \pi \in (a_*, b_*), \end{cases}$$

and the subrisk, $r_*(\pi) = r(\pi; a_*, b_*)$ *is given by,*

$$r(\pi; a_*, b_*) = \begin{cases} R(\pi; a_*, b_*) & \pi < b_*; \\ S(\pi; b_*) & \pi \geq b_*, \end{cases}$$

with,

$$S(\pi; b) = e(\pi) + d(\pi; b);$$

$$d(\pi; b) = C(\lambda^1(1-\pi) + \lambda^0\pi)\,(1 + N_b(\pi)),$$

where $N_b(\pi)$ *is the integer such that* $\tilde{x}(b) - \tilde{x}(\pi) - 1 \leq N_b(\pi) < \tilde{x}(b) - \tilde{x}(\pi)$, *and,*

$$x(\pi) = \log[\frac{\pi}{1-\pi}]; \quad \tilde{x} = \frac{x}{\log \frac{\lambda^1}{\lambda^0}}; \quad C = c\,\frac{\lambda^1 - \lambda^0}{\lambda^0\lambda^1}.$$

The definition of the subrisk, is completed by specifying,

$$R(\pi; a, b) = d(\pi; b) + \bar{D}(\pi; b) + K(a, b)\,\bar{H}(\pi; b);$$

$$\bar{D}(\pi; b) = \lambda^1(1-\pi)\,D_0(\pi; b) + \lambda^0\pi\,D_1(\pi; b);$$

$$\bar{H}(\pi; b) = \lambda^1(1-\pi)\,H_0(\pi; b) + \lambda^0\pi\,H_1(\pi; b),$$

with H_i, D_i, $i = 0, 1$ *defined by,*

$$H_i(\pi; b) = e^{-\nu_i\tilde{x}(\pi)} \sum_{n=0}^{N_b(\pi)} \frac{(-1)^n}{n!}[(\tilde{x}(b) - \tilde{x}(\pi) - n)\nu_i e^{-\nu_i}]^n;$$

$$D_i(\pi; b) = -Ce^{\nu_i(\tilde{x}(b)-\tilde{x}(\pi)-1)} \sum_{n=0}^{N_b(\pi)-1} e^{-\nu_i n} \sum_{m=0}^{n} \frac{(-1)^m}{m!}[(\tilde{x}(b) - \tilde{x}(\pi) - n - 1)\nu_i]^m,$$

where,

$$\nu_i = \frac{\lambda^i \log \frac{\lambda^1}{\lambda^0}}{\lambda^1 - \lambda^0}; \qquad i = 0, 1.$$

The constant $K(a,b)$ is defined via,

$$K(a,b) = \frac{e(a) - [d(a;b) + \bar{D}(a;b)]}{\bar{H}(a;b)}.$$

The pair (a_*, b_*) is the unique solution to

$$r'(b_*^-; a_*, b_*) = e'(b_*) + \frac{c}{b_*(1-b_*)};$$
$$r'(a_*; a_*, b_*) = e'(a_*).$$

Note, that the 'empty sum equals zero', and '$0^0 = 1$' conventions are used.

Proof:

Except for the explicit form given for the subrisk, the proof is contained in the preceding discussion. Note that the conditions (A1), (A2), and (T) are trivially satisfied because $0 < c(\lambda^1 - \lambda^0) < \infty$. Also note that $\tau_* < \infty\, P_\pi - a.s.$, $\forall\, \pi \in [0,1]$, in view of Theorem 1.1.1. As for the subrisk, in view of Theorem 1.2.1 it is only necessary to show that it satisfies 4.2.20 and is continuous for all $\pi < b$. The semicolons are dropped here for notational convenience. First note that $r(a) = R(a) = e(a)$, due to the definition of $K(a,b)$. Next note that,

$$d(\pi) = C(\frac{\lambda^1 - \lambda^0}{\lambda^1 \lambda^0})\,(\lambda^1 - (\lambda^1 - \lambda^0)\pi)\,(1-1) = 0, \quad \forall \pi \in [b, \frac{u+1}{u+b}b)$$

and so, $S(\pi) = e(\pi)$ on this set. Hence, the boundary conditions of 4.2.20 have been satisfied. Also observe that,

$$-\pi(1-\pi)\,d'(\pi) + (u+\pi)[d(\frac{u+1}{u+\pi}\pi) - d(\pi)] = -c \qquad \forall \pi \in (0,1)$$

where the derivative is taken from the right, as usual. Next, one can show, albeit tedious, that,

$$U\bar{D} \equiv U\bar{H} \equiv 0. \tag{4.2.31}$$

Thus far, have argued that the subrisk satisfies 4.2.20. It only remains to demonstrate the continuity on $\pi < b$. Clearly, $R(\pi)$ is continuous for all π except possibly at those $\pi < b$ where,

$$N_b(\pi) - N_b(\pi^-) = -1. \tag{4.2.32}$$

It is not too difficult to see that $H_i(\pi)$, $i = 0, 1$, is continuous at these points. On the other hand, $d(\pi)$ is clearly discontinuous at these points. A simple calculation shows that,

$$d(\pi) - d(\pi^-) = -C(\lambda^1(1-\pi) + \lambda^0\pi),$$

at a discontinuity. On the other hand at the same discontinuity one can show that,

$$D_i(\pi) - D_i(\pi^-) = C \qquad i = 0, 1.$$

From this it is easy to see that $\bar{D}(\pi)$ is also discontinuous, but of the same magnitude and opposite sign as $d(\pi)$. Therefore, the sum of the two is continuous on $(0, b)$. The existence and uniqueness of the solution to 4.2.30 is given in Appendix II. The proof is complete. ∎

Given the form of the risk in the theorem, it is not hard to see why explicit results are few and far between in the sequential analysis of jump-type processes, although analogous results have been obtained for the Wald problem dealing with Poisson processes [DVORETZKY &c].

Unlike the diffusion formulation in section 4.1, there is no easy extension of the time-homogeneous result to the most general case. An obvious reason is that in the diffusion case the drift factors out of the characteristic operator (or equivalently the Dynkin function $d(\pi)$), whereas in the Poisson case the rates do not factor out completely since u is 'tangled up' in $\tilde{U}$. As a result, in general when the rates are stochastic, the state space of the process must be enlarged to include the rate. This therefore leads to a partial functional differential equation for the risk in the two variables, π, and u (see 4.2.7). In addition, since the π-boundary is also stochastic in this case, one is forced to consider the (π, u)-boundary of the (π_t, u_t) process. This has the immediate consequence that in general, one must extend the notion of threshold policies from intervals to include open sets.

There is however one extension that can be obtained from the result above with

little difficulty. This is the case when n_t is given as in 4.1.1 and when the rate processes $\{\lambda_t^i : t \geq 0\}$ obey,

$$\lambda_t^i = \lambda^i \lambda_t, \qquad t \geq 0; \quad i = 0, 1; \quad \lambda^1 > \lambda^0. \tag{4.2.33}$$

Obviously (see 4.2.1), $\{\lambda_t : t \geq 0\}$ must be an $\mathcal{F}_t$-predictable process such that $\lambda_t > 0$ P_π-a.s. Under these assumptions and the detectability conditions (A1) and (A2) one obtains the following theorem.

Theorem 4.2.2 *Assume (A1) and (A2) hold. In the problem of Bayesian sequential hypothesis testing, based on the partial observation process (see 4.2.1, 4.2.33),*

$$n_t = \int_0^t (\theta \lambda_s^1 + (1-\theta)\lambda_s^0)\, ds + m_t \qquad t \geq 0,$$

with running cost,

$$E_\pi[\int_0^\tau c_s\, ds] = E_\pi[\int_0^\tau c\,(\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\, ds] \quad c > 0,$$

and terminal cost $e(\pi_\tau)$, the threshold policy, $u_* = (\tau_*, \delta_*)$, given in the homogeneous case, is Bayesian. Moreover, the risk and the thresholds are exactly the same.

Proof:

First note that (T) follows from (A1). The computation of the risk is given by,

$$\begin{aligned} \rho(\pi) &= \inf_{\tau \in \mathcal{T}} E_\pi[\int_0^\tau c_s ds + e(\pi_\tau)] \\ &= \inf_{\tau \in \mathcal{T}} E_\pi[\int_0^\tau c(\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\, ds + e(\pi_\tau)] \\ &= \inf_{\tau \in \mathcal{T}} E_\pi[-\int_0^\tau (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\tilde{U} r_*(\pi_s)\, ds + e(\pi_\tau)], \end{aligned} \tag{4.2.34}$$

with r_* as in Theorem 4.2.1 and $\tilde{U}$ the operator as in 4.2.20. Hence,

$$\rho(\pi) = r_*(\pi) + \inf_{\tau \in \mathcal{T}} E_\pi[e(\pi_\tau) - r_*(\pi_\tau)] = r_*(\pi), \tag{4.2.35}$$

but this is the same risk as in the homogeneous case, and so the result is shown. ∎

This concludes the section and chapter.

APPENDIX I

Suppose one is given a measurable space $(\Omega, \mathcal{F})$ upon which two distinguished probability measures P_i, $i = 0, 1$, are defined. Also suppose one is given some arbitrary (P_i-completed) sub-σ-algebra $\mathcal{O}_t \subseteq \mathcal{F}$, $i = 0, 1$.

Next define the family of probability measures, $\{P_\pi : 0 \leq \pi \leq 1\}$, via,

$$P_\pi\{A\} = \pi P_1\{A\} + (1-\pi)P_0\{A\} \qquad \forall A \in \mathcal{F},\ \pi \in [0,1].$$

Finally, suppose there is an $\mathcal{F}$-measurable random variable $\theta : \Omega \longrightarrow \{0, 1\}$ such that,

$$P_\pi\{\theta = 1\} = \pi \quad , \quad P_\pi\{\theta = 0\} = 1 - \pi, \tag{A.1.1}$$

i.e., $\theta = 1$ with prior probability π and vice versa.

Given the above set-up, consider first the problem of expressing $E_\pi[\gamma|\mathcal{O}_t]$ in terms of P_i-conditional expectations, $i = 0, 1$, where γ is some $\mathcal{F}$-measurable random variable and $\pi \in [0, 1]$. Now if $\pi = 0, 1$, the problem is trivial, so suppose $\pi \in (0, 1)$. Noticing that $P_i \ll P_\pi$ for $i = 0, 1$, $\pi \in (0, 1)$, it follows that one may define the Radon-Nikodym derivatives,

$$R^i(\omega) = \frac{dP_i}{dP_\pi}(\omega) \qquad \omega \in \Omega,\ i = 0, 1,\ \pi \in (0, 1),$$

and then there holds,

$$E_\pi[R^i|\mathcal{O}_t] = \frac{dP_i^t}{dP_\pi^t} = R_t^i \qquad i = 0, 1, \tag{A.1.2}$$

where P_i^t, $i = 0, 1$, P_π^t are the $\mathcal{O}_t$-restrictions of the respective measures. Given this set-up, one can prove the following theorem.

Theorem A.1.1 *Let* $\pi \in [0, 1]$. *Then,*

$$E_\pi[\gamma|\mathcal{O}_t] = \pi R_t^1 E_1[\gamma|\mathcal{O}_t] + (1-\pi)R_t^0 E_0[\gamma|\mathcal{O}_t] \quad P_\pi\text{-a.s.}.$$

Proof: Let $A \in \mathcal{O}_t$ and compute,

$$\begin{aligned}\int_A E_\pi[\gamma|\mathcal{O}_t]\,dP_\pi &= \int_A \gamma\,dP_\pi \\ &= \int_A \gamma\frac{dP_\pi}{dP_\pi}\,dP_\pi \\ &= \int_A \gamma\frac{\pi dP_1 + (1-\pi)dP_0}{dP_\pi}\,dP_\pi,\end{aligned} \tag{A.1.3}$$

using the Radon-Nikodym theorem. Continuing from the last line,

$$\begin{aligned}\int_A E_\pi[\gamma|\mathcal{O}_t]\,dP_\pi &= \pi\int_A \gamma\,dP_1 + (1-\pi)\int_A \gamma\,dP_0 \\ &= \pi\int_A E_1[\gamma|\mathcal{O}_t]\,dP_1 + (1-\pi)\int_A E_0[\gamma|\mathcal{O}_t]\,dP_0.\end{aligned} \tag{A.1.4}$$

Now for $i = 0, 1$ compute,

$$\int_A E_i[\gamma|\mathcal{O}_t]\,dP_i = \int_A E_i[\gamma|\mathcal{O}_t]\,dP_i^t = \int_A E_i[\gamma|\mathcal{O}_t]\frac{dP_i^t}{dP_\pi^t}\,dP_\pi^t. \tag{A.1.5}$$

Combining A.1.2–5 yields,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\,dP_\pi = \pi\int_A E_1[\gamma|\mathcal{O}_t]R_t^1\,dP_\pi + (1-\pi)\int_A E_0[\gamma|\mathcal{O}_t]R_t^0\,dP_\pi,$$

and hence,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\,dP_\pi = \int_A \{\pi R_t^1 E_1[\gamma|\mathcal{O}_t] + (1-\pi)R_t^0 E_0[\gamma|\mathcal{O}_t]\}\,dP_\pi.$$

Since $A \in \mathcal{O}_t$ is arbitrary, it follows that,

$$E_\pi[\gamma|\mathcal{O}_t] = \pi R_t^1 E_1[\gamma|\mathcal{O}_t] + (1-\pi)R_t^0 E_0[\gamma|\mathcal{O}_t], \tag{A.1.6}$$

except on $\mathcal{O}_t$ sets of P_π-measure zero. ∎

Now suppose that one takes $\gamma = \theta\gamma'$, θ the binary random variable defined in the beginning, and γ' any $\mathcal{F}$-measurable random variable. Then note,

$$\int_A E_i[\theta\gamma'|\mathcal{O}_t]\,dP_i = \int_A \theta\gamma'\,dP_i = \int_{A\cap\{\theta=1\}} \gamma'\,dP_i \quad i = 0, 1,$$

and hence,

$$\int_A E_i[\theta\gamma'|\mathcal{O}_t]\,dP_i = \begin{cases} 0 & i=0 \\ \int_A \gamma'\,dP_i & i=1. \end{cases}$$

Thus, for this choice of γ A.1.6 becomes,

$$E_\pi[\theta\gamma'|\mathcal{O}_t] = \pi R_t^1 E_1[\gamma'|\mathcal{O}_t]. \qquad (A.1.7)$$

Now A.1.7 will be used to connect the *a posteriori* probability process to the likelihood ratio, where the *a posteriori* probability is defined as,

$$\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\} \qquad \pi \in [0,1], \qquad (A.1.8)$$

and the likelihood ratio is given by,*

$$\Lambda_t(\omega) = \frac{dP_1^t}{dP_0^t}(\omega) = E_0[\frac{dP_1}{dP_0}|\mathcal{O}_t] \quad \forall\omega \in \Omega, P_0 - a.s., \qquad (A.1.9)$$

assuming of course that $P_1 \ll P_0$. Here, $\{\mathcal{O}_t : t \geq 0\}$ takes on the role of a filtration on $(\Omega, \mathcal{F})$, either right-continuous or discrete (see section 2.1). Letting $\gamma' \equiv 1$ in A.1.7 one immediately obtains,

$$\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\} = E_\pi[\theta|\mathcal{O}_t] = \pi\frac{dP_1^t}{dP_\pi^t} \quad \pi \in (0,1). \qquad (A.1.10)$$

Using the Radon-Nikodym Theorem successively and observing that $P_1^t \ll P_0^t$, one obtains the following sequence of steps,

$$\begin{aligned} \pi_t = \pi\frac{dP_1^t}{\pi dP_1^t + (1-\pi)dP_0^t} &= \pi\frac{\frac{dP_1^t}{dP_0^t}}{1-\pi+\pi\frac{dP_1^t}{dP_0^t}} \\ &= \frac{\frac{\pi}{1-\pi}\Lambda_t}{1+\frac{\pi}{1-\pi}\Lambda_t} \\ &= \phi(\Lambda_t), \end{aligned} \qquad (A.1.11)$$

where the last line serves to define the mapping $\phi : [0,\infty] \longrightarrow [0,1]$. From A.1.11 one obtains,

$$\Lambda_t = \phi^{-1}(\pi_t) = \frac{1-\pi}{\pi}\frac{1-\pi_t}{\pi_t}. \qquad (A.1.12)$$

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* The symbol $\frac{dP^1}{dP^0}$ can be interpreted as the ratio of probabililty masses if necessary.

Finally, for completeness note that for $\pi = 1$ that,

$$\pi_t = E_1[\theta|\mathcal{O}_t] = \pi \cdot 1 = 1,$$

while for $\pi = 0$,

$$\pi_t = E_0[\theta|\mathcal{O}_t] = \pi \cdot R_t^1 = 0.$$

From A.1.11 and A.1.12 it is clear that threshold policies based upon $\{\pi_t : t \geq 0\}$ and $\{\Lambda_t : t \geq 0\}$ are equivalent in terms of optimality, only their thresholds differ.

APPENDIX II

In this appendix, it is shown that there exists a unique solution (a_*, b_*) to the system of equations,

$$r(b^-; a, b) = e(b); \qquad (A.2.1.a)$$

$$r'(a; a, b) = e'(a), \qquad (A.2.1.b)$$

with $r(\pi; a, b)$ and $e(\pi)$ as given in Theorem 4.2.1 (see also 4.2.30). First suppose that a solution (a_*, b_*) to A.2.1 exists, then from Theorem 1.2.1 and its corollary it follows that $r_*(\pi) = r(\pi; a_*, b_*) \in C^2(0, b_*)$. From this and Theorem 3.2.1 it follows that r_* is concave on $(0, b_*)$. This implies, in view of Theorem 3.2.2, Lemma 4.2.1, and A.2.1 that the conditions (C1), (C2), and (C3) hold. Finally, invoking Theorem 3.1.2 shows that the solution is unique. Thus, it is only necessary to show that at least one solution to A.2.1 exists.

The approach to accomplish this will be to show that for every $a \in [0, 1]$ there exists a $b_1(a) \in [0, 1]$, $b_1(a) > a$ such that A.2.1.a becomes,

$$r(b_1^-(a); a, b_1(a)) = e(b_1(a)). \qquad (A.2.2)$$

In addition, it will be shown that the mapping $a \mapsto b_1(a)$ is continuous and onto $(0, 1)$. Following that it will be demonstrated using A.2.1.b that there exists a continuous mapping, $a \mapsto b_2(a)$, satisfying,

$$r'(a; a, b_2(a)) = e'(a), \qquad (A.2.3)$$

with $b_2(0) = 1$, and such that the equation $b_2(a) = a$ has a solution for $0 < a < 1$. Consequently, the two curves $b_1(a), b_2(a)$ must cross, i.e., there exists an a_0 satisfying $b_1(a_0) = b_2(a_0)$. Choosing $b_* = b_1(a_0)$, and $a_* = a_0$ then yields a pair satisfying A.2.1.

To begin, from Theorem 4.2.1 there follows,

$$r(\pi; a, b) = d(\pi; b) + \bar{D}(\pi; b) + K(a, b)\bar{H}(\pi; b) \quad \forall \pi < b. \tag{A.2.4}$$

Letting $\pi \uparrow b$ and noting $N_b(b^-) = 0$, after some algebra one obtains,

$$r(b^-; a, b) = C(\lambda^1(1-b) + \lambda^0 b) + K(a, b)\left(\frac{1-b}{b}\right)^u (1-b)(\lambda^0 + \lambda^1), \tag{A.2.5}$$

where by definition $K(a, b)$ is

$$K(a, b) = \frac{e(a) - [d(a; b) + \bar{D}(a; b)]}{\bar{H}(a; b)}. \tag{A.2.6}$$

In the proof of Theorem 4.2.1, it was shown that $\bar{H}(\pi; b)$ and $[d(\pi; b) + \bar{D}(\pi; b)]$ are continuous for all $\pi < b$. Thus $K(a, b)$ is at least continuous in its first argument and one can verify albeit tedious that the mapping $a \mapsto b(a)$ defined implicitly by A.2.2 and A.2.5 is continuous. Now, for any $a \in [0, 1]$, suppose $b = b_1(a)$ is chosen so that $[\Sigma^{-1}b, b) \ni a$. Then from A.2.6 there follows,

$$K(a, b) = \frac{e(a) - C(\lambda^1(1-a) + \lambda^0 a)}{(1-a)(\lambda^0 + \lambda^1)} \left(\frac{a}{1-a}\right)^u. \tag{A.2.7}$$

After some simplification, using A.2.5-7 one obtains,

$$r(b^-; a, b) = C(\lambda^1(1-b) + \lambda^0 b) + [e(a) - C(\lambda^1(1-a) + \lambda^0 a)] \left(\frac{1-b}{1-a}\right) \left(\frac{a(1-b)}{(1-a)b}\right)^u. \tag{A.2.8}$$

From A.2.8 it is clear then that two solutions to A.2.2 are given by $b_1(0) = 0$, and $b_1(1) = 1$. Thus, it has been shown that $a \mapsto b_1(a)$ is continuous and onto $(0, 1)$.

The second half of the argument deals with A.2.1.b. First, suppose that for any $a \in [0, 1]$, the inequality $a < \Sigma^{-1}b$ is respected by $b = b_2(a)$. Then from A.2.4, one obtains the (right) derivative,

$$r'(\pi; a, b) = d'(\pi; b) + \bar{D}'(\pi; b) + K(a, b)\bar{H}'(\pi; b) \quad \pi < \Sigma^{-1}b. \tag{A.2.9}$$

Recall from Theorem 4.2.1 that, $U\bar{D}(\pi;b) \equiv U\bar{H}(\pi;b) \equiv 0$, and so,

$$r'(\pi;a,b) = d'(\pi;b) + \frac{u+\pi}{\pi(1-\pi)}\left[\bar{D}(\Sigma\pi;b) - \bar{D}(\pi;b) + K(a,b)\Big(\bar{H}(\Sigma\pi;b) - \bar{H}(\pi;b)\Big)\right]. \tag{A.2.10}$$

Substituting $\pi = a$ into A.2.10 and the result into A.2.1.b yields,

$$\bar{D}(\Sigma a;b) - \bar{D}(a;b) + K(a,b)\Big(\bar{H}(\Sigma a;b) - \bar{H}(a;b)\Big) = \frac{a(1-a)}{u+a}\Big(e'(a) - d'(a;b)\Big). \tag{A.2.11}$$

The last line can be simplified in two ways. First, a direct computation of the right derivative $d'(\pi;b)$ yields,

$$d'(\pi;b) = -C(\lambda^1 - \lambda^0)(1 + N_b(\pi)). \tag{A.2.12}$$

Also, from A.2.2 and A.2.5 there holds,

$$K(a,b) = \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{(1-b)(\lambda^0 + \lambda^1)}\left(\frac{b}{1-b}\right)^u. \tag{A.2.13}$$

Combining A.2.11-13 with a little algebra yields an equation whose left-hand side is,

$$(1-b)^{u+1}\left[\bar{D}(\Sigma a;b) - \bar{D}(a;b)\right] + \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{\lambda^0 + \lambda^1} b^u \left[\bar{H}(\Sigma a;b) - \bar{H}(a;b)\right], \tag{A.2.14}$$

and whose right-hand side is,

$$\frac{a(1-a)}{u+a}e'(a)(1-b)^{u+1} + \frac{a(1-a)}{u+a}C(\lambda^1 - \lambda^0)(1-b)^{u+1}(1 + N_b(a)). \tag{A.2.15}$$

Now notice that $(a,b) = (0,1)$ solves both sides (recall $\Sigma 0 = 0$). The only troublesome term is in A.2.15, $a(1-b)^{u+1}N_b(a)$, which goes to zero since,

$$\begin{aligned} 0 \le a(1-b)^{u+1}N_b(a) &< a(1-b)^{u+1}\big(\tilde{x}(b) - \tilde{x}(a)\big) \\ &= \tilde{a}(1-b)^{u+1}\log\left(\frac{b(1-a)}{(1-b)a}\right) \\ &\le \tilde{a}(1-b)^{u+1}\left(\frac{b(1-a)}{(1-b)a} - 1\right) \\ &= \tilde{b}(1-a)(1-b)^u - \tilde{a}(1-b)^{u+1} \overset{a\downarrow 0, b\uparrow 1}{\longrightarrow} 0. \end{aligned} \tag{A.2.16}$$

Thus far have shown $b_2(0) = 1$, and from A.2.10 it is clear that $a \mapsto b_2(a)$ is continuous, arguing as before. The last thing to show is that there exists an a_0 such that $0 < b_2(a_0) = a_0 < 1$, i.e., that the curve $b_2(a)$ intercepts the line $b = a$ in the (a, b)-plane. For if this is shown, the two curves $b_1(a)$ and $b_2(a)$ must cross, yielding a solution to the problem. To show $b_2(a_0) = a_0$ for some a_0, consider A.2.4 for all b in the vicinity of a. In fact, for any $a \in [0, 1]$, choose $b = b_2(a)$ to satisfy $[\Sigma^{-1}b, b) \ni a$, then from A.2.4

$$\begin{aligned} r(\pi; a, b) &= d(\pi; b) + K(a, b)\bar{H}(\pi; b) \\ &= C\big(\lambda^1(1-\pi) + \lambda^0\pi\big) + K(a, b)\left(\frac{1-\pi}{\pi}\right)^u (1-\pi)(\lambda^0 + \lambda^1), \end{aligned} \tag{A.2.17}$$

and so,

$$r'(\pi; a, b) = -C(\lambda^1 - \lambda^0) - K(a, b)\left(\frac{1-\pi}{\pi}\right)^u \left(\frac{u+\pi}{\pi}\right)(\lambda^0 + \lambda^1). \tag{A.2.18}$$

Substituting for $K(a, b)$ from A.2.13 under the condition $a \in [\Sigma^{-1}b, b)$ yields,

$$r'(\pi; a, b) = -C(\lambda^1 - \lambda^0) - \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{1-b}\left(\frac{b(1-\pi)}{(1-b)\pi}\right)^u \left(\frac{u+\pi}{\pi}\right). \tag{A.2.19}$$

Thus, setting $r'(a; a, b) = e'(a)$ and using A.2.19 one gets,

$$-a(1-b)C(\lambda^1 - \lambda^0) - (u+a)\Big[e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)\Big]\left(\frac{b(1-a)}{(1-b)a}\right)^u = a(1-b)e'(a). \tag{A.2.20}$$

Evaluating A.2.20 on the line $b = a$ gives,

$$-a(1-a)C(\lambda^1 - \lambda^0) - (u+a)\Big[e(a) - C\big(\lambda^1(1-a) + \lambda^0 a\big)\Big] = a(1-a)e'(a). \tag{A.2.21}$$

After some algebraic simplification A.2.21 can be rewritten,

$$a(u+1) - c = 0, \tag{A.2.22}$$

i.e., $a = \frac{c(\lambda^1 - \lambda^0)}{\lambda^1}$. So if $c < u + 1$, then a solution to A.2.1.b exists on the line $a = b$. In this case, there exists a solution (a_*, b_*) solving A.2.1, as argued previously. Notice that the condition $c < u + 1$ holds automatically for all $c \leq 1$. On the other hand, this condition can also always be obtained by the simple change of time scale. This completes the argument.

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